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Valter Moretti

Analytical Mechanics

Classical, Lagrangian and Hamiltonian Mechanics, Stability Theory, Special Relativity

Springer

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Preface

This textbook aims at introducing readers, primarily students enrolled in undergraduate Mathematics or Physics courses, to the topics and methods of classical Mathematical Physics, together with a mathematical formulation of the *theory of special relativity*. Special attention is devoted to *classical mechanics*, its *Lagrangian formulation*—including an introduction to *Lyapunov stability*, and to the *Hamiltonian formulation*, together with a few important results such as the *Liouville theorem* and *Poincaré's recurrence theorem*. The general purpose is to present the logical-mathematical structure of physical theories, by introducing them in an axiomatic way, and start from a limited number of physical assumptions. To do so, we use the tools of analysis and elementary differential geometry. Certain traditional topics—such as the detailed treatment of rigid bodies—are presented only in their foundational aspects, given that there are several excellent and modern textbooks on analytical mechanics that address this type of subject and more, both in depth and rigorously; for example, [BRSG16] and [Bis16]. We have, on the other hand, insisted on topics that have had a major impact on theoretical and mathematical physics beyond analytical mechanics, such as the Galilean symmetry (equivariance) of classical dynamics and the Poincaré symmetry of relativistic dynamics, or the broad relationship between symmetries and constants of motion, the independence of mathematical objects from choices (think reference frames) or the possibility of describing dynamics in a global way while still working in local coordinates.

This book is the result of 20+ years of teaching undergraduate mathematics and physics at Trento University, starting from the one-year course formerly known as *rational mechanics*, that nowadays is almost inevitably compressed into one semester to fit the current 3-year degree. Based on this experience, the text has been conceived to be flexible and thus adapt to different curricula and to the needs of a variety of students and instructors. Undergraduates may find the abstract formalism daunting at first, but they will definitely reap the rewards of our approach throughout. Instructors may select only certain materials, should they want to simplify matters, or follow the path we set out and take advantage of the more advanced sections.

The author's research work does not dwell on analytical mechanics nor classical mathematical physics, but rather mathematical and foundational aspects of quantum theories, both relativistic and non-relativistic. In a sense, this fact has been greatly beneficial to the preparation of the text (which was born out of lecture notes and has now become a structured, proper textbook). On the one hand, this expertise has prevented the excessive use of abstract formalism, which has become (alas) inadequate to the structure of current undergraduate courses, even though the book has kept to the highest didactical standards, appropriate for undergraduate classes in mathematics and physics. On the other hand, the author's interest in other areas of mathematical physics allowed for a critical understanding of the foundational aspects of classical mechanics, a perspective that permeates the entire text. Here mechanics is not seen as an isolated, albeit important, mathematical chapter of theoretical physics, but rather it is viewed as a source of some of the ideas that lie at the heart of modern theoretical and mathematical physics. For this reason, the mathematical formulation of the elementary notions of space, time, spacetime, frame systems, etc. has been addressed critically. We have provided the most suitable logical-mathematical description and clarified the limits beyond which mathematical formalism ceases to describe physical phenomenology. The author believes that, in mathematical physics, the *physical content* of a mathematical statement should always be made transparent. The mathematical tools employed to describe a physical phenomenon should be efficient, and should neither suggest nor say more than what Physics requires. Based on this, for example, it is imperative to explain why it is crucial that the topology used to describe classical Physics is *Hausdorff*; or to what end one assumes that spacetime has a smooth structure; what is the true physical meaning of the *law of inertia* once we forego the traditional tautological rhetoric; what it means, concretely even though in an idealised context, that a ruler is *ideal*; finally, what we should understand when we say, today, that the time and space of classical Physics are *absolute*, without giving up on the *Galilean invariance* that sanctions the equivalence of the rest spaces and temporal axes of all inertial frame systems. In the same spirit, as we consider the subject of great conceptual importance, the book discusses at length and with different levels of formality the formidable and mysterious relationship between dynamical symmetries and the associated constants of motion. Such a relationship has, on the one hand, proven to be so profound that it has *withstood all the revolutions of twentieth-century physics*, as we shall briefly see when we introduce the Lagrangian formalism in special relativity. On the other hand, this same relationship has become a key tool to *define* fundamental notions; for instance, linear and angular momenta, but also novel, more abstract quantities, contextually inaccessible to classical mechanics, such as *quantum mechanics* and the theory of *elementary particles*.

The first complement wrapping the book up offers a crash course on the theory of ODEs and systems of differential equations, on smooth manifolds as well. This compendium includes the most significant proofs, for instance the local and global existence and uniqueness theorem for ODE systems.

The second complement works out the mathematical structure of special relativity, introduced axiomatically in Chap. 10, starting from the physical principles that underpin it.

The Appendices summarise basic notions of analysis, point-set topology and differential geometry. The final section contains solutions and hints for solving the exercises proposed.

We should point out that the sequence in which various physical and mathematical concepts are presented does not usually correspond to their historical development, nor do the terms we use have the same meaning they had when they were first introduced. For example, Newtonian Mechanics, as discussed in Chap. 3, does not reflect the presentation in Newton's Principia but is the result of a modern approach, itself based on other people's earlier reworkings after Newton, like E. Mach. A critical historical reconstruction of the fundamental concepts of classical mechanics together with much information on the wealth of contributors, from Aristoteles to Hamilton, passing through Galilei, Newton, Euler and Lagrange, can be found in [Bis16]. The concise presentation of special relativity in Chap. 10 is completely geometrical in nature and does not delve into the difficult and genuinely physical problems such as the synchronisation of events at different places,

Prerequisites and Reference Textbooks

The backgrounds required for reading most of the material are differential and integral calculus in one and several variables, elementary notions of Geometry and Linear Algebra, basic facts from point-set Topology and the fundamentals of Physical Mechanics. The core theory of ODE systems should be known (Existence and uniqueness theorems and little else). In any case, the Complement (Chap. 14) contains all the technical results on systems of differential equations that are used in the book including proofs, with minor exceptions. The parts (chapters, section, propositions, remarks etc.) labelled AC ("Advanced Content") need more sophisticated mathematical tools, especially regarding Differential Geometry. At any arte, all technical notions are briefly recalled before they are invoked, and the Appendices contain a detailed summary with several proofs of the more advanced concepts. The only notions that are not recapped, and appear in a small number of applications in Chap. 12 and Appendix B, regard measure theory, for which we refer to [Rud78].

Among the books with a constructive didactical purpose are the already mentioned [Bis16], and [BRSG16]. As standard references we recommend [Gol50], [Arn92] and [FaMa02] for more advanced material, complements and exercises; their content clearly goes beyond any Italian undergraduate lecture course on Analytical Mechanics. Passing to the advanced texts, beside the classical [AbMa78], an excellent modern book dedicated to the geometric formulation of classical theories, in particular Hamiltonian ones, is the recent [RuSc13]. We finally recommend the abridged [Cardin15] as advanced text for readers interested in modern results on Symplectic Geometry and its applications to classical Mechanics. A superlative introductory textbook on Special (and General) Relativity is Rindler's [Rin06].

Notations and Conventions

(0) Sections, theorems, proofs and exercises marked with AC ("Advanced Content") are not fundamental to an introductory course, because they refer to more advanced (in particular, mathematical) topics based on the complement and the two appendices on differential geometry at the end of the text. These parts become instead important to those wishing to strengthen the formalism.

(1) – The symbol := means *equal, by definition, to*;
– $A \subset B$ includes the possibility $A = B$ (in other books our $\subset$ would be denoted by $\subseteq$);
– formulas of the type

$$\sum_k a_k + b \;,$$

where b does *not* depend on the summation index, should be interpreted as

$$b + \sum_k a_k \;,$$

unless otherwise stated.
– the Kronecker delta will be written as δ_{ij}, δ^{ij} or δ^i_j indifferently, all having the standard meaning: 0 if $i \neq j$ and 1 if $i = j$.

(2) Vectors in affine and Euclidean spaces or $\mathbb{R}^n$ are written in boldface, e.g. $\mathbf{u}$; the exceptions are vectors in Σ_t^{3N} and vector fields on manifolds (in Appendices A and B in particular) where a normal font is often used. In Chap. 10, dedicated to special relativity, we will not use boldface for vectors in Minkowski spacetime, except for pseudo-orthonormal triples.

(3) Products of vectors.

– The positive-definite inner product of vectors is always denoted by $\mathbf{v} \cdot \mathbf{u}$ and occasionally by $(\mathbf{v}|\mathbf{u})$.
– The indefinite inner product of special relativity is indicated by $g(V, F)$.
– The cross product is written $\mathbf{u} \wedge \mathbf{v}$. The same symbol is used to express the exterior product of forms, and the context should clarify which meaning $\wedge$ has.

(4) Almost everywhere the coordinates and components of (contravariant) vectors are written in the standard tensor notation with upper indices:

$$x^1, x^2, x^3, \ldots, x^n \,, \qquad \mathbf{v} = \sum_{j=1}^{n} v^j \mathbf{e}_j \,.$$

Matrices associated with linear operators follow the same tensorial convention; for example, the generic entry of an orthogonal matrix R is written $R^i{}_j$.

(5) In accordance with the standard tensorial notation, summation over certain indices will be expressed by pairs at different height, for instance

$$x'^j = \sum_{k=1}^{3} R^j{}_k x^k, \quad \frac{dx^j}{dt} = \sum_{k=1}^{2n} S^{jk} \frac{\partial \mathscr{H}(t, \mathbf{x}(t))}{\partial x^k}, \quad \omega = -\frac{1}{2} \sum_{i,j=1}^{2n} S_{ij} dx^i \wedge dx^j .$$

The only exceptions occur in a few sections in Chap. 5 and Sect. 7.3, where we will exclusively use lower indices to prevent the notation from becoming too cumbersome.

(6) The conventions used for the *symplectic matrix* are as follows:

$$S := \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right] ,$$

and

$$S = [S_{ij}]_{i,j=1,\ldots,2n} \,, \quad S^{ij} := S_{ij} \,.$$

(7) We will often denote Lagrangian and Hamiltonian coordinates in the following compact form

$$(t, q) := (t, q^1, \ldots, q^n) \,, \quad (t, q, \dot{q}) := (t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) \,,$$
$$(t, q, p) := (t, q^1, \ldots, q^n, p_1, \ldots, p_n) \,.$$

(8) The wedge product $\wedge$ between p-forms is defined so that:

$$dx^1 \wedge \cdots \wedge dx^p \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^p} \right) = 1 \,.$$

(9) If M is a differentiable manifold and we define on M a C^k vector field X, a C^k field of differential forms Ω or a function $f : M \to N$ of class $C^k(M; N)$, we implicitly assume that M (and N in the last case) are of class C^r with $r \geq k$.

(10) The *Hamiltonian flow* is denoted by various symbols throughout the book depending on the level of formality and on the context:

   (a) Φ for Hamiltonian systems on $\mathbb{R} \times \mathbb{R}^{2n}$,
   (b) $\Phi^{(Z)}$ for Hamiltonian systems on phase spacetime $F(\mathbb{V}^{n+1})$, where Z is the *Hamiltonian dynamic vector field*,
   (c) φ in the autonomous case on phase space $\mathbb{F}$, regarding the *Poincaré Theorem*,
   (d) $\varphi^{(W)}$ for autonomous Hamiltonian systems on symplectic manifolds, where W is a symplectic dynamic vector field.

   To make things worse $\Phi^{(Z)}$ is also employed to indicate the *dynamic vector field* on the *spacetime of kinetic states* $A(\mathbb{V}^{n+1})$ in the *Lagrangian formulation*. At any rate, there will be no confusion since the two contexts will always be kept distinct.

(11) In Chap. 10, the components of *four-vectors* in Minkowski frames are labelled by Greek indices. For example, V^μ with $\mu = 0, 1, 2, 3$. When we consider only the *spatial components*, we will use Roman letters instead, for instance V^a with $a = 1, 2, 3$. We shall not make use of the popular *summation convention for repeated indices*.

(12) We shall adopt the following conventions regarding real bilinear forms $a : V \times V \to \mathbb{R}$ on real vector spaces V. a is called **positive semi-definite** when $a(\mathbf{v}, \mathbf{v}) \geq 0$ for every $\mathbf{v} \in V$, and **positive definite** if it is positive semi-definite and $a(\mathbf{v}, \mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$.

   A recurring situation is that in which a is the bilinear form given by a linear map $A : V \to V$, meaning $a(\mathbf{u}, \mathbf{v}) := \mathbf{u} \cdot A\mathbf{v}$ where $\cdot$ is a positive-definite inner product on V and A is self-adjoint: $A\mathbf{v} \cdot \mathbf{u} = \mathbf{v} \cdot A\mathbf{u}$. Then a is positive semi-definite if and only if the eigenvalues of A are all non-negative; it is positive definite if and only if the eigenvalues of A are strictly positive.

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Trento, Italy
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Valter Moretti

General Framework of Analytical Mechanics

Analytical mechanics is first and foremost the mathematical formulation, or better, a collection of mathematical formulations, of classical mechanics produced from the eighteenth century to the present day in order to extend and make rigorous Newton's formulation of mechanics. Analytical mechanics is the attempt to isolate a possibly small number of definitions and axioms, expressible in the thorough language of geometry and analysis, from which one should deduce several properties of physical systems, especially their dynamics. The accent is therefore placed on the rigour and the mathematical conciseness, as is typical in *mathematical physics*, rather than on the physical relevance of results obtained with little care for mathematical rigour, which is the typical approach of *theoretical physics*. (The latter should by no means be regarded with contempt, considering it has produced not only fundamental physical theories, but novel mathematical ideas as well!) At the same time, the rigorous approach has eventually given rise to physically important theoretical results: it suffices to mention *Jacobi's theorem*, *Noether's theorem*, *Liouville's theorem* and *Poincaré's recurrence theorem*. Part of the methods and results ensuing from the classical setup have been subsequently subsumed under relativistic theories. The *theory of special relativity* may still be considered, from a certain perspective, a part of (evidently non-classical) analytical mechanics, and as such we treat it in this book.

From the point of view of physics, it is important to stress that the description of reality in terms of *classical* analytical mechanics has clear-cut limits of applicability, and generally speaking, it should be considered an approximation of some other deeper theory. Indeed, it is inadequate in at least two respects.

(1) The classical description no longer holds in the regime of relative velocities comparable to the speed of light/strong gravitational fields/cosmological distances and times. In these contexts, the most satisfying description known at the moment is given by the *Theory of Special Relativity*, which we shall introduce in Chap. 10 mathematically, and in Complement 15 under a more physical lens,

and by the *Theory of General Relativity*,[1] of which classical mechanics is an approximation. As we shall discuss in the sequel, axiomatically, the revolution initiated by relativity has shown how the classical metric structures (lengths and time intervals) are actually *specific to the chosen frame system*, but at the same time they form part of the metric structure of an *absolute* spacetime carrying special symmetry features (at least until we can neglect gravity, or describe it semi-classically) represented by the so-called *Lorentz-Poincaré group*. The ensuing spacetime geometry has proved to be the mathematical language necessary to deal with phenomena of general physical relevance, like the notion of *causality*. The implications of this novel viewpoint have been astonishingly fruitful and have influenced crucially the development of twentieth century physics. Relativity theory has built, jointly with quantum mechanics, the *language itself* and the *paradigm* of one hundred years of research work in theoretical physics. These theories are the foundations of the physical theories of the twenty-first century.

(2) The classical description stops being adequate, roughly speaking, also for microscopic systems (at the molecular scale or smaller). In such contexts the best description is provided by *quantum mechanics* (and at high energies, by *quantum field theory*), of which, once again, classical mechanics is an approximation. Whilst the mathematical language of relativistic theories is still the one of differential geometry, the mathematical language of quantum theories is that of *functional analysis* (*Hilbert spaces* and *operator algebras* in particular). Geometry still exists here, but is almost invariably infinite dimensional. The exception are the contexts such as *quantum information*, where finite-dimensional Differential Geometry still plays an important part.

Classical mechanics, on the other hand, works perfectly well for the most common applications, but not only those. It is remarkable to remind that the *Apollo* missions' enterprise that brought mankind to the moon was entirely conceived within the framework of classical mechanics: all models were built, and all calculations were performed, in a classical regime.

The general scheme of physical theories is far from completed, given that quantum theory and relativity theory do not amalgamate coherently. There are in particular several conceptual problems in trying to reconcile the quantum description with that of general relativity: at present we lack an exhaustive and coherent mathematical description of the physical structure of what is in existence.

[1] There remain unresolved issues in cosmology, also within relativistic theories, in particular regarding the so-called problem of *dark energy and dark matter*.

Contents