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On some intrinsic differentiability properties for absolutely continuous functions between Carnot groups and the area formula



Andrea Pinamonti*, Francesco Serra Cassano, Kilian Zambanini

Dipartimento di Matematica, Università di Trento, Via Sommarive, 14, 38123 Povo TN, Italy

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ABSTRACT

We discuss Q -absolutely continuous functions between Carnot groups, following Malý's definition for maps of several variables ([43]). Such maps enjoy nice regularity properties, like continuity, Pansu differentiability a.e., weak differentiability and an area formula. Furthermore, we extend Stein's result concerning the sharp condition for continuity and differentiability a.e. of a Sobolev map in terms of the integrability of the weak gradient: more precisely, we prove that a Sobolev map between Carnot groups with horizontal gradient of its sections uniformly bounded in $L^{Q,1}$ admits a representative which is Q -absolutely continuous.

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1. Introduction

In this paper we study Sobolev regularity, continuity and Pansu differentiability for maps $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ satisfying suitable intrinsic notions of absolute continuity or

* Corresponding author.

E-mail addresses: andrea.pinamonti@unitn.it (A. Pinamonti), francesco.serracassano@unitn.it (F. Serra Cassano), kilian.zambanini@unitn.it (K. Zambanini).

bounded variation, called *Q-absolute continuity* and *bounded Q-variation*, where $\mathbb{G}_1$ and $\mathbb{G}_2$ are Carnot groups and Ω is an open set in $\mathbb{G}_1$. In particular, we will apply such a study for extending in this setting an analogous of the remarkable Euclidean Stein's result [55] concerning the sharp condition providing continuity and differentiability a.e. in terms of integrability of the weak gradient, thus paving the way to the study of the gradient regularity of solutions to the sub-Laplace Poisson equation, see e.g. [34]. Moreover we will prove an area formula for *Q-absolutely continuous* maps between Carnot group.

The definition and main properties of a Carnot group are collected in section 2. Let us quickly recall here that a *Carnot group* $\mathbb{G}$ is a simply connected nilpotent Lie group which can be represented as $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ equipped with:

- a family of *intrinsic dilations* $(\delta_\lambda)_{\lambda>0}: \mathbb{G} \rightarrow \mathbb{G}$, which are automorphisms of $\mathbb{G}$;
- a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, called *horizontal subbundle*, which generates by commutation the whole $T\mathbb{G}$;
- a *Carnot-Carathéodory* (or, also, *control*) *metric* d_c associated to the horizontal subbundle $H\mathbb{G}$, which is left-invariant and homogeneous with respect to the intrinsic dilations.

The Haar measure of $\mathbb{G}$ turns out to be the n -dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ and the metric (Hausdorff) dimension of $(\mathbb{G}, d_c)$ coincides with its homogeneous dimension Q . If $\mathbb{G}$ is commutative, then $\mathbb{G} \equiv \mathbb{R}^n$ as Euclidean vector space; if $\mathbb{G}$ is not commutative, then $Q > n$. In particular, we will see that the homogeneous dimension Q will play the same role of the topological dimension n in the Euclidean setting.

A Carnot group is a sub-Riemannian structure, where several classical differential notions and results of the Euclidean/Riemannian geometry do not apply (see [25,26]). Thus new intrinsic notions are needed in this setting, such as, for instance, the ones of differentiable and Sobolev regular function between Carnot groups (see Definitions 2.5 and 2.18, respectively). Nevertheless, from the intrinsic point of view, some important analogous results to Euclidean setting still hold for maps between Carnot groups, such as a Rademacher differentiability result due to Pansu (see Theorem 2.6) and a Sobolev-Morrey embedding theorem (see, Theorem 2.21).

Let us stress that the intrinsic notion of differentiability for mappings between Carnot groups, and the related results, cannot be always recovered within the theory of differentiation for mappings between metric spaces (see Section B).

The notions of *n-dimensional absolutely continuous function* and *n-variation* for a map $f: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ were initially introduced by Malý in [43]. They have been studied in several papers (see, for instance, [13,29–31]). An historical account as far as the concept of absolute continuity for functions can be found in [43] and the references therein.

These notions can also be introduced in a metric measure setting, thus it is immediate to extend them to maps between Carnot groups. In the following the balls involved are the ones with respect to the CC distance d_c .

Definition 1.1. Let $\mathbb{G}_1 = (\mathbb{R}^n, \cdot, d_1)$ and $\mathbb{G}_2 = (\mathbb{R}^m, *, d_2)$ be two Carnot groups, let Q be the homogeneous dimension of $\mathbb{G}_1$ (see (1)) and let $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$, with Ω open set in $\mathbb{G}_1$.

- (i) f is said to be *Q -absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for each finite disjoint family of open balls $U_1, \dots, U_k$ contained in Ω ,

$$\text{if } \sum_{i=1}^k \mathcal{L}^n(U_i) < \delta \text{ then } \sum_{i=1}^k (\text{osc}_{U_i} f)^Q < \varepsilon$$

where

$$\text{osc}_U f := \text{diam}_{\mathbb{G}_2}(f(U)).$$

- (ii) The *Q -variation* of f is defined as

$$V_Q(f, \Omega) := \sup \left\{ \sum_{i=1}^k (\text{osc}_{U_i} f)^Q : \{U_1, \dots, U_k\} \text{ disjoint family of balls in } \Omega \right\}.$$

We say that f has bounded Q -variation if $V_Q(f, \Omega) < \infty$.

We will denote by $BV^Q(\Omega, \mathbb{G}_2)$ the class of functions $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ with bounded Q -variation and with $AC^Q(\Omega, \mathbb{G}_2)$ the class of Q -absolutely continuous functions. We use the notation $BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$ for functions which are locally of bounded Q -variation and $AC_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$ for the family of functions which are locally Q -absolutely continuous. It actually holds that $AC^Q(\Omega, \mathbb{G}_2) \subset BV^Q(\Omega, \mathbb{G}_2)$ if $\Omega \subset \mathbb{G}_1$ is a bounded open set (see Proposition 3.1).

The study of absolutely continuous functions has been extended from the Euclidean framework to the one of metric measures spaces: firstly, the study of real-valued functions defined on measure metric spaces (see, for instance, [50,51]), and then the one of metric space-valued maps (see, for instance, [61,62,44,35]). We are going to compare later our results with the previous ones.

Note that also in this context the homogeneous dimension Q of the Carnot group $\mathbb{G}_1$ plays the same role of the Euclidean dimension. Indeed, we will see that, roughly speaking, the class $AC_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$ locally lies between the class of Sobolev maps which admit weak horizontal gradient with summability Q , that is the class $W_{\text{loc}}^{1,Q}(\Omega, \mathbb{G}_2)$ (see Definition 2.18), and the union over $p > Q$ of the spaces $W_{\text{loc}}^{1,p}(\Omega, \mathbb{G}_2)$. Moreover it shares good properties with the maps in $W_{\text{loc}}^{1,p}(\Omega, \mathbb{G}_2)$ with $p > Q$.

In section 3 we begin our study by stressing some simple classes of functions contained in $AC^Q(\Omega, \mathbb{G}_2)$. In particular we will see that, if f satisfies the so-called (RR) condition, then $f \in AC^Q(\Omega, \mathbb{G}_2)$ and we will show that, if f is a Sobolev map with exponent $p > Q$, that is $f \in W^{1,p}(\Omega, \mathbb{G}_2)$, then $f \in AC^Q(\Omega, \mathbb{G}_2)$ (see Theorem 3.6). Moreover

we will prove that maps with finite Q -variation between Carnot groups, that is maps in $BV^Q(\Omega, \mathbb{G}_2)$, are Pansu-differentiable a.e. and admit weak horizontal gradient (see Theorems 3.7 and 3.10). As a consequence, since $AC_{\text{loc}}^Q(\Omega, \mathbb{G}_2) \subset BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$, also functions in $AC^Q(\Omega, \mathbb{G}_2)$ share the same properties.

In section 4 we will prove that a Sobolev function $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ admitting weak gradient in the Lorenz space $L^{Q,1}(\Omega)$ admits a representative in $AC^Q(\Omega, \mathbb{G}_2)$ (Theorem 4.2), extending in this way Stein's result on Euclidean spaces. Moreover we will discuss this result also for maps in a suitable class of Orlicz-Sobolev spaces between Carnot groups (see Appendix A and in particular Theorem 6.7).

In section 5 we will show an area formula for functions in $AC^Q(\Omega, \mathbb{G}_2)$ (see Theorem 5.7 and Corollary 5.8), which extends the one proved independently in [39] and in [59] for Lipschitz functions.

Eventually in Appendix B we will collect, for sake of completeness, some relevant notions and results about the differentiability and Sobolev regularity of metric spaces-valued maps defined on a metric measure space, in order to compare our results with the previous ones.

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2. Preliminaries

2.1. Carnot groups

A *Carnot group* $(\mathbb{G}, \cdot)$ is a simply connected and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields has dimension n and admits a stratification of step $s \in \mathbb{N}$, that is

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where the vector spaces $V_1, \dots, V_s \subset \mathfrak{g}$ satisfy

$$V_i = [V_1, V_{i-1}] \quad \text{for } i = 2, \dots, s, \quad [V_1, V_s] = \{0\}.$$

We set $m_i = \dim(V_i)$ and $h_i = \sum_{j=1}^i m_j$ for $i = 1, \dots, s$. We fix an *adapted basis* of $\mathfrak{g}$, i.e. a basis $X_1, \dots, X_n$ such that

$X_{h_{i-1}+1}, \dots, X_{h_i}$ is a basis of V_i , $i = 1, \dots, s$.

The subspace V_1 of $\mathfrak{g}$ is also called *horizontal layer*, while $X_1, \dots, X_{m_1}$ are the *generating vector fields* of the group. We endow the algebra $\mathfrak{g}$ with the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ that makes the basis $X_1, \dots, X_n$ orthonormal. Exploiting the exponential identification $p = \exp(\sum_{i=1}^n p_i X_i)$, we can identify $\mathbb{G}$ with $\mathbb{R}^n$, endowed with the group law determined by the Campbell–Hausdorff formula. In particular, the identity $e \in \mathbb{G}$ corresponds to $0 \in \mathbb{R}^n$ and the *inversion map* becomes $\iota(p) = p^{-1} = -p$ for any $p \in \mathbb{G}$. Moreover, it is not restrictive to assume that $X_i(0) = e_i$ for any $i = 1, \dots, n$. Therefore, by left-invariance, we get

$$X_i(p) = d\tau_p e_i, \quad i = 1, \dots, n,$$

where $\tau_p: \mathbb{G} \rightarrow \mathbb{G}$ is the *left-translation* by $p \in \mathbb{G}$, i.e. $\tau_p(q) = p \cdot q$ for any $q \in \mathbb{G}$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_{m_1}$ plays a particularly important role in the theory and it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span} \{X_1(x), \dots, X_{m_1}(x)\}, \quad x \in \mathbb{G}.$$

For any $i = 1, \dots, n$, the *degree* $d(i) \in \{1, \dots, s\}$ of the basis vector field X_i is $d(i) = j$ if and only if $X_i \in V_j$. The group *dilations* $(\delta_\lambda)_{\lambda \geq 0}: \mathbb{G} \rightarrow \mathbb{G}$ are hence given by

$$\delta_\lambda(p) = \delta_\lambda(p_1, \dots, p_n) = (\lambda p_1, \dots, \lambda^{d(i)} p_i, \dots, \lambda^s p_n) \quad \text{for all } p \in \mathbb{G}.$$

The Haar measure of the group $\mathbb{G}$ coincides with the n -dimensional Lebesgue measure $\mathcal{L}^n$. The homogeneity property $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$ holds for any measurable set $E \subset \mathbb{G}$, where

$$Q := \sum_{i=1}^s i \dim(V_i) \in \mathbb{N} \tag{1}$$

is the *homogeneous dimension* of $\mathbb{G}$. For notational convenience, we will use also the shorthand $|E| = \mathcal{L}^n(E)$.

In the following proposition (see [20]) we collect some fairly elementary properties of the group operation.

Proposition 2.1. *The group product has the form*

$$x \cdot y = x + y + \mathcal{Q}(x, y), \quad \forall x, y \in \mathbb{R}^n \tag{2}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n): \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and each $\mathcal{Q}_i$ is a homogeneous polynomial of degree $d(i)$ i.e.

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{d(i)} \mathcal{Q}_i(x, y), \quad \forall x, y \in \mathbb{G}.$$

Moreover $\mathcal{Q}$ is anti-symmetric, which means

$$\mathcal{Q}_j(x, y) = -\mathcal{Q}_j(-y, -x),$$

and also

$$\begin{aligned} \mathcal{Q}_1(x, y) &= \dots = \mathcal{Q}_{m_1}(x, y) = 0, \\ \mathcal{Q}_j(x, 0) &= \mathcal{Q}_j(0, y) = 0, \\ \mathcal{Q}_j(x, x) &= \mathcal{Q}_j(x, -x) = \mathcal{Q}_j(-x, x) = 0, \quad \text{for } m_1 < j \leq n, \\ \mathcal{Q}_j(x, y) &= \mathcal{Q}_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}}), \quad \text{if } 1 < i \leq s \quad \text{and} \quad j \leq h_i. \end{aligned} \tag{3}$$

Definition 2.2. An absolutely continuous curve $\gamma: [a, b] \rightarrow \mathbb{G}$ is horizontal if there exist $u_1, \dots, u_{m_1} \in L^1[a, b]$ such that

$$\gamma'(t) = \sum_{j=1}^{m_1} u_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [a, b].$$

We define the *horizontal length* of a horizontal curve γ by $L(\gamma) = \int_a^b |u(t)| dt$, where $u = (u_1, \dots, u_{m_1})$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{m_1}$.

Chow-Rashevskii Theorem asserts that any two points of $\mathbb{G}$ can be connected by horizontal curves [7, Theorem 9.1.3]. Actually, any two points can be connected by a geodesic, despite such a curve can be not unique [45, Theorem 1.4.4]. This allows us to define the *Carnot-Carathéodory distance* (*CC distance*) as follows.

Definition 2.3. The *CC distance* between $x, y \in \mathbb{G}$ is

$$d_c(x, y) := \inf\{L(\gamma) \mid \gamma: [0, 1] \rightarrow \mathbb{G} \text{ horizontal joining } x \text{ to } y\}.$$

This CC metric induces the Euclidean topology and it is an *invariant distance*, since it satisfies $d_c(zx, zy) = d_c(x, y)$ (left translation invariance) and $d_c(\delta_r(x), \delta_r(y)) = r d_c(x, y)$ (homogeneity) for $x, y, z \in \mathbb{G}$ and $r > 0$. Every invariant distance d induces a *homogeneous norm*, simply defining $\|x\| := d(x, 0)$. We denote the CC open ball with center x and radius r by $U_r(x)$ or by $U(x, r)$ and in particular we use the shorthand $U_1 = U(0, 1)$. When convenient, we will consider the closed CC ball of center x and radius r , denoted by $B(x, r)$. It can be proved that $\mathcal{L}^n(U_r(x)) = r^Q \mathcal{L}^n(U_1(0))$, where Q is the homogeneous dimension of $\mathbb{G}$ (see (1)). Therefore Q turns out to be the Hausdorff dimension of the metric measure space $(\mathbb{G}, d_c, \mathcal{L}^n)$. In particular note that, if the step of $\mathbb{G}$ is $s \geq 2$, then $Q > n$, which means that the Hausdorff metric dimension of $\mathbb{G}$ is larger than its topological dimension n .

Following [20, Thm. 5.1], we also introduce the left-invariant and homogeneous distance $d_\infty(p, q) = d_\infty(q^{-1} \cdot p, 0)$ for $p, q \in \mathbb{G}$, where, identifying $\mathbb{G} = \mathbb{R}^n = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s}$ and letting $\pi_{\mathbb{R}^{m_i}} : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$ to be the canonical projection for $i = 1, \dots, s$,

$$d_\infty(p, 0) = \max\{\varepsilon_i |\pi_{\mathbb{R}^{m_i}}(p)|_{\mathbb{R}^{m_i}}^{1/i} : i = 1, \dots, s\} \quad \text{for all } p \in \mathbb{G}, \quad (4)$$

with constants $\varepsilon_1 = 1$ and $\varepsilon_i \in (0, 1)$ for all $i = 2, \dots, s$ depending on the structure of $\mathbb{G}$. We use the shorthand $\|p\|_\infty = d_\infty(p, 0)$ for $p \in \mathbb{G}$. Consequently, for $p \in \mathbb{G}$ and $r > 0$, we define the open and closed balls with respect to this metric as

$$U_\infty(p, r) = \{q \in \mathbb{G} : d_\infty(q, p) < r\}, \quad B_\infty(p, r) = \{q \in \mathbb{G} : d_\infty(q, p) \leq r\}.$$

2.2. Differential calculus on Carnot groups

This section is taken from [20] and [38, Chapter 3]; we refer to [40] for a generalization of these contents. The notion of *P-differentiability* for functions acting between Carnot groups was introduced by Pansu in [47].

Definition 2.4. Let $\mathbb{G}_1, \mathbb{G}_2$ be Carnot groups, with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$ and dilations $\delta_\lambda^1, \delta_\lambda^2$. We say that $L : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *H-linear*, or is a *homogeneous homomorphism*, if L is a group homomorphism such that

$$L(\delta_\lambda^1 g) = \delta_\lambda^2 L(g), \quad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0.$$

Definition 2.5. Let $\Omega \subset \mathbb{G}_1$ be open. We say that $f : \Omega \rightarrow \mathbb{G}_2$ is *P-differentiable* at $x_0 \in \Omega$ if there is a *H-linear* function $d_P f(x_0) : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that

$$\|(d_P f(x_0)(x_0^{-1} \cdot x))^{-1} \cdot f(x_0)^{-1} \cdot f(x)\|_2 = o(\|x_0^{-1} \cdot x\|_1), \quad \text{as } \|x_0^{-1} \cdot x\|_1 \rightarrow 0$$

where $o(t)/t \rightarrow 0$ as $t \rightarrow 0^+$. The *H-linear* map $d_P f(x_0)$ is called the *Pansu's differential* of f at x_0 .

The fundamental result where P-differentiability applies is the following generalization of Rademacher theorem and it is due to P. Pansu [47]. Other possible proofs of this result were obtained by S.K. Vodopyanov [59] and by V. Magnani [39], see also [45]. Pansu's theorem was later extended by Magnani in [40], in which a general homogeneous group as target space is considered. In [42] the authors proved the result for Banach space valued mappings (with the Radon-Nikodym property on the first layer); see also the subsequent works [41, 48].

Theorem 2.6 (Pansu [47]). Let $\mathbb{G}_1 = (\mathbb{R}^n, \cdot)$ and $\mathbb{G}_2 = (\mathbb{R}^m, *)$ be two Carnot groups equipped with two CC metrics denoted, respectively, by d_1 and d_2 . Let $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a Lipschitz continuous function with Ω an open set. Then f is P-differentiable at $\mathcal{L}^n$ -a.e. in Ω .

A version of the classical Stepanov's theorem [56] holds for maps between Carnot groups, which extends Pansu-Rademacher theorem (see [59, Theorem 3.1]).

Theorem 2.7 (*Stepanov's theorem for Carnot groups*). Let $\mathbb{G}_1 = (\mathbb{R}^n, \cdot)$ and $\mathbb{G}_2 = (\mathbb{R}^m, *)$ be two Carnot groups equipped with two CC metrics denoted, respectively, by d_1 and d_2 . Let $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ and assume that

$$\text{Lip}(f)(p) := \limsup_{q \rightarrow p} \frac{d_2(f(q), f(p))}{d_1(q, p)} < \infty \text{ for } \mathcal{L}^n\text{-a.e. } p \in \Omega. \quad (5)$$

Then f is P -differentiable at $\mathcal{L}^n$ -a.e. $p \in \Omega$.

Let $X_1, \dots, X_{m_1}$ be a generating family of vector fields. For $j = 1, \dots, m_1$, we say that $f : \mathbb{G} \rightarrow \mathbb{R}$ is *differentiable along* X_j at a point x_0 if the map $\lambda \mapsto f(\tau_{x_0}(\delta_\lambda e_j)) = f(\exp(\lambda X_j)(x_0))$ is differentiable at $\lambda = 0$, where e_j is the j -th vector of the canonical basis of $\mathbb{R}^n$. In other words, f is differentiable along X_j at x_0 if there exists

$$X_j f(x_0) := \lim_{t \rightarrow 0} \frac{f(\exp(t X_j)(x_0)) - f(x_0)}{t}.$$

Even if f is not differentiable, we can still define the discrete difference quotients:

Definition 2.8. Let $f : \Omega \rightarrow \mathbb{R}$, $i = 1, \dots, m_1$. We define the *difference quotient* at x in direction X_i as

$$D_i^h f(x) := \frac{f(\exp(h X_i)(x)) - f(x)}{h}$$

defined for $x \in \Omega' \subset \subset \Omega$ with $0 < |h| < \text{dist}(\Omega', \partial\Omega)$.

Notice that, if $f : \Omega \subset \mathbb{G} \rightarrow \mathbb{R}$ is P -differentiable at a point x_0 , then it is also differentiable along X_j at x_0 and

$$X_j f(x_0) = d_P f(x_0)(e_j).$$

For any function $f : \mathbb{G} \rightarrow \mathbb{R}$ for which the partial derivatives $X_j f$ exist, we define the *horizontal gradient* of f , denoted by $\nabla_{\mathbb{G}} f$, as

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m_1} (X_i f) X_i.$$

We will represent the horizontal gradient in coordinates $\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_{m_1} f)$.

We conclude this section by recalling the notion of convolution in Carnot groups; we address the interested reader to [16, Proposition 1.20] for the proof.

Lemma 2.9. Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group and let $\varrho \in \mathbf{C}^\infty(\mathbb{R}^n)$ be such that $0 \leq \varrho \leq 1$, $\int_{\mathbb{R}^n} \varrho d\mathcal{L}^n = 1$, $\text{spt} \varrho \subset U_1$ and $\varrho(x^{-1}) = \varrho(-x) = \varrho(x)$ for all $x \in \mathbb{R}^n$. Let us denote

$$\varrho_\varepsilon(x) := \varepsilon^{-Q} \varrho(\delta_{1/\varepsilon}(x)) \quad x \in \mathbb{R}^n; \quad (6)$$

$$(f \star \varrho_\varepsilon)(x) := \int_{\mathbb{R}^n} \varrho_\varepsilon(x \cdot y^{-1}) f(y) d\mathcal{L}^n(y) = \int_{\mathbb{R}^n} \varrho_\varepsilon(y) f(y^{-1} \cdot x) d\mathcal{L}^n(y) \quad (7)$$

Then

- (i) if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $f \star \varrho_\varepsilon \in \mathbf{C}^\infty(\mathbb{R}^n)$ and $f \star \varrho_\varepsilon \rightarrow f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$;
- (ii) $\text{spt } f \star \varrho_\varepsilon \subset \text{spt } f \cdot B(0, \varepsilon)$;
- (iii) $X_j(\varphi \star \varrho_\varepsilon) = X_j \varphi \star \varrho_\varepsilon$ for any $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}^n)$ and each $j = 1, \dots, m_1$;
- (iv) $\int_{\mathbb{R}^n} (f \star \varrho_\varepsilon) g d\mathcal{L}^n = \int_{\mathbb{R}^n} (g \star \varrho_\varepsilon) f d\mathcal{L}^n$ for every $f \in L^\infty(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$;
- (v) if $f \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}^0(\Omega)$ for a suitable open set $\Omega \subset \mathbb{R}^n$ then $f \star \varrho_\varepsilon \rightarrow f$ uniformly on compact subsets of Ω as $\varepsilon \rightarrow 0$.

2.3. Lorentz space

Let $\Omega \subset \mathbb{G} \equiv \mathbb{R}^n$ be an open set, let Q denote the homogeneous dimension of $\mathbb{G}$ and let $A := \mathcal{L}^n(\Omega)$. If $g : \Omega \rightarrow \bar{\mathbb{R}}$ is a measurable function, $\mathcal{L}^n$ -a.e. finite, we define its *distribution function* $\lambda_g : [0, \infty) \rightarrow [0, \infty]$ as

$$\lambda_g(s) := \mathcal{L}^n(\{x \in \Omega : |g(x)| > s\})$$

and the *nonincreasing rearrangement* $g^* : [0, \infty) \rightarrow [0, \infty]$ of g as

$$g^*(t) := \inf \{s \geq 0 : \lambda_g(s) \leq t\}.$$

It is well-known that λ_g and g^* are nonincreasing and right-continuous. Moreover g^* is the nonincreasing function which is equimeasurable with $|g|$, namely

$$\mathcal{L}^n(\{x \in \Omega : |g(x)| > s\}) = \mathcal{L}^1(\{t \in \mathbb{R}^+ : g^*(t) > s\}) \quad \text{for all } s > 0.$$

The *Lorentz space* $L^{Q,1}(\Omega)$ is defined as the class of all measurable functions $g : \Omega \rightarrow \bar{\mathbb{R}}$, $\mathcal{L}^n$ -a.e. finite, for which the quasi-norm

$$\|g\|_{L^{Q,1}(\Omega)} := \int_0^A t^{\frac{1-Q}{Q}} g^*(t) dt$$

is finite. Note that it is not a norm since the triangle inequality may fail. A systematic study of Lorentz spaces is carried out in [54] whereas a simpler recent account can be found in [31].

Let us recall some properties of the Lorentz space $L^{Q,1}(\Omega)$ whose proof can be found in [6] and [31].

Proposition 2.10.

(i) For any measurable function $g : \Omega \rightarrow \mathbb{R}$,

$$\|g\|_{L^{Q,1}(\Omega)} = Q \int_0^\infty \lambda_g(s)^{1/Q} ds.$$

(ii) $L^{Q,1}(\Omega) \subset L^Q(\Omega)$; moreover, if $|\Omega| < \infty$, it holds that $L^p(\Omega) \subset L^{Q,1}(\Omega)$ for each $p > Q$.

(iii) If $|g_1| \leq |g_2|$ a.e. in Ω , then

$$\|g_1\|_{L^{Q,1}(\Omega)} \leq \|g_2\|_{L^{Q,1}(\Omega)}.$$

Notice that, in view of point (ii) of the previous Proposition, the Lorentz space $L^{Q,1}(\Omega)$ is, locally, an intermediate space between $L^Q(\Omega)$ and $L^p(\Omega)$ for all $p > Q$.

Let us also recall the following remarkable characterization of $L^{Q,1}(\Omega)$ in terms of Orlicz integrability conditions, contained in [31, Corollary 2.4], which we will need later.

Theorem 2.11. *Let g be a measurable function on Ω . Then the following properties are equivalent:*

(i) $g \in L^{Q,1}(\Omega)$.

(ii) There is a positive nonincreasing function $\varphi \in L^{1/Q}((0, \infty))$ such that

$$\int_\Omega F_\varphi(|g|) dx < \infty, \tag{8}$$

where

$$F_\varphi(s) := \begin{cases} s \varphi(s)^{\frac{1}{Q}-1} & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases} \tag{9}$$

2.4. Sobolev spaces on Carnot groups

In this section we introduce and study L^p and Sobolev spaces for maps between Carnot groups.

2.4.1. Sobolev functions from a Carnot group to $\mathbb{R}$

Sobolev spaces for scalar functions defined on a Carnot group $\mathbb{G}$ are constructed in the usual way by considering distributional derivatives along the basis $X_1, \dots, X_{m_1}$ of

the horizontal layer of $\mathbb{G}$: a function $g \in L^1_{loc}(\Omega)$ is the *weak (or distributional) derivative of f along X_i* if

$$\int_{\Omega} g \varphi \, dx = - \int_{\Omega} f X_i \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

As usual we will write $X_i f$ also to denote the weak derivative of f along X_i and $\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_{m_1} f)$ for the *weak horizontal gradient*.

Definition 2.12. Let Ω be an open subset of $\mathbb{G} \equiv \mathbb{R}^n$ and $1 \leq p \leq \infty$. We define the (horizontal) Sobolev space $W_{\mathbb{G}}^{1,p}(\Omega)$ as the space of all the functions with finite norm

$$\|u\|_{W_{\mathbb{G}}^{1,p}} := \|u\|_p + \|\nabla_{\mathbb{G}} u\|_p, \quad (10)$$

where $\nabla_{\mathbb{G}} u$ denotes the horizontal distributional gradient of u .

The following result is well-known, see e.g. [16], [19], [23].

Proposition 2.13. *For any $1 \leq p \leq \infty$, $(W_{\mathbb{G}}^{1,p}(\Omega), \|\cdot\|_{W_{\mathbb{G}}^{1,p}})$ is a Banach space, reflexive if $1 < p < \infty$. Moreover, $W_{\mathbb{G}}^{1,2}(\Omega)$ is a Hilbert space.*

Another way to define the $W_{\mathbb{G}}^{1,p}$ with $1 \leq p < \infty$ is to take the closure of C^∞ functions w.r.t. the norm (10). As in the Euclidean case, the two approaches are equivalent. This was obtained independently in [19, Theorem 1.2] and [23, Theorem A.2]. However, the method goes back to the classical result by Friedrichs [22]. The result can be stated as follows.

Theorem 2.14. *Let $1 \leq p < \infty$. We have*

$$C^\infty(\Omega) \cap W_{\mathbb{G}}^{1,p}(\Omega) \text{ is dense in } W_{\mathbb{G}}^{1,p}(\Omega).$$

For functions in $W_{\mathbb{G}}^{1,p}(\Omega)$, the difference quotients are uniformly bounded by the L^p norm of the weak gradient. This is expressed by the following Lemma, whose Euclidean counterpart is a classical result (see for instance [24]).

Lemma 2.15. *Let $f \in W_{\mathbb{G}}^{1,p}(\Omega)$. Then, for every $i = 1, \dots, m_1$ and for every $\Omega' \subset\subset \Omega$ it holds*

$$\|D_i^h f\|_{L^p(\Omega')} \leq \|X_i f\|_{L^p(\Omega)}$$

for every $0 < |h| < \text{dist}(\Omega', \partial\Omega)$.

Proof. By Theorem 2.14, it suffices to prove the statement for $f \in C^\infty(\Omega) \cap W_{\mathbb{G}}^{1,p}(\Omega)$. Fix $i = 1, \dots, m_1$, $\Omega' \subset\subset \Omega$ and $0 < |h| < \text{dist}(\Omega', \partial\Omega)$. Consider $x \in \Omega'$ and define

$$\gamma_x(s) := \exp(hsX_i)(x).$$

Then γ_x is a horizontal curve satisfying $\gamma_x(0) = x$, $\gamma_x(1) = \exp(hX_i)(x)$ and, by construction, $\dot{\gamma}_x(s) = hX_i(\gamma_x(s))$. Then the result follows by a direct computation:

$$\begin{aligned} \|D_i^h f\|_{L^p(\Omega')}^p &= \int_{\Omega'} \left| \frac{f(\exp(hX_i)(x)) - f(x)}{h} \right|^p dx = \int_{\Omega'} \left| \frac{f(\gamma_x(1)) - f(\gamma_x(0))}{h} \right|^p dx \\ &= \frac{1}{|h|^p} \int_{\Omega'} \left| \int_0^1 \frac{d}{ds} (f \circ \gamma_x)(s) ds \right|^p dx \\ &= \frac{1}{|h|^p} \int_{\Omega'} \left| \int_0^1 \langle \nabla_{\mathbb{G}} f(\gamma_x(s)), \dot{\gamma}_x(s) \rangle_{\gamma_x(s)} ds \right|^p dx \\ &= \frac{1}{|h|^p} \int_{\Omega'} \left| \int_0^1 hX_i f(\gamma_x(s)) ds \right|^p dx \leq \int_{\Omega'} \int_0^1 |X_i f(\gamma_x(s))|^p ds dx \\ &= \int_0^1 \int_{\Omega'} |X_i f(\gamma_x(s))|^p dx ds = \int_0^1 \int_{\Omega' \cdot \delta_{hs}(e_i)} |X_i f(z)|^p dz ds \\ &\leq \int_0^1 \int_{\Omega} |X_i f(z)|^p dz ds = \|X_i f\|_{L^p(\Omega)}^p. \quad \square \end{aligned}$$

Proposition 2.16. Let $1 \leq p < +\infty$. Let $f : \Omega \subset \mathbb{G} \rightarrow \mathbb{R}$ and suppose that

- (i) f is P -differentiable a.e. in Ω ;
- (ii) $f \in W_{\mathbb{G},\text{loc}}^{1,p}(\Omega)$.

Then

$$X_i f(x) = \lim_{t \rightarrow 0} \frac{f(\exp(tX_i)(x)) - f(x)}{t} \text{ for a.e. } x \in \Omega$$

for each $i = 1, \dots, m_1$, where $X_i f$ denotes the distributional derivative of f along the vector field X_i .

Proof. Let us first show that, since $f \in W_{\mathbb{G},\text{loc}}^{1,p}(\Omega)$, then

$$D_i^h f \rightarrow X_i f \quad \text{in } L_{\text{loc}}^p(\Omega).$$

Let $\Omega' \subset\subset \Omega$ and $u \in C^\infty(\Omega) \cap W_{\mathbb{G}}^{1,p}(\Omega)$. We write

$$D_i^h f - X_i f = D_i^h(f - u) + D_i^h u - X_i u + X_i(u - f)$$

and using the previous lemma for h sufficiently small

$$\|D_i^h f - X_i f\|_{L^p(\Omega')} \leq \|D_i^h u - X_i u\|_{L^p(\Omega')} + 2\|X_i(u - f)\|_{L^p(\Omega)}.$$

Now the claim follows by Theorem 2.14, since for $u \in C^\infty(\Omega)$ it is well known that $D_i^h u \rightarrow X_i u$ (which coincides with the classical directional derivative) *uniformly* on compact sets. On the other hand, by assumption (i), $D_i^h f$ converges pointwise almost everywhere to the classical derivative of f along the vector field X_i . Hence the conclusion follows. $\square$

We conclude this section by recalling a well-known estimate on the Riesz potential that will be helpful later on. The proof easily follows using the Poincaré inequality in Carnot groups [28], [18] and [21] and we omit it.

Theorem 2.17. *Let $U = U(x, r)$ and let $f : \mathbb{G} \rightarrow \mathbb{R}$ be a continuously differentiable function. Then there exists $C > 0$, independent of U and f , such that*

$$|f(x) - f_U| \leq C \int_U \frac{|\nabla_{\mathbb{G}} f(y)|}{d(x, y)^{Q-1}} dy \quad (11)$$

for all $x \in U$, where $f_U := \int_U f dx$. Moreover, if $f \in W_{\mathbb{G}, \text{loc}}^{1,1}(U)$, then (11) holds for a.e. $x \in U$ and $\nabla_{\mathbb{G}} f(y)$ denotes the weak horizontal gradient of f at y .

2.4.2. Sobolev functions between Carnot groups

We are now going to introduce the notions of L^p functions and Sobolev functions with values in a Carnot group. These concepts actually work for mappings between general metric spaces and can be introduced in several different ways (see Section B). A possible approach relies on *upper gradients* (this is for instance the case discussed in [35]), while here we use the definition by *sections of mappings*. These notions go back to Ambrosio [3] and Reshetnyak [52] for functions defined on open sets of $\mathbb{R}^n$ or a Riemannian manifold and have been extended by Vodop'yanov [58] to the setting of Carnot groups. A recent account containing different approaches to this argument is due to Kleiner, Müller and Xie [32,33].

Definition 2.18. Let $\mathbb{G}_1 = (\mathbb{R}^n, \cdot, d_1)$ and $\mathbb{G}_2 = (\mathbb{R}^m, \star, d_2)$ be two Carnot groups and let $\Omega \subset \mathbb{G}_1$ be an open set.

- (i) We say that a map $f : \Omega \rightarrow \mathbb{G}_2$ belongs to $L^p(\Omega, \mathbb{G}_2)$, and we will write $f \in L^p(\Omega, \mathbb{G}_2)$, if $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is measurable and the map $\mathbb{R}^n \ni x \mapsto d_2(0, f(x))$ is in $L^p(\Omega)$.
- (ii) We say that a map $f : \Omega \rightarrow \mathbb{G}_2$ belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{G}_2)$, and we will write $f \in W^{1,p}(\Omega, \mathbb{G}_2)$, if
- (•) $f \in L^p(\Omega, \mathbb{G}_2)$;
 - (•) for all $z \in \mathbb{G}_2$ the functions $u_z : \Omega \rightarrow \mathbb{R}$

$$u_z(x) := d_2(z, f(x)) \text{ if } x \in \Omega, \quad (12)$$

are in $W_{\mathbb{G}_1, \text{loc}}^{1,p}(\Omega)$;

- (•) there exists a nonnegative function $g \in L^p(\Omega)$ (independent of z) such that

$$|\nabla_{\mathbb{G}_1} u_z| \leq g \text{ a.e. in } \Omega, \text{ for each } z \in \mathbb{G}_2. \quad (13)$$

In the following we will refer to a function u_z as a section of f .

As usual, we will denote by $W_{\text{loc}}^{1,p}(\Omega, \mathbb{G}_2)$ the set of functions $f : \Omega \rightarrow \mathbb{G}_2$ such that $f \in W^{1,p}(\Omega', \mathbb{G}_2)$ for each $\Omega' \subset \subset \Omega$.

Remark 2.19. The stability condition requested in (13) with respect to the family of 1-Lipschitz maps $\varphi : \mathbb{G}_2 \rightarrow \mathbb{R}$, $\varphi(y) := d_2(z, y)$, can be extended to all Lipschitz maps between $\mathbb{G}_2$ and $\mathbb{R}$. Indeed the following characterization holds (see [58, Proposition 1] and [32, Proposition 2.31], see also Lemma 7.4): $f \in W^{1,p}(\Omega, \mathbb{G}_2)$ if and only if there exists a nonnegative function $g \in L^p(\Omega)$ such that, for each Lipschitz function $\varphi : \mathbb{G}_2 \rightarrow \mathbb{R}$, the composition map $\varphi \circ f : \Omega \rightarrow \mathbb{R}$ is in $W_{\mathbb{G}_1, \text{loc}}^{1,p}(\Omega)$ and $|\nabla_{\mathbb{G}_1}(\varphi \circ f)| \leq \text{Lip}(\varphi) g$ a.e. in Ω .

Notice that, exploiting this remark, it is easy to see that Definition 2.18 does not depend on the specific distance d_2 on $\mathbb{G}_2$ which appears in (12). In fact, suppose $\tilde{d}_2$ is another invariant distance on $\mathbb{G}_2$. Then d_2 and $\tilde{d}_2$ determine the same Sobolev spaces. Indeed, since d_2 and $\tilde{d}_2$ are both invariant, they are equivalent as distances (see [7, Corollary 5.1.5]), namely there exists a constant $L > 1$ such that

$$L^{-1}d_2(x, y) \leq \tilde{d}_2(x, y) \leq Ld_2(x, y) \quad (14)$$

for every $x, y \in \mathbb{G}_2$. Suppose now $f \in W^{1,p}(\Omega, \mathbb{G}_2)$ according to the distance d_2 . Then

- The map $\mathbb{R}^n \ni x \mapsto \tilde{d}_2(0, f(x))$ is in $L^p(\Omega)$ by (14);
- For every $z \in \mathbb{G}_2$, the map $\varphi_z : (\mathbb{G}_2, d_2) \rightarrow (\mathbb{R}, |\cdot|)$ defined by $\varphi_z(y) := \tilde{d}_2(z, y)$ is L -Lipschitz since

$$|\varphi_z(x) - \varphi_z(y)| = |\tilde{d}_2(z, x) - \tilde{d}_2(z, y)| \leq \tilde{d}_2(x, y) \leq Ld_2(x, y).$$

Hence we can apply Remark 2.19 and deduce that the composition $\tilde{d}_2(z, f(\cdot)) = \varphi_z \circ f \in W_{\mathbb{G}_1, \text{loc}}^{1,p}(\Omega)$;

- Again by the previous considerations and Remark 2.19, we conclude that $|\nabla_{\mathbb{G}_1}(\varphi_z \circ f)| \leq Lg$ a.e. in Ω for each $z \in \mathbb{G}_2$.

Thus $f \in W^{1,p}(\Omega, \mathbb{G}_2)$ with respect to $\tilde{d}_2$: this shows that Definition 2.18 does not depend on the specific invariant distance on $\mathbb{G}_2$.

P-differentiability and Morrey embedding theorem for real-valued Sobolev spaces on Carnot groups when $p > Q$ (see, for instance, [46] and [23, Theorem 1.11]) can be extended to Sobolev maps between Carnot groups. Here Q always denotes the homogeneous dimension of $\mathbb{G}_1$.

Theorem 2.20 (*P-differentiability with $p > Q$, [60, Corollary 2]*). *Let $f \in W^{1,p}(\Omega, \mathbb{G}_2)$ with $p > Q$. Then f is P-differentiable a.e. in Ω .*

The following theorem can be found in [33, Lemma 2.7].

Theorem 2.21 (*Morrey-Sobolev embedding with $p > Q$*). *Let $f : U(x, r) \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ and $p > Q$. There is a positive constant C_p (depending on p) such that, if $f \in W^{1,p}(U(x, r), \mathbb{G}_2)$ and g is as in (13), then f has a continuous representative and for each $y \in U(x, r)$ it satisfies*

$$d_2(f(x), f(y)) \leq C_p r^{1-\frac{Q}{p}} \left(\int_{U(x,r)} g^p d\mathcal{L}^n \right)^{1/p}. \quad (15)$$

3. Absolutely continuous functions between Carnot groups and some properties

Let us collect in this section some interesting properties of Q -absolutely continuous functions and functions of bounded Q -variation.

Let us begin to observe that, on bounded open sets (with finite measure would suffice), the class of Q -absolutely functions is contained in the one of bounded Q -variation functions.

Proposition 3.1. *Let $\Omega \subset \mathbb{G}_1$ be a bounded open set. Then $AC^Q(\Omega, \mathbb{G}_2) \subset BV^Q(\Omega, \mathbb{G}_2)$.*

Proof. The strategy of the proof is inspired by [35, Lemma 2.16], where the authors prove a local version of this result for abstract metric measure spaces. Let's denote by $A := \mathcal{L}^n(\Omega) < +\infty$ and let $f \in AC^Q(\Omega, \mathbb{G}_2)$. Fix $\varepsilon = 1$: by Q -absolute continuity, there exists $\delta > 0$ such that, for each finite disjoint family of open balls $U_1, \dots, U_k$ contained in Ω ,

$$\text{if } \sum_{i=1}^k \mathcal{L}^n(U_i) < 2\delta \text{ then } \sum_{i=1}^k (\text{osc}_{U_i} f)^Q < 1. \quad (16)$$

Let $n \in \mathbb{N}$ be sufficiently large so that $A < n\delta$. Let now $\mathcal{F} = \{U_1, \dots, U_k\}$ be a disjoint family of balls in Ω and write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\begin{aligned} \mathcal{F}_1 &= \{U_j \in \mathcal{F} : \mathcal{L}^n(U_j) \geq \delta\}, \\ \mathcal{F}_2 &= \{U_j \in \mathcal{F} : \mathcal{L}^n(U_j) < \delta\}. \end{aligned}$$

Notice that the elements in $\mathcal{F}_1$ are at most n and their radii are equibounded. For any $U = U(x, r) \in \mathcal{F}_1$ and any $y \in U$ consider a geodesic joining x and y . On such a curve pick points $x = x_0, x_1, \dots, x_{N-1}, x_N = y$ such that x_i and x_{i+1} live in a common small ball (contained in U) of measure less than 2δ . Note that it is possible to find N independent of y and U since all the elements in $\mathcal{F}_1$ have equibounded radii. By (16) we can now estimate

$$\begin{aligned} d_2(f(x), f(y))^Q &\leq (d_2(f(x), f(x_1)) + \dots + d_2(f(x_{N-1}), f(y)))^Q \\ &\leq C(Q)(d_2(f(x), f(x_1))^Q + \dots + d_2(f(x_{N-1}), f(y))^Q) \leq C(Q)N := K. \end{aligned}$$

By the triangular inequality the same estimate holds (with a greater constant $\tilde{K}$) for any $y, z \in U$. We conclude that $\sum_{U \in \mathcal{F}_1} (\text{osc}_U f)^Q \leq n\tilde{K}$.

Collect instead the elements of $\mathcal{F}_2$ into at most n sets $\mathcal{F}_{2,1}, \dots, \mathcal{F}_{2,n}$ with $\sum_{U \in \mathcal{F}_{2,j}} \mathcal{L}^n(U) < 2\delta$, so that by (16)

$$\sum_{U \in \mathcal{F}_2} (\text{osc}_U f)^Q = \sum_{j=1}^n \sum_{U \in \mathcal{F}_{2,j}} (\text{osc}_U f)^Q < n$$

Hence $V_Q(f, \Omega) < n(\tilde{K} + 1)$ and the thesis follows. $\square$

It is clear that a Q -absolutely continuous function is actually continuous; here we provide a first example of a class of Q -absolutely continuous maps.

Remark 3.2. Let $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be Lipschitz continuous. Then f turns out to be Q -absolutely continuous. Indeed,

$$\text{osc}_{U_r(x)}(f) := \text{diam}_{\mathbb{G}_2}(f(U_r(x))) \leq \text{Lip}(f) \text{diam}_{\mathbb{G}_1}(U_r(x))$$

for each $x \in \mathbb{G}_1$ and $r > 0$. Using the previous inequality we immediately get that there exists $L_1 > 0$ such that

$$\text{osc}_U(f)^Q \leq (2 \text{Lip}(f))^Q r^Q = L_1 |U| \quad (17)$$

for each (open) CC open ball $U \subset \mathbb{G}_1$ and the conclusion follows.

Definition 3.3. A function $f : \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ satisfies the condition (RR) if there is a nonnegative function $w \in L^1(\Omega)$, called weight, such that

$$\operatorname{osc}_{U_r(x)}(f)^Q \leq \int_{U_r(x)} w \, dy \quad (18)$$

for every ball $U_r(x) \subset \Omega$, being Q the homogeneous dimension of $\mathbb{G}_1$.

Remark 3.4. The acronym (RR) stands for Rado and Reichelderfer [49], who introduced a similar condition for proving an area formula in $\mathbb{R}^n$. The condition (RR), as stated in Definition 3.3, is a generalization to Carnot groups of the one due to Malý [43].

Remark 3.5. It is immediate to check that if $f : \Omega \rightarrow \mathbb{G}_2$ satisfies the condition (RR), then $f \in AC^Q(\Omega, \mathbb{G}_2)$, by the absolute continuity of the Lebesgue integral. The viceversa is known in Euclidean spaces for functions $f \in AC^n(\Omega, \mathbb{R}^m)$ [13, Thm. 9] and it is still open for maps between Carnot groups.

As observed in Remark 3.2, Lipschitz functions are Q -absolutely continuous. Also Sobolev functions with a high order of summability share the same property. This result was shown in [43, Thm. 4.1] for the Euclidean case and the proof can be carried out in the same way by exploiting Morrey-Sobolev embedding Theorem 2.21. Here we mention an alternative approach which relies on an equivalent definition of Sobolev spaces through upper gradients.

Theorem 3.6. Let $\Omega \subset \mathbb{G}_1$ be an open bounded set and let $f \in W^{1,p}(\Omega, \mathbb{G}_2)$ with $p > Q$, where Q is the homogeneous dimension of $\mathbb{G}_1$. Then, for a suitable representative, $f \in AC^Q(\Omega, \mathbb{G}_2)$.

Proof. By the following Corollary 7.6, we can assume $f \in N^{1,p}(\Omega : \mathbb{G}_2)$, where $N^{1,p}(\Omega : \mathbb{G}_2)$ denotes the Newtonian Sobolev space between Ω and $\mathbb{G}_2$ (see Appendix B). Applying [35, Theorem 1.4] we immediately get the conclusion. $\square$

We are now interested in studying some differentiability properties of absolutely continuous maps between Carnot groups. We start with the following

Theorem 3.7 (Differentiability a.e.). Let $f \in BV^Q(\Omega, \mathbb{G}_2)$. Then

- (i) f is P -differentiable for a.e. $p \in \Omega$;
- (ii) $\operatorname{Lip}(f) \in L^Q(\Omega)$, where $\operatorname{Lip}(f)$ is the function defined in (5).

Remark 3.8. Note that Pansu's differentiability for maps in $BV^Q(\Omega, \mathbb{G}_2)$ does not follow from differentiability result à la Cheeger-Kleiner proved in [35, Proposition 1.3] for maps between metric spaces, of which the target metric space is biLipschitz embedded into a

Banach space satisfying the Radon- Nikodym property (RNP). Indeed it is well-known that Carnot groups may not be biLipschitz embedded into a Banach space satisfying (RNP) (see Appendix B.1)

Proof. We can follow the same proof of [43, Thm. 3.3] with suitable changes. For the sake of completeness, we collect here the main steps. Let $g : \Omega \rightarrow \mathbb{R}$ be a nonnegative upper semicontinuous function such that

$$0 \leq g(x) \leq [Lip(f)(x)]^Q \quad \text{for all } x \in \Omega.$$

For any $x \in \{g > 0\}$, there exists a unique integer $k = k(x)$ such that $2^k \leq g(x) < 2^{k+1}$. Furthermore, since g is upper semicontinuous, there exists $0 < r = r(x) < 1$ such that $U(x, 5r) \subset \{g < 2^{k+1}\}$. In addition, from the fact that $g(x) \leq [Lip(f)(x)]^Q$, we can assume, up to reduce r , that

$$2^k r^Q \leq g(x) r^Q < 2 \left(\operatorname{osc}_{U(x,r)} f \right)^Q. \quad (19)$$

Applying Vitali's covering Lemma, we can extract a disjoint (hence at most countable) family $\{U(x_i, r_i)\}_i \subset \{U(x, r(x))\}$ of balls such that

$$\{g > 0\} \subset \bigcup_{x \in \{g > 0\}} U(x, r(x)) \subset \bigcup_i U(x_i, 5r_i).$$

We define, for all $k \in \mathbb{Z}$, $I_k := \{i : k(x_i) \geq k\}$ and $E_k := \{x \in \Omega : 2^k \leq g(x) < 2^{k+1}\}$. Notice that $E_k \subset \bigcup_{i \in I_k} U(x_i, 5r_i)$. In fact, let $x \in E_k$. In particular $g(x) > 0$, hence $x \in U(x_i, 5r_i)$ for some i . If it were $i \notin I_k$, then $k(x_i) < k$, which implies $x \in U(x_i, 5r_i) \subset \{g < 2^k\}$, contradicting $x \in E_k$. Thus we get

$$\begin{aligned} \mathcal{L}^n(E_k) &\leq \sum_{i \in I_k} \mathcal{L}^n(U(x_i, 5r_i)) \leq C \sum_{i \in I_k} r_i^Q \\ &\stackrel{(19)}{\leq} C \sum_{i \in I_k} 2^{-k(x_i)} \left(\operatorname{osc}_{U(x_i, r_i)} f \right)^Q. \end{aligned}$$

Finally we use this estimate to bound

$$\begin{aligned} \int_{\Omega} g(x) \, dx &\leq \sum_{k=-\infty}^{+\infty} \int_{E_k} g(x) \, dx \leq \sum_{k=-\infty}^{+\infty} 2^{k+1} \mathcal{L}^n(E_k) \\ &\leq C \sum_{k=-\infty}^{+\infty} \sum_{i \in I_k} 2^{k+1-k(x_i)} \left(\operatorname{osc}_{U(x_i, r_i)} f \right)^Q \\ &= C \sum_{i=1}^{+\infty} \sum_{k \leq k(x_i)} 2^{k+1-k(x_i)} \left(\operatorname{osc}_{U(x_i, r_i)} f \right)^Q \end{aligned}$$

$$= 4C \sum_{i=1}^{+\infty} (\operatorname{osc}_{U(x_i, r_i)} f)^Q \leq CV_Q(f, \Omega),$$

where we exploited the properties of the Q -variation and the fact that $\sum_{k \leq k(x_i)} 2^k = 2^{k(x_i)+1}$. It follows that

$$\int_{\Omega} [Lip(f)(x)]^Q dx = \sup \left\{ \int_{\Omega} g(x) dx : g \text{ u.s.c., } 0 \leq g \leq Lip(f)^Q \right\} \leq CV_Q(f, \Omega).$$

Thus (ii) follows and also that $Lip(f)(x) < \infty$ for a.e. $x \in \Omega$. From Theorem 2.7, (i) follows, too. $\square$

We will now focus our attention to the case when $\mathbb{G}_2 = (\mathbb{R}^m, +)$. Note that in this case, since $\mathbb{R}^m$ is abelian, a map $f = (f_1, \dots, f_m) : \Omega \subset \mathbb{G} \rightarrow \mathbb{R}^m$ is Q -absolutely continuous (respectively of finite Q -variation) if and only if each component $f_i : \Omega \subset \mathbb{G} \rightarrow \mathbb{R}$ is Q -absolutely continuous (respectively of finite Q -variation), for each $i = 1, \dots, m$. Thus we can reduce the study to real-valued functions defined on an open set of a Carnot group. We will denote

$$\begin{aligned} BV_{\mathbb{G}}^Q(\Omega) &:= BV^Q(\Omega, \mathbb{R}), \quad AC_{\mathbb{G}}^Q(\Omega) := AC^Q(\Omega, \mathbb{R}), \\ BV_{\mathbb{G}; \text{loc}}^Q(\Omega) &:= BV_{\text{loc}}^Q(\Omega, \mathbb{R}), \quad AC_{\mathbb{G}; \text{loc}}^Q(\Omega) := AC_{\text{loc}}^Q(\Omega, \mathbb{R}). \end{aligned}$$

The following theorem can be proved mutatis mutandis as its Euclidean counterpart, and the proof relies on standard arguments.

Theorem 3.9. *If $f \in L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega)$ let us denote*

$$\|f\|_{Q, BV} := \|f\|_{L^Q(\Omega)} + V_Q(f, \Omega)^{1/Q}.$$

Then $(L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega), \|\cdot\|_{Q, BV})$ is a Banach space. Moreover $L^Q(\Omega) \cap AC_{\mathbb{G}}^Q(\Omega)$ is a closed linear subspace of $L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega)$.

Proof. It is easy to see that $\|\cdot\|_{Q, BV}$ is a norm on $L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega)$. Let us prove that it is a Banach norm. Let $(f_h)_h \subset (L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega), \|\cdot\|_{Q, BV})$ be a Cauchy sequence. It is sufficient to prove that there exist $f \in L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega)$ and a subsequence $(f_{h_k})_k$ satisfying

$$f_{h_k} \rightarrow f \text{ in } L^Q(\Omega), \text{ as } k \rightarrow \infty, \quad (20)$$

and

$$\lim_{k \rightarrow \infty} V_Q(f_{h_k} - f, \Omega) = 0. \quad (21)$$

Since $(f_h)_h \subset (L^Q(\Omega) \parallel \cdot \|_{L^Q(\Omega)})$ is also a Cauchy sequence, we can suppose there exists a function $f^* \in L^Q(\Omega)$ such that (20) holds with $f \equiv f^*$ and $h \equiv h_k$. Let us now prove that, up to a subsequence,

$$(f_h(x))_h \text{ is a Cauchy sequence in } \mathbb{R} \quad \text{for every } x \in \Omega. \quad (22)$$

Indeed, we can assume that, up to a subsequence, there exists a $\mathcal{L}^n$ -negligible set $N \subset \Omega$ such that

$$f_h(x) \rightarrow f^*(x) \in \mathbb{R}, \text{ as } h \rightarrow \infty, \text{ for each } x \in \Omega \setminus N. \quad (23)$$

Since $(f_h)_h$ is a Cauchy sequence with respect to the norm $\parallel \cdot \|_{Q,BV}$, we can assume that, for each $\varepsilon > 0$ there exists $\bar{h} = \bar{h}(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} & \sum_{i=1}^m |f_h(x^{(i)}) - f_k(x^{(i)}) - f_h(y^{(i)}) + f_k(y^{(i)})|^Q \\ & \leq \sum_{i=1}^m \text{osc}_{U_i}(f_h - f_k)^Q \\ & \leq V_Q(f_h - f_k, \Omega) < \varepsilon, \end{aligned} \quad (24)$$

for each $h, k > \bar{h}$, for any finite disjoint family of balls $U_1, \dots, U_m$ in Ω , for each $x^{(i)}, y^{(i)} \in U_i$ and $i = 1, \dots, m$.

Fix now a ball $U(x, r) \subset \Omega$. By the density of $\Omega \setminus N$ in Ω , there exists $x_0 \in U(x, r) \cap (\Omega \setminus N)$. By (24), the sequence

$$g_h(x) := f_h(x) - f_h(x_0) \quad h \in \mathbb{N},$$

is a Cauchy sequence. On the other hand, by (23), the sequence $(f_h(x_0))_h$ is a Cauchy sequence. Thus (22) follows. From (22), it follows that there exists a function $f : \Omega \rightarrow \mathbb{R}$ defined as

$$f(x) := \lim_{h \rightarrow \infty} f_h(x) \quad \text{for each } x \in \Omega,$$

and, by (23),

$$f(x) = f^*(x) \quad \text{for each } x \in \Omega \setminus N.$$

In particular, (20) immediately follows. Let us now prove that $f \in BV_{\mathbb{G}}^Q(\Omega)$, that is,

$$V_Q(f, \Omega) < \infty. \quad (25)$$

By passing to the limit in (24) as $k \rightarrow \infty$, we get, for each $\varepsilon > 0$ there exists $\bar{h} = \bar{h}(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{i=1}^m |f_h(x^{(i)}) - f(x^{(i)}) - f_h(y^{(i)}) + f(y^{(i)})|^Q \leq \varepsilon,$$

for each $h > \bar{h}$, for any finite disjoint family of balls $U_1, \dots, U_m$ in Ω , for each $x^{(i)}, y^{(i)} \in U_i$ and $i = 1, \dots, m$. Taking the supremum on each ball U_i , and then the supremum on all finite, disjoint family of balls in Ω , it follows that, for each $\varepsilon > 0$ there exists $\bar{h} = \bar{h}(\varepsilon) \in \mathbb{N}$ such that

$$V_Q(f_h - f, \Omega) \leq \varepsilon \text{ for each } h > \bar{h}. \quad (26)$$

Observe now that $f \mapsto V_A(f, \Omega)^{1/Q}$ satisfies the triangle inequality. Thus, from (26), if $h > \bar{h}$

$$\begin{aligned} V_A(f, \Omega)^{1/Q} &\leq V_A(f - f_h, \Omega)^{1/Q} + V_A(f_h, \Omega)^{1/Q} \\ &\leq \varepsilon^{1/Q} + V_A(f_h, \Omega)^{1/Q}, \end{aligned}$$

and (25) follows since $V_A(f_h, \Omega)^{1/Q} < \infty$. Applying again (26), (21) follows, too.

Finally let us show that $L^Q(\Omega) \cap AC_{\mathbb{G}}^Q(\Omega)$ is a closed linear subspace of $(L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega), \|\cdot\|_{Q, BV})$. Let $(f_h)_h \subset L^Q(\Omega) \cap AC_{\mathbb{G}}^Q(\Omega)$ and assume there exists $f \in L^Q(\Omega) \cap BV_{\mathbb{G}}^Q(\Omega)$ and $\lim_{h \rightarrow \infty} \|f_h - f\|_{Q, BV} = 0$. In particular note (26) holds. Now observe that, by (26), if $h > \bar{h}$, given a finite, disjoint family of balls $U_1, \dots, U_m$ in Ω ,

$$\begin{aligned} \left(\sum_{i=1}^m \text{osc}_{U_i}(f)^Q \right)^{1/Q} &\leq \left(\sum_{i=1}^m \text{osc}_{U_i}(f - f_h)^Q \right)^{1/Q} + \left(\sum_{i=1}^m \text{osc}_{U_i}(f_h)^Q \right)^{1/Q} \\ &\leq V_Q(f - f_h)^{1/Q} + \left(\sum_{i=1}^m \text{osc}_{U_i}(f_h)^Q \right)^{1/Q} \\ &\leq \varepsilon^{1/Q} + \left(\sum_{i=1}^m \text{osc}_{U_i}(f_h)^Q \right)^{1/Q}. \end{aligned} \quad (27)$$

Since each $f_h \in AC_{\mathbb{G}}^Q(\Omega)$ for each h , by (27), it follows that $f \in AC_{\mathbb{G}}^Q(\Omega)$, too. $\square$

Theorem 3.7 states that BV^Q -functions are P-differentiable pointwise almost everywhere. We will now prove that such maps enjoy additional differentiability properties, namely they are weakly differentiable according to Definition 2.18. More precisely,

Theorem 3.10 (Weak differentiability). *If $Q > 1$, then*

$$BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2) \subset W_{\text{loc}}^{1, Q}(\Omega, \mathbb{G}_2).$$

Remark 3.11. Note that from [35, Theorem 1.2] it follows that $BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2) \subset D_{\text{loc}}^{1, Q}(\Omega : \mathbb{G}_2)$, where $D_{\text{loc}}^{1, Q}(\Omega : \mathbb{G}_2)$ denotes the (local) *Dirichlet class* of maps between metric

spaces $(\Omega, d_1, \mathcal{L}^n)$ and $(\mathbb{G}_2, d_2)$ according to [27, Chap. 7] (see Appendix B.2). It is also straightforward from definition that $BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2) \subset L_{\text{loc}}^\infty(\Omega, \mathbb{G}_2)$. By definition, $D_{\text{loc}}^{1,Q}(\Omega : \mathbb{G}_2) \cap L_{\text{loc}}^Q(\Omega, \mathbb{G}_2) = N_{\text{loc}}^{1,Q}(\Omega : \mathbb{G}_2)$, where $N_{\text{loc}}^{1,Q}(\Omega : \mathbb{G}_2)$ denotes the (local) *Newtonian Sobolev space* of maps between the previous metric spaces. On the other hand, from Corollary 7.6, it follows that $N_{\text{loc}}^{1,Q}(\Omega : \mathbb{G}_2) = W_{\text{loc}}^{1,Q}(\Omega, \mathbb{G}_2)$. Therefore Theorem 3.10 can be shown also by this strategy.

Here we propose an alternative proof of Theorem 3.10 with respect to the one in [35, Theorem 1.2], by using the intrinsic convolution in Carnot groups and in the same spirit of [43, Theorem 3.2]. In order to prove Theorem 3.10 we need the following technical lemma. Point (i) establishes a uniform estimate on the distance between the images of integer points after a right-translation. Point (ii) provides a way to cover $\mathbb{G}$ with arbitrarily small balls centered at suitable dilations of the integer lattice.

Lemma 3.12.

(i) *For every $R_0 > 0$, it holds that*

$$\sigma_0 := \inf \{d_c(\xi \cdot p, \eta \cdot p) : \xi, \eta \in \mathbb{Z}^n, \xi \neq \eta, p \in B(0, R_0)\} > 0. \quad (28)$$

(ii) *Given a positive integer k , let $\mathbf{Z}_k$ be the set of all points $z \in \mathbb{G} \equiv \mathbb{R}^n$ such that $\delta_k(z)$ has integer coordinates, that is,*

$$\mathbf{Z}_k := \delta_{1/k}(\mathbb{Z}^n). \quad (29)$$

Then there exists a positive constant $\sigma_1 > 0$ such that

$$\cup_{z \in \mathbf{Z}_k} U(z, \sigma_1/k) = \mathbb{G}.$$

Proof. (i) Since d_c and d_∞ are equivalent distances, it is sufficient to show that

$$\sigma_0^* := \inf \{d_\infty(\xi \cdot p, \eta \cdot p) : \xi, \eta \in \mathbb{Z}^n, \xi \neq \eta, p \in B_\infty(0, R_0)\} > 0. \quad (30)$$

By contradiction, if $\sigma_0^* = 0$, there exist three sequences $(\xi_h)_h, (\eta_h)_h$ in $\mathbb{Z}^n$, $(p_h)_h$ in $B_\infty(0, R_0)$, with $\xi_h \neq \eta_h$ and $\lim_{h \rightarrow \infty} d_\infty(\xi_h \cdot p_h, \eta_h \cdot p_h) = \lim_{h \rightarrow \infty} \|p_h^{-1} \cdot \eta_h^{-1} \cdot \xi_h \cdot p_h\|_\infty = 0$. Let $x_h := \eta_h^{-1} \cdot \xi_h$. We observe that $(x_h)_h$ is bounded in $(\mathbb{R}^n, d_\infty)$. In fact, for all h ,

$$\begin{aligned} \|x_h\|_\infty &= \|p_h \cdot p_h^{-1} \cdot x_h \cdot p_h \cdot p_h^{-1}\|_\infty \\ &\leq \|p_h\|_\infty + \|p_h^{-1} \cdot x_h \cdot p_h\|_\infty + \|p_h^{-1}\|_\infty \\ &\leq 2R_0 + \|p_h^{-1} \cdot x_h \cdot p_h\|_\infty. \end{aligned}$$

Hence, up to a subsequence, we can assume that $p_h \rightarrow p_*$ and $x_h \rightarrow x_*$ as $h \rightarrow \infty$. This implies, by the continuity of the norms and of the group operation, that

$$0 = \lim_{h \rightarrow \infty} \|p_h^{-1} \cdot x_h \cdot p_h\|_\infty = \|p_*^{-1} \cdot x_* \cdot p_*\|_\infty, \quad (31)$$

$$\|x_*\|_{\mathbb{R}^n} = \lim_{h \rightarrow \infty} \|\eta_h^{-1} \cdot \xi_h\|_{\mathbb{R}^n}. \quad (32)$$

From (31) it follows that $x_* = 0$.

Let us now show that

$$\|\eta^{-1} \cdot \xi\|_{\mathbb{R}^n} \geq 1 \text{ for each } \xi, \eta \in \mathbb{Z}^n, \text{ with } \xi \neq \eta, \quad (33)$$

thus a contradiction will follow from (32). Let us represent now $\xi, \eta \in \mathbb{Z}^n$ by means of the layer representation

$$\xi = [\xi^1, \xi^2, \dots, \xi^s] \text{ and } \eta = [\eta^1, \eta^2, \dots, \eta^s],$$

with $\xi^i, \eta^i \in \mathbb{Z}^{m_i}$ ($i = 1, \dots, s$) and, since $\xi \neq \eta$, we can assume that $\xi^{i_0} \neq \eta^{i_0}$ for a suitable i_0 . Here we recall that $\mathfrak{g} = V_1 \oplus \dots \oplus V_s$ and $m_i = \dim(V_i)$. Meanwhile, we can represent the element $\eta^{-1} \cdot \xi$ as

$$\begin{aligned} \eta^{-1} \cdot \xi &= [(\eta^{-1} \cdot \xi)^1, (\eta^{-1} \cdot \xi)^2, \dots, (\eta^{-1} \cdot \xi)^s], \\ (\eta^{-1} \cdot \xi)^1 &= \xi^1 - \eta^1, (\eta^{-1} \cdot \xi)^i = \xi^i - \eta^i + \mathcal{Q}^i(-\eta^1, \dots, -\eta^{i-1}, \xi^1, \dots, \xi^{i-1}) \end{aligned}$$

with $i = 2, \dots, s$ and $\mathcal{Q}^i := (\mathcal{Q}_{h_{i-1}+1}, \dots, \mathcal{Q}_{h_i})$. It is clear that if $\xi^1 - \eta^1 \neq 0$, then $\|\eta^{-1} \cdot \xi\|_{\mathbb{R}^n} \geq \|\xi^1 - \eta^1\|_{\mathbb{R}^{m_1}} \geq 1$ and (33) holds. If $\xi^1 - \eta^1 = 0$, then, by the properties of $\mathcal{Q}_i$'s (see (3)) $\mathcal{Q}^2(-\eta^1, \xi^1) = \mathcal{Q}^2(-\xi^1, \xi^1) = 0$, which implies $\|\eta^{-1} \cdot \xi\|_{\mathbb{R}^n} \geq \|(\eta^{-1} \cdot \xi)^2\|_{\mathbb{R}^{m_2}} = \|\xi^2 - \eta^2\|_{\mathbb{R}^{m_2}}$. Thus if $\xi^2 - \eta^2 \neq 0$, (33) holds. Otherwise, in the baddest case, it could happen that $\xi^1 - \eta^1 = \dots = \xi^{i_0-1} - \eta^{i_0-1} = 0$. This implies that $\mathcal{Q}^{i_0}(-\eta^1, \dots, -\eta^{i_0-1}, \xi^1, \dots, \xi^{i_0-1}) = \mathcal{Q}^{i_0}(-\xi^1, \dots, -\xi^{i_0-1}, \xi^1, \dots, \xi^{i_0-1}) = 0$. Thus $\|\eta^{-1} \cdot \xi\|_{\mathbb{R}^n} \geq \|(\eta^{-1} \cdot \xi)^{i_0}\|_{\mathbb{R}^{m_{i_0}}} = \|\xi^{i_0} - \eta^{i_0}\|_{\mathbb{R}^{m_{i_0}}} \geq 1$ and (33) holds.

(ii) It is well-known that it is possible to construct a paving in a Carnot group (see, for instance, [17, Sect. 2.3]). More precisely, if $Y := \{x = (x_1, \dots, x_n) : -1/2 < x_i \leq 1/2\}$ denote the unit cube in $\mathbb{R}^n$ centered at the origin, denote

$$Q_\xi^\varepsilon := \delta_\varepsilon(\tau_\xi(Y)) \text{ if } \varepsilon > 0, \xi \in \mathbb{Z}^n.$$

Then, it holds that, for each $\varepsilon > 0$, the family of sets $\{Q_\xi^\varepsilon : \xi \in \mathbb{Z}^n\}$ is a paving of $\mathbb{G} = (\mathbb{R}^n, \cdot)$, that is, the sets are disjoint and

$$\mathbb{R}^n = \cup_{\xi \in \mathbb{Z}^n} Q_\xi^\varepsilon. \quad (34)$$

It is easy to see that there exist $\sigma_1^*, \sigma_1 > 0$ such that

$$Y \subset U_\infty(0, \sigma_1^*) \subset U(0, \sigma_1). \quad (35)$$

Thus, choosing $\varepsilon = 1/k$, from (34) and (35), we get the desired conclusion. $\square$

Proof of Theorem 3.10. 1st step. Let us first show our conclusion in the case $\mathbb{G}_1 \equiv \mathbb{G}$ and $\mathbb{G}_2 = \mathbb{R}$. Proof's strategy will be the same as [43, Theorem 3.2]. We will point out in the following the main changes and differences which need in the setting of Carnot groups. First assume that $f \in BV_{\mathbb{G}}^Q(\mathbb{R}^n) \cap L^Q(\mathbb{R}^n)$. Let σ_1 be as in (ii) of the previous lemma and find σ_0 as in (i) for $R_0 = 2\sigma_1$. This means that, for each $y \in B(0, 2\sigma_1)$,

$$\sigma_0 \leq \inf\{d_c(\xi \cdot y, \eta \cdot y) : \xi, \eta \in \mathbb{Z}^n, \xi \neq \eta\}.$$

This also implies that, for every integer k and every $y \in B(0, 2\sigma_1/k)$,

$$\frac{\sigma_0}{k} \leq \inf\{d_c(p \cdot y, q \cdot y) : p, q \in \mathbf{Z}_k, p \neq q\}.$$

Hence, for every $y \in B(0, 2\sigma_1/k)$, the family of CC balls $\{U(z \cdot y, \sigma_0/3k) : z \in \mathbf{Z}_k\}$ is disjoint. In addition, up to increase σ_1 , we can assume $\sigma_1 \geq \sigma_0$.

Let $(\varrho_k)_k$ be the sequence of mollifiers defined in (6), where $\varepsilon = 1/k$, $\varrho_k \equiv \varrho_{1/k}$ and suppose that $\text{spt}(\varrho) \subset B(0, \sigma_0/6)$. Observe that it holds that

$$\nabla_{\mathbb{G}} \varrho_k(x) = k^{Q+1} \nabla_{\mathbb{G}} \varrho(\delta_k(x)) \text{ for each } x \in \mathbb{R}^n, k \in \mathbb{N}, \quad (36)$$

$$\int_{\mathbb{R}^n} X_i \varrho_k(r_x(y)) dy = 0 \text{ for each } x \in \mathbb{R}^n, i = 1, \dots, m_1, \quad (37)$$

$$\begin{aligned} \nabla_{\mathbb{G}}(g \star \varrho_k)(x) &= \int_{\mathbb{R}^n} \nabla_{\mathbb{G}} \varrho_k(r_x(y^{-1})) g(y) dy \\ &= \int_{\mathbb{R}^n} \nabla_{\mathbb{G}} \varrho_k(w^{-1}) g(\ell_x(w)) dw \\ &\text{for each } g \in L_{\text{loc}}^1(\mathbb{R}^n), x \in \mathbb{R}^n. \end{aligned} \quad (38)$$

Indeed (36) follows by the definition of ϱ_k and homogeneity of intrinsic gradient $\nabla_{\mathbb{G}} := (X_1, \dots, X_{m_1})$; (37) follows by the properties of ϱ_k , the structure of the horizontal vector fields X_i and the right-invariance of n -dimensional Lebesgue measure $\mathcal{L}^n$; (38) follows by the definition of convolution and by the left invariance of the horizontal vector fields.

Thus, given $y \in B(0, 2\sigma_1/k)$, by (37), (38) and the fact that the family $\{U(z \cdot y, \sigma_0/3k) : z \in \mathbf{Z}_k\}$ is disjoint,

$$\begin{aligned}
& \sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx = \\
&= \sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int \left| \int_{\mathbb{R}^n} \nabla_{\mathbb{G}} \varrho_k(w^{-1}) f(\tau_x(w)) dw \right|^Q dx \\
&= \sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int \left| \int_{\mathbb{R}^n} \nabla_{\mathbb{G}} \varrho_k(w^{-1}) (f(\tau_x(w)) - f(x)) dw \right|^Q dx \\
&= \sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int \left| k^{Q+1} \int_{\mathbb{R}^n} \nabla_{\mathbb{G}} \varrho_k(\delta_k(w^{-1})) (f(\tau_x(w)) - f(x)) dw \right|^Q dx \\
&= \sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int \left| k^{Q+1} \int_{B(0, \sigma_0/6k)} \nabla_{\mathbb{G}} \varrho_k(\delta_k(w^{-1})) (f(\tau_x(w)) - f(x)) dw \right|^Q dx \\
&\leq c_1 \sum_{z \in \mathbf{Z}_k} (\text{osc}_{U(z \cdot y, \sigma_0/3k)} f)^Q \leq c_1 V_Q(f, \mathbb{R}^n).
\end{aligned}$$

By integrating the previous inequality with respect to $y \in B(0, 2\sigma_1/k)$, it follows that there exists a positive constant $c_2 > 0$ such that

$$\begin{aligned}
& k^Q \int_{B(0, 2\sigma_1/k)} \left(\sum_{z \in \mathbf{Z}_k U(z \cdot y, \sigma_0/6k)} \int |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx \right) dy \\
& \leq c_2 V_Q(f, \mathbb{R}^n) \text{ for each positive integer } k.
\end{aligned} \tag{39}$$

Notice also that, writing $g(x) = |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q$, we get, for each $z \in \mathbb{G}$,

$$\begin{aligned}
\int_{U(z, \sigma_1/k)} g(x) dx &= \int_{U(0, \sigma_1/k)} g(z \cdot x) dx \\
&= c_3 k^Q \int_{U(0, \sigma_0/6k)} \left(\int_{U(0, \sigma_1/k)} g(z \cdot x) dx \right) dy \\
&= c_3 k^Q \int_{U(0, \sigma_0/6k)} \left(\int_{U(0, \sigma_1/k) \cdot y^{-1}} g(z \cdot x \cdot y) dx \right) dy \\
&\leq c_3 k^Q \int_{U(0, \sigma_0/6k)} \left(\int_{B(0, 2\sigma_1/k)} g(z \cdot x \cdot y) dx \right) dy
\end{aligned}$$

$$\begin{aligned}
&= c_3 k^Q \int_{B(0, 2\sigma_1/k)} \left(\int_{U(0, \sigma_0/6k)} g(z \cdot x \cdot y) dy \right) dx \\
&= c_3 k^Q \int_{B(0, 2\sigma_1/k)} \left(\int_{U(z \cdot x, \sigma_0/6k)} g(y) dy \right) dx.
\end{aligned}$$

In these passages we performed some changes of variables (which are allowed being the n -dimensional Lebesgue measure both left and right invariant), we noticed that $U(0, \sigma_1/k) \cdot y^{-1} \subset B(0, 2\sigma_1/k)$ for every $y \in U(0, \sigma_0/6k)$ and applied Fubini-Tonelli theorem.

Let us now notice that, by applying Lemma 3.12 (ii) and the previous estimates, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx \leq \sum_{z \in \mathbf{Z}^k U(z, \sigma_1/k)} \int |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx \\
&\leq c_3 k^Q \int_{B(0, 2\sigma_1/k)} \left(\sum_{z \in \mathbf{Z}^k U(z \cdot y, \sigma_0/6k)} \int |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx \right) dy.
\end{aligned} \tag{40}$$

From (39) and (40), we get that there exists $c_4 > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla_{\mathbb{G}}(f \star \varrho_k)(x)|^Q dx \leq c_4 V_Q(f, \mathbb{R}^n) < \infty, \forall k \in \mathbb{N}. \tag{41}$$

Since $W_{\mathbb{G}}^{1,Q}(\mathbb{R}^n)$ is a reflexive space and $f \star \varrho_k \rightarrow f$ in $L^Q(\mathbb{R}^n)$, as $k \rightarrow \infty$, we can infer that $f \in W_{\mathbb{G}}^{1,Q}(\mathbb{R}^n)$. In the general case $f \in BV_{\mathbb{G};\text{loc}}^Q(\Omega)$, it is sufficient to show that $f \varphi \in W_{\mathbb{G}}^{1,Q}(\Omega)$ for each $\varphi \in \mathbf{C}_c^\infty(\Omega)$ which is a cutoff function in a small neighborhood of a point. Using zero-extension, since $f \varphi \in BV_{\mathbb{G}}^Q(\mathbb{R}^n) \cap L^Q(\mathbb{R}^n)$, by the previous argument, $f \varphi \in W_{\mathbb{G}}^{1,Q}(\mathbb{R}^n)$ and we are done.

2nd step. Let us now show the conclusion in the general case. We assume $f \in BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$ and prove that $f \in W^{1,Q}(\Omega', \mathbb{G}_2)$ for every $\Omega' \subset\subset \Omega$. Let $u_z : \Omega \rightarrow \mathbb{R}$ be the function in (12). Then, by the triangle inequality for distances, it follows that

$$\text{osc}_{U_{d_1}(x,r)}(u_z) \leq \text{osc}_{U_{d_1}(x,r)}(f)$$

for each ball $U_{d_1}(x, r) \subset \Omega$ and $z \in \mathbb{G}_2$. In particular, $u_z \in BV_{\mathbb{G};\text{loc}}^Q(\Omega)$ and

$$V_Q(u_z, \Omega) \leq V_Q(f, \Omega) \text{ for each } z \in \mathbb{G}_2.$$

From Theorem 3.7 and the previous step, $u_z : \Omega \rightarrow \mathbb{R}$ is P-differentiable a.e. and $u_z \in W_{\mathbb{G}_1;\text{loc}}^{1,Q}(\Omega)$ (implying $u_z \in W_{\mathbb{G}_1}^{1,Q}(\Omega')$). Thus, by Proposition 2.16,

$$|X_i u_z(x)| = \left| \lim_{t \rightarrow 0} \frac{u_z(\exp(tX_i)(x)) - u_z(x)}{t} \right| \\ \leq \text{Lip}(u_z)(x) \leq \text{Lip}(f)(x)$$

for a.e. $x \in \Omega$ and for each $z \in \mathbb{G}_2$. Since $\Omega' \subset\subset \Omega$ and $f \in BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$, by Theorem 3.7 it follows that $\text{Lip}(f) \in L^Q(\Omega')$. To conclude notice that $f \in L^Q(\Omega', \mathbb{G}_2)$: indeed $f \in BV_{\text{loc}}^Q(\Omega, \mathbb{G}_2)$, so it is locally bounded and $\Omega' \subset\subset \Omega$. Hence $f \in W^{1,Q}(\Omega', \mathbb{G}_2)$ according to Definition 2.18. $\square$

4. Sobolev maps between Carnot groups with horizontal gradient of sections in $L^{Q,1}$

In this section we deal with the sharp condition on the summability of the weak gradient of a Sobolev function which ensures continuity and differentiability almost everywhere. It is well known that these properties hold for maps in the Sobolev space $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ provided that $p > n$ (in the limit case $p = n$ the answer is negative, see also Remark 4.3). Such a result has been extended to the case of Carnot groups (see Theorem 2.20 and Theorem 3.6). In his article [55], Elias M. Stein sharpened the previous result on Euclidean spaces: if a weakly differentiable function admits weak gradient in the Lorentz space $L^{n,1}(\mathbb{R}^n)$, then it is continuous and differentiable a.e. This is a remarkable improvement since, by Proposition 2.10, $L^{n,1}$ is locally an intermediate space between L^n and L^p (for all $p > n$). The approach of Stein makes use of the machinery of harmonic analysis, while later Kauhanen, Koskela and Malý [31] obtained an alternative proof exploiting tools of real analysis and, in particular, the already discussed notion of n -absolutely continuous functions. Next we will follow this last strategy in order to discuss Stein's result for maps between Carnot groups.

Continuity and differentiability have been already studied also for real valued Sobolev functions on metric measure spaces with upper gradient in the Lorentz space $L^{Q,1}$, where Q denotes the Hausdorff dimension of the metric space (see, for instance, [51,50,63,61,62]). Here we are going to prove that a Sobolev map $f: \Omega \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$, with horizontal gradients of its sections $u_z: \Omega \rightarrow \mathbb{R}$ uniformly bounded in $L^{Q,1}(\Omega)$ (see (12)), actually satisfies the condition (RR), if Q denotes the homogeneous dimension of $\mathbb{G}_1$.

Let us first recall an estimate of the Riesz potential in Carnot groups in terms of the Orlicz conditions, which extends the Euclidean one first introduced in [31, Theorem 3.1].

Theorem 4.1. *Let $\mathbb{G} \equiv \mathbb{R}^n$ be a Carnot group, d the associated CC distance, Q its homogeneous dimension and $\Omega \subset \mathbb{G}$. Let g be a nonnegative measurable function on Ω and let φ be a nonincreasing positive function on $(0, \infty)$. If $F_\varphi(s)$ is defined as in (9), then for any $z \in \Omega$ and any measurable set $E \subset \Omega$, we have the inequality*

$$\left(\int_E d(x, z)^{1-Q} g(x) dx \right)^Q \leq C_Q \left(\int_0^\infty \varphi^{\frac{1}{Q}} ds \right)^{Q-1} \int_E F_\varphi(g) dx$$

where $C_Q := (1 + |\partial U(0, 1)|_{\mathbb{G}}(\mathbb{G}))^Q$ and $|\partial U(0, 1)|_{\mathbb{G}}(\mathbb{G})$ denotes the $\mathbb{G}$ -perimeter of the CC open ball $U(0, 1)$.

Proof. The proof follows the Euclidean one in [31, Theorem 3.1], with suitable changes. We provide here the details for the reader's convenience.

Up to a left translation, we can assume $z = 0$. Moreover, we can suppose the integral on the left-hand side to be positive, since otherwise there is nothing to prove. For notational simplicity, let's write

$$I_1 = \int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds, \quad I_2 = \int_E F_{\varphi}(g(x)) dx. \quad (42)$$

Similarly as before, we will assume $I_1, I_2 < +\infty$, otherwise the proof is over.

Let $0 < c < +\infty$ be an arbitrary value such that $c \leq \int_E \|x\|^{1-Q} g(x) dx$, where $\|\cdot\|$ denotes the homogeneous norm associated to the distance d .

Let $S := \{(x, s) \in E \times \mathbb{R} : 0 < s < g(x)\}$. We can write, by Fubini-Tonelli theorem

$$\int_E \|x\|^{1-Q} g(x) dx = \int_S \|x\|^{1-Q} dx ds.$$

Define

$$\lambda = \frac{I_2^{1/Q}}{c I_1^{1/Q}}$$

and

$$S' = \left\{ (x, s) \in S : \varphi(s) < \left(\frac{\|x\|}{c\lambda} \right)^Q \right\} \quad S'' = S \setminus S'.$$

If $(x, s) \in S'$, then $\|x\|^Q > c^Q \lambda^Q \varphi(s) \geq c^Q \lambda^Q \varphi(g(x))$ since φ is nonincreasing. Hence we estimate

$$\begin{aligned} \int_{S'} \|x\|^{1-Q} dx ds &\leq c^{1-Q} \lambda^{1-Q} \int_{S'} \varphi^{\frac{1-Q}{Q}}(g(x)) dx ds \\ &\leq c^{1-Q} \lambda^{1-Q} \int_E \left(\int_0^{g(x)} ds \right) \varphi^{\frac{1-Q}{Q}}(g(x)) dx \\ &= c^{1-Q} \lambda^{1-Q} \int_E F_{\varphi}(g(x)) dx = c^{1-Q} \lambda^{1-Q} I_2. \end{aligned} \quad (43)$$

Now we bound the integral over S'' by using Fubini-Tonelli theorem and the integration formula in polar coordinates in Carnot groups (see, for instance, [46, Corollary 4.6]):

$$\begin{aligned}
 \int_{S''} \|x\|^{1-Q} dx ds &\leq \int_0^{+\infty} \left(\int_{B(0, c\lambda\varphi^{1/Q}(s))} \|x\|^{1-Q} dx \right) ds \\
 &= \int_0^{+\infty} \left(\int_0^{c\lambda\varphi^{1/Q}(s)} \left(\int_{\partial U(0,1)} \|\delta_r(y)\|^{1-Q} r^{Q-1} d|\partial U(0,1)|_{\mathbb{G}}(y) \right) dr \right) ds \\
 &\leq |\partial U(0,1)|_{\mathbb{G}}(\mathbb{G}) \int_0^{+\infty} c\lambda\varphi^{\frac{1}{Q}}(s) ds = |\partial U(0,1)|_{\mathbb{G}}(\mathbb{G}) c\lambda I_1.
 \end{aligned} \tag{44}$$

Combining (43) and (44), we get

$$\begin{aligned}
 c &\leq \int_E \|x\|^{1-Q} g(x) dx = \int_{S'} \|x\|^{1-Q} dx ds + \int_{S''} \|x\|^{1-Q} dx ds \\
 &\leq c^{1-Q} \lambda^{1-Q} I_2 + |\partial U(0,1)|_{\mathbb{G}}(\mathbb{G}) c\lambda I_1 \\
 &= (1 + |\partial U(0,1)|_{\mathbb{G}}(\mathbb{G})) I_1^{\frac{Q-1}{Q}} I_2^{\frac{1}{Q}},
 \end{aligned}$$

which implies

$$c^Q \leq C_Q I_1^{Q-1} I_2 = C_Q \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_E F_\varphi(g(x)) dx.$$

Since this inequality holds for every $c \leq \int_E \|x\|^{1-Q} g(x) dx$, we get the conclusion. $\square$

Theorem 4.2. Let $\mathbb{G}_1 \equiv \mathbb{R}^n$, $\mathbb{G}_2 \equiv \mathbb{R}^m$, $\Omega \subset \mathbb{G}_1$ be an open set and let Q denote the homogeneous dimension of $\mathbb{G}_1$. Assume that $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{G}_2)$ and the function g in (13) belongs to $L^{Q,1}(\Omega)$. Then, a suitable representative of f is continuous and satisfies the condition (RR). In particular $f \in AC^Q(\Omega, \mathbb{G}_2)$ and then it is P -differentiable a.e. in Ω .

Proof. 1st step. First assume $\mathbb{G}_1 = \mathbb{G}$, $\mathbb{G}_2 = \mathbb{R}$ with $f \in W_{\mathbb{G}, \text{loc}}^{1,1}(\Omega)$ and $|\nabla_{\mathbb{G}} f| \in L^{Q,1}(\Omega)$. We stress that this case was already studied in the general case of measure metric spaces (see, for instance, [61, Thm. 6.6]) by extending the strategy exploited in [31]. However we are going to repeat this strategy since we will use it in the second step, which is still open to our knowledge.

First of all notice that we can apply Theorem 2.11 inferring that, since $|\nabla_{\mathbb{G}} f| \in L^{Q,1}(\Omega)$, there exists a nonnegative nonincreasing function $\varphi : (0, +\infty) \rightarrow [0, +\infty)$ such that

$$\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds < +\infty, \quad (45)$$

$$\int_{\Omega} F_{\varphi}(|\nabla_{\mathbb{G}} f(x)|) dx < +\infty. \quad (46)$$

Let's apply now Theorem 2.17, finding a negligible set N such that

$$|f(z) - f_U| \leq C \int_U \frac{|\nabla_{\mathbb{G}} f(y)|}{d(z, y)^{Q-1}} dy \quad (47)$$

for each CC open ball $U \subset \Omega$ and $z \in U \setminus N$. This implies that, for every CC open ball $U \subset \Omega$,

$$(\text{osc}_{U \setminus N} f)^Q \leq \int_U w(x) dx, \quad (48)$$

where

$$w(x) = 2^Q C^Q C_Q \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} F_{\varphi}(|\nabla_{\mathbb{G}} f(x)|).$$

In order to prove (48) we can assume the left-hand side to be positive, and choose $0 \leq c < \text{osc}_{U \setminus N} f$. By definition of oscillation and the triangular inequality, there exists $z \in U \setminus N$ such that $c \leq 2|f(z) - f_U|$. Combining this fact with (47) and Theorem 4.1, we infer that

$$\begin{aligned} c^Q &\leq 2^Q C^Q \left(\int_U \frac{|\nabla_{\mathbb{G}} f(y)|}{d(z, y)^{Q-1}} dy \right)^Q \\ &\leq 2^Q C^Q C_Q \left(\int_0^{\infty} \varphi^{\frac{1}{Q}} ds \right)^{Q-1} \int_U F_{\varphi}(|\nabla_{\mathbb{G}} f(x)|) dx \\ &= \int_U w(x) dx. \end{aligned}$$

By arbitrariness of $0 \leq c < \text{osc}_{U \setminus N} u$, we conclude that (48) is satisfied. As a consequence, f is locally uniformly continuous on $\Omega \setminus N$ and can be extended to a continuous function $\tilde{f}$ on the whole space Ω which satisfies the (RR) condition with weight w .

2nd step. Let us now prove the general case. For given $z \in \mathbb{G}_2 \equiv \mathbb{R}^m$ let $u_z : \Omega \rightarrow \mathbb{R}$ denote a section of f , that is, the function defined in (12). By definition $u_z \in W_{\mathbb{G}_1, \text{loc}}^{1,1}(\Omega)$

and satisfies (13) with $g \in L^{Q,1}(\Omega)$. Thus, by the 1st step, there exists a continuous function $\tilde{u}_z : \Omega \rightarrow \mathbb{R}$ and a negligible set $N_z \subset \Omega$, found as in (47), such that

$$\tilde{u}_z(x) = u_z(x) = d_2(z, f(x)) \text{ for each } x \in \Omega \setminus N_z. \quad (49)$$

Now we proceed again as in the 1st step: since $g \in L^{Q,1}(\Omega)$, we know that there exists a nonnegative nonincreasing function $\varphi : (0, +\infty) \rightarrow [0, +\infty)$ such that

$$\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds < +\infty, \quad (50)$$

$$\int_{\Omega} F_{\varphi}(g(x)) dx < +\infty. \quad (51)$$

We want to show that

$$(\text{osc}_{U \setminus N_z} \tilde{u}_z)^Q \leq \tilde{C} \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_U F_{\varphi}(g(x)) dx \quad (52)$$

for each CC open ball $U \subset \Omega$ and each $z \in \mathbb{G}_2$. If the left-hand side is 0, clearly there is nothing to prove. Otherwise pick any $0 \leq c < \text{osc}_{U \setminus N_z} \tilde{u}_z$. This implies that there exists $x \in U \setminus N_z$ such that $c \leq 2|\tilde{u}_z(x) - (\tilde{u}_z)_U|$. Hence, using Theorem 4.1,

$$\begin{aligned} c^Q &\leq 2^Q |\tilde{u}_z(x) - (\tilde{u}_z)_U|^Q \stackrel{(47)}{\leq} 2^Q C^Q \left(\int_U \frac{|\nabla_{\mathbb{G}_1} \tilde{u}_z(y)|}{d_1(x, y)^{Q-1}} dy \right)^Q \\ &\leq 2^Q C^Q \left(\int_U \frac{g(y)}{d_1(x, y)^{Q-1}} dy \right)^Q \\ &\leq 2^Q C^Q C_Q \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_U F_{\varphi}(g(x)) dx. \end{aligned}$$

Since this estimate holds for every $0 \leq c < \text{osc}_{U \setminus N_z} \tilde{u}_z$, we get (52).

From the fact that $\tilde{u}_z$ is continuous on Ω , the same estimate holds replacing $\text{osc}_{U \setminus N_z} \tilde{u}_z$ with $\text{osc}_U \tilde{u}_z$. Let $N = \cup_{z \in \mathbb{Q}^m} N_z$. Then N is still negligible and, by (49), it follows that

$$\begin{aligned} |d_2(z, f(x)) - d_2(z, f(y))|^Q &\leq (\text{osc}_U \tilde{u}_z)^Q \\ &\leq \tilde{C} \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_U F_{\varphi}(g(x)) dx \end{aligned} \quad (53)$$

for each $x, y \in U \setminus N$ and $z \in \mathbb{Q}^m$, for each CC open ball $U \subset \Omega$. Now observe that, by the triangle inequality and density of $\mathbb{Q}^m$ in $\mathbb{G}_2 \equiv \mathbb{R}^m$, it follows that

$$\sup_{z \in \mathbb{Q}^m} |d_2(z, f(x)) - d_2(z, f(y))| = d_2(f(x), f(y)) \quad (54)$$

for each $x, y \in U \setminus N$. Thus, from (53) and (54), it follows that

$$d_2(f(x), f(y))^Q \leq \tilde{C} \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_U F_\varphi(g(x)) dx, \quad (55)$$

for each $x, y \in U \setminus N$ and each CC open ball $U \subset \Omega$. This implies that $f : \Omega \setminus N \rightarrow \mathbb{G}_2$ is locally uniformly continuous and, since $\Omega \setminus N$ is dense in Ω , it can be extended to a (unique) continuous function $f : \Omega \rightarrow \mathbb{G}_2$. Moreover, from (55), it follows also that

$$(\text{osc}_U f)^Q \leq \tilde{C} \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} \int_U F_\varphi(g(x)) dx,$$

for each CC open ball $U \subset \Omega$. This means that f satisfies the condition (RR) with weight

$$w(x) = \tilde{C} \left(\int_0^{+\infty} \varphi^{\frac{1}{Q}}(s) ds \right)^{Q-1} F_\varphi(g(x)). \quad \square$$

Remark 4.3. Let us recall that Theorem 4.2 is sharp already in the Euclidean case. Indeed, Stein [55] pointed out that, if $\nabla f \notin L^{n,1}$, then both continuity and differentiability of f may fail. For instance, an explicit example is given by the function $u(x) := \log(\log(1/|x|))$ on $B(0, 1/3)$ extended in a smooth way on $\mathbb{R}^n$ with compact support. By a direct computation it is easy to prove that $u \in W^{1,n}(\mathbb{R}^n)$ and moreover $|\nabla u| \in L^{n,q}$ for every $q > 1$ (see [54] for the definition of general Lorentz spaces). If $\{q_k\}_{k \in \mathbb{N}}$ is an enumeration of the rational points in $\mathbb{R}^n$, we set $f_k(x) = u(x - q_k)$ and define

$$f(x) = \sum_{k=0}^{+\infty} 2^{-k} f_k(x).$$

Then such a map f , and its weak gradient, have the same summability properties of u , but it is essentially unbounded (thus discontinuous and not differentiable) around any point. Hence Theorem 4.2 is sharp also in the refined class of Lorentz spaces.

Remark 4.4. Note that Theorem 4.2 is not invertible, in the following sense: even though Ω is a bounded open set of an Euclidean space $\mathbb{R}^n$, it is well known that a function

$f \in AC^n(\Omega)$ need not admit weak gradient ∇f in $L^{n,1}(\Omega)$ (see, for instance, [30, Theorem 3.7]).

5. Area formula and change of variables for absolutely continuous functions between Carnot groups

An important property of Lipschitz functions between Euclidean spaces is the so called area formula, relating the Hausdorff measure of the image of a Lipschitz map (counted with multiplicity) to the integral of the Jacobian on the domain. Such a formula holds also in the setting of Carnot groups (Theorem 5.3); we will extend this result to the case of absolutely continuous functions (see Theorem 5.7).

5.1. Area formula between Carnot groups

The content of this section mainly relies on [39].

An important tool in Euclidean geometric measure theory is the so-called *area formula* for Lipschitz maps, which turns to be true also for Lipschitz maps between Carnot groups.

Definition 5.1. [H-Jacobian] Given two Carnot groups $(\mathbb{G}_i, d_i)$, $i = 1, 2$, endowed with an invariant distance d_i , let $L : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a H -linear map. Let Q denote the homogeneous dimension of $\mathbb{G}_1$. The *horizontal jacobian* of L is defined by

$$J_Q(L) := \frac{\mathcal{H}_{d_2}^Q(L(B_1))}{\mathcal{H}_{d_1}^Q(B_1)}$$

where B_1 denotes the unit open ball in $(\mathbb{G}_1, d_1)$ and for $i = 1, 2$, $\mathcal{H}_{d_i}^Q$ denotes the Hausdorff measure w.r.t. d_i .

Remark 5.2. In the previous definition it is possible to replace B_1 with any other measurable subset of $\mathbb{G}_1$ of positive finite measure, by the H -linearity of the map L .

The following Theorem is taken from [39, Theorem 4.4].

Theorem 5.3. (Area formula for Lipschitz maps) *Let $f : E \subset (\mathbb{G}_1, d_1) \rightarrow (\mathbb{G}_2, d_2)$ be a Lipschitz continuous function with E measurable set. Then*

$$\int_E J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} N(f, E, y) d\mathcal{H}_{d_2}^Q(y)$$

where $N(f, E, \cdot)$ denotes the multiplicity function of f relatively to E , that is $N(f, E, y) = \#\{f^{-1}(y) \cap E\} \in \mathbb{N} \cup \{+\infty\}$.

Remark 5.4. The original definition of P-differentiability (cf. Definition 2.5) requires E to be an open subset of a Carnot group $\mathbb{G}_1$. However, it is possible to extend this notion to the set of density points of a general measurable set. In particular, the Pansu's differential, if it exists, is uniquely determined at a density point of E . Moreover, a generalization of Theorem 2.6 holds: it has been proved by Magnani in [39, Theorem 3.9] that a Lipschitz map $f : E \rightarrow \mathbb{G}_2$ is $\mathcal{H}_{d_1}^Q$ a.e. P-differentiable, where $E \subset \mathbb{G}_1$ is measurable, while Q and d_1 denote the homogeneous dimension and the CC distance of $\mathbb{G}_1$ respectively.

Aim of this chapter is to extend Theorem 5.3 to the case of maps in $AC^Q(\Omega, \mathbb{G}_2)$, which are not necessarily Lipschitz continuous. The following useful result is proved in [35, Lemma 2.12] in the general setting of metric measure spaces.

Proposition 5.5 (*Lusin's (N) condition for AC^Q*). *Let $\mathbb{G}_1 \equiv \mathbb{R}^n$ and $\mathbb{G}_2 \equiv \mathbb{R}^m$ be Carnot groups and let $\Omega \subset \mathbb{G}_1$ be an open set. Let $f \in AC^Q(\Omega, \mathbb{G}_2)$, where Q denotes the homogeneous dimension of $\mathbb{G}_1$. Then, for every measurable subset $E \subset \Omega$, it holds*

$$\mathcal{L}^n(E) = 0 \implies \mathcal{H}^Q(f(E)) = 0.$$

The next result is inspired by Theorem 3.1.9 in [15] and adapted to our setting.

Theorem 5.6. *Let $\Omega \subset \mathbb{G}_1 \equiv \mathbb{R}^n$ be an open set and $A \subset \Omega$ be measurable. If $f : \Omega \rightarrow \mathbb{G}_2 \equiv \mathbb{R}^m$ has the property that*

$$\limsup_{y \rightarrow x} \frac{d_2(f(y), f(x))}{d_1(y, x)} < +\infty \quad \text{for all } x \in A,$$

then there exists a countable measurable partition $\{A_k\}_k$ of A such that $f|_{A_k}$ is Lipschitz continuous.

Proof. For every $j \in \mathbb{N}$, we define

$$C_j := \{z \in \Omega : d_2(f(y), f(z)) \leq j d_1(y, z) \text{ for every } y \in U_{d_1}(z, 1/j)\}. \quad (56)$$

It follows that A is contained in $\cup_j C_j$. Indeed, let $a \in A$, so that

$$\limsup_{y \rightarrow a} \frac{d_2(f(y), f(a))}{d_1(y, a)} < +\infty.$$

This implies that there exists $j \in \mathbb{N}$ and $\bar{r} > 0$ sufficiently small such that $\frac{d_2(f(y), f(a))}{d_1(y, a)} < j$ for every $y \in U_{d_1}(a, \bar{r})$. Hence, up to increase j , we deduce that the previous inequality holds for every $y \in U_{d_1}(a, \frac{1}{j})$, which means that $a \in C_j$.

Notice that each C_j is closed in Ω . In fact, let $x_n \in C_j$ which converge to $x \in \Omega$ as

$n \rightarrow \infty$ and let $y \in U_{d_1}(x, \frac{1}{j})$. If $n \in \mathbb{N}$ is chosen sufficiently large, we can assume $\{x, y\} \subset U_{d_1}(x_n, 1/j)$. Hence

$$\begin{aligned} d_2(f(y), f(x)) &\leq d_2(f(y), f(x_n)) + d_2(f(x_n), f(x)) \\ &\leq j d_1(y, x_n) + j d_1(x_n, x) \end{aligned}$$

and, passing to the limit as $n \rightarrow \infty$, we get $d_2(f(y), f(x)) \leq j d_1(y, x)$, so that $x \in C_j$. Exploiting the Lindelöf property of $\mathbb{G}_1 \equiv \mathbb{R}^n$, we can write C_j as a countable union of closed sets C_j^i , for $i \in \mathbb{N}$, each of which of diameter less than $\frac{1}{j}$. If we relabel $\{C_k\}_k = \{C_j^i\}_{j,i}$, it follows that $f|_{C_k}$ is Lipschitz by (56). To conclude, define $A_k := C_k \cap A$ which are still measurable and notice that we can suppose the sets A_k to be disjoint (hence a partition of A) by the usual differences procedure. $\square$

We now combine Theorem 5.6 and Theorem 5.3 to get an area formula for absolutely continuous maps between Carnot groups.

Theorem 5.7. *Let $\Omega \subset \mathbb{G}_1$ be open and $f \in AC^Q(\Omega, \mathbb{G}_2)$. For every measurable subset E of Ω , we have*

$$\int_E J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} N(f, E, y) d\mathcal{H}_{d_2}^Q(y) \quad (57)$$

where J_Q is the Jacobian introduced in Definition 5.1, while $N(f, E, \cdot)$ denotes the multiplicity function of f relatively to E .

Proof. We note that, by Theorem 3.7, $\text{Lip}(f)$ is finite almost everywhere, which means

$$\text{Lip}(f)(x) := \limsup_{y \rightarrow x} \frac{d_2(f(y), f(x))}{d_1(y, x)} < +\infty \quad \text{for } \mathcal{L}^n - a.e. x \in \Omega.$$

We can therefore apply Theorem 5.6 and write Ω as a disjoint union of measurable subsets

$$\Omega = A_0 \cup \bigcup_{k=1}^{+\infty} A_k,$$

with $\mathcal{L}^n(A_0) = 0$ and $f|_{A_k}$ Lipschitz.

Moreover we know, again by Theorem 3.7, that f is P-differentiable almost everywhere. In the negligible set where f does not satisfy this condition, we can define $d_P f(x)$ to be for instance the zero map.

Let now E be a measurable subset of Ω and denote $E_k := E \cap A_k$. Applying Theorem 5.3 we get, for every $k \geq 1$,

$$\int_{E_k} J_Q(d_P f|_{E_k}(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} N(f, E_k, y) d\mathcal{H}_{d_2}^Q(y).$$

Moreover this identity trivially holds for $k = 0$ in view of Lemma 5.5. Note that, if x is a density point of E_k and $f : \Omega \rightarrow \mathbb{G}_2$ is P-differentiable at $x \in E_k$, then also $f|_{E_k}$ is P-differentiable at x in the sense of Remark 5.4 and the two differentials coincide. Hence, summing over $k \in \mathbb{N}$ we find, by a simple application of monotone convergence,

$$\begin{aligned} \int_E J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) &= \sum_{k=0}^{+\infty} \int_{E_k} J_Q(d_P f|_{E_k}(x)) d\mathcal{H}_{d_1}^Q(x) \\ &= \sum_{k=0}^{+\infty} \int_{\mathbb{G}_2} N(f, E_k, y) d\mathcal{H}_{d_2}^Q(y) = \int_{\mathbb{G}_2} N(f, E, y) d\mathcal{H}_{d_2}^Q(y). \quad \square \end{aligned}$$

Through standard arguments of approximation by simple functions, it is easy to extend formula (57) to a more general form:

Corollary 5.8. *If $f \in AC^Q(\Omega, \mathbb{G}_2)$, then for any measurable subset E of Ω and each summable function $u : E \rightarrow \mathbb{R}$, it holds*

$$\int_E u(x) J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} \sum_{x \in f^{-1}(y)} u(x) d\mathcal{H}_{d_2}^Q(y).$$

As a straightforward consequence of Theorem 4.2 and Theorem 5.7, we get the following

Corollary 5.9. *Let $\Omega \subset \mathbb{G}_1$ be an open set and let $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{G}_2)$ such that the function g in (13) belongs to $L^{Q,1}(\Omega)$. Then a suitable representative of f satisfies*

$$\int_E J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} N(f, E, y) d\mathcal{H}_{d_2}^Q(y)$$

for every measurable subset $E \subset \Omega$. Moreover, for every $u : E \rightarrow \mathbb{R}$ summable,

$$\int_E u(x) J_Q(d_P f(x)) d\mathcal{H}_{d_1}^Q(x) = \int_{\mathbb{G}_2} \sum_{x \in f^{-1}(y)} u(x) d\mathcal{H}_{d_2}^Q(y).$$

Appendix A. Orlicz-Sobolev spaces on Carnot groups

Orlicz spaces arise naturally in many applications in real, harmonic or functional analysis. Such spaces were introduced by the Polish mathematician Władysław Orlicz in 1932 and are generalizations of the classical L^p spaces. Indeed, the role played by the

function t^p is replaced by more general convex maps, called N -functions.

We recall now the main definitions and basic properties of Orlicz spaces: see for instance [1], [36] or [37] for an exhaustive treatment.

Definition 6.1. A map $A : [0, +\infty) \rightarrow [0, +\infty)$ is called an N -function if it is convex, vanishes only at 0 and has the following limit properties:

$$\lim_{t \rightarrow 0^+} \frac{A(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{A(t)}{t} = +\infty.$$

It is easy to verify that a map satisfying these properties is actually continuous on $[0, +\infty)$ and strictly increasing. Given an N -function $A : [0, +\infty) \rightarrow [0, +\infty)$, its *Young conjugate* $\tilde{A}$ is defined as the N -function

$$\tilde{A}(t) := \sup_{s > 0} \{st - A(s)\}.$$

Examples of N -functions and associated conjugates are for instance

$$\begin{aligned} A(t) &= \frac{t^p}{p}, & \tilde{A}(t) &= \frac{t^{p'}}{p'} & \left(1 < p < +\infty, \quad p' = \frac{p}{p-1} \right); \\ A(t) &= e^t - t - 1, & \tilde{A}(t) &= (1+t) \log(1+t) - t. \end{aligned}$$

Definition 6.2. Let $\Omega \subset \mathbb{R}^n$ be measurable and let A be an N -function. For every measurable map $u : \Omega \rightarrow \mathbb{R}$, we define its *Luxemburg norm* as

$$\|u\|_{L^A(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

The *Orlicz space* associated to the N -function A is defined as

$$L^A(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \|u\|_{L^A(\Omega)} < +\infty\}.$$

Equipped with $\|\cdot\|_{L^A(\Omega)}$, the Orlicz space $L^A(\Omega)$ is a Banach space. Moreover, an interesting relation between an N -function and its Young conjugate is expressed by the following Hölder-type inequality:

$$\int_{\Omega} u(x)v(x) dx \leq 2\|u\|_{L^A(\Omega)}\|v\|_{L^{\tilde{A}}(\Omega)}.$$

Notice that, when considering $A(t) := t^p$, for $1 < p < +\infty$, we get that $L^A(\Omega)$ coincides with the classical $L^p(\Omega)$ space. Moreover in this case we get $\|\cdot\|_{L^A(\Omega)} = p^{-1/p}\|\cdot\|_{L^p(\Omega)}$.

In order to establish an inclusion relation similar to the classical one for Lebesgue spaces on domain with finite measure, we introduce two new concepts: if A and B are N -functions, we will say that B *dominates* A *globally* if there exists a positive constant k such that $A(t) \leq B(kt)$ for all $t \geq 0$. If this estimate holds only for large values of t , we will say that B *dominates* A *near infinity*. An interesting result is that the embedding $L^B(\Omega) \hookrightarrow L^A(\Omega)$ holds if and only if either B dominates A globally or B dominates A near infinity and $|\Omega| < +\infty$. Hence functions which are globally equivalent, i.e. each one dominates the other globally, determine the same Orlicz space (and clearly the same holds if we require that they are equivalent near infinity and $|\Omega| < +\infty$).

Note that, if $1 < p < q < +\infty$, then the map $B(t) := t^q$ dominates the function $A(t) := t^p$ near infinity, so that $L^q(\Omega) \subset L^p(\Omega)$ if Ω has finite measure.

In the same spirit of Sobolev space, it is now straightforward to define Orlicz-Sobolev functions by means of Orlicz integrability conditions on the weak derivatives. We give this definition directly in the general setting of Carnot groups:

Definition 6.3. Let $\Omega \subset \mathbb{G} \equiv \mathbb{R}^n$ be open and let A be an N -function. We define the *Orlicz-Sobolev space*

$$W_{\mathbb{G}}^{1,A}(\Omega) := \{u \in L^A(\Omega) : \exists X_i u \in L^A(\Omega) \quad i = 1, \dots, m_1\},$$

where $X_i u$ are the distributional derivatives of u computed along the generating vector fields of the group. We also set

$$\|u\|_{W_{\mathbb{G}}^{1,A}(\Omega)} := \|u\|_{L^A(\Omega)} + \|\nabla_{\mathbb{G}} u\|_{L^A(\Omega)}.$$

Equipped with such a norm, it is easy to see that $(W_{\mathbb{G}}^{1,A}(\Omega), \|\cdot\|_{W_{\mathbb{G}}^{1,A}(\Omega)})$ is a Banach space. If $\mathbb{G}$ is the Euclidean $\mathbb{R}^n$, we will use the notation $W^{1,A}(\Omega)$.

The classical embedding theorems for Sobolev spaces have been extended also to Orlicz-Sobolev spaces in the Euclidean setting. In particular, the condition $p > n$ has to be replaced by a more general requirement on the growth of the N -function which induces the space. In [1] or [14] it has been proven that, if $\Omega \subset \mathbb{R}^n$ is sufficiently regular and A is an N -function, then $W^{1,A}(\Omega)$ is embedded in the space of continuous and bounded maps provided that the condition

$$\int_1^{+\infty} \frac{A^{-1}(t)}{t^{1+1/n}} dt < +\infty \quad (58)$$

is satisfied. In the case of $A(t) = t^p$, we recover the classical embedding theorem for $p > n$. However, as remarked in [11], this condition is not sharp, since it can be improved by requiring

$$\int_1^{+\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < +\infty, \quad (59)$$

which is a weaker assumption than (58) (see [57]). Indeed, G. Talenti proved in the same work that (59) is a sufficient condition for boundedness (see also [12]), while A. Cianchi showed in [11] that it is also the sharp condition for continuity. Moreover the same author proved in [2] that this condition is also the correct one to ensure differentiability almost everywhere.

The strategy of Cianchi relies on rearrangement tools, while in the following we will provide an alternative way to extend this result in the setting of Carnot groups, following the approach introduced by Kauhanen, Koskela and Malý in [31], in the same spirit of Theorem 4.2.

In order to reach this result, we need first to extend Definition 6.3 to the case of group-valued maps, inspired by Definition 2.18. This generalization is not straightforward as for Euclidean spaces (in which it suffices to work in components), since a Carnot group cannot be seen in general as direct product of subgroups.

Definition 6.4. Let $\mathbb{G}_1 \equiv \mathbb{R}^n, \mathbb{G}_2 \equiv \mathbb{R}^m$ be Carnot groups. Let $\Omega \subset \mathbb{G}_1$ be open and let A be an N -function.

- (i) We say that a function $u : \Omega \rightarrow \mathbb{G}_2$ belongs to the space $L^A(\Omega, \mathbb{G}_2)$ if $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is measurable and the map $u_0 : \Omega \rightarrow \mathbb{R}$ defined by $u_0(x) = d_2(0, u(x))$ is in $L^A(\Omega)$;
- (ii) We say that a function $u : \Omega \rightarrow \mathbb{G}_2$ belongs to the Orlicz-Sobolev space $W^{1,A}(\Omega, \mathbb{G}_2)$ if
 - $u \in L^A(\Omega, \mathbb{G}_2)$;
 - for all $z \in \mathbb{G}_2$, the functions $u_z : \Omega \rightarrow \mathbb{R}$ defined by

$$u_z(x) := d_2(z, u(x)) \quad \text{if } x \in \Omega \quad (60)$$

are in $W_{\mathbb{G}_1, \text{loc}}^{1,A}(\Omega)$ (according to Definition 6.3);

- There exists a nonnegative function $g \in L^A(\Omega)$ (which does not depend on z) such that

$$|\nabla_{\mathbb{G}_1} u_z| \leq g \quad \text{a.e. in } \Omega, \quad \text{for each } z \in \mathbb{G}_2.$$

We provide now the natural extension of Theorem 3.6 and Theorem 4.2 to the case of Orlicz-Sobolev maps: under a sufficient growth condition on A , functions in the space $W^{1,A}(\Omega, \mathbb{G}_2)$ are Q -absolutely continuous and so enjoy nice regularity properties.

The following Lemma represents the key step in order to apply the strategy proposed by the authors in [31] also to the case of Orlicz-Sobolev spaces.

Lemma 6.5. *Let $\Omega \subset \mathbb{G} \equiv \mathbb{R}^n$ be open with $\mathcal{L}^n(\Omega) < +\infty$. Let A be an N -function satisfying*

$$\int_1^{+\infty} \left(\frac{t}{A(t)} \right)^{\frac{1}{Q-1}} dt < +\infty, \quad (61)$$

and let g be a nonnegative function in the space $L^A(\Omega)$. Then there exists a nonincreasing function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\int_{\Omega} F_{\varphi}(g(x)) dx < +\infty \quad (62)$$

and

$$\int_0^{+\infty} \varphi^{1/Q}(t) dt < +\infty. \quad (63)$$

Proof. Notice first that we can assume that

$$\int_0^1 \left(\frac{t}{A(t)} \right)^{\frac{1}{Q-1}} dt < +\infty, \quad (64)$$

since it is always possible to replace A with another N -function which is equivalent near infinity (so that it determines the same Orlicz space) and decays sufficiently slow to 0. Since $g \in L^A(\Omega)$, there exists for sure $\bar{\lambda} > 0$ such that $\int_{\Omega} A(\bar{\lambda}g(x)) dx < +\infty$. Define $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ as

$$\varphi(t) := \left(\frac{A(\bar{\lambda}t)}{t} \right)^{\frac{Q}{1-Q}}.$$

By definition φ is positive and also nonincreasing. In fact, by convexity of A , we know that $A(\lambda t_1 + (1-\lambda)t_2) \leq \lambda A(t_1) + (1-\lambda)A(t_2)$ for every $\lambda \in (0, 1)$, $t_1, t_2 \in [0, +\infty)$. Consider $0 < t < s$ and apply this inequality with $t_1 = \bar{\lambda}s$, $t_2 = 0$, $\lambda = \frac{t}{s}$. We get $A(\bar{\lambda}t) \leq \frac{t}{s}A(\bar{\lambda}s)$ and, by simple rearrangements, $\varphi(t) \geq \varphi(s)$. Moreover we can calculate

$$\begin{aligned} \int_{\Omega} F_{\varphi}(g(x)) dx &= \int_{\{g>0\}} g(x) \left[\left(\frac{A(\bar{\lambda}g(x))}{g(x)} \right)^{\frac{Q}{1-Q}} \right]^{\frac{1-Q}{Q}} dx \\ &= \int_{\Omega} A(\bar{\lambda}g(x)) dx < +\infty \end{aligned}$$

and also, exploiting (61) and (64),

$$\int_0^{+\infty} \varphi^{1/Q}(t) dt = \int_0^{+\infty} \left(\frac{A(\bar{\lambda}t)}{t} \right)^{\frac{1}{1-Q}} dt = \bar{\lambda}^{\frac{Q}{1-Q}} \int_0^{+\infty} \left(\frac{t}{A(t)} \right)^{\frac{1}{Q-1}} dt < +\infty. \quad \square$$

Remark 6.6. It is proved in [12] that condition (61) is indeed equivalent to the following:

$$\int_0^{+\infty} \frac{\tilde{A}(t)}{t^{1+Q'}} dt < +\infty, \quad (65)$$

which clearly extends (59) in the case of Carnot groups of homogeneous dimension Q .

Under condition (61), or equivalently (65), functions in the space $W^{1,A}(\Omega, \mathbb{G}_2)$ are Q -absolutely continuous. Actually, only an Orlicz bound on the weak horizontal gradient of the sections is needed. This is expressed by the following result.

Theorem 6.7. *Let $\mathbb{G}_1 \equiv \mathbb{R}^n$, $\mathbb{G}_2 \equiv \mathbb{R}^m$ be Carnot groups and let $\Omega \subset \mathbb{G}_1$ open. Suppose also that $\mathcal{L}^n(\Omega) < +\infty$ and denote by Q the homogeneous dimension of $\mathbb{G}_1$. Let A be an N -function satisfying (61). If $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{G}_2)$ and there exists a function $g \in L^A(\Omega)$ such that*

$$|\nabla_{\mathbb{G}} u_z| \leq g \quad \text{for all } z \in \mathbb{G}_2,$$

where u_z are defined as in (60), then there exists a representative of f which is continuous and satisfies the (RR) condition. In particular $f \in AC^Q(\Omega, \mathbb{G}_2)$ and is P -differentiable a.e. in Ω .

Proof. The proof of this result can be achieved using the same strategy exploited in Theorem 4.2. In particular the existence of a nonincreasing function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ satisfying properties (62) and (63) is ensured by Lemma 6.5 instead of Theorem 2.11. The other steps of the proof are unchanged. $\square$

Remark 6.8. Equivalently one can observe that, combining Lemma 6.5 and the characterization given by Theorem 2.11, a function g as in the assumptions belongs actually to the Lorentz space $L^{Q,1}(\Omega)$ and then apply Theorem 4.2.

We deduce that, under condition (61), Orlicz-Sobolev maps between Carnot groups are actually in $AC^Q(\Omega, \mathbb{G}_2)$: hence they share properties like continuity, P -differentiability a.e. (Theorem 3.7) and area formula (Theorem 5.7 and Corollary 5.8).

Appendix B. Differentiation for maps between metric spaces and metric space-valued Sobolev maps

B.1. Differentiation for maps between metric spaces

In this section we will assume that the metric measure space $X \equiv (X, d_X, \mu)$ admits a p -Poincaré inequality with $p \geq 1$, the ambient measure μ is doubling and $Y \equiv (Y, d_Y)$ is a (separable) metric space.

Let us recall that a general differentiation theory has been carried out for maps $f : X \rightarrow Y$, whenever Y can be isometrically embedded into a Banach space V satisfying the *Radon-Nikodym property* (RNP) (see [9,10]). We recall that a Banach space V satisfies (RNP) if each Lipschitz function $f : \mathbb{R} \rightarrow V$ is differentiable a.e. The first pioneering result is due to Cheeger [8] when $Y = \mathbb{R}$, who was able to endow the metric measure space X by a so-called *Cheeger differentiable structure* and proved a Rademacher differentiability theorem for Lipschitz function in this context. This result was remarkably extended by Cheeger and Kleiner to maps with target space $Y = V$ satisfying (RNP). Then, whenever the target space Y can be bi-Lipschitz embedded into a Banach space V satisfying (RNP), a differentiation theory can be carried out also for maps $f : \Omega \subset X \rightarrow Y \subset V$ and one can get a Stepanov's type theorem (see, for instance, [62]). Moreover metric space-valued Sobolev functions can be introduced by means of the notion of *upper gradient* in this setting as in [27, Chap. 7]).

This approach may not work, as concern the differentiability issue, if the target space is a Carnot group. Indeed, as showed in [9, Theorem 6.1], the simplest Carnot group $\mathbb{H}^1$ does not admit a bi-Lipschitz embedding into any Banach space satisfying (RNP).

B.2. Metric space-valued Sobolev maps

As far as the notion of metric space-valued Sobolev mapping is concerned, different equivalent approaches can be introduced. Let us point out here two ones related to our arguments: the approach by *embedding of the target space* and the one by *sections of mappings*. From now on we can weaken the assumptions on the involved metric spaces. More precisely, we assume that $X \equiv (X, d_X, \mu)$ is a metric measure space with ambient measure μ *finite on bounded sets*, that is $\mu(E) < \infty$ for each bounded $E \subset X$, and $Y \equiv (Y, d_Y, y_0)$ is a pointed complete and separable metric space. Moreover, we will denote by $B(x, r)$ and $U(x, r)$ the closed and open ball of X centered at x with radius r , respectively.

Approach by embedding of the target metric space: This approach extends the one due to Shanmugalingam [53] who introduced the notion of *Newtonian Sobolev space* $N^{1,p}(X)$ for real-valued functions defined on X and it was carried out in [27, Chap. 7]. Let us briefly recall here the main notions and results. We first introduce vector-valued Sobolev functions on metric spaces. If V is a Banach space and $1 \leq p < \infty$, we say that a mapping $f : X \rightarrow V$ belongs to the *Dirichlet space* $D^{1,p}(X : V)$ if f is measurable and

it possesses a p -integrable p -weak upper gradient (see [27, Section 6.2]). When $V = \mathbb{R}$, we simply denote $D^{1,p}(X) := D^{1,p}(X : \mathbb{R})$. It is clear that $D^{1,p}(X : V)$ is a vector space. We denote as *Sobolev space of V -valued functions on X* the space $N^{1,p}(X : V) := D^{1,p}(X : V) \cap L^p(X, V)$ and $N^{1,p}(X : V)$ turns out to be a Banach space with respect to the norm

$$\|u\|_{N^{1,p}(X:V)} := \|u\|_{L^p(X,V)} + \|\varrho_u\|_{L^p(X)},$$

where ϱ_u denotes the minimal p -weak upper gradient of u guaranteed by [27, Chap. 6 and Theorem 6.3.20].

Consider now the *Kuratowski isometric embedding* of Y into $V := \ell^\infty(Y)$, with respect to the base point y_0 . More precisely, let $\ell^\infty(Y)$ denote the Banach space of bounded functions from Y to $\mathbb{R}$, equipped with the norm

$$\|\varphi\|_{\ell^\infty(Y)} := \sup_{y \in Y} |\varphi(y)| \quad \text{if } \varphi \in \ell^\infty(Y).$$

Then the mapping

$$\begin{aligned} Y \ni y &\mapsto \varphi_y \in \ell^\infty(Y) \\ \varphi_y(z) &:= d(z, y) - d(y_0, z) \quad z \in Y \end{aligned}$$

is an isometric embedding (see [27, Theorem 4.1]). According to [27], we can introduce the Sobolev and Dirichlet classes of mappings between metric spaces X to Y (depending on the base point y_0) as follows:

$$N^{1,p}(X : Y) := \{f \in N^{1,p}(X : \ell^\infty(Y)) : f(x) \in Y \text{ } \mu\text{-a.e. } x \in X\}$$

and similarly

$$D^{1,p}(X : Y) := \{f \in D^{1,p}(X : \ell^\infty(Y)) : f(x) \in Y \text{ } \mu\text{-a.e. } x \in X\}.$$

Moreover we denote as $N_{\text{loc}}^{1,p}(X : Y)$ ($D_{\text{loc}}^{1,p}(X : Y)$, respectively) the class of all maps $f : X \rightarrow \ell^\infty(Y)$ such that each $x \in X$ has a small radius $r > 0$ for which $f|_{U(x,r)} \in N^{1,p}(U(x,r) : Y)$ ($f|_{U(x,r)} \in D^{1,p}(U(x,r) : Y)$, respectively). Eventually we denote as $L_{\text{loc}}^p(X, \mu)$ the class of measurable maps $f \rightarrow \mathbb{R}$ such that $\int_E |f|^p d\mu < \infty$ for each bounded set $E \subset X$. Let us recall the following characterizations of $N^{1,p}(X : Y)$ and $D^{1,p}(X : Y)$.

Theorem 7.1. ([27, Theorem 7.1.20]) *Let V be a Banach space and let $f : X \rightarrow V$ be a measurable function. The following conditions are equivalent:*

- (i) *f has a μ -representative in the Dirichlet class $D^{1,p}(X : V)$;*

- (ii) there exists a p -integrable Borel function $\varrho : X \rightarrow [0, \infty]$ with the following property: for each 1-Lipschitz function $\varphi : V \rightarrow \mathbb{R}$ there exists a μ -representative f_φ of the function $\varphi \circ f$ in $D^{1,p}(X)$ so that the minimal upper gradient ϱ_{f_φ} of f_φ satisfies $\varrho_{f_\varphi} \leq \varrho$ μ -a.e. in X .

Theorem 7.2. ([27, Proposition 7.1.38]) A map $f : X \rightarrow Y$ is in $D^{1,p}(X : Y)$ if and only if there is a non-negative p -integrable Borel function ϱ on X such that whenever $\gamma : [0, \ell(\gamma)] \rightarrow X$ is a non constant rectifiable curve in X

$$d_Y(f(\gamma(\ell(\gamma))), f(\gamma(0))) \leq \int_{\gamma} \varrho \, ds,$$

where $\ell(\gamma)$ denotes the length of γ and

$$\int_{\gamma} \varrho \, ds := \int_0^{\ell(\gamma)} \varrho(\gamma(t)) \, dt.$$

Moreover, if $\mu(X)$ is finite, then $f \in N^{1,p}(X : Y)$ if and only if $f \in D^{1,p}(X : Y)$ and the function $X \ni x \mapsto d_Y(y_0, f(x))$ is p -integrable for some (and hence every) $y_0 \in Y$.

Theorem 7.2 shows that there is an intrinsic way of determining membership in $D^{1,p}(X : Y)$ and $N_{\text{loc}}^{1,p}(X : Y)$ that is independent of the embedding of Y into a Banach space. Instead, membership in $N^{1,p}(X : Y)$ (as well as in $L^p(X : Y)$) depends on the specific embedding and, in particular, on the base point y_0 .

Approach by sections of mappings: It is an extension of the one exploited in Definition 2.18, first introduced by Ambrosio [3], for the case of BV functions, and then by Reshetnyak [52], for the case of Sobolev functions. More precisely, first one provides the notion of a suitable real-valued Sobolev space $W^{1,p}(X)$ and, for each $u \in W^{1,p}(X)$ a notion of modulus of the “weak gradient” $|\nabla_X u|$ is introduced. For instance, by using the previous Newtonian Sobolev space $N^{1,p}(X)$, the minimal p -weak upper gradient ϱ_u plays the role of $|\nabla_X u|$ (see, for instance, [4,5]). In the following we assume that $W^{1,p}(X) = N^{1,p}(X)$. Then, given a mapping $f : X \rightarrow Y$, we say that $f \in W^{1,p}(X, Y)$ if

- the map $X \ni x \mapsto d_Y(y_0, f(x))$ belongs to $L^p(X, \mu)$;
- for each $y \in Y$, the section $u_y : X \rightarrow \mathbb{R}$,

$$u_y(x) := d_Y(y, f(x)) \text{ if } x \in X \text{ belongs to } D^{1,p}(X) \quad (66)$$

and there exists $g \in L^p(X, \mu)$ such that

$$\varrho_{u_y} \leq g \text{ } \mu\text{-a.e. in } X, \text{ for each } y \in Y, \quad (67)$$

denoting by ϱ_u the minimal p -weak upper gradient of $u \in D^{1,p}(X)$. A function g as in (67) will be called a p -weak upper gradient of f .

As usual, let us denote by $W_{\text{loc}}^{1,p}(X, Y)$ the class of all maps $f : X \rightarrow Y$ such that each $x \in X$ has a bounded open neighborhood U_x for which $f|_{U_x} \in W^{1,p}(U_x, Y)$.

As a consequence of Theorem 7.1, for fixed y_0 , we get the following coincidence of spaces:

Theorem 7.3. *Let $1 < p < \infty$. Then $W^{1,p}(X, Y) = N^{1,p}(X : Y)$.*

Before the proof of Theorem 7.3, we will introduce a preliminary result, which is an extension of [32, Proposition 2.31] to metric spaces.

Lemma 7.4. *Let $1 < p < \infty$, $f \in W^{1,p}(X, Y)$ and let $\varphi : Y \rightarrow \mathbb{R}$ be Lipschitz. Then*

- (i) $X \ni x \mapsto \varphi(f(x)) - \varphi(y_0)$ is in $L^p(X, \mu)$.
- (ii) If $f_\varphi := \varphi \circ f$, then $f_\varphi - \varphi(y_0) \in N^{1,p}(X)$ and f_φ admits a minimal p -weak upper gradient $\varrho_{f_\varphi} \leq L g$ μ -a.e. in X , being g the function in (67) and $L := \text{Lip}(\varphi)$.

Proof. (i) By definition of $W^{1,p}(X, Y)$, the map $X \ni x \mapsto d_Y(y_0, f(x))$ is in $L^p(X, \mu)$. Being φ Lipschitz,

$$|\varphi(f(x)) - \varphi(y_0)| \leq L d_Y(y_0, f(x)) \text{ for each } x \in X,$$

and we get the desired conclusion.

(ii) Let $Z := \{z_i : i \in \mathbb{N}\} \subset Y$ be a dense set. Let us denote

$$\tilde{f}_\varphi := f_\varphi - \varphi(y_0) : X \rightarrow \mathbb{R} \text{ and } u_i := \varphi(z_i) + L u_{z_i} : X \rightarrow \mathbb{R},$$

where u_z is the section of f defined in (66). Note that we can represent f_φ as

$$\varphi(f(x)) = \inf_{i \in \mathbb{N}} u_i(x) \text{ for each } x \in X. \quad (68)$$

Indeed, being φ Lipschitz,

$$|\varphi(f(x)) - \varphi(z_i)| \leq L d_Y(z_i, f(x)).$$

This means that

$$\varphi(f(x)) \leq \varphi(z_i) + L d_Y(z_i, f(x)) = \varphi(z_i) + L u_{z_i}(x) = u_i(x).$$

Moreover, consider a sequence $(i_k)_{k \in \mathbb{N}}$ such that $z_{i_k} \rightarrow f(x)$. We get

$$\begin{aligned} |u_{i_k}(x) - \varphi(f(x))| &= |\varphi(z_{i_k}) + L d_Y(z_{i_k}, f(x)) - \varphi(f(x))| \\ &\leq |\varphi(z_{i_k}) - \varphi(f(x))| + |L d_Y(z_{i_k}, f(x))| \leq 2L d_Y(z_{i_k}, f(x)) \rightarrow 0, \end{aligned}$$

which implies (68). Thus, from (68), it also follows that

$$\tilde{f}_\varphi(x) = \varphi(f(x)) - \varphi(y_0) = \inf_{i \in \mathbb{N}} (\tilde{u}_i(x)), \quad (69)$$

where $\tilde{u}_i := u_i - \varphi(y_0)$. Note also that $\tilde{u}_i \in D^{1,p}(X) \cap L^p_{\text{loc}}(X, \mu)$ with

$$\varrho_{\tilde{u}_i} \leq Lg \text{ } \mu\text{-a.e. in } X \text{ for each } i \in \mathbb{N} \quad (70)$$

where $\varrho_{\tilde{u}_i}$ denotes the minimal p -weak upper gradient of $\tilde{u}_i$ and g a p -weak upper gradient of f , that is a function satisfying in (67). Let $v_j := \inf_{1 \leq i \leq j} \tilde{u}_i$. From [27, Proposition 7.1.8], $D^{1,p}(X)$ is a lattice, that is $v_j \in D^{1,p}(X)$ for each j and

$$\varrho_{v_j} \leq Lg \text{ } \mu\text{-a.e. in } X, \text{ for each } j. \quad (71)$$

Now observe that, since

$$\tilde{f}_\varphi \leq v_j \leq v_1 \text{ for each } j, \quad (72)$$

from (69) and (72), by applying Lebesgue dominated convergence theorem, it follows that

$$v_j \rightarrow \tilde{f}_\varphi \text{ in } L^p_{\text{loc}}(X, \mu) \text{ as } j \rightarrow \infty. \quad (73)$$

Let $x_0 \in X$ and $\Omega_n := U(x_0, n)$, if $n \in \mathbb{N}$. Then $(\Omega_n)_n$ is an increasing sequence of open balls of X and $X = \cup_{n=1}^\infty \Omega_n$. From (73) and (71), by the locality of Sobolev functions in $N^{1,p}(X)$ (see [27, Lemma 7.3.22]), there exists

$$\hat{f} \in N^{1,p}(X) \text{ such that } \hat{f} = \tilde{f}_\varphi \text{ } \mu\text{-a.e. in } X \quad (74)$$

and $\hat{f}$ admits a p -weak upper gradient ϱ_∞ in X satisfying

$$\varrho_\infty \leq Lg \text{ } \mu\text{-a.e. in } X. \quad (75)$$

By (74) and (75), it follows that $\tilde{f}_\varphi$ admits a μ -representative $\hat{f} \in N^{1,p}(X)$ with minimal p -weak upper gradient $\varrho_{\hat{f}} \leq Lg \text{ } \mu\text{-a.e. in } X$. Thus we get the desired conclusion. $\square$

Proof of Theorem 7.3. Let us first show the inclusion

$$N^{1,p}(X : Y) \subset W^{1,p}(X, Y).$$

Let $f \in N^{1,p}(X : Y)$. From Theorem 7.2,

$$f \in N^{1,p}(X : Y) \subset D^{1,p}(X : Y) \subset D^{1,p}(X : \ell^\infty(Y)), \quad (76)$$

and the map

$$X \ni x \mapsto d_Y(y_0, f(x)) \text{ is in } L^p(X, \mu).$$

For a given $y \in Y$, let $\varphi_y : Y \rightarrow \mathbb{R}$ be the function, $\varphi_y(z) := d_Y(y, z)$. Then by applying Theorem 7.1 (ii)

$$f_y := \varphi_y \circ f \in D^{1,p}(X) \text{ and } \varrho_{f_y} \leq \varrho_f \mu\text{-a.e. in } X, \text{ for each } y \in Y. \quad (77)$$

Thus, combining (76) and (77), it follows that $f \in W^{1,p}(X, Y)$.

Let us now show the inclusion

$$W^{1,p}(X, Y) \subset N^{1,p}(X : Y).$$

Let $f \in W^{1,p}(X, Y)$. By definition, we can assume $X \ni x \mapsto d_Y(y_0, f(x))$ is in $L^p(X, \mu)$. By applying Lemma 7.4 and Theorem 7.1 (ii), we can conclude that there exists a μ -representative of f such that $f \in N^{1,p}(X : Y)$. $\square$

We are going to apply Theorem 7.3 in the setting of Carnot groups. In particular, for maps between Carnot groups, we have chosen to fix $y_0 = 0_{\mathbb{G}_2}$ (see Definition 2.18). Let us first recall the coincidence between the Newtonian Sobolev space and the horizontal Sobolev space of real valued functions, which is well-known (see, for instance, [32, Proposition C.12]).

Theorem 7.5. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{G}$ be an open set of a Carnot group $\mathbb{G} \equiv \mathbb{R}^n$. Then*

$$N_{\mathbb{G}}^{1,p}(\Omega) = W_{\mathbb{G}}^{1,p}(\Omega)$$

where $N_{\mathbb{G}}^{1,p}(\Omega)$ and $W_{\mathbb{G}}^{1,p}(\Omega)$ denote the Newtonian Sobolev space defined before between metric spaces $X = (\Omega, d_c, \mathcal{L}^n)$ and $Y = \mathbb{R}$ and the horizontal Sobolev space defined in Definition 2.12, respectively.

As an immediate consequence of Theorems 7.3 and 7.5 and Definition 2.18, the following result follows (see also [32, Proposition C.13])

Corollary 7.6. *Let $\Omega \subset \mathbb{G}$ be an open set of a Carnot group $\mathbb{G} \equiv \mathbb{R}^n$ and let $Y \equiv (Y, d_Y, y_0)$ be a pointed complete separable metric space. Then*

$$N^{1,p}(\Omega : Y) = W^{1,p}(\Omega, Y)$$

where $N^{1,p}(\Omega : Y)$ and $W^{1,p}(\Omega, Y)$ denote the Newtonian space between metric space $X = (\Omega, d_c, \mathcal{L}^n)$ and Y , and the metric space-valued Sobolev space defined by sections introduced before, respectively.

Data availability

No data was used for the research described in the article.

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