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INCOMPLETE PAIRWISE COMPARISON MATRICES AND OPTIMIZATION TECHNIQUES

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Abstract

Pairwise comparison matrices (PCMs) play a key role in multi-criteria decision-making, especially in the analytic hierarchy process. It could be necessary for an expert to compare alternatives based on various criteria. However, for a variety of reasons, such as lack of time or insufficient knowledge, it may happen that the expert cannot provide judgments on all pairs of alternatives. In this case, an incomplete pairwise comparison matrix is formed.

In the first research part, an optimization algorithm is proposed for the optimal completion of an incomplete PCM. It is intended to numerically minimize a constrained eigenvalue problem, in which the objective function is difficult to write explicitly in terms of variables. Numerical simulations are carried out to examine the performance of the algorithm. The simulation results show that the proposed algorithm is capable of solving the minimization of the constrained eigenvalue problem.

In the second part, a comparative analysis of eleven completion methods is studied. The similarity of the eleven completion methods is analyzed on the basis of numerical simulations and hierarchical clustering. Numerical simulations are performed for PCMs of different orders considering various numbers of missing comparisons. The results suggest the existence of a cluster of five extremely similar methods, and a method significantly dissimilar from all the others.

In the third part, the filling in patterns (arrangements of known comparisons) of incomplete PCMs based on their graph representation are investigated under given conditions: regularity, diameter and number of vertices, but without prior information. Regular and quasi-regular graphs with minimal diameter are proposed. Finally, the simulation results indicate that the proposed graphs indeed provide better weight vectors than alternative graphs with the same number of comparisons. This research problem's contributions include a list of (quasi-)regular graphs with diameters of 2 and 3, and vertices from 5 up to 24.

Keywords: Pairwise comparisons, Incomplete pairwise comparison matrix, Analytic hierarchy process, Perron eigenvalue, Consistency, Diameter, Regular graph.

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List of Publications

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Nomenclature

AHP Analytic Hierarchy Process

ANP Analytic Network Process

BWM Best-Worst Method

DEMATEL DEcision MAking Trial and Evaluation Laboratory

ELECTRE ÉLimination Et Choix Traduisant la REalité

MACBETH Measuring Attractiveness by a Categorical Based Evaluation Technique

MAVT Multi-Attribute Value Theory

MCDA Multi-Criteria Decision Analysis

MCDM Multi-Criteria Decision Making

PAPRIKA Potentially All Pairwise RanKings of all possible Alternatives

PCM Pairwise Comparison Matrix

PROMETHEE Preference Ranking Organization METHod for Enriched Evaluation

Chapter 1

Introduction

1.1 Motivation

Pairwise comparison is a key concept in multi-criteria decision analysis (MCDA) (Greco et al., 2016; Triantaphyllou et al., 1998). The quantitative application of pairwise comparisons in decision analysis was pioneered by Thurstone (1927). Since then, Saaty expanded the idea of pairwise comparison through a widely-used multi-criteria decision-making method, the *analytic hierarchy process (AHP)* (Saaty, 1977, 1980).

In recent years, AHP has been regarded as an active area of research in multi-criteria decision analysis (Wallenius et al., 2008). The AHP is a methodology in MCDA for the theory of relative measurement, which utilizes a hierarchy for organizing and analyzing complex decision problems. The ultimate objective (goal) is defined at the top of the hierarchy. The criteria (if necessary, subcriteria) are positioned beneath the goal (in the middle level). All the alternatives are placed at the bottom level (Saaty, 2004).

The AHP uses a *pairwise comparison matrix (PCM)* to reveal the relative importance of alternatives and criteria. It is mainly useful to transform judgments into relative weights of importance for both objective and subjective assessments. It makes appropriate for decision problems when one wants to select the best alternative or to calculate the weights of the decision alternatives. The assumption is that decision-makers respond with a numerical answer to the query, "How many times is the ith alternative more important/preferable than the jth alternative?" and then incorporate it into a square matrix $\mathbf{A} = (a_{ij})_{n\times n}$ of size n, where $a_{ij} > 0$

and $a_{ji} = 1/a_{ij}$, $\forall i, j$. The goal of comparing alternatives in pairs is to find a priority vector (a vector of weights) in order to evaluate preferences or produce a ranking. In other words, if $X = \{x_1, x_2, \ldots, x_n\}$ is a set of alternatives, then the preference relation on $X \times X$ is represented by a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n \times n}$ and the aim is to determine a weight vector $\mathbf{w}$ on X:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mapsto \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \tag{1.1}$$

where a_{ij} indicates the relative measure of how much the alternative x_i is preferred to alternative x_j. A simple rule that governs how to convert a weight vector into a ranking vector is: the larger the weight, the higher location of the alternative in the ranking (Saaty, 1994).

A set of n alternatives requires a maximum of $n(n-1)/2$ paired comparisons to be made. For instance, experts perform fifteen comparisons for six alternatives, twenty-one comparisons for seven alternatives, and so on. The number of paired comparisons increases as n increases. Thus, when the number of alternatives is larger, the difficulty of comparing the whole $n(n-1)/2$ comparisons is inevitable. Due to this reason, several weighting methods were proposed (Lin, 2007).

Preferences can be modeled using partial information, the so-called *incomplete pairwise comparison matrices*. Some of the data/entries could be missing due to a number of reasons, such as decision makers' insufficient knowledge of particular alternatives or criteria, lack of time due to the abundance of alternatives, or the decision makers' unwillingness to provide all of the pairwise judgments (Harker, 1987a).

Furthermore, in the AHP, if there are n alternatives and m criteria, a decision-maker must compare the total of $\frac{n(n-1)}{2} \times m + \frac{m(m-1)}{2}$ paired comparisons. Therefore, utilizing all the paired comparisons is costly and time-consuming for rather large m and n. Carmone Jr et al. (1997) conducted an interesting Monte Carlo investigation of incomplete PCMs to ascertain how the AHP is affected by decreased sets of pairwise comparisons. The result suggests that half of the known $n(n-1)/2$ comparisons are sufficient to achieve a satisfactory result. This insight

led researchers to look for weighting methods using fewer pairwise comparisons. As a result, several methods have been proposed based on the following goals: (i) to derive the priority vectors. That means, a priority vector can be derived from an incomplete PCM without knowing the whole comparisons (e.g., Mazurek and Kułakowski (2022); Xu (2004b); Kułakowski (2020)); and (ii) to estimate the missing values in the incomplete PCMs using the data from the ones that are previously known. There are many methods to estimate the missing values in incomplete PCMs (e.g., Fedrizzi and Giove (2007); Bozóki et al. (2010); Zhou et al. (2018); Herrera-Viedma et al. (2007a)).

Incomplete PCMs have attracted the attention of many scholars since its introduction by Harker (1987c) due to a frequent and relevant issue when uncertainty is involved in the pairwise judgements. According to a recent study, incomplete PCMs are considered to be a hot research topic in AHP (Alrasheedi, 2019; Ureña et al., 2015). Researchers face a hurdle in developing an optimization algorithm that completes partial matrices into totally coherent matrices. Proposing an algorithm that produces minimal inconsistency is of our great interest.

Besides the optimal completion of incomplete PCMs, various features of incomplete PCMs and their applications have been explored in literature. For instance, inconsistency measures for incomplete PCMs (Kułakowski and Talaga, 2020; Szybowski et al., 2020), inconsistency thresholds (Ágoston and Csató, 2022a), applications for ranking purpose in sports (Temesi et al., 2023; Bozóki et al., 2016; Csató, 2013), and the optimal completion procedures and paths to determine which comparisons should be made and which ones can be omitted (Fedrizzi and Giove, 2013; Bozóki and Szádoczki, 2022).

The importance of pairwise comparisons is not only limited to AHP, it can also be applied in other multi-criteria decision-making methods, such as the ANP (Analytic Network Process) (Saaty, 2004), MAVT (Multi-Attribute Value Theory) (Keeney and Raiffa, 1993), the BWM (best-worst method (Rezaei, 2015), ELECTRE (ÉLimination Et Choix Traduisant la REalité) (Roy, 1968; Figueira et al., 2016), PROMETHEE (Preference Ranking Organization METHod for Enriched Evaluation) (Brans and Vincke, 1985; Brans and De Smet, 2016), MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) (Figueira et al., 2005), and PAPRIKA (Potentially All Pairwise RanKings of all

possible Alternatives) (Hansen and Ombler, 2008).

Multi-criteria decision-making (MCDM) methods have numerous real-world applications (Eltarabishi et al., 2020; Triantaphyllou et al., 1998; Vaidya and Kumar, 2006). In particular, many industrial engineering applications require an optimal decision based on the assessment of several alternatives in terms of various criteria. MCDM models provide a viable solution to effectively quantify the pertinent data (Triantaphyllou et al., 1995). Some of the industrial engineering applications that call for MCDM techniques include:

- Integrated manufacturing (Putrus, 1990; Yurdakul, 2004): For instance, to determine the best alternative (e.g., best computer system) in order to upgrade the computer system of a computer integrated manufacturing (CIM) facility with respect to the decision criteria like cost, performance characteristics (i.e., CPU speed, memory capacity, RAM, etc.), availability of software, maintenance, expendability, etc.

- Layout design (Cambron and Evans, 1991; Hadi-Vencheh and Mohamadghasemi, 2013; Eraslan et al., 2020): e.g., In the literature (Eraslan et al., 2020), AHP and ELECTRE methods have been used to evaluate an appropriate office layout design (three office alternatives are compared) for employees' job satisfaction at the work place based on the nine main criteria: working safety, dust, smell, light, working position, noise, working area, position of tools, and position of materials.

- Flexible manufacturing systems (FMSs) (Amini and Asoodar, 2016; Tabucanon et al., 1994; Özgürler et al., 2011; Maniya and Bhatt, 2011; Yadav and Jayswal, 2018; Wabalickis, 1988): e.g., (i) To select the most appropriate tractor using AHP based on five main criteria: price, maintenance, production power, ergonomic and model (Amini and Asoodar, 2016); and (ii) To select an appropriate machine (e.g., conventional machines or numerical control machines) for flexible manufacturing systems based on criteria, such as machine procedures, lead time, labor cost, and operation shift (Tabucanon et al., 1994).

- Evaluation of technology investment decisions (Ghasempour et al., 2019; Boucher and MacStravic, 1991): e.g., Prioritize projects (for selecting solar

plants site and technology) based on criteria, such as risk and return.

1.2 Goal and research questions

The goal of this thesis is to propose an efficient optimization algorithm, study the comparative analysis of some selected completion methods to identify similarities, highlight the differences, and make future recommendations, and provide a systematic filling in pattern designs for incomplete PCMs that the decision-maker decides which pairwise comparisons should be performed first in order to make a better optimal decision.

Answers to the following research questions will help to achieve this goal.

Question 1: Which optimization algorithm gives a better optimal completion of the Perron-Frobenius eigenvalue problem subject to interval constraints? How fast and efficient the proposed algorithm is? The partial information in the pairwise comparison matrices could be optimally estimated by employing various techniques of optimization. Optimizing the Perron-Frobenius eigenvalue problem appears to yield a small inconsistency (Shiraishi et al., 1998; Bozóki et al., 2010).

Question 2: How similar are the different completion methods with respect to the completed matrices between methods? Is there a significant difference in the completion methods? What comparison criteria are most appropriate? There are several completion methods in the literature. We aim at studying the similarities of the different methods from a numerical point of view.

Question 3: For incomplete PCMs, which filling in patterns are recommended to estimate the preferences in the best way under our assumptions (with given conditions: a number of vertices, regularity, and diameter)? In other words, how to provide a systematic collection of incomplete pairwise comparisons' patterns in an optimal way with properties/parameters: a number of criteria (vertices), regularity, and diameter? What are the challenges and limitations of this research? The main objective of this research problem is to examine which pairwise comparisons should be done first to get a decent estimate of the decision makers' preferences depending on the whole set of comparisons without prior information about the alternatives to be compared.

1.3 Outline of the thesis

The structure of the thesis is highlighted below in Figure 1.1. Chapter 1 presents the motivation, specific goals and objectives of the research, as well as the research questions that the thesis aims to answer. Chapter 2 presents an overview of the state of the art for complete and incomplete preference relations in multi-criteria decision making. This chapter discusses the concepts and theories related to multi-criteria decision making, including the different types of preference relations, incomplete pairwise comparison matrices, and various optimization techniques (methods) for deriving weights and completing incomplete matrices. It aims at presenting a foundation for the research conducted in later chapters of the thesis. Chapter 3 summarizes the research results and their contributions: Section 3.1 (Publication I) summarizes the implementation of Nelder-Mead algorithm on the constrained eigenvalue minimization problem. Section 3.2 (Publication II) summarizes the comparative study of eleven completion methods for incomplete PCMs. The similarities and the differences between completion methods are discussed from the numerical point of view through the hierarchical cluster analysis. Section 3.3 (Publication III) summarizes the filling in pattern designs of incomplete PCMs, to determine the optimal set of paired comparisons, using parameters: regularity, minimal diameter and number of vertices. Finally, Chapter 4 concludes the thesis by discussing the limitations of the research and presenting potential directions for future research.

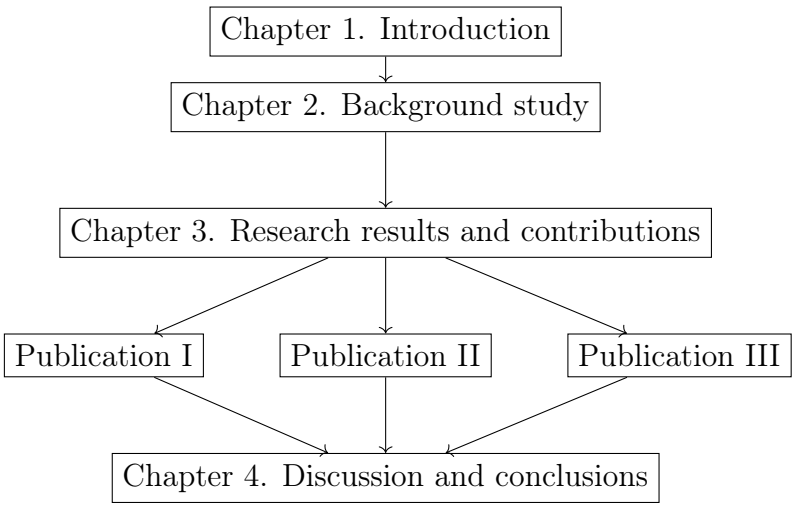


Figure 1.1: Structure of the thesis

Chapter 2

Background study

This chapter presents (in)complete preference relations, decision-making process, different completion methods for incomplete preference relations, and terminologies for the optimal filling in strategies related to the research problems from the state of the art.

2.1 Decision-making theory

Decision-making is a natural process that all humans can engage in for everyday life, which differs from a simple choice (choosing the fastest route to school/work or the best meal selection) to more complex ones that require considerable thought. For instance, one can describe a house/flat he/she wants to rent in terms of the rental cost, size/area, distance from work, etc. Hence, making a choice gets more difficult when multiple alternatives and criteria are being considered (see Example 1). Decision-making under conditions of various, usually conflicting criteria in the light of ranking or selection between alternatives (including people) is referred to *multi-criteria decision-making (MCDM)*, also known as *multi-criteria decision analysis (MCDA)* (Xu and Yang, 2001; Mardani et al., 2015).

Example 1 *Consider the following decision problem that contains five alternatives (Flat 1, Flat 2, Flat 3, Flat 4, Flat 5) with respect to four criteria (size, rental cost, distance from work and number of bedrooms) in order to choose/rent an ideal flat in Trento, Italy. This example is intended to show the readers that AHP has been suggested for a flat selection problem and its hierarchy is shown*

in Figure 2.1.

	Size of flat (in m^2)	**Rental cost (in Euro)**	**Distance from work (in km)**	**Number of bedrooms**
Flat 1	40	600	7	1
Flat 2	45	550	10	1
Flat 3	50	620	6	1
Flat 4	60	800	5	2
Flat 5	70	1000	0.3	2

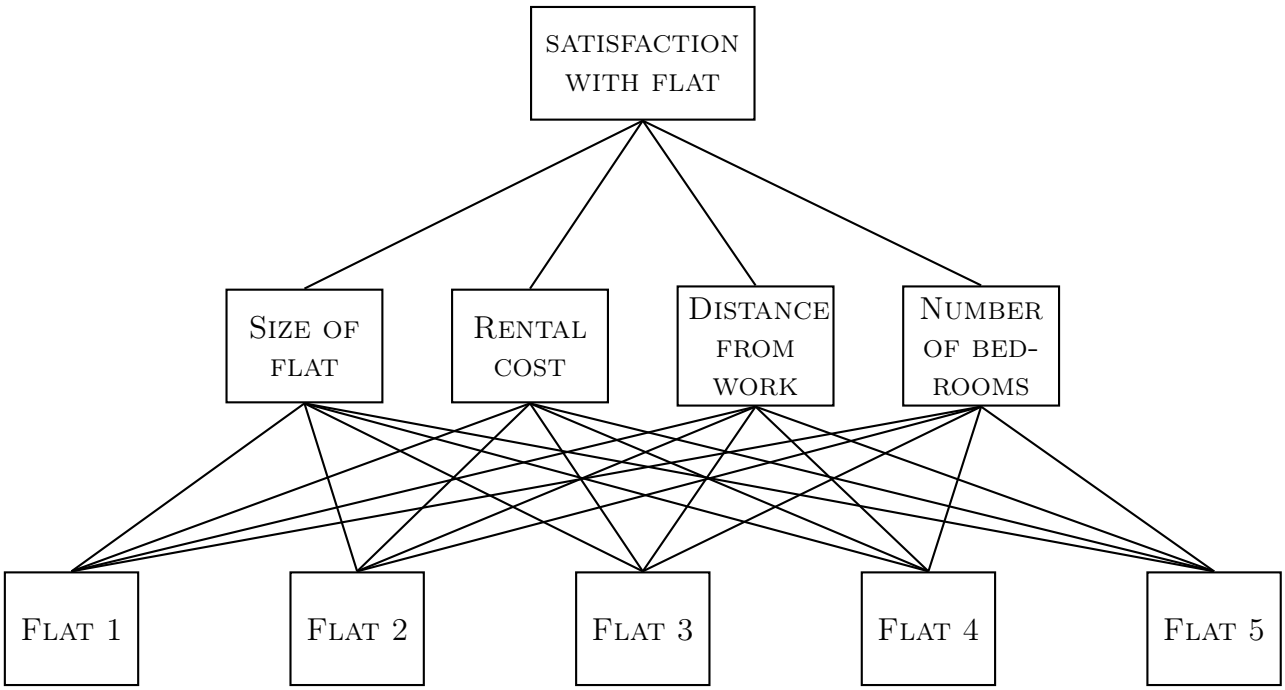


Figure 2.1: Hierarchy for the flat selection problem

The concept of a decision process is often connected to terminologies, such as *decision-maker*, *decision analysis*, *alternatives* and *criteria*. A *decision-maker* can be a person or entity responsible for making a decision, or in charge of a decision-making process, whereas *decision analysis* is a structured process to make a rational decision with the aid of mathematical models, i.e., to provide the decision-maker with a strategy for assessing the prospective effects of a choice using scientific methods. A set of alternatives can be evaluated or ranked using a set of criteria. *Alternatives* are options that a decision-maker can compare to each other, whereas *criteria* are the objects or attributes that a decision-maker uses to evaluate the preferences of these alternatives (Brunelli, 2011).

Multi-criteria decision analysis is often applied in various settings such as industrial engineering (Triantaphyllou et al., 1995), business analytics (Yalcin et al.,

2022), government agencies (Kurth et al., 2017) and healthcare (Marsh et al., 2017). The commonality across all MCDA applications is that they all require ranking or selecting alternatives by taking into account a number of different criteria together.

History of MCDA

The practice of MCDM/A is assumed to have its roots in an issue that required mediation/arbitration or negotiation, which is related to the life of King Solomon (1011–931 BC) (Köksalan et al., 2013). Solomon was a man of great wisdom and made his kingdom of Israel very wealthy. In the bible, a famous example of Solomon's judgement was reported as: Solomon resolved a quarrel with wisdom when two women came to him and that argued over who was the true mother of a baby. One mother's baby passed away in the middle of the night after being crushed by her rolling over on it while she slept. Each claimed to be the mother of the living child. He recommended cutting the living child in half with a sword, each woman to receive half. Then, the true mother was revealed to Solomon because she was willing to hand over her child to the lying woman, as sad a decision as it was. Solomon then ruled to give the child to the compassionate woman, claiming her to be the child's true mother.

The foundation for contemporary decision analysis probably trace back to the famous American statesman Benjamin Franklin by his well-known decision-making process in a letter sent to Joseph Priestley (London, 1772). In his letter, first, he advises that the pros and cons should be described as they occur over a period of time (during three or four days) consideration rather than all at a single time. He then weighs them. Third, he determines the balance after assigning weights and makes a decision accordingly. The other modern decision analysis goes back to the formal analysis of utility theory by von Neumann and Morgenstern (1940s), albeit many other significant early contributions can be found (Köksalan et al., 2011, 2013). Furthermore, the development of decision analysis in its current form was largely influenced by the work of Raiffa, Schlaifer, and Howard (1950s) (Rellstab, 1992). The acronym MCDM (Multiple Criteria Decision-Making) became more well-known by the work of Stanley Zionts in 1979 as a result of his article: "MCDM – If not a Roman numeral, then what?" (Zionts, 1979).

MCDA software tools are quite helpful for applications when there are many criteria and alternatives, and the weighting methods are more sophisticated. There are software tools based on pairwise comparisons, such as ExpertChoice (Forman et al., 1983) and SuperDecisions (Adams and Saaty, 2003). Nowadays, MCDA software is mostly supported by web-based programs (e.g., PriEsT and FITradeoff), which are free software tools, and other resources can be located on the website of International Society on MCDM (Softwares, 2023) and in the literature (Weistroffer and Li, 2016). Note that MCDM and MCDA can be used interchangeably.

2.1.1 Multi-criteria decision-making methods

Making decisions may include a lot of data and can be challenging. Thus, MCDM methods can be helpful in such situations. We need decision-making methods to avoid or reduce the chance of making poor decisions. The advantage of employing decision-making methods include the following (Kulakowski, 2020; Figueira et al., 2016):

- analyzing the situation and identifying the possible alternatives and their consequences;
- organizing and/or structuring the decision making-process to improve coherence among alternatives, objectives and goals;
- proposing a common framework by identifying the actors, stakeholders and experts;
- developing recommendations based on the results of the computational procedures and the decision process;
- participating in the legalization of the ultimate decision.

There are many MCDA methods listed in the literature with their own characteristics (Cinelli et al., 2022, 2020; Triantaphyllou et al., 1998). One may find 205 and more MCDA methods on the website of "MCDA Methods Selection Software (MCDA-MSS)" that aids experts in answering: Which MCDA method is the most suitable for a given decision-making problem?

MCDA methods can be classified as *compensatory* and *outranking* methods (Hwang

and Yoon, 1981). Compensatory methods compute aggregated global weights for each alternative (e.g., AHP (Saaty, 1977) and TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) (Hwang and Yoon, 1981)). In contrast, outranking methods result in ordinal ranking without calculating global weight or global utility from the comparison of multiple alternatives (e.g., ELECTRE (ÉLimination Et Choix Traduisant la REalité) (Roy, 1968; Figueira et al., 2016) and PROMETHEE (Preference Ranking Organization METHod for Enriched Evaluation) (Brans and Vincke, 1985; Brans and De Smet, 2016)). In this strategy, alternatives that do not meet a specific criterion are eliminated.

Furthermore, in the field of *multi-criteria decision analysis*, there has been two schools: namely, the *European school* and the *American school* (Lootsma, 1990). For historical reasons, the European school is also known as the *French school*, which is due to the proposal of ELECTRE, PROMETHEE and related methods mainly in the French-speaking-regions. ELECTRE (Roy, 1968), PROMETHEE (Brans and Vincke, 1985) and other several outranking methods represent the European school, whereas Saaty's AHP (Saaty, 1977) and MAVT (Keeney and Raiffa, 1993) represent the American school. Both schools are focused on the same issue: the assessment of a limited number of alternatives (pairwise comparisons) in light of a limited number of conflicting criteria by a single decision-maker. However, they use different kinds of subjective judgments. For instance, the AHP can always be used to determine the preferred alternative and to rank the available alternatives. In contrast, ELECTRE is often unable to completely rank the available alternatives. It is even sometimes unable to choose the alternative that is most preferred (Lootsma, 1990).

In addition, unlike other methods (including MAVT and TOPSIS), AHP and ANP work well with *intangibles*. The AHP allows one to build scales for tangibles and intangibles so that priorities can be derived. It enables the development of the relative scales of measurement required for measuring intangibles, which are prevalent in many scientific fields and multi-criteria decision-making (Saaty, 2008b; Saaty and Shang, 2011). A comparative analysis of some MCDM methods (including both schools) was conducted by (Triantaphyllou, 2000; Wallenius et al., 2008; Macharis et al., 2004; Fernandes et al., 2015).

Nevertheless, this thesis emphasizes the American school or AHP approach and

is not aimed at comparing the MCDM methodologies with AHP.

2.1.2 AHP and Operations Research

To answer the question: Which academic field does AHP fall under?, some authors believe that it is found at the intersection of decision analysis and operations research (Brunelli, 2015). In fact, decision analysis is a sub-field of decision-making theory based on mathematical methods and is intended to help decision-makers in selecting from a set of predetermined alternatives (Keeney and Raiffa, 1993), while the development of mathematical methods for making optimal decisions is at the core of *operations research* (Triantaphyllou et al., 1998). According to Merriam-Webster dictionary, *Operations research* is defined as *the application of scientific and especially mathematical methods to the study and analysis of problems involving complex systems.* As long as AHP looks for advanced mathematical methods for the study and analysis of complex decision problems, it is fair to say that AHP belongs to the field of *operations research.*

2.2 Preference relations

Preference relations are frequently used in decision-making models to represent information for decision-making problems. A preference relation can be described mathematically as follows.

Definition 1 (Preference relation (Alonso et al., 2004)) *A preference relation R on the set $X = \{x_1, \ldots, x_n\}$ is characterized by a function $\mu_R : X \times X \to D$, where D is the domain preference degrees' representation given by the decision maker for each pair of alternatives.*

In this section, the three preference relations: multiplicative PCMs, additive PCMs and reciprocal relations (or fuzzy preference relations) are presented in details, albeit there are several kinds of preference relations (Xu, 2007b).

2.2.1 Multiplicative pairwise comparison matrices

Pairwise judgments can be represented into a mathematical framework termed as a *pairwise comparison matrix (PCM)* when applied to a reference set X of cardinality n, where $X = \{x_1, x_2, \ldots, x_n\}$.

Comparisons can be made using a numerical scale that expresses how much more significant or dominant one element is over another element according to the criterion to which they are being compared. (Saaty, 1980) adopted a discrete scale $\{1/9, \ldots, 1/2, 1, 2, \ldots, 9\}$ in order to associate between the verbal judgements and numerical values for the matrix entries a_{ij} of $\mathbf{A}$ for all $i, j = 1, 2, \ldots, n$, because of human cognitive limits (Miller, 1956). According to Miller, the human brain cannot process more than *seven plus or minus two* alternatives at once. Different scales, other than Saaty's scale, have been proposed in literature, such as transitive scale by Ji and Jiang (2003), balanced scale by Pöyhönen et al. (1997) and others (Dodd et al., 1995; Lootsma, 1993; Salo and Hämäläinen, 1997). Nevertheless, in this thesis, the Saaty fundamental scale is chosen and its associated verbal expressions are presented in Table 2.1.

Table 2.1: Saaty's fundamental scale and verbal expression

Scale	Linguistic Expression (scale meaning)
1	Equal importance
2	Weak or slight importance
3	Moderate importance
4	Above moderate
5	Strong importance
6	Above strong
7	Very strong importance
8	Very, very strong
9	Extreme importance
$1.1 - 1.9$	If the alternatives are close together.
Reciprocals of the above number	If alternative x_i is assigned to the above number when compared to alternative x_j, then alternative x_j has the reciprocal value when compared with alternative x_i.

Definition 2 (Multiplicative pairwise comparison matrix (Saaty, 1977)) *A positive and reciprocal real matrix* $\mathbf{A} = (a_{ij})_{n \times n}$, *i.e.* $a_{ij} > 0$ *and* $a_{ji} = 1/a_{ij}$ *for all* $i, j = 1, 2, \ldots, n$, *is known as a* pairwise comparison matrix (PCM). *From the reciprocity property, it follows that* $a_{ii} = 1$.

Intuitively, entry a_{ij} in $\mathbf{A}$ indicates the ratio between two positive weights w_i and w_j such that $a_{ij} \approx w_i/w_j$, where w_i and w_j, are the weights corresponding to the ith and the jth elements of the reference set, respectively.

Example 2 *Consider the following multiplicative PCM* $\mathbf{A}$ *that was collected*

from an audience of roughly 30 people, using consensus to make each judgment, about drink consumption in the USA, to decide "Which drink is consumed more in the USA and how much more than another drink?" (Saaty, 2008a).

$$\mathbf{A} = \begin{array}{c} \\ \text{Coffee} \\ \text{Wine} \\ \text{Tea} \\ \text{Beer} \\ \text{Soda} \\ \text{Milk} \\ \text{Water} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{Coffee} & \text{Wine} & \text{Tea} & \text{Beer} & \text{Soda} & \text{Milk} & \text{Water} \end{array} \\ \begin{bmatrix} 1 & 9 & 5 & 2 & 1 & 1 & 1/2 \\ 1/9 & 1 & 1/3 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/5 & 2 & 1 & 1/3 & 1/4 & 1/3 & 1/9 \\ 1/2 & 9 & 3 & 1 & 1/2 & 1 & 1/3 \\ 1 & 9 & 4 & 2 & 1 & 2 & 1/2 \\ 1 & 9 & 3 & 1 & 1/2 & 1 & 1/3 \\ 2 & 9 & 9 & 3 & 2 & 3 & 1 \end{bmatrix} \end{array}$$

According to Saaty's interpretations, for instance, $a_{12} = 9$ indicates that "Coffee" is extremely preferred to "Wine". It can also be verified that using the right eigenvector method (Saaty, 1977) the priority vector from the decision matrix is

$$\mathbf{w} = (0.18, 0.02, 0.04, 0.12, 0.19, 0.13, 0.33).$$

Thus, the preferences are: Water $\succ$ Soda $\succ$ Coffee $\succ$ Milk $\succ$ Beer $\succ$ Tea $\succ$ Wine. And "Water" is much more consumed than other drinks. The symbol "$\succ$" means "is preferred to".

Reciprocity is a necessary condition to meet the minimal coherence of $\mathbf{A}$. However, in order to ascertain whether the pairwise judgments contained in a PCM reflect rational choices, it is useful to ask experts to discriminate with a suitable level of rationality. Hence, the consistency condition is linked to the condition of rationality.

2.2.2 Ordinal and cardinal consistency

The consistency condition is important to assure the trustworthiness/reliability of the final decision (ranking result) (Brunelli, 2018). There are two types of consistencies: *ordinal consistency* and *cardinal consistency*.

Ordinal consistency implies the transitivity of logical inferences (Tversky, 1969). For instance, if alternative x_i is preferred to alternative x_h and x_h is preferred to alternative x_j, then alternative x_i should be preferred to alternative x_j. Or equivalently, it can be seen as the three-way transitivity of preferences. That means, a pairwise comparison matrix $\mathbf{A}$ is said to be ordinally consistent if there

is no three-way cycle in a PCM $\mathbf{A}$ (Gass, 1998). Meaning that, the preferences $x_i \succ x_h \succ x_j \succ x_i$ form a three-way cycle, and hence transitivity fails. Here, the symbol "$\succ$" represents a *strict preference* relation. In fact, there are other levels of rationality (coherence conditions) (Brunelli and Cavallo, 2020).

Definition 3 (Ordinal consistency (Wu and Tu, 2021)) *A pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be ordinally consistent if the following four conditions are satisfied for all $i, h, j = 1, 2, \ldots, n$:*

(i) $a_{ih} > 1$ *and* $a_{hj} > 1 \implies a_{ij} > 1$*;*

(ii) $a_{ih} > 1$ *and* $a_{hj} = 1 \implies a_{ij} > 1$*;*

(iii) $a_{ih} = 1$ *and* $a_{hj} > 1 \implies a_{ij} > 1$*; and*

(iv) $a_{ih} = 1$ *and* $a_{hj} = 1 \implies a_{ij} = 1$.

Note that, based on the relationships between alternatives, $a_{ih} > 1$ can be interpreted as $x_i \succ x_h$, and $a_{ih} = 1$ means $x_i \sim x_h$, where the symbols "$\succ$" and "$\sim$" refer to "preference" and "indifferent" relations, respectively.

Cardinal consistency indicates the intensity-based rationality of preferences. In this thesis, we stick to cardinal consistency.

Definition 4 (Cardinal consistency (Saaty, 1977)) *A pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is* consistent *if and only if the cardinal transitivity $a_{ij} = a_{ih}a_{hj}$ holds for all $i, j, h = 1, 2, \ldots, n$. A PCM $\mathbf{A}$ is called* inconsistent *when there is at least one triple of indices (i, j, h) such that the previous relation is not satisfied.*

Proposition 1 (Saaty (1977)) *There exists a vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ such that*

$$a_{ij} = \frac{w_i}{w_j},\ \forall i, j \tag{2.1}$$

if and only if the multiplicative PCM $\mathbf{A} = (a_{ij})_{n\times n}$ is consistent.

If $\mathbf{A}$ is consistent, it is straightforward to calculate a vector $\mathbf{w}$ using the geometric mean method (Crawford and Williams, 1985):

$$w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}},\ i = 1, 2, \ldots, n.$$

Example 3 *Let us consider the following pairwise comparison matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 9 & 5 \\ 1/3 & 1 & 3 & 1 \\ 1/9 & 1/3 & 1 & 1/3 \\ 1/5 & 1 & 3 & 1 \end{pmatrix}.$$

It can be verified that $\mathbf{A}$ *is not consistent as* $a_{13}a_{34} \neq 5 = a_{14}$*. Consequently, there is no vector* $\mathbf{w} = (w_1, \ldots, w_n)$ *that satisfies Eq.* (1)*. However, if* $a_{14} = 3$*, instead of* 5*, then* $\mathbf{A}$ *becomes consistent and the normalized vector* $\mathbf{w} = (\frac{9}{16}, \frac{3}{16}, \frac{1}{16}, \frac{3}{16})$.

Proposition 1 is an alternative characterization of cardinal consistency, where the weights w_i and w_j can be derived through the row geometric means (see related papers, e.g., Crawford (1987); Bortot and Pereira (2013); Csató (2019)).

Definition 5 (Inconsistency Ratio) *Saaty's inconsistency ratio (CR) (Saaty, 1977, 1980) is defined by*

$$CR(\mathbf{A}) = \frac{\lambda_{max}(\mathbf{A}) - n}{(n-1)RI_n} \tag{2.2}$$

where $\lambda_{max}(\mathbf{A})$ *is the Perron eigenvalue of the complete PCM* $\mathbf{A}$*, and* RI_n *is the random inconsistency index associated with a matrix order* n *as stated in Table 2.2, which is the average CI of random matrices using Saaty's discrete scale.*

Table 2.2: Random index RI_n values (Alonso and Lamata, 2006)

n	3	4	5	6	7	8	9	10
RI_n	0.5245	0.8815	1.1086	1.2479	1.3417	1.4056	1.4499	1.4854

CR serves as a measure of inconsistency for the judgments in PCMs. The estimated inconsistency of the judgments increases with the large value of CR. In addition, because it is unavoidable that some degree of inconsistency must be accepted, Saaty (1977) proposed a 10% cut-off rule to specify the set of acceptable PCMs. It follows that PCMs with $CR < 0.1$ are accepted. Otherwise, PCMs must be revised.

Another well-known inconsistency index in the literature is the Koczkodaj inconsistency index (denoted as $KI(\mathbf{A})$) (Koczkodaj, 1993; Duszak and Koczkodaj,

1994), which is defined as

$$KI(\mathbf{A}) = \max\left\{1 - \min\left\{\frac{a_{ih}a_{hj}}{a_{ij}}, \frac{a_{ij}}{a_{ih}a_{hj}}\right\}, 1 \leq i < h < j \leq n\right\} \tag{2.3}$$

where $\mathbf{A} = (a_{ij})_{n\times n}$ is a complete PCM. Koczkodaj's index measures the highest local inconsistency of a PCM (that is, the maximum of all possible triads). There are $n(n-1)(n-2)/6$ possible triads for a given $n \times n$ PCM (Duszak and Koczkodaj, 1994). Furthermore, the comparative analysis of Saaty's and Koczkodaj's inconsistencies was provided by Bozóki and Rapcsák (2008).

There are many different levels of inconsistency indices, such as geometric consistency index (Crawford and Williams, 1985; Aguarón and Moreno-Jiménez, 2003), Golden-Wang consistency index (Golden and Wang, 1989), Barzilai's relative error index (Barzilai, 1998), Harmonic consistency index (Stein and Mizzi, 2007), and others (Brunelli, 2018). Moreover, many authors have examined the issue of inconsistency measures in the literature (Bortot et al., 2022; Cavallo, 2020; Mazurek, 2022; Brunelli, 2017; Fedrizzi et al., 2002; Bortot and Pereira, 2013; Brunelli and Fedrizzi, 2015).

2.2.3 Additive pairwise comparison matrices

Additive PCMs are alternative representations of pairwise comparisons to model decision-making problems.

Definition 6 (Additive pairwise comparison matrix (Barzilai, 1998)) *A matrix* $\mathbf{P} = (p_{ij})_{n\times n}$ *is said to be* additive pairwise comparison matrix *if and only if the preferences satisfy the condition of reciprocity (additive reciprocity)*

$$p_{ij} = -p_{ji},\ \forall i, j.$$

Note that $p_{ij} \in (-\infty, \infty),\ \forall i, j$, i.e., the domain of additive PCM is the real line. Preference representation $p_{ij} > 0$ indicates that x_i is strictly preferred to x_j, whereas $p_{ij} < 0$ expresses the opposite preference and the diagonal entries $a_{ii} = 0$ means that x_i and x_j are indifferent.

The consistency condition (additive consistency) is represented by

$$p_{ih} = p_{ij} + p_{jh},\ \forall i, j, h. \tag{2.4}$$

The additive PCM is consistent if and only if the consistency condition is satisfied. Moreover, there exists a priority vector $\mathbf{u} = (u_1, \ldots, u_n)$ such that

$$p_{ij} = u_i - u_j,\ \forall i, j \tag{2.5}$$

if and only if the additive PCM is consistent. More information related to deriving priority vectors from complete additive PCMs can be found in the literature (Barzilai and Golany, 1990).

Even if the thesis solely focuses on multiplicative PCMs and reciprocal relations, it is important to see the relations between the three types of preference representations. Transformation of multiplicative PCMs into additive PCMs could be possible through the logarithmic function $\log_b(a_{ij}) = p_{ij}$, $\forall i, j$ and for $b > 0$, and the converse transformation (additive into multiplicative) is done using the inverse: $b^{p_{ij}} = a_{ij}$, $\forall i, j$ (Brunelli, 2015, p. 47).

Example 4 *Consider a consistent multiplicative PCM:*

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{4} & 2 \\ 4 & 1 & 8 \\ \frac{1}{2} & \frac{1}{8} & 1 \end{pmatrix}.$$

Then, one can find the following skew-symmetric additive matrix by using the base-2 logarithm:

$$\mathbf{P} = \begin{pmatrix} \log_2 1 & \log_2 \frac{1}{4} & \log_2 2 \\ \log_2 4 & \log_2 1 & \log_2 8 \\ \log_2 \frac{1}{2} & \log_2 \frac{1}{8} & \log_2 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}.$$

Moreover, the additive matrix is consistent (i.e., $-2 + 3 = 1$*) from Eq.* (2.4)*.*

2.2.4 Reciprocal relations

Another well-known preference relation type is called *reciprocal relations* which is also named as *fuzzy preference relations* (Tanino, 1984; Fedrizzi and Giove,

2007; Cabrerizo et al., 2014; Herrera-Viedma et al., 2007b).

Definition 7 (Reciprocal relation) *A matrix* $\mathbf{R} = (r_{ij})_{n\times n}$ *is called a reciprocal relation if* $r_{ij} \in [0,1]$ $(i,j = 1,2,\ldots,n)$ *and* $r_{ji} = 1 - r_{ij}$. *It follows that,* $r_{ii} = 0.5$, $\forall i$.

Intuitively, r_{ij} indicates the degree of preference between alternatives i and j. That means, the value $r_{ij} = 0.5$ refers to an indifference between alternative i and alternative j, $r_{ij} > 0.5$ refers to alternative i is preferred over alternative j, $r_{ij} = 1$ refers to alternative i is absolutely preferred over alternative j and $r_{ij} = 0$ has the reverse meaning.

Definition 8 (Additively consistent reciprocal relation (Tanino, 1984)) *A reciprocal relation* $\mathbf{R} = (r_{ij})_{n\times n}$ *is additively consistent if and only if*

$$r_{ij} = r_{ih} + r_{hj} - 0.5, \;\; i,j,h = 1,\ldots,n, \tag{2.6}$$

that is, Tanino's (Tanino, 1984) additive consistency property holds.

Proposition 2 (Tanino (1984)) *There exists a priority vector* $\mathbf{v} = (v_1,\ldots,v_n)$ *such that*

$$r_{ij} = \frac{1}{2} + \frac{1}{2}(v_i - v_j) \text{ and } |v_i - v_j| \leq 1, \;\; \forall i,j \tag{2.7}$$

if and only if the reciprocal relation $\mathbf{R}$ *is additively consistent.*

The transformation function that converts multiplicative PCMs into reciprocal relations, and vice versa, is presented below.

A function $g : [1/9, 9] \to [0,1]$ such that

$$r_{ij} = g(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}) \tag{2.8}$$

converts multiplicative PCMs into reciprocal relations and its inverse $a_{ij} = g^{-1}(r_{ij}) = 9^{2r_{ij}-1}$ gives back to an equivalent representation of multiplicative PCM (Fedrizzi, 1990).

Example 5 *Consider again the multiplicative PCM in Example 4. Its transfor-*

mation into reciprocal relations is

$$\mathbf{R} = \begin{pmatrix} 0.5 & \frac{1}{2}(1+\log_9 \frac{1}{4}) & \frac{1}{2}(1+\log_9 2) \\ \frac{1}{2}(1+\log_9 4) & 0.5 & \frac{1}{2}(1+\log_9 8) \\ \frac{1}{2}(1+\log_9 \frac{1}{2}) & \frac{1}{2}(1+\log_9 \frac{1}{8}) & 0.5 \end{pmatrix} \approx \begin{pmatrix} 0.5000 & 0.1845 & 0.6577 \\ 0.8155 & 0.5000 & 0.9732 \\ 0.3423 & 0.0268 & 0.5000 \end{pmatrix}.$$

Note that *multiplicative consistency* is an alternative type of consistency condition for reciprocal relations.

Definition 9 (Multiplicatively consistent reciprocal relation) *A reciprocal relation $\mathbf{R} = (r_{ij})_{n\times n}$ is multiplicatively consistent if it satisfies the following condition (Shimura, 1973):*

$$\frac{r_{ih}}{r_{hi}} = \frac{r_{ij}}{r_{ji}}\frac{r_{jh}}{r_{hj}}, \ \forall i, j, h. \tag{2.9}$$

An equivalent consistency condition to (2.9), which has a similar structure to the consistency of multiplicative PCMs and additive PCMs, was formulated by Chiclana et al. (2008) as follows:

$$r_{ih} = \frac{r_{ij}r_{jh}}{r_{ij}r_{jh} + (1-r_{ij})(1-r_{jh})}, \ \forall i, j, h. \tag{2.10}$$

Proposition 3 (Tanino (1984)) *There exists a priority vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that*

$$r_{ij} = \frac{w_i}{w_i + w_j}, \ \forall i, j \tag{2.11}$$

if and only if the reciprocal relation $\mathbf{R}$ is multiplicatively consistent.

The simplest method for deriving a priority vector $\mathbf{w}$ associated with reciprocal relations can be found in (Fedrizzi and Brunelli, 2010). The procedure is similar to the geometric mean method for multiplicative PCMs. That is,

$$w_i = \sqrt[n]{\prod_{j=1}^{n}\left(\frac{r_{ij}}{1-r_{ij}}\right)}, i = 1, 2, \ldots, n. \tag{2.12}$$

A multiplicative PCM $\mathbf{A} = (a_{ij})_{n\times n}$ can be transformed into multiplicatively

consistent reciprocal relations $\mathbf{R} = (r_{ij})_{n\times n}$ using the function

$$r_{ij} = \frac{a_{ij}}{1 + a_{ij}},\ \forall i, j \tag{2.13}$$

and its inverse $a_{ij} = \frac{r_{ij}}{1-r_{ij}}$, or $a_{ij} = \frac{r_{ij}}{r_{ji}}$ gives an equivalent representation of multiplicative PCM.

Example 6 *Considering the consistent PCM in (4), one can find the following multiplicatively consistent reciprocal relation using the transformation (2.13):*

$$\mathbf{R} \approx \begin{pmatrix} 0.5000 & 0.2000 & 0.6667 \\ 0.8000 & 0.5000 & 0.8889 \\ 0.3333 & 0.1111 & 0.5000 \end{pmatrix}.$$

Its corresponding priority vector using (2.12) *is* $\mathbf{w} = (0.7938, 3.1749, 0.3606)$.

An alternative approach to preference representation was that described by Cavallo and D'Apuzzo (2009) and refers to a general group structure. The idea is to create a unified framework and offer a useful consistency index so that various PCM types can be defined. However, this approach is not considered in the thesis.

2.2.5 Other types of preference relations

In addition to the above preference relations, there are various types of preference relations that have been proposed in the literature, such as

(i) Linguistic preference relations (Xu, 2005a; Cabrerizo et al., 2013; Li et al., 2018; Liu et al., 2017; Wu et al., 2015; Herrera and Herrera-Viedma, 2000): Let $L = \{\ell_\alpha | \alpha = -t, \ldots, -1, 0, 1, \ldots, t\}$ be a finite and totally ordered discrete term set with odd and small cardinality, such as 7 and 9, where ℓ_α represents a linguistic variable. For instance, a set L with 9 terms could be provided as follows: $L = \{\ell_{-4} =$ extremely poor; $\ell_{-3} =$ very poor; $\ell_{-2} =$ poor; $\ell_{-1} =$ slightly poor; $\ell_0 =$ fair; $\ell_1 =$ slightly good; $\ell_2 =$ good; $\ell_3 =$ very good; $\ell_4 =$ extremely good$\}$. Usually L satisfies the following conditions: (i) the set is ordered ($\ell_\alpha > \ell_\beta$ if $\alpha > \beta$); (ii) there is a negation operator ($neg(\ell_t) = \ell_{-t}$). To preserve all the given information, the discrete term set L is extended to a continuous term set $\hat{L} = \{\ell_\alpha | \alpha \in [-p, p]\}$, where

$p(p > t)$ is a sufficiently large positive integer. The term $\ell_\alpha \in L$ is called original linguistic term, and $\ell_\alpha \in \hat{L}$ is called a virtual linguistic term (Xu, 2005a). A decision maker often uses the original linguistic terms to evaluate alternatives and the virtual linguistic terms in operation. Consider any two linguistic virtual terms $\ell_\alpha, \ell_\beta \in \hat{L}$, then their operational laws are expressed as follows (Xu, 2004a): (i) $\ell_\alpha \oplus \ell_\beta = \ell_{\alpha+\beta}$; (ii) $\lambda\ell_\alpha = \ell_{\lambda\alpha}, \lambda \in [0, 1]$. A *linguistic preference relation* on the set X can be represented by a decision matrix $\mathbf{B} = (b_{ij})_{n\times n} \subset X \times X$ with

$$b_{ij} \in \hat{L},\ b_{ij} + b_{ji} = \ell_0,\ b_{ii} = \ell_0,\ i, j = 1, \ldots, n$$

where b_{ij} denotes the preference degree or intensity of the alternative x_i over x_j. The preference degrees in the linguistic preference relation $\mathbf{B}$ are interpreted as: $b_{ij} > \ell_0$ indicates alternative x_i is preferred to x_j, $b_{ij} < \ell_0$ indicates alternative x_j is preferred to x_i and $b_{ii} = \ell_0$ indicates indifference between x_i and x_j.

(ii) Intuitionistic fuzzy preference relations (Xu, 2007a; Xu and Liao, 2015; Szmidt and Kacprzyk, 1998): An intuitionistic preference relation $\dot{\mathbf{R}}$ was defined by Xu (2007a) as a preference structure, whose entries are intuitionistic fuzzy numbers, denoted as $\dot{r}_{ij} = (u_{ij}, v_{ij})$ with $u_{ij}, v_{ij} \in [0, 1],\ 0 \leq u_{ij} + v_{ij} \leq 1,\ u_{ij} = v_{ji},\ u_{ii} = v_{ii} = 0.5,\ i, j = 1, \ldots, n$. u_{ij} indicates the preference degree of the alternative x_i to alternative x_j (i.e., membership degree); v_{ij} indicates the degree to which alternative x_i is not preferred to alternative x_j (i.e., non-membership degree). The hesitancy degree (also known as indeterminacy degree) is represented by $\pi_{ij} = 1 - u_{ij} - v_{ij}$.

(iii) Hesitant preference relations (Zhang and Wu, 2014a,b; Xia et al., 2013; Xia and Xu, 2011, 2013):

(a) Hesitant multiplicative preference relations: This type of preference relation was developed by Zhang and Wu (2014a) to represent hesitant information using a $1 - 9$ Saaty's scale. A decision maker may hesitate between some numerical preferences in $[1/9, 9]$ due to time constraints and/or lack of knowledge. In this case, the preference degree of the alternatives x_i over x_j can be represented by a hesitant element (an entry of a decision ma-

trix), such as $\{8, 9\}$, $\{1/8, 1/7\}$, $\{1/9, 1/8, 1/5\}$ and so on. For instance, in group-decision making, some experts may provide 7, some provide 8 and others provide 9. The experts may not be able to convince one another to change their views. In such cases, the degree of alternatives can be represented by a matrix element $\{7, 8, 9\}$ (Zhang and Wu, 2014a,b).

(b) Hesitant fuzzy preference relations: The preference degrees of alternatives are represented using the $0.1 - 0.9$ scale, instead of $1 - 9$ scale, in hesitant fuzzy sets (Xia et al., 2013; Xia and Xu, 2011, 2013).

(iv) Triangular fuzzy multiplicative preference relations (Van Laarhoven and Pedrycz, 1983): A preference relation $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be *triangular fuzzy multiplicative preference relation* if it satisfies the following conditions: $a_{ij} = [a_{ij}^L, a_{ij}^M, a_{ij}^U]$, $a_{ij}^L a_{ji}^U = a_{ij}^M a_{ji}^M = a_{ij}^U a_{ji}^L = 1$, $a_{ij}^U \geq a_{ij}^M \geq a_{ij}^L > 0$, $a_{ii}^L = a_{ii}^M = a_{ii}^U = 1$ for all $i, j = 1, \ldots, n$, where a_{ij} indicates the triangular fuzzy preference degree of the alternative x_i over x_j, a_{ij}^L, a_{ij}^M and a_{ij}^U are the lower, medium and upper limits of a_{ij}, respectively.

(v) Triangular fuzzy additive preference relations (Xu, 2002): a triangular fuzzy additive preference relation $\mathbf{R} = (r_{ij})_{n\times n}$ satisfies the following conditions. $r_{ij} = [r_{ij}^L, r_{ij}^M, r_{ij}^U]$, $r_{ij}^L + r_{ji}^U = r_{ij}^M + r_{ji}^M = r_{ij}^U + r_{ji}^L = 1$, $r_{ij}^U \geq r_{ij}^M \geq r_{ij}^L \geq 0$, $r_{ii}^L = r_{ii}^M = r_{ii}^U = 0.5$ for all $i, j = 1, \ldots, n$, where r_{ij} indicates the triangular fuzzy preference degree of the alternative x_i over x_j, r_{ij}^L, r_{ij}^M and r_{ij}^U are the lower, medium and upper limits of r_{ij}, respectively.

(vi) Interval-valued preference relations:

(a) Interval fuzzy preference relations (Xu, 2004c; Genç et al., 2010): A preference relation $\mathbf{R} = (r_{ij})_{n\times n}$ is said to be an *interval fuzzy preference relation* if $r_{ij} = [r_{ij}^L, r_{ij}^U]$, $r_{ji} = [r_{ji}^L, r_{ji}^U]$, $r_{ij}^L + r_{ji}^U = r_{ij}^U + r_{ji}^L = 1$, $r_{ij}^U \geq r_{ij}^L \geq 0$, $r_{ii}^L = r_{ii}^U = 0.5$ for all $i, j = 1, \ldots, n$, where r_{ij} represents the interval-valued preference degree of the alternative x_i over x_j, r_{ij}^L and r_{ij}^U are the lower and upper limits of r_{ij}, respectively.

(b) Interval multiplicative preference relations (Saaty and Vargas, 1987; Salo and Hämäläinen, 1992): a preference relation $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be an *interval multiplicative preference relation* if $a_{ij} = [a_{ij}^L, a_{ij}^U]$, $a_{ij}^L a_{ji}^U = a_{ij}^U a_{ji}^L = 1$, $a_{ij}^U \geq a_{ij}^L > 0$, $a_{ii}^L = a_{ii}^U = 1$ for all $i, j = 1, \ldots, n$, where a_{ij}

represents the interval-valued preference degree of the alternative i over j, a_{ij}^L and a_{ij}^U are the lower and upper limits of a_{ij}, respectively.

(vii) Self-confident fuzzy preference relation: A preference relation $\mathbf{A} = (a_{ij}, s_{ij}^*)_{n\times n}$ having two components is said to be *self-confident fuzzy preference relation* if the first component a_{ij} represents the preference degree or intensity of the alternatives x_i over x_j, and the second component s_{ij}^* represents the self-confidence level associated with a_{ij} (Dong et al., 2019; Zhu et al., 2021).

(viii) Many other preference relations that account for uncertainty can be found in the literature (Xu, 2007a,b; Xia and Xu, 2013; Xia et al., 2013; Xia and Xu, 2011; Ureña et al., 2015).

Nevertheless, these types of preference relations are not considered in the thesis.

2.2.6 Weighting methods for complete PCMs

Hereinafter, we use the name *pairwise comparison matrix* instead of *multiplicative pairwise comparison matrix*.

The main aim of using a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is to obtain a priority vector (weight vector) $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}_+^n$ so that the ratio between weights adequately represent the matrix entries, i.e., $a_{ij} \approx \frac{w_i}{w_j}$, $\forall i, j$. Any algorithm or procedure that derives a priority vector $\mathbf{w}$ from a PCM $\mathbf{A}$ is referred to as a *weighting method or prioritization method*. There are several weighting methods to derive a priority vector from a complete pairwise comparison matrix, where all entries of the matrix are known (Golany and Kress, 1993; Choo and Wedley, 2004; Lin, 2007; Kulakowski, 2020). Among these, the two most widely-used weighting methods are the *eigenvector method* (EVM) developed by (Saaty, 1977), and the *geometric mean method* (GMM) proposed by Crawford and Williams (1985).

The EVM provides a priority vector (right eigenvector) $\mathbf{w}$ corresponding to the maximum eigenvalue of a PCM $\mathbf{A}$ such that

$$\mathbf{A}\mathbf{w} = \lambda_{max}\mathbf{w} \tag{2.14}$$

where λ_{max} is the maximum eigenvalue of $\mathbf{A}$ (also known as Perron root or the

Perron-Frobenius eigenvalue). By the Perron-Frobenius theorem, the priority vector exists and is unique up to a scalar multiple. Moreover, the vector $\mathbf{w}$ is usually assumed to be normalized, i.e., $\sum_{i=1}^{n} w_i = 1$.

Theorem 1 (Perron-Frobenius Theorem) *If a square matrix is positive, then it has a unique real maximum eigenvalue and an associated positive right eigenvector.*

The GMM is the most straightforward method for complete PCMs, where each weight w_i is the geometric mean of the ith row of $\mathbf{A} = (a_{ij})_{n\times n}$ (Crawford and Williams, 1985):

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij}}}{\sum_{h=1}^{n}\left(\sqrt[n]{\prod_{j=1}^{n} a_{hj}}\right)}, \; i = 1, 2, \ldots, n. \tag{2.15}$$

This formula also helps to find a unique optimal solution for the *logarithmic least squares (LLS) problem* (Kwiesielewicz, 1996):

$$\begin{aligned} \min_{(w_1,\ldots,w_n)} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log a_{ij} - \log\frac{w_i}{w_j}\right)^2 \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \; \forall i. \end{cases} \end{aligned} \tag{2.16}$$

with an appropriate normalization. Note that the LLS problem formulation starts from the most commonly used consistency condition $a_{ij} = a_{ih}a_{hj}, \forall i, h, j$. If a PCM $\mathbf{A}$ is consistent, then $a_{ij} = \frac{w_i}{w_j}, \; \forall i, j$. On the other hand, if a PCM $\mathbf{A}$ is not consistent, the following expression measures the error e_{ij} between the judgment values a_{ij}:

$$e_{ij} = a_{ij}\frac{w_j}{w_i}, \; \forall i, j. \tag{2.17}$$

Equivalently, it can be rewritten as

$$\log e_{ij} = \log a_{ij} + \log\frac{w_j}{w_i}, \; \forall i, j. \tag{2.18}$$

Thus, the optimization problem (2.16) was formulated from (2.18) by considering the objective function as the sum of errors (i.e., $\sum_{i=1}^{n}\sum_{j=1}^{n}(\log e_{ij})^2$).

The EVM and GMM provide the same priority vector when PCMs are consistent. The similarity between these two prioritization methods can be found in the literature (Kułakowski et al., 2022; Mazurek et al., 2022).

2.3 Incomplete pairwise comparison matrices

In real-world applications, incomplete data could appear due to the unpredictable or complicated nature of decision-making settings. We refer to a PCM as being incomplete when one or more of its elements are absent. These elements are absent because of many reasons, such as a shortage of time and/or limited knowledge to make all the $n(n-1)/2$ comparisons (Harker, 1987a; Tang et al., 2018), or impossibility to compare some of the alternatives (e.g. in sports (Bozóki et al., 2016)).

Definition 10 (Incomplete Pairwise Comparison Matrix) *A PCM* $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ *is said to be an incomplete PCM if* $a_{ji} = 1/a_{ij}$ *if* a_{ij} *is known, otherwise* $a_{ji} = a_{ij} = *$*, where* $*$ *denotes the missing elements. Alternatively, it can be expressed in the form:*

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & * & \cdots & a_{1n} \\ * & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & \cdots & 1 \end{pmatrix}. \tag{2.19}$$

2.3.1 Graph representations of incomplete PCMs

Since the 1940s, graphs have been used to show paired comparisons in literature (Kendall and Smith, 1940). In fact, it has now gained popularity in the literature due to the widespread use of incomplete PCMs (Blanquero et al., 2006; Csató, 2015; Gass, 1998; Wang and Takahashi, 1998; Bozóki et al., 2016; Kułakowski et al., 2019).

It is customary to visualize the structure of an incomplete pairwise comparison matrix using an associated directed or undirected graph. In the thesis, the only thing that matters is whether there is a comparison between the two components. As a result, we employ undirected graphs, where the vertex labels the criteria or

alternatives. If the decision-makers compare the two individual items, then there is an edge between the two vertices.

Definition 11 (Undirected graph) *For a given incomplete PCM* $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$, *an associated undirected graph* G *to* $\hat{\mathbf{A}}$ *is defined as*

$$G := (V, E), \tag{2.20}$$

where $V = \{1, 2, \ldots, n\}$ *denotes the set of vertices (nodes), and* E *denotes the set of undirected edges* $\{i, j\}$ *(pairs of vertices) corresponding to the already assigned comparisons,* $E = \{\{i, j\} |\, a_{ij}\, is\, known, i, j = 1, 2, \ldots, n\,; i \neq j\}$.

With the exception of the diagonal entries, this means that if the matrix entry a_{ij} is already known, the edge is allocated either from node i to node j or from node j to node i. Unknown entries will not be given an edge. It should be noted that matrix $\hat{\mathbf{A}}$ performs a role that is similar to the adjacency matrix of graph G.

Definition 12 (Connected graph) *Two vertices* u *and* v *of undirected graph* G *are said to be connected if the graph* G *contains a path from* u *to* v. *A graph is called connected if every pair of vertices in the graph is connected. Otherwise, the graph* G *is called disconnected.*

For our investigation, a key attribute of the graph (2.20) will be its degree of connectivity. In actuality, we only take connected graphs corresponding to incomplete PCMs into account. In other words, if an incomplete $n \times n$ PCM corresponds to a non-connected graph, it simply implies that there are at least two nonempty subsets of the reference set's members, and none of them are compared (directly or indirectly) with any element of the other subset. This issue is obviously unimportant for practical considerations, so we don't take it into account.

The association between the missing comparisons in the incomplete PCM and its corresponding connected graph is as follows. To have a connected system of comparisons, the minimum number of comparisons needed is $n - 1$. A complete graph, having $n - 1$ degrees per node, can be used to represent a complete PCM of size n, and of course, it is connected. However, the connectedness of the corresponding graph to an incomplete PCM varies depending on the number of

missing comparisons (k). Considering k as the number of missing elements only in the upper triangle of a PCM, the graph is always connected if $k \leq n-2$, and possibly connected if there are at most $k = n(n-1)/2-(n-1) = (n-1)(n-2)/2$ missing elements. In fact, if a graph is connected along with $k = (n-1)(n-2)/2$, then the graph becomes a spanning tree. The graph is disconnected when $k > (n-1)(n-2)/2$, because the number of known entries in the incomplete matrix is less than $n-1$.

Note that the undirected graph G is a spanning tree if it is connected without cycles. Each spanning tree with n vertices has exactly $n-1$ edges (comparisons).

Example 7 *Consider a* 5×5 *incomplete PCM with four missing comparisons where the missing comparisons are represented by '*$*$*' as follows:*

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & \frac{1}{3} & 5 & 2 & * \\ 3 & 1 & * & * & \frac{1}{4} \\ \frac{1}{5} & * & 1 & * & 9 \\ \frac{1}{2} & * & * & 1 & 7 \\ * & 4 & \frac{1}{9} & \frac{1}{7} & 1 \end{pmatrix}.$$

The undirected graph representation of the given incomplete PCM is depicted in Fig. 2.2.

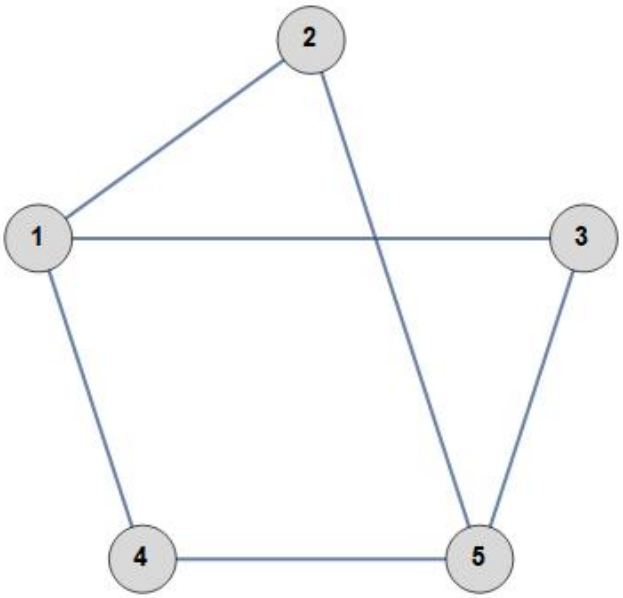


Figure 2.2: Undirected graph representation of $\hat{\mathbf{A}}$ in Example 7.

The connectedness of the associated undirected graph is an important property

for our numerical simulations as the solutions of some completion methods significantly depend on it. That means, the existence and uniqueness of solutions are influenced by the connectedness of the graph.

2.3.2 (Quasi-)regular graphs for incomplete PCMs

In this subsection, the notions of *s-(quasi-)regularity*, *graph diameter* and a twisted product of two graphs are described.

Definition 13 (*s*-regular graph) *When every vertex has degree s, the graph is said to be s-regular graph or s-edged connected graph, which means that every vertex in the graph has s neighbors.*

The s-regularity essentially means that there are no distinguishable vertices, such as in the case of the star graph. Semantically, s-regularity indicates "how many comparisons have to be made?" because every vertex has a degree of s.

In sports contests like the Swiss system or other tournaments (Csató, 2013; Ólafsson, 1990; Biró et al., 2017; Kujansuu et al., 1999), where each player or team plays an equal number of games in the first phase, regularity produces a form of symmetry that is also desirable (before the knockout stages). As a result, the comparisons' representing graph is regular (Csató, 2017).

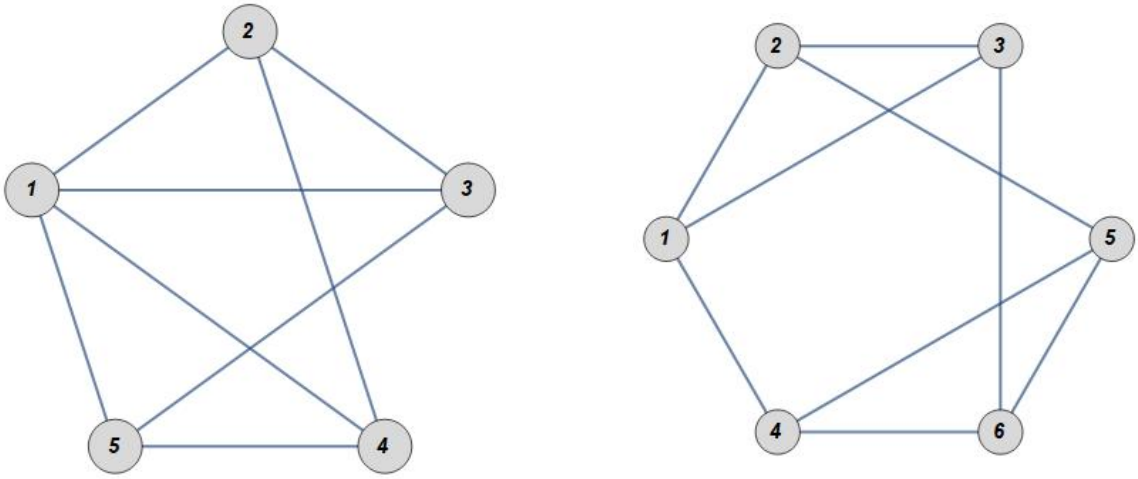
Definition 14 (*s*-quasi-regular graph) *When every vertex in a graph has degree s, but exactly one vertex has degree $s+1$, the graph is said to be s-quasi-regular.*

Example 8 *Consider 3−(quasi-)regular graphs with different vertices in Figure 2.3. Vertex 1 in Figure 2.3 (a) has degree 4, while the remaining vertices have equal degree 3. Hence, it is* 3-quasi-regular *graph. In contrast, in Figure 2.3 (b), all vertices have degree 3, and hence it is* 3-regular *graph.*

Definition 15 (Diameter of a graph) *The length of the longest shortest path between any two vertices of a graph G is called a diameter of a graph G, represented by d and computed as*

$$d = \max_{u,v \in V} \ell(u, v),$$

where V represents the set of vertices of G and $\ell(.,.)$ is the length of the shortest



(a) 3-quasi-regular graph

(b) 3-regular graph

Figure 2.3: The graph representation of 3-quasi-regular and 3-regular designs with different vertices. Reprinted from Publication III.

path between two vertices of G (that is, the distance between them).

For instance, in Figure 2.3, the diameter is 2 in each subfigure.

A twisted product of two graphs (Bermond et al., 1982, 1984) is one of the graph construction techniques that we extensively used to find the (quasi-)regular graphs and it is defined as follows.

Definition 16 (Twisted product of two graphs (Bermond et al., 1982)) *Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs. Let $\overrightarrow{E}$ be the set of arcs in an arbitrary orientation of G. For each arc $(i, j) \in \overrightarrow{E}$, let $f_{(i,j)}$ be a one to one mapping from V' to V'. The twisted product of two graphs G and G', denoted by $G * G'$ is defined as follows:*

(a) The vertex set of $G * G'$ is the Cartesian product $V \times V'$;

(b) There is an edge between vertices (i, i') and (j, j') if and only if

either $i = j$ and $\{i', j'\} \in E'$, or $(i, j) \in \overrightarrow{E}$ and $j' = f_{(i,j)}(i')$.

In general, the twisted product of two (quasi-)regular graphs can be used to construct new families of (quasi-)regular graphs with interesting properties. For example, if G and G' are both regular graphs, then $G * G'$ is also regular. Additionally, if G and G' are both connected, then $G * G'$ is also connected.

Example 9 *The product of K_2 (a complete graph with 2 vertices) and K_3 (a complete graph with 3 vertices) gives a 3-regular graph with vertices $n = 6$,*

degree $s = 3$ *and diameter* $d = 2$*, denoted as* X_6*. That is,* $K_2 * K_3 = X_6$*.*

Example 10 *Let* $G = K_n$ *be a complete graph and* $G' = X_8$ *be the graph on 8 vertices with degree 3, then the product of the two graphs* $K_n * X_8$ *has* $8n$ *vertices, degree* $n + 2$ *and diameter 2 (Bermond et al., 1982).*

2.4 Completion methods for incomplete PCMs

There are two different techniques/methods for estimating the missing values of an incomplete PCM: iterative methods (e.g., Herrera-Viedma et al. (2007a); Alonso et al. (2008)) and optimization methods (e.g., Fedrizzi and Giove (2007); Bozóki et al. (2010); Triantaphyllou (1995); Zhang et al. (2012)). Some of them were established based on the reduction of Saaty's inconsistency ratio (e.g., see subsection 2.4.1).

2.4.1 Optimization algorithms for Perron eigenvalue minimization problem

In this subsection, three optimization techniques are presented to solve a Perron eigenvalue minimization problem. The optimization algorithms are implemented to reduce the inconsistency ratio CR or, equivalently, to give the best λ_{max}-optimal completion.

Considering Saaty's CR index Saaty (1977, 1980), the Perron eigenvalue optimization problem is the following:

$$\underset{\mathbf{x}>\mathbf{0}}{\arg\min} \quad \lambda_{\max}(\hat{\mathbf{A}}(\mathbf{x})) \tag{2.21}$$

where $\hat{\mathbf{A}}(\mathbf{x})$ is an incomplete PCM with k unknown variables (missing comparisons) such that $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, and $\lambda_{max}(\hat{\mathbf{A}}(\mathbf{x}))$ denotes the Perron eigenvalue function of $\hat{\mathbf{A}}(\mathbf{x})$. Moreover, a solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*)$ to problem (2.21) is referred to as an optimal completion of $\hat{\mathbf{A}}$.

Initially, a problem of this nature—the generalization of the eigenvector method to a specific term of the characteristic polynomial—was considered by Shiraishi et al. (1998); Shiraishi and Obata (2002). More recent research has focused on the Perron eigenvalue (λ_{max}) minimization (2.21) of incomplete PCMs (Bozóki

et al., 2010; Ábele-Nagy, 2015; Tekile, 2019; Tekile et al., 2021).

The objective function in (2.21) is non-convex. However, it can be converted into a strictly convex function using exponential parametrization: $x \mapsto e^t$, $\forall x \in \mathbb{R}_+$(Bozóki et al., 2010). This is a good property for the uniqueness of the solution.

Example 11 *Let us consider a* 3×3 *incomplete PCM with one missing comparison:*

$$\hat{\mathbf{B}} = \begin{pmatrix} 1 & 2 & x \\ \frac{1}{2} & 1 & 5 \\ \frac{1}{x} & \frac{1}{5} & 1 \end{pmatrix}.$$

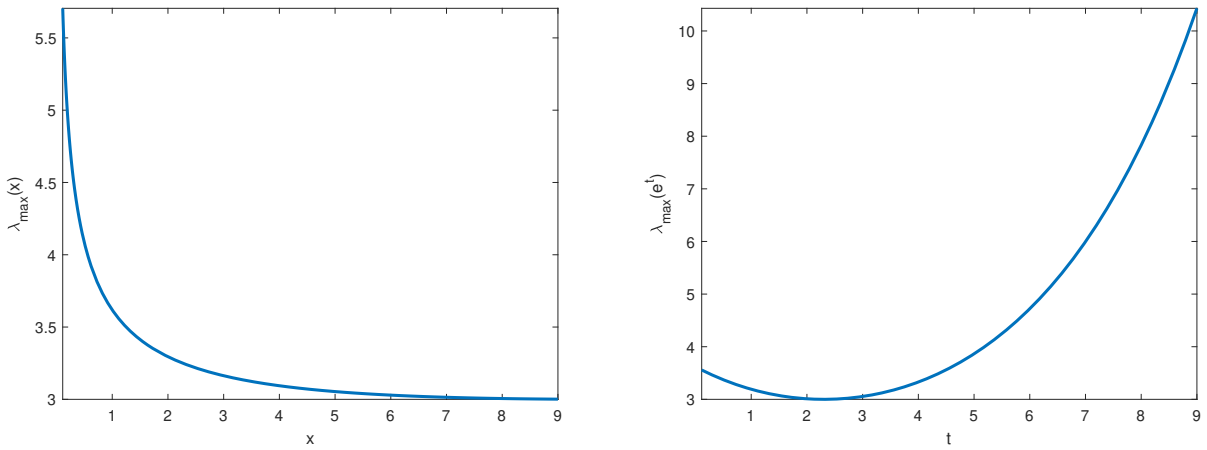
Since the PCM $\hat{\mathbf{B}}$ is of order 3, the analytic formula for λ_{max} can be easily found as follows (Rao Tummala and Ling, 1998):

$$\lambda_{max}(\hat{\mathbf{B}}(x)) = 1 + \sqrt[3]{\frac{x}{10}} + \sqrt[3]{\frac{10}{x}}$$

and using the exponential parametrization $x = e^t$, one obtains

$$\lambda_{max}(\hat{\mathbf{B}}(e^t)) = 1 + \sqrt[3]{\frac{e^t}{10}} + \sqrt[3]{\frac{10}{e^t}}.$$

Thus, the transformation of the non-convexity of $\lambda_{max}(\hat{\mathbf{B}}(x))$ into the convexity of $\lambda_{max}(\hat{\mathbf{B}}(e^t))$ can be depicted in Figure 2.4.



(a) Non-convexity of the function $\lambda_{max}(x)$ (b) Strict convexity of the function $\lambda_{max}(e^t)$

Figure 2.4: The transformation of the non-convexity of $\lambda_{max}(\hat{\mathbf{B}}(x))$ into the strict convexity of $\lambda_{max}(\hat{\mathbf{B}}(e^t))$ in Example 11. Note that $\lambda_{max}(x)$ and $\lambda_{max}(\hat{\mathbf{B}}(x))$ are used interchangeably.

Proposition 4 (Bozóki et al. (2010)) *The connectedness of the undirected graph corresponding to the incomplete PCM is a necessary and sufficient condition for the uniqueness of the eigenvalue minimization problem* (2.21).

Cyclic coordinates method

The method of *cyclic coordinates* using MATLAB's solver `fminbnd` was suggested by Bozóki et al. (2010) in order to find the best λ_{max}-optimal completion. This method minimizes the objective function in (2.21) *cyclically* with respect to the coordinate variables. In each step, a function of a single variable is minimized (i.e., one variable at a time is solved). The aim is to minimize λ_{max} so that a complete PCM $\mathbf{A}$ could be obtained.

In reference to (Bozóki et al., 2010), the basic steps of the method are described as follows. Let $x_q \in \mathbb{R}_+$ be the missing comparisons (unknown variables) for $q = 1, 2, \ldots, k$, and k denotes the number of unknown variables. Let $x_q^{(m)}$ also denote the value of x_q in the mth step of the iteration. Each iteration of the method consists of k steps. Thus, x_1 is changed first by keeping the rest of the associated variables fixed, then x_2 and so forth until x_k. The procedure is then repeated beginning with x_1 until the halting condition is met.

The cyclic coordinates algorithm is provided in Algorithm 1. Furthermore, the global convergence of the cyclic coordinates algorithm was stated and proved in the literature (Luenberger and Ye, 1984, pp. 266–267).

Algorithm 1 Cyclic coordinates algorithm for $\arg\min_{\mathbf{x}>\mathbf{0}} \lambda_{\max}(\hat{\mathbf{A}}(\mathbf{x}))$

Input: An incomplete PCM $\hat{\mathbf{A}}$ with k missing comparisons.
Output: A complete PCM $\mathbf{A}$.

1: *Set* $m \leftarrow 0$, *and* $x_q^{(0)} \leftarrow 1\ \forall q$ ▷ Assign initial values.
2: **while** $\max_{q=1,2,..,k} ||x_q^{(m)} - x_q^{(m-1)}|| > Tolerance$ **do** ▷ Stopping criteria.
3: Choose $q \in \{1, 2, .., k\}$
4: $x_q^{(m)} \leftarrow \arg\min_{x_q} \lambda_{max}(\hat{\mathbf{A}}(x_1^{(m)}, ..., x_{q-1}^{(m)}, x_q, x_{q+1}^{(m-1)}, ..., x_k^{(m-1)}))$ ▷ Solves one variable at a time.
5: Repeat step 4 for all $q \in \{1, 2, .., k\}$, and then go to step 6
6: $m \leftarrow m + 1$ ▷ increment for the mth step of the iteration.
7: **end while**

Newton's method

Newton's method is an optimization algorithm with derivative information that is used to determine an optimal solution. It can be used as both in univariate case (using the first derivative), and in multivariate case (using the second derivative). In the multivariate situation, if the objective function f is twice differentiable, then the gradient vector $\nabla f(\mathbf{x})$ and the inverse of the Hessian matrix $\mathbf{H}(\mathbf{x})$ are involved.

The univariate and multivariate Newton methods were employed by Ábele-Nagy (2015) to solve the Perron eigenvalue minimization problem. In the univariate case, the author optimized only one variable at a time using the cyclic coordinates with Newton method. In contrast, in the multivariate case, all the variables can be optimized simultaneously instead of cyclically optimizing one variable at a time.

Here, the multivariate Newton's algorithm is presented as follows (Algorithm 2) for solving (2.21). The optimization procedure is as follows. For the sake of convexity behavior, exponential parametrization is used. Let $x_q = e^{t_q}$ be the (i,j)th missing comparison, and objective function $f(\mathbf{t}) = \lambda_{max}(e^{t_1}, e^{t_2}, ..., e^{t_k})$, where k is the number of missing comparisons, $q = 1, \ldots, k$. The derivatives $\frac{\partial \lambda_{max}(\mathbf{x})}{\partial x_q}$ and $\frac{\partial^2 \lambda_{max}(\mathbf{x})}{(\partial x_q)^2}$ are known (known as Harker derivatives) (Harker, 1987b). The goal is to minimize f. To do this, we employ Newton's multivariate method in the rth iteration:

$$\mathbf{t}^{(r+1)} = \mathbf{t}^{(r)} - \gamma[\mathbf{H}_f(\mathbf{t}^{(r)})]^{-1} \nabla f(t^{(r)}), \tag{2.22}$$

where $\mathbf{H}_f(\mathbf{t}^{(r)})$ represents the Hessian matrix of $f(\mathbf{t})$, $\nabla f(\mathbf{t}^{(r)})$ is the gradient vector of $f(\mathbf{t})$, and γ is Newton's step size.

One could refer Harker's first and second derivative formulas to compute the gradient vector $\nabla f(\mathbf{t}) = (\frac{\partial f}{\partial t_1}, ..., \frac{\partial f}{\partial t_k})$ and Hessian matrix $\mathbf{H}_f(\mathbf{t})$, respectively, in the literature (Ábele-Nagy, 2015; Tekile, 2017; Harker, 1987b).

The halting criteria must be specified for $\mathbf{x}$ but not for $\mathbf{t}$ because slight changes in $\mathbf{t}$ result in significant variations in $\mathbf{x}$.

Algorithm 2 Newton's algorithm for $\arg\min_{\mathbf{x}>\mathbf{0}} \lambda_{\max}(\hat{\mathbf{A}}(\mathbf{x}))$

Input: An incomplete PCM $\hat{\mathbf{A}}$ with k missing comparisons.
Output: A complete PCM $\mathbf{A}$.

1: Choose starting values of $\mathbf{t}$ and *step size* γ ▷ Initial value must be assigned to $\mathbf{t}$.
2: *Set* $r \leftarrow 0$ ▷ Initial iteration.
3: **while** $\max_{q=1,2,..,k} ||x_q^{(r)} - x_q^{(r-1)}|| > Tolerance$ **do** ▷ Stopping criteria.
4: $\mathbf{t}^{(r+1)} \leftarrow \mathbf{t}^{(r)} - \gamma[\mathbf{H}_f(\mathbf{t}^{(r)})]^{-1}\nabla f(\mathbf{t}^{(r)})$ ▷ Variables are optimized simultaneously.
5: $r \leftarrow r + 1$. ▷ Increments for the rth iteration.
6: **end while**

Gradient descent algorithm

The gradient descent algorithm is a first-order derivative optimization process that moves in steps proportional to the negative of the gradient function at the current point. It is also referred to as the *steepest descent algorithm.*

The *gradient descent algorithm* was implemented by Tekile (2017, 2019) for solving the Perron eigenvalue minimization problem (2.21). A unique optimal solution is guaranteed with the exponential parametrization, $x_q = e^{t_q}$, $q = 1, \ldots, k$. The algorithm implementation procedure (Algorithm 3) is as follows. Let $f(\mathbf{t}) = \lambda_{max}(e^{t_1}, e^{t_2}, ..., e^{t_k})$, where k represents the number of unknown variables. The goal is to minimize $f(\mathbf{t})$. The gradient vector $\nabla f(\mathbf{t}) = (\frac{\partial f}{\partial t_1}, ..., \frac{\partial f}{\partial t_k})$ could be computed using Harker's first derivative formula (Harker, 1987b).

Algorithm 3 Gradient descent algorithm for $\arg\min_{\mathbf{x}>\mathbf{0}} \lambda_{\max}(\hat{\mathbf{A}}(\mathbf{x}))$

Input: An incomplete PCM $\hat{\mathbf{A}}$ with k missing comparisons.
Output: A complete PCM $\mathbf{A}$.

1: Choose starting values of $\mathbf{t}$ and step size γ ▷ Inital value must be assigned to $\mathbf{t}$.
2: *Set* $r \leftarrow 0$ ▷ Initial iteration.
3: **while** $\max_{q=1,2,..,k} ||x_q^{(r)} - x_q^{(r-1)}|| > Tolerance$ **do** ▷ Stopping criteria.
4: $\mathbf{d}^r \leftarrow -\nabla f(\mathbf{t}^r)$ ▷ Gradient vector.
5: *If* $\mathbf{d}^r = 0$, *then* stop. Otherwise, go to step 6
6: $\mathbf{t}^{r+1} \leftarrow \mathbf{t}^r + \gamma\mathbf{d}^r$ ▷ Gradient descent formula.
7: $r \leftarrow r + 1$ ▷ Increments for the rth iteration.
8: **end while**

The global convergence of the gradient descent algorithm was stated and proved in the literature (Freund, 2004, pp. 3–5).

2.4.2 Other completion methods for incomplete PCMs

The optimal completion of incomplete PCMs by optimizing quantities other than λ_{max} has been given the same, or even more, attention. These methods can be categorized as optimization methods and iterative methods (Ureña et al., 2015; Kulakowski, 2020). Some of the methods have the common objective of completing incomplete PCMs, which are called *completion methods*. On the other hand, the other methods—the so-called *prioritization methods*—have a goal to derive priority vectors of incomplete PCMs without first completing them. Nevertheless, it is also possible to estimate the values of the missing comparisons using the acquired weights as a ratio w_i/w_j. Thus, sometimes, we call them completion methods as seen in a broader manner (see also Chapter 3.2). Some of the completion methods (and prioritization methods) for incomplete PCMs are presented as follows.

Geometric mean method for incomplete PCMs

Kułakowski (2020) provided an extension of the geometric mean method (GMM) for an incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$. The idea is: first, the missing entries $a_{ij} = *$ are replaced by the ratios/variables w_i/w_j in $\hat{\mathbf{A}}$, say the substituted matrix $\mathbf{A}^*$, and then using the geometric mean of the ith row of $\mathbf{A}^*$:

$$w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}},\ i = 1, \ldots, n$$

yields a system of nonlinear equations. Finally, the system of nonlinear equations boils down to a system of linear equations through logarithmic transformations, and then a solution is obtained.

The above idea/procedure leads to the following ranking algorithm.

Step 1. Construct an auxiliary matrix $\mathbf{B} = (b_{ij})_{n\times n}$ from the given incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$, such that

$$b_{ij} = \begin{cases} 1, & \text{if } i \neq j \ \&\ a_{ij} = * \\ 0, & \text{if } i \neq j \ \&\ a_{ij} \neq * \\ n - k_i, & \text{if } i = j \end{cases}$$

where k_i represents the number of missing comparisons in the ith row of $\hat{\mathbf{A}}$ and $'*'$ indicates the missing comparisons. Then, a vector of constant terms

$$\mathbf{v} = \begin{pmatrix} \sum_{\substack{j=1 \\ a_{1j} \neq *}}^{n} \log a_{1j} \\ \vdots \\ \vdots \\ \sum_{\substack{j=1 \\ a_{nj} \neq *}}^{n} \log a_{nj} \end{pmatrix}.$$

Step 2. Solve the following system of linear equations, with $\hat{\mathbf{w}} = (\hat{w_1}, \ldots, \hat{w_n})^T$:

$$\mathbf{B}\hat{\mathbf{w}} = \mathbf{v}. \tag{2.23}$$

and create a weight vector $\mathbf{w} = (w_1, \ldots, w_n)^T$ in the form:

$$\mathbf{w} = (e^{\hat{w_1}}, e^{\hat{w_2}}, \ldots, \ldots, e^{\hat{w_n}})^T.$$

Step 3. Normalize $\mathbf{w}$ and obtain the resulting vector $\mathbf{w}_{GM}$:

$$\mathbf{w}_{GM} = \left(\frac{w_1}{\sum_{j=1}^{n} w_j}, \frac{w_2}{\sum_{j=1}^{n} w_j}, \ldots, \frac{w_n}{\sum_{j=1}^{n} w_j} \right)^T.$$

Note that the solution for Eq. (2.23) exists if the graph associated with the incomplete PCM $\hat{\mathbf{A}}$ is connected. For more details, see (Kułakowski, 2020; Kulakowski, 2020).

In general, it is worth noting that the geometric mean method (Kułakowski, 2020), the incomplete logarithmic least squares method (Bozóki et al., 2010) and the *geometric mean of all spanning trees* for incomplete PCMs (Siraj et al., 2012; Mazurek and Kułakowski, 2022) result in the same weight vectors (and the same completed matrix by replacing the missing comparisons with the ratio w_i/w_j) (Bozóki and Tsyganok, 2019; Kułakowski, 2020).

Weighted least absolute error method

A priority vector $\mathbf{w} = (w_1, \ldots, w_n)$ from a given incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n \times n}$ can be derived from the following linear optimization problem (Takeda and Yu,

1995):

$$\min_{(w_1,\ldots,w_n)} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} \chi(a_{ij}) \cdot \epsilon_{ij}$$

$$\text{s.t.} \begin{cases} \chi(a_{ij}) \cdot |a_{ij}w_j - w_i| \leq \chi(a_{ij}) \cdot \epsilon_{ij} \\ \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \ i = 1, \ldots, n. \end{cases}$$

where ϵ_{ij} is a nonnegative variable and

$$\chi(a_{ij}) = \begin{cases} 1, & a_{ij} \text{ is known} \\ 0, & a_{ij} \text{ is unknown.} \end{cases}$$

A lexicographically optimal completion method for incomplete PCMs

Recently, Ágoston and Csató (2022b) proposed a lexicographical-based completion method for incomplete PCMs using the well-known method of Koczkodaj's inconsistency index (Koczkodaj, 1993) and the concept of nucleolus in cooperative game theory (Schmeidler, 1969) so that a non-uniqueness nature of the optimization problem in (Koczkodaj et al., 1999) could be adjusted. The proposed algorithm identifies the missing comparisons in a way that the inconsistencies of the triads could be lexicographically minimal and gives a unique optimal completion. However, the associated graph with incomplete PCM must be connected in order for the lexicographically optimal completion to be unique. For more details, see (Ágoston and Csató, 2022b).

There are other several prioritization methods (or completion methods) for incomplete PCMs in literature, such as Least Squares Method (LSM) (Chu et al., 1979; Chen and Triantaphyllou, 2001), Weighted Least Squares Method (WLSM) (Chu et al., 1979; Chen and Triantaphyllou, 2001), Xu goal programming method (Xu, 2004b), Xu eigenproblem method (Xu, 2005b), and many others (Yuan et al., 2023; Ureña et al., 2015; Alrasheedi, 2019; Mazurek and Kułakowski, 2022; Brunelli et al., 2007; Faramondi et al., 2020; Yuan et al., 2023) (see also Chapter 3.2).

2.5 Simplex-based optimization techniques

In this section, simplex search algorithms are discussed, in particular, the Nelder-Mead algorithm which has connection with Publication I in Chapter 3.

Definition 17 (Simplex) *A simplex S in $\mathbb{R}^k$ can be defined as the convex hull of $k+1$ vertices.*

Example 12 *A simplex in $\mathbb{R}$ is a* line segment*; a simplex in $\mathbb{R}^2$ is a* triangle*; a simplex in $\mathbb{R}^3$ is a* tetrahedron*, and so on.*

Spendley et al. (1962) proposed the first simplex-based optimization algorithm. This algorithm employs only two types of simplex transformations—*reflection* and *shrinkage*—to create a new simplex in each step. The reflection step is away from the vertex with the maximum function value, whereas the shrinkage step is towards the vertex with the minimum function value. In these transformations, throughout all iterations, the angles between each simplex's edges remain the same. As a consequence, the working simplex can vary in size but not shape. Later, the method of Spendley et al. (1962) was modified by Nelder and Mead (1965) by including two additional transformations—*expansion* and *contraction*—that enable the working simplex to change both its size and shape (Powell, 1998; Singer and Nelder, 2009).

The Nelder-Mead algorithm is one of the most popular direct search methods (Lewis et al., 2000; Wright, 1996) for unconstrained nonlinear optimization problems. It is appropriate for the minimization of functions of several variables without derivative information or with discontinuity. For k-dimensional vectors (a function of k variables), the algorithm employs a simplex with $k+1$ non-degenerate points (vertices) to determine the optimal point based on the four possible steps of transformations: *reflection*, *expansion, contraction* and *shrink*. Their respective scalar parameters (coefficients) can be denoted as: reflection (ρ), expansion (χ), contraction (γ) and shrink (σ). They must satisfy the following conditions: $\rho > 0$, $\chi > 1$, $\chi > \rho$, $0 < \gamma < 1$, $0 < \sigma < 1$ (Lagarias et al., 1998; Nelder and Mead, 1965; Nocedal and Wright, 2006).

The actual implementation of the Nelder-Mead algorithm is usually reasonable, despite rarely becoming stuck in a non-stationary point. However, when stag-

nation happens, it can be overcome by restarting the algorithm (Kelley, 1999a). The convergence of the algorithm for low and high dimensional problems were investigated in the literature (Lagarias et al., 1998, 2012; McKinnon, 1998; Gao and Han, 2012). In addition, the algorithm's complexity analysis for a single iteration has been addressed in the literature (Singer and Singer, 1999, 2004).

Another version of Nelder-Mead algorithm for unconstrained optimization problem was implemented by Gao and Han (2012) using adaptive parameters for higher-dimensional problems, but they have not explicitly stated the convergence properties of the algorithm.

Furthermore, Mehta and Dasgupta (2012) proposed a modified Nelder-Mead algorithm for solving a general nonlinear constrained optimization problem that handles linear and nonlinear (in)equality constrains. The authors evaluated the performance of their algorithm by examining several benchmark problems and comparing it with various methods such as, α constrained method with mutation (Takahama and Sakai, 2005); Genetic algorithm (Deb et al., 2000); Bees algorithm (Pham et al., 2005), and found that, in terms of effectiveness and efficiency, it is comparable to such algorithms. Nevertheless, we consider a different approach to dealing with the interval constraints (in Chapter 3, Publication I).

The Nelder-Mead algorithm has a wide variety of real-world applications, especially in chemistry, engineering, and medicine (Wright, 1996). It has also been used and integrated into various software programs: for example, MATLAB's `fminsearch` (The MathWorks, 2021) and Python (Kochenderfer and Wheeler, 2019).

Chapter 3

Research results and contributions

This chapter summarizes the objectives of the publications and their respective contributions.

3.1 Publication I

Publication I represents, "Constrained Eigenvalue Minimization of Incomplete Pairwise Comparison Matrices by Nelder-Mead Algorithm" (Tekile et al., 2021). The goal of the research was to propose an efficient optimization algorithm that numerically minimizes the Perron eigenvalue (a maximum eigenvalue function) subject to interval constraints, where the objective function could not be expressed explicitly in terms of variables.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k_+$, and $f(\mathbf{x}) \coloneqq \lambda_{max}(\hat{\mathbf{A}}(\mathbf{x}))$. The constrained eigenvalue minimization problem is expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & l_q \leq x_q \leq u_q, \ q = 1, 2, \ldots, k \end{aligned} \tag{3.1}$$

where $\hat{\mathbf{A}}(\mathbf{x})$ is an incomplete PCM that contains k unknown variables (missing comparisons), and l_q and u_q are the lower and upper bounds for the variable x_q, respectively. Moreover, we consider a restriction $1/9 \leq l_q$, $u_q \leq 9$ for all $q = 1, 2, \ldots, k$. The restriction $[1/9, 9]$ is due to Saaty's proposal and to comply with AHP formulation.

The constrained eigenvalue problem (3.1) is an extension of the eigenvector

method for incomplete PCMs based on the concept of Shiraishi et al. (1998), which is used to find an optimal completion with minimal CR inconsistency. In fact, the eigenvector $\mathbf{x}$ that corresponds to the minimal λ_{max} in (3.1) provides a priority vector. However, the uniqueness of a solution to the problem (3.1) is not guaranteed because of the non-convexity nature of the objective function. Hence, by transforming the objective function into convexity, the connectedness of the graph associated with the incomplete PCM $\hat{\mathbf{A}}$ remains a necessary and sufficient condition for uniqueness, which was proved by Bozóki et al. (2010).

Furthermore, it frequently occurs that a decision maker feels more appropriate to communicate their pairwise comparisons as intervals rather than as a precise numerical value because of the inherent uncertainty in articulating preferences (e.g., Salo and Hämäläinen (1992); Saaty and Vargas (1987)). Therefore, interval judgments provide the preference evaluation with important flexibility by indicating a range for the relative importance of the preferences.

In general, it is noted that the maximum eigenvalue function (Perron eigenvalue) cannot be explicitly written as an analytic function of the variables $x_1, x_2, \ldots, x_k$. In light of this, we proposed the Nelder-Mead algorithm (Nelder and Mead, 1965; Lagarias et al., 1998) to solve the constrained optimization problem, where the objective function is *strictly convex* using an *exponential parametrization*. However, in practice, the algorithm works for an unconstrained optimization problem. Thus, the techniques of coordinate transformation (Tepljakov, 2017; Oldenhuis, 2009) needed to be applied in order to change the constrained problem (3.1) into an unconstrained problem. Moreover, we transform the initial point $\mathbf{x}_0$ of the constrained problem (3.1) into the initial point $\mathbf{z}_0$ of the unconstrained problem so that an initial simplex with $k+1$ vertices for the unconstrained problem could be constructed. And again, after the unconstrained problem is solved, we need to transform its solution into the solution of the constrained problem (3.1) through another inverse coordinate transformation.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k_+$ be the primal k-dimensional variable vector corresponding to the constrained problem, and $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in \mathbb{R}^k$ be a new search vector corresponding to the unconstrained problem. Then, the constrained problem (3.1) into an unconstrained problem $\min_{\mathbf{z}\in\mathbb{R}^k} f(\mathbf{z})$, the coordinate transformation T could be expressed by a function $T : [\frac{1}{9}, 9] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ such that

$\frac{1}{9} \leq \ell_q, u_q \leq 9$ and

$$T(x_q) = \arcsin\left(2(x_q - \ell_q)/(u_q - \ell_q) - 1\right),\ q = 1, 2, \ldots, k. \tag{3.2}$$

In short, T can be expressed as $T(x_q) = z_q$ such that

$$z_q = \arcsin\left(2(x_q - \ell_q)/(u_q - \ell_q) - 1\right),\ q = 1, 2, \ldots, k. \tag{3.3}$$

The inverse coordinate transformation T_{inv} (transformation from the solution of unconstrained problem to the solution of the constrained problem) is expressed as $T_{inv} : \mathbb{R} \to [\frac{1}{9}, 9]$ such that $T_{inv}(z_q) = x_q$, where $\frac{1}{9} \leq \ell_q, u_q \leq 9$ and

$$x_q = \ell_q + \frac{1}{2}(u_q - \ell_q)(\sin(z_q) + 1),\ q = 1, 2, \ldots, k. \tag{3.4}$$

Note that, using the coordinate transformation T in (3.2), the initial vertex $\mathbf{z}_0$ can be calculated from the initial vertex $\mathbf{x}_0$ of the constrained problem (3.1). Let $\mathbf{z}_{0,q}$ be the qth component of $\mathbf{z}_0$, and $\mathbf{x}_{0,q}$ be the qth component of $\mathbf{x}_0$ such that $l_q \leq \mathbf{x}_{0,q} \leq u_q$. Then,

$$\mathbf{z}_{0,q} = \arcsin\big(2(\mathbf{x}_{0,q} - l_q)/(u_q - l_q) - 1\big),\ q = 1, 2, \ldots, k. \tag{3.5}$$

However, it is a good idea to shift the initial coordinate values by 2π, i.e.

$$\mathbf{z}_{0,q} = 2\pi + \arcsin\left(2(\mathbf{x}_{0,q} - l_q)/(u_q - l_q) - 1\right),\ q = 1, 2, \ldots, k. \tag{3.6}$$

The addition of 2π to the initial values of $\mathbf{z}_{0,i}$ is a way to shift the values to ensure that the initial simplex region has a non-zero diameter. This is necessary because if the initial simplex region has a vanishingly small diameter, the Nelder-Mead algorithm may not be able to converge to a good solution.

The standard Nelder-Mead algorithm (Lagarias et al., 1998) along with coordinates transformation techniques (3.3) and (3.4), from now on we call it simply the *Nelder-Mead algorithm (4)*, was successfully implemented to the constrained problem (3.1). The algorithm uses the standard values $\rho = 1,\ \chi = 2,\ \gamma = 1/2$, and $\sigma = 1/2$, which are frequently used in the literature (Nelder and Mead, 1965; Lagarias et al., 1998; Kelley, 1999b; Baudin, 2010; The MathWorks, 2021).

The Nelder-Mead algorithm starts with an initial simplex with $k+1$ nondegenerate vertices $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{k+1}$ around a given initial point $\mathbf{z}_0$. Vertex $\mathbf{z}_1$ can be chosen at random. In practice, however, the most common choice of $\mathbf{z}_1$ is $\mathbf{z}_1$=$\mathbf{z}_0$ to make a proper restarts of the algorithm (Singer and Nelder, 2009). Next, the remaining k vertices are generated using a step-size s_i in the direction of the unit vector $\mathbf{e}_i = (0, 0, \ldots, 1, \ldots, 0) \in \mathbb{R}^k$, which forms a right-angled simplex at $\mathbf{z}_0$: $\mathbf{z}_{i+1} = \mathbf{z}_0 + s_i\mathbf{e}_i$, $\forall i = 1, \ldots, k$ such that

$$s_i = \begin{cases} 0.05 & if \;\; \mathbf{z}_{0,i} \neq 0 \\ 0.00025 & if \;\; \mathbf{z}_{0,i} = 0 \end{cases} \tag{3.7}$$

where $\mathbf{z}_{0,i}$ is the ith component of $\mathbf{z}_0$ (The MathWorks, 2021).

The initial simplex S_0 is a convex hull of $k+1$ vertices $\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{z}_{k+1}$. The vertices of S_0 are ordered according to the increasing function values:

$$f(\mathbf{z}_1) \leq f(\mathbf{z}_2) \leq \ldots \leq f(\mathbf{z}_k) \leq f(\mathbf{z}_{k+1}). \tag{3.8}$$

Vertex $\mathbf{z}_1$ is considered as the best vertex (vertex that has the smallest function value) and $\mathbf{z}_{n+1}$ as the worst vertex (vertex that has the largest function value). The centroid $\overline{\mathbf{z}}$ is computed as $\overline{\mathbf{z}} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{z}_i$, which is the average of the non-worst k vertices, i.e. all points (vertices) except for $\mathbf{z}_{k+1}$. Moreover, at each iteration of the algorithm, the simplex vertices are ordered as $\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{z}_{k+1}$ according to the increasing values of the objective function.

Given that `TolZ`, `TolFun`, `MaxIter`, and `MaxFunEvals`, the algorithm terminates when one of the following three conditions is met:

(C1) $\max_{1\leq i\leq k} ||\mathbf{z}_{i+1} - \mathbf{z}_1||_\infty \leq$ `TolZ` and $\max_{1\leq i\leq k} |f(\mathbf{z}_{i+1}) - f(\mathbf{z}_1)| \leq$ `TolFun`;

(C2) The maximum number of iterations (`MaxIter`) has been reached;

(C3) The maximum number of function evaluations (`MaxFunEvals`) has been reached.

We focused on PCMs of size 4 and above because PCMs of size 3 already have an analytical formula (Shiraishi and Obata, 2021), making the optimal completion straightforward. Illustrative examples are provided below to the optimal

Algorithm 4 Nelder-Mead algorithm for $\min_{x_q \in [l_q, u_q]} f(\mathbf{x})$. Reprinted from Publication I.

Set an initial point/vertex $\mathbf{x}_0$ that satisfies the given interval constraints of (3.1).
Apply the coordinate transformation T from (3.3) to form unconstrained problem $\min_{\mathbf{z} \in \mathbb{R}^k} f(\mathbf{z})$.
Compute an initial simplex S_0 with vertices $\{\mathbf{z}_i\}_{i=1,\ldots,k+1}$ for the unconstrained problem.
Compute $f_i = f(\mathbf{z}_i)$, $i = 1, 2, \ldots, k+1$.
Sort the vertices in S_0 with an increasing objective function values.
while $\max_i ||\mathbf{z}_{i+1} - \mathbf{z}_1||_\infty >$ `TolZ` & $\max_i |f_{i+1} - f_1| >$ `TolFun` **do**
  if count $>$ `MaxIter` or count $>$ `MaxFunEvals` **stop**. ▷ Stopping criteria.
  count = count + 1 ▷ Counts the number of iterations and function evaluations.
  $\overline{\mathbf{z}} \leftarrow \frac{1}{k} \sum_{i=1}^{k} \mathbf{z}_i$ ▷ Calculate centroid.
  $\mathbf{z}_r \leftarrow (1+\rho)\overline{\mathbf{z}} - \rho \mathbf{z}_{k+1}$ ▷ **Reflection**
  $f_r \leftarrow f(\mathbf{z}_r)$ ▷ Function value at $\mathbf{z}_r$.
  if $f_r < f_1$ **then**
    $\mathbf{z}_e \leftarrow (1+\rho\chi)\overline{\mathbf{z}} - \rho\chi \mathbf{z}_{k+1}$ ▷ **Expansion**
    $f_e \leftarrow f(\mathbf{z}_e)$
    if $f_e < f_r$ **then**
      $\mathbf{z}_{k+1} \leftarrow \mathbf{z}_e$ ▷ Accept $\mathbf{z}_e$ and replace the worst vertex $\mathbf{z}_{k+1}$ with $\mathbf{z}_e$
    else
      $\mathbf{z}_{k+1} \leftarrow \mathbf{z}_r$ ▷ Accept $\mathbf{z}_r$ and replace the worst vertex $\mathbf{z}_{k+1}$ with $\mathbf{z}_r$
    end if
  else if $f_1 \leq f_r < f_k$ **then**
    $\mathbf{z}_{k+1} \leftarrow \mathbf{z}_r$ ▷ Accept $\mathbf{z}_r$ and replace $\mathbf{z}_{k+1}$ with $\mathbf{z}_r$
  else if $f_k \leq f_r < f_{k+1}$ **then**
    $\mathbf{z}_{oc} \leftarrow (1+\rho\gamma)\overline{\mathbf{z}} - \rho\gamma \mathbf{z}_{k+1}$ ▷ **Outside contraction**
    $f_{oc} \leftarrow f(\mathbf{z}_{oc})$
    if $f_{oc} \leq f_r$ **then**
      $\mathbf{z}_{k+1} \leftarrow \mathbf{z}_{oc}$ ▷ Accept $\mathbf{z}_{oc}$ and replace $\mathbf{z}_{k+1}$ with $\mathbf{z}_{oc}$
    else
      Compute k new vertices $\mathbf{z}_i = \mathbf{z}_1 + \sigma(\mathbf{z}_i - \mathbf{z}_1)$, $i = 2, \ldots, k+1$ ▷ **Shrink**
      $f_i \leftarrow f(\mathbf{z}_i)$, $i = 2, \ldots, k+1$ ▷ Compute $f_i = f(\mathbf{z}_i)$, $i = 2, \ldots, k+1$
    end if
  else
    $\mathbf{z}_{ic} \leftarrow (1-\rho)\overline{\mathbf{z}} + \rho \mathbf{z}_{k+1}$ ▷ **Inside contraction**
    $f_{ic} \leftarrow f(\mathbf{z}_{ic})$
    if $f_{ic} < f_{k+1}$ **then**
      $\mathbf{z}_{k+1} \leftarrow \mathbf{z}_{ic}$ ▷ Accept $\mathbf{z}_{ic}$ and replace $\mathbf{z}_{k+1}$ with $\mathbf{z}_{ic}$
    else
      Compute k new vertices $\mathbf{z}_i = \mathbf{z}_1 + \sigma(\mathbf{z}_{i,old} - \mathbf{z}_1)$, $i = 2, \ldots, k+1$ ▷ **Shrink**
      $f_i \leftarrow f(\mathbf{z}_i) \, \forall i = 2, \ldots, k+1$.
    end if
  end if
  $S \leftarrow \{\mathbf{z}_i\}_{i=1,\ldots,k+1}$. ▷ Update simplex S with vertices $\{\mathbf{z}_i\}_{i=1,\ldots,k+1}$.
  Compute $f_i = f(\mathbf{z}_i)$, $i = 1, 2, \ldots, k+1$.
  Sort the $k+1$ vertices of the simplex S with an increasing objective function values.
end while
Select the best vertex and hence it will be an optimal solution for the unconstrained problem.
Apply the inverse coordinate transformation T_{inv} from (3.4) to get an optimal solution for the constrained problem.
Output: An optimal solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*)$ for the constrained problem (3.1), and the corresponding objective value $f(\mathbf{x}^*)$.

completion of the *incomplete* PCMs using the *Nelder-Mead algorithm.*

Example 13 *Consider the incomplete PCM* $\hat{\mathbf{A}}$ *with matrix size 4 (Tekile et al., 2021):*

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & * & 1/3 & 1 \\ * & 1 & 1/9 & * \\ 3 & 9 & 1 & 3 \\ 1 & * & 1/3 & 1 \end{pmatrix}$$

where $*$ denotes the missing comparisons. Equivalently, the above incomplete PCM with unknown variables x_1 and x_2 can be rewritten as

$$\hat{\mathbf{A}}(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & 1/3 & 1 \\ 1/x_1 & 1 & 1/9 & x_2 \\ 3 & 9 & 1 & 3 \\ 1 & 1/x_2 & 1/3 & 1 \end{pmatrix}$$

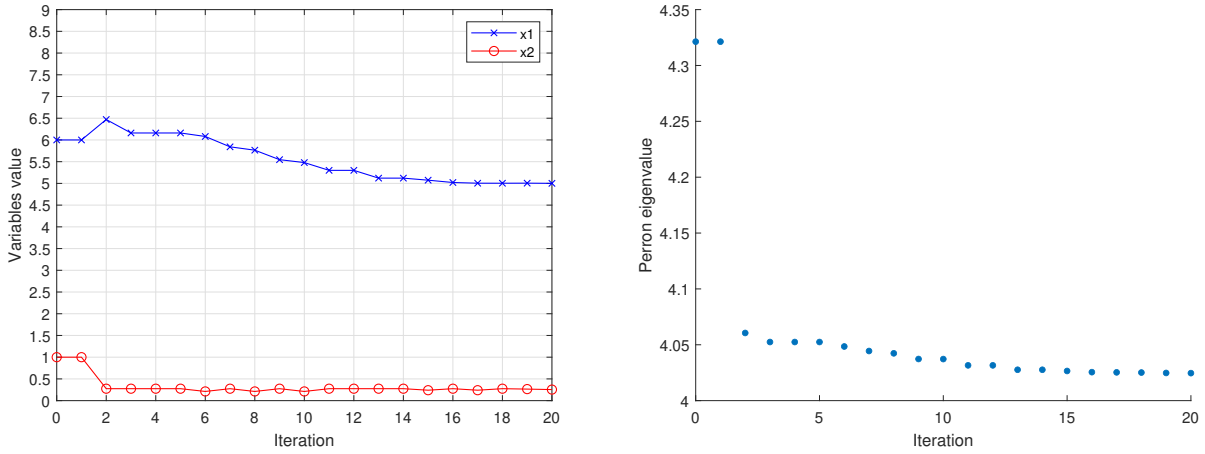
where $\mathbf{x} = (x_1, x_2)$.

As was previously noted, it might be advantageous for the expert to convey his/her preferences as intervals. Hence, the eigenvalue minimization problem can be constructed with interval constraints as follows:

$$\begin{aligned} \min_{\mathbf{x}>0} \quad & \lambda_{max}(\hat{\mathbf{A}}(\mathbf{x})) \\ \text{s.t.} \quad & 5 \leq x_1 \leq 7 \\ & 1/9 \leq x_2 \leq 9. \end{aligned} \tag{3.9}$$

It is obvious that by utilizing the consistency condition, we can obtain $x_1 = a_{13}a_{32} = 3$, and $x_2 = a_{23}a_{34} = \frac{1}{3}$ or $x_2 = a_{14}/a_{12} = \frac{1}{3}$ (as $a_{12}a_{24} = a_{14}$). However, within the interval $[5, 7]$, applying the Nelder-Mead algorithm to the minimization problem (3.9) with initial value $\mathbf{x}_0 = (6, 1)$, the algorithm arrives at the solution $x_1 = 5$ and $x_2 = 0.2582$ with $\lambda_{max} = 4.0246$ ($CR = 0.0093$). It should be noted that obtaining a consistent matrix is no longer achievable due to the constraint on x_1 (by Definition 4). The reduction of its objective function (Perron eigenvalue) by the algorithm and the evolution of the variables are shown in Figure 3.1. In this example, the given values for termination are: `TolZ` $= 10^{-4}$, `TolFun` $= 10^{-4}$.

As can be seen in Figure 3.1 (**b**), within the first few iterations, there is a quick reduction of the objective function.



(**a**) (**b**)

Figure 3.1: The values of x_1 and x_2, and the first 20 iterations of the algorithm to solve (3.9): (**a**) Evolution of the variables with respect to iterations; (**b**) Perron eigenvalue vs. Iteration.

Example 14 *Consider the* 7×7 *incomplete PCM* $\hat{\mathbf{A}}$ *with six unknown variables* $(x_1, \ldots, x_6) = \mathbf{x}$ *after replacing the missing comparisons (Tekile et al., 2021):*

$$\hat{\mathbf{A}}(\mathbf{x}) = \begin{pmatrix} 1 & 1/3 & x_1 & 1 & 1/4 & 2 & x_5 \\ 3 & 1 & 1/2 & x_2 & x_3 & 3 & 3 \\ 1/x_1 & 2 & 1 & 4 & 5 & 6 & 5 \\ 1 & 1/x_2 & 1/4 & 1 & 1/4 & 1 & 2 \\ 4 & 1/x_3 & 1/5 & 4 & 1 & x_4 & 1 \\ 1/2 & 1/3 & 1/6 & 1 & 1/x_4 & 1 & x_6 \\ 1/x_5 & 1/3 & 1/5 & 1/2 & 1 & 1/x_6 & 1. \end{pmatrix}$$

Let us first define a constrained eigenvalue minimization as follows in order to achieve completion while adhering to the constraint restriction $[1/9, 9]$:

$$\begin{aligned} \min \quad & \lambda_{max}(\hat{\mathbf{A}}(\mathbf{x})) \\ \text{s.t.} \quad & 1/9 \leq x_q \leq 9 \ \ for \ \ q = 1, 2, 3, 4, 5, 6. \end{aligned} \tag{3.10}$$

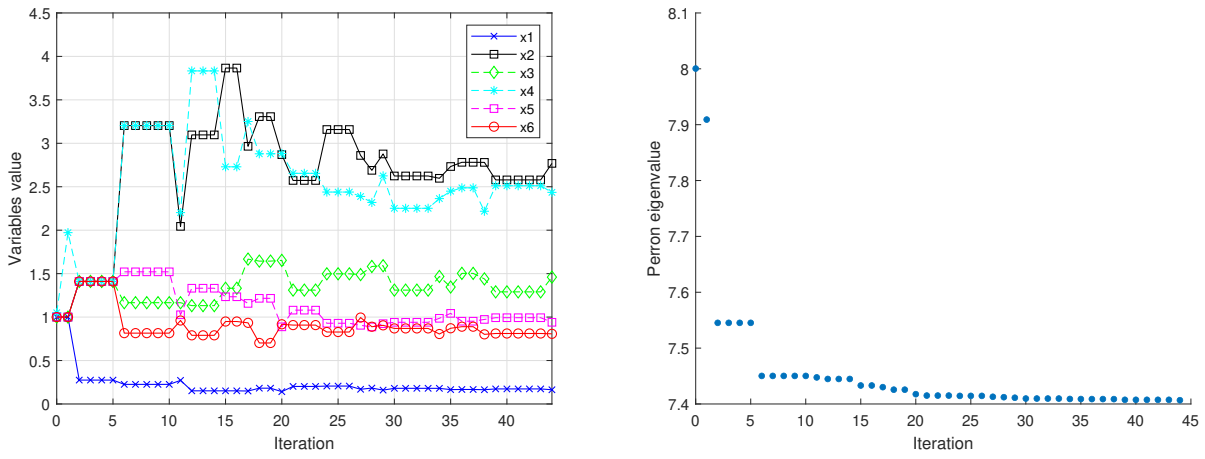
Then applying the algorithm to minimization problem (3.10) with initial value

$\mathbf{x}_0 = (1, 1, 1, 1, 1, 1)$, the optimal solution is

$$\mathbf{x}^* = (0.1618, 2.7207, 1.3465, 2.5804, 0.8960, 0.7731),$$

$$\lambda_{max}(\hat{\mathbf{A}}(\mathbf{x}^*)) = 7.4067.$$

The completed matrix $\mathbf{A}$ is acceptable, referring to Saaty's 0.1 cut-off rule because $CR = 0.0504$. Moreover, the first 44 iterations and values of the variables for each iteration are provided in Figure 3.2. The left-side graph in Figure 3.2 shows the evolution of the variables with respect to the number of iterations. As can be seen, the variables do not converge monotonically. This is mainly due to the algorithm's *contraction* step. For example, when we look at the value of x_2 at iterations 11, 18, and 27, it fluctuates. Also, the value of λ_{max} at iterations 2 and 6 drops significantly due to the *expansion* step. In this example, we used the values for termination: `TolZ` $= 10^{-4}$, `TolFun` $= 10^{-4}$. There is no significant difference in the λ_{max} values after the 44-th iteration.



(**a**) (**b**)

Figure 3.2: The values of $x_1, \ldots, x_6$, and the first 44 iterations of the algorithm for minimization (3.10): (**a**) Evolution of the variables with respect to iterations; (**b**) Perron eigenvalue vs. Iteration. Reprinted from Publication I.

The convergence analysis of the proposed algorithm lacks a precise statement in literature (Price et al., 2002; Gao and Han, 2012). For instance, if the optimization problem is strictly convex with a function of one variable, then the global convergence is guaranteed. However, for two variables, it may converge to a non-stationary point, although the problem is strictly convex (McKinnon, 1998).

Thus, the numerical simulations were needed in order to clarify the performance of the algorithm in the case of our constrained optimization problem (3.1).

The performance of the algorithm was examined through extensive numerical simulations to show how well the suggested algorithm fills an incomplete PCM and could validate its performance by varying the number of missing comparisons. The case of connected undirected graphs corresponding to incomplete PCMs (for each matrix size $n = 4, 5, 6, 7, 8, 9, 10$) was considered to carry out the simulation. The simulation results were measured by, a widely-used inconsistency measure, Saaty's inconsistency ratio (CR).

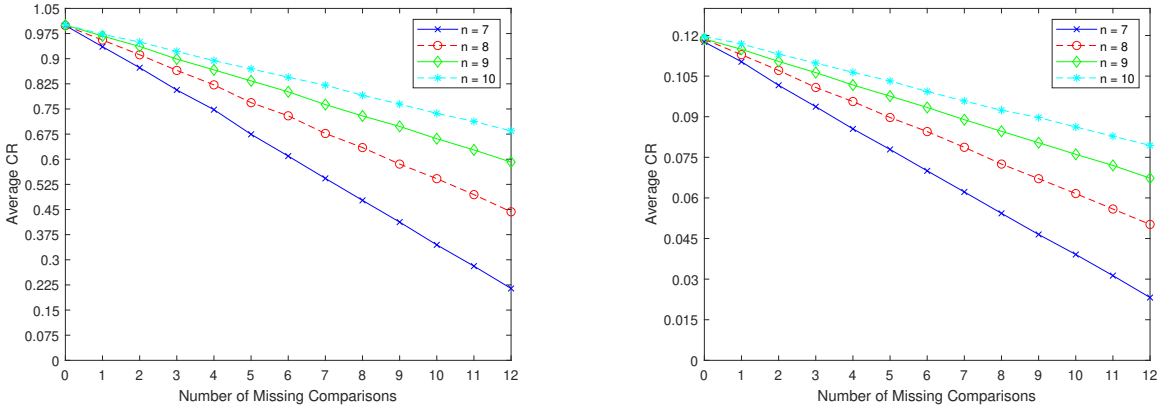
Two classes of random complete PCMs were chosen that result in different levels of inconsistencies:

(i) Random PCMs from the Saaty scale $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$; and

(ii) Random consistent PCMs on $[1/9, 9]$ with a slight change by Hadamard multiplicative perturbation using the *log-normal* distribution ($\mu = 0$, $\sigma = 0.65$) which result in inconsistent PCMs with more realistic CR values close to Saaty's threshold 0.1.

In contrast to type (i) matrices, type (ii) matrices represent more reasonable real-world instances. All in all, an incomplete PCM is constructed by removing one or more entries at random using a uniform distribution. Meanwhile, a test has to be done in order to verify whether the associated graph is connected. In turn, the proposed algorithm is put into practice.

Among the interesting results in Publication I, Figure 3.3 depicts that the algorithm effectively executes the optimal completion of incomplete PCMs. Additionally, the algorithm gives more consistent PCMs with more incomplete data (as k increases and nearer to $n(n-1)/2$).



(a) Results from the Saaty Scale

(b) Results from the perturbed matrices

Figure 3.3: Average CR of 10,000 PCMs versus the number of missing comparisons k with respect to the matrix size n. Reprinted from Publication I.

In general, our simulation results based on the $10,000$ generated PCMs, suggest the following:

- the solution obtained is unique if and only if the undirected graph associated with the incomplete PCM is connected;
- the proposed algorithm can estimate the missing values in the incomplete PCMs;
- the algorithm provides more consistent PCMs with more incomplete data, while the graphs are connected;
- the algorithm is fast and efficient.

3.2 Publication II

Publication II represents, "A numerical comparative study of completion methods for pairwise comparisons matrices" (Tekile et al., 2023). The goal of the research was to analyze the completion methods from the numerical point of view in order to identify the similarities and highlight the possible differences between methods based on the logarithmic distance measure (using the completed matrices) as a comparison criterion.

The eleven completion methods, denoted as M1,. . .,M11, have been selected based on the common aim of completing incomplete preferences in accordance with some appropriate criteria, e.g., the uniqueness of a solution. The first three methods

M1, M2, M3 are inconsistency based optimization methods, the next three methods M4, M5, M6 are based on minimization of judgments' biases/errors, other three methods M7, M8, M9 are non-optimization-based algorithms, and the last two M10, M11 are prioritization methods, however, they can be considered as completion methods in a 'broad' sense because the missing values can be estimated using weights.

For quick reference, the methods considered for the comparative analysis are provided below.

Method M1. $\lambda_{\max}$-based optimal completion method
Method M2. c_3-based optimal completion method
Method M3. ρ-based optimal completion method
Method M4. A method of δ-based local inconsistency indicator
Method M5. ϵ-based Least Absolute Error (LAE) method
Method M6. ϵ-based Least Squares Method (LSM)
Method M7. Connecting paths method
Method M8. Alonso et al.'s method
Method M9. DEMATEL-based optimal completion method
Method M10. Harker's eigenvalue methods
Method M11. Incomplete logarithmic least squares method (ILLSM)

Let $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ be an incomplete PCM of size n, and $x_1, \ldots, x_k$ denote k missing comparisons. Then the 11 completion methods are described as follows.

Method M1. $\lambda_{\max}$-based optimal completion method

The missing comparisons in the incomplete PCM $\hat{\mathbf{A}}$ are estimated as the argument solving the minimization of the Perron-Frobenius eigenvalue function ($\lambda_{\max}$) within the limits of the interval $[1/9, 9]$, i.e.,

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & \lambda_{\max}(\hat{\mathbf{A}}(\mathbf{x})) \\ \text{s.t.} \quad & 1/9 \leq x_q \leq 9,\ q = 1, 2, \ldots, k \end{aligned} \tag{3.11}$$

where $\mathbf{x} = (x_1, \ldots, x_k)$ and k is the number of missing comparisons. Different optimization techniques have been successfully applied to (3.11) due to the non-

analytic nature of $\lambda_{\max}$ (Bozóki et al., 2010; Tekile et al., 2021; Ábele-Nagy, 2015; Tekile, 2019). And then, Saaty's CR is used as a measure of inconsistency.

Method M2. c_3-based optimal completion method

In the characterstic polynomial of a PCM $\mathbf{A}$, the coefficient c_3 of λ^{n-3} is utilized as a measure of inconsistency index (Shiraishi et al., 1998), and given as

$$c_3(\mathbf{A}) = \sum_{i=1}^{n-2} \sum_{h=i}^{n-1} \sum_{j=h+1}^{n} \left(2 - \left(\frac{a_{ih}a_{hj}}{a_{ij}} + \frac{a_{ij}}{a_{ih}a_{hj}} \right) \right).$$

The value of c_3 is non-positive for $n \geq 3$, and equals zero if and only if the PCM is consistent. Thus, Obata et al. (1999) observed that the consistency of $\hat{\mathbf{A}}$ is maximized by solving the analytic function of its variables $x_1, x_2, \ldots, x_k$, writen as $c_3(\hat{\mathbf{A}}(\mathbf{x}))$, subject to the interval constraint $[1/9, 9]$, i.e.,

$$\begin{aligned} \arg\max_{\mathbf{x}} \quad & c_3(\hat{\mathbf{A}}(\mathbf{x})) \\ \text{s.t.} \quad & 1/9 \leq x_q \leq 9, \ q = 1, 2, \ldots, k \end{aligned} \tag{3.12}$$

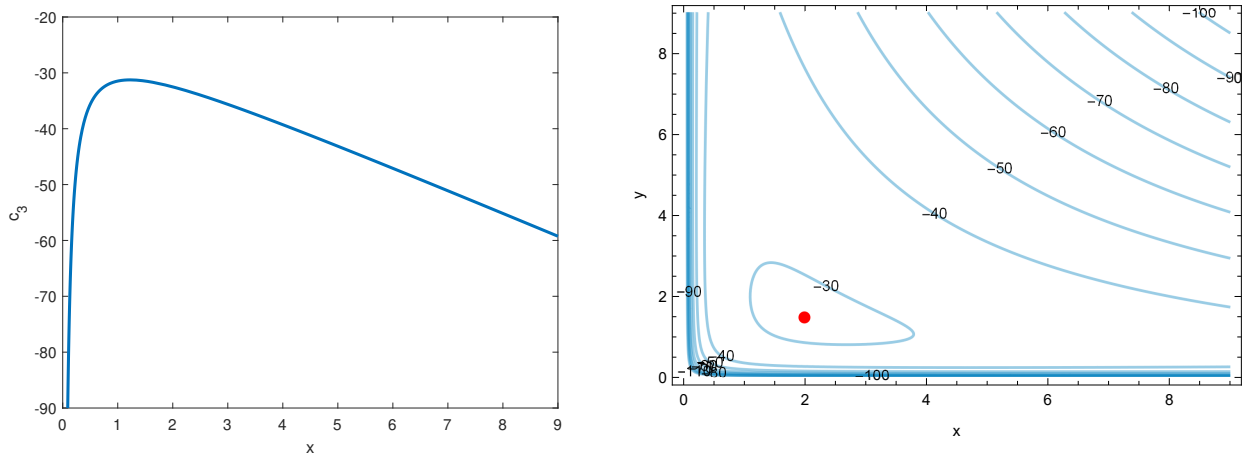
where $\mathbf{x} = (x_1, \ldots, x_k)$ and k is the number of missing comparisons.

It is worth noting that convex optimization techniques are effective to solve (3.12) due to the concavity nature of the objective function c_3 with respect to max (Shiraishi and Obata, 2002). An illustrative example is provided below to show the relation between c_3 and the missing comparison.

Example 15 *Consider the incomplete PCM* $\hat{\mathbf{A}}$ *with one missing comparison, denoted as* x*:*

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & x & 3 & 1 \\ 1/x & 1 & 1/2 & 4 \\ 1/3 & 2 & 1 & 9 \\ 1 & 1/4 & 1/9 & 1 \end{pmatrix}.$$

The estimated value of the missing comparison x using (3.12) is $x \approx 1.2247$ with $\max(c_3) \approx -31.2571$. The graph of $c_3(x) = -4.1667x - 6.2500/x - 21.0509$ and a contour plot of c_3 with two missing comparisons are shown in Figure 3.4.



(**a**) (**b**)

Figure 3.4: (**a**) Graph of c_3 with one missing comparison a_{12}; (**b**) Contour plot of c_3 with two missing comparisons $a_{12} = x$ and $a_{24} = y$, from Example 15. The red point in the bottom contour line at $(2, 1.5)$ indicates the maximum value.

Method M3. ρ-based optimal completion method

Fedrizzi and Giove (2007) proposed a method for incomplete reciprocal relation $\hat{\mathbf{R}} = (r_{ij})_{n\times n}$ based on the consistency condition (2.6), which results in the following inconsistency index ρ:

$$\rho = \sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{j=1}^{n}(r_{ih} + r_{hj} - r_{ij} - 0.5)^2.$$

Thus, the estimated values for the missing comparisons in $\hat{\mathbf{R}}$ are obtained by minimizing the inconsistency index ρ, i.e.,

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & \rho(\hat{\mathbf{R}}(\mathbf{x}) \\ \text{s.t.} \quad & 0 \le x_q \le 1,\ q = 1, 2, \ldots, k \end{aligned} \tag{3.13}$$

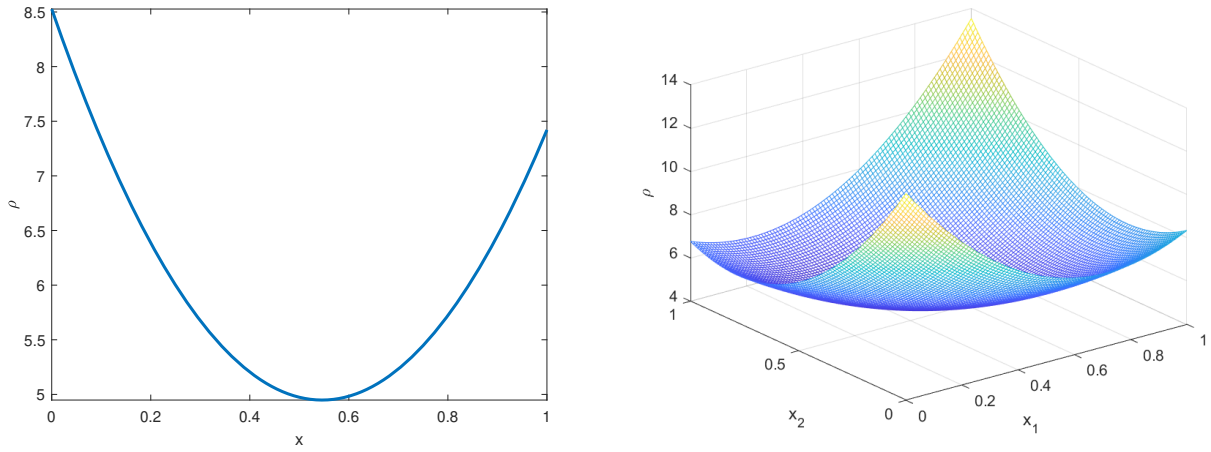
where $\hat{\mathbf{R}} = (r_{ij})_{n\times n}$ is an incomplete reciprocal relation, $\mathbf{x} = (x_1, \ldots, x_k)$ and k is the number of missing comparisons.

After the completion of the incomplete reciprocal relation by this method using optimization problem (3.13), the transformation (2.8) is utilized in order to find a complete PCM $\mathbf{A} = (a_{ij})_{n\times n}$ within the interval $[1/9, 9]$.

Example 16 *Consider the following incomplete reciprocal relation with one missing comparison, denoted as x, which was transformed from Example 15:*

$$\hat{\mathbf{R}}(x) = \begin{pmatrix} 0.5000 & x & 0.7500 & 0.5000 \\ 1-x & 0.5000 & 0.3423 & 0.8155 \\ 0.2500 & 0.6577 & 0.5000 & 1.0000 \\ 0.5000 & 0.1845 & 0 & 0.5000 \end{pmatrix}.$$

Then solving the minimization problem (3.13), the estimated value for the missing comparison r_{12} is $x \approx 0.5461$. The graph of $\rho(\hat{\mathbf{R}}(x))$ is depicted in Figure 3.5.



(**a**) (**b**)

Figure 3.5: Graphs of ρ from Example 16: (**a**) with one missing comparison r_{12}; (**b**) with two missing comparisons r_{12} and r_{23}.

Method M4. A method of δ-based local inconsistency indicator

Ergu and Kou (2013) and Ergu et al. (2011) considered the consistency condition $a_{ij} = a_{ih}a_{hj},\ \forall i, j, h$ of a PCM $\mathbf{A}$ in order to formulate a local index

$$\delta_{ij}(\mathbf{A}) = \frac{1}{n}\sum_{h=1}^{n} a_{ih}a_{hj} - a_{ij}, \quad \forall i, j. \tag{3.14}$$

That is, $\delta_{ij}(\mathbf{A}) = 0$ if all the indirect comparisons of i with j through h agree with the pairwise comparison a_{ij}.

Following this, the values of the missing comparisons, or the values of the variables

$x_1, x_2, \ldots, x_k$, of an incomplete PCM $\hat{\mathbf{A}}(\mathbf{x})$ could be obtained using the following optimization problem (Ergu and Kou, 2013):

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\delta_{ij}(\hat{\mathbf{A}}(\mathbf{x}))\Big)^2 \\ \text{s.t.} \quad & 1/9 \le x_q \le 9,\ q = 1, 2, \ldots, k. \end{aligned} \tag{3.15}$$

The expression $\delta_{ij}(\hat{\mathbf{A}}(\mathbf{x}))$ can be considered as an indicator for the (i, j)th position of an incomplete PCM $\hat{\mathbf{A}}(\mathbf{x})$.

Method M5. ϵ-based Least Absolute Error (LAE) method

Ergu et al. (2014) considered three formulations related to the well-known consistency results of a PCM $\mathbf{A}$:

i) a PCM $\mathbf{A}$ is consistent if and only if there exists a (weight) vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j}, \quad \forall i, j.$$

ii) In the consistent case, the ith component of $\mathbf{w}$ is related to the entries on the ith row of matrix $\mathbf{A}$ by means of the following function,

$$w_i = \left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}.$$

iii) In the consistent case, $a_{ij}a_{ji} = 1$. Let the quantity be c_{ij} such that $c_{ij} = a_{ij}a_{ji}$ to quantify the inconsistency related to the (i, j)th comparison of $\mathbf{A}$. Then, c_{ij} is rewritten as

$$c_{ij} = a_{ij} \cdot a_{ji} = \frac{w_i}{w_j} \cdot a_{ji} = \frac{\left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}}{\left(\prod_{h=1}^{n} a_{jh}\right)^{\frac{1}{n}}} \cdot a_{ji}.$$

Then, following the formulation (iii) above, and noting that in the consistent case

$c_{ij} = 1$, Ergu et al. (2016) proposed a local error,

$$\epsilon_{ij}(\mathbf{A}) = c_{ij} - 1 = \frac{\left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}}{\left(\prod_{h=1}^{n} a_{jh}\right)^{\frac{1}{n}}} \cdot a_{ji} - 1, \quad \forall i, j. \tag{3.16}$$

The indicator $\epsilon_{ij}(\mathbf{A}) = 0$ if all the indirect comparisons of i with j through h agree with the pairwise comparison a_{ij}.

Considering the incomplete PCM $\hat{\mathbf{A}}(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}_+^k$, the values of the unknown variables are computed by minimizing the sum of absolute values of the errors $\epsilon_{ij}(\hat{\mathbf{A}}(\mathbf{x}))$. That is, by solving the following optimization problem of Least Absolute Error (LAE):

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\epsilon_{ij}(\hat{\mathbf{A}}(\mathbf{x}))\right| \\ \text{s.t.} \quad & 1/9 \leq x_q \leq 9,\ q = 1, 2, \ldots, k. \end{aligned} \tag{3.17}$$

Method M6. ϵ-based Least Squares Method (LSM)

Based on Eq. (3.16), Ergu et al. (2016) proposed the optimization problem of the Least Squares Method (LSM) as follows, where the sum of the *squares* of each error $\epsilon_{ij}(\hat{\mathbf{A}}(\mathbf{x}))$ as a function of unknown variables $x_1, \ldots, x_k$ is minimized. That is,

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\epsilon_{ij}(\hat{\mathbf{A}}(\mathbf{x}))\right)^2 \\ \text{s.t.} \quad & 1/9 \leq x_q \leq 9,\ q = 1, 2, \ldots, k. \end{aligned} \tag{3.18}$$

Method M7. Connecting paths method

It is natural to fill in the missing comparisons a_{ij} of an incomplete PCM $\hat{\mathbf{A}}$ by taking the average of the intensities of all connecting paths that connect alternatives/criteria i to j if the associated graph to $\hat{\mathbf{A}}$ is connected (Harker,

1987c).

For the known comparisons a_{ih} and a_{hj}, under the consistent situation, a_{ij} can be obtained using the expression $a_{ij} = a_{ih}a_{hj}$. The pair a_{ih} and a_{hj}, $h \neq i, j$, is called an *elementary connecting path* of a_{ij} (Chen et al., 2015).

The value of the missing comparison a_{ij} of the incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ could be obtained by taking the geometric mean of intensities of all rth connecting paths of a_{ij}, denoted $CP(a_{ij})_r$, connecting the two alternatives i and j in $\hat{\mathbf{A}}$ (Chen and Triantaphyllou, 2001), i.e.,

$$a_{ij} = \left(\prod_{r=1}^{N} CP(a_{ij})_r\right)^{\frac{1}{N}} \tag{3.19}$$

where

- $CP(a_{ij})_r$ represents the rth connecting path of missing comparison a_{ij} and computed as

$$CP(a_{ij})_r : \; a_{ij} = a_{i,h_1} \cdot a_{h_1,h_2} \cdot a_{h_2,h_3} \cdots a_{h_t,j}, \tag{3.20}$$

  for the known comparisons a_{i,h_t} and $a_{h_t,j}$, and indices $i, j, h_1, \ldots, h_t \in \{1, \ldots, n\}$, $1 \leq t \leq n-2$; and

- N is the total number of connecting paths.

Finally, using the reciprocal property $a_{ji} = 1/a_{ij}$, the missing comparisons of the lower triangular part of the PCM $\hat{\mathbf{A}}$ are obtained.

In addition, for the sake of comparability of the method with the other completion methods, the interval restriction $[1/9, 9]$ is taken into account. Assuming the estimated value as a_{ij}: if $a_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $a_{ij} > 9$, then $a_{ij} = 9$.

Example 17 *Consider the following incomplete PCM with 1 missing comparison:*

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 3 & * \\ 1/2 & 1 & 1/2 & 4 \\ 1/3 & 2 & 1 & 5 \\ * & 1/4 & 1/5 & 1 \end{pmatrix}.$$

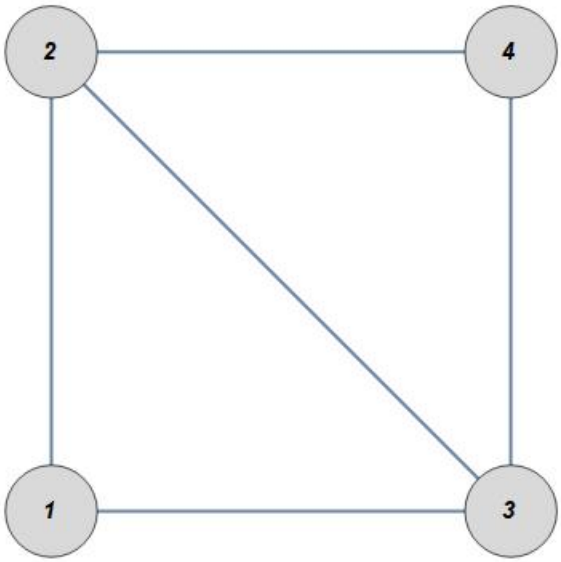


Figure 3.6: Undirected graph representation of $\hat{\mathbf{A}}$.

The undirected graph representation of this incomplete PCM is shown in Figure 3.6.

The four connecting paths for the missing entry a_{14} *are:* $1-2-3-4$, $1-2-4$, $1-3-2-4$, *and* $1-3-4$. *Then, using Eqs.* (3.19) *and* (3.20)*:*

(i) $CP(a_{14})_1 = a_{12} \cdot a_{23} \cdot a_{34} = 2 \cdot 1/2 \cdot 5 = 5$

(ii) $CP(a_{14})_2 = a_{12} \cdot a_{24} = 2 \cdot 4 = 8$

(iii) $CP(a_{14})_3 = a_{13} \cdot a_{32} \cdot a_{24} = 3 \cdot 2 \cdot 4 = 24$

(iv) $CP(a_{14})_4 = a_{13} \cdot a_{34} = 3 \cdot 5 = 15$

Therefore,

$$a_{14} = \sqrt[4]{CP(a_{14})_1 \cdot CP(a_{14})_2 \cdot CP(a_{14})_3 \cdot CP(a_{14})_4} = 10.9545.$$

However, in this example, the value of the missing entry a_{14} *must be* 9 *as the value* 10.9545 *is out of the interval* $[1/9, 9]$ *for the sake of comparison with the other methods.*

When matrix size becomes large, the number of connecting paths will be very large (e.g., with $n = 8$ and 3 missing comparisons, we can find 1064 connecting paths). In addition, in the literature (Chen and Triantaphyllou, 2001), for a PCM of size ten, the number of possible connecting paths to be considered might be equal to $109,600$ and hence this condition is mentioned as a drawback of the

method due to its computational cost.

Method M8. Alonso et al.'s method

Herrera-Viedma et al. (2007b) developed a method for estimating the missing values of incomplete reciprocal relations (fuzzy preference relations). Following this, Alonso et al. (2008) adapted an equivalent estimation procedure for incomplete PCMs. Thus, the missing values of an incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ can be estimated as follows, based on two different requirements.

I. Define the sets that identify which missing comparisons can be estimated at each phase of the algorithm.

i. $V = \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\}$, where V represents the set of pairs of alternatives (pairwise comparisons) without considering diagonal entries of $\hat{\mathbf{A}}$;

ii. $MV = \{(i,j) \in V \mid a_{ij} \text{ is unknown}\}$, where MV represents the set of missing values (missing comparisons) of $\hat{\mathbf{A}}$;

iii. $EV = V \backslash MV$, where EV represents the set of estimated values (known comparisons) of $\hat{\mathbf{A}}$ for which the decision-maker provides preference values (except the diagonal entries);

iv. The set of intermediate alternatives j (when $j \neq i,k$) that can be used to estimate the missing comparison a_{ik} $(i \neq k)$:

$H_{ik}^1 = \{j \neq i,k \mid (i,j),(j,k) \in EV\}$;

$H_{ik}^2 = \{j \neq i,k \mid (j,k),(j,i) \in EV\}$;

$H_{ik}^3 = \{j \neq i,k \mid (i,j),(k,j) \in EV\}$.

Then, we compute the following: EMV_h (the set of missing comparisons that must be estimated in iteration h of the algorithm), KV_h (the set of all known values (known comparisons) in iteration h), and UV_h (the set of unknown values (missing comparisons) in iteration h).

$$KV_h = EV \cup \left(\bigcup_{\ell=0}^{h-1} EMV_\ell\right); UV_h = MV \setminus \left(\bigcup_{\ell=0}^{h-1} EMV_\ell\right);$$

$$EMV_h = \{(i,k) \in UV_h \mid \exists j \in \{H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3\}\}.$$

Note that $EMV_0 = \emptyset$, $KV_1 = EV$, and EV will be replaced by KV_h to calculate H_{ik}^1, H_{ik}^2 and H_{ik}^3 in iteration h with updated PCM $\hat{\mathbf{A}}$.

II. Develop an expression for estimating a particular missing comparison cp'_{ik} in iteration h:

$$cp'_{ik} = \left(\prod_{\ell \in K} \left(\prod_{j \in H_{ik}^{\ell}}^{n} ca_{ik}^{j\ell} \right)^{1/\#H_{ik}^{\ell}} \right)^{1/\#K} \tag{3.21}$$

where $K = \{\ell \in \{1,2,3\} \mid H_{ik}^{\ell} \neq \emptyset\}$, $ca_{ik}^{j1} = a_{ij}a_{jk}$, $ca_{ik}^{j2} = a_{jk}/a_{ji}$, and $ca_{ik}^{j3} = a_{ij}/a_{kj}$. Moreover, we restrict the values of a_{ik} in the range $[1/9, 9]$: if $cp'_{ik} < 1/9$, then $a_{ik} = 1/9$; else if $cp'_{ik} > 9$, then $a_{ik} = 9$; else $a_{ik} = cp'_{ik}$.

The general algorithm for the missing comparisons estimation in iteration h is presented in Algorithm 5 (Herrera-Viedma et al., 2007b; Alonso et al., 2008).

Algorithm 5 Iterative algorithm in step h

Input: Incomplete PCM $\hat{\mathbf{A}}$.
Output: Complete PCM $\mathbf{A}$.
Step 1: $EMV_0 = \emptyset$ and $h = 1$
Step 2: **while** $EMV_h \neq \emptyset$
$h = h + 1$;
For every $(i,k) \in EMV_h$, calculate cp'_{ik} in Eq. (3.21), where
(a) $K = \emptyset$;
(b) $H_{ik}^1 = \{j \neq i,k \mid (i,j),(j,k) \in KV_h\}$; if $(\#H_{ik}^1 \neq 0)$ then $K = K \cup \{1\}$;
(c) $H_{ik}^2 = \{j \neq i,k \mid (j,k),(j,i) \in KV_h\}$; if $(\#H_{ik}^2 \neq 0)$ then $K = K \cup \{2\}$;
(d) $H_{ik}^3 = \{j \neq i,k \mid (i,j),(k,j) \in KV_h\}$; if $(\#H_{ik}^3 \neq 0)$ then $K = K \cup \{3\}$.
Let $a_{ik} = cp'_{ik}$.
end while
Step 3: Restrict a_{ik} in the range $[1/9, 9]$: if $cp'_{ik} < 1/9$, then $a_{ik} = 1/9$; else if $cp'_{ik} > 9$, then $a_{ik} = 9$; else $a_{ik} = cp'_{ik}$.
Step 4: Display a complete PCM $\mathbf{A}$.

To estimate all missing values (missing comparisons), a sufficient condition for Algorithm 5 is also provided in Proposition 5.

Proposition 5 (Zhang et al. (2014)) *The iterative algorithm (Algorithm 5) estimates all missing values of an incomplete PCM if the set of* $n-1$ *non-leading diagonal elements of the incomplete PCM is known (i.e., each alternative must be compared at least once).*

The iterative algorithm terminates when h reaches the maximum iteration, i.e., $EMV_{maxIter} = \emptyset$. Since the connected graphs are considered for our numerical simulations (Proposition 5 is satisfied), the estimation of all missing comparisons with the algorithm is achievable. Thus, the algorithm terminates after all the missing comparisons are obtained. For more details, see (Alonso et al., 2008).

Method M9. DEMATEL-based optimal completion method

DEMATEL (DEcision-MAking Trial and Evaluation Laboratory) has been demonstrated in the literature to be a useful method for evaluating interrelationships and causal analysis, allowing decision-makers to categorize the involving factors of a system into cause and effect groups. Moreover, unlike the AHP, it assumes that all criteria are mutually dependent and influence other criteria. The comparison scales of the DEMATEL method are 0 (no influence), 1 (low influence), 2 (high influence), and 3 (very high influence). Using these comparison scales, the comparative judgments of a decision-maker are expressed in the form of a direct-relation matrix in which the principal diagonal entries are zero (Falatoonitoosi et al., 2013; Lin et al., 2011; Wu and Chang, 2015; Lin and Wu, 2008).
Zhou et al. (2018) proposed a DEMATEL-based completion method for incomplete PCMs. It is an iterative algorithm that enables a decision-maker to directly estimate the missing comparisons in the incomplete PCM.

The method typically has four fundamental phases to estimate the missing values of an incomplete PCM on the interval $[1/9, 9]$:
Step 1. Transform the incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ into a direct-relation matrix $\mathbf{D}_r$ by replacing all missing comparisons with zero, and compute the normalized matrix $\mathbf{N}$. The direct-relation matrix $\mathbf{D}_r$ is written as $\mathbf{D}_r = (d_{ij})_{n\times n}$, where $d_{ij} = a_{ij}$ if a_{ij} is already known and $d_{ij} = 0$ if a_{ij} is missing. The normalized matrix $\mathbf{N}$ can be obtained from $\mathbf{N} = \frac{\mathbf{D}_r}{m}$, such that

$$m = \max\left\{\max_i \sum_{j=1}^{n} d_{ij}, \max_j \sum_{i=1}^{n} d_{ij}\right\}.$$

The matrix $\mathbf{N}$ is normalized. That is, all entries of $\mathbf{N}$ are in $[0, 1]$.
Step 2. Transform the direct-relation matrix $\mathbf{D}_r$ into total-relation matrix $\mathbf{T}_r$

using the formula

$$\mathbf{T}_r = \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}$$

where $\mathbf{I}$ is an identity matrix of size n and $\mathbf{N}$ is a normalized matrix. The total-relation matrix captures both direct and indirect relationships between influencing factors and provides information on how one factor affects another. By computing the sum of the rows and columns of $\mathbf{T}_r$, $R_i(i = 1, \ldots, n)$ and $C_i(i = 1, ..., n)$, respectively, the relevance of each influencing factor can be identified. Additionally, the sign of $R_i - C_i$ can categorize each factor/criterion as either a 'cause' $(R_i - C_i > 0)$ or an 'effect' $(R_i - C_i < 0)$ (Lin et al., 2011; Zhou et al., 2018). It is important to note that the total-relation matrix $\mathbf{T}_r$ can be derived in the following way:

$$\begin{aligned}
\mathbf{T}_r &= \sum_{p=1}^{\infty} \mathbf{N}^p \\
&= \lim_{p\to\infty} (\mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^p) \\
&= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} + \mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^{p-1}) \\
&= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}(\mathbf{I} - \mathbf{N})(\mathbf{I} + \mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^{p-1}) \\
&= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}(\mathbf{I} - \mathbf{N}^p) \\
&= \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}
\end{aligned} \tag{3.22}$$

because $\mathbf{N}^p$ converges to a zero matrix as $p \to \infty$.

Step 3. Transform the total-relation matrix $\mathbf{T}_r = (t_{ij})_{n\times n}$ into a complete PCM $\mathbf{C} = (c_{ij})_{n\times n}$ with:

$$c_{ij} = \sqrt{\frac{t_{ij}}{t_{ji}}} \text{ for } i < j.$$

Note that $\mathbf{C}$ satisfies the reciprocal condition $c_{ji} = 1/c_{ij}$, $\forall i, j$, and also $c_{ii} = 1$, $\forall i$.

Step 4. The missing entries a_{ij} of the incomplete PCM $\hat{\mathbf{A}}$ are estimated as $a_{ij} = c_{ij}\,(i, j = 1, 2, \ldots, n)$. Moreover, we restrict the values of missing comparisons to be in the interval $[1/9, 9]$: if $c_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $c_{ij} > 9$, then $a_{ij} = 9$ by keeping the known entries of the incomplete PCM $\mathbf{A}$ unchanged. Now

$\mathbf{A}$ will be the completed matrix from the method.

In the event that the expression $\mathbf{I} - \mathbf{N}$ lacks inverse, the calculation of the total-relation matrix $\mathbf{T}_r$ may not be possible, and the method may fail to achieve its goal. However, the authors suggested an alternative approach of using $\mathbf{T}_r = \mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^p$, where p is a positive integer greater than or equal to 5 in most cases, as an approximation of the total-relation matrix. This is because the consistency ratio (CR) of the complete PCM $\mathbf{A}$ decreases gradually as p increases from 5. For more details, see (Zhou et al., 2018).

Method M10. Harker's eigenvalue method

Harker's eigenvalue method (Harker, 1987a) completes an incomplete PCM $\hat{\mathbf{A}} = (a_{ij})_{n\times n}$ by constructing an auxiliary matrix $\mathbf{C} = (c_{ij})_{n\times n}$:

$$c_{ij} = \begin{cases} 1 + k_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \ \& \ a_{ij} = * \\ a_{ij}, & \text{otherwise} \end{cases}$$

where k_i represents the number of missing comparisons in the ith row of $\hat{\mathbf{A}}$ and $'*'$ indicates the missing comparisons. Then, the method computes the maximum eigenvalue $\lambda_{\max}$ of $\mathbf{C}$ and its corresponding right eigenvector $\mathbf{w} = (w_1, \ldots, w_n)$ such that

$$\lambda_{\max}\mathbf{w} = \mathbf{C}\mathbf{w}. \tag{3.23}$$

The weight vector $\mathbf{w}$ is normalized (i.e., $\sum_{i=1}^{n} w_i = 1$), and its ratio w_i/w_j replaces the missing comparisons of the incomplete PCM $\hat{\mathbf{A}}$.

Furthermore, the values of missing comparisons are restricted to the interval $[1/9, 9]$ in order to make it comparable with the other completion methods, i.e., if $a_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $a_{ij} > 9$, then $a_{ij} = 9$ without changing the known comparisons of the incomplete PCM $\hat{\mathbf{A}}$.

Example 18 *Consider the* 5×5 *incomplete PCM* $\hat{\mathbf{A}}$ *with four missing compar-*

isons:

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & * & 3 & * & * \\ * & 1 & 1/9 & * & 5 \\ 1/3 & 9 & 1 & 3 & 1/2 \\ * & * & 1/3 & 1 & 8 \\ * & 1/5 & 2 & 1/8 & 1 \end{pmatrix}.$$

Then, the auxiliary matrix corresponding to $\hat{\mathbf{A}}$ *is*

$$\mathbf{C} = \begin{pmatrix} 4 & 0 & 3 & 0 & 0 \\ 0 & 3 & 1/9 & 0 & 5 \\ 1/3 & 9 & 1 & 3 & 1/2 \\ 0 & 0 & 1/3 & 3 & 8 \\ 0 & 1/5 & 2 & 1/8 & 2 \end{pmatrix}.$$

The weight vector (eigenvector from (3.23)*) is*

$$\mathbf{w} = (0.243, 0.130, 0.295, 0.218, 0.114).$$

Thus, the completed matrix by Harker's eigenvalue method will be

$$\mathbf{A} = \begin{pmatrix} 1 & \mathbf{1.871} & 3 & \mathbf{1.117} & \mathbf{2.132} \\ \mathbf{0.534} & 1 & 1/9 & \mathbf{0.597} & 5 \\ 1/3 & 9 & 1 & 3 & 1/2 \\ \mathbf{0.896} & \mathbf{1.676} & 1/3 & 1 & 8 \\ \mathbf{0.469} & 1/5 & 2 & 1/8 & 1 \end{pmatrix}.$$

Method M11. Incomplete logarithmic least squares method (ILLSM)

The incomplete logarithmic least squares problem (Bozóki et al., 2010) associated with an incomplete PCM $\hat{\mathbf{A}}(a_{ij})_{n\times n}$ is a nonlinear optimization problem and expressed as

$$\min_{\mathbf{w}>\mathbf{0}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\chi(a_{ij})\left(\log a_{ij} - \log\frac{w_i}{w_j}\right)^2 \tag{3.24}$$

$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ \frac{1}{9} \leq \frac{w_i}{w_j} \leq 9,\ i,j = 1,\ldots,n. \end{cases} \tag{3.25}$$

where

$$\chi(a_{ij}) = \begin{cases} 1, & a_{ij} \text{ is known} \\ 0, & a_{ij} \text{ is unknown.} \end{cases}$$

In the objective function, the unknown comparisons are given values of zero. Then, after minimization, the ratio w_i/w_j in the weight vector $\mathbf{w} = (w_1, \ldots, w_n)$ estimates the values of the missing comparisons in $\hat{\mathbf{A}}$. The constraints $\frac{1}{9} \leq \frac{w_i}{w_j} \leq 9,\ i,j = 1,\ldots,n$ in the optimization problem (3.24) enforce the values of the missing entries in the interval $\frac{1}{9}, 9$, for the sake of comparability with the other methods.

Theorem 2 (Bozóki et al. (2010)) *The connectedness of graph G associated with the incomplete PCM is a necessary and sufficient condition for the uniqueness of the optimal solution of the incomplete logarithmic least squares problem.*

3.2.1 Numerical Simulations and Hierarchical Clustering

To measure the closeness between two completed matrices, the Manhattan distance $\mathcal{D}$ is used:

$$\mathcal{D}(\mathbf{A},\mathbf{B}) = \|\log(\mathbf{A}) - \log(\mathbf{B})\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|\log(a_{ij}) - \log(b_{ij})| \tag{3.26}$$

where $\mathbf{A} = (a_{ij})_{n\times n}$ and $\mathbf{B} = (b_{ij})_{n\times n}$ are the completed matrices obtained by applying two different methods to the same incomplete matrix $\hat{\mathbf{A}} = (\hat{a}_{ij})_{n\times n}$ of order n.

It is reasonable to apply the logarithm to each matrix entry before computing the distance between two PCMs, although various distance formulas are provided in the literature. This is due to the fact that, for instance, the distance between judgments $\frac{1}{8}$ and $\frac{1}{9}$ should be the same as that between judgments 8

and 9 (Fedrizzi, 1990, pp. 233–235). Notice that log stands for natural logarithm. Moreover, the formula (3.26) was used by Mazurek et al. (2021) with the exception of the logarithmic transformations.

From now on, the following notation will be utilized:

- $\mathcal{D}$ denotes the distance in Eq. (3.26);
- $\mathcal{D}^*$ denotes the mean distance in Eq. (3.26) computed over a number of instances.

Two completion methods are more similar when the mean distance $\mathcal{D}^*$ is smaller.

Perturbed consistent PCMs were generated to carry out the numerical simulations for $n = 4, \ldots, 8$ along with various numbers of missing comparisons. A consistent PCM $\mathbf{A} = (a_{ij})_{n \times n}$ on $[1/9, 9]$ is generated by using $a_{ij} \leftarrow\!\shortmid w_i/w_j$, where $(w_1, \ldots, w_n)$ is a randomly generated vector with $w_i \in [1, 9]$. Then the consistent matrix is modified using a random perturbation $a_{ij} \leftarrow\!\shortmid a_{ij} \cdot \beta$, $\beta \sim$ from Lognormal distribution Lognormal$(0, \sigma^2)$ with $\sigma = 0.7$, where the distribution of the CR of the perturbed PCMs has an expected value very close to 0.1. We believe that these inconsistent PCMs are more relevant than randomly generated matrices, anyway, which are also analyzed in Publication II. Once the upper triangle has been completed in this manner, reciprocity is used to create the lower triangle.

To construct an incomplete PCM of order n, one or more comparisons are randomly removed from a complete PCM independently and using a uniform distribution in the upper triangle, replacing them with unknowns. Then, starting with the upper triangle, the lower triangle is created to obtain the reciprocals of the unknowns. Finally, an incomplete PCM will be constructed. A test is performed in order to check whether the corresponding undirected graph associated with each incomplete PCM is connected or not. It is important to note that the uniqueness of the solution for the optimization-based methods depends on the connectedness of the associated undirected graph. Moreover, there are some methods that consider the graph's connectedness as a sufficient condition for finding a solution (e.g., methods M7, M8, M9 and M10).

Furthermore, an agglomerative hierarchical clustering algorithm is applied using

a single-linkage method (Johnson, 1967; Everitt et al., 2011), which is simple and suitable for our goal. The algorithm merges the two clusters W and Z at $\min\{d^*_{st} |\ s \in W\ , t \in Z\}$ where d^*_{st} is the distance between elements. That is, the most similar clusters are merged in the distance matrix, and deleting rows and columns are kept so that old clusters are joined into new ones using the minimum distance.

Let $\mathbf{D}^* = (d^*_{st})_{11\times 11}$ be a distance matrix, where the entries d^*_{st} of the matrix $\mathbf{D}^*$ are the mean distances between the eleven methods: M_s and M_t $(s, t = 1, \ldots, 11)$. Then, the basic steps of the agglomerative hierarchical clustering are the following:

(1) Start with $m = 11$ clusters, each object making a single cluster of its own, and then m decreases by 1 at each step;

(2) The most two similar clusters are joined using the minimum distance;

(3) The distance matrix $\mathbf{D}^*$ is updated;

(4) Steps (2) and (3) are repeated until only one cluster remains.

3.2.2 Results and discussion

Figure 3.7 reports the mean distances between methods by means of heatmaps. The color of each heatmap cell is described by the colorbar (on the rightmost side) subject to the ranges of the distance values. Dark-blue represents larger values and light-blue represents smaller values. For instance, in Figure 3.7, greater values in the distance matrix can be found in the cells of the heatmap linked to method M4, and are shaded with dark-blue, in each subfigure, in an L-shaped pattern. Since the distance matrix is symmetric, the heatmaps only depict the values in the upper triangle. Additionally, the two methods M3 and M5 yield results that are quite similar for all distances taken into account.

Dendrograms are used to display the hierarchical clustering of the eleven completion methods. The dendrogram is built in the bottom-up hierarchy, which is also known as agglomerative hierarchical clustering. For instance, as depicted in Figure 3.8, the methods M3 and M5 are linked with the smallest height, hence they are the most similar; the set of methods {M3, M5, M11} is the next most

similar methods; M1 and M2 are the third most similar methods; the cluster {M1, M2, M3, M5, M11} represents the fourth most similar methods, and so on.

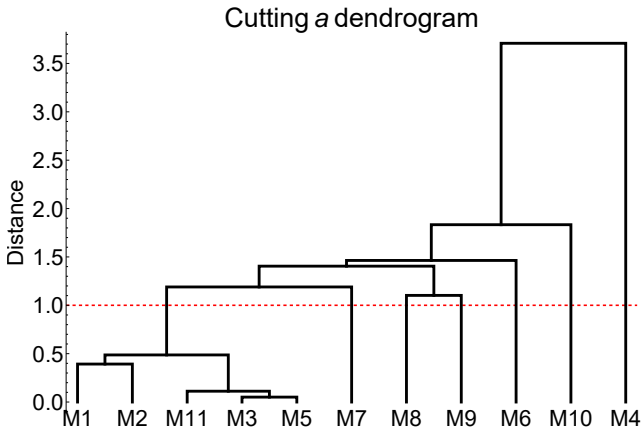


Figure 3.8: A sample of cutting a dendrogram into seven clusters: {M1,M2, M3,M5,M11}, {M7}, {M8,M9}, {M6}, {M10}, {M4}. Reprinted from Publication II.

Simulations were performed by further varying the order of PCMs and the number of missing comparisons. When we cut all dendrograms into a constant of seven clusters, as shown in Figure 3.9, five methods—M1, M2, M3, M5, M11—are always found in one cluster. This indicates that these five methods are extremely similar, irrespective of the order of the matrix or the number of missing entries, with an exception made for the very specific case with limited relevance $k = 1$. That is, the similarity among the completion methods does not change significantly for the increasing k and/or n, except for the very specific case $k = 1$. At $k = 1$ while n increases, the five methods M3, M5, M7, M8 and M11 are the most similar. On the other hand, method M4 is typically associated with greater mean distances. In fact, cluster M4 has the highest height across all of the dendrograms, making this method an outlier.

Furthermore, if we try to analyze the discrepancy of the completion method M4 results to the others, one question may come up in one's mind: 'Why the discrepancy of the M4 results?' It is fair to reason out that it could be related to the possibility that biases, upon which method M4 is based, may differ greatly from inconsistencies. For instance, consider the entry $a_{15} = 6$ of the following

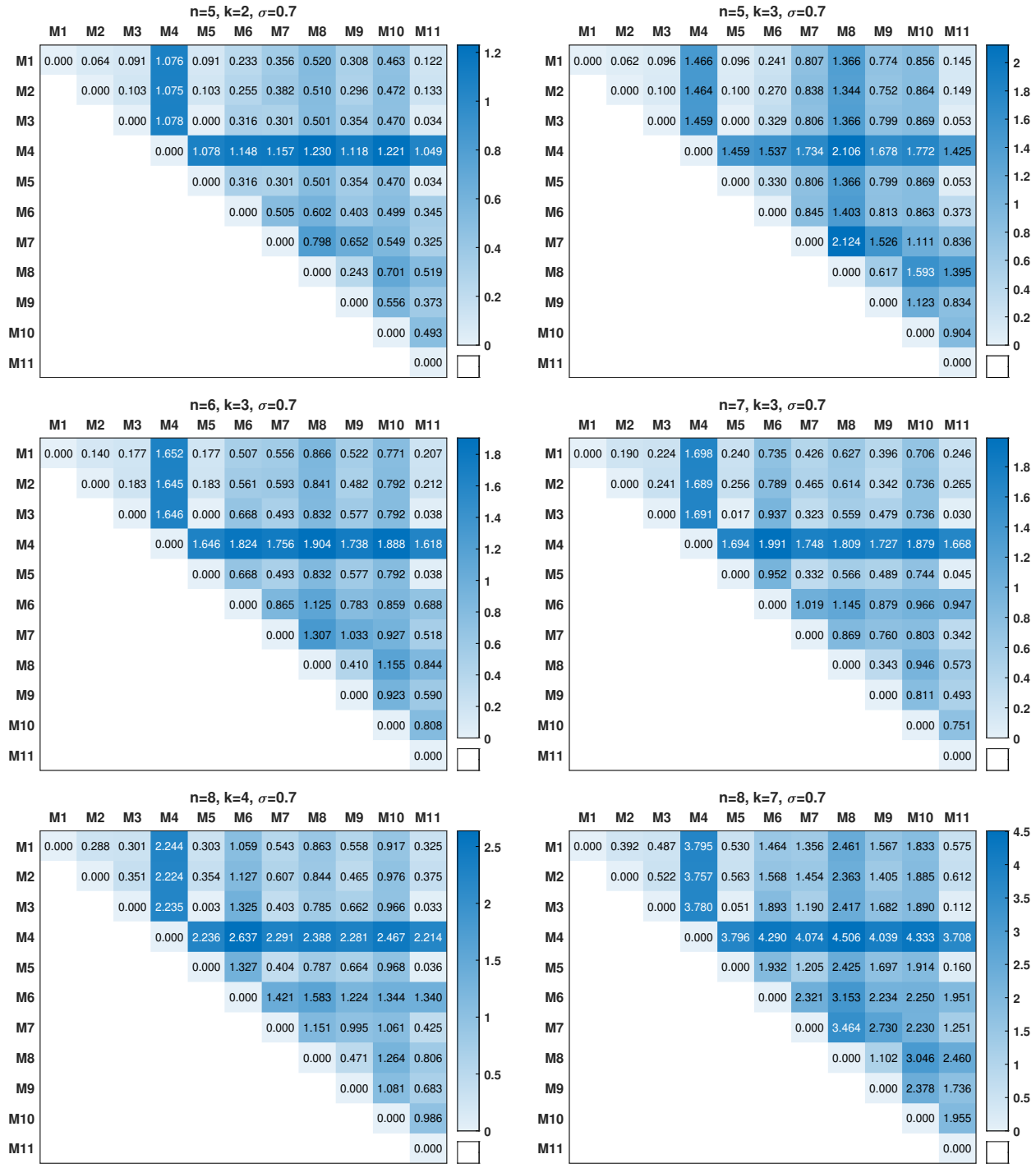


Figure 3.7: Heatmaps and distances $\mathcal{D}^*$ over 1000 instances using $\sigma = 0.7$ for different values of n and k. Reprinted from Publication II.

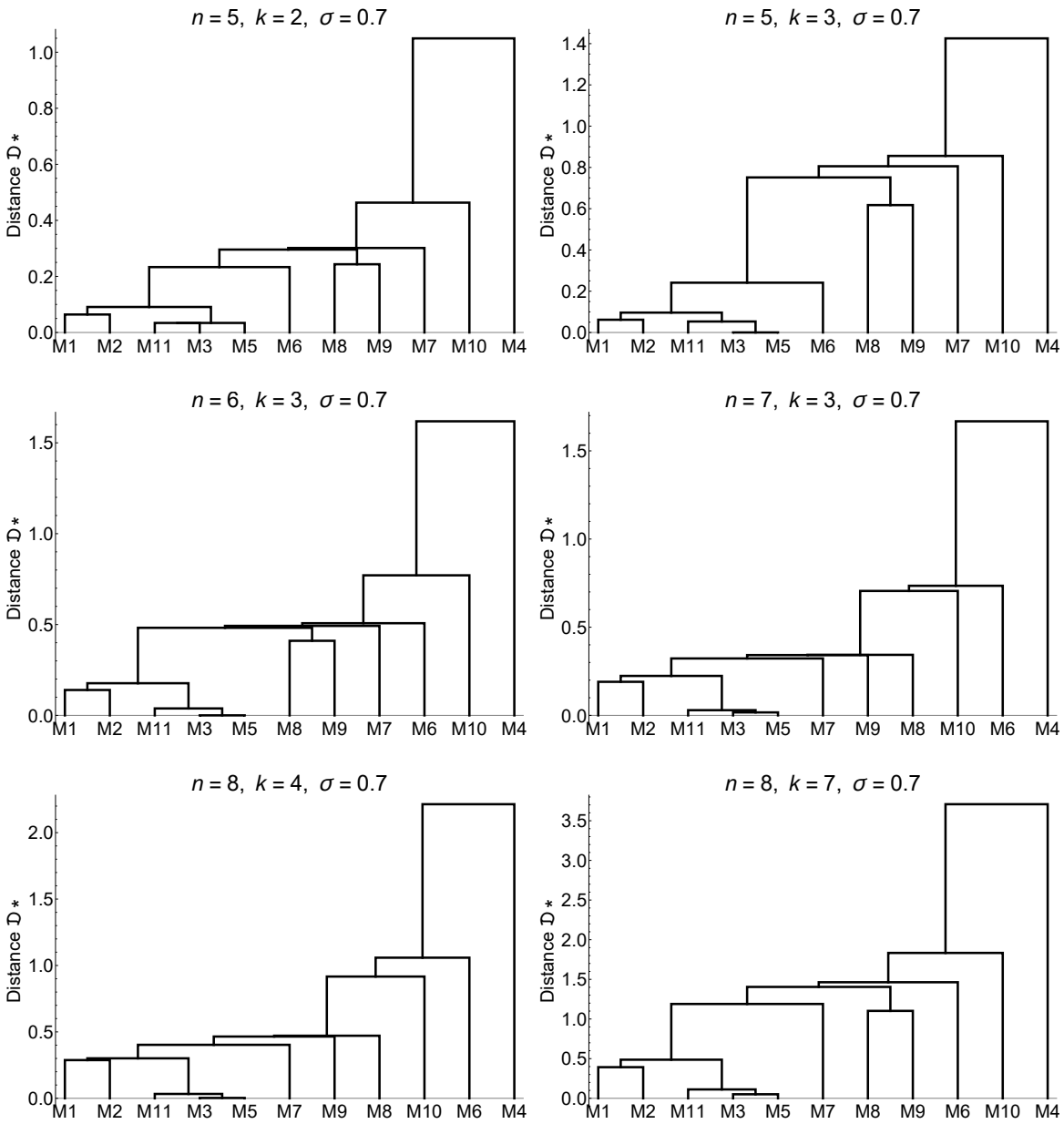


Figure 3.9: Hierarchical clustering of the eleven completion methods corresponding to Figure 3.7. Reprinted from Publication II.

PCM

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 & 1 & \mathbf{6} \\ \frac{1}{4} & 1 & 5 & 4 & \frac{1}{2} \\ \frac{1}{7} & \frac{1}{5} & 1 & 2 & 1 \\ 1 & \frac{1}{4} & \frac{1}{2} & 1 & 9 \\ \mathbf{\frac{1}{6}} & 2 & 1 & \frac{1}{9} & 1 \end{pmatrix}.$$

If we examine its value with those found through non-trivial indirect comparisons $a_{1j}a_{j5},\ \forall j \neq 1, 5$, we get

$$a_{12}a_{25} - a_{15} = 2 - 6 = -4$$
$$a_{13}a_{35} - a_{15} = 7 - 6 = 1$$
$$a_{14}a_{45} - a_{15} = 9 - 6 = 3$$

which draws attention to the existence of inconsistencies. Nevertheless, when they are added up using Eq. (3.14), they provide a null bias because the errors indeed exist but cancel one another out. The difference between the ideas of bias and inconsistency makes the selection of the completion method even more crucial.

We verified that, through numerical simulations, the 11 completion methods yield the same completed (and consistent) PCM when only considering incomplete PCMs corresponding to connected graphs, if the initial complete PCMs are all consistent.

It is important to show whether the similarity between two completion methods is related with the similarity of the weight vectors extracted from the complete PCMs. Hence, the weight vector (normalized eigenvector) is calculated from the completed PCM $\mathbf{C}$ using the right eigenvector method:

$$\lambda_{\max}\mathbf{w} = \mathbf{C}\mathbf{w}$$

where $\lambda_{\max}$ is the the maximum eigenvalue of $\mathbf{C}$. Then, the Manhattan distance $\hat{d}_{s,t}$ between two weight vectors is calculated:

$$\hat{d}_{s,t}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} |u_i - v_i| \tag{3.27}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$ and $\mathbf{v} = (v_1, \ldots, v_n)^T$ are weight vectors derived from the same completed PCM by applying two different methods M_s and M_t ($s, t = 1, 2, \ldots, 11$), whereas $\mathcal{D}_{s,t}$ (see Eq. (3.26)) represents the distance between two completed matrices from two different methods M_s and M_t. It is worth noting that all weight vectors are derived from the completed matrices (after completion) and normalized ($\sum_{i=1}^{n} u_i = 1$ and $\sum_{i=1}^{n} v_i = 1$).

We considered the two extreme cases: the most similar methods (M1 and M2), and significantly dissimilar (M1 and M4). Scatter plots are provided in Figures 3.10 and 3.11, respectively. Spearman's rank correlation coefficients suggest that the two variables (distance between completed matrices versus distance between their weight vectors) have a strong co-monotonic relationship.

All calculations and visualizations were carried out using MathWorks MATLAB R2021b and Wolfram Mathematica 12.1.

3.3 Publication III

Publication III represents, "Filling in pattern designs for incomplete pairwise comparison matrices: (quasi-)regular graphs with minimal diameter" (Szádoczki et al., 2022). The goal of the research was to develop an innovative approach to filling in pattern designs using graphs that shows the patterns of the comparisons that have to be made, based on the concept of diameter which was missing in the relevant literature, and that contains a collection of (quasi-)regular graphs with diameters of 2 and 3, degrees of 3, 4 and 5, and vertices from 5 up to 24.

The arrangements of the known comparisons are particularly important, and prior to the decision-making process, the set of comparisons is carefully planned. We consider the items to be compared in a symmetric manner without any further prior information. Every item is compared to the same number of elements, which creates a sort of symmetry. However, we avoid situations when two items are compared only indirectly via a very long path. Thus, we search for graphs with a minimum diameter (d parameter) for a particular number of vertices (n) and regularity levels (s). When the d parameter is small, our system of comparisons becomes more reliable or stable. The s parameter is also important to show us how many comparisons have to be made and avoid some cases for a given criteria

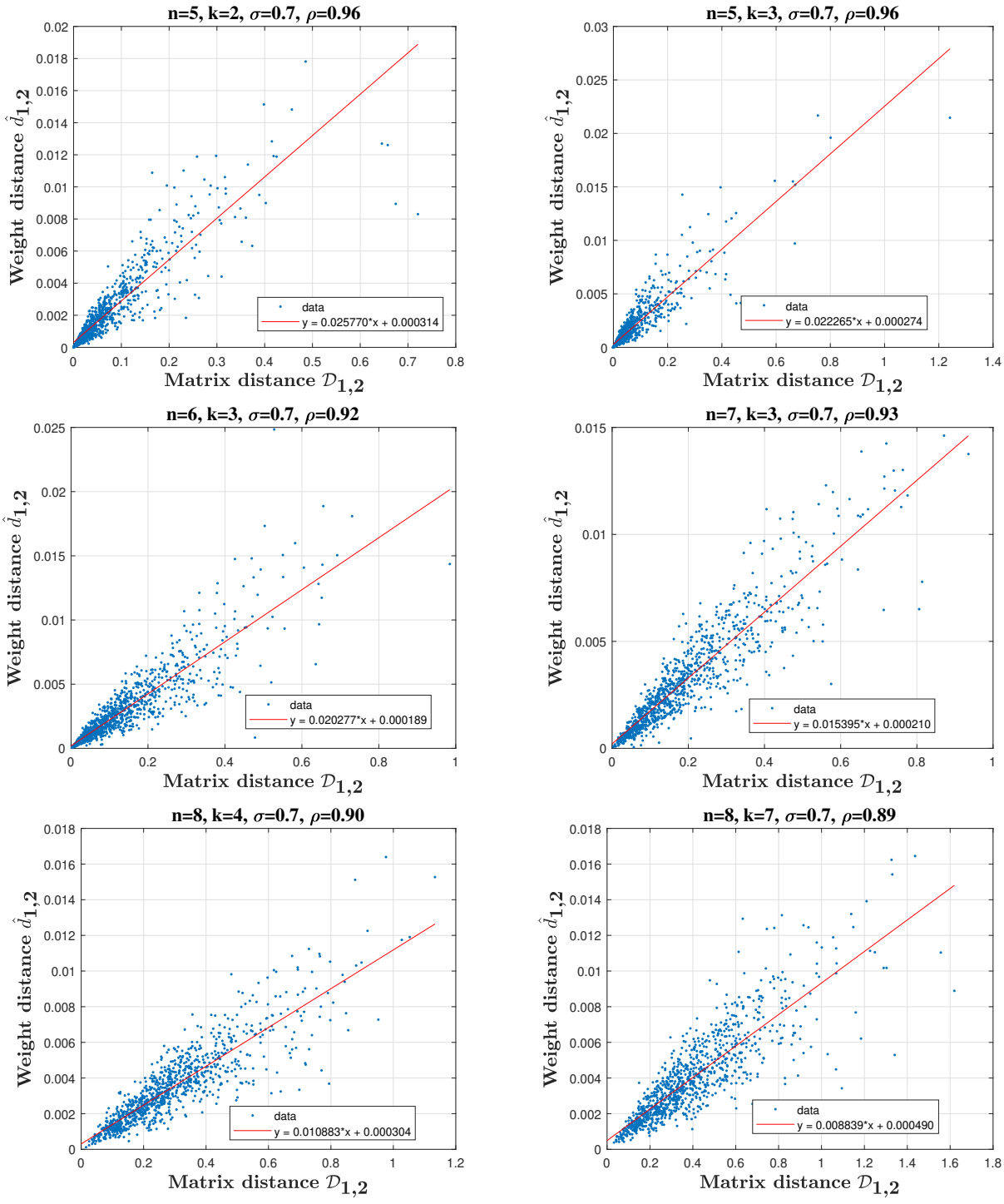


Figure 3.10: The relation between distances from the completed PCMs ($\mathcal{D}_{1,2}$) and distances from weight vectors ($\hat{d}_{1,2}$) computed on 1000 perturbed PCMs with $\sigma = 0.7$. The indices $\{1, 2\}$ indicate the methods M1 and M2, respectively. The value of ρ indicates the Spearman rank correlation coefficient. Reprinted from Publication II.

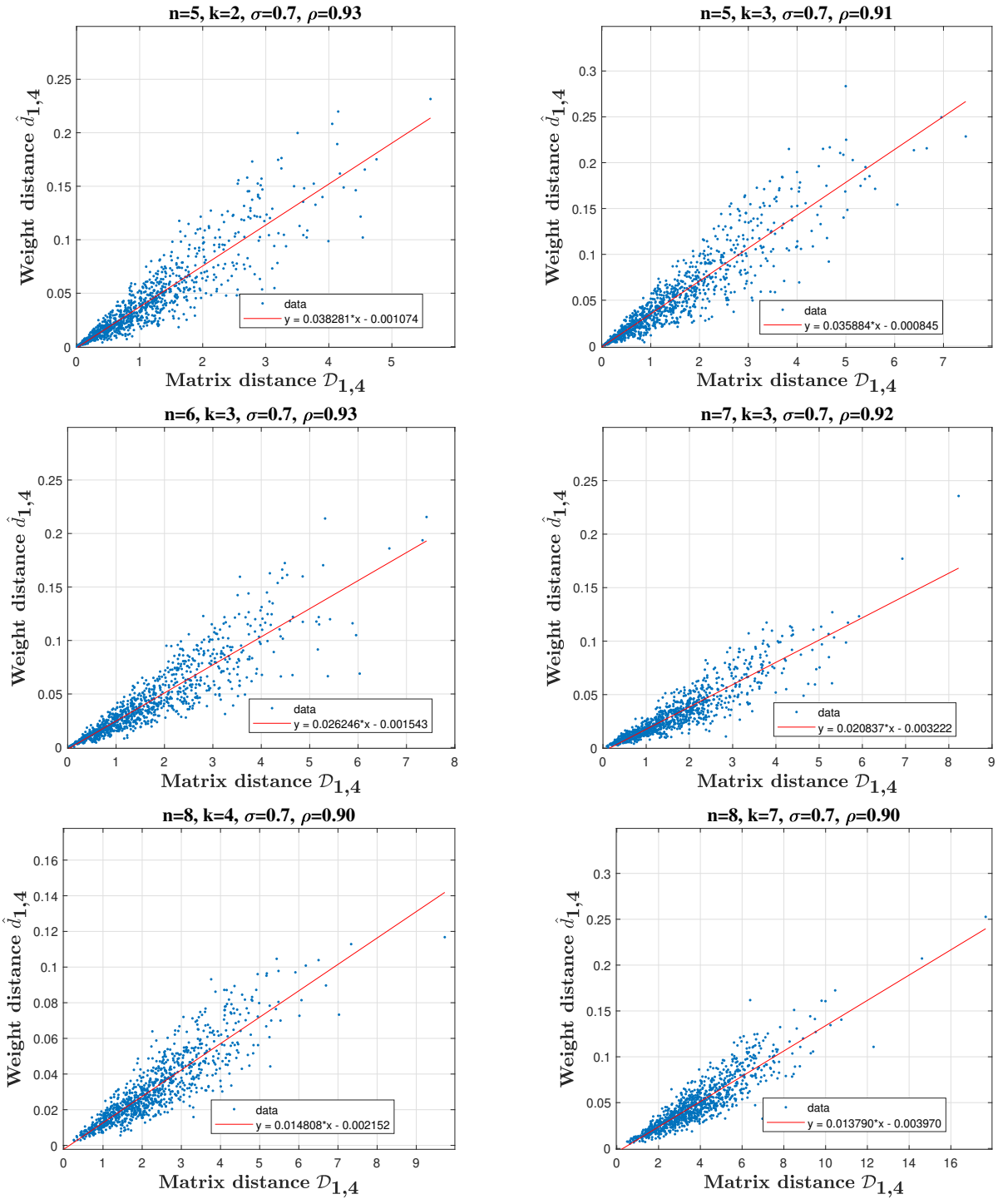


Figure 3.11: The relation between distances from the completed PCMs ($\mathcal{D}_{1,4}$) and distances from weight vectors ($\hat{d}_{1,4}$) computed on 1000 perturbed PCMs with $\sigma = 0.7$. The indices $\{1, 4\}$ indicate the methods M1 and M4, respectively. The value of ρ indicates the Spearman rank correlation coefficient. Reprinted from Publication II.

(vertices) because every vertex has a degree of s and hence the number of edges is $ns/2$. Hence, if the decision-maker wants to create a PCM as quickly as possible, he/she should use a small s value. However, usually there should be a trade-off between the parameters because for many criteria n, the smaller regularity s will result in a more fragile system of comparisons, or a larger diameter d.

Computational and constructing methods have been used to determine the graph(s) with minimal diameter d for a given (n, s) pair. Using these methods, it was simple to identify which s is the minimum required to reach a certain d for a given n. We identified that the suitable values for the regularity at the selected upper bound of $n = 24$ are $s = 3, 4, 5$ and the suitable values for the graph's diameter are $d = 2, 3$. For many (n, s) pairs, the diameter $d = 1$ would logically imply a complete graph and hence a complete PCM, neither of which are important to us.

For a general MCDM problem, the *completion ratio* may provide more information as an indicator that shows how far we are from the 'extreme' case, in which the decision-makers must perform all the $\frac{n(n-1)}{2}$ comparisons rather than considering only $\frac{ns}{2}$ comparisons in case of regular graphs or $\frac{ns+1}{2}$ comparisons in case of quasi-regular graphs. Therefore, the completion ratio is defined as follows:

$$c = \begin{cases} \frac{ns/2}{n(n-1)/2} & \text{if } n \text{ or } s \text{ is even} \\ \frac{(ns+1)/2}{n(n-1)/2} & \text{if } n \text{ and } s \text{ are odd} \end{cases} \tag{3.28}$$

that will be calculated for every instance.

The different construction techniques that have been used for the collection of our graphs for a given (n, s, d) pair are the following.

1. We used the built-in graphs in Wolfram Mathematica (Wolfram Research, 2020) as a starting reference point and the ones having the minimal diameter will be selected.

2. We used nauty and Traces (McKay and Piperno, 2014) and IGraph/M (Horvát, 2020) as construction methods for smaller and middle-sized graphs to generate all the possible (quasi-)regular graphs. Then the required ones could be selected. For instance, consider the following result, which has been found by using the first technique nauty and Traces. As can be seen

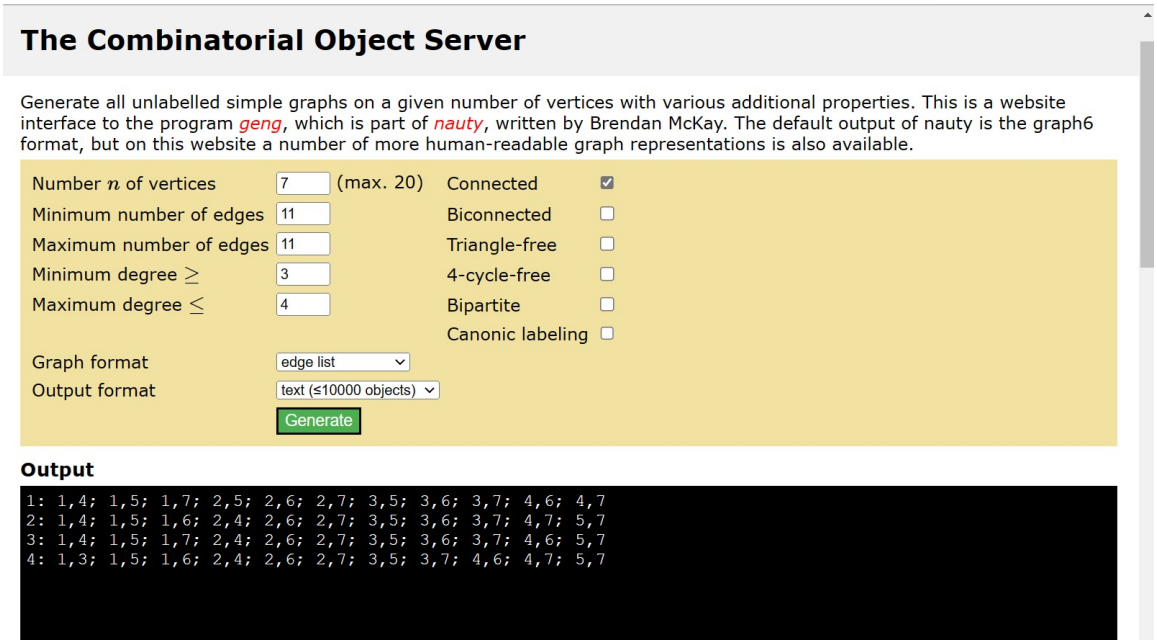


Figure 3.12: The construction of 3-quasi-regular graph on 7 vertices with the help of *nauty geng* (COS++, 2021).

in Figure 3.12, if we provide the number of vertices $n = 7$, minimum degree $s = 3$, maximum degree 4 (because of quasi-regular graph), maximum and minimum number of edges $(ns+1)/2 = (7\times 3+1)/2 = 11$ (see Eq. (3.28)), then we can find 4 possible graphs quickly (displayed in the output as edge list). In this example, all of the four generated graphs are 3-quasi-regular graphs with diameter 2.

3. We collected the well-known graphs from different literature, such as the Petersen graph (Holton and Sheehan, 1993), Wagner graph (Maharry and Robertson, 2016) and prism graph (Pratt, 1996).

4. We used several construction techniques to generate all the possible regular graph designs for larger graphs, such as the twisted product, integer linear programming or merging and extending techniques with the help of some existing graphs, although many of these cases were challenging and time-consuming.

We defined s-quasi-regularity, which are our findings and the first to be used in this context. Most of the quasi-regular graphs have been constructed using a twisted product of two graphs (see Definition 16), while few have been found using *integer linear programming* or strategies for extending and merging with the help of some existing graphs. For instance, the 5-quasi-regular graph on 21

vertices has been found using the method of twisted product $K_3 * X_7$, where K_3 is a complete graph with three vertices, whereas X_7 is a graph with a diameter of 2 on 7 vertices in which all of the vertices have degree 3 with the exception of one which has a degree of 4. The 5-regular graph with 22 vertices has been obtained using the following *binary integer programming*:

Let $V = \{1, \ldots, 22\}$ be the set of vertices, and let $P = \{i \in V, j \in V : i < j\}$ be the pairs of vertices. For $(i, j) \in P$, let $X_{i,j}$ be a binary decision variable that indicates whether (i, j) is an edge. For $(i, j) \in P$ and $h \in V \setminus \{i, j\}$, let $Y_{i,j,h}$ be a binary decision variable that indicates whether h is a common neighbor of i and j. For $(i, j) \in P$, let $SLACK_{i,j}$ be a slack variable.

$$\min \sum_{(i,j)\in P} SLACK_{i,j}$$

$$\sum_{(i,j)\in P:h\in\{i,j\}} X_{i,j} = 5 \qquad \text{for } h \in V \quad (1)$$

$$X_{i,j} + \sum_{h\in V\setminus\{i,j\}} Y_{i,j,h} + SLACK_{i,j} \geq 1 \qquad \text{for } (i,j) \in P \quad (2)$$

$$Y_{i,j,h} \leq X_{i,h}[\text{if } i < h] + X_{h,i}[\text{if } h < i] \text{ for } (i,j) \in P \text{ and } h \in V \setminus \{i,j\} \quad (3)$$

$$Y_{i,j,h} \leq X_{j,h}[\text{if } j < h] + X_{h,j}[\text{if } h < j] \text{ for } (i,j) \in P \text{ and } h \in V \setminus \{i,j\} \quad (4)$$

$$X_{i,j},\ X_{i,h},\ X_{h,i},\ X_{h,j},\ X_{j,h},\ Y_{i,j,h},\ SLACK_{i,j} \in \{0,1\} \text{ for all } i,j,h \in V \quad (5)$$

5-regularity is enforced by constraint (1). Diameter 2 is enforced by constraint (2). Constraints (3) and (4) enforce that $Y_{i,j,h} = 1$ implies h is a neighbor of i and j, respectively. The existence of the required graph is guaranteed if and only if the binary integer programming has a solution with $SLACK_{i,j} = 0$, $\forall$ $(i, j) \in P$. Reprinted from Publication III.

The problem has 5082 variables, 9493 constraints, and 1 objective function. Its solution can be found using standard optimization software/solver, such as *Gurobi with AMPL* or *Gurobi with Python*.

Table 3.1 shows graphs with the minimal values of regularity $s = 3$ and diameters $d = 2$ (left-sided graphs) and $d = 3$ (right-sided graphs) from vertices of $n = 5$ up to $n = 16$. Graphs with vertices $n = 1, 2, 3, 4$ are not considered because of the

following reasons: (i) For $n \leq 3$, the 3-regularity is not possible; (ii) For $n = 4$, the graph diameter is 1, because this is a complete graph and forms a complete PCM. It is also worth noting that 3-regular graphs are only examined when n is even, while the 3-quasi-regular graphs (the closest graphs to 3-regularity) are examined when n is odd. Furthermore, it should be noted that the completion ratio (c) is decreasing while n increases. Thus, we should stress the fact that

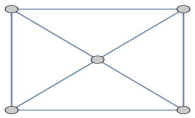
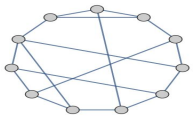
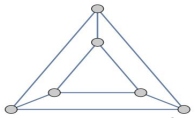
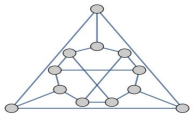
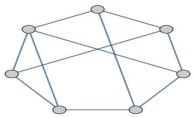
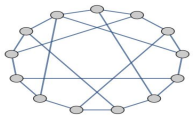
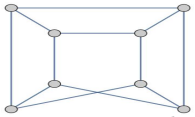
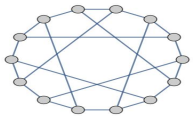
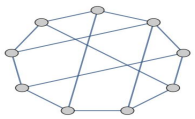
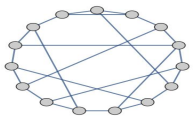
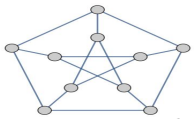
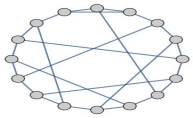
s=3 d=2	Graph	Further Information	s=3 d=3	Graph	Further Information
n=5		• $c = 8/10 = 0.8$ • 1 graph	n=11		• $c = 17/55 \approx 0.309$ • 134 graphs
n=6	 3-prism graph ($C_3 \times K_2$)	• $c = 9/15 = 0.6$ • 2 graphs	n=12	 Tietze graph	• $c = 18/66 \approx 0.273$ • 34 graphs
n=7		• $c = 11/21 \approx 0.524$ • 4 graphs	n=13		• $c = 20/78 \approx 0.256$ • 353 graphs
n=8	 Wagner graph	• $c = 12/28 \approx 0.429$ • 2 graphs	n=14	 Heawood graph	• $c = 21/91 \approx 0.231$ • 34 graphs
n=9		• $c = 14/36 \approx 0.389$ • 2 graphs	n=15		• $c = 23/105 \approx 0.219$ • 290 graphs
n=10	 Petersen graph	• $c = 15/45 \approx 0.333$ • 1 graph	n=16		• $c = 24/120 = 0.2$ • 14 graphs

Table 3.1: 3-(quasi-)regular graphs on n vertices with minimal diameter $d = 2, 3$. Reprinted from Publication III.

there are very few graphs for each pair (n, s) with the smallest diameter.

When n is becoming large, the diameter goes up but the completion ratio is still decreasing in n. However, the completion ratio is increasing in s. We exam-

ined the interesting graphs with different cases, which can assist anyone with an MCDM problem in determining which comparisons need to be made. The summary of the results is provided in Table 3.2, which contains the number of possible (quasi-)regular graphs for a given (n, s, d) pair. Graphs with diameter $d = 2$ are represented by *lightgray*, while graphs with diameter $d = 3$ are represented by *gray*. '$\geq$' represents that there are at least as many possible graphs. An open problem, when $n = 23$ and $s = 5$ with diameter $d = 3$, is represented by '?'. The remaining graphs and their corresponding filling in pattern designs can be seen in Publication III (Szádoczki et al., 2022), provided as a supplementary material to the paper in the Appendix.

	s		
n	**3**	**4**	**5**
5	1		
6	2		
7	4		
8	2		
9	2		
10	1		
11	134	37	
12	34	26	
13	353	10	
14	34	1	
15	290	1	
16	14		≥ 3
17	51		≥ 1
18	1		≥ 1
19	4		≥ 1
20	1		≥ 1
21		≥ 3	≥ 1
22		≥ 1	≥ 1
23		≥ 1	?
24		≥ 1	≥ 1

Table 3.2: Results' summary: the number of s-(quasi-)regular graphs on n vertices with diameter d. Lightgray represents $d = 2$ and gray represents $d = 3$, '$\geq$' refers that there are at least as many possible graphs. '?' indicates that it is still an open problem. Reprinted from Publication III.

Furthermore, an extensive numerical simulation has been carried out to validate our recommended filling in pattern designs. As for the priority vectors derivation techniques from the incomplete PCMs, we applied the two well-known completion methods: Incomplete Logarithmic Least Squares Method (ILLSM)

and Eigenvector Method based on the CR-minimal completion (Bozóki et al., 2010). We used two metrics (distance measures)—the Euclidean distance and the Chebyshev distance—to calculate the differences between the weight vectors derived from a certain filling in designs and from a complete PCMs. For each of the examined filling in pattern's parameter combinations (n, s, d), 1000 perturbed PCMs were used at three different levels of inconsistency (perturbation) to compare various filling in patterns with the recommended ones. Finally, the results suggested that the (quasi-)regular graphs with minimal diameter provide the closest priority vectors on average than the other alternative graphs with the same number of comparisons. For more detailed methodology and results of all parameter combinations, see the Appendix of Publication III.

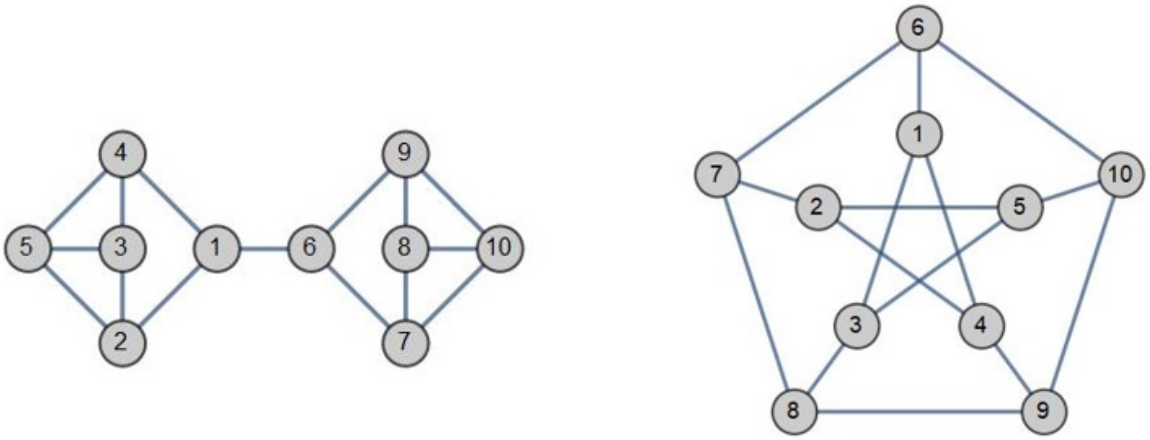
3.3.1 Motivational example

This example illustrates the importance of the diameter for better filling in designs having 10 alternatives. Among 3-regular graphs on 10-vertices, the Petersen graph has the smallest diameter, which is 2, while the Alternative 3-regular graph has a diameter 5.

A numerical simulation has also been performed to compare the two different filling in patterns (see Table 3.13) represented by the two graphs (see Figure 3.13). In contrast to the alternative graph's filling in patterns, the simulation results suggested that the Petersen graph's filling in patterns typically has small errors and better performance (small standard deviation). As a result, it appears that our recommendation performs effectively in the context of pairwise comparison matrices.

The main contribution of the research project was a systematic collection of filling pattern designs for incomplete PCMs through graph representations of PCMs. Our results—the proposed (quasi)-regular graphs with minimal diameter—can be used for both theorists and practitioners in multi-criteria decision-making, given in several formats: graph, adjacency matrix, list of edges and 'Graph6' code (provided in the appendix of Publication III). For instance, practitioners can utilize our recommended graphs as a 'recipe' in designing questionnaires based on pairwise comparisons. Our findings can also be utilized in group decision-making as we treat the individual preferences to be compared in a symmetric way and

that we don't need to assume any prior knowledge. Although our findings were presented using paired comparison matrices, their applicability is much broader. For instance, the problem of tournament design is brought on by the ranking of sport players or teams based on their matches: which pairs should compete against one another without having prior information about their strengths?



(a) An alternative 3-regular graph

(b) Our recommended graph: the Petersen graph

Figure 3.13: Two 3-regular graphs. Reprinted from Publication III.

	1	2	3	4	5	6	7	8	9	10
1	1	a_{12}		a_{14}		a_{16}				
2	$\frac{1}{a_{12}}$	1	a_{23}		a_{25}					
3		$\frac{1}{a_{23}}$	1	a_{34}	a_{35}					
4	$\frac{1}{a_{14}}$		$\frac{1}{a_{34}}$	1	a_{45}					
5		$\frac{1}{a_{25}}$	$\frac{1}{a_{35}}$	$\frac{1}{a_{45}}$	1					
6	$\frac{1}{a_{16}}$					1	a_{67}		a_{69}	
7						$\frac{1}{a_{67}}$	1	a_{78}		a_{710}
8							$\frac{1}{a_{78}}$	1	a_{89}	a_{810}
9						$\frac{1}{a_{69}}$		$\frac{1}{a_{89}}$	1	a_{910}
10							$\frac{1}{a_{710}}$	$\frac{1}{a_{810}}$	$\frac{1}{a_{910}}$	1

	1	2	3	4	5	6	7	8	9	10
1	1		a_{13}	a_{14}		a_{16}				
2		1		a_{24}	a_{25}		a_{27}			
3	$\frac{1}{a_{13}}$		1		a_{35}			a_{38}		
4	$\frac{1}{a_{14}}$	$\frac{1}{a_{24}}$		1					a_{49}	
5		$\frac{1}{a_{25}}$	$\frac{1}{a_{35}}$		1					a_{510}
6	$\frac{1}{a_{16}}$					1	a_{67}			a_{610}
7		$\frac{1}{a_{27}}$				$\frac{1}{a_{67}}$	1	a_{78}		
8			$\frac{1}{a_{38}}$				$\frac{1}{a_{78}}$	1	a_{89}	
9				$\frac{1}{a_{49}}$				$\frac{1}{a_{89}}$	1	a_{910}
10					$\frac{1}{a_{510}}$	$\frac{1}{a_{610}}$			$\frac{1}{a_{910}}$	1

Table 3.3: Two different filling in patterns represented by the graphs in Figure 3.13 based on the known entries a_{ij} of the PCM provided. The pattern corresponding to the Alternative graph can be seen to the left side, while the Petersen graph's filling in design is shown on the right side of the PCM. Reprinted from Publication III.

Chapter 4

Discussion and conclusions

4.1 Discussion

4.1.1 Summary of research findings and their implications

The thesis has addressed three research problems/questions related to incomplete pairwise comparison matrices (PCMs).

In Section 3.1 (Publication I), our study focused on an application of the Nelder-Mead algorithm for solving a constrained eigenvalue minimization problem using the coordinate transformation techniques. Several numerical simulations were conducted to evaluate the performance of the algorithm in tackling the minimization problem subject to interval constraints. The simulations results indicated that the algorithm has the ability to effectively estimate the missing values. Moreover, the solution obtained is unique, provided that the undirected graph associated with the incomplete PCM is connected. Additionally, the proposed algorithm addresses a gap in the existing literature by producing an optimal point that can lie on the boundary of the interval, unlike other existing methods, such as the cyclic coordinates method (Bozóki et al., 2010), which returns the optimal point within the interval's interior—as opposed to the boundary—due to the slow convergence of the method. The findings of the study can aid decision-makers in completing partial information in PCMs. Consequently, the suggested algorithm can be easily incorporated and transformed into tools for multi-criteria decision-making, enabling practitioners to use it conveniently.

In Section 3.2 (Publication II), we analyzed the eleven completion methods by

examining the completed matrices numerically and comparing the results. Hierarchical cluster analysis was utilized to identify the similarities and highlight differences between the methods. The significance of conducting numerical studies to analyze and compare various completion methods becomes more important when there is a lack of a comprehensive axiomatic framework. Surprisingly, there was recently no proof demonstrating the similarity or differences between various completion methods in terms of the minimal diversity between the completed matrices. It was unknown whether these methods can be used interchangeably with minimal variation in results from a practical perspective. From the numerical perspective, the main contributions of the second research project are: (i) The results indicate that five methods are more similar than the others, while one method differs greatly from the others; (ii) The study has pointed out a difference in the results obtained from bias-based methods and inconsistency-based methods. Specifically, the fourth method showed significant divergence from the other inconsistency-based methods. This highlights the importance of understanding the distinction between bias and inconsistency when selecting completion methods. It also presents an avenue for further research to explore this difference; (iii) The strength of the relationship between the distance between completed matrices and the distance between their weight vectors can be inferred from Spearman's rank correlation coefficients, which indicate a strong co-monotonic relationship between the two variables.

In Section 3.3 (Publication III), we provided a systematic collection of filling in patterns (arrangements of known elements) of incomplete pairwise comparisons with the help of undirected graphs under given conditions: diameter, regularity and number of vertices. The minimal diameter was considered to be the key concept in our research, which was missing in the relevant literature. The proposed graphs, which have minimal diameter and are (quasi-)regular, are not only relevant in terms of graph theory but also demonstrate their significance in multicriteria decision-making. They have potential applications in group decision-making (Oliva et al., 2019) with diverse individual preferences. Moreover, our results using pairwise comparisons can be applied to ranking sports players or teams for tournament design without prior knowledge of their strengths. The results of the study are most applicable to these specific types of MCDM problems

and may not be directly generalizable to other types of preference relations. However, the approach presented in the thesis, which uses (quasi-)regular graphs to fill in the patterns of incomplete PCMs, could potentially be adapted and applied to other types of decision-making problems that involve pairwise comparisons.

4.1.2 Limitations and future research directions

It is important to acknowledge the limitations of our research design and consider how they may affect the validity and generalizability of our findings. Our research study relied on a numerical simulation approach instead of an empirical approach, despite the fact that we employed randomly generated PCMs and perturbed to be somewhat representative of real-world instances. Empirical research provides actual data from real-world observations or experiments, which can be used to test hypotheses and validate assumptions (e.g., Bozóki et al. (2013)). However, it can be time-consuming, expensive, and subject to biases and errors. On the other hand, numerical simulations involve creating mathematical models or algorithms to generate hypothetical pairwise comparisons based on specified parameters or assumptions. They can be used to explore the properties of different types of PCMs, evaluate the sensitivity of different methods, and generate data for mathematical/statistical analyses. However, they rely on assumptions and simplifications that may not accurately reflect real-world preferences. To address these limitations, we may consider incorporating empirical research in future studies.

In Section 3.1 (Publication I), while conducting simulations, we did not encounter any issues with the convergence of the algorithm. However, it is worth noting that we relied on numerical simulations rather than theorems to evaluate its convergence, which could be a limitation of our study. Further investigation may be necessary to address this potential limitation.

In Section 3.2 (Publication II), we conducted a comparative analysis using cardinal consistency, but did not consider ordinal consistency in our analysis. As a result, we did not include other methods that rely on ordinal consistency (e.g., Yuan et al. (2023)). Additionally, it may be a subject of future research to examine the reliability of weight vectors derived from incomplete PCMs employing completion (and prioritization) methods.

In Section 3.3 (Publication III), based on the numerical simulations results, it appears that the proposed (quasi-)regular graphs with minimal diameter outperform other alternative graphs with same number of comparisons, as per the simulation design used in the study. Nevertheless, this does not exclude the chance that there may be other simulation strategies/designs that are even more advantageous, which we have not yet identified. Additionally, while our numerical simulations provide some evidence to support the confirmation of our recommendations, it would be nice to prove our conclusions as a theorem. Furthermore, the examination of filling in pattern designs with vertices of $n = 23$, regularity of $s = 5$, and diameter of 2 remains unresolved and necessitates additional research. The computational capacity for simulations also limited the number of alternatives that can be considered. Moreover, exploring the robustness of the results between various regularity levels, between different diameters, and larger minimal diameters may require further study and could be a focus of future research.

4.2 Conclusions

The thesis recognizes the importance and relevance of incomplete PCMs and their generalization to complete PCMs (i.e., with null missing comparisons). The methods and optimization techniques developed for incomplete matrices are valuable tools for decision-making in practical settings, where incomplete information is often the norm rather than the exception. Assume that, *CPCMs* represents the set of all complete pairwise comparison matrices. *CGPCMs* represents the set of all pairwise comparison matrices with connected graphs, but may not be complete PCM. Moreover, *(Possibly) Incomplete PCMs* represents the set of all incomplete pairwise comparison matrices, including those where the graph of comparisons may not be connected. Figure 4.1 depicts the inclusion relations '$\subseteq$' between the sets of PCMs:

$$\text{CPCMs} \subseteq \text{CGPCMs} \subseteq \text{(Possibly) Incomplete PCMs.}$$

To sum up, the thesis dealt with three research problems that are associated with incomplete PCMs and offers answers/solutions to overcome them. The study of the first research problem proposed an efficient optimization algorithm, the Nelder-Mead algorithm, for solving the constrained eigenvalue minimization

problem with interval constraints. The proposed algorithm effectively estimates missing values and produces a unique optimal solution that can lie on the boundary of the interval, addressing a gap in the existing literature.

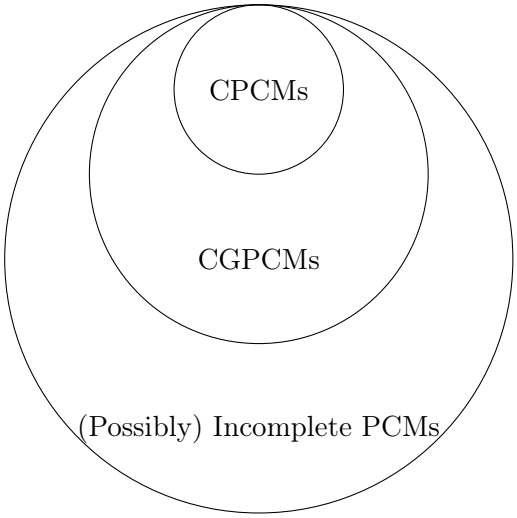


Figure 4.1: The inclusion relations between the sets of PCMs.

The second research problem analyzed eleven completion methods using numerical simulations and hierarchical cluster analysis. The study found that five methods were more similar than others, one method differed greatly, and there was a difference between bias-based and inconsistency-based methods. The study highlighted the importance of understanding this distinction when selecting completion methods and inferred a strong relationship between the distance between completed matrices and the distance between their weight vectors. The study's findings provide valuable insights for selecting appropriate completion methods and suggest avenues for further research.

The third research problem focused on developing an innovative approach to filling in pattern designs using graphs for incomplete PCMs. We aimed at identifying the minimum value of diameter required to reach a certain regularity for a given number of vertices. The study emphasized the importance of the arrangement of known comparisons in the decision-making process, and provided different construction techniques for generating (quasi-)regular graphs with minimal diameter for a given (n, s) pair. The completion ratio has also been defined as an indicator that shows how far decision-makers are from performing all comparisons. The different completion methods, such as ILLSM (Incomplete Logarithmic Least Squares Method) and Eigenvector Method for incomplete PCMs, were used to

derive the priority vectors from the incomplete PCMs to perform the numerical simulations. Finally, the research presented a systematic approach to filling in pattern designs for incomplete PCMs, which can assist decision-makers with an MCDM problem in determining which comparisons need to be made. We provided a collection of (quasi-)regular graphs with diameters of 2 and 3, degrees of 3, 4, and 5, and vertices from 5 up to 24. Furthermore, we validated our recommended filling in pattern designs through an extensive numerical simulations.

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