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A Quaternionic Bernstein Theorem

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Abstract. We prove a four-dimensional version of a Bernstein's theorem, with complex polynomials being replaced by quaternionic polynomials. Moreover, using an Almansi-type decomposition of polynomials, we formulate the quaternionic Bernstein's inequality in terms of four-dimensional zonal harmonics and Gegenbauer polynomials.

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1. Introduction

The famous Bernstein's inequality for complex polynomials (first established in this form by M. Riesz in 1914) states that:

Theorem. (A) *If* $p(z)$ *is a complex polynomial of degree* d *and* $\max_{|z|=1}|p(z)| = M$, *then* $|p'(z)| \leq d\,M$ *for* $|z|=1$, *with equality holding if and only if* $p(z)$ *is a multiple of the power* z^d.

Recently [15], Bernstein's inequality has been proved for quaternionic polynomials with coefficients on one side. The inequality in the complex case can be deduced from a more general theorem, proved by Bernstein [3] in 1930.

Theorem. (B) *Let* $p(z)$ *and* $q(z)$ *be two complex polynomials with degree of* $p(z)$ *not exceeding that of* $q(z)$. *If* $q(z)$ *has all its zeros in* $\{|z| \leq 1\}$ *and* $|p(z)| \leq |q(z)|$ *for* $|z|=1$, *then* $|p'(z)| \leq |q'(z)|$ *for* $|z|=1$.

It is then natural to pose the following question: *Is it possible to extend Theorem (B) to quaternionic polynomials?*

This short note gives an answer to this question. We show that a quaternionic version of Theorem (B) holds true only after imposing an assumption on the second polynomial (Theorem 2.1). We must require that the quaternionic polynomial $Q \in \mathbb{H}[X]$ on the right-hand side of the inequality has every coefficients belonging to a fixed commutative subalgebra of $\mathbb{H}$, i.e., to a

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isomorphic copy of $\mathbb{C}$. We also show in Proposition 2.7 that the assumption made on Q in Theorem 2.1 is necessary. This restricted version of the Bernstein Theorem is, however, sufficient to deduce, as in the complex case, the quaternionic Bernstein's inequality: if $P \in \mathbb{H}[X]$ is a quaternionic polynomial of degree d, then the sup-norms satisfy $\|P'\| \leq d\|P\|$ (Corollary 2.4).

In Sect. 3, we restate the inequality in terms of four-dimensional zonal harmonics and Gegenbauer polynomials. To obtain this form, we use results from [12] to obtain an Almansi-type decomposition of a quaternionic polynomial.

We refer the reader to [5,6,9] for definitions and properties concerning the algebra $\mathbb{H}$ of quaternions and many aspects of the theory of quaternionic *slice-regular* functions, a class of functions which includes polynomials and convergent power series, and more generally for *slice* functions. The ring $\mathbb{H}[X]$ of quaternionic polynomials is defined by fixing the position of the coefficients with respect to the indeterminate X (e.g., on the right) and by imposing commutativity of X with the coefficients when two polynomials are multiplied together (see, e.g., [11, Sect. 16]). Given two polynomials $P, Q \in \mathbb{H}[X]$, let $P \cdot Q$ denote the product obtained in this way. A direct computation (see [11, Sect. 16.3]) shows that if $P(x) \neq 0$, then

$$(P \cdot Q)(x) = P(x)Q(P(x)^{-1}xP(x)), \tag{1}$$

while $(P \cdot Q)(x) = 0$ if $P(x) = 0$. In particular, if P has real coefficients, then $(P \cdot Q)(x) = P(x)Q(x)$. In this setting, a (left) root or zero of a polynomial $P(X) = \sum_{h=0}^{d} X^h a_h$ is an element $x \in \mathbb{H}$, such that $P(x) = \sum_{h=0}^{d} x^h a_h = 0$.

A subset A of $\mathbb{H}$ is called *circular*, or *axially symmetric*, if, for each $x \in A$, A contains the whole set (a 2-sphere if $x \notin \mathbb{R}$, a point if $x \in \mathbb{R}$)

$$\mathbb{S}_x = \{pxp^{-1} \in \mathbb{H} \mid p \in \mathbb{H}^*\}, \tag{2}$$

where $\mathbb{H}^* := \mathbb{H} \setminus \{0\}$. In particular, for any imaginary unit $I \in \mathbb{H}$, $\mathbb{S}_I = \mathbb{S}$ is the 2-sphere of all imaginary units in $\mathbb{H}$. It is well known (see, e.g., [5, Sect. 3.3]) that if $P \not\equiv 0$, the zero set $V(P)$ consists of isolated points or isolated 2-spheres of the form (2).

2. A Bernstein-Type Theorem

Let $I \in \mathbb{S}$ and let $\mathbb{C}_I \subset \mathbb{H}$ be the real subalgebra generated by I, i.e., the complex plane generated by 1 and I. If $\mathbb{C}_I$ contains every coefficient of $P \in \mathbb{H}[X]$, then we say that P is a $\mathbb{C}_I$*-polynomial*. Every $\mathbb{C}_I$-polynomial P is *one-slice-preserving*, i.e., $P(\mathbb{C}_I) \subseteq \mathbb{C}_I$. If this property holds for two imaginary units I, J, with $I \neq \pm J$, then it holds for every unit and P is called *slice-preserving*. This happens exactly when all the coefficients of P are real.

Let $P(X) = \sum_{k=0}^{d} X^k a_k$ be a polynomial of degree $d \geq 1$ with quaternionic coefficients. Let $P'(X) = \sum_{k=1}^{d} X^{k-1} k a_k$ be the derivative of P. For every $I \in \mathbb{S}$, let $\pi_I : \mathbb{H} \to \mathbb{H}$ be the orthogonal projection onto $\mathbb{C}_I$ and $\pi_I^{\perp} = id - \pi_I$. Let $P^I(X) := \sum_{k=1}^{d} X^k a_{k,I}$ be the $\mathbb{C}_I$-polynomial with coefficients $a_{k,I} := \pi_I(a_k)$.

We denote by $\mathbb{B} = \{x \in \mathbb{H} \mid |x| < 1\}$ the unit ball in $\mathbb{H}$ and by $\mathbb{S}^3 = \{x \in \mathbb{H} \mid |x| = 1\}$ the unit sphere.

We recall that a quaternionic polynomial, as any slice-regular function, satisfies the maximum modulus principle [5, Theorem 7.1]. Let

$$\|P\| = \max_{|x|=1} |P(x)| = \max_{|x|\leq 1} |P(x)|$$

denote the sup-norm of the polynomial $P \in \mathbb{H}[X]$ on $\mathbb{B}$. Given $y \in \mathbb{S}^3$, let us denote

$$M_y(P) := \max_{z\in\mathbb{S}_y} |P(z)|, \quad m_y(P) := \min_{z\in\mathbb{S}_y} |P(z)|.$$

Theorem 2.1. (Bernstein-type theorem) *Let $P, Q \in \mathbb{H}[X]$ be two quaternionic polynomials with degree of P not exceeding that of Q. Assume that there exists $I \in \mathbb{S}$, such that Q is a $\mathbb{C}_I$-polynomial. If $V(Q) \subseteq \overline{\mathbb{B}}$ and $|P(x)| \leq |Q(x)|$ for $x \in \mathbb{S}^3$, then $|P'(x)| \leq |Q'(x)|$ for $x \in \mathbb{S}^3 \cap \mathbb{C}_I$. For every $x = \alpha + J\beta \in \mathbb{S}^3$, if P' is not identically zero on $\mathbb{S}_x$, it holds*

$$|P'(x)| \leq \frac{\sqrt{2}M_x(P')}{\sqrt{M_x(P')^2 + m_x(P')^2}} \max\{|Q'(x_I)|, |Q'(\overline{x}_I)|\} \tag{3}$$

with $x_I = \alpha + I\beta$. Moreover, it holds $\|P'\| \leq \sqrt{2}\|Q'\|$.

Before proving the theorem, we state a technical lemma about a norm estimate that holds for quaternionic polynomials and more generally for any continuous slice function.

Lemma 2.2. *Let $P \in \mathbb{H}[X]$, $y \in \mathbb{S}^3$ with P not identically zero on $\mathbb{S}_y$. Let $I \in \mathbb{S}$ be fixed. Then, it holds*

$$|P(x)| \leq \frac{\sqrt{2}M_y(P)}{\sqrt{M_y(P)^2 + m_y(P)^2}} \max\{|P(x_I)|, |P(\overline{x}_I)|\}$$

for every $x = \alpha + K\beta \in \mathbb{S}_y$, where $x_I = \alpha + I\beta \in \mathbb{S}_y \cap \mathbb{C}_I$.

Proof. Let $M := M_y(P)$, $m := m_y(P)$. We may assume that y is not real. Since P is a slice function, it can be expressed as $P(x) = P_s^o(x) + \operatorname{Im}(x)P_s'(x)$, where P_s^o and P_s' are constant functions on $\mathbb{S}_y$ (the *spherical value* and the *spherical derivative* of P, respectively [7, Sect. 3.3]). Let $\langle u, v\rangle$ denote the Euclidean scalar product of $u, v \in \mathbb{H}$. Then, for every $x \in \mathbb{S}_y$, it holds

$$|P(x)|^2 = C + 2\langle v, \operatorname{Im}(x)\rangle,$$

where $C = |P_s^\circ(y)|^2 + |\operatorname{Im}(y)|^2|P_s'(y)|^2$ and $v := P_s^\circ(y)\overline{P_s'(y)}$. If $v \in \mathbb{C}_J$, then we get as in [9, Lemma 5.3] that

$$M = \max_{\mathbb{S}_y\cap\mathbb{C}_J} |P(x)| \text{ and } m = \min_{\mathbb{S}_y\cap\mathbb{C}_J} |P(x)|.$$

Therefore, it holds $M^2 = C + 2|\operatorname{Im}(v)||\operatorname{Im}(y)|$, $m^2 = C - 2|\operatorname{Im}(v)||\operatorname{Im}(y)|$, whence $M^2 + m^2 = 2C$. Let $\tilde{x} \in \mathbb{S}_y \cap \mathbb{C}_I = \{x_I, \overline{x}_I\}$ be such that $\langle v, \operatorname{Im}(\tilde{x})\rangle \geq 0$. Then

$$\max\{|P(x_I)|^2, |P(\overline{x}_I)|^2\} = |P(\tilde{x})|^2 \geq C = \frac{M^2 + m^2}{2},$$

whence

$$M^2 \le \frac{2M^2}{M^2+m^2} \max\{|P(x_I)|^2, |P(\overline{x}_I)|^2\},$$

which is equivalent to the thesis. □

Example 2.3. A simple example illustrating Lemma 2.2 is given by the linear polynomial $P(X) = 2X - j - k$. Let $y = i$. Then, $\mathbb{S}_i = \mathbb{S}$ and $M_i(P) = \left|P(-(j+k)/\sqrt{2})\right| = 2+\sqrt{2}$, $m_i(P) = \left|P((j+k)/\sqrt{2})\right| = 2-\sqrt{2}$. The inequality of the Lemma is then

$$|P(x)| \le \frac{1+\sqrt{2}}{\sqrt{3}} \max\{|P(x_I)|, |P(\overline{x}_I)|\} \quad \text{for every } x \in \mathbb{S}.$$

Let I be orthogonal to $(j+k)/\sqrt{2}$; for example, $I = i$. Then

$$\max\{|P(x_i)|, |P(\overline{x}_i)|\} = |P(i)| = \sqrt{6},$$

showing that the constant $\frac{1+\sqrt{2}}{\sqrt{3}}$ is the best one in the estimate above.

Proof of Theorem 2.1. Let $\lambda \in \mathbb{H}$ with $|\lambda| > 1$ and set $R := Q - P\lambda^{-1} \in \mathbb{H}[X]$. The polynomials Q and $R^I = Q - (P\lambda^{-1})^I$ are $\mathbb{C}_I$-polynomials, and then, they can be identified with elements of $\mathbb{C}_I[X]$, with $\deg(R^I) \le \deg(Q)$. For every $x \in \mathbb{C}_I$, it holds

$$|R^I(x) - Q(x)| = |(P\lambda^{-1})^I(x)| = |\pi_I((P\lambda^{-1})(x))| \le |(P\lambda^{-1})(x)| = \frac{|P(x)|}{|\lambda|}.$$

If $x \in \mathbb{S}^3 \cap \mathbb{C}_I = \{x \in \mathbb{C}_I \mid |x| = 1\}$, then

$$|R^I(x) - Q(x)| \le \frac{|P(x)|}{|\lambda|} \le \frac{|Q(x)|}{|\lambda|} \le |Q(x)|. \tag{4}$$

In view of Rouché's Theorem for polynomials in $\mathbb{C}_I[X]$, R^I and Q have the same zeros in the disc $\{x \in \mathbb{C}_I \mid |x| < 1\}$. Moreover, if $|x| = 1$ and $Q(x) = 0$, the inequality (4) gives $R^I(x) = 0$. Since $\deg(R^I) \le \deg(Q)$ and $V(Q) \subseteq \overline{\mathbb{B}}$, we get that $V(R^I) \cap \mathbb{C}_I \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_I$. From the complex Gauss–Lucas Theorem, we get $V(R') \cap \mathbb{C}_I \subseteq V((R^I)') \cap \mathbb{C}_I \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_I$.

Now, let $x \in \mathbb{C}_I$ with $|x| > 1$ be fixed and define $\lambda := Q'(x)^{-1}P'(x) \in \mathbb{H}$. Observe that $Q'(x) \ne 0$ again from the complex Gauss–Lucas Theorem applied to the polynomial Q considered as element of $\mathbb{C}_I[X]$. If $|\lambda| > 1$, the polynomial $R = Q - P\lambda^{-1} \in \mathbb{H}[X]$ defined as above has zero derivative at x: $R'(x) = Q'(x) - P'(x)\lambda^{-1} = 0$, contradicting what obtained before. Therefore, it must be $|\lambda| \le 1$, i.e., $|P'(x)|/|Q'(x)| \le 1$ for all $x \in \mathbb{C}_I$ with $|x| > 1$. By continuity, $|P'(x)| \le |Q'(x)|$ for all $x \in \mathbb{C}_I$ with $|x| = 1$.

To prove (3), we apply Lemma 2.2 to P' and use the inequalities $|P'(x_I)| \le |Q'(x_I)|$, $|P'(\overline{x}_I)| \le |Q'(\overline{x}_I)|$. The last statement follows from a general property of slice functions (see again [9, Lemma 5.3]): since Q' is a $\mathbb{C}_I$-polynomial, its maximum modulus on the 2-sphere $\mathbb{S}_x$ is attained at one of the points $x_I = \alpha + I\beta$, $\overline{x}_I = \alpha - I\beta$ of the intersection $\mathbb{S}_x \cap \mathbb{C}_I$. □

Corollary 2.4. (Bernstein's inequality) *If $P \in \mathbb{H}[X]$ is a quaternionic polynomial of degree d, then $\|P'\| \le d\|P\|$.*

Proof. Let $M = \|P\|$ and apply the previous theorem to $P(X)$ and $Q(X) = MX^d$. Since Q is slice-preserving, the first inequality in the thesis of Theorem 2.1 holds for every $I \in \mathbb{S}$. □

Remark 2.5. The proof of Theorem 2.1 makes use of the complex Gauss–Lucas Theorem. One could hope to obtain a better estimate by means of a quaternionic Gauss–Lucas Theorem. Unfortunately, this last result is valid only for a small class of quaternionic polynomials, as it has been showed in [8].

The inequality of Corollary 2.4 is best possible with equality holding if and only if P is a multiple of the power X^d. One implication is immediate. If $P(X) = X^d a$, with $a \in \mathbb{H}$ and $d \geq 1$, then $\|P'\| = \|dX^{d-1}a\| = d|a| = d\|P\|$. We show the converse.

Proposition 2.6. *If $P \in \mathbb{H}[X]$ is a quaternionic polynomial of degree d, and $|P'(y)| = d\|P\|$ at a point $y \in \mathbb{S}^3$, then $P(X) = X^d a$, for an $a \in \mathbb{H}$ with $|a| = \|P\|$.*

Proof. We can assume that $P(X)$ is not constant. Let $b = P'(y)^{-1}$ and set $Q(X) := P(X)b = \sum_{k=1}^{d} X^k a_k$. Then, $Q'(y) = 1$, $\|Q\| = 1/d$ and $\|Q'\| \leq 1$. Let $I \in \mathbb{S}$, such that $\mathbb{C}_I \ni y$. Then

$$1 = Q'(y) = \textstyle\sum_k k y^{k-1} a_k = \pi_I(Q'(y)) = \sum_k k y^{k-1} \pi_I(a_k) = (Q^I)'(y).$$

If $x \in \mathbb{C}_I \cap \mathbb{S}^3$, it holds

$$\begin{aligned} \left|(Q^I)'(x)\right| &= \left|\sum_k k x^{k-1} \pi_I(a_k)\right| = \left|\pi_I\left(\sum_k k x^{k-1} a_k\right)\right| \leq \left|\sum_k k x^{k-1} a_k\right| \\ &= |Q'(x)| \leq 1. \end{aligned}$$

This means that the $\mathbb{C}_I$-polynomial Q^I, considered as an element of $\mathbb{C}_I[X]$, satisfies the equality in the classic Bernstein's inequality. The same inequality implies that

$$1 = \max_{x \in \mathbb{C}_I \cap \mathbb{S}^3} |(Q^I_{|\mathbb{C}_I})'(x)| \leq d \max_{x \in \mathbb{C}_I \cap \mathbb{S}^3} |Q^I_{|\mathbb{C}_I}(x)| \leq d\|Q\| = 1,$$

i.e., $\max_{x \in \mathbb{C}_I \cap \mathbb{S}^3} |Q^I_{|\mathbb{C}_I}(x)| = 1/d$. Therefore, the restriction of Q^I to $\mathbb{C}_I$ coincides with the function $x^d c$, with $c \in \mathbb{C}_I$, $|c| = 1/d$

$$Q^I(x) = \sum_{k=1}^{d} x^k \pi_I(a_k) = x^d c \text{ for every } x \in \mathbb{C}_I.$$

This implies that $\pi_I(a_d) = c$, $\pi_I(a_k) = 0$ for each $k = 1, \ldots, d-1$ and Q can be written as $Q(X) = X^d c + \widetilde{Q}(X)$, with the coefficients of $\widetilde{Q}$ belonging to $\mathbb{C}_I^{\perp} = \pi_I^{\perp}(\mathbb{H})$. When $x \in \mathbb{C}_I \cap \mathbb{S}^3$, $\widetilde{Q}(x) \in \mathbb{C}_I^{\perp}$, and then

$$\frac{1}{d^2} \geq |Q(x)|^2 = |x^d c|^2 + |\widetilde{Q}(x)|^2 = \frac{1}{d^2} + |\widetilde{Q}(x)|^2.$$

This inequality forces $\widetilde{Q}$ to be the zero polynomial, and then, $P(X) = Q(X)b^{-1} = X^d c b^{-1}$. □

We now show that in Theorem 2.1, the assumption on Q to be one-slice-preserving is necessary.

Proposition 2.7. *Let*

$$P(X) = (X-i)\cdot(X-j)\cdot(X-k), \quad Q(X) = 2X\cdot(X-i)\cdot(X-j).$$

Then, $V(Q) = \{0, i\} \subseteq \overline{\mathbb{B}}$ *and* $|P(x)| \leq |Q(x)|$ *for every* $x \in \mathbb{S}^3$*, but there exists* $y \in \mathbb{S}^3$*, such that* $|P'(y)| > |Q'(y)|$.

Proof. By a direct computation, we obtain

$$\begin{aligned} P(X) &= X^3 - X^2(i+j+k) + X(i-j+k) + 1, \\ Q(X) &= 2X^3 - 2X^2(i+j) + 2Xk, \\ P'(X) &= 3X^2 - 2X(i+j+k) + i - j + k, \\ Q'(X) &= 6X^2 - 4X(i+j) + 2k. \end{aligned}$$

Let $P_1(X) = X - k$, $Q_1(X) = 2X$, $P_2(X) = (X-j)\cdot P_1(X)$, $Q_2(X) = (X-j)\cdot Q_1(X)$. Then, $P(X) = (X-i)\cdot P_2(X)$ and $Q(X) = (X-i)\cdot Q_2(X)$. For every $x \in \mathbb{S}^3 \setminus \{j\}$, using formula (1), we get

$$|P_2(x)| = |x-j||(x-j)^{-1}x(x-j) - k| \leq 2|x-j| = |x-j||2x| = |Q_2(x)|.$$

Since $P_2(j) = Q_2(j) = 0$, the inequality holds also at j. From this, we obtain, for each $x \in \mathbb{S}^3 \setminus \{i\}$

$$\begin{aligned} |P(x)| &= |x-i||P_2((x-i)^{-1}x(x-i))| \leq |x-i||Q_2((x-i)^{-1}x(x-i))| \\ &= |Q(x)|. \end{aligned}$$

Since P and Q vanish at i, $|P(x)| \leq |Q(x)|$ for every $x \in \mathbb{S}^3$.

Let $y = \frac{1}{10}\left(1 + 9i + 4j - \sqrt{2}k\right) \in \mathbb{S}^3$. An easy computation gives

$$|P'(y)|^2 = \frac{7}{25}(5+\sqrt{2}) \simeq 1.80, \quad |Q'(y)|^2 = \frac{4}{25}(10 - 3\sqrt{2}) \simeq 0.92.$$

□

3. Bernstein Inequality and Zonal Harmonics

Since the restriction of a complex variable power z^m to the unit circumference is equal to $\cos(m\theta) + i\sin(m\theta)$, the classic Bernstein inequality for complex polynomials can be restated in terms of trigonometric polynomials. In this section, we show that a similar interpretation is possible in four dimensions, by means of an Almansi-type decomposition of quaternionic polynomials and its relation with zonal harmonics in $\mathbb{R}^4$.

Quaternionic polynomials, as any slice-regular function, are biharmonic with respect to the standard Laplacian of $\mathbb{R}^4$ [12, Theorem 6.3]. In view of Almansi's Theorem (see e.g. [1, Proposition 1.3]), the four real components of such polynomials have a decomposition in terms of a pair of harmonic functions. The results of [12] can be applied to obtain a refined decomposition of the polynomial in terms of the quaternionic variable.

Let $\mathcal{Z}_k(x, a)$ denote the real four-dimensional *(solid) zonal harmonic* of degree k with pole $a \in \mathbb{S}^3$ (see, e.g., [2, Ch.5]). The symmetry properties of zonal harmonics imply that $\mathcal{Z}_k(x,a) = \mathcal{Z}_k(x\overline{a},1)$ for every $x \in \mathbb{H}$ and any $a \in \mathbb{S}^3$. Moreover, it holds [12, Corollary 6.7(d)]

$$x^k = \widetilde{\mathcal{Z}}_k(x) - \overline{x}\,\widetilde{\mathcal{Z}}_{k-1}(x) \quad \text{for every } x \in \mathbb{H} \text{ and } k \in \mathbb{N}, \tag{5}$$

where $\widetilde{\mathcal{Z}}_k(x)$ is the real-valued zonal harmonic defined by $\widetilde{\mathcal{Z}}_k(x) := \frac{1}{k+1}\mathcal{Z}_k(x,1)$ for any $k \geq 0$ and by $\widetilde{\mathcal{Z}}_{-1} := 0$.

In the following, we will consider polynomials in the four real variables x_0, x_1, x_2, x_3 of the form $A(x) = \sum_{k=0}^{d} \widetilde{\mathcal{Z}}_k(x)a_k$, with quaternionic coefficients $a_k \in \mathbb{H}$. They will be called *zonal harmonic polynomials with pole 1*. All these polynomials have an axial symmetry with respect to the real axis: for every orthogonal transformation T of $\mathbb{H} \simeq \mathbb{R}^4$ fixing 1, it holds $A \circ T = A$.

Proposition 3.1. (Almansi-type decomposition) *Let $P \in \mathbb{H}[X]$ be a quaternionic polynomial of degree $d \geq 1$. There exist two zonal harmonic polynomials A, B with pole* 1*, of degrees d and $d-1$, respectively, such that*

$$P(x) = A(x) - \overline{x}B(x) \quad \textit{for every } x \in \mathbb{H}. \tag{6}$$

The restrictions of A and B to the unit sphere $\mathbb{S}^3$ are spherical harmonics depending only on $x_0 = \mathrm{Re}(x)$.

Proof. Let $P(X) = \sum_{k=0}^{d} X^k c_k$. Formula (6) follows immediately from (5) setting

$$A(x) = \sum_{k=0}^{d} \widetilde{\mathcal{Z}}_k(x)c_k \text{ and } B(x) = \sum_{k=0}^{d-1} \widetilde{\mathcal{Z}}_k(x)c_{k+1}.$$

The restriction of $\widetilde{\mathcal{Z}}_k(x)$ to the unit sphere $\mathbb{S}^3$ is equal to the Gegenbauer (or Chebyshev of the second kind) polynomial $C_k^{(1)}(x_0)$, where $x_0 = \mathrm{Re}(x)$ (see [12, Corollary 6.7(e)]). This property implies immediately the last statement. □

Remark 3.2. See [13,14] for an extension of the Almansi decomposition to polynomials or more generally slice-regular functions on quaternions and Clifford algebras.

Thanks to the previous decomposition, the quaternionic Bernstein inequality of Corollary 2.4 can be restated in terms of Gegenbauer polynomials $C_k^{(1)}(x_0)$. Let $d \in \mathbb{N}$. For any $(d+1)$-uple $\alpha = (a_0, \ldots, a_d) \in \mathbb{H}^{d+1}$, let $Q_\alpha : \mathbb{S}^3 \to \mathbb{H}$ be defined by

$$Q_\alpha(x) := \sum_{k=0}^{d} (C_k^{(1)}(x_0) - \overline{x}\,C_{k-1}^{(1)}(x_0))a_k$$

for any $x = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{S}^3$ (where we set $C_{-1}^{(1)} := 0$). Being the restriction to $\mathbb{S}^3$ of the quaternionic polynomial $P(X) = \sum_{k=0}^{d} X^k a_k$, which has biharmonic real components on $\mathbb{H}$, Q_α is a quaternionic valued *spherical biharmonic* of degree d (see, e.g., [10]).

Corollary 3.3. *Let* $\alpha=(a_0,\dots,a_d)$ *and* $\alpha'=(a_1,2a_2,\dots,ka_k,\dots,da_d,0)\in\mathbb{H}^{d+1}$. *Then, it holds*

$$\text{if}\quad |Q_\alpha(x)|=\left|\sum_{k=0}^{d}\left(C_k^{(1)}(x_0)-\overline{x}\,C_{k-1}^{(1)}(x_0)\right)a_k\right|\le M \text{ for every } x\in\mathbb{S}^3,$$

$$\text{then}\quad |Q_{\alpha'}(x)|=\left|\sum_{k=0}^{d-1}\left(C_k^{(1)}(x_0)-\overline{x}\,C_{k-1}^{(1)}(x_0)\right)(k+1)a_{k+1}\right|\le dM$$

for every $x\in\mathbb{S}^3$.

Proof. Let $P(X)=\sum_{k=0}^d X^k a_k$. From formula (5), it follows that the restriction of P' to the unit sphere is the spherical biharmonic $Q_{\alpha'}$. Corollary 2.4 permits to conclude. □

Remark 3.4. Let $P\in\mathbb{H}[X]$ be a polynomial with Almansi-type decomposition $P(x)=A(x)-\overline{x}B(x)$ and let $y=\alpha+J\beta\in\mathbb{S}^3$, $\alpha,\beta\in\mathbb{R},\beta>0$. Let $v=A(y)\overline{B(y)}$. It follows from general properties of slice functions [9, Lemma 5.3] that if $v\in\mathbb{R}$, then $|P|_{|\mathbb{S}_y}$ is constant, while if $v\notin\mathbb{R}$, then the maximum modulus of P on the 2-sphere $\mathbb{S}_y\subset\mathbb{S}^3$ is attained at the point $\alpha+I\beta$, with $I=\operatorname{Im}(v)/|\operatorname{Im}(v)|$, while the minimum modulus is attained at $\alpha-I\beta$. In principle, this reduces the problem of maximizing or minimizing the modulus of P on the unit sphere (or ball) to a one-dimensional problem.

Example 3.5. Consider the polynomial $P(X)=(X-i)\cdot(X-j)\cdot(X-k)$ of Proposition 2.7. Since the first four zonal harmonics are

$$\widetilde{\mathcal{Z}}_0(x)=1,\ \widetilde{\mathcal{Z}}_1(x)=2x_0,\ \widetilde{\mathcal{Z}}_2(x)=3x_0^2-x_1^2-x_2^2-x_3^2,$$
$$\widetilde{\mathcal{Z}}_3(x)=4x_0(x_0^2-x_1^2-x_2^2-x_3^2),$$

the Almansi-type decomposition of P is $P(x)=A(x)-\overline{x}B(x)$, with

$$\begin{aligned}A(x)&=\widetilde{\mathcal{Z}}_3(x)+\widetilde{\mathcal{Z}}_0(x)+(i+k)\left(\widetilde{\mathcal{Z}}_1(x)-\widetilde{\mathcal{Z}}_2(x)\right)-j\left(\widetilde{\mathcal{Z}}_1(x)+\widetilde{\mathcal{Z}}_2(x)\right)\\&=(1+4x_0^3-4x_0x_1^2-4x_0x_2^2-4x_0x_3^2)+(i+k)(2x_0-3x_0^2+x_1^2+x_2^2+x_3^2)\\&\quad-j(2x_0+3x_0^2-x_1^2-x_2^2-x_3^2),\\B(x)&=(3x_0^2-x_1^2-x_2^2-x_3^2)+i(1-2x_0)-j(1+2x_0)+k(1-2x_0)\end{aligned}$$

harmonic polynomials. Their restrictions to $\mathbb{S}^3$ are the spherical harmonics

$$\begin{aligned}A_{|\mathbb{S}^3}(x)&=(1-4x_0+8x_0^3)+i(1+2x_0-4x_0^2)+j(1-2x_0-4x_0^2)\\&\quad+k(1+2x_0-4x_0^2),\\B_{|\mathbb{S}^3}(x)&=(-1+4x_0^2)+i(1-2x_0)-j(1+2x_0)+k(1-2x_0).\end{aligned}$$

Following the observation made in Remark 3.4, since $\operatorname{Im}(A(y)\overline{B(y)})=4((\alpha-1)i+\alpha k)$, where $\alpha=\operatorname{Re}(y)$, $y\in\mathbb{S}^3$, one can find the 2-sphere $\mathbb{S}_y\subset\mathbb{S}^3$ where the maximum modulus of P is attained. A direct computation gives $\operatorname{Re}(y)=(1-\sqrt{19})/6\sim-0.56$ and the corresponding maximum value $\|P\|\sim4.70$ attained at the point $\tilde{y}=(1-\sqrt{19})/6-i(5+\sqrt{19})/12+k(1-\sqrt{19})/12$ of $\mathbb{S}^3$.

Some of the results presented in this note can be generalized to the general setting of real alternative *-algebras, where polynomials can be defined and share many of the properties valid on the quaternions (see [7]). The polynomials of Proposition 2.7 can be defined every time the algebra contains an Hamiltonian triple i, j, k, i.e., when the algebra contains a subalgebra isomorphic to $\mathbb{H}$ (see [4, Sect. 8.1]). This is true, e.g., for the algebra of octonions and for the Clifford algebras with signature $(0, n)$, with $n \geq 2$. In all such algebras, one can repeat the previous proofs and get the analog of Theorem 2.1, as well as of the Bernstein inequality (see also [15] for this last result).

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Declarations

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