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Inversion maps and torus actions on rational homogeneous varieties

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Abstract
Complex projective algebraic varieties with $\mathbb{C}^*$-actions can be thought of as geometric counterparts of birational transformations. In this paper we describe geometrically the birational transformations associated to rational homogeneous varieties endowed with a $\mathbb{C}^*$-action with no proper isotropy subgroups.

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Contents

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1 Introduction

The transition from classic projective to modern birational geometry in the 1980s left algebraic geometry with the task of understanding brand new birational transformations such as flips and flops. An interesting idea, that sprung out of the work of Reid [14], Thaddeus [17, 18] and Włodarczyk [19], is that these transformations may be understood by means of actions of tori: given a birational map ψ between complex projective varieties, there exists an algebraic variety X (not proper, in general) admitting a $\mathbb{C}^*$-action such that ψ is the map induced among two GIT-quotients of X by $\mathbb{C}^*$.

On the other hand, if one starts with a projective variety X admitting a nontrivial $\mathbb{C}^*$-action, then one may consider the geometric quotients of the action and the induced birational map among them. Among these quotients, two distinguished ones are particularly relevant: starting with the general point $x \in X$, we call *sink* and *source* of the action the (unique) fixed-point components Y_-, Y_+ that contain, respectively, $\lim_{t\to 0} t^{-1} \cdot x$ and $\lim_{t\to 0} t \cdot x$. Considering the *Białynicki-Birula cells*,

$$X^{\pm}(Y_{\pm}) := \left\{ x \in X : \lim_{t\to 0} t^{\pm 1} \cdot x \in Y_{\pm} \right\},$$

the quotients $\mathcal{G}_{\pm} := \left(X^{\pm}(Y_{\pm}) \setminus Y_{\pm}\right)/\mathbb{C}^*$ are geometric quotients of X and, in the case in which X is smooth (see [1, 7]), $\mathcal{G}_{\pm}$ are weighted projective bundles over $Y_{\pm}$. The nonempty intersection $X^-(Y_-) \cap X^+(Y_+)$ defines a birational map $\psi_a : X^-(Y_-) \dashrightarrow X^+(Y_+)$ that descends via the quotient by $\mathbb{C}^*$ to a birational map

$$\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+,$$

that we call the *birational map induced by the* $\mathbb{C}^*$*-action*. This map encodes many geometric properties of X; for instance, it is equivariant with respect to the action of the centralizer of $\mathbb{C}^*$ in the group of automorphisms of X. When the group of automorphisms of X is large (for instance if X is rational homogeneous), the list of possible associated birational maps is expected to be small.

Note that when X is smooth and the $\mathbb{C}^*$-action has no finite stabilizers (in our language we say that the action is *equalized*), $\mathcal{G}_{\pm}$ are smooth varieties. One can describe ψ as a sequence of Atiyah flips, blowups and blowdowns (cf. [12]).

The goal of this paper is to study equalized $\mathbb{C}^*$-actions on some rational homogeneous varieties (*RH variety*, for short) of Picard number one G/P (G semisimple algebraic group, P maximal parabolic subgroup) and to describe their associated birational maps. A key property of these transformations is that they are equivariant with respect to a certain reductive subgroup $G_0 \subset G$.

Example 1.1 The simplest example is the projectivization of the inversion map of matrices

$$\psi : \mathbb{P}(M_{n\times n}) \dashrightarrow \mathbb{P}(M_{n\times n}), \qquad [A] \longmapsto [A^{-1}],$$

which is PGL(n)-equivariant. It was already noted by Thaddeus in [18, Section 4] that this birational map is induced by a certain $\mathbb{C}^*$-action on X, the Grassmannian of n-dimensional subspaces of a $2n$-dimensional complex vector space. The restriction of this action to a certain $\mathbb{C}^*$-invariant subvariety $(\mathbb{P}^1)^n \subset X$ has the standard Cremona transformation

$$\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}, \qquad (x_1 : \ldots : x_n) \longmapsto \left(x_1^{-1} : \ldots : x_n^{-1}\right),$$

as the associated birational map. This map, which can be thought of as the projectivization of the inversion of diagonal $n \times n$ matrices, is equivariant with respect to the action of an algebraic torus $(\mathbb{C}^*)^{n-1}$.

Thaddeus' observation—and its analogous formulation in the case of the inversion of symmetric matrices, cf. [9, Lemma 3.6]—can be extended to other $\mathbb{C}^*$-actions on rational homogeneous varieties. Roughly speaking, we expect the associated birational maps to behave like "inversion maps" for certain algebraic structures on $X^{\pm}(Y_{\pm})$.

This idea is particularly clear in the case in which the action is equalized and the extremal fixed-point components $Y_{\pm}$ are isolated points, i.e. in the case that the induced birational map is a Cremona transformation:

$$\psi : \mathcal{G}_- = \mathbb{P}(T_{X,Y_-}) \dashrightarrow \mathbb{P}(T_{X,Y_+}) = \mathcal{G}_+.$$

In this situation we will prove the following:

Theorem 1.2 *Consider an equalized $\mathbb{C}^*$-action on an RH variety X of Picard number one with isolated sink and source, let $\psi : \mathbb{P}(T_{X,Y_-}) \dashrightarrow \mathbb{P}(T_{X,Y_+})$ be the Cremona transformation induced by the action. Then there exists a unique structure of Jordan algebra on T_{X,Y_-} and a linear isomorphism $\alpha : T_{X,Y_+} \to T_{X,Y_-}$, such that $\jmath := \mathbb{P}(\alpha) \circ \psi$ is the projectivization of the corresponding inversion map in T_{X,Y_-}.*

The precise way in which ψ can be constructed upon the map $\jmath$ will be described in Sect. 4. Our result can be considered a reformulation of [15, 2.21] into the language of $\mathbb{C}^*$-actions. Note that in the same text the author shows how the inverse map completely determines the structure of Jordan algebra and that any simple Jordan algebra appears this way. A key point in the proof of Theorem 1.2 will be the fact that the $\mathbb{C}^*$-actions considered in the statement are completely determined by a grading on the Lie algebra $\mathfrak{g}$ of G; the hypotheses of Theorem 1.2 can be rephrased by saying that the grading is *short* (see Sect. 2.4.1) and *balanced* (see Sect. 4 for the precise definition).

Given a short and balanced grading on a simple Lie algebra $\mathfrak{g}$, one may consider the actions induced on the other RH G-varieties of Picard number one (not only those on which the sink and the source are isolated) and study the induced birational maps. As in Theorem 1.2, up to composition with a biregular map, we obtain a birational involution of the corresponding geometric quotient $\mathcal{G}_-$, which in this case is the projectivization of a homogeneous vector bundle over an RH variety. When G is of classical type, we study explicit descriptions of the birational map ψ, obtaining the following (we refer to Sect. 2.3 for the notations):

Theorem 1.3 *Let G be a simple algebraic group whose Lie algebra $\mathfrak{g}$ is of classical type and admits a short and balanced grading. Let X be an RH G-variety of Picard number one endowed with the $\mathbb{C}^*$-action determined by the grading (as in Sect. 2.4.1). Assume that the extremal fixed point components of the action are not isolated points. Then the induced birational transformation $\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+$ is a small $\mathbb{Q}$-factorial modification, uniquely determined by the geometric quotients $\mathcal{G}_{\pm}$ of X (up to automorphisms of $\mathcal{G}_{\pm}$). Their complete list is the following:*

- *$X = \mathrm{A}_{2n-1}(k)$ is the Grassmannian of k-dimensional linear subspaces in a 2n-dimensional vector space V, $k \leq n$, endowed with a $\mathbb{C}^*$-action producing a weight decomposition $V = V_- \oplus V_+$ with $\dim V_\pm = n$. The extremal fixed point components $Y_\pm$ are Grassmannians $\mathrm{A}_{n-1}(k)$ of k-dimensional linear subspaces of $V_\pm$, and $\mathcal{G}_\pm \simeq \mathbb{P}\big(\mathcal{S}_\pm^\vee \otimes V_\mp\big)$, where $\mathcal{S}_\pm$ denote the corresponding universal bundles (of rank k) on $Y_\pm$.*
- *$X = \mathrm{C}_n(k)$ (resp. $X \simeq \mathrm{D}_n(k)$ with $k \leq n-2$ or $X \simeq \mathrm{D}_n(n)$, n even) is the symplectic (resp. orthogonal) Grassmannian of isotropic subspaces in a 2n-dimensional vector space V, endowed with a $\mathbb{C}^*$-action producing a weight decomposition $V = V_- \oplus V_+$ with $\dim V_\pm = n$, $V_\pm$ isotropic. Then $Y_\pm$ are Grassmannians $\mathrm{A}_{n-1}(k)$ of k-dimensional linear subspaces of $V_\pm$ and $\mathcal{G}_\pm \simeq \mathbb{P}\big(\mathcal{N}_{Y_-|X}\big)$, where the normal bundles $\mathcal{N}_{Y_\pm|X}$ are non-trivial extensions of*

$$0 \to (\mathcal{S}_\pm \otimes \mathcal{Q}_\pm)^\vee \longrightarrow \mathcal{N}_{Y_\pm|X} \longrightarrow \mathcal{C}_\pm \to 0,$$

and, being $\mathcal{S}_\pm$, $\mathcal{Q}_\pm$ the corresponding universal bundles (of rank k and $n-k$, respectively) on $Y_\pm$,

$$\mathcal{C}_\pm := \begin{cases} S^2\mathcal{S}_\pm^\vee & \text{if } X \simeq \mathrm{C}_n(k), \\ \wedge^2\mathcal{S}_\pm^\vee & \text{if } X \simeq \mathrm{D}_n(k). \end{cases}$$

- *$X = \mathrm{D}_n(n-1)$ is the orthogonal Grassmannian of isotropic subspaces in a 2n-dimensional vector space V (n even), endowed with a $\mathbb{C}^*$-action producing a weight decomposition $V = V_- \oplus V_+$ with $\dim V_\pm = n$, $V_\pm$ isotropic. Then $Y_\pm \simeq \mathrm{A}_{n-1}(n-1)$ are projective spaces parametrizing hyperplanes in $V_\pm$, and $\mathcal{G}_\pm \simeq \mathbb{P}\big(\bigwedge^2 T_{Y_\pm}(-2)\big)$.*
- *$X = \mathrm{B}_n(k)$ (resp. $X = \mathrm{D}_n(k)$) is the orthogonal Grassmannian of isotropic subspaces of a $(2n+1)$-dimensional (resp. 2n-dimensional) vector space V, endowed with a $\mathbb{C}^*$-action producing a weight decomposition $V = V_- \oplus V_0 \oplus V_+$ (corresponding to weights $-1, 0, 1$, respectively), $\dim(V_\pm) = 1$. Then $Y_\pm \simeq \mathrm{B}_{n-1}(k-1)$ (resp. $Y_\pm \simeq \mathrm{D}_{n-1}(k-1)$ if $k \leq n-2$, $(Y_-, Y_+) \simeq (\mathrm{D}_{n-1}(n-2), \mathrm{D}_{n-1}(n-1))$ if $k = n-1$, and $(Y_-, Y_+) \simeq (\mathrm{D}_{n-1}(n-1), \mathrm{D}_{n-1}(n-2))$ if $k = n$) are orthogonal Grassmannians of isotropic subspaces in V_0, and $\mathcal{G}_\pm \simeq \mathbb{P}\big(\mathcal{Q}_\pm\big)$, where $\mathcal{Q}_\pm$ are the corresponding universal bundles on $Y_\pm$.*

Besides the cases of simple Lie algebras of classical type, only the exceptional Lie algebra $\mathfrak{e}_7$ admits a short and balanced grading, that we will later denote by σ_7. The birational maps $\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+$ induced by the corresponding $\mathbb{C}^*$-action on the RH varieties $\mathrm{E}_7(k)$ can still be described in terms of certain universal bundles. We refer to Sect. 5.2 for the precise definitions of the vector bundles involved.

Theorem 1.4 *Consider the short and balanced grading σ_7 on the Lie algebra $\mathfrak{e}_7$. Then the corresponding $\mathbb{C}^*$-action H_7 on the RH variety with Picard number one $\mathrm{E}_7(k)$ is such that $Y_- \simeq \mathrm{E}_6(k)$, $Y_+ \simeq \mathrm{E}_6(s(k))$ (where $s(k)$ is symmetric node for the non-trivial automorphism of the Dynkin diagram E_6 and $s(7) = 0$) and*

$$\psi : \mathbb{P}(\mathcal{Q}_k) \dashrightarrow \mathbb{P}\left(\mathcal{Q}'_{s(k)}\right),$$

where $\mathcal{Q}_k$, $\mathcal{Q}'_{s(k)}$ are the quotient bundles over $Y_\pm$ arising, respectively, from the short exact sequences

$$0 \to \mathcal{S}_k \longrightarrow V(\omega_6) \otimes \mathcal{O}_{Y_-} \longrightarrow \mathcal{Q}_k \to 0,$$
$$0 \to \mathcal{S}'_{s(k)} \longrightarrow V(\omega_1) \otimes \mathcal{O}_{Y_+} \longrightarrow \mathcal{Q}'_{s(k)} \to 0.$$

The contents of Theorems 1.3 and 1.4 are summarized in Table 1 below.

Table 1 Birational transformations induced by a short and balanced grading on an RH variety X of Picard number one

X	Conditions	σ	Y_-	$\mathcal{G}_-$	Y_+	$\mathcal{G}_+$
$\mathrm{A}_{2n-1}(k)$	$k \le n$	σ_n	$\mathrm{A}_{n-1}(k)$	$\mathbb{P}\left(\mathcal{S}_-^\vee \otimes V_+\right)$	$\mathrm{A}_{n-1}(k)$	$\mathbb{P}\left(\mathcal{S}_+^\vee \otimes V_-\right)$
$\mathrm{B}_n(k)$		σ_1	$\mathrm{B}_{n-1}(k-1)$	$\mathbb{P}(\mathcal{Q}_-)$	$\mathrm{B}_{n-1}(k-1)$	$\mathbb{P}(\mathcal{Q}_+)$
$\mathrm{C}_n(k)$	$k \le n-1$	σ_n	$\mathrm{A}_{n-1}(k)$	Proposition 5.5	$\mathrm{A}_{n-1}(k)$	Proposition 5.5
$\mathrm{C}_n(n)$		σ_n	pt	$\mathbb{P}\left(S^2 V_+\right)$	pt	$\mathbb{P}\left(S^2 V_-\right)$
$\mathrm{D}_n(k)$	$k \le n-2$	σ_1	$\mathrm{D}_{n-1}(k-1)$	$\mathbb{P}(\mathcal{Q}_-)$	$\mathrm{D}_{n-1}(k-1)$	$\mathbb{P}(\mathcal{Q}_+)$
		σ_n	$\mathrm{A}_{n-1}(k)$	Proposition 5.5	$\mathrm{A}_{n-1}(k)$	Proposition 5.5
$\mathrm{D}_n(n-1)$		σ_1	$\mathrm{D}_{n-1}(n-2)$	$\mathbb{P}(\mathcal{Q}_-)$	$\mathrm{D}_{n-1}(n-1)$	$\mathbb{P}(\mathcal{Q}_+)$
	n even	σ_n	$\mathrm{A}_{n-1}(n-1)$	$\mathbb{P}\left(\bigwedge^2 T_{\mathrm{A}_{n-1}(n-1)}(-2)\right)$	$\mathrm{A}_{n-1}(n-1)$	$\mathbb{P}\left(\bigwedge^2 T_{\mathrm{A}_{n-1}(n-1)}(-2)\right)$
$\mathrm{D}_n(n)$		σ_1	$\mathrm{D}_{n-1}(n-1)$	$\mathbb{P}(\mathcal{Q}_-)$	$\mathrm{D}_{n-1}(n-2)$	$\mathbb{P}(\mathcal{Q}_+)$
	n even	σ_n	pt	$\mathbb{P}\left(\bigwedge^2 V_+\right)$	pt	$\mathbb{P}\left(\bigwedge^2 V_-\right)$
$\mathrm{E}_7(k)$		σ_7	$\mathrm{E}_6(k)$	$\mathbb{P}(\mathcal{Q}_k)$	$\mathrm{E}_6(s(k))$	$\mathbb{P}\left(\mathcal{Q}'_{s(k)}\right)$

Outline

The structure of the paper is the following. We start with a section on background material on $\mathbb{C}^*$-actions on RH varieties (Sect. 2). Then in Sect. 3 we study the fixed-point components of an equalized $\mathbb{C}^*$-action on an RH variety of classical type via projective geometry. In Sect. 4 we study Cremona transformations induced by $\mathbb{C}^*$-actions on RH varieties, proving Theorem 1.2. Finally, Sect. 5 is devoted to the proof of Theorems 1.3 and 1.4.

2 Preliminaries

Throughout this paper, unless otherwise stated, all the varieties will be projective, smooth and defined over the field of complex numbers. Given a vector bundle $\mathcal{E}$ over such a variety, we denote by $\mathbb{P}(\mathcal{E})$ its homothetical projectivization, that is

$$\mathbb{P}(\mathcal{E}) := \operatorname{Proj}\left(\bigoplus_{m\geq 0} S^m\mathcal{E}^\vee\right).$$

2.1 Notation and basic facts on $\mathbb{C}^*$-actions

Let X be a smooth projective variety endowed with a nontrivial $\mathbb{C}^*$-action. We will follow the notations and conventions in [11].

- We will denote by $X^{\mathbb{C}^*}$ the set of fixed-points of the action, and $\mathcal{Y}$ the set of irreducible components of $X^{\mathbb{C}^*}$.
- Given a point $x \in X$, we denote by $x_\pm := \lim_{t\to 0} t^{\pm 1}\cdot x \in X^{\mathbb{C}^*}$ the *sink* x_- and the *source* x_+ of the orbit $\mathbb{C}^*\cdot x$.
- The only components $Y_-, Y_+ \in \mathcal{Y}$ such that, for a general $x \in X$, $\lim_{t\to 0} t^{\pm 1}\cdot x \in Y_\pm$ are called *sink* and *source* of the action, respectively.
- Given $Y \in \mathcal{Y}$ – which is a smooth subvariety of X – the normal bundle of Y in X will be denoted $\mathcal{N}_{Y|X}$. It decomposes as a direct sum of two subbundles

  $$\mathcal{N}_{Y|X} = \mathcal{N}^-(Y)\oplus\mathcal{N}^+(Y),$$

  on which $\mathbb{C}^*$ acts with negative and positive weights, respectively; their ranks are denoted by $\nu^\pm(Y)$. The two summands $\mathcal{N}^\pm(Y)$ are $\mathbb{C}^*$-equivariantly isomorphic to subsets of X, the so-called *Białynicki-Birula cells* $X^\pm(Y)\subset X$, defined as

  $$X^\pm(Y) := \left\{x\in X : \lim_{t\to 0} t^{\pm 1}\cdot x \in Y\right\}.$$

  Note that $X^\pm(Y_\pm)$ are open subsets in X.
- The action is *equalized* if the weights of the action of $\mathbb{C}^*$ on $\mathcal{N}_{Y|X}$ are equal to ± 1 for every $Y\in\mathcal{Y}$; equivalently, the action has no nontrivial isotropy subgroups.
- We get a birational map $\psi_a : \mathcal{N}^-(Y_-) \dashrightarrow \mathcal{N}^+(Y_+)$ defined as the composition

  $$\mathcal{N}^-(Y_-)\simeq X^-(Y_-) \hookleftarrow X^-(Y_-)\cap X^+(Y_+) \hookrightarrow X^+(Y_+)\simeq\mathcal{N}^+(Y_+).$$

  If the $\mathbb{C}^*$-action is equalized, the map descends via the quotient by homotheties to a birational map

  $$\psi : \mathbb{P}\left(\mathcal{N}_{Y_-|X}\right) \dashrightarrow \mathbb{P}\left(\mathcal{N}_{Y_+|X}\right). \tag{1}$$

Note that this map sends a general point of $\mathbb{P}\left(\mathcal{N}_{Y_-|X}\right)$ corresponding to the tangent direction at $x_- \in Y_-$ of the closure of an orbit $\mathbb{C}^* \cdot x$ ($x \in X$ general) to its tangent direction at $x_+ \in Y_+$. The maps ψ_a and (in the equalized case) ψ are called the *birational maps associated with the* $\mathbb{C}^*$-action on X.

- Given an ample line bundle L on X, a linearization of the $\mathbb{C}^*$-action on L exists. Then $\mathbb{C}^*$ acts on the fibers of $L|_Y$ by multiplication with a character, which we denote by $\mu_L(Y)$. Up to multiplication with a character, we may assume that $\mu_L(Y_-) = 0$; then it follows that $\mu_L(Y) > 0$ for every $Y \in \mathcal{Y} \setminus \{Y_-\}$ and that the maximum value δ of μ_L is achieved at the source Y_+. We set:

$$Y_r := \bigsqcup_{\mu_L(Y)=r} Y.$$

Denoting $V := H^0(X, L)^\vee$, the $\mathbb{C}^*$-action induces a weight decomposition

$$V = \bigoplus_{r \in \mathbb{Z}} V_r.$$

If L is very ample, we may write $Y_r = \mathbb{P}(V_r) \cap X$ for every r.

- Note that $\mu_{rL} = r\mu_L$. We will be interested in the case in which the Picard number of X is one and we will assort the fixed-point components of the $\mathbb{C}^*$-action by their weights with respect to the ample generator of the Picard group of the variety.
- Given the closure C of an orbit $\mathbb{C}^* \cdot x$ satisfying that $x_- \in Y$, $x_+ \in Y'$, it follows that

$$C \cdot L = (\mu_L(Y') - \mu_L(Y))t,$$

where t is the weight of $\mathbb{C}^*$ on T_{C,x_+}; if the action is equalized, then $t = 1$ for every x.

Remark 2.1 In the case in which the $\mathbb{C}^*$-action is equalized and the Picard number of X is equal to one, which will be our main interest, the map ψ may be decomposed as a sequence of Atiyah flips and, in certain situations, a blowup and a blowdown (cf. [12]).

2.2 Preliminaries on adjoint groups of simple Lie algebras

The main characters in this paper will be the equalized $\mathbb{C}^*$-actions on rational homogeneous varieties (*RH varieties*, for short) of Picard number one. Here, we will introduce some notations we will use regarding these varieties and the corresponding $\mathbb{C}^*$-actions.

Throughout this paper, G will denote the adjoint group of a simple Lie algebra $\mathfrak{g}$. We consider a Cartan subgroup $H \subset G$, a Borel subgroup $H \subset B \subset G$ and their corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We denote by Φ the root system of G with respect to H, by $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi^+ \subset \Phi$ a set of positive simple roots of $\mathfrak{g}$ (determined by the choice of B), by $\omega_1, \ldots, \omega_n$ the corresponding fundamental weights, by $W = \mathrm{N}_G(H)/H$ the Weyl group of G, and by $s_1, \ldots, s_n \in W$ the reflections associated to these positive simple roots. We will denote by $\mathcal{D}$ the Dynkin diagram of $\mathfrak{g}$, whose nodes are numbered by $D = \{1, \ldots, n\}$ (we follow the numbering convention of [3, Planche I-IX]), in one-to-one correspondence with the set of positive simple roots Δ. We will denote by $w_0 \in \mathrm{N}_G(H) \subset G$ an element in the class of the longest element of the Weyl group $W = \mathrm{N}_G(H)/H$ (with respect to the choice of $\Delta \in \Phi$).

Each G-homogeneous variety of Picard number one is completely determined by the choice of a node k in $\mathcal{D}$, since this determines uniquely a fundamental weight ω_k of the simple Lie algebra $\mathfrak{g}$ associated with $\mathcal{D}$. The closed G-orbit in the projectivization $\mathbb{P}(V(\omega_k))$ of the irreducible representation $V(\omega_k)$ associated with ω_k is an RH G-variety of Picard number one, that we denote by $\mathcal{D}(k)$. Alternatively, given the node k of the Dynkin diagram $\mathcal{D}$ and the subgroup $W_k \subset W$ generated by the reflections s_i, $i \neq k$, the subgroup $P := BW_kB \subset G$ is a parabolic subgroup containing B, and

$$G/P \simeq \mathcal{D}(k).$$

We stress out that, the subgroup P is completely determined by its Lie algebra $\mathfrak{p} \subset \mathfrak{g}$, whose Cartan decomposition is:

$$\mathfrak{p} = \underbrace{\bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta \oplus \mathfrak{h}}_{\mathfrak{b}} \oplus \bigoplus_{\substack{\beta \in \Phi^+: \\ \sigma_k(\beta)=0}} \mathfrak{g}_{-\beta}. \tag{2}$$

Here $\sigma_k : \mathrm{M}(H) \to \mathbb{Z}$ denotes the *height map* determined by sending α_j to 1 if $j = k$, and to 0 otherwise, where $\mathrm{M}(H)$ is the group of characters of H, cf. [16].

Notation 2.2 Consider the Dynkin diagram $\mathcal{D}$, whose nodes are numbered by elements of $D = \{1, \ldots, n\}$. We define the RH varieties $\mathcal{D}(0)$ and $\mathcal{D}(n+1)$ to be isolated points.

2.3 Preliminaries on RH varieties of classical type

2.3.1 Projective description

For the reader's convenience, we include here the standard projective descriptions of the RH varieties of Picard number one of classical type. Moreover, we introduce some notations about them that will be useful later on.

- For every $k \in \{1, \ldots, n\}$ the variety $\mathrm{A}_n(k)$ is the *Grassmannian* of k-dimensional subspaces of the $(n+1)$-dimensional vector space V, which is the standard representation of the Lie algebra of type A_n.
- For $k \in \{1, \ldots, n\}$ the variety $\mathrm{B}_n(k)$ is the *orthogonal Grassmannian* parametrizing k-dimensional vector subspaces of a $(2n+1)$-dimensional vector space V that are isotropic with respect to a fixed bilinear symmetric form of maximal rank. In particular, $\mathrm{B}_n(1)$ is a $(2n-1)$-dimensional smooth quadric.
- For $k \in \{1, \ldots, n\}$, $\mathrm{C}_n(k)$ is the *symplectic Grassmannian*, which parametrizes k-dimensional vector subspaces of a $2n$-dimensional vector space V that are isotropic with respect to a fixed bilinear skew-symmetric form of maximal rank. In particular, $\mathrm{C}_n(1) \simeq \mathbb{P}(V)$.
- For the description of the varieties $\mathrm{D}_n(k)$, called *orthogonal Grassmannians* as well, we consider a $2n$-dimensional vector space V equipped with a bilinear symmetric form of maximal rank. Then, for $k \leq n-2$, $\mathrm{D}_n(k)$ parametrizes k-dimensional vector subspaces of V that are isotropic with respect to such a symmetric form. In particular $\mathrm{D}_n(1)$ is a $(2n-2)$-dimensional smooth quadric. The varieties $\mathrm{D}_n(n-1)$, $\mathrm{D}_n(n)$ parametrize the two families of n-dimensional isotropic subspaces of V. Any $(n-1)$-dimensional isotropic subspace of V is the intersection of precisely two isotropic subspaces of dimension n, one in each family $\mathrm{D}_n(n-1)$, $\mathrm{D}_n(n)$. The orthogonal Grassmannian of $(n-1)$-dimensional isotropic subspaces of V, denoted by $\mathrm{D}_n(n-1, n)$ has Picard number two (and so it will not be considered in this paper), with two contractions:

$$\begin{array}{ccc} & \mathrm{D}_n(n-1,n) & \\ {\scriptstyle \mathbb{P}^{n-1}} \swarrow & & \searrow {\scriptstyle \mathbb{P}^{n-1}} \\ \mathrm{D}_n(n-1) & & \mathrm{D}_n(n). \end{array}$$

2.3.2 Universal bundles

Grassmannians (standard, orthogonal and symplectic) come equipped with some universal bundles that we will now describe.

- The Grassmannian $\mathrm{A}_n(k)$ supports two universal vector bundles $\mathcal{S}, \mathcal{Q}$ of rank k and $n+1-k$, respectively. The former is the subbundle of the trivial bundle, whose fiber over an element $[W] \in \mathrm{A}_n(k)$ is the corresponding subspace $W \subset V$. Then $\mathcal{Q}$ is defined as the cokernel of the inclusion $\mathcal{S} \hookrightarrow V \otimes \mathcal{O}_{\mathrm{A}_n(k)}$, so that we have the short exact sequence:

$$0 \to \mathcal{S} \longrightarrow V \otimes \mathcal{O}_{\mathrm{A}_n(k)} \longrightarrow \mathcal{Q} \to 0.$$

  It is then well known that we have a decomposition of the tangent bundle of $\mathrm{A}_n(k)$ as

$$T_{\mathrm{A}_n(k)} \simeq \mathcal{S}^\vee \otimes \mathcal{Q}.$$

- In the case of *isotropic Grassmannians* (i.e. orthogonal or symplectic), we may consider the pullback of the two universal bundles on standard Grassmannians via the natural inclusions $\mathrm{B}_n(k) \subset \mathrm{A}_{2n+1}(k)$, $\mathrm{C}_n(k), \mathrm{D}_n(k) \subset \mathrm{A}_{2n}(k)$ (for $k \leq n-2$ in the D_n-case), $\mathrm{D}_n(n-1), \mathrm{D}_n(n) \subset \mathrm{A}_{2n}(n)$. In order to consider all these cases together, we start with a finite-dimensional vector space V endowed with a nondegenerate symmetric or skew-symmetric isomorphism $q : V \to V^\vee$. For every $k < \dim V/2$, consider the variety $X \subset \mathrm{A}_{\dim V-1}(k)$ parametrizing isotropic k-dimensional subspaces of V and the restrictions of the universal bundle – that we still denote by $\mathcal{S}$ and $\mathcal{Q}$ – on this isotropic Grassmannian. The fact that the subspaces parametrized by X are q-isotropic implies that the composition

$$\mathcal{S} \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{q} V^\vee \otimes \mathcal{O}_X \longrightarrow \mathcal{S}^\vee$$

  is zero, and so we have an injection $\mathcal{S} \to \mathcal{Q}^\vee$, whose cokernel we denote by $\mathcal{K}$. The isomorphism q induces a (symmetric or skew-symmetric, respectively) isomorphism $\mathcal{K} \simeq \mathcal{K}^\vee$ and so, summing up, we have a commutative diagram with short exact rows and columns:

$$\begin{array}{ccccc} \mathcal{S} & \longrightarrow & \mathcal{Q}^\vee & \longrightarrow & \mathcal{K} \simeq \mathcal{K}^\vee \\ \| & & \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & V \otimes \mathcal{O} \overset{q}{\simeq} V^\vee \otimes \mathcal{O} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{S}^\vee & = & \mathcal{S}^\vee \end{array} \tag{3}$$

  Furthermore, it is known for example by [8, Proposition 5.1 and 5.4] that the tangent bundle of X fits into a short exact sequence:

$$0 \to \mathcal{S}^\vee \otimes \mathcal{K} \longrightarrow T_X \longrightarrow \mathcal{C} \to 0 \tag{4}$$

where

$$\mathcal{C} = \begin{cases} \bigwedge^2 \mathcal{S}^\vee & \text{if } \mathcal{D} = \mathrm{B}_{(\dim V-1)/2}, \mathrm{D}_{\dim V/2}, \\ S^2\mathcal{S}^\vee & \text{if } \mathcal{D} = \mathrm{C}_{\dim V/2}. \end{cases}$$

Note that in the case where X parametrize maximal isotropic subspaces, i.e. $X = \mathrm{C}_n(n), \mathrm{D}_n(n-1), \mathrm{D}_n(n)$, the bundle $\mathcal{K}$ is equal to zero, so with the above notation we will have $T_X \simeq \mathcal{C}$ (cf. [8, Section 3.1]).

2.4 Equalized $\mathbb{C}^*$-actions on RH varieties

Let us now describe briefly equalized $\mathbb{C}^*$-actions on rational homogeneous varieties of Picard number one as above. We will use the notation introduced in Sect. 2.1.

2.4.1 Short gradings

Following [5, 10], equalized $\mathbb{C}^*$-actions on RH varieties are given by the choice of a *short grading* on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

Up to the adjoint action of an element of W, we may assume that $\mathfrak{b} \subset \mathfrak{g}_0 \oplus \mathfrak{g}_+$, and the grading will then be completely determined by the choice of an index $i \in D = \{1, \dots, n\}$ such that the corresponding height map $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$ satisfies that $\sigma_i(\Phi) = \{-1, 0, 1\}$. We will denote

$$\mathfrak{p}_\pm := \mathfrak{g}_0 \oplus \mathfrak{g}_\pm,$$

which are parabolic subalgebras of $\mathfrak{g}$, corresponding to opposite parabolic subgroups $P_\pm \subset G$, cf. [2, Section 4.8].

Remark 2.3 We have $B \subset P_+ = BW_iB$, so that, with the marked Dynkin diagram notation, $G/P_+ \simeq \mathcal{D}(i)$ and $P_\pm$ have a common Levi part G_0 with Lie algebra $\mathfrak{g}_0$.

The reductive subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ decomposes as $\mathfrak{g}^\perp \oplus \mathfrak{h}_i$, where $\mathfrak{h}_i$ is a 1-dimensional vector subspace of the Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{g}^\perp$ is semisimple. The subalgebras $\mathfrak{g}^\perp, \mathfrak{g}_0, \subset \mathfrak{g}$ and $\mathfrak{h}_i \subset \mathfrak{h}$ correspond to subgroups $G^\perp \subset G_0 \subset G$ and $H_i \subset H$. The Dynkin diagram $\mathcal{D}^\perp$ associated with $G^\perp$ can be obtained by deleting the node i from the Dynkin diagram $\mathcal{D}$.

Moreover, H_i can be described as the image of the induced injective map $\sigma_i^* : \mathbb{C}^* \to H$. In the sequel we will directly identify H_i with $\mathbb{C}^*$ via σ_i^*.

Then the following statement follows from [5]:

Proposition 2.4 *With the above notation, the H_i-action on each rational homogeneous G-variety $X = G/P$ is equalized, if and only if the grading induced by σ_i on $\mathfrak{g}$ is short.*

Remark 2.5 The inclusion $\imath : G^\perp \hookrightarrow G$, together with the choice of the Cartan subgroup $H^\perp := G^\perp \cap H$, define a group homomorphism $\imath^* : \mathrm{M}(H) \to \mathrm{M}(H^\perp)$ sending $\{\alpha_j : j \neq i\}$ to the set of positive simple roots of $G^\perp$ corresponding to the choice of the Borel subgroup $B^\perp := G^\perp \cap B$ and sending the fundamental weight ω_i to zero. We denote by $\alpha_j \in \mathrm{M}(H^\perp)$ the image of $\alpha_j \in \Delta$ via $\imath^*$ for every $j \neq i$.

Table 2 Short gradings of simple Lie algebras

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7
σ_i	σ_i for $i = 1, \ldots, n$	σ_1	σ_n	$\sigma_1, \sigma_{n-1}, \sigma_n$	σ_1, σ_6	σ_7

The following table describes the list of possible height maps that induce a short grading on $\mathfrak{g}$:

For the RH varieties of classical type, equalized actions have explicit descriptions, that we will present in detail in Sect. 3.

2.4.2 Fixed-point components

We will now describe the fixed-point component of an equalized H_i-action as in Sect. 2.4.1 on an RH variety $G/P \simeq \mathcal{D}(k)$ as in Sect. 2.2, with $i, k \in D$. For every $w \in W$, we will denote

$$P_w^\perp := G^\perp \cap \operatorname{conj}_w(P)$$

which is a parabolic subgroup of $G^\perp$. In particular, $P^\perp := P_e^\perp$, where $e \in W$ is the neutral element. Such a parabolic subgroup is completely determined by its Lie algebra:

$$\mathfrak{p}_w^\perp = \left(\bigoplus_{\substack{\beta\in\Phi^+: \\ \sigma_i(w(\beta))=0}} \mathfrak{g}_{w(\beta)} \right) \oplus \left(\operatorname{Ad}_w(\mathfrak{h}) \cap \mathfrak{h}^\perp \right) \oplus \left(\bigoplus_{\substack{\beta\in\Phi^+: \\ \sigma_k(\beta)=0 \\ \sigma_i(w(\beta))=0}} \mathfrak{g}_{-w(\beta)} \right) \subset \mathfrak{g}^\perp. \tag{5}$$

Following [10, Corollary 3.10], the fixed-point components of the H_i-action are $G^\perp$-homogeneous varieties. Since each fixed-point component $Y \subset G/P$ must contain a fixed-point for the H-action (which are the points of the form wP for $w \in W$, see [4, Section 3.4]), Y can then be written as:

$$Y = G^\perp / P_w^\perp.$$

As a rational homogeneous $G^\perp$-variety, $Y \simeq \mathcal{D}^\perp(J)$, for some $J \subset D \setminus \{i\}$.

The Picard group of $G/P \simeq \mathcal{D}(k)$ is generated by the homogeneous line bundle L determined by the fundamental weight ω_k. Then, following [10, Corollary 3.8], the $\mathbb{C}^*$-action admits a linearization on L such that the L-weight of the fixed-point component passing by wP is

$$\mu_L(G^\perp / P_w^\perp) = \sigma_i(\omega_k - w(\omega_k)).$$

The minimum and maximum value of μ_L are achieved at $w = e$ and $w = w_0$. So we may conclude that the sink and the source of the H_i-action are, respectively:

$$Y_- := G^\perp / P^\perp, \qquad Y_+ := G^\perp / P_{w_0}^\perp. \tag{6}$$

Remark 2.6 In particular, with the marked Dynkin diagram notation, if $i \neq k$,

$$Y_- \simeq \mathcal{D}^\perp(k), \qquad Y_+ \simeq \mathcal{D}^\perp(w_0(k)),$$

where $w_0(k)$ denotes the node of the Dynkin diagram $\mathcal{D}^\perp \subset \mathcal{D}$ corresponding to the positive simple root $-w_0(\alpha_k) \in \Delta$.

2.4.3 The normal bundle of the extremal fixed-point components

We end up this section by describing the normal bundles of the sink and source $Y_\pm$ of the H_i-action into the rational homogeneous variety $X = G/P \simeq \mathcal{D}(k)$, which, by the Białynicki-Birula theorem, are isomorphic to two $\mathbb{C}^*$-invariant open sets of X.

Lemma 2.7 *With the above notation, the normal bundles of the sink and the source $Y_\pm \subset X$ of the H_i-action on X are the homogeneous $G^\perp$-bundles*

$$\mathcal{N}_{Y_-|X} \simeq G^\perp \times^{P^\perp} N_-, \qquad \mathcal{N}_{Y_+|X} \simeq G^\perp \times^{P^\perp_{w_0}} N_+.$$

where N_-, N_+ are, respectively, the $P^\perp$, $P^\perp_{w_0}$-submodules of $\mathfrak{g}$ defined by:

$$N_- := \bigoplus_{\substack{\beta\in\Phi^+\\ \sigma_i(\beta)>0\\ \sigma_k(\beta)>0}} \mathfrak{g}_{-\beta} \subset \mathfrak{g}_-, \qquad N_+ := \bigoplus_{\substack{\beta\in\Phi^+\\ \sigma_i(\beta)>0\\ \sigma_k(w_0(\beta))<0}} \mathfrak{g}_{\beta} \subset \mathfrak{g}_+. \tag{7}$$

Moreover the $H^\perp$-weight of every subspace $\mathfrak{g}_\beta$ is equal to $\iota^*\beta$.

Proof We will do the proof in the case of Y_-, being Y_+ analogous. Note that the action of G_0 on Y_- lifts to $\mathcal{N}_{Y_-|X}$, so we may write

$$\mathcal{N}_{Y_-|X} = G_0 \times^{P_0} N_-.$$

where $N_- = \mathcal{N}_{Y_-|X,eP}$ and $P_0 := G_0 \cap P$. In order to compute N_- as a P_0-module, we start from the Cartan decomposition of $\mathfrak{g}$ and note that:

$$\mathfrak{g} = \left(\bigoplus_{\beta\in\Phi^+} \mathfrak{g}_\beta\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\beta\in\Phi^+} \mathfrak{g}_{-\beta}\right),$$

$$\mathfrak{p} = \left(\bigoplus_{\beta\in\Phi^+} \mathfrak{g}_\beta\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_k(\beta)=0}} \mathfrak{g}_{-\beta}\right),$$

$$\mathfrak{g}_0 = \left(\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_i(\beta)=0}} \mathfrak{g}_\beta\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_i(\beta)=0}} \mathfrak{g}_{-\beta}\right),$$

$$\mathfrak{g}_0 \cap \mathfrak{p} = \left(\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_i(\beta)=0}} \mathfrak{g}_\beta\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_k(\beta)=0\\ \sigma_i(\beta)=0}} \mathfrak{g}_{-\beta}\right).$$

Then we can conclude that N_- is isomorphic as a P_0-module to the quotient $(\mathfrak{g}/\mathfrak{p})/(\mathfrak{g}_0/\mathfrak{g}_0\cap\mathfrak{p})$, hence to

$$\bigoplus_{\substack{\beta\in\Phi^+:\\ \sigma_i(\beta)>0\\ \sigma_k(\beta)>0}} \mathfrak{g}_{-\beta}.$$

To reduce the action on N_- from P_0 to $P^\perp$ and compute the $H^\perp$-weights, we apply the group homomorphism $\iota^* : \mathrm{M}(H) \to \mathrm{M}(H^\perp)$, see Remark 2.5. □

3 Equalized actions on Grassmannians

For every case of Table 2 of classical type, we describe in this section the fixed-point component of the $\mathbb{C}^*$-action H_i induced by the height map σ_i on each RH G-variety X of Picard number one. We denote by L the generator of the Picard group. We proceed as in [5, Section 4] (see also [11, Example 7.12]): we first describe the H_i-action on the standard representation V and on its closed G-orbit in $\mathbb{P}(V)$, then we study the induced H_i-action on the other G-homogeneous varieties, see Sect. 2.3.1 for the notation.

3.1 A_n-varieties

We consider the H_i-action on $\mathrm{A}_n(k)$, the Grassmannian of k-dimensional subspaces of the $(n+1)$-dimensional vector space V, corresponding to the standard representation $V(\omega_1)$. For simplicity we will assume that $k \leq n-k+1$. The H_i-action on $\mathrm{A}_n(1) \simeq \mathbb{P}(V)$ is induced by the linear action on V given by

$$t \cdot (x_0, \dots, x_n) = (tx_0, \dots, tx_{i-1}, x_i, \dots, x_n),$$

on a certain set of coordinates. In particular, we get a decomposition $V = V_- \oplus V_+$, where

$$V_- = \{x_i = \ldots = x_n = 0\}, \qquad V_+ = \{x_0 = \ldots = x_{i-1} = 0\}.$$

Then the H_i-action on $\mathrm{A}_n(k)$ is obtained as the induced action on the Grassmannians of k-subspaces of V. In every case, the only fixed-point component of weight s, for every s, is:

$$\mathrm{A}_{i-1}(k-s) \times \mathrm{A}_{n-i}(s),$$

respectively, which is the product of two Grassmannians of vector subspaces of V_- and V_+. By Notation 2.2, we recall that the factor $\mathrm{A}_{i-1}(k-s)$ (resp. $\mathrm{A}_{n-i}(s)$) is a point if $s = k, k-i$ (resp. $s = 0, n-i+1$). Hence we have three possible cases according to the values of i, k and n:

	s	Y_-	Y_s	Y_+
$i \leq k$	$[k-i, k]$	$\mathrm{A}_{n-i}(k-i)$	$\mathrm{A}_{i-1}(k-s) \times \mathrm{A}_{n-i}(s)$	$\mathrm{A}_{n-i}(k)$
$i \in [k, n-k+1]$	$[0, k]$	$\mathrm{A}_{i-1}(k)$	$\mathrm{A}_{i-1}(k-s) \times \mathrm{A}_{n-i}(s)$	$\mathrm{A}_{n-i}(k)$
$i \geq n-k+1$	$[0, n-i+1]$	$\mathrm{A}_{i-1}(k)$	$\mathrm{A}_{i-1}(k-s) \times \mathrm{A}_{n-i}(s)$	$\mathrm{A}_{i-1}(i-n+k-1)$

Note that Y_- is given by the point $[V_-]$ if and only if $i = k$. On the other hand Y_+ is the point $[V_+]$ if and only if $i = n-k+1$. In the rest of the cases, the extremal fixed-point components of the action are positive dimensional Grassmannians.

Remark 3.1 Composing the H_i-action with the map $t \mapsto t^{-1}$ and with the permutation $(x_0, \dots, x_n) \mapsto (x_n, \dots, x_0)$, we obtain the H_{n-i}-action. In other words, we may assume without loss of generality that $i \leq n-i+1$, and in particular we may discard the third case in the above table.

3.2 B_n-varieties

We consider the H_1-action on $B_n(k)$, the orthogonal Grassmannian of k-dimensional subspaces of the $(2n+1)$-dimensional standard representation V, isotropic with respect to a given nondegenerate symmetric form. We may choose coordinates $(x_0, \ldots, x_{2n})$ (with respect to a certain basis $\{e_0, \ldots, e_{2n}\}$) on V so that this form is represented by a symmetric (block) matrix:

$$\left(\begin{array}{c|c|c} 0 & I_n & 0 \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

where I_n denote the identity $n \times n$ matrix. In particular, $B_n(1) \subset \mathbb{P}(V)$ is the quadric hypersurface Q^{2n-1} given by the equation

$$x_0 x_n + \ldots + x_{n-1} x_{2n-1} + x_{2n}^2 = 0.$$

In this case, the H_1-action on every $B_n(k)$ is induced by the linear action on V:

$$t \cdot (x_0, \ldots, x_{2n}) = (t x_0, x_1, \ldots, x_{n-1}, t^{-1} x_n, x_{n+1}, \ldots, x_{2n-1}, x_{2n}), \tag{8}$$

giving a decomposition $V = V_- \oplus V_0 \oplus V_+$, where we have

$$\begin{aligned} V_- &= \{x_1 = \ldots = x_{2n} = 0\}, \\ V_0 &= \{x_0 = x_n = 0\}, \\ V_+ &= \{x_0 = \ldots = x_{n-1} = x_{n+1} = \ldots = x_{2n} = 0\}. \end{aligned}$$

The vector subspace $V_0 \subset V$ inherits a nondegenerate symmetric bilinear form, whose associated Lie algebra (which is of type B_{n-1}) is precisely $\mathfrak{g}^{\perp}$. We have two cases:

	$[\mu_L(Y_-), \mu_L(Y_+)]$	Y_-	$Y_{(\mu_L(Y_+)+\mu_L(Y_-))/2}$	Y_+
$k < n$	$[0, 2]$	$B_{n-1}(k-1)$	$B_{n-1}(k)$	$B_{n-1}(k-1)$
$k = n$	$[0, 1]$	$B_{n-1}(n-1)$	$\emptyset$	$B_{n-1}(n-1)$

The sink and the source of the action parametrize isotropic subspaces of V containing, respectively, V_- and V_+; every such subspace intersects V_0 in a $(k-1)$-dimensional isotropic subspace of V_0. When $k < n$ we also have an inner fixed-point component, parametrizing k-dimensional isotropic subspaces of V_0.

We note here that the value of $\mu_L(Y_+)$ changes because the ample generator L of $\mathrm{Pic}(B_n(k))$ is the Plücker line bundle for $k < n$, and its square root for $k = n$.

3.3 C_n-varieties

We consider the H_n-action on $C_n(k)$, the symplectic Grassmannian of k-dimensional subspaces of the $2n$-dimensional standard representation V, isotropic with respect to a given nondegenerate skew-symmetric form; we may assume this form to be given, in a certain set of coordinates, by the matrix

$$\left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

The H_n-action on every $\mathrm{C}_n(k)$ is induced from the linear action on V given by:

$$t \cdot (x_0, \dots, x_{2n-1}) = (tx_0, \dots, tx_{n-1}, x_n, \dots, x_{2n-1}). \tag{9}$$

In other words, it is the restriction to $\mathrm{C}_n(k)$ of the H_n-action on the Grassmannian $\mathrm{A}_{2n-1}(k)$. The fixed-point components of $\mathrm{C}_n(k)$ can be computed by intersecting the fixed-point components of $\mathrm{A}_{2n-1}(k)$ with this variety. Note also that the skew-symmetric form induces an isomorphism $q : V_- \to V_+^\vee$, that extends to isomorphisms between Grassmannians of subspaces of V_- and subspaces of V_+ of complementary dimension. With this information at hand one may compute fixed-point components of the H_n-action on $\mathrm{C}_n(k)$:

	$[\mu_L(Y_-), \mu_L(Y_+)]$	Y_-	Y_s	Y_+
$k < n$	$[0, k]$	$\mathrm{A}_{n-1}(k)$	$\mathrm{A}_{n-1}(k-s, n-s)$	$\mathrm{A}_{n-1}(n-k)$
$k = n$	$[0, n]$	pt	$\mathrm{A}_{n-1}(n-s)$	pt

We note that, in the case $k = n$, every inner fixed-point component $Y_s \simeq \mathrm{A}_{n-1}(n-s)$ is the image of the map

$$\mathrm{A}_{n-1}(n-s) \longrightarrow \mathrm{A}_{n-1}(n-s) \times \mathrm{A}_{n-1}(s) \subset \mathrm{A}_{2n-1}(n), \qquad W \longmapsto (W, \ \ker q(W)),$$

which in particular tells us that the restriction to $Y_s \simeq \mathrm{A}_{n-1}(n-s)$ of the Plücker embedding of $\mathrm{C}_n(n)$ is not the minimal embedding but its second symmetric power.

3.4 D_n-varieties

We consider now equalized actions on the orthogonal Grassmannian of k-dimensional subspaces of the $2n$-dimensional standard representation V, isotropic with respect to a given nondegenerate symmetric form. We choose coordinates $(x_0, \dots, x_{2n-1})$ in V (with respect to a basis $\{e_0, \dots, e_{2n-1}\}$) so that this form is given by the matrix:

$$\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right).$$

We have three equalized $\mathbb{C}^*$-actions, coming from the action on V that take the form:

$$\begin{aligned} H_1 : &\qquad t \cdot (x_0, \dots, x_{2n-1}) = (tx_0, x_1, \dots, x_{n-1}, t^{-1}x_n, x_{n+1}, \dots, x_{2n-1}), \\ H_{n-1} : &\qquad t \cdot (x_0, \dots, x_{2n-1}) = (tx_0, \dots, tx_{n-2}, x_{n-1}, x_n, \dots, x_{2n-2}, tx_{2n-1}), \\ H_n : &\qquad t \cdot (x_0, \dots, x_{2n-1}) = (tx_0, \dots, tx_{n-2}, tx_{n-1}, x_n, \dots, x_{2n-2}, x_{2n-1}). \end{aligned}$$

Remark 3.2 Up to reordering the coordinates x_{n-1}, x_{2n-1} we have only two types of equalized $\mathbb{C}^*$-actions, H_1 and H_n.

3.4.1 H_1-action

The case of the H_1-action on the varieties $\mathrm{D}_n(k)$ is analogous to the case B_n, Sect. 3.2, and the fixed-point components are Grassmannians of isotropic subspaces in V_0, with respect to the restriction of the symmetric form in V. We get here four cases:

Note that, as in the B_n-case, the value of $\mu_L(Y_+)$ changes because the ample generator L of $\mathrm{Pic}(\mathrm{D}_n(k))$ is the Plücker line bundle for $k < n-1$, and its square root for $k = n-1$ or $k = n$.

	$[\mu_L(Y_-), \mu_L(Y_+)]$	Y_-	$Y_{(\mu_L(Y_+)+\mu_L(Y_-))/2}$	Y_+
$k < n-2$	$[0, 2]$	$\mathrm{D}_{n-1}(k-1)$	$\mathrm{D}_{n-1}(k)$	$\mathrm{D}_{n-1}(k-1)$
$k = n-2$	$[0, 2]$	$\mathrm{D}_{n-1}(n-3)$	$\mathrm{D}_{n-1}(n-2, n-1)$	$\mathrm{D}_{n-1}(n-3)$
$k = n-1$	$[0, 1]$	$\mathrm{D}_{n-1}(n-2)$	$\emptyset$	$\mathrm{D}_{n-1}(n-1)$
$k = n$	$[0, 1]$	$\mathrm{D}_{n-1}(n-1)$	$\emptyset$	$\mathrm{D}_{n-1}(n-2)$

3.4.2 H_n-action

The case is analogous to the H_n-action on the varieties of type C_n. In fact, the linear action on V is the same as (9). We will denote again

$$V_- = \{x_n = \ldots = x_{2n-1} = 0\}, \qquad V_+ = \{x_0 = \ldots = x_{n-1} = 0\}.$$

They are isotropic subspaces of maximal dimension and, up to renumbering the nodes of D_n we may assume that $V_- \in \mathrm{D}_n(n)$. We will then distinguish two cases:

(A) n is even and $V_+ \in \mathrm{D}_n(n)$.
(B) n is odd and $V_+ \in \mathrm{D}_n(n-1)$.

Hence for the H_n-action on $\mathrm{D}_n(k)$ we have five cases:

	$[\mu_L(Y_-), \mu_L(Y_+)]$	Y_-	Y_s	Y_+
$k < n-1$	$[0, k]$	$\mathrm{A}_{n-1}(k)$	$\mathrm{A}_{n-1}(k-s, n-s)$	$\mathrm{A}_{n-1}(n-k)$
$k = n-1$ (A)	$[1, n/2]$	$\mathrm{A}_{n-1}(n-1)$	$\mathrm{A}_{n-1}(n-1-2s)$	$\mathrm{A}_{n-1}(1)$
$k = n$ (A)	$[0, n/2]$	pt	$\mathrm{A}_{n-1}(2s)$	pt
$k = n-1$ (B)	$[0, (n-1)/2]$	$\mathrm{A}_{n-1}(n-1)$	$\mathrm{A}_{n-1}(n-1-2s)$	pt
$k = n$ (B)	$[0, (n-1)/2]$	pt	$\mathrm{A}_{n-1}(2s)$	$\mathrm{A}_{n-1}(1)$

4 $\mathbb{C}^*$-actions and inversion in Jordan algebras: balanced gradings

This section is devoted to study equalized $\mathbb{C}^*$-actions on rational homogeneous varieties whose associated birational map is a Cremona transformation of a projective space, and we will prove that they can be understood in the language of Jordan algebras.

Consider a 1-dimensional torus $H_i \subset H \subset G$, determined by a height map σ_i producing a short grading (see Sect. 2.4.1):

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

Again, let $w_0 \in \mathrm{N}_G(H) \subset G$ be an element in the class of the longest element of the Weyl group W of G (see Sect. 2.2).

Definition 4.1 The short grading of $\mathfrak{g}$ determined by $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$ is said to be *balanced* if $t^{-1} = \mathrm{conj}_{w_0}(t)$ for every $t \in H_i$. In this case we will say that the action of H_i on every RH variety G/P is *balanced*.

Note that this property does not depend on the choice of the particular representative $w_0 \in \mathrm{N}_G(H)$. In particular, denoting by L_g the left multiplication by g for every $g \in G$, we have the following trivial statement:

Table 3 Balanced short gradings of simple Lie algebras

$\mathfrak{g}$	A_{2n-1}	B_n	C_n	D_n	D_{2n}	E_7
σ_i	σ_n	σ_1	σ_n	σ_1	$\sigma_{2n}, \sigma_{2n-1}$	σ_7

Lemma 4.2 *With the notations as above, if the short grading of $\mathfrak{g}$ determined by $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$ is balanced, then for every parabolic subgroup $P \subset G$ and for every $t \in H_i$ the following diagram is commutative:*

$$\begin{array}{ccc} G/P & \xrightarrow{L_{t^{-1}}} & G/P \\ {\scriptstyle L_{w_0}}\downarrow & & \downarrow{\scriptstyle L_{w_0}} \\ G/P & \xrightarrow[L_t]{} & G/P \end{array}$$

Then we may prove the following:

Lemma 4.3 *The complete list of balanced short gradings on simple Lie algebras is given in the following table:*

Proof We will use that, being G the adjoint group of $\mathfrak{g}$, the adjoint representation of G is faithful. Then the balanced condition is equivalent to saying that

$$\mathrm{Ad}_{w_0} \circ \mathrm{Ad}_t \circ \mathrm{Ad}_{w_0^{-1}} = \mathrm{Ad}_{t^{-1}} \quad \text{for every } t \in H_i,$$

as endomorphisms of $\mathfrak{g}$. Since they trivially coincide on the Cartan subalgebra $\mathfrak{h}$, the two maps above are equal if they coincide on $\mathfrak{g}_\beta$ for every $\beta \in \Phi$. Then the balanced condition is equivalent to saying that H_i is contained in the following subgroup of H:

$$H' := \left\{ t \in H \mid t^{w_0(\beta)} = t^{-\beta}, \text{ for every } \beta \in \Phi \right\}. \tag{10}$$

Here we are denoting by w_0 the automorphism of $\mathrm{M}(H)$ induced by w_0, that is the class of w_0 in the Weyl group W. It is a well-known fact (see for example [3]) that w_0 coincides with $-\mathrm{id}$ in the cases in which the Dynkin diagram $\mathcal{D}$ of $\mathfrak{g}$ has no non-trivial automorphisms and in the case D_{2n}. We conclude that in these cases $H' = H$, and so every short grading is balanced.

We are then left with the cases A_n, D_{2n-1} and E_6, in which w_0 is equal to the composition of $-\mathrm{id}$ with the homomorphism induced by the permutation of the positive simple roots Δ given by the only nontrivial automorphism s of $\mathcal{D}$. Then a straightforward computation in each case shows that the complete list of balanced short gradings is the one given in Table 3. □

In particular, consider the parabolic subgroups $P_\pm \subset G$ defined by H_i and the corresponding short grading, we obtain the following consequence:

Corollary 4.4 *With the notation above, consider the H_i-action on a rational homogeneous G-variety G/P. Then the following conditions are equivalent:*

(i) the H_i-action on G/P is equalized and has isolated extremal fixed-points;
(ii) $P = P_+$ and the grading of $\mathfrak{g}$ defined by H_i is short and balanced.

Proof We have already noted that the H_i-action is equalized on every G/P if and only if it is given by a short grading of $\mathfrak{g}$ (Proposition 2.4). On the other hand, consider the H_i-action on G/P. From Sect. 2.4.2 we know that the sink of the action is isolated if and only if $G^{\perp} \subset P$, which holds if and only if $P = P_+$. If, moreover, the action is balanced, then by definition, the point $w_0 P_+$ (which is always contained in the source of the H_i-action on G/P_+) will be an isolated fixed-point, as well. Finally, assume that the H_i-action on G/P_+ is equalized and has $w_0 P_+$ as an isolated source. From the description of the equalized actions of Sect. 3 for the classical cases, [5, Proposition 5.2] for the E_6-varieties and [11, Theorem 8.9] for the E_7-variety, we get that our conditions are satisfied precisely in the cases listed in Table 3, i.e. in the cases in which the grading is balanced. □

We will now study the birational map associated with the $\mathbb{C}^*$-action induced by a balanced short grading as above, linking it with an inversion map through the map L_{w_0}.

The unipotent radicals of $P_\pm$ – that we denote by $G_\pm \subset P_\pm$ – have $\mathfrak{g}_\pm$ as Lie algebras. Being G the adjoint group of a semisimple Lie algebra, $G_\pm \subset G$ consists only of nilpotent elements and then it is known that the exponential maps $\exp : \mathfrak{g}_\pm \to G_\pm$ are isomorphisms of varieties (see [6, Proposition 1.2]).

Lemma 4.5 *The composition of* $\exp : \mathfrak{g}_- \to G_- \subset G$ *with the projection onto* G/P_+ *is an open immersion, that we denote by* $\mathrm{ex} : \mathfrak{g}_- \hookrightarrow G/P_+$. *Its image, which we still denote by* $\mathfrak{g}_-$*, is an open neighborhood of* $eP_+ \in G/P_+$.

Proof First of all, $\mathfrak{g}_-$ and G/P_+ have the same dimension. At this point, denoted by $\pi : G \to G/P_+$ the quotient map, let $U \subset G$ be an open set. By definition, $\pi(U)$ is open in G/P_+ if and only if $\pi^{-1}(\pi(U))$ is open in G:

$$\pi(U) = \{uP : u \in U\}, \qquad \pi^{-1}(\pi(U)) = \{up : u \in U,\ p \in P_+\} = U \cdot P_+.$$

By [2, p. 86], $G_- \cdot P_+$ is open in G, hence $\pi(G_-)$ is open in G/P_+. It remains to prove that $\pi(G_-) \simeq G_-$, but this follows from the fact that, since G_- is the unipotent radical of the opposite parabolic group of P_+ (see [2, p. 88]), we have $G_- \cap P_+ = \{e\}$. We have obtained that $\mathfrak{g}_- \simeq G_- \simeq \pi(G_-) \subset G/P_+$ as an open subset. □

Furthermore, denoting by $L_t : G/P_+ \to G/P_+$ the left multiplication with t, for every $t \in H_i$, we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{g}_- & \xrightarrow{\;t^{-1}\;} & \mathfrak{g}_- \\ {\scriptstyle \mathrm{ex}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ex}} \\ G/P_+ & \xrightarrow{\;L_t\;} & G/P_+ \end{array}$$

where the upper horizontal map is the multiplication with t^{-1}. In other words, $\mathfrak{g}_-$ can be identified with a $\mathbb{C}^*$-invariant open subset of G/P_+, and the action of $\mathbb{C}^*$ on $\mathfrak{g}_-$ is the inverse of the homothetical action.

On the other hand, we have another open immersion:

$$\mathrm{ex}' : \mathfrak{g}_+ \longrightarrow G/P_+, \qquad x \longmapsto \exp(x) w_0 P_+,$$

whose image is a $\mathbb{C}^*$-invariant open neighborhood (denoted by $\mathfrak{g}_+$) of $w_0P_+ \in G/P_+$, that fits into the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g}_- & \xrightarrow{\mathrm{Ad}_{w_0}} & \mathfrak{g}_+ \\ {\scriptstyle \mathrm{ex}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ex}'} \\ G/P_+ & \xrightarrow{L_{w_0}} & G/P_+. \end{array}$$

Intersecting $\mathfrak{g}_-$ and $\mathfrak{g}_+$ inside G/P_+, and restricting Ad_{w_0} to $\mathrm{Ad}_{w_0}^{-1}(\mathfrak{g}_- \cap \mathfrak{g}_+)$ we get a birational map that we denote

$$\jmath : \mathfrak{g}_- \dashrightarrow \mathfrak{g}_-.$$

By construction, we have a commutative diagram:

$$\begin{array}{ccc} & & \mathfrak{g}_- \\ & \overset{\jmath}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{Ad}_{w_0}} \\ \mathfrak{g}_- & \underset{\psi_a}{\searrow} & \\ & & \mathfrak{g}_+. \end{array} \tag{11}$$

We note that, by Lemma 4.2 and the fact that the inclusions $\mathfrak{g}_\pm \subset G/P_+$ are $\mathbb{C}^*$-invariant, $\jmath$ satisfies:

$$\jmath(tx) = t^{-1}x, \text{ for all } t \in H_i, \ x \in \mathfrak{g}_-. \tag{12}$$

so that the inversion map can be thought of as a birational automorphism counterpart of the birational map ψ_a induced by the H_i-action (see Sect. 2.1).

The following statement is a consequence of [15, 2.21].

Proposition 4.6 *With the above notation, if the short grading on $\mathfrak{g}$ determined by σ_i : $\mathrm{M}(H) \to \mathbb{Z}$ is balanced, then there exists a unique structure of Jordan algebra on $\mathfrak{g}_-$ having $\jmath$ as an inverse map.*

Proof Following [15], the statement follows from the fact that the map $\jmath$ defines a *J-structure* on $\mathfrak{g}_-$, that determines completely a structure of Jordan algebra on $\mathfrak{g}_-$ whose inverse map is $\jmath$. Furthermore, [15, 2.21] tells us that in our case one needs to check three conditions (A,B,C). Condition (A) follows from the fact that our grading is equalized, and (B) is precisely Equation (12). The last condition, (C), translated into our notation says that G_0 acts with an open orbit on $\mathfrak{g}_-$, and this follows by direct application of [16, Theorem 2.1]. □

This concludes the proof of Theorem 1.2.

5 Proof of Theorems 1.3 and 1.4

We have already seen in Sect. 4 that, when a $\mathbb{C}^*$-action on an RH variety G/P is determined by a balanced short grading (and its sink and source are isolated), the induced birational map (which is a Cremona transformation) is determined, up to composition with a projectivity, by the inverse map of a *unique* Jordan algebra structure on the tangent space of G/P.

Suppose now that the sink and source are not isolated, then the birational map $\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+$ induced by the action has as domain and codomain (which are, by the balancedness hypothesis, isomorphic) the projectivizations of two homogeneous vector bundles on two

RH varieties of Picard number one. In particular, $\mathcal{G}_\pm$ have Picard number two. Moreover, it has been proved in [12, Section 4 and Corollary 4.12 (iii) in particular] that $\mathcal{G}_\pm$ are Mori Dream Spaces, isomorphic in codimension one. More precisely, following [12], we may state:

Proposition 5.1 *Let G be a simple algebraic group, whose Lie algebra $\mathfrak{g}$ admits a short and balanced grading. Let $X = G/P$ be an RH variety of Picard number one, endowed with the $\mathbb{C}^*$-action associated to the grading. Assume moreover that the sink and the source are not isolated fixed-points. Then:*

- *The quotients $\mathcal{G}_\pm$ are Mori dream spaces, corresponding to two different nef chambers of $\overline{\mathrm{Mov}(\mathcal{G}_-)} = \overline{\mathrm{Mov}(\mathcal{G}_+)}$.*
- *There are as many nef chambers of $\overline{\mathrm{Mov}(\mathcal{G}_-)}$ as $\delta := \mu_L(Y_+) - \mu_L(Y_-)$, where L denotes the ample generator of $\mathrm{Pic}(X)$. Each nef chamber corresponds to a geometric quotient of X by the action of $\mathbb{C}^*$, more precisely*

$$\mathcal{G}_k := \mathrm{Proj}\left(\bigoplus_{m\in 2\mathbb{Z}_{\geq 0}} \mathrm{H}^0\left(X, L^m\right)_{((2k-1)m/2)}\right), \qquad k = 1, \ldots, \delta.$$

- *We have $\mathcal{G}_1 = \mathcal{G}_-$ and $\mathcal{G}_\delta = \mathcal{G}_+$. Each of the nef cones of $\mathcal{G}_\pm$ contains an extremal ray of $\mathrm{Mov}(\mathcal{G}_-)$.*
- *The map $\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+$ can be factorized as the sequence of the birational maps:*

$$\mathcal{G}_1 = \mathcal{G}_- \dashrightarrow \mathcal{G}_2 \dashrightarrow \cdots \dashrightarrow \mathcal{G}_\delta = \mathcal{G}_+,$$

which are Atiyah flips corresponding to wall crossings in $\overline{\mathrm{Mov}(\mathcal{G}_-)}$.

In particular, this immediately implies the following:

Corollary 5.2 *In the situation of Proposition 5.1, the birational map $\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+$ is a small $\mathbb{Q}$-factorial modification, uniquely determined by $\mathcal{G}_\pm$.*

We will finish the Section with the proofs of Theorem 1.3 and Theorem 1.4, which is done by determining the normal bundles of $Y_\pm$ in X in each case.

5.1 Proof of Theorem 1.3

By Corollary 5.2, the map ψ is in each case determined by the varieties $\mathcal{G}_\pm$, which we will describe in each case. We will proceed case-by-case according to Table 3. Besides, we have included here some remarks, in which we provide further geometric features of ψ.

5.1.1 H_n-action on Grassmannians

We consider the equalized and balanced H_n-action on the Grassmannian $\mathrm{A}_{2n-1}(k)$. We refer to Sect. 3.1 for the notations and results therein contained. In particular, given a the $2n$-dimensional vector space V, the H_n-action provides a decomposition $V = V_- \oplus V_+$ such that $\dim V_\pm = n$. The extremal fixed-point components of the action on $\mathrm{A}_{2n-1}(k)$ are:

$$Y_\pm = \left\{[W] \in \mathbb{P}\left(\bigwedge^k V_\pm\right) : W \subset V_\pm\right\} \simeq \mathrm{A}_{n-1}(k); \qquad (13)$$

they parametrize k-dimensional vector subspaces contained in $V_\pm$, respectively. We denote by $\mathcal{S}_\pm$, $\mathcal{Q}_\pm$ the universal bundles over $Y_\pm$ of rank k and $n-k$, respectively.

If $k = n$, ψ is a Cremona transformation (see Sect. 4), which can be explicitly described (see [18, Section 4], [10, Section 4]) as the projectivization of the inversion of linear maps. More generally:

Proposition 5.3 *With the notation as above, consider the equalized and balanced H_n-action on the Grassmannian $\mathrm{A}_{2n-1}(k)$. Then $Y_\pm \simeq \mathrm{A}_{n-1}(k)$ and*

$$\mathcal{N}_{Y_\pm|\mathrm{A}_{2n-1}(k)} \simeq \mathcal{S}_\pm^\vee \otimes V_\mp.$$

Proof In order to compute the normal bundles $\mathcal{N}_{Y_\pm|\mathrm{A}_{2n-1}(k)}$, we restrict the universal bundles $\mathcal{S}$, $\mathcal{Q}$ of $\mathrm{A}_{2n-1}(k)$ on the fixed-point components $Y_\pm$, obtaining:

$$\begin{aligned} \mathcal{S}|_{Y_\pm} &\simeq \mathcal{S}_\pm, \\ \mathcal{Q}|_{Y_\pm} &\simeq \mathcal{Q}_\pm \oplus \left(V_\mp \otimes \mathcal{O}_{Y_\pm}\right), \end{aligned}$$

where $\mathcal{S}_\pm$, $\mathcal{Q}_\pm$ are the universal bundles of $Y_\pm$. Consider the short exact sequences

$$0 \to T_{Y_\pm} \longrightarrow T_{\mathrm{A}_{2n-1}(k)}|_{Y_\pm} \longrightarrow \mathcal{N}_{Y_\pm|\mathrm{A}_{2n-1}(k)} \to 0;$$

together with the equalities

$$\begin{aligned} T_{Y_\pm} &\simeq \mathcal{S}_\pm^\vee \otimes \mathcal{Q}_\pm, \\ T_{\mathrm{A}_{2n-1}(k)}|_{Y_\pm} &\simeq \left(\mathcal{S}_\pm^\vee \otimes \mathcal{Q}_\pm\right) \oplus \left(\mathcal{S}_\pm^\vee \otimes \left(V_\mp \otimes \mathcal{O}_{Y_\pm}\right)\right); \end{aligned}$$

then we get

$$\mathcal{N}_{Y_\pm|\mathrm{A}_{2n-1}(k)} \simeq \mathcal{S}_\pm^\vee \otimes V_\mp.$$

□

Remark 5.4 Following [18, Section 4], for every $u \le k$ we define $\mathrm{Sec}_u(\mathcal{S}_\pm, V_\mp)$ to be the variety of secant $(u-1)$-dimensional linear spaces to the relative Segre embedding

$$\mathbb{P}\left(\mathcal{S}_\pm^\vee\right) \times \mathbb{P}\left(V_\mp\right) \subset \mathbb{P}\left(\mathcal{S}_\pm^\vee \otimes V_\mp\right).$$

Thaddeus describes such a variety as the space of linear maps $f : W_\pm \to V_\mp$ (where $W_\pm \subset V_\pm$ is a linear subspace of dimension k) of rank up to $u \le k$.

Our map ψ is then defined on the open set

$$\mathbb{P}\left(\mathcal{S}_\pm^\vee \otimes V_\mp\right) \setminus \mathrm{Sec}_{k-1}(\mathcal{S}_\pm, V_\mp),$$

which corresponds to the linear maps $f : W_\pm \to V_\mp$ of maximal rank.

Assume that $f : W_- \to V_+$ is an injective linear map from a k-dimensional linear subspace $W_- \subset V_-$ and denote $W_+ := f(W_-)$. Consider the inverse $f^{-1} : W_+ \to W_-$ and compose it with the inclusion $W_- \hookrightarrow V_-$, to a map $\bar{f} : W_+ \to V_-$. Then ψ can be described as follows:

$$\psi : \mathbb{P}\left(\mathcal{S}_-^\vee \otimes V_+\right) \dashrightarrow \mathbb{P}\left(\mathcal{S}_+^\vee \otimes V_-\right), \qquad [f] \longmapsto [\bar{f}].$$

Note that, if $k = 1$, then $\mathcal{S}_\pm^\vee = \mathcal{O}_{Y_\pm}(1)$. So $\mathbb{P}(\mathcal{O}_{Y_\pm}(1) \otimes V_\mp) = \mathbb{P}(V_-) \times \mathbb{P}(V_+)$, and the map ψ is the identity.

5.1.2 H_n-action on isotropic Grassmannians

We merge the cases of the equalized and balanced H_n-actions on the symplectic Grassmannian $X \simeq \mathrm{C}_n(k)$ or the even orthogonal Grassmannian $X \simeq \mathrm{D}_n(k)$, with n even if $k = n-1, n$. As shown in Sects. 3.3 and 3.4.2, which we refer for the notations and the results therein, the fixed-point components of the action are RH varieties of type A_{n-1}.

With the exception of the case $X \simeq \mathrm{D}_n(n-1)$, n even, the sink and the source of the H_n-action on X, as in (13), are Grassmannians parametrizing k-dimensional linear subspaces of $V_\pm$, respectively:

$$Y_\pm = \left\{ [W] \in \mathbb{P}\left(\bigwedge^k V_\pm \right) : W \subset V_\pm \right\} \simeq \mathrm{A}_{n-1}(k).$$

As in Sect. 5.1.1, $Y_\pm$ support two universal bundles, of rank k and $n-k$ respectively, that we denote by $\mathcal{S}_\pm$, $\mathcal{Q}_\pm$. Moreover, the isomorphism

$$V_+ \simeq V_-^\vee$$

(see Sect. 2.3.2) allows us to identify $Y_\pm$ with the Grassmannians of $(n-k)$-dimensional vector subspaces of $V_\mp$. Then, restricting Diagram (3) to Y_-, for instance, we get:

$$\begin{array}{ccccc} \mathcal{S}_- & \longrightarrow & (V_- \otimes \mathcal{O}_{Y_-}) \oplus \mathcal{Q}_-^\vee & \longrightarrow & \mathcal{K}_{|Y_-} \\ \| & & \downarrow & & \downarrow \\ \mathcal{S}_- & \longrightarrow & (V_- \oplus V_-^\vee) \otimes \mathcal{O}_{Y_-} & \longrightarrow & (V_-^\vee \otimes \mathcal{O}_{Y_-}) \oplus \mathcal{Q}_- \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{S}_-^\vee & = & \mathcal{S}_-^\vee \end{array} \tag{14}$$

so that

$$\mathcal{K}_{|Y_-} \simeq \mathcal{Q}_- \oplus \mathcal{Q}_-^\vee. \tag{15}$$

In the case $X \simeq \mathrm{D}_n(n-1)$, n even, the sink and the source of the action are projective spaces $\mathrm{A}_{n-1}(n-1)$, parametrizing $(n-1)$-dimensional subspaces of V_- and V_+, respectively. The restriction of Diagram (3) to Y_- provides:

$$\begin{array}{ccccc} \mathcal{S}_{|Y_-} & = & \mathcal{S}_{|Y_-} & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \\ \Omega_{Y_-}(1) \oplus \mathcal{O}_{Y_-}(-1) & \longrightarrow & (V_- \oplus V_-^\vee) \otimes \mathcal{O}_{Y_-} & \longrightarrow & \mathcal{O}_{Y_-}(1) \oplus T_{Y_-}(-1) \\ \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathcal{S}_{|Y_-}^\vee & = & \mathcal{S}_{|Y_-}^\vee. \end{array} \tag{16}$$

We will study now the normal bundles of $Y_\pm$ in X (which completely determine the induced birational map $\psi : \mathbb{P}\left(\mathcal{N}_{Y_-|X}\right) \dashrightarrow \mathbb{P}\left(\mathcal{N}_{Y_+|X}\right)$ by Corollary 5.2). The case $k = n$ (that is, the case in which ψ is a Cremona transformation) has been explicitly studied in [10, Sections 4.3 and 4.4] (see also [18, Appendix 10 and 11]) which show that:

- If $X \simeq \mathrm{C}_n(n)$ is a Lagrangian Grassmannian, then the birational map (1) is the inversion of symmetric tensors:

$$\psi : \mathbb{P}\left(S^2 V_+\right) \dashrightarrow \mathbb{P}\left(S^2 V_-\right) \simeq \mathbb{P}\left(S^2 V_+^\vee\right).$$

- If $X \simeq \mathrm{D}_n(n)$, n even, is a spinor variety, then the birational map (1) is the inversion of skew-symmetric tensors:

$$\psi : \mathbb{P}\left(\bigwedge^2 V_+\right) \dashrightarrow \mathbb{P}\left(\bigwedge^2 V_-\right) \simeq \mathbb{P}\left(\bigwedge^2 V_+^\vee\right).$$

To describe the normal bundles of $Y_\pm$ in X in the remaining cases, we will consider separately the case in which X parametrizes non-maximal isotropic subspaces of V and the case $X \simeq \mathrm{D}_n(n-1)$, n even. We start by proving the following.

Proposition 5.5 *With the notation as above, consider the equalized and balanced H_n-action on $X \simeq \mathrm{C}_n(k)$ with $k < n$ or $X \simeq \mathrm{D}_n(k)$ with $k < n-1$. Then $Y_\pm \simeq \mathrm{A}_{n-1}(k)$ and the normal bundles $\mathcal{N}_{Y_\pm|X}$ are non-trivial extensions*

$$0 \to (\mathcal{S}_\pm \otimes \mathcal{Q}_\pm)^\vee \longrightarrow \mathcal{N}_{Y_\pm|X} \longrightarrow \mathcal{C}_\pm \to 0,$$

where

$$\mathcal{C}_\pm = \begin{cases} S^2\mathcal{S}_\pm^\vee & \text{if } X \simeq \mathrm{C}_n(k), \\ \bigwedge^2 \mathcal{S}_\pm^\vee & \text{if } X \simeq \mathrm{D}_n(k). \end{cases}$$

Proof Combining the fact that $T_{Y_\pm} \simeq \mathcal{S}_\pm^\vee \otimes \mathcal{Q}_\pm$ with (4), we obtain:

$$\begin{array}{ccccc} \mathcal{S}_\pm^\vee \otimes \mathcal{Q}_\pm & \xrightarrow{\simeq} & T_{Y_\pm} & & \\ \downarrow & & \downarrow & & \\ \mathcal{S}_\pm^\vee \otimes \mathcal{K}|_{Y_\pm} & \longrightarrow & T_X|_{Y_\pm} & \longrightarrow & \mathcal{C}_\pm \\ \downarrow & & \downarrow & & \| \\ \mathcal{S}_\pm^\vee \otimes \mathcal{K}|_{Y_\pm}/\mathcal{Q}_\pm & \longrightarrow & \mathcal{N}_{Y_\pm|X} & \longrightarrow & \mathcal{C}_\pm. \end{array} \tag{17}$$

We conclude by noting that $\mathcal{K}|_{Y_\pm}/\mathcal{Q}_\pm = \mathcal{Q}_\pm^\vee$, which follows from the fact that $\mathrm{Hom}(\mathcal{Q}, \mathcal{Q}^\vee) = 0$. □

Finally we consider the case of the H_n-action on $X \simeq \mathrm{D}_n(n-1)$.

Proposition 5.6 *With the notations as above, consider the equalized and balanced H_n-action on $\mathrm{D}_n(n-1)$, n even. Then $Y_\pm \simeq \mathrm{A}_{n-1}(n-1)$ are projective spaces, and*

$$\mathcal{N}_{Y_\pm|\mathrm{D}_n(n-1)} \simeq \bigwedge^2 T_{Y_\pm}(-2).$$

Proof We will do the proof in the case of Y_-, being the case of Y_+ analogous. By (4) we have that $T_{\mathrm{D}_n(n-1)} \simeq \bigwedge^2 \mathcal{S}^\vee$. Using Diagram (16) we then get:

$$T_{\mathrm{D}_n(n-1)}|_{Y_-} \simeq \bigwedge^2 \mathcal{S}^\vee|_{Y_-} \simeq \bigwedge^2 \left(\mathcal{O}_{Y_-}(1) \oplus T_{Y_-}(-1)\right) \simeq T_{Y_-} \oplus \bigwedge^2 T_{Y_-}(-2).$$

Then we have a short exact sequence:

$$0 \to T_{Y_-} \longrightarrow T_{Y_-} \oplus \bigwedge^2 T_{Y_-}(-2) \longrightarrow \mathcal{N}_{Y_-|\mathrm{D}_n(n-1)} \to 0. \tag{18}$$

We claim that $\mathrm{Hom}(T_{Y_-}, \bigwedge^2 T_{Y_-}(-2)) = 0$, so that, in particular we conclude that $\mathcal{N}_{Y_\pm|\mathrm{D}_n(n-1)} \simeq \bigwedge^2 T_{Y_\pm}(-2)$. If there was a non-trivial morphism $T_{Y_-} \to \bigwedge^2 T_{Y_-}(-2)$, twisting it with $\mathcal{O}_{Y_-}(-1)$, we would obtain a non-trivial map

$$T_{Y_-}(-1) \longrightarrow \bigwedge^2 T_{Y_-}(-3).$$

Since $T_{Y_-}(-1)$ is globally generated, this would imply that $\mathrm{H}^0(Y_-, \bigwedge^2 T_{Y_-}(-3)) \neq 0$. On the other hand, we have that

$$\bigwedge^2 T_{Y_-}(-3) \simeq \bigwedge^{n-3} \Omega_{Y_-}(n-3)$$

and that $H^0(Y_-, \bigwedge^{n-3} \Omega_{Y_-}(n-3)) = 0$ by Bott formula (cf. [13, p.8]), leading to a contradiction. □

5.1.3 H_1-action on orthogonal Grassmannians

We consider here together the cases of the equalized and balanced H_1-actions on the orthogonal Grassmannian $X \simeq \mathrm{B}_n(k)$ or $X \simeq \mathrm{D}_n(k)$. We refer to Sects. 3.2 and 3.4.1 for the notation and preliminary results. In particular, the fixed-point components are RH varieties of type B_{n-1} or D_{n-1}, respectively (that can be identified with orthogonal Grassmannians of isotropic subspaces in $V_0 \subset V$), and the possible pairs (Y_-, Y_+) are:

$$\begin{array}{ll}
(\mathrm{B}_{n-1}(k-1), \mathrm{B}_{n-1}(k-1)) & \text{if } X \simeq \mathrm{B}_n(k), \\
(\mathrm{D}_{n-1}(k-1), \mathrm{D}_{n-1}(k-1)) & \text{if } X \simeq \mathrm{D}_n(k) \text{ with } k \leq n-2, \\
(\mathrm{D}_{n-1}(n-2), \mathrm{D}_{n-1}(n-1)) & \text{if } X \simeq \mathrm{D}_n(n-1), \\
(\mathrm{D}_{n-1}(n-1), \mathrm{D}_{n-1}(n-2)) & \text{if } X \simeq \mathrm{D}_n(n).
\end{array}$$

As orthogonal Grassmannians of $(k-1)$-dimensional isotropic subspaces of V_0, $Y_\pm$ support two universal bundles $\mathcal{S}_\pm$, $\mathcal{Q}_\pm$ of rank $k-1$ and $\dim V_0 - (k-1)$, respectively, fitting in short exact sequences

$$0 \to \mathcal{S}_\pm \longrightarrow V_0 \otimes \mathcal{O}_{Y_\pm} \longrightarrow \mathcal{Q}_\pm \to 0.$$

The case $k = 1$ gives us the Cremona transformation as described in Sect. 4; we will provide later a projective description of the map ψ obtained in this case (cf. Remark 5.8). More generally, the following statement holds:

Proposition 5.7 *With the notation as above, consider the equalized and balanced H_1-action on the orthogonal Grassmannian $X \simeq \mathrm{B}_n(k)$ or $X \simeq \mathrm{D}_n(k)$. Then*

$$\mathcal{N}_{Y_\pm|X} \simeq \mathcal{Q}_\pm.$$

Proof We will prove the statement for the sink Y_-, being the case of Y_+ analogous. We start by writing Y_- as the RH $G^\perp$-variety $G^\perp/P^\perp$ (Sect. 2.4.2); then, since both $\mathcal{N}_{Y_-|X}$ and $\mathcal{Q}_-$ are homogeneous $G^\perp$-bundles (Lemma 2.7), it is enough to show that their fibers at $eP^\perp$ are isomorphic as $P^\perp$-modules.

On one hand, Lemma 2.7 tells us that

$$\mathcal{N}_{Y_-|X,eP^\perp} = N_- := \bigoplus_{\beta\in S} \mathfrak{g}_\beta$$

where $S = \left\{\beta \in \Phi^- : \sigma_1(\beta) < 0,\ \sigma_k(\beta) < 0\right\}$. Using the notation of Bourbaki (cf. [3, Planche II,IV]), there exists a basis $\{\epsilon_i,\ i = 1,\dots n\}$ of $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ such that the positive simple roots of $\mathfrak{g}$ can be written as:

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } i < n \text{ and } \alpha_n = \begin{cases} \epsilon_n & \text{if } \mathfrak{g} = \mathrm{B}_n, \\ \epsilon_{n-1} + \epsilon_n & \text{if } \mathfrak{g} = \mathrm{D}_n, \end{cases}$$

hence we can rewrite S as

$$S = \left\{-\epsilon_1 + \epsilon_j\right\}_{j=k+1}^n \cup \left\{-\epsilon_1 - \epsilon_j\right\}_{j=2}^n \cup \begin{cases} \{-\epsilon_1\} & \text{if } \mathfrak{g} = \mathrm{B}_n \\ \emptyset & \text{if } \mathfrak{g} = \mathrm{D}_n \end{cases}$$

in the case $X \not\simeq \mathrm{D}_n(n-1)$ and

$$S = \{-\epsilon_1 + \epsilon_n\} \cup \left\{-\epsilon_1 - \epsilon_j\right\}_{j=2}^{n-1}.$$

otherwise.

Since the first fundamental weight ω_1 is ϵ_1, by Remark 2.5 the $H^\perp$-weights of the action are

$$\left\{\epsilon_j\right\}_{j=k+1}^n \cup \left\{-\epsilon_j\right\}_{j=2}^n \cup \begin{cases} \{0\} & \text{if } \mathfrak{g} = \mathrm{B}_n \\ \emptyset & \text{if } \mathfrak{g} = \mathrm{D}_n \end{cases}$$

when $X \not\simeq \mathrm{D}_n(n-1)$ and

$$\{\epsilon_n\} \cup \left\{-\epsilon_j\right\}_{j=2}^{n-1}$$

otherwise.

On the other hand, consider the basis $\mathcal{B} = \{e_i\}$ of V introduced in Sects. 3.2 and 3.4. Then $\mathcal{B} \setminus \{e_0, e_n\}$ is a basis for V_0; in particular, for $i < n$, the $H^\perp$-weight of e_i is ϵ_{i+1}, the $H^\perp$-weight of e_{n+i} is $-\epsilon_{i+1}$ and, if $\mathfrak{g} = \mathrm{B}_n$, the $H^\perp$-weight of e_{2n} is 0. We denote

$$V_0 \supset W := \begin{cases} \langle e_1, \dots, e_{k-1}\rangle & \text{if } X \not\simeq \mathrm{D}_n(n-1), \\ \langle e_1, \dots, e_{n-2}, e_{2n-1}\rangle & \text{if } X \simeq \mathrm{D}_n(n-1), \end{cases}$$

we have that

$$\mathcal{Q}_- = G^\perp \times^{P^\perp} V_0/W.$$

Then a straightforward computation shows that the decomposition of V_0/W on $H^\perp$-eigenspaces is the same as the one obtained above. □

Remark 5.8 In the case $k = 1$, the induced map ψ is a Cremona transformation determined by the inversion in a Jordan algebra (see Sect. 4). Furthermore, $\psi : \mathbb{P}(\mathfrak{g}_-) \to \mathbb{P}(\mathfrak{g}_+)$ is a linear isomorphism that can be projectively described as follows. The symmetric bilinear form defining the Lie algebra $\mathfrak{g}$ determines a quadric $Q = \mathcal{D}(1)$ ($\mathcal{D} = \mathrm{B}_n$ or D_n) in the projectivization of the standard representation $V = V(\omega_1)$ of $\mathfrak{g}$. Then $\mathfrak{g}_\pm$ correspond to the tangent spaces of Q at sink and source $Y_\pm$, which are isolated points. Consider a tangent direction $[v_-] \in \mathbb{P}(\mathfrak{g}_-)$, it determines a line ℓ_- passing by Y_-. The plane $\ell_- + Y_+$ intersects

Q on a ($\mathbb{C}^*$-invariant) conic, which is smooth at Y_+; the tangent direction to the conic at Y_+ is $\psi([v_-])$.

Let us now describe projectively also the cases in which $k > 1$.

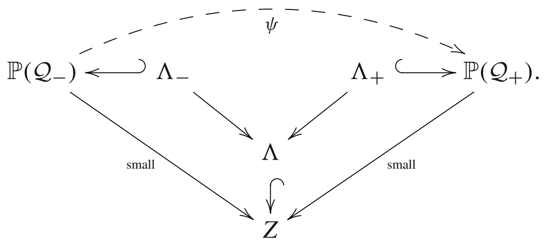
Remark 5.9 In the cases in which X is a spinor variety, i.e. $X \simeq \mathrm{B}_n(n), \mathrm{D}_n(n-1), \mathrm{D}_n(n)$, there are no inner fixed-point components, hence the induced birational transformation ψ in the statement above is an isomorphism. In the remaining cases, $\psi : \mathbb{P}(\mathcal{Q}_-) \dashrightarrow \mathbb{P}(\mathcal{Q}_+)$ is an Atiyah flip (cf. [11, 12]), that can be described projectively as follows.

Again, we identify $Y_\pm$ with two orthogonal Grassmannians of $(k-1)$-dimensional isotropic subspaces of V_0. Then the bundle $\mathbb{P}(\mathcal{Q}_-)$ can be identified with the family of pairs (W_-, W_0), where $W_- \in Y_-$, and $W_0 \subset V_0$ is a subspace of dimension k containing W_-. A similar description applies to $\mathbb{P}(\mathcal{Q}_+)$.

Given two pairs $(W_-, W_0) \in \mathbb{P}(\mathcal{Q}_-)$, $(W_+, W_0') \in \mathbb{P}(\mathcal{Q}_+)$, we have that ψ sends (W_-, W_0) to (W_+, W_0') if $W_0 = W_0'$ and the set of isotropic vectors of W_0 is precisely $W_- \cup W_+$. The maps ψ, ψ^{-1} are not defined, respectively, in the sets

$$\Lambda_\pm := \{(W_\pm, W_0) \in \mathbb{P}(\mathcal{Q}_\pm) : W_0 \subset V_0 \text{ isotropic}\} \subset \mathbb{P}(\mathcal{Q}_\pm).$$

Denote $Z \simeq \mathrm{A}_{\dim V_0 - 1}(k)$, the Grassmannian of k-dimensional subspaces of V_0. We have an obvious (small) contraction $\mathbb{P}(\mathcal{Q}_\pm) \to Z$, sending $(W_\pm, W_0)$ to W_0; the image of $\Lambda_\pm$ via this contraction, that we denote by $\Lambda \subset Z$, is the orthogonal Grassmannian of isotropic k-dimensional subspaces of V_0. Then the map ψ is precisely a flip for these two contractions:



5.2 Proof of Theorem 1.4

To complete the description of the birational maps associated to a short and balanced grading, we are left with the case of E_7 and the grading given by the height map σ_7. The corresponding H_7-action on the variety $\mathrm{E}_7(7)$ has isolated sink and source, and the associated birational map $\psi : \mathbb{P}(T_{\mathrm{E}_7(7), Y_-}) \dashrightarrow \mathbb{P}(T_{\mathrm{E}_7(7), Y_+})$ is a Cremona transformation $\mathbb{P}^{26} \dashrightarrow \mathbb{P}^{26}$. Following [11, Section 8], this is the special Cremona transformation whose exceptional locus is the Cartan variety $\mathrm{E}_6(6) \simeq \mathrm{E}_6(1)$, which is the 16-dimensional Severi variety.

Let us now consider the induced H_7-action on varieties $\mathrm{E}_7(k)$, $k < 7$. As in the cases of classical type, the induced birational map is completely determined by the normal bundles $\mathcal{N}_{Y_\pm | \mathrm{E}_7(k)}$ (cf. Corollary 5.2). The extremal fixed-point components of the action $Y_\pm \subset \mathrm{E}_7(k)$, which are RH E_6-varieties (see Table 4 below), can be obtained by applying directly the arguments in Sect. 2.4.2.

Note that in each case, the markings of the Dynkin diagram E_6 for the sink and the source are symmetric with respect to the nontrivial automorphism of the E_6 diagram (cf. Remark 2.6) that we denote by s:

$$s(1) = 6, \quad s(2) = 2, \quad s(3) = 5, \quad s(4) = 4, \quad s(5) = 3, \quad s(6) = 1.$$

Table 4 Extremal fixed-point components of the H_7-action in $E_7(k)$, $k < 7$

k	Y_-	Y_+	$\dim(Y_\pm)$	$\operatorname{rank}(N_{Y_\pm, E_7(k)})$
1	$E_6(1)$	$E_6(6)$	16	17
2	$E_6(2)$	$E_6(2)$	21	21
3	$E_6(3)$	$E_6(5)$	25	22
4	$E_6(4)$	$E_6(4)$	29	24
5	$E_6(5)$	$E_6(3)$	25	25
6	$E_6(6)$	$E_6(1)$	16	26

Notation 5.10 In order to include the case of $E_7(7)$, we define $s(7) := 0$.

The variety $E_6(6)$ can be embedded via L, the generator of its Picard group, into the projectivization of the (27-dimensional) representation $V(\omega_6)$, so that $V(\omega_6)^\vee = H^0(E_6(6), L)$. On the other hand, for every $k \leq 6$, the variety $E_6(k)$ can be thought of as a family of projective spaces in $\mathbb{P}(V(\omega_6))$. In fact, for $k = 6$ this is given simply by the embedding $E_6(6) \subset \mathbb{P}(V(\omega_6))$. For $k \neq 1, 6$ the natural projections of E_6-varieties

$$\begin{array}{ccccc} & & E_6(k,6) & & \\ & \swarrow^{p_k} & & \searrow & \overset{q_k}{\searrow} \\ E_6(k) & & & & E_6(6) \hookrightarrow \mathbb{P}(V(\omega_6)) \end{array}$$

present $E_6(k)$ as a family of $\mathbb{P}^5$'s, $\mathbb{P}^4$'s, $\mathbb{P}^2$'s, and $\mathbb{P}^1$'s, in the cases $k = 2, 3, 4, 5$, respectively. Finally, the variety $E_6(1)$ parametrizes smooth 8-dimensional quadrics in $E_6(6)$; considering the linear span of these quadrics, we may also think of $E_6(1)$ as the parameter space of a family of (9-dimensional) projective subspaces of $\mathbb{P}(V(\omega_6))$:

$$\begin{array}{ccc} & E_6(1,6) & \\ \swarrow^{p_1} & & \searrow^{q_1} \\ E_6(1) & & \mathbb{P}(V(\omega_6)). \end{array}$$

These families of (9, 5, 4, 2, 1-dimensional) projective subspaces of $\mathbb{P}(V(\omega_6))$ are the projectivizations of the following E_6-homogeneous vector bundles:

$$\mathcal{S}_k := p_{k*} q_k^* \mathcal{O}_{\mathbb{P}(V(\omega_6))}(1).$$

This is a subbundle of the trivial vector bundle $V(\omega_6) \otimes \mathcal{O}_{E_6(k)}$, and we denote the corresponding cokernel:

$$\mathcal{Q}_k := \left(V(\omega_6) \otimes \mathcal{O}_{E_6(k)}\right) / \mathcal{S}_k.$$

Analogously, if we start with $E_6(1) \subset \mathbb{P}(V(\omega_1))$, then we can describe the other E_6-homogeneous varieties of Picard number one as families of projective spaces in $\mathbb{P}(V(\omega_1))$:

$$\begin{array}{ccc} & E_6(1,k) & \\ \swarrow^{p'_k} & & \searrow^{q'_k} \\ E_6(k) & & \mathbb{P}(V(\omega_1)). \end{array}$$

In particular, we have a homogeneous vector bundle $\mathcal{S}'_k := p'_{k*}q'^{*}_k\mathcal{O}_{\mathbb{P}(V(\omega_1))}(1)$ over $\mathrm{E}_6(k)$ which is subbundle of the trivial vector bundle $V(\omega_1) \otimes \mathcal{O}_{\mathrm{E}_6(k)}$, and we denote the corresponding cokernel as $\mathcal{Q}'_k$.

Proposition 5.11 *With the notations as above, consider the equalized and balanced H_7-action on $\mathrm{E}_7(k)$. Then $Y_- \simeq \mathrm{E}_6(k)$, $Y_+ \simeq \mathrm{E}_6(s(k))$, and*

$$\mathcal{N}_{Y_-|\mathrm{E}_7(k)} = \mathcal{Q}_k, \qquad \mathcal{N}_{Y_+|\mathrm{E}_7(k)} = \mathcal{Q}'_{s(k)}.$$

Sketch of the proof The proof is analogous to the one in Proposition 5.7. The homogeneous bundles $\mathcal{Q}_k$, $\mathcal{N}_{Y_-|\mathrm{E}_7(k)}$ (respectively $\mathcal{Q}'_{s(k)}$, $\mathcal{N}_{Y_+|\mathrm{E}_7(k)}$) are completely determined by their fibers at a point, which are $P^\perp$-modules. One can check that these modules are isomorphic by computing their $H^\perp$-weights, for instance by using `SageMath` software. □

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Declarations

Conflict of interest We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work.

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