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Optimal control of stochastic differential equations with dynamical boundary conditions

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In this paper we investigate the optimal control problem for a class of stochastic Cauchy evolution problem with non standard boundary dynamic and control. The model is composed by an infinite dimensional dynamical system coupled with a finite dimensional dynamics, which describes the boundary conditions of the internal system. In other terms, we are concerned with non standard boundary conditions, as the value at the boundary is governed by a different stochastic differential equation.

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1 Setting of the problem

Our model is a one dimensional semilinear diffusion equation in a confined system, where interactions with extremal points cannot be disregarded. The extremal points have a mass and the boundary potential evolves with a specific dynamic. Stochasticity enters through fluctuations and random perturbations both in the inside as on the boundaries; in particular, in our model we assume that the control process is perturbed by a noisy term.

There is a growing literature concerning such problems; we shall mention the paper [2] where a problem in a domain $\mathcal{O} \subset \mathbb{R}^n$ is concerned; the authors cite as an example an SPDE with stochastic perturbations which appears in connection with random fluctuations of the atmospheric pressure field. As opposite to ours, however, that paper is not concerned with control problems. Quite recently, the authors became aware of the paper [1] where a different application to some generalized Lamb model is proposed.

The internal dynamic is described by a stochastic evolution problem in the unit interval $D = [0, 1]$

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x,u(t,x)) + g(t,x,u(t,x))\dot{W}(t,x) \tag{1}$$

which we write as an abstract evolution problem on the space $L^2(0,1)$

$$\mathrm{d}u(t) = A_m u(t) + F(t,u(t))\,\mathrm{d}t + G(t,u(t))\,\mathrm{d}W(t), \tag{2}$$

where the leading operator is $A_m = \partial_x^2$ with domain $D(A_m) = H^2(0,1)$. We assume that f and g are real valued mappings, defined on $[0,T]\times[0,1]\times\mathbb{R}$, which verify some boundedness and Lipschitz continuity assumptions.

The boundary dynamic is governed by a finite dimensional system which follows a (ordinary, two dimensional) stochastic differential equation

$$\partial_t v_i(t) = -b_i v_i(t) + \partial_\nu u(t,i) + h_i(t)\dot{V}_i(t), \qquad i = 0,1$$

where b_i are positive numbers and $h_i(t)$ are bounded, measurable functions; ∂_ν is the normal derivative on the boundary, and coincides with $(-1)^i\partial_x$ for $i=0,1$. For notational semplicity, we introduce the 2×2 diagonal matrices $B = \mathrm{diag}(b_0,b_1)$ and $h(t) = \mathrm{diag}(h_0(t),h_1(t))$. There is a constraint

$$Lu = v$$

which we interpret as the operator evaluating boundary conditions; the system is coupled by the presence, in the second equation, of a feedback term C that is an unbounded operator

$$Cu = \begin{pmatrix} \partial_x u(0) \\ -\partial_x u(1) \end{pmatrix}.$$

The idea is to rewrite the problem in an abstract form for the vector $\mathbf{u} = \begin{pmatrix} u(\cdot) \\ v \end{pmatrix}$ on the space $\mathcal{X} = L^2(0,1)\times\mathbb{R}^2$, that is

$$\begin{cases} \mathrm{d}\mathbf{u} = \mathbb{A}\mathbf{u}(t) + \mathbb{F}(t,\mathbf{u}(t))\,\mathrm{d}t + \mathbb{G}(t,\mathbf{u}(t))\,\mathrm{d}\mathbb{W}(t) \\ \mathbf{u}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases} \tag{3}$$

Our main concern is to study spectral properties of the matrix operator

$$\mathbb{A} = \begin{pmatrix} A_m & 0 \\ C & B \end{pmatrix}$$

on the domain

$$D(\mathbb{A}) = \{\mathbf{u} \in D(A_m)\times\mathbb{R}^2 \,:\, Lu = v\}.$$

Theorem 1. *$\mathbb{A}$ is the infinitesimal generator of a strongly continuous, analytic semigroup of contractions $\mathrm{e}^{t\mathbb{A}}$, self-adjoint and compact.*

We shall prove the above theorem in Section 2. Further, we shall prove that $\mathbb{A}$ is a self-adjoint operator with compact resolvent, which implies that the generated semigroup is Hilbert-Schmidt. Moreover, we can characterize the complete, orthonormal system of eigenfunctions associated to $\mathbb{A}$.

Let us fix a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$; on this space we define $W(t)$, that is a space-time Wiener process taking values in X and $V(t) =$

$(V_1(t), V_2(t))$, that is a $\mathbb{R}^2$-valued Wiener process, such that $W(t,x)$ and $V(t)$ are independent.

As a corollary to Theorem 1, using standard results for infinite dimensional stochastic differential equations, compare [3, Theorem 7.4], we obtain the following existence result

Theorem 2. *For any initial condition* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in X \times \mathbb{R}^2$ *there exists a unique process* $\mathbf{u} \in L^2_F(0,T; X \times \mathbb{R}^2)$ *such that*

$$\mathbf{u}(t) = \mathrm{e}^{t\mathbb{A}} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t \mathrm{e}^{t\mathbb{A}} \mathbb{F}(\mathbf{u}(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{t\mathbb{A}} \mathbb{G}(\mathbf{u}(s))\,\mathrm{d}\mathbb{W}(s),$$

that is by definition a mild solution of (3).

The abstract semigroup setting we propose in this paper allows us to obtain an optimal control synthesis for the above evolution problem with boundary control and noise. This means that we assume a boundary dynamics of the form:

$$\partial_t v(t) = bv(t) - \partial_\nu u(t,\cdot) + h(t)[z(t) + \dot{V}(t)] \tag{4}$$

where $z(t)$ is the control process and takes values in a given subset of $\mathbb{R}^2$.

As before, we can rewrite the system – defined by the internal evolution problem (1) and the dynamical boundary conditions described by (4) – in the following abstract form

$$\begin{cases} \mathrm{d}\mathbf{u}^z_t = \mathbb{A}\mathbf{u}^z_t\,\mathrm{d}t + \mathbb{F}(t, \mathbf{u}^z_t)\,\mathrm{d}t + \mathbb{G}(t, \mathbf{u}^z_t)[Pz_t\,\mathrm{d}t + \mathrm{d}\mathbf{W}_t] \\ \mathbf{u}_{t_0} = \mathbf{u}_0. \end{cases} \tag{5}$$

Here, $P : \mathbb{R}^2 \to \mathcal{X}$ denotes the immersion of the boundary space in the product space $\mathcal{X} = L^2(0,1) \times \mathbb{R}^2$.

The aim is to choose a control process z, within a set of admissible controls, in such a way to minimize a cost functional of the form

$$J(t_0, u_0, z) = \mathbb{E} \int_{t_0}^T \lambda(s, \mathbf{u}^z_s, z_s)\,\mathrm{d}s + \mathbb{E}\phi(\mathbf{u}^z_T) \tag{6}$$

where λ and ϕ are given real functions. In our setting, altough the control lives in a finite dimensional space, we obtain an abstract optimal control problem in infinite dimensions. Such type of problems has been exhaustively studied by Fuhrman and Tessitore in [8]. The control problem is understood in the usual weak sense (see [7]). We prove that if f and g are sufficiently regular then the abstract control problem, under suitable assumptions on λ and ϕ, can be solved and we can characterize optimal controls by a feedback law (see Theorem 17 and compare Theorem 7.2 in [8]).

Theorem 3. *In our assumptions, there exists an admissible control* $\{\bar{z}_t,\ t \in [0,T]\}$ *taking values in a bounded subset of* $\mathbb{R}^2$*, such that the closed loop equation:*

$$\begin{cases} d\overline{\mathbf{u}}_\tau = \mathbb{A}\overline{\mathbf{u}}_\tau\,d\tau + \mathbb{G}(\overline{\mathbf{u}}_\tau)P\Gamma(\overline{\mathbf{u}}_\tau, \mathbb{G}(\overline{\mathbf{u}}_\tau)^*\nabla_x v(\tau, \overline{\mathbf{u}}_\tau))\,d\tau \\ \qquad\qquad + \mathbb{F}(\overline{\mathbf{u}}_\tau)\,d\tau + \mathbb{G}(\overline{\mathbf{u}}_\tau)\,d\mathbf{W}_\tau, \qquad \tau \in [t_0, T], \\ \overline{\mathbf{u}}_{t_0} = \mathbf{u}_0 \in \mathcal{X}. \end{cases} \tag{7}$$

admits a solution $\overline{\mathbf{u}}$ *and the couple* $(\overline{z}, \overline{\mathbf{u}})$ *is optimal for the control problem.*

Stochastic boundary value problems are already present in the literature, see the paper [11] and the references therein; in those papers, the approach to the solution of the system is more similar to that in [2]. We also need to mention the paper [5] for a one dimensional case where the boundary values are set equal to a white noise mapping.

2 Generation properties

Let $X = L^2(0,1)$ be the Hilbert space of square integrable real valued functions defined on $D = [0,1]$ and $\mathcal{X} = X \times \mathbb{R}^2$. In this section we consider the following initial-boundary value problem on the space $\mathcal{X}$

$$\begin{cases} \frac{d}{dt}u(t) = A_m u(t) \\ v(t) = Lu(t) \\ \frac{d}{dt}v(t) = Bv(t) - Cu(t) \\ u(0) = u_0 \in X, \quad v(0) = v_0 \in \mathbb{R}^2. \end{cases} \tag{8}$$

In the above equation, A_m is an unbounded operator with maximal domain

$$A_m = \partial_x^2, \quad D(A_m) = H^2(0,1);$$

B is a diagonal matrix with negative entries $(-b_0, -b_1)$.

Let $C : D(C) \subset X \to \partial X$ the feedback operator, defined on $D(C) = H^1(0,1)$ as

$$Cu = \begin{pmatrix} \partial_x u(0) \\ -\partial_x u(1) \end{pmatrix}.$$

The boundary evaluation operator L is the mapping $L : X \to \mathbb{R}^2$ given by

$$Lu = \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}.$$

Its inverse is the Dirichlet mapping $D_\lambda^{A,L} : \mathbb{R}^2 \to D(A_m)$

$$D_\lambda^{A,L}\phi = u(x) \in D(A_m) : \qquad \begin{cases} (\lambda I - A_m)u(x) = 0, \\ Lu = \phi. \end{cases}$$

As proposed in [10], we define a *mild solution* of (8) a function $\mathbf{u} \in C([0,T];\mathcal{X})$ such that

$$\begin{cases} u(t) = u_0 + A_m \int_0^t u(s)\,\mathrm{d}s, & t \in [0,T] \\ v(t) = v_0 + B\int_0^t v(s)\,\mathrm{d}s + C\int_0^t u(s)\,\mathrm{d}s. \end{cases}$$

In order to use semigroup theory to study equation (8), we consider a matrix operator describing the evolution with feedback on the boundary

$$\mathbb{A} = \begin{pmatrix} A_m & 0 \\ C & B \end{pmatrix}$$

on the domain

$$D(\mathbb{A}) = \{\mathbf{u} \in D(A_m) \times \mathbb{R}^2 \,:\, Lu = v\}.$$

Then a mild solution for equation (8) exists if and only if $\mathbb{A}$ is the generator of a strongly continuous semigroup.

The above definition of the domain $D(\mathbb{A})$ puts in evidence the relation between the first and the second component of the vector $\mathbf{u}$. There is a different characterization that is sometimes useful in the applications.

Let us define the operator A_0 as $A_0 = A_m$ on $D(A_0) = \{u \in D(A_m) \,:\, Lu = 0\}$. We can then write the domain of $\mathbb{A}$ as

$$D(\mathbb{A}) = \{\mathbf{u} \in D(A_m) \times \partial X \,:\, u - D_0^{A,L} v \in D(A_0)\}.$$

The operator $\mathbb{A}$ can be decomposed as the product

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & -D_0^{A,L} \\ C & I \end{pmatrix}$$

Then, according to `Engel` [6], $\mathbb{A}$ is called a one-sided K-coupled matrix-valued operator.

Proof of Theorem 1

In this section we apply form theory in order to prove generation property of the operator $\mathbb{A}$, compare the monograph [13].

Proposition 4. $\mathbb{A}$ *is the infinitesimal generator of a strongly continuous, analytic semigroup of contractions, self-adjoint and compact.*

We will give the proof in two steps. First of all we will consider the following form:

$$a(\mathbf{u}, \mathbf{v}) = \int_0^1 u'(x) v'(x)\,\mathrm{d}x + b_0\, u(0)\, v(0) + b_1\, u(1)\, v(1)$$

on the domain

$$V = \left\{\mathbf{u} = (u, \alpha) \in H^1(0,1) \times \mathbb{R}^2 \mid u(0) = \alpha_0, u(1) = \alpha_1\right\}$$

and we will show that it is densely defined, closed, positive, symmetric and continue. Moreover, the operator associated with the form a is $(\mathbb{A}, D(\mathbb{A}))$ defined above. According to [13], this implies that the operator $\mathbb{A}$ is self-adjoint and generates a contraction semigroup $\mathrm{e}^{t\mathbb{A}}$ on $\mathcal{X}$ that is analytic of angle $\frac{\pi}{2}$. Then we will show the self-adjointness and the compactness of the semigroup $\mathrm{e}^{t\mathbb{A}}$. To see this, we will refer to [9].

Let us begin with the properties of the form a.

Lemma 5. *The form a is densely defined, closed, positive, symmetric and continue.*

Proof. By assumption, since b_0 and b_1 are positive real numbers, it follows that in particular a is symmetric and positive.

It is clear that V is a linear subspace of $\mathcal{X}$. Observe that V is dense in $\mathcal{X}$ if any $\mathbf{u} \in \mathcal{X}$ can be approximated with elements of V. Consider $(u, \alpha) \in L^2[0,1] \times \mathbb{R}^2$.

Since $C_c^\infty[0,1]$ is dense in $L^2(0,1)$ it follows that for all $\varepsilon > 0$ there exists $v \in C_c^\infty[0,1]$ such that

$$|u - v|_{L^2[0,1]} \le \frac{\varepsilon}{3}.$$

Now let $\rho_0(x)$ be a symmetric function in $C_c^\infty(\mathbb{R})$ with support in $B_\varepsilon(0)$, $\rho_0(0) = 1$ and $\int_{\mathbb{R}} \rho_0(x)\,\mathrm{d}x = \varepsilon/3$. Finally, let $\rho_1(x) = \rho_0(x-1)$. Then, if we define the function $\rho = v + \alpha_0\,\rho_0\Big|_{[0,1]} + \alpha_1\,\rho_1\Big|_{[0,1]}$, we have:

$$\begin{aligned} |u - \rho|_{L^2[0,1]} &\le |u - v|_{L^2[0,1]} + |\alpha_0\rho_0|_{L^2[0,1]} + |\alpha_1\rho_1|_{L^2[0,1]} \le \\ &\le \max\{1, \alpha_0, \alpha_1\}\,\varepsilon. \end{aligned}$$

Morever, $\rho(0) = \alpha_0$ and $\rho(1) = \alpha_1$. Thus

$$|(u, \alpha) - (\rho, \rho(0), \rho(1))|_{\mathcal{X}} \le M\varepsilon$$

for a suitable M. This shows that V is dense in $\mathcal{X}$.

In order to check closedness and continuity of a, observe first that the norm induced by a on the space V is equivalent to the norm given by the inner product

$$(\mathbf{u}, \mathbf{v})_V = \int_0^1 [u'(x)v'(x) + u(x)v(x)]\,\mathrm{d}x + u(1)v(1) + u(0)v(0).$$

In fact, if we set $b = b_0 + b_1$, we have

$$\|\mathbf{u}\|_a = \sqrt{a(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_V^2}$$

so that

$$\|\mathbf{u}\|_a^2 \le 2\,\|u\|_{H^1(0,1)}^2 + 2b\left[u(0)^2 + u(1)^2\right] \le \max\{2, 2b\}\,\|\mathbf{u}\|_V^2.$$

Now observe that V becomes a Hilbert space when equipped with the inner product defined above since V is a closed subspace of $H^1(0,1) \times \mathbb{R}^2$. Then a is closed.

Finally, a is continuous. To see this, take $\mathbf{u}, \mathbf{v} \in V$; then

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\le \int_0^1 |u'(x)v'(x)|\,\mathrm{d}x + b\,[|u(0)|\,|v(0)| + |u(1)|\,|v(1)|] \\ &\le \|u\|_{H^1(0,1)}\,\|v\|_{H^1(0,1)} + b\,[|u(0)|\,|v(0)| + |u(1)|\,|v(1)|] \\ &\le \|\mathbf{u}\|_V\,\|\mathbf{v}\|_V \le M\,\|\mathbf{u}\|_a\,\|\mathbf{v}\|_a \end{aligned}$$

by the Cauchy-Schwartz inequality. □

Lemma 6. *The operator associated with a is $(\mathbb{A}, D(\mathbb{A}))$ defined above.*

Proof. Denote by $(\mathcal{C}, D(\mathcal{C}))$ the operator associated with a. By definition, $\mathcal{C}$ is given by

$$\begin{aligned} D(\mathcal{C}) &= \{\mathbf{f} \in V \mid \exists \mathbf{g} \in \mathcal{X} \text{ s.t. } a(\mathbf{f}, \mathbf{g}) = (\mathbf{g}, \mathbf{h})_{\mathcal{X}} \forall \mathbf{h} \in V\} \\ \mathcal{C}\mathbf{f} &= -\mathbf{g}. \end{aligned}$$

Let us first show that $\mathbb{A} \subset \mathcal{C}$. Take $\mathbf{f} \in D(\mathbb{A})$. Then for all $\mathbf{h} \in V$

$$\begin{aligned} a(\mathbf{f}, \mathbf{h}) &= \int_0^1 f'(x)h'(x)\,\mathrm{d}x + b_0 f(0)h(0) + b_1 f(1)h(1) \\ &= f'(x)h(x)|_0^1 - \int_0^1 f''(x)h(x)\,\mathrm{d}x + b_0 f(0)h(0) + b_1 f(1)h(1) \\ &= f'(1)h(1) - f'(0)h(0) - \int_0^1 f''(x)h(x)\,\mathrm{d}x + b_0 f(0)h(0) + b_1 f(1)h(1). \end{aligned}$$

At the same time, if we set $\alpha = (f(0), f(1))$, $\beta = (h(0), h(1))$, we have

$$\begin{aligned} (\mathbb{A}\mathbf{f}, \mathbf{h}) = (Af, h)_{L^2(0,1)} &+ (Cf + B\alpha, \beta)_{\mathbb{R}^2} = \\ &= \int_0^1 f''(x)h(x)\,\mathrm{d}x + f'(0)h(0) - f'(1)h(1) \\ &\quad - b_0 f(0)h(0) - b_1 f(1)h(1) = -a(\mathbf{f}, \mathbf{g}). \end{aligned}$$

The last equality shows that $\mathbb{A} \subset \mathcal{C}$.

To check the converse inclusion $\mathcal{C} \subset \mathbb{A}$ take $\mathbf{f} \in D(\mathcal{C})$. By definition, there exists $\mathbf{g} \in \mathcal{X}$ such that

$$a(\mathbf{f}, \mathbf{h}) = (\mathbf{g}, \mathbf{h})_{\mathcal{X}}, \quad \forall \mathbf{h} \in V$$

that is,

$$\int_0^1 f'(x)h'(x)\,\mathrm{d}x = \int_0^1 g(x)h(x)\,\mathrm{d}x.$$

Now choose $\mathbf{h} = (h, \alpha) \in V$ such that the function h belongs to $H_0^1(0,1)$ (the existence of such a function is ensured by the continuous embedding of $H_0^1(0,1)$ in $H^1(0,1)$). Then by the last equality we can derive that $f' \in H^1(0,1)$ and g is the weak derivative of f': it follows that $f' \in H^1(0,1)$ and we conclude that $f \in H^2(0,1)$. Integrating by parts as in the proof of the first inclusion we see that

$$\begin{aligned} a(\mathbf{f}, \mathbf{h}) &= \int_0^1 f'(x)h'(x)\,\mathrm{d}x + b_0 f(0)h(0) + b_1 f(1)h(1) \\ &= f'(x)h(x)|_0^1 - \int_0^1 f''(x)h(x)\,\mathrm{d}x + b_0 f(0)h(0) + b_1 f(1)h(1) \\ &= (-\mathbb{A}\mathbf{f}, \mathbf{h}) = (\mathbf{g}, \mathbf{h}), \quad \forall \mathbf{h} \in V. \end{aligned}$$

This implies that $\mathbb{A}\mathbf{f} = -\mathbf{g}$, and the proof is complete. □

Corollary 7. *The operator* $(\mathbb{A}, D(\mathbb{A}))$ *is self-adjoint and dissipative. Moreover it has compact resolvent.*

Proof. The self-adjointness of $\mathbb{A}$ follows by [13] (Proposition 1.24) and the dissipativity is obsvious. Since $D(\mathbb{A}) \subset H^2(0,1) \times \mathbb{R}^2$, the operator $\mathbb{A}$ has compact resolvent and the claim follows. □

Taking into account the above corollary, it follows that $\mathbb{A}$ generates a contraction semigroup $(\mathrm{e}^{t\mathbb{A}})_{t\geq 0}$ on $\mathcal{X}$ that is analytic of angle $\pi/2$ and self-adjoint. Finally, by [9, Corollary XIX.6.3] we obtain that $\mathrm{e}^{t\mathbb{A}}$ is compact for all $t > 0$.

Thus we have just proved Proposition 4.

Remark 1. *By the Spectral Theorem [9, Chapter XIX, Corollary 6.3] it follows that there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{X}$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of real negative numbers $\lambda_n \leq 0$, such that $e_n \in D(\mathbb{A})$, $\mathbb{A}e_n = \lambda_n e_n$ and $\lim_{n\to\infty} \lambda_n = -\infty$. Moreover, $\mathbb{A}$ is given by*

$$\mathbb{A}\mathbf{u} = \sum_{n=1}^{\infty} \lambda_n (\mathbf{u}, e_n) e_n, \quad \mathbf{u} \in D(\mathbb{A})$$

and

$$\mathrm{e}^{t\mathbb{A}}\mathbf{u} = \sum_{n=1}^{\infty} e^{\lambda_n t} (\mathbf{u}, e_n) e_n, \quad \mathbf{u} \in \mathcal{X}.$$

2.1 Spectral properties of the matrix operator

We shall now apply Theorem 2.5 in `Engel`[6] in order to describe the spectrum of $\mathbb{A}$. According to that result

$$\sigma(\mathbb{A}) \subseteq \sigma(A_0) \cup \sigma(B) \cup S \tag{9}$$

where

$$S = \{\lambda \in \rho(A_0) \cap \rho(B) \,:\, \mathrm{Det}(F(\lambda)) = 0\}. \tag{10}$$

The matrix $F(\lambda)$ is defined as

$$F(\lambda) = I - (\lambda - B) L_\lambda K_\lambda R(\lambda, B)$$

where the operators L_λ and K_λ are given by

$$L_\lambda = -BR(\lambda, B)R(0, B)C, \qquad K_\lambda = -A_0 R(\lambda, A_0) D_0^{A,L}.$$

Notice that the matrix $F(\lambda)$ can also be written as

$$F(\lambda) = I + CA_0 R(\lambda, A_0) D_0^{A,L} R(\lambda, B).$$

Remark 2. *In case when the feedback operator matrix C is identically zero, the above construction implies that $S = \emptyset$.*

Determining the set S

In the following, we construct explicitly the set S. The idea is to construct the matrix $F(\lambda)$ and compute its determinant.

We have to distinguish two cases. If $\lambda < 0$ we have

$$\mathrm{Det}(F(\lambda)) = 1 + \sqrt{-\lambda}\frac{\cos(\sqrt{-\lambda})}{\sin(\sqrt{-\lambda})}\left(\frac{1}{\lambda + b_0} + \frac{1}{\lambda + b_1}\right) + \frac{\lambda}{(\lambda + b_0)(\lambda + b_1)}$$

We note that the equation $\mathrm{Det}(F(\lambda)) = 0$ has infinite solutions $\{\lambda_j\}_{j\in\mathbb{N}}$ and every λ_j belongs to the interval $(-\pi^2(j+1)^2, -\pi^2 j^2)$.

Each λ_j is eigenvalue of the operator $\mathbb{A}$ corresponding to the eigenfunction $\phi_j = (e_j(x), e_j(0), e_j(1))$ where

$$e_j(x) = \frac{\sqrt{-\lambda_j}B_j}{b_0+\lambda_j}\cos\sqrt{-\lambda_j}x + B_j \sin\sqrt{-\lambda_j}x.$$

for a normalizing constant $0 < B_j < \frac{1+\sqrt{-\lambda_j}}{-1+\sqrt{-\lambda_j}}$.

If $\lambda > 0$ then

$$\mathrm{Det}(F(\lambda)) = 1 + \sqrt{\lambda}\frac{\left(1+\mathrm{e}^{2\sqrt{\lambda}}\right)}{\left(-1+\mathrm{e}^{2\sqrt{\lambda}}\right)}\left(\frac{1}{b_0+\lambda}+\frac{1}{b_1+\lambda}\right)+\frac{\lambda}{(b_0+\lambda)(b_1+\lambda)}.$$

We note that $\mathrm{Det}(F(\lambda)) > 0$ for every $\lambda > 0$. This means that there are not elements λ strictly positive in S. Moreover the eigenvalues of $\mathbb{A}$ in S are all negative.

Remark 3. *It is possible to verify directly with some computation that the eigenvalues of A are not eigenvalues of $\mathbb{A}$.*

Further, the same happens in general with the eigenvalues of B, except in case b_0 and b_1 satisfy an explicit relation. In any case, also if b_0 and b_1 happen to belong to $\sigma(\mathbb{A})$, they are in a finite number and do not affect its behaviour.

Therefore, with no loss of generality, in the following we may and do assume that all the eigenvalues of $\mathbb{A}$ are contained in S.

Theorem 8. *In the above assumptions the semigroup $\mathrm{e}^{t\mathbb{A}}$ is Hilbert-Schmidt, that is,*

$$\sum_{i=1}^{\infty} |\mathrm{e}^{t\mathbb{A}}\phi_i|^2_{L^2(0,1)\times\mathbb{R}^2} < \infty \tag{11}$$

for any orthonormal basis $\{\phi_i\}$ of $L^2(0,1)\times\mathbb{R}^2$.

Proof. In order to prove that the semigroup $\mathrm{e}^{t\mathbb{A}}$ is Hilbert-Schmidt, it is enough verify inequality (11) for an orthonormal basis. Let $\{\phi_i\}$ be the orthonormal sequence of eigenfunctions of the operator $\mathbb{A}$ described in Remark 1. Then

$$\sum_{i=1}^{\infty} |e^{t\mathbb{A}}\phi_i|^2_{L^2(0,1)\times\mathbb{R}^2} = \sum_{i=1}^{\infty} e^{2t\lambda_i}$$

where λ_i are the eigenvalues of the operator $\mathbb{A}$. By (9) it follows that

$$\sum_{i=1}^{\infty} e^{2t\lambda_i} \le \sum_{i:\,\lambda_i\in\sigma(A)} e^{2t\lambda_i} + \sum_{i:\,\lambda_i\in\sigma(B)} e^{2t\lambda_i} + \sum_{i:\,\lambda_i\in S} e^{2t\lambda_i}.$$

But, by Remark 3 we have that

$$\sum_{i=1}^{\infty} e^{2t\lambda_i} \le \sum_{i:\,\lambda_i\in\sigma(B)} e^{2t\lambda_i} + \sum_{i:\,\lambda_i\in S} e^{2t\lambda_i} < \infty$$

because the first of the last two series is a finite sum while the second one converges since the eigenvalues λ_i in S are asymptotic to $-\pi^2 i^2$.

$\square$

3 The abstract problem

In this section we are concerned with problem (3): we introduce the relevant assumptions and formulate the main existence and uniqueness result for its solution.

Let $\mathbf{W} = (W, V)$ be the Wiener process taking values in $= L^2(0,1) \times \mathbb{R}^2$. We denote $\{\mathcal{F}_t,\ t \in [0,T]\}$ the natural filtration of $\mathbf{W}$, augmented with the family $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}_T$:

$$\mathcal{F}_t = \sigma(\mathbf{W}(s)\ :\ s \in [0,t]) \vee \mathcal{N}.$$

The filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions.

Define $\mathbb{F} : [0,T] \times \mathcal{X} \to \mathcal{X}$ such that, for every $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{X}$,

$$\mathbb{F}(t, \mathbf{u}) = \mathbb{F}\left(t, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} F(t,u) \\ 0 \end{pmatrix}$$

where $F(t,u)(\xi) = f(t, \xi, u(\xi))$.

Let $\mathbb{G}$ be the mapping from $[0,T] \times \mathcal{X}$ with values into $L(\mathcal{X})$ (i.e. the space of linear operators from $\mathcal{X}$ to $\mathcal{X}$) such that, for every $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y \\ \eta \end{pmatrix} \in \mathcal{X}$,

$$\mathbb{G}\left(t, \begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot \begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} G_1(t,u)\, y \\ G_2(t,v)\, \eta \end{pmatrix}$$

where

$$(G_1(t,u)\, y)(\xi) = g(t, \xi, u(\xi)) y(\xi) \quad \text{and} \quad (G_2(t,v) \cdot \eta) = h(t)\, \eta;$$

we stress that h is a diagonal matrix.

Therefore, we are concerned with the following abstract problem

$$\begin{cases} \mathrm{d}\mathbf{u}_t = \mathbb{A}\mathbf{u}_t\, \mathrm{d}t + \mathbb{F}(t, \mathbf{u}_t)\, \mathrm{d}t + \mathbb{G}(t, \mathbf{u}_t) \mathrm{d}\mathbf{W}_t \\ \mathbf{u}_{t_0} = \mathbf{u}_0 \end{cases} \tag{12}$$

on which we formulate the following assumptions.

Assumption 9.

(i) $f : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$, *is a measurable mapping, bounded and Lipschitz continuous in the last component*

$$|f(t,x,u)| \le K, \quad |f(t,x,u) - f(t,x,v)| \le L|u - v|.$$

for every $t \in [0,T]$, $x \in [0,1]$, $u, v \in \mathbb{R}$.

(ii) $g : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$, *is a measurable mapping such that*

$$|g(t,x,u)| \le K, \quad |g(t,x,u) - g(t,x,v)| \le L|u-v|$$

for every $t \in [0,T]$, $x \in [0,1]$, $u,v \in \mathbb{R}$.

(iii) $h : [0,T] \to M(2,2)$ *is a bounded measurable mapping verifying* $|h(t)| \le K$ *for every* $t \in [0,T]$.

Existence and uniqueness of solutions for (12) is a standard result in the literature, see for instance the monograph [3]. In order to apply the known results, we shall verify that the nonlinear coefficients $\mathbb{F}$ and $\mathbb{G}$ satisfy suitable Lipschitz continuous conditions. That will be enough to prove the existence of a *mild solution* which is a process $\mathbf{u}_t$ adapted to the filtration $\mathcal{F}_t$ satisfying the following integral equation

$$\mathbf{u}_t = e^{t\mathbb{A}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{F}(s,\mathbf{u}_s)\,\mathrm{d}s + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{G}(s,\mathbf{u}_s)\,\mathrm{d}\mathbf{W}s. \tag{13}$$

Proposition 10. *Under Assumptions 9(i)–(iii), the following hold:*

1. *the mapping* $\mathbb{F} : \mathcal{X} \to \mathcal{X}$ *is measurable and satisfies, for some constant* $L > 0$,
$$|\mathbb{F}(t,\mathbf{u}) - \mathbb{F}(t,\mathbf{v})|_{\mathcal{X}} \le L|\mathbf{u}-\mathbf{v}|_{\mathcal{X}} \quad \mathbf{u},\mathbf{v} \in \mathcal{X}.$$
2. $\mathbb{G}$ *is a mapping* $[0,T] \times \mathcal{X} \to L(\mathcal{X})$ *such that*
   a. *for every* $\mathbf{v} \in \mathcal{X}$ *the map* $\mathbb{G}(\cdot,\cdot)\mathbf{v} : [0,T] \times \mathcal{X} \to \mathcal{X}$ *is measurable,*
   b. $e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}) \in L_2(\mathcal{X})$ *–the space of Hilbert Schmidt operators from* $\mathcal{X}$ *to* $\mathcal{X}$*– for every* $s > 0$, $t \in [0,T]$ *and* $\mathbf{u} \in \mathcal{X}$, *and*
   c. *for every* $s > 0$, $t \in [0,T]$ *and* $\mathbf{u}.\mathbf{v} \in \mathcal{X}$ *we have*
$$|e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})|_{L_2(\mathcal{X})} \le L\,s^{-1/2}\,(1 + |\mathbf{u}|_{\mathcal{X}}), \tag{14}$$
$$|e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}) - e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{v})|_{L_2(\mathcal{X})} \le L\,s^{-1/2}|\mathbf{u}-\mathbf{v}|_{\mathcal{X}}, \tag{15}$$
$$|\mathbb{G}(t,\mathbf{u})|_{L(\mathcal{X})} \le L\,(1 + |\mathbf{u}|_{\mathcal{X}}), \tag{16}$$
   for a constant $L > 0$.

Proof. 1. We have, for $\mathbf{u} = \begin{pmatrix} u \\ x \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v \\ y \end{pmatrix}$

$$|\mathbb{F}(t,\mathbf{u}) - \mathbb{F}(t,\mathbf{v})|_{\mathcal{X}} = |F(t,u) - F(t,v)|_X \le L|u-v|_X \le L|\mathbf{u}-\mathbf{v}|_{\mathcal{X}}.$$

2. Condition (16) follows from the definition of $\mathbb{G}$ and the Assumptions 9 (ii)-(iii) on g and h.

   Now we prove condition (14). Let $\{\phi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{X}$. Then

$$\begin{aligned}
|e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})|^2_{L_2(\mathcal{X})} &= \sum_{j,k} |< e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})\phi_j, \phi_k >|^2_{\mathcal{X}} \\
&= \sum_{j,k} |< \mathbb{G}(t,\mathbf{u})\phi_j, e^{s\mathbb{A}}\phi_k >|^2_{\mathcal{X}} \\
&\le |\mathbb{G}(t,\mathbf{u})|^2_{L(\mathcal{X})}\,|e^{s\mathbb{A}}|^2_{L_2(\mathcal{X})} \le L^2(1 + |\mathbf{u}|^2_{\mathcal{X}})|e^{s\mathbb{A}}|^2_{L_2(\mathcal{X})}.
\end{aligned}$$

Using Theorem 8,

$$|e^{s\mathbb{A}}|^2_{L_2(\mathcal{X})} \approx \sum_{n=1}^{\infty} \mathrm{e}^{-2sn^2} \approx \frac{1}{\sqrt{s}}$$

where $f(t) \approx g(t)$ means that $f(s)/g(s) = O(1)$ as $s \to 0$; this verifies (14).

In order to prove the last statement (15), we take the orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ consisting of eigenvectors of $\mathbb{A}$ (see Remark 1). We recall that $\phi_k = (e_k(x), e_k(0), e_k(1))$ where

$$e_k(x) = B_k \frac{\sqrt{-\lambda_k}}{b_0 + \lambda_k} \cos \sqrt{-\lambda_k}x + B_k \sin \sqrt{-\lambda_k}x.$$

We have

$$|e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}) - e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{v})|^2_{L_2(\mathcal{X})} = \sum_{j,k} | < e^{s\mathbb{A}}[\mathbb{G}(t,\mathbf{u}) - \mathbb{G}(t,\mathbf{v})]\phi_j, \phi_k > |^2_{\mathcal{X}} =$$

$$= \sum_{j,k} | < \mathbb{G}(t,\mathbf{u}) - \mathbb{G}(t,\mathbf{v})\phi_j, e^{s\mathbb{A}}\phi_k > |^2_{\mathcal{X}} = \sum_k e^{2s\lambda_k} \, |\mathbb{G}(t,\mathbf{u}) - \mathbb{G}(t,\mathbf{v})\phi_k|^2.$$

But, for $\mathbf{u} = \begin{pmatrix} u \\ x \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v \\ y \end{pmatrix}$, by the definition of the operator $\mathbb{G}$, we have

$$|\mathbb{G}(t,\mathbf{u}) - \mathbb{G}(t,\mathbf{v})\phi_k|^2_{\mathcal{X}} = \int_0^1 |g(t,x,u(x)) - g(t,x,v(x))|^2 |e_k(x)|^2 dx \leq$$

$$\leq \int_0^1 K^2 |u(x) - v(x)|^2 dx \leq K^2 |\mathbf{u} - \mathbf{v}|^2_{\mathcal{X}}$$

since the function g is Lipschitz and $|e_k(x)| \leq B_k$ is uniformly bounded in k. Consequently

$$|e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}) - e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{v})|_{L_2(\mathcal{X})} \leq \left(\sum_k e^{2t\lambda_k} \right)^{1/2} K |\mathbf{u} - \mathbf{v}|_{\mathcal{X}} \leq$$

$$\leq |e^{s\mathbb{A}}|_{L_2(\mathcal{X})} K |\mathbf{u} - \mathbf{v}|_{\mathcal{X}}$$

which concludes the proof.

□

Proposition 11. *Under the assumptions 9, for every $p \in [2,\infty)$ there exists a unique process $\mathbf{u} \in L^p(\Omega; C([0,T];\mathcal{X}))$ solution of (12).*

Proof. We can apply Theorem 5.3.1 in [4]. In fact by Proposition 4 the operator $\mathbb{A}$ generates a strongly continuous semigroup $\{e^{t\mathbb{A}}\}$ of bounded linear operators in the Hilbert space $\mathcal{X}$. Moreover, for this theorem to apply we need to verify that coefficients $\mathbb{F}$ and $\mathbb{G}$ satisfy conditions (14)—(16), which follows from Proposition 10.

□

4 Stochastic control problem

After some preliminaries, in this section we are concerned with an abstract control problem in infinite dimension. We settle the problem in the framework of weak control problems (see [7]).

We aim to control the evolution of the system by the boundary. This means that we assume a boundary dynamic of the form:

$$\partial_t v(t) = bv(t) - \partial_\nu u(t,\cdot) + h(t)[z(t) + \dot{V}(t)] \tag{17}$$

where $z(t)$ is the control process. We require that $z \in L^2(\Omega \times [0,T]; \mathbb{R}^2)$.

As in the previous section we can write the system

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x,u(t,x)) + g(t,x,u(t,x))\dot{W}(t,x) \\ \partial_t v(t) = bv(t) - \partial_\nu u(t,\cdot) + h(t)[z(t) + \dot{V}(t)] \end{cases} \tag{18}$$

in the following abstract form

$$\begin{cases} \mathrm{d}\mathbf{u}_t^z = \mathbb{A}\mathbf{u}_t^z \,\mathrm{d}t + \mathbb{F}(t,\mathbf{u}_t^z)\,\mathrm{d}t + \mathbb{G}(t,\mathbf{u}_t^z)[Pz_t\,\mathrm{d}t + \mathrm{d}\mathbf{W}_t] \\ \mathbf{u}_{t_0} = \mathbf{u}_0 \end{cases} \tag{19}$$

where $P : \mathbb{R}^2 \to \mathcal{X}$ is the immersion of the boundary space in the product space $\mathcal{X} = X \times \mathbb{R}^2$. Equation (19), in the framework of stochastic optimal control problem, is called the controlled state equation associated to an admissible control system. We recall that, in general, fixed $t_0 \geq 0$ and $u_0 \in \mathcal{X}$, an *admissible control system* (a.c.s) is given by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{\mathbf{W}_t\}_{t\geq 0}, z)$ where

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space,
- $\{F_t\}_{t\geq 0}$ is a filtration in it, satisfying the usual conditions,
- $\{\mathbf{W}_t\}_{t\geq 0}$ is a Wiener process with values in $\mathcal{X}$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$,
- z is a process with values in a space K, predictable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies the constraint: $z(t) \in \mathcal{Z}$, $\mathbb{P}$-a.s., for almost every $t \in [t_0, T]$, where $\mathcal{Z}$ is a suitable domain of K.

In our case the space K coincide with $\mathbb{R}^2$.

To each a.c.s. we associate the mild solution $\mathbf{u}^z$ of state equation the mild solution $\mathbf{u}^z \in C([t_0,T]; L^2(\Omega; \mathcal{X}))$ of the state equation. We introduce the functional cost

$$J(t_0, u_0, z) = \mathbb{E}\int_{t_0}^T \lambda(s, \mathbf{u}_s^z, z_s)\,\mathrm{d}s + \mathbb{E}\phi(\mathbf{u}_T^z) \tag{20}$$

We consider the problem of minimizing the functional J over all admissible control systems (which is known in the literature as the weak formulation of the control problem); any a.c.s. that minimizes J –if it exists– is called optimal for the control problem. We define in classical way the Hamiltonian function relative to the above problem

$$\psi : [0,T] \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

setting

$$\psi(t, \mathbf{u}, \mathbf{w}) = \inf_{z \in \mathcal{Z}} \{\lambda(t, \mathbf{u}, z) + < \mathbf{w}, Pz >\} \tag{21}$$

and we define the following set

$$\Gamma(t, \mathbf{u}, \mathbf{w}) = \{z \in \mathcal{Z} : \lambda(t, \mathbf{u}, z) + < \mathbf{w}, Pz >= \psi(t, \mathbf{u}, z)\}$$

We consider the Hamilton-Jacobi-Bellman equation associated to the control problem

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \psi(t, x, v(t,x), \mathbb{G}(t,x)^* \nabla_x v(t,x)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T],\ x \in \mathcal{X}, \\ v(T,x) = \phi(x). \end{cases} \tag{22}$$

where the operator $\mathcal{L}_t$ is defined by

$$\mathcal{L}_t[\phi](x) = \frac{1}{2} \mathrm{Trace} \left(\mathbb{G}(t,x) \mathbb{G}(x)^* \nabla^2 \phi(x) \right) + < \mathbb{A}x, \nabla \phi(x) > .$$

Under suitable assumptions, if we let v denote the unique solution of (22) then we have $J(t,x,z) \geq v(t,x)$ and the equality holds if and only if the following feedback law is verified by z and $\mathbf{u}^z_\sigma$:

$$z(\sigma) = \Gamma(\sigma, \mathbf{u}^z_\sigma, \mathbb{G}(\sigma, \mathbf{u}^z_\sigma)^* \nabla_x v(\sigma, \mathbf{u}^z_\sigma)).$$

Thus, we can characterize optimal controls by a feedback law.

This class of stochastic control problems, in infinite dimensional setting, has been studied by Fuhrman and Tessitore [8] (we refer to Theorem 7.2 in that paper for precise statements and additional results).

In order to characterize optimal controls by a feedback law we have to require that the abstract operators $\mathbb{F}$ and $\mathbb{G}$ satisfy further regularity conditions.

We will prove that, under suitable assumptions on the functions f and g in the problem (18), the abstract operators fit the required conditions.

We impose that the operators $\mathbb{F}$ and $\mathbb{G}$ are Gâteaux differentiable. This notion of differentiability is weaker than the differentiability in the Fréchet sense.

We recall that for a mapping $F : X \to V$, where X and V denote Banach spaces, the directional derivative at point $x \in X$ in the direction $h \in X$ is defined as

$$\nabla F(x; h) = \lim_{s \to 0} \frac{F(x + sh) - F(x)}{s},$$

whenever the limit exists in the topology of V. F is called Gâteaux differentiable at point x if it has directional derivative in every direction at point x and there exists an element of $L(X,V)$, denoted $\nabla F(x)$ and called Gâteaux derivative, such that $\nabla F(x; h) = \nabla F(x)h$ for every $h \in X$.

Definition 12. *We say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X;V)$ if it is continuous, Gâteaux differentiable on X, and $\nabla F : X \to L(X, V)$ is strongly continuous.*

The last requirement in the definition above means that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Note that $\nabla F : X \to L(X,V)$ is not continuous in general if $L(X,V)$ is endowed with the norm operator topology; clearly, if this happens then F is Fréchet differentiable on X. Membership of a map in $\mathcal{G}^1(X,V)$ may be conveniently checked as shown in the following lemma.

Lemma 13. *A map $F : X \to V$ belongs to $\mathcal{G}^1(X,V)$ provided the following conditions hold:*

i) the directional derivatives $\nabla F(x;h)$ exist at every point $x \in X$ and in every direction $h \in X$;

ii) for every h, the mapping $\nabla F(\cdot;h) : X \to V$ is continuous;

iii) for every x, the mapping $h \mapsto \nabla F(x;h)$ is continuous from X to V.

When F depends on additional arguments, the previous definitions and properties have obvious generalizations.

The following assumptions are necessary in order to provide Gâteaux differentiability for the coefficients of the abstract formulation.

Assumption 14. *For a.a. $t \in [0,T]$, $\xi \in [0,1]$ the functions $f(t,\xi,\cdot)$ and $g(t,\xi,\cdot)$ belong to the class $C^1(\mathbb{R})$.*

Proposition 15. *Under assumptions 9 and 14, for every $s > 0$, $t \in [0,T]$,*

$$\mathbb{F}(t,\cdot) \in \mathcal{G}^1(\mathcal{X},\mathcal{X}), \qquad \mathrm{e}^{s\mathbb{A}}\mathbb{G}(t,\cdot) \in \mathcal{G}^1(\mathcal{X}, L_2(\mathcal{X})).$$

Proof. The first statement is an immediate consequence of the fact that $f(t,\xi,\cdot) \in C^1(\mathbb{R},\mathbb{R})$. In order to prove that $\mathrm{e}^{s\mathbb{A}}\mathbb{G}(t,\cdot)$ belongs to the class $\mathcal{G}^1(\mathcal{X}, L_2(\mathcal{X}))$ we use the continuous differentiability of g and an argument similar to that used in the proof of Proposition 10.

We note that, for $\mathbf{u} = \binom{u}{x}$ and $\mathbf{v} = \binom{v}{y}$, the gradient operator $\nabla_{\mathbf{u}} \left(e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})\right)\mathbf{v}$ is an Hilbert Schmidt operator that maps

$$\mathbf{w} = \begin{pmatrix} w \\ p \end{pmatrix} \mapsto e^{s\mathbb{A}} \begin{pmatrix} g_u(t,\cdot,u(\cdot))w(\cdot)v(\cdot) \\ 0 \end{pmatrix} = e^{s\mathbb{A}} \left(\nabla_u(\mathbb{G}(t,\mathbf{u})\mathbf{v})(\mathbf{w})\right)$$

In fact, we have

$$
\begin{aligned}
&\lim_{r\to 0}\left\|\frac{e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}+r\mathbf{v})-e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})}{r}-\nabla_u e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})\mathbf{v}\right\|_{L_2(\mathcal{X})}\\
&\quad=\lim_{r\to 0}\sum_{j,k}<\frac{e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u}+r\mathbf{v})-e^{s\mathbb{A}}\mathbb{G}(t,\mathbf{u})}{r}\,\phi_j-e^{s\mathbb{A}}\left(\nabla_u(\mathbb{G}(t,\mathbf{u})\mathbf{v})\phi_j\right),\phi_k>^2\\
&\quad=\lim_{r\to 0}\sum_{j,k}<\left(\frac{\mathbb{G}(t,\mathbf{u}+r\mathbf{v})-\mathbb{G}(t,\mathbf{u})}{r}-\nabla_u\mathbb{G}(t,\mathbf{u})\mathbf{v}\right)\phi_j,e^{s\mathbb{A}}\phi_k>^2\\
&\quad=\lim_{r\to 0}\sum_{k}e^{2s\lambda_k}\left|\left(\frac{\mathbb{G}(t,\mathbf{u}+r\mathbf{v})-\mathbb{G}(t,\mathbf{u})}{r}-\nabla_u\mathbb{G}(t,\mathbf{u})\mathbf{v}\right)\phi_k\right|^2_{\mathcal{X}}\\
&\quad=\lim_{r\to 0}\sum_{k}e^{2s\lambda_k}\int_0^1\left|\frac{g(t,u(\xi)+rv(\xi))-g(t,u(\xi))}{r}e_k(\xi)-g_u(t,u(\xi))v(\xi)e_k(\xi)\right|^2d\xi\\
&\quad\le c\lim_{r\to 0}\sum_{k}e^{2s\lambda_k}\int_0^1\left|\frac{g(t,u(\xi)+rv(\xi))-g(t,u(\xi))}{r}-g_u(t,u(\xi))v(\xi)\right|^2d\xi\\
&\quad=c\lim_{r\to 0}\sum_{k}e^{2s\lambda_k}\int_0^1\left|\int_0^1\left[g_u(t,u(\xi)+\alpha rv(\xi))-g_u(t,u(\xi))\right]d\alpha\, v(\xi)\right|^2d\xi
\end{aligned}
$$

and, by dominated convergence, this limit is equal to zero. In similar way we can prove the points $(ii)-(iii)$ of Lemma 13 to obtain the thesis. □

In order to prove the main result of this section we require the following hypothesis.

Assumption 16.

(i) *λ is measurable and for a.e. $t\in[0,T]$, for all $\boldsymbol{u},\boldsymbol{u}'\in\mathcal{X}$, $z\in\mathcal{Z}$*

$$|\lambda(t,\boldsymbol{u},z)-\lambda(t,\boldsymbol{u}',z)|\le C|1+\boldsymbol{u}+\boldsymbol{u}'|^m|\boldsymbol{u}-\boldsymbol{u}'|$$

$$|\lambda(t,0,z)|\le C$$

for suitable $C\in\mathbb{R}^+$, $m\in\mathbb{N}$;

(ii) *$\mathcal{Z}$ is a Borel and bounded subset of $\mathbb{R}^2$;*

(iii) *$\phi\in\mathcal{G}^1(\mathcal{X},\mathbb{R})$ and, for every $\sigma\in[0,T]$, $\psi(\sigma,\cdot,\cdot)\in\mathcal{G}^{1,1}(\mathcal{X}\times\mathcal{X},\mathbb{R})$;*

(iv) *for every $t\in[0,T]$, $\mathbf{u},\mathbf{w},\mathbf{h}\in\mathcal{X}$*

$$|\nabla_{\mathbf{u}}\psi(t,\mathbf{u},\mathbf{w})h|+|\nabla_{\mathbf{u}}\phi(\mathbf{u})\mathbf{h}|\le L|\mathbf{h}|(1+|\mathbf{u}|)^m;$$

(v) *for all $t\in[0,T]$, for all $\mathbf{u}\in\mathcal{X}$ and $\mathbf{w}\in\mathcal{X}$ there exists a unique $\Gamma(t,\mathbf{u},\mathbf{w})\in\mathcal{Z}$ that realizes the minimum in (21). Namely*

$$\lambda(t,\mathbf{u},\Gamma(t,\mathbf{u},\mathbf{w}))+<\mathbf{w},P\Gamma(t,\mathbf{u},\mathbf{w})>=\psi(t,\mathbf{u},\mathbf{w})$$

Moreover $\Gamma\in C([0,T]\times\mathcal{X}\times\mathbb{R}^2;\mathcal{Z})$.

Theorem 17. *Suppose that assumptions 9, 14 and 16 hold. For all a.c.s. we have* $J(t_0, u_0, z) \geq v(t_0, u_0)$ *and the equality holds if and only if the following feedback law is verified by* z *and* $\mathbf{u}^z$*:*

$$z(\sigma) = \Gamma(\sigma, \mathbf{u}^z_\sigma, G(\sigma, \mathbf{u}^z_\sigma)^* \nabla_x v(\sigma, \mathbf{u}^z_\sigma)), \quad \mathbb{P} - \text{a.s. for a.a. } \sigma \in [t_0, T]. \tag{23}$$

Finally there exists at least an a.c.s. for which (23) holds. In such a system the closed loop equation:

$$\begin{cases} d\bar{\mathbf{u}}_\tau = \mathbb{A}\bar{\mathbf{u}}_\tau \, d\tau + \mathbb{G}(\tau, \bar{\mathbf{u}}_\tau) P \Gamma(\tau, \bar{\mathbf{u}}_\tau, \mathbb{G}(\tau, \bar{\mathbf{u}}_\tau)^* \nabla_x v(\tau, \bar{\mathbf{u}}_\tau)) \, d\tau \\ \qquad\qquad + \mathbb{F}(\tau, \bar{\mathbf{u}}_\tau) \, d\tau + \mathbb{G}(\tau, \bar{\mathbf{u}}_\tau) \, d\mathbf{W}_\tau, \qquad \tau \in [t_0, T], \\ \bar{\mathbf{u}}_{t_0} = \mathbf{u}_0 \in \mathcal{X}. \end{cases} \tag{24}$$

admits a solution $\bar{u}$ *and if* $\bar{z}(\sigma) = \Gamma(\sigma, \bar{\mathbf{u}}_\sigma, G(\sigma, \bar{\mathbf{u}}_\sigma)^* \nabla_x v(\sigma, \bar{\mathbf{u}}_\sigma))$ *then the couple* $(\bar{z}, \bar{\mathbf{u}})$ *is optimal for the control problem.*

Proof. By Proposition 4 we know that $\mathbb{A}$ generates a strongly continuous semigroup of linear operators $e^{t\mathbb{A}}$ on $\mathcal{X}$. The assumption 9 ensures that the statements in Proposition 10 hold. Moreover, the assumption 14 guarantees that the results in Proposition 15 are true. Finally, these conditions together with Assumption 16 allow us to apply Theorem 7.2 in [8] and to perform the synthesis of the optimal control. □

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