Asian J. Math.
Vol. 27, No. 3, pp. 375–404, June 2023

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TERRACINI LOCUS FOR THREE POINTS ON A SEGRE VARIETY*

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Abstract. We introduce the notion of *r-th Terracini locus* of a variety and we compute it for at most three points on a Segre variety.

Key words. Terracini loci, Segre varieties, zero-dimensional schemes.

Mathematics Subject Classification. 14N07, 15A69.

1. Introduction. The celebrated Terracini Lemma [48, 6] is a well-known and extremely powerful result in Algebraic Geometry that allows to compute the dimensions of r-th secant varieties of a given variety X in terms of the dimensions of the sum of tangent spaces at r generic points of X. If X is the embedding of a variety Y into a projective space via a complete linear system $\mathcal{L}$, then the codimension of the r-th secant variety of X is equal to $h^0(Y, I_Z \otimes \mathcal{L})$ where Z is a 0-dimensional scheme of r double generic fat points (cf. e.g. [25, 30, 38]). The classical apolarity theory [37, 42] is the very well-known example in which the variety X is $Y = \mathbb{P}^n$ embedded via $\mathcal{O}(d)$ for which Alexander-Hirschowitz completely classified dimensions of all secant varieties to any Veronese variety (cf. [8]). Another complete classification is for secant varieties of Segre-Veronese embedding of products of $Y = (\mathbb{P}^1)$'s via $\mathcal{O}(d_1, \dots, d_k)$ due to Laface-Postinghel (cf. [44]). Recently Galuppi-Oneto determined dimensions of secant varieties in the case of Segre-Veronese embedding of $Y = \mathbb{P}^m \times \mathbb{P}^n$ in bidegree (d_1, d_2) for all $d_1, d_2 \geq 3$ (cf. [34]). There is a vast literature in this field (see e.g. [1, 2, 3, 4, 11, 12, 14, 16, 23, 27, 35, 36] and references therein) but almost nothing has been said for the case in which the 0-dimensional scheme of double fat points is not necessarily supported on generic points. Clearly if the points are not generic, the equivalence between $h^0(Y, I_Z \otimes \mathcal{L})$ and the codimension of the secant variety of X is not valid anymore. Indeed, for r general points $P_1, \dots, P_r \in X$ and for a generic $Q \in \langle P_1, \dots, P_r \rangle$, Terracini lemma states that $\dim\langle T_{P_1}X, \dots, T_{P_r}X\rangle$ is the dimension of the variety $\sigma_r(X)$, while for non-general $P_1, \dots, P_r \in X$ one can only say that $\dim\langle T_{P_1}X, \dots, T_{P_r}X\rangle \leq \dim \sigma_r(X)$. This phenomenon is related to the fact that $\operatorname{codim}\langle T_{P_1}X, \dots, T_{P_r}X\rangle = h^0(Y, I_Z \otimes \mathcal{L})$ which may be higher than the one for generic points. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and tensorize it by $\mathcal{L}$:

$$0 \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_Z \longrightarrow 0.$$

If $h^1(Y, \mathcal{L}) = 0$, then we get the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{I}_Z \otimes \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}|_Z) \longrightarrow H^1(Y, \mathcal{I}_Z \otimes \mathcal{L}) \longrightarrow 0.$$

*Received January 26, 2022; accepted for publication March 10, 2023. The authors were partially supported by GNSAGA of INDAM. Santarsiero was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 445466444.

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We will always take $(Y, \mathcal{L})$ such that $h^1(Y, \mathcal{L}) = 0$. Therefore, one gets

$$h^0(Y, \mathcal{L}) - h^0(Y, \mathcal{I}_Z \otimes \mathcal{L}) = h^0(Z, \mathcal{L}|_Z) - h^1(Y, \mathcal{I}_Z \otimes \mathcal{L})$$

that is to say that the dimension of the span of embedding of Z via $\mathcal{L}$ can be computed as

$$h^0(Z, \mathcal{L}|_Z) - h^1(\mathcal{I}_Z \otimes \mathcal{L}) - 1.$$

From this one easily see the role played by the $h^1(\mathcal{I}_Z \otimes \mathcal{L})$ in controlling the value of $h^0(Y, \mathcal{I}_Z \otimes \mathcal{L})$.

In this paper we fix our attention on the case of Segre varieties, i.e. the embedding of $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ via $\mathcal{O}(1, \ldots, 1)$ and Z a scheme of either 2 or 3 double fat points. We will define the key object that we will call the r-th Terracini locus that will essentially contain all the subsets of r points for which both $h^0(\mathcal{I}_Z(1, \ldots, 1)) > 0$ and $h^1(\mathcal{I}_Z(1, \ldots, 1)) > 0$.

We like to point out the geometric importance of the r-th Terracini locus. Consider the open part of the r-th Abstract Secant variety $Abs_r(X)$ of a Segre Variety $X \subset \mathbb{P}^N$, namely

$$Abs_r^0(X) := \{(Q, (P_1, \ldots, P_r)) \in \mathbb{P}^N \times X_{\mathrm{reg}}^r \,|\, Q \in \langle P_1, \ldots, P_r \rangle \cong \mathbb{P}^{r-1}\}.$$

Note that in the definition of $Abs_r^0(X)$ we only take $P_1, \ldots, P_r$ linearly independent. If one considers the first projection on $\mathbb{P}^N$ one gets that $Abs_r^0(X)$ projects onto what we can call an "open part" of the r-th secant variety of X, namely $\sigma_r^0(X) := \{Q \in \mathbb{P}^N \,|\, Q \in \langle P_1, \ldots, P_r \rangle \cong \mathbb{P}^{r-1}, \text{ where all } P_i \in X\}$:

$$T_r : Abs_r^0(X) \longrightarrow \sigma_r^0(X).$$

We call such a projection T_r the r-th *Terracini map*. The differential of the r-th Terracini map is defined on each point of $Abs_r^0(X)$ and the r-th Terracini locus is nothing else than a measure of the degeneracy of such a linear map. Remark that any point of $Abs_r^0(X)$ is smooth since X is smooth. Notice that the Terracini map is a differential only when defined on the open part $Abs_r^0(X)$, i.e. before taking its Zariski closure. Moreover, we remark that the rank of the differential of the r-th Terracini map a priori depends both on the points $P_1, \ldots, P_r \in X$ and on $Q \in \langle P_1, \ldots, P_r \rangle$.

We like to point out that the interest of studying the behaviour of a set of fixed, not necessarily generic, points is not only a purely, however extremely interesting, mathematical speculation but it is object of study also in applied and numerical fields (cf. e.g. [21, 31, 18, 32, 50, 11, 47, 20]).

1.1. A numerical point of view. Working with tensors coming from applied problems measurement errors may occur. Moreover, working with a machine, one is forced to use non-exact arithmetic and, even if we start with an exact tensor, round-off errors may occur due to the possibly inexact representation of the given tensor into the machine.

Therefore, when running algorithms that involve tensors coming from actual applications, the actual input is a perturbed tensor and the error representation one is starting with may be amplified when performing algorithms.

The condition number of a function measures the rate of error that happens to the output element conditioned to a small change on the element in the domain (see

(1) below for a formal definition). Moreover, a problem is said to be well-conditioned if it has a small condition number and it is ill-conditioned otherwise. In this last case one says that the problem is sensitive to small perturbations.

Terracini loci are involved when measuring the sensitivity of a tensor rank decomposition, also called CPD: canonical polyadic decomposition (cf. e.g. [33]).

Denote by $T^{-1}_{r,(p_1,\dots,p_r)}$ the local inverse of T_r at $(p_1,\dots,p_r)\in(X)^r$. If the differential $\mathrm{d}_{(p_1,\dots,p_r)}T_r$ of T_r at $(p_1,\dots,p_r)$ is invertible, then a local inverse exists at $(p_1,\dots,p_r)$. Moreover, we recall that $\left(\mathrm{d}_{(p_1,\dots,p_r)}T_r\right)^{-1}=\mathrm{d}_qT^{-1}_{r,(p_1,\dots,p_r)}$. To define the condition number of a r-uple $(p_1,\dots,p_r)\in(X)^r$, we follow the spectral characterization of [21, Theorem 1.1].

Denote by U_i an orthonormal basis of the affine tangent space $T_{p_i}X$ for all $i=1,\dots,r$ and let $U=[U_1\cdots U_r]$. We recall that the spectral norm $\|U\|_2$ of U is the largest singular value $\zeta(U)$ of U, i.e. $\zeta(U)$ is the square root of the biggest eigenvalue of UU^*.

If T_r is locally invertible at $(p_1,\dots,p_r)$ then

$$\|\left(\mathrm{d}_{(p_1,\dots,p_r)}T_r\right)^{-1}\|_2=\|U^{-1}\|_2=\zeta(U^{-1})=\frac{1}{\min\{\lambda\mid\lambda\text{ is a singular value of }U\}}.$$

In this case they define the condition number of $(p_1,\dots,p_r)$ as

$$\kappa(p_1,\dots,p_r):=\|\left(\mathrm{d}_{(p_1,\dots,p_r)}T_r\right)^{-1}\|_2.$$

Otherwise, if $\mathrm{d}T_r$ is not invertible at $(p_1\dots,p_r)$, then U has an eigenvalue equal to 0, which is also the smallest singular value of U, and in this case we set $\kappa(p_1,\dots,p_r)=\infty$. The *condition number* of the r-uple $(p_1,\dots,p_r)$ is

$$\kappa(p_1,\dots,p_r):=\begin{cases}\|\left(\mathrm{d}_{(p_1,\dots,p_r)}T_r\right)^{-1}\|_2 & \text{if }\mathrm{d}T_r\text{ is invertible at }(p_1,\dots,p_r),\\ \infty & \text{otherwise.}\end{cases}\tag{1}$$

The condition number of a tensor rank decomposition is therefore a measure of the sensitivity of the decomposition itself under errors perturbations. One would like to avoid points of $(X)^r$ whose condition number is infinite since in these cases to a unique element $q\in\mathbb{P}^N$ correspond different r-uples in $(X)^r$ and this behaviour generates ambiguity in the interpretations of the results when performing algorithms of tensor rank decomposition.

In [21], the authors defined the *ill-posed set* of a decomposition $(p_1,\dots,p_r)$ as

$$\Sigma_{\mathbb{P}}=\{(p_1,\dots,p_r)\in(X)^r\colon\kappa(p_1,\dots,p_r)=\infty\}.$$

This locus contains precisely all r-uples $(p_1,\dots,p_r)\in(X)^r$ for which the differential of the map T_r has not maximal rank. Therefore, the distinction between the r-th Terracini locus of a multiprojective space and the ill-posed locus $\Sigma_{\mathbb{P}}$ relies on

- considering the r rank-1 tensors as a set of points instead of a tuple;
- working with the minimal multiprojective space containing the r points.

Indeed, after passing from tuples to sets, the variety $\Sigma_{\mathbb{P}}$ contains the Terracini locus, and when restricted to the minimal multiprojective space containing the r points, they are equal.

Remark 1.1. Even though we will work under minimality assumption for the multiprojective space containing a set of points, the result we will achieve in this paper are interesting from a numerical point of view: in [31] the authors proved that the condition number of a CPD does not change under Tucker compression of the CPD itself, which is the analogous of considering the minimal multiprojective space containing both the tensor and all its possible rank decompositions.

In the first section of this paper we introduce the notation and we show that the second Terracini locus is empty, meaning that the differential of the second Terracini map has everywhere full rank. The second section is a technical one where we concentrate all the core lemmas that will be needed in the sequel in order to prove our main theorem (Theorem 5.9) that is a complete description of the 3-rd Terracini locus. Section 4 is a crucial section where we show all the examples that will turn out the only cases in which the 3-rd Terracini locus will be not empty. Section 5 is devoted to the proof of the main theorem that essentially will be a discussion on why the already highlighted examples in Subsections 4.1 and 4.2 are the only nonempty 3-rd Terracini loci.

In summary: We describe all classes of Terracini 3-loci for concise tensors.
In each case we describe several geometric properties.

Let $Z \subset Y$ be a scheme of $r \geq 2$ double points embedded via Segre in an n-dimensional multiprojective space Y. In the last section we compute the maximal value $\max_{n>0,r\geq 2}\{h^1(\mathcal{I}_Z(1,\dots,1)) > 0\}$. We will show that

$$h^1(\mathcal{I}_Z(1,\dots,1)) \leq (r-1)(n+1)$$

and that equality holds if and only if $Y = \mathbb{P}^n$. Since $h^0(\mathbb{P}^n, \mathcal{I}_{\mathbb{P}^n}(1)) = 0$, we compute the maximal value of such dimension providing that also $h^0(\mathcal{I}_Z(1,\dots,1)) > 0$. Finally we prove that for any multiprojective space Y of dimension $n \geq 3$, one can always find $r \geq 3$ points $S \subset Y$ belonging to the corresponding r-th Terracini locus. This conclusion might open to further investigation of the introduced locus.

2. Notation. We work over an algebraically closed field $\mathbb{K}$ of characteristic 0. In the following we will always deal with a multiprojective space of $k > 0$ factors Y of the form

$$Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}.$$

NOTATION 2.1. Let $V_1, \dots, V_k$ be $\mathbb{K}$-vector spaces of dimensions $n_1+1, \dots, n_k+1$ respectively. Denote by ν the Segre embedding of Y, which is defined as

$$\begin{aligned} \nu\colon \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k) &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_k) \\ ([v_1], \dots, [v_k]) &\mapsto [v_1 \otimes \cdots \otimes v_k]. \end{aligned}$$

We will denote the Segre variety of Y by

$$X := \nu(Y).$$

NOTATION 2.2. We denote the projection of Y onto the i-th factor by

$$\pi_i\colon Y \longrightarrow \mathbb{P}^{n_i}.$$

Fix $Y_i := \mathbb{P}^{n_1} \times \cdots \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k}$, for some $1 \le i \le k$. By η_i we denote the map that projects Y onto Y_i forgetting the i-th factor, i.e.

$$\eta_i \colon Y \longrightarrow Y_i.$$

The Segre embedding of Y_i is denoted via

$$\nu_i \colon Y_i \longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_k).$$

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n. Fix $r \ge 2$. The r-th *secant variety* $\sigma_r(X)$ of X is the Zariski closure of all $(r-1)$-planes spanned by r linearly independent points of X. Namely

$$\sigma_r(X) := \overline{\bigcup_{p_1,\dots,p_r \in X} \langle p_1, \dots, p_r \rangle}.$$

Remark that $\dim \sigma_r(X) \le \min\{r(n+1)-1, N\}$. If the previous inequality is strict, the r-th secant variety of X is said to be *defective* with *defect* $\min\{r(n+1)-1, N\} - \dim \sigma_r(X)$.

NOTATION 2.3. For any smooth $p \in Y$, denote by $(2p, Y)$ the first infinitesimal neighbourhood of p in Y, i.e. the closed subscheme of Y with $(\mathcal{I}_{p,Y})^2$ as its ideal sheaf. For any finite set $S \subset Y$ let $(2S, Y) := \cup_{p,S}(2p, Y)$. We often write $2p$ and $2S$ instead of $(2p, Y)$ and $(2S, Y)$ if the dependence from Y is clear.

REMARK 2.1. Fix two projective varieties $A \supseteq B$ and let $p \in A_{\mathrm{reg}} \cap B_{\mathrm{reg}}$. Thus both $(2p, A)$ and $(2p, B)$ are defined and $\deg(2p, A) = \dim A + 1$ and $\deg(2p, B) = \dim B + 1$. Set $x := \dim A$ and $y := \dim B$. We claim that

$$(2p, B) = (2p, A) \cap B \text{ (scheme-theoretic intersection).} \tag{2}$$

Indeed, let R be the local ring $\mathcal{O}_{A,p}$ and $I \subseteq \mathcal{O}_{A,p}$ the ideal such that $R/I = \mathcal{O}_{B,p}$. Let μ denote the maximal ideal of the local ring $\mathcal{O}_{A,p}$. Note that μ/I is the maximal ideal of R/I. Since both sides of the equality (2) are zero-dimensional, both sides of (2) are the spectrum of R by some ideals and we need to prove that these ideals are equal. Since p is a smooth point of B, there are regular generators $u_1, \dots, u_y$ of μ/I. Take $v_i \in \mu$, $i = 1, \dots, y$, such that $v_i/I = u_i$. Since p is a smooth point both of A and of B, there are $x-y$ elements v_j, $y+1 \le j \le v_x$, such that they generate I and $v_1, \dots, v_x$ generate μ ([46, Prop. 22 at p. 77]). The scheme $(2p, A) \cap B$ is associated to the ideal of R generated by all $v_i v_j$, $1 \le i \le j \le x$, and all v_j, $j = y+1, \dots, v_x$. The scheme $(2p, B)$ is associated to the ideal of R/I generated by all $u_i u_j$, $1 \le i \le j \le y$, and hence to the ideal of R generated by all $v_i v_j$, $1 \le i \le j \le y$, and all v_j, $j = y+1, \dots, v_x$. Obviously these two ideals of R are equal.

Remark that if $W \subset Y$ is a proper multiprojective subspace and $p \in W$, then $(2p, W) \neq (2p, Y)$ as schemes. In fact $\deg(2p, W) = \dim W + 1$ and $\deg(2p, Y) = \dim Y + 1$. However by Remark 2.1, $(2p, W) = (2p, Y) \cap W$ (scheme-theoretic intersection) and hence $(2p, W) \subseteq (2p, Y)$. Thus for any finite set $S \subset W$ one has $(2S, W) = (2S, Y) \cap W$.

NOTATION 2.4. For $k > 0$ fix the following notation:

- $\varepsilon_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^k$ is the k-uple given by all 0's but 1 in the i-th position;

- $\hat{\varepsilon}_i := (1, \dots, 1, 0, 1, \dots, 1) \in \mathbb{N}^k$ is the k-uple given by all 1's but 0 in the i-th position;
- ε_I is the k-uple having 1's in the places indexed by the finite set $I \subset \{1, \dots, k\}$ and 0's everywhere else;
- $\hat{\varepsilon}_I$ is the k-uple having 0's in the places indexed by I and 1's everywhere else.

Let $Z \subset Y$ be a 0-dimensional scheme. Fix a finite set $I \subset \{1, \dots, k\}$ and $H \in |\mathcal{O}(\varepsilon_I)|$. Denote by $\mathrm{Res}_H(Z)$ the residue of Z with respect to H, i.e. the 0-dimensional scheme defined by the ideal sheaf $\mathcal{I}_Z : \mathcal{I}_H$. By $H \cap Z$ denote the scheme-theoretic intersection of Z and H. The residual exact sequence of Z with respect to H is

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_H(Z)}(\hat{\varepsilon}_I) \longrightarrow \mathcal{I}_Z(1, \dots, 1) \longrightarrow \mathcal{I}_{H \cap Z, H}(1, \dots, 1) \longrightarrow 0,$$

(cf. [9, Definition 2.1]). Since throughout the paper we will deal with double points, we recall how to adapt the residual exact sequence in this case.

Take $Z := (2S, Y)$, where $S \subset Y$ is a finite set, consider a proper subset $S' \subset S$ such that $S' = S \cap H$ and denote by $S'' := S \setminus S'$. In this case, the scheme-theoretic intersection of Z and H is $(2S', H)$, while $\mathrm{Res}_H(Z)$ is the zero-dimensional scheme of what is left once we specialized $(2S', Y)$ into H, namely

$$\mathrm{Res}_H(Z) = S' \cup (2S'', Y).$$

Since we will use the restricted exact sequence only with 0-dimensional schemes, we recall the restriction sequence of Z with respect to Y, namely

$$0 \longrightarrow \mathcal{I}_Z(1, \dots, 1) \longrightarrow \mathcal{O}_Y(1, \dots, 1) \longrightarrow \mathcal{O}_Z(1, \dots, 1) \longrightarrow 0.$$

Since this exact sequence is defined for any embedding of Y, one can also use it for any line bundle given by $\mathcal{O}(\varepsilon_I)$ instead of the one given by $\mathcal{O}(1, \dots, 1)$, i.e.

$$0 \longrightarrow \mathcal{I}_Z(\varepsilon_I) \longrightarrow \mathcal{O}_Y(\varepsilon_I) \longrightarrow \mathcal{O}_Z(\varepsilon_I) \longrightarrow 0.$$

The corresponding cohomology exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{I}_Z(\varepsilon_I)) \longrightarrow H^0(Y, \mathcal{O}_Y(\varepsilon_I)) \longrightarrow H^0(Y, \mathcal{O}_Z(\varepsilon_I)) \longrightarrow H^1(Y, \mathcal{I}_Z(\varepsilon_I)) \longrightarrow 0$$

shows that the dimension of the subspace

$$\langle \nu(Z) \rangle = \mathbb{P}\left(H^0(Y, \mathcal{I}_Z(\varepsilon_I))^\perp\right)$$

is equal to $h^0(\mathcal{O}_Z(\varepsilon_I)) - h^1(\mathcal{I}_Z(\varepsilon_I)) - 1$.

From this one easily sees the role played by the $h^1(\mathcal{I}_Z(\varepsilon_I))$ in controlling the dependence of the multilinear forms passing through Z and therefore the speciality of Z.

Since it will be used several times later, let us clearly state the above discussion in the following lemma.

LEMMA 2.5. *Let Y be a multiprojective space of $k \geq 1$ factors and let $S \subset Y$ be a set of r points. The restriction exact sequence of $(2S, Y)$ in Y gives*

$$\deg(2S, Y) + h^0\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) = h^0(\mathcal{O}_Y(1, \dots, 1)) + h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right).$$

REMARK 2.2. For the sake of clarity, we also recall that for a generic $S \subset Y$ with $\#S = r$, the quantity $\dim\langle\nu(S)\rangle$ is equal to $h^0(\mathcal{O}_Y(1,\dots,1)) - h^0(\mathcal{I}_S(1,\dots,1))$, while $\dim\sigma_r(X) = h^0(\mathcal{O}_Y(1,\dots,1)) - h^0\big(\mathcal{I}_{(2S,Y)}(1,\dots,1)\big)$.

NOTATION 2.6. For any zero-dimensional scheme $Z \subset Y$ set

$$\delta(Z,Y) := h^1(\mathcal{I}_Z(1,\dots,1)).$$

If $W \subseteq Y$ is a multiprojective subspace such that $Z \subset W$, set $\delta(Z,W) := h^1(W,\mathcal{I}_{Z,W}(1,\dots,1))$. We remark that $\delta(Z,W) = \delta(Z,Y)$ since $h^i(\mathcal{I}_W(1,\dots,1)) = 0$ for $i = 1,2$. Sometimes we will write $\delta(Z)$ instead of $\delta(Z,Y)$, when the dependence on Y is clear.

In particular for any finite set $S \subset W$ there are defined the integers $\delta((2S,Y),Y)$, $\delta((2S,W),W)$ and $\delta((2S,W),Y)$. Set

$$\delta(2S,Y) := \delta((2S,Y),Y) \text{ and } \delta(2S,W) := \delta((2S,W),W).$$

Clearly $\delta(2S,W) = \delta((2S,W),Y)$. For the specific case of double fat points, $\delta(2S,Y)$ will be called the Terracini defect of S in Y (see Definition 2.8).

REMARK 2.3. By using the residue exact sequence with $H \in |\mathcal{O}_Y(\varepsilon_i)|$, $i \in \{1,\dots,k\}$, for a finite set $S \subset H$ such that $h^0(H,\mathcal{I}_{S\cap H,H}(1,\dots,1)) = 0$ one has $\delta(2S,Y) = \delta(2S,H) + h^0(\mathcal{I}_S(\hat{\varepsilon}_i))$, since $\mathrm{Res}_H(2S) = S$.

NOTATION 2.7. Let Y be any multiprojective space of k factors, with $k > 0$. For all positive integers r, denote by $S(Y,r)$ the set of all subsets of Y with cardinality r.

Let $S \in S(Y,r)$ be a set of $r > 0$ distinct points. The *minimal multiprojective space containing* S is $Y' := \mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_{k'}} \subseteq Y$ where $\mathbb{P}^{n'_i} := \langle\pi_i(S)\rangle$, $i = 1,\dots,k'$ and $k' \le k$. The integer $k' \le k$ is the maximum integer such that $\#\pi_i(S) > 1$ for all $i \le k$. Clearly $\mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_{k'}} \times \{o_{k'+1}\} \times \cdots \times \{o_k\} \cong \mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_{k'}}$ and the convention of forgetting factors of dimension 0 will be used even after a suitable permutation, hence in general Y' is such that $k' \le k$ and $n'_i > 0$ for all i's.
Let $S \subset Y$ be a finite set. We remark that working with the minimal multiprojectve space $Y' \subset Y$ containing S is a harmless assumption that allows us to work with smaller multiprojective spaces. Moreover in the forthcoming Lemma 3.2 we describe the relations between the values $\delta(S,Y')$ and $\delta(S,Y)$.

We have now introduced all the necessary tools to define the **r-th Terracini locus** that will be the main actor of the present paper.

DEFINITION 2.8. For all positive integers r and for any multiprojective space Y, define

$$\mathbb{T}_1(Y,r) := \left\{S \in S(Y,r) \mid h^0\big(\mathcal{I}_{(2S,Y)}(1,\dots,1)\big) > 0 \text{ and } \delta(2S,Y) > 0\right\}.$$

We will call the *r-th Terracini locus* $\mathbb{T}(Y,r)$ of all r-uple of points of Y the set

$$\boxed{\mathbb{T}(Y,r) := \{S \in \mathbb{T}_1(Y,r) \,|\, Y \text{ is the minimal multiprojective space containing } S\}.}$$

For any $S \in \mathbb{T}_1(Y,r)$ we call the integer $\delta(2S,Y)$ introduced in Notation 2.6 the *r-th Terracini defect of S in Y* or the *defect of $2S$ in Y*.

If $Y' \subseteq Y$ is the minimal multiprojective space containing a set S, the integer $\delta(2S, Y') := h^1\left(Y', \mathcal{I}_{(2S,Y')}(1,\dots,1)\right)$ is called the *absolute r-th Terracini defect of* S.

Set $S = \{p_1,\dots,p_r\} \subset Y$. Notice that the condition $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 0$ means that $\langle T_{p_1}X,\dots,T_{p_r}X\rangle$ does not fill the ambient space, and that the condition $\delta(2S,Y) > 0$ means that the tangent spaces $T_{p_1}X,\dots,T_{p_r}X$ are linearly independent.

2.1. The 2-nd Terracini locus is empty. In this subsection we prove that no sets of two distinct points $S \subset Y$ such that Y is the minimal multiprojective space containing S, is contained in the 2-nd Terracini locus $\mathbb{T}(Y,2)$.

PROPOSITION 2.9. *The 2-nd Terracini locus $\mathbb{T}(Y,2)$ is empty for any multiprojective space Y.*

Proof. Let $S \in S(Y,2)$ be such that Y is the minimal multiprojective space containing S. So $\#\pi_i(S) = 2$ for all i's and $Y \cong (\mathbb{P}^1)^k$ for some $k \geq 1$. By definition of $\mathbb{T}(Y,2)$, we need to prove that either $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = 0$ or $h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = 0$. Clearly if $k = 1$ then $h^0\left(\mathcal{I}_{(2S,\mathbb{P}^1)}(1)\right) = 0$. If $k = 2$, then $h^0\left(\mathcal{I}_{(2S,Y)}(1,1)\right) = 0$ since S can be seen as a general subset of 2 distinct points by the action of $(\mathrm{Aut}(\mathbb{P}^1))^2$ and a general 2×2 matrix has rank 2.

Let $k \geq 3$. Let E be the set of all $A \subset Y$ such that $\#A = \#\pi_i(A) = 2$ for all i's. The group $(\mathrm{Aut}(\mathbb{P}^1))^k$ acts transitively on E. Thus S may be considered as a general subset of Y with cardinality 2. Hence in this case $\dim \sigma_2(X) = h^0\left(\mathcal{O}_Y(1,\dots,1)\right) - h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right)$ (cf. Remark 2.2) and since $\dim \sigma_2(X) = 2k+1$ for all $k \geq 3$ (cf. [43, 49]), Terracini's lemma gives $h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = 0$. □

Let's point out some consequence of Proposition 2.9 in terms of identifiability of rank 2 tensors. Since we are dealing with finite subsets S of two distinct points, the minimal multiprojective space containing S is $Y = (\mathbb{P}^1)^k$ for some $k \geq 1$, which is equivalent to say that $\#\pi_i(S) = 2$ for all i's. Thus we may look at $S := \{p_1, p_2\}$ as a general set of two distinct points thanks to the action of $(\mathrm{Aut}(\mathbb{P}^1))^k$.
The emptiness of the 2-nd Terracini locus $\mathbb{T}(Y,2)$ means that the differential of the map $T_2 : Abs_r^0(X) \longrightarrow \sigma_r^0(X)$ has full rank for any $X = \nu((\mathbb{P}^1)^k)$, with arbitrary $k \geq 2$. Since we are working with general points, the condition

$$h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 0$$

corresponds to prescribe that the 2-nd secant variety $\sigma_2(X)$ does not fill the ambient space. This condition together with

$$h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 0$$

are equivalent to ask that the dimension of the tangent space $T_q\sigma_2(X)$ at a general $q \in \mathbb{P}^{2^k-1}$ such that $q \in \langle \nu(p_1), \nu(p_2)\rangle$, is strictly less than $2(k+1) - 1$.

3. Main lemmas.

REMARK 3.1. If $A \subset B \subset Y$ are zero-dimensional schemes, then

$$\delta(A,Y) \leq \delta(B,Y) \leq \delta(A,Y) + \deg(B) - \deg(A). \tag{3}$$

Indeed the first inequality is clear since $A \subset B$. Moreover we remark that if $A \subset B$ then $h^0\left(\mathcal{I}_B(1,\dots,1)\right) \leq h^0\left(\mathcal{I}_A(1,\dots,1)\right)$. So by the restriction exact sequences of both A and B with respect to Y (cf. Lemma 2.5), we get the second inequality. In particular for all $S' \subset S \subset Y$ we have

$$\delta(2S', Y) \leq \delta(2S, Y) \leq \delta(2S', Y) + (\#S - \#S')(\dim Y + 1). \quad (4)$$

Definition 3.1. We say that a finite set $S \subset Y$ is *minimally Terracini* if $\delta(2S, Y) > 0$ and $\delta(2S', Y) = 0$ for all $S' \subsetneq S$.

By Remark 3.1, if $S' \subset S$ is a scheme of r double points, the r-th Terracini defect $\delta(2S', Y)$ is smaller than $\delta(2S, Y)$. It is natural to wonder what happens if instead we fix the finite set $S \in S(W, r)$ and we compare the behaviour of the two r-th Terracini defects $\delta(2S, W)$ and $\delta(2S, Y)$, where $W \subsetneq Y$ is a smaller multiprojective space. In this case, since Y is no longer the minimal multiprojective space containing $S \subset W \subsetneq Y$, the r-th Terracini defect $\delta(2S, Y)$ may be bigger than $\delta(2S, W)$.

In the following key lemma we explicit this behaviour and we can consider the forthcoming result as a sort of concision for the Terracini locus of a finite set $S \subset Y$. More precisely, in case 3.2 of Lemma 3.2 we give an upper bound for $\delta(2S, Y)$ via $\delta(2S, W)$. Case 3.2 can be considered as a strong version of concision because the achievement of equality $\delta(2S, W) = \delta(2S, Y)$ is telling that the defect of $2S$ is independent of the number of factors of the multiprojective space where S is embedded.

Lemma 3.2. *Let $W \subsetneq Y$ be multiprojective spaces. Let $S \subset W$ be a finite set. Then:*
- (a) $\delta(2S, W) \leq \delta(2S, Y) \leq \delta(2S, W) + (\#S - 1)(\dim Y - \dim W)$.
- (b) *If $Y = W \times Y'$, with Y' a multiprojective space of positive dimension and $\nu(S)$ is linearly independent, then $\delta(2S, W) = \delta(2S, Y)$.*

Proof. Since the restriction map $H^0(Y, \mathcal{O}_Y(1, \ldots, 1)) \longrightarrow H^0(W, \mathcal{O}_W(1, \ldots, 1))$ is surjective and $(2S, W) \subseteq (2S, Y)$, the first inequality of part 3.2 is the first inequality of (3). So we just need to prove the second inequality of 3.2: we will do it by induction on the integer $\dim Y - \dim W$.

First assume $\dim Y = \dim W + 1$. Thus there is $i \in \{1, \ldots, k\}$ such that $W \in |\mathcal{O}_Y(\varepsilon_i)|$. Note that $W \cap (2S, Y) = (2S, W)$ and that $\mathrm{Res}_W(2S, Y) = S$. Thus the residual exact sequence of W gives the following exact sequence

$$0 \longrightarrow \mathcal{I}_S(\hat{\varepsilon}_i) \longrightarrow \mathcal{I}_{(2S,Y)}(1, \ldots, 1) \longrightarrow \mathcal{I}_{(2S,W)}(1, \ldots, 1) \longrightarrow 0. \quad (5)$$

Since the restriction map $H^0(Y, \mathcal{O}_Y(1, \ldots, 1)) \longrightarrow H^0(W, \mathcal{O}_W(1, \ldots, 1))$ is surjective, $h^1\left(Y, \mathcal{I}_{(2S,W)}(1, \ldots, 1)\right) = h^1\left(W, \mathcal{I}_{(2S,W)}(1, \ldots, 1)\right)$. Since S is a finite set, $h^i(\mathcal{L} \otimes \mathcal{O}_S) = 0$ for all $i > 0$ and $\mathcal{L}$ line bundles on Y. The long cohomology exact sequence of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(\hat{\varepsilon}_i) \longrightarrow \mathcal{O}_Y(\hat{\varepsilon}_i) \longrightarrow \mathcal{O}_S(\hat{\varepsilon}_i) \longrightarrow 0$$

gives $h^2(\mathcal{I}_S(\hat{\varepsilon}_i)) = h^2(\mathcal{O}_Y(\hat{\varepsilon}_i)) = 0$. Indeed, we remark that the vanishing of h^2 holds also when $\dim Y = 2$, since in this case by duality we have that $h^2(\mathcal{O}_Y(\hat{\varepsilon}_i)) = h^0(\mathcal{O}_Y(\omega_Y \otimes -\hat{\varepsilon}_i))$. Since $h^1(\mathcal{O}_Y(\hat{\varepsilon}_i)) = 0$ and $\mathcal{O}_Y(\hat{\varepsilon}_i)$ is globally generated, $h^1(\mathcal{I}_S(\hat{\varepsilon}_i))) \leq \#S - 1$. Thus (5) gives part 3.2. Note that we have $\delta(2S, W) = \delta(2S, Y)$ if $h^1(\mathcal{I}_S(\hat{\varepsilon}_i)) = 0$.

Now assume $\dim Y \geq \dim W + 2$. We can always find a multiprojective space M such that $W \subsetneq M \subseteq Y$ and in particular we take $M \in |\mathcal{O}_Y(\varepsilon_i)|$ for some i. The inductive step follows by applying the codimension one case to the inclusion $M \subset Y$ and we conclude by applying the inductive assumption on the inclusion $W \subset M$.

Assume that W is isomorphic to a factor of Y, say $Y \cong W \times Y'$. We will show 3.2 by induction on the number of factors of Y'. Assume $Y = W \times \mathbb{P}^m$ for some $m > 0$, where $W \cong W' \times \{o\}$ for some $o \in \mathbb{P}^m$ and some positive dimensional multiprojective space W'. We will work by induction on $m \geq 1$.
First assume $m = 1$, so $W \in |\mathcal{O}_Y(\varepsilon_2)|$ and in particular $W = \pi_2^{-1}(o)$ where $o \in \mathbb{P}^1$. We remark that the Segre embedding ν_2 of W can be seen as the restriction to W of the Segre embedding of Y. Thus $\nu(S)$ is linearly independent if and only if $\nu_2(S)$ is linearly independent. Note that the linear independence of $\nu_2(S)$ is equivalent to $h^1(\mathcal{I}_S(1,0)) = 0$ because $\pi_2(S) = \{o\}$. Since we already proved part 3.2 and $h^1(\mathcal{I}_S(1,0)) = 0$ we get the result.

Assume now $m \geq 2$ and fix $H \in |\mathcal{O}_Y(\varepsilon_2)|$ containing W. By induction we get $\delta(2S, W) = \delta(2S, H)$. Since H is a divisor of Y and $h^1(\mathcal{I}_S(0,1)) = 0$ we get the result by applying the base case of 3.2.

Assume now Y has $k \geq 3$ factors, i.e. $Y \cong W \times Y'$ where Y' is a multiprojective space with at least two factors. Let $\mathbb{P}^{n_k}$ be the last factor of Y, again we will show the result by induction on $n_k \geq 1$. If $n_k = 1$, one can always find $M \in |\mathcal{O}_Y(\varepsilon_k)|$ containing W and by induction we get $\delta(2S, W) = \delta(2S, M)$. We remark as before that the Segre embedding of $\nu(S)$ is linearly independent if and only if $\nu_k(S)$ is linearly independent and this is equivalent to say that $h^1(\mathcal{I}_S(\hat{\varepsilon}_k)) = 0$. Since $M = \pi_k^{-1}(o)$, for some $o \in \mathbb{P}^1$ we get the result by applying 3.2.

Assume now $n_k \geq 2$, and take some $M \in |\mathcal{O}_Y(\varepsilon_k)|$ containing W. By induction $\delta(2S, W) = \delta(2S, M)$, since $h^1(\mathcal{I}_S(\hat{\varepsilon}_k)) = 0$ and M is a divisor of Y we get $\delta(2S, M) = \delta(2S, Y)$ by 3.2. □

Lemma 3.3. *Let $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and $Y' \subseteq Y$ with $Y' := \mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ for some $m_i > 0$. Let $S \subset Y'$ be a finite subset such that Y' is the minimal multiprojective space containing S and suppose that both $\pi_{1|S}$ and $\pi_{2|S}$ are injective and both $\pi_1(S)$ and $\pi_2(S)$ are linearly independent. Then $m_1 = m_2 = \#S - 1$ and $h^1\left(Y', \mathcal{I}_{(2S,Y')}(1,1)\right) = h^1\left(Y, \mathcal{I}_{(2S,Y)}(1,1)\right)$.*

Proof. Since $\pi_i(S)$ is linearly independent and Y' is the minimal multiprojective space containing S, we have $m_1 = m_2 = \#S - 1$. Moreover since $h^0(\mathcal{I}_S(1,0)) = h^0(\mathcal{I}_S(0,1)) = 0$, the same holds for the corresponding h^1's. To conclude it is sufficient to use the proof of part 3.2 of Lemma 3.2. More precisely, we can consider the following exact sequence

$$0 \longrightarrow \mathcal{I}_S(0,1) \longrightarrow \mathcal{I}_{(2S,Y)}(1,1) \longrightarrow \mathcal{I}_{(2S,Y')}(1,1) \longrightarrow 0$$

and since $h^1(\mathcal{I}_S(0,1)) = 0$ we get $h^1\left(Y', \mathcal{I}_{(2S,Y')}(1,1)\right) = h^1\left(Y, \mathcal{I}_{(2S,Y)}(1,1)\right)$. □

We recall here the Horace Differential Lemma ([7, 9], see also [19]).

Lemma 3.4 (Horace Differential Lemma [7, 9]). *Let M be an integral projective variety, D an integral effective Cartier divisor of M and $\mathcal{L}$ a line bundle on M such that $h^i(\mathcal{L}) = 0$ for all $i > 0$ and $h^1(\mathcal{L}(-D)) = 0$. Set $n := \dim M$. Let $Z \subsetneq M$ be a closed subscheme. Suppose $h^1\left(M, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D)\right) = 0$ and $h^1\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) = 0$. Fix $i \in \{0,1\}$. To prove that a general union A of Z and one double point satisfies $h^i(\mathcal{I}_A \otimes \mathcal{L}) = 0$ it is sufficient to prove that $h^i\left(\mathcal{I}_{\mathrm{Res}_D(Z)\cup(2o,D)} \otimes \mathcal{L}(-D)\right) = 0$ and $h^i\left(D, \mathcal{I}_{(Z\cap D)\cup\{o\}} \otimes \mathcal{L}_{|D})\right) = 0$, where o is a general point of D. Since o is general in D, $h^1\left(D, \mathcal{I}_{(Z\cap D)\cup\{o\}} \otimes \mathcal{L}_{|D})\right) = 0$ if and* *only if* $h^1\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) = 0$ *and* $h^0\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) > 0$. *The same trick works for a general union of* Z *and finitely many double points.*

Proposition 3.5. *Write* $Y = \mathbb{P}^{n_1} \times Y_1$ *as in Notation 2.2. For any fixed point* $o \in \mathbb{P}^{n_1}$ *and closed subscheme* $Z_1 \subset Y_1$, *let* $Z' := \{o\} \times Z_1 \subset Y$. *Then*

$$\dim\langle\nu(Z')\rangle = (n_1+1)(\dim\langle\nu_1(Z_1)\rangle + 1) - 1.$$

Proof. By assumption $h^0\left(Y_1, \mathcal{I}_{Z_1,Y_1}(1,\dots,1)\right) = h^0(\mathcal{O}_{Y_1}(1,\dots,1)) - \dim\langle\nu_1(Z_1)\rangle - 1$. The Künneth formula gives

$$h^0\left(\mathcal{I}_{Z'}(1,\dots,1)\right) = (n_1+1)\left(h^0(\mathcal{O}_{Y_1}(1,\dots,1)) - \dim\langle\nu_1(Z_1)\rangle - 1\right).$$

Since $h^0\left(\mathcal{O}_Y(1,\dots,1)\right) = (n_1+1)h^0\left(\mathcal{O}_{Y_1}(1,\dots,1)\right)$, we get the result. □

Proposition 3.6. *Fix a finite set* $S \subset Y$. *Assume that there exists an index* $i \in \{1,\dots,k\}$ *for which the projection* $\eta_{i|S} : S \longrightarrow Y_i$ *is injective. If* $\delta(2\eta_i(S), Y_i) = 0$, *then also* $\delta(2S, Y) = 0$.

Proof. With no loss of generality we may assume $i = 1$. Set $S' := \eta_1(S)$, $s := \#S$ and $m := n_2 + \cdots + n_k = \dim Y_1$. The submersion $\eta_1 : Y \longrightarrow Y_1$ has the property that $\eta_1^*(O_{Y_1}(1,\dots,1)) \cong \mathcal{O}_Y(\hat{\varepsilon}_1)$ and this isomorphism induces an isomorphism of global sections. By assumption $2S'$ imposes $s(m+1)$ independent conditions on $H^0\left(Y_1, \mathcal{O}_{Y_1}(1,\dots,1)\right)$. Thus the scheme $\eta_1^{-1}(2S')$ imposes $s(m+1)$ independent conditions on $H^0\left(\mathcal{O}_Y(\hat{\varepsilon}_1)\right)$. The scheme $\eta_1^{-1}(S')$ is the union of s disjoint varieties isomorphic to $\mathbb{P}^{n_1}$ and embedded by ν as linear spaces and $\eta_1^{-1}(2S')$ is the union of the first infinitesimal neighborhoods of the corresponding $\mathbb{P}^{n_1}$'s in Y. By Proposition 3.5 the scheme $(2\eta_1^{-1}(S'), Y)$ imposes $s(n_1+1)(m+1)$ independent conditions on $H^0\left(\mathcal{O}_Y(1,\dots,1)\right)$, i.e. the s connected components of $\eta_1^{-1}(2S')$ span linearly independent linear spaces. For each $o \in S$ the scheme $\eta_i^{-1}(2o') = 2\eta_i^{-1}(o')$, $o' := \eta_1(o)$, contains the double point $(2o, Y)$. In the Segre embedding the scheme $\nu((2o, Y))$ gives $\dim Y + 1$ independent conditions. Since the s subspaces spanned by the connected components of $\nu(\eta_1^{-1}(2S'))$ are linearly independent, $\nu(2S, Y)$ is linearly independent, i.e. $\delta(2S, Y) = 0$. □

4. The examples. The following examples will be crucial for the main theorem.

4.1. Example: two of the three points share the last $k-1$ coordinates. In the first example we work over $Y = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$, where $m \in \{1,2\}$ and $k \geq 3$. We consider a set of three distinct points $S \subset Y$ with $S := \{a,b,c\}$ such that a and b share all the last $k-1$ coordinates and we request that Y is the minimal multiprojective space containing S.

Example 4.1. *Let* $Y = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ *for some* $k \geq 3$, *with* $m \in \{1,2\}$. *Define* $S := \{a,b,c\} \subset Y$ *be such that*

$a := (a_1, u_2, \dots, u_k), b := (b_1, u_2, \dots, u_k), c := (c_1, \dots, c_k)$, *with*
$a_1, b_1, c_1 \in \mathbb{P}^m$ *such that* $a_1 \neq b_1$ *and* $u_i \neq c_i$ *for all* $i > 1$.

Moreover if $m = 2$ *assume also* $\dim\langle\pi_1(S)\rangle = 2$.

In the following proposition we prove that a necessary and sufficient condition for a set S of three points as in Example 4.1 to lie in the third Terracini locus $\mathbb{T}(Y, 3)$ is that $k \geq 4$.

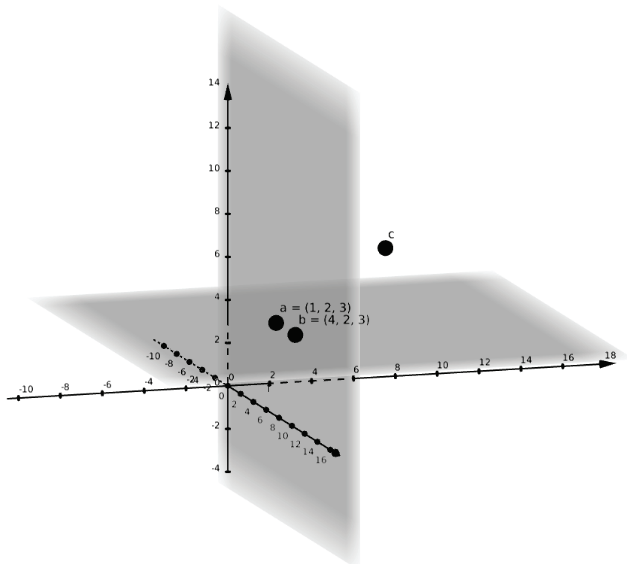

FIG. 1. *Picture of Example 4.1 with $m = 1$.*

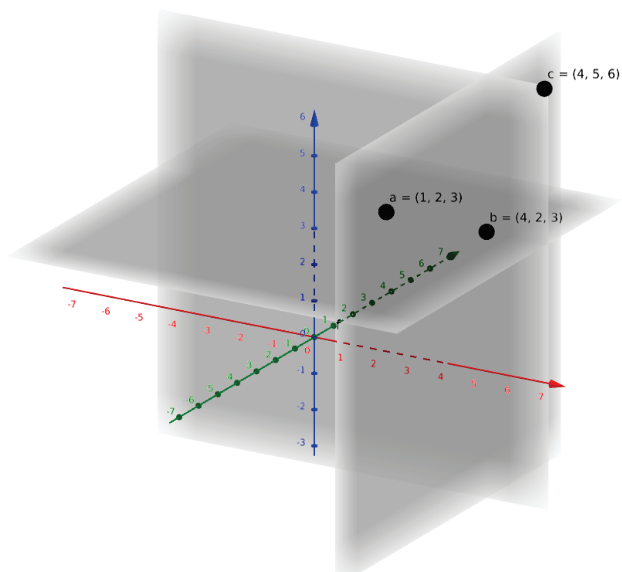

FIG. 2. *Pseudo-picture of Example 4.1 with $m = 2$ (the red axis is a $\mathbb{P}^2$)*

PROPOSITION 4.1. *Let $Y = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ for some $k \geq 3$, with $m \in \{1, 2\}$. Let $S \subset Y$ be as in Example 4.1. Then $S \in \mathbb{T}(Y, 3)$ if and only if $k \geq 4$.*

Proof. We remark that since $\#\pi_i(S) \geq 2$ for all i's and $\#\pi_1(S) = 3$ if $m = 2$, then Y is the minimal multiprojective space containing S. If we consider the subset

$$S' := \{a, b\}$$

of S we may apply Remark 3.1 and have that $\delta(2S, Y) \geq \delta(2S', Y)$. Since $S' \subset \mathbb{P}^m \times \{u_2\} \times \cdots \times \{u_k\} \subset Y$, one can use case 3.2 of Lemma 3.2, with $W := \mathbb{P}^m$, to get

$$\delta(2S', Y) = m + 1.$$

Thus in order to see if $S \in \mathbb{T}(Y, 3)$, it suffices to understand whether

$$h^0\left(\mathcal{I}_{(2S,Y)}(1, \ldots, 1)\right) > 0.$$

- If $k \geq 4$ then $h^0(\mathcal{O}_Y(1, \ldots, 1)) = (m+1)2^{k-1} > 3(m+k) = \deg(2S, Y)$ so we easily have that if $k \geq 4$ then $S \in \mathbb{T}(Y, 3)$.
- Let now $k = 3$. If we show that in this case none of the sets $S \subset Y$ as above belongs to $\mathbb{T}(Y, 3)$ we will be done.
  Remark that by assumption $u_i \neq c_i$ for $i = 2, 3$. To determine whether $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) > 0$ or not, we distinguish two cases depending on m being equal to either 1 or 2.
  (a) Assume $m = 2$, i.e. $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.
  Since $h^0(\mathcal{O}_Y(\varepsilon_1)) = 3$, there exists $H \in |\mathcal{I}_{\{a,c\}}(\varepsilon_1)|$ and remark that

$$H \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

  Since $\langle \pi_1(S) \rangle = \mathbb{P}^2$, then $H \cap S = \{a, c\}$. Consider the residual exact sequence of S with respect to H:

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_H(2S,Y)}(0, 1, 1) \longrightarrow \mathcal{I}_{(2S,Y)}(1, 1, 1) \longrightarrow \mathcal{I}_{H \cap (2S,Y)}(1, 1, 1) \longrightarrow 0.$$

  Since H is smooth then $H \cap (2S, Y) = (2(S \cap H), H) = (2a \cup 2c, H)$ and the residue of $(2S, Y)$ with respect to H is

$$\mathrm{Res}_H(2S, Y) = \{a, c\} \cup (2b, Y).$$

  Remark that $h^0\left(\mathcal{I}_{\mathrm{Res}_H(2S,Y)}(0, 1, 1)\right) = h^0\left(Y_1, \mathcal{I}_{\eta_1(\mathrm{Res}_H(2S,Y))}(1, 1)\right)$.
  Since $\pi_i(a) = \pi_i(b) \neq \pi_i(c)$ for $i = 2, 3$, then $\eta_1(\mathrm{Res}_H(2S, Y)) = \eta_1(\{a, c\} \cup (2b, Y)) = \eta_1(c) \cup (2\eta_1(b), Y_1)$.
  In order to compute $h^0\left(Y_1, \mathcal{I}_{\eta_1(c) \cup (2\eta_1(b), Y_1)}(1, 1)\right)$, we have to look at the hyperplanes of $\mathbb{P}^3$ containing both $\nu_1(\eta_1(c))$ and $T_{\nu_1(\eta_1(b))}\nu_1(Y_1)$. Note that the tangent space $T_{\nu_1(\eta_1(b))}\nu_1(Y_1)$ is spanned by the union of two lines through $\nu_1(b)$, i.e. the set of all $x \in Y_1$ with $\pi_2(x) = \pi_2(b)$ and the set of all $y \in Y_1$ with $\pi_3(y) = \pi_3(b)$. Thus, since $u_i \neq c_i$ for $i = 2, 3$, there are no such hyperplanes, hence

$$h^0\left(Y_1, \mathcal{I}_{\eta_1(\mathrm{Res}_H(2S,Y))}(1, 1)\right) = 0.$$

  So by the residual sequence of S with respect to H recalled above, it is sufficient to prove that $h^0\left(H, \mathcal{I}_{(2a \cup 2c, H)}(1, 1, 1)\right) = 0$.
  Since $\langle \pi_1(S) \rangle = \mathbb{P}^2$ then $\pi_i(a) \neq \pi_i(c)$ for $i = 1, 2, 3$. Since $H \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ thus $\{a, c\}$ is in the open orbit of $\nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ for the action of $(\mathrm{Aut}(\mathbb{P}^1))^3$ on H. Since $\sigma_2(\nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = \mathbb{P}^7$ (cf. e.g. [25, 26, Proposition 2.3]), we have

$$h^0\left(H, \mathcal{I}_{(2\{a,c\},H)}(1, 1, 1)\right) = 0.$$

(b) Assume $m = 1$, i.e. $Y = (\mathbb{P}^1)^3$.
Fix $H \in |\mathcal{I}_a(\varepsilon_3)|$. Since $u_i \neq c_i$ for $i = 2, 3$ and H is smooth we have that $H \cap S = \{a, b\}$ and $\mathrm{Res}_H(2S, H) = \{a, b\} \cup (2c, Y)$. As in the last part of step (a), we remark that $h^0(\mathcal{I}_{\mathrm{Res}_H(2S,Y)}(1, 1, 0)) = h^0(Y_3, \mathcal{I}_{\eta_3(\{a,b\} \cup (2c,Y))})$ and in order to compute it we have to look at the hyperplanes of $\mathbb{P}^3$ containing both $T_{\nu_3(\eta_3(c))}\nu_3(Y_3)$ and $\nu_3(\eta_3(\{a, b\}))$.
So $h^0(\mathcal{I}_{\mathrm{Res}_H(2S,Y)}(1, 1, 0)) = 0$. Moreover, identifying $\nu(H)$ with a smooth quadric surface, by looking at case $k = 2$ of the proof of Proposition 2.9 we get $h^0\left(\mathcal{I}_{(\{a,b\},H)}(1, 1)\right) = 0$, therefore $h^0\left(H, \mathcal{I}_{(2S,Y)}(1, 1, 1)\right) = 0$.

Thus any set of points $S \subset Y$ constructed as above is in the 3-rd Terracini locus $\mathbb{T}(Y, 3)$ if and only if $k \geq 4$. □

4.2. Example: two of the three points share the last $k - 2$ coordinates. In the second example, we work over $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, where $n_1, n_2 \in \{1, 2\}$. We consider $S \subset Y$, with $S := \{u, v, o\}$ such that u and v share just the last $k - 2$ components and we request that Y is the minimal multiprojective space containing S.

Example 4.2. *Let* $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, *where* $n_1, n_2 \in \{1, 2\}$ *and* $k \geq 3$. *Let* $S := \{o, u, v\}$ *where*

$$u = (u_1, u_2, u_3 \ldots , u_n), v = (v_1, v_2, u_3, \ldots , u_n), o = (o_1, \ldots , o_n) \text{ with}$$
$$\langle u_i, v_i\rangle := L_i \cong \mathbb{P}^1 \text{ for } i = 1, 2 \text{ and } o_j \neq u_j \text{ for all } j = 3, \ldots , k.$$

Moreover if $n_i = 2$ *assume also that* $o_i \notin L_i$ *for* $i = 1, 2$.

Remark 4.1. Example 4.1 is not a particular case of Example 4.2. To fix the ideas let $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and take $S, S' \subset Y$ as in Examples 4.1 and 4.2 respectively. Then $S = \{a, b, c\}$ with

$$a = (a_1, a_2, a_3), b = (b_1, a_2, a_3), c = (c_1, c_2, c_3) \text{ such that}$$
$$a_i \neq c_i \text{ for all } i = 2, 3 \text{ and } \langle a_1, b_1, c_1\rangle \cong \mathbb{P}^2,$$

while $S' = \{o, u, v\}$ with

$$u = (u_1, u_2, u_3), v = (v_1, v_2, u_3), o = (o_1, o_2, o_3) \text{ such that}$$
$$u_3 \neq o_3 \text{ and } \langle u_1, v_1, o_1\rangle \cong \mathbb{P}^2.$$

Notice that S' cannot be as in Example 4.1 even if $o_2 \in \{u_2, v_2\}$.

Taking $S \subset Y$ as in Example 4.2, we will prove that

- if $k \geq 4$ then $S \in \mathbb{T}(Y, 3)$;
- if $k = 3$ and $n_1 = n_2 = 2$ then $S \in \mathbb{T}(Y, 3)$;
- if $k = 3$ and either $\{n_1, n_2\} = \{1, 2\}$ or $n_1 = n_2 = 1$ then we need to add more restrictive conditions to the points of S in order to get $S \in \mathbb{T}(Y, 3)$.

The previous properties are an immediate consequence of the following, more precise, result.

Proposition 4.2. *Let* $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, *where* $n_1, n_2 \in \{1, 2\}$ *and* $k \geq 3$. *Let* $S = \{o, u, v\} \subset Y$ *be as in Example 4.2. Set*

$$Y' := L_1 \times L_2 \times \{\pi_3(u)\} \times \cdots \times \{\pi_k(u)\} \subset Y.$$

Then

(i) $2 \le \delta(2S, Y) \le 5$.
(ii) *If* $k \ge 4$ *then* $\delta(2S, Y) = 2$ *and* $h^0\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) = 2 + (n_1 + 1)(n_2 + 1)2^{k-2} - 3(n_1 + n_2 + k - 1) > 0$.
(iii) *If* $k = 3$ *and* $n_1 = n_2 = 2$ *then* $\delta(2S, Y) = 2$ *and* $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) = 2$.
(iv) *If* $k = 3$ *and* $n_1 = n_2 = 1$ *then* $4 \le \delta(2S, Y) \le 5$ *and* $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) > 0$ *if and only if* $\pi_i(u) = \pi_i(o)$ *and* $\pi_h(v) = \pi_h(o)$ *for some* $i, h \in \{1, 2\}$.
(v) *If* $k = 3$ *and* $\{n_1, n_2\} = \{1, 2\}$ *then* $\delta(2S, Y) \ge 3$ *and* $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) > 0$ *if and only if* $\pi_2(o) \in \pi_2(S')$.

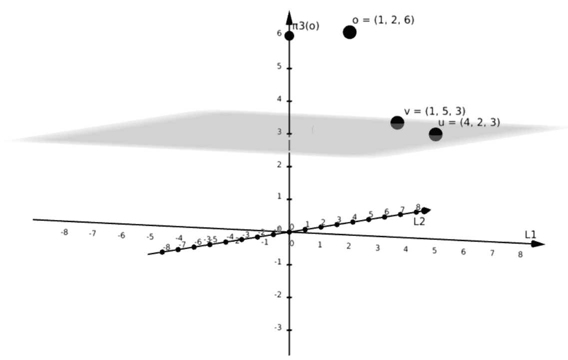

FIG. 3. *Picture of Proposition 4.2.*

Proof. Remark that $S' := \{u, v\} \subset Y'$, and Y' is actually the minimal multiprojective space containing S' while Y is the minimal multiprojective space containing S. Part 3.2 of Lemma 3.2 gives $h^1\left(Y, \mathcal{I}_{(2S',Y)}(1, \dots, 1)\right) = h^1\left(Y', \mathcal{I}_{(2S',Y')}(1, 1)\right)$ and hence $h^1\left(\mathcal{I}_{(2S',Y)}(1, \dots, 1)\right) = h^1\left(\mathcal{I}_{(2S',Y')}(1, 1, 0, \dots, 0)\right)$. Proposition 2.9 (or rather its proof for $k = 2$) and Lemma 3.3 give $h^1\left(Y', \mathcal{I}_{(2S',Y')}(1, 1)\right) = 2$.

Moreover, we remark that in each case we immediately recover the value of $h^0(\mathcal{I}_{(2S,Y)}(1, \dots, 1))$ once we computed $\delta(2S, Y)$ (cf. Lemma 2.5).

(ii) Assume $k \ge 4$. Let M be the only element of $|\mathcal{I}_o(\varepsilon_k)|$. The residual exact sequence of M gives $h^1(Y, \mathcal{I}_{(2S,Y)}(1, \dots, 1)) \le h^1(Y, \mathcal{I}_{(2S',Y)\cup\{o\}}(\hat{\varepsilon}_k))$. Let G be the subgroup of $\mathrm{Aut}(\mathbb{P}^{n_1}) \times \mathrm{Aut}(\mathbb{P}^{n_2}) \times \mathrm{Aut}(\mathbb{P}^1)^{k-2}$ fixing pointwise Y'. The group G has an open orbit U and $o \notin U$. Since $k \ge 4$, we get that $h^0(Y, \mathcal{I}_{(2S',Y)\cup\{o\}}(\hat{\varepsilon}_k)) = h^0(Y, \mathcal{I}_{(2S',Y)}(\hat{\varepsilon}_k)) - 1$. Thus $h^1(Y, \mathcal{I}_{(2S',Y)\cup\{o\}}(\hat{\varepsilon}_k)) = h^1(Y, \mathcal{I}_{(2S',Y)}(\hat{\varepsilon}_k))$. We saw at the beginning of the proof of the proposition that $h^1\left(\mathcal{I}_{(2S',Y)}(1, \dots, 1)\right) = h^1\left(\mathcal{I}_{(2S',Y')}(1, 1, 0, \dots, 0)\right) = h^1\left(Y', \mathcal{I}_{(2S',Y')}(1, 1)\right) = 2$ and this concludes the proof.

(iii) Assume $k = 3$ and $n_1 = n_2 = 2$. Let $M \in |\mathcal{O}_Y(\varepsilon_3)|$ containing o and consider the residual exact sequence of M

$$0 \to \mathcal{I}_{(2S',Y)\cup\{o\}}(1, 1, 0) \to \mathcal{I}_{(2S,Y)}(1, 1, 1) \to \mathcal{I}_{(2o,Y)\cap M}(1, 1, 1).$$

We have $h^1(\mathcal{I}_{(2S,Y)}(1, 1, 1)) \le h^1(\mathcal{I}_{(2S',Y)\cup\{o\}}(1, 1, 0))$ since $h^1(\mathcal{I}_{(2o,Y)}(1, 1, 1)) = 0$. Since $h^0\left(Y, \mathcal{I}_{(2S',Y')}(1, 1, 0)\right) > 0$ and o is in the open orbit U, we have $h^0\left(Y, \mathcal{I}_{(2S',Y')\cup\{o\}}(1, 1, 0)\right) = h^0\left(Y, \mathcal{I}_{(2S',Y')}(1, 1, 0)\right) - 1$. Therefore we have $h^1\left(Y, \mathcal{I}_{(2S',Y')\cup\{o\}}(\hat{\varepsilon}_3)\right) = h^1\left(Y, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_3)\right)$. The result follows since at the beginning of the proof of the proposition we showed that $h^1\left(\mathcal{I}_{(2S',Y)}(1, \dots, 1)\right) = h^1\left(\mathcal{I}_{(2S',Y')}(1, 1, 0, \dots, 0)\right) = h^1\left(Y', \mathcal{I}_{(2S',Y')}(1, 1)\right) = 2$.

(iv) Assume $k = 3$ and $n_1 = n_2 = 1$.
Since $h^0(\mathcal{O}_Y(1,1,1)) = 8$ and $\deg(2S, Y) = 12$, by Lemma 2.5, we have $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 4 + h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)$, so $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) \geq 4$. To conclude this case it is sufficient to show the following

CLAIM 4.2.1. *With the notation as above,* $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) > 0$ *if and only if* $\pi_i(u) = \pi_i(o)$ *for some* $i \in \{1,2\}$ *and* $\pi_j(v) = \pi_j(o)$ *for some* $j \in \{1,2\}$. *In this case* $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 1$ *and* $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 5$.

Proof. Take $H \in |\mathcal{I}_{\{u\}}(\varepsilon_3)|$. Since $\pi_3(u) = \pi_3(v)$, then $H \cap S = S'$. Since H is smooth, $(2S, Y) \cap H = (2S', H)$ and $\mathrm{Res}_H(2S, Y) = S' \cup \{2o\}$. We identify $\nu(H)$ with a smooth quadric surface $Q \subset \mathbb{P}^3$. Since a tangent plane to a smooth quadric surface Q is tangent to Q at a unique point, then we have the vanishing of $h^0\left(H, \mathcal{I}_{(2S,Y)\cap H,H}(1,1,1)\right)$.
Consider the residual exact sequence of S with respect to H:

$$0 \longrightarrow \mathcal{I}_{S'\cup\{2o\}}(1,1,0) \longrightarrow \mathcal{I}_{(2S,Y)}(1,1,1) \longrightarrow \mathcal{I}_{(2S,Y)\cap H,H}(1,1,1) \longrightarrow 0. \quad (6)$$

Since $h^0\left(H, \mathcal{I}_{(2S,Y)\cap H,H}(1,1,1)\right) = 0$, then

$$h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = h^0\left(\mathcal{I}_{S'\cup\{2o\}}(1,1,0)\right).$$

Moreover $h^0\left(\mathcal{I}_{S'\cup\{2o\}}(1,1,0)\right) = h^0\left(Y_3, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_3)}(1,1)\right)$ and we can think of $\nu_3(Y_3)$ as a smooth quadric surface. Since $T_{\nu_3(o)}\nu_3(Y)$ is a plane $h^0\left(Y_3, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_3)}(1,1)\right) \leq 1$.
Now $h^0\left(Y_3, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_3)}(1,1)\right) = 1$ if and only if both $\nu_3(\eta_3(u))$ and $\nu_3(\eta_3(u))$ are contained in $\nu_3(Y_3) \cap T_{\nu_3(\eta_3(o))}(\nu_3(Y_3))$. We remark that $T_{\nu_3(\eta_3(o))}(\nu_3(Y_3))$ is spanned by the union of two lines through $\nu_3(\eta_3(o))$, i.e. the image by ν_3 of the set of all $x \in Y_3$ with $\pi_1(x) = \pi_1(o)$ and the set of all $y \in Y_3$ with $\pi_2(y) = \pi_2(o)$. Hence Claim 4.2.1 is just a translation of this observation. ☐

(v) Assume $\{n_1, n_2\} = \{1,2\}$ and $k = 3$.
With no loss of generality we may assume $n_1 = 2$ and $n_2 = 1$. Since $h^0(\mathcal{O}_Y(1,1,1)) = 12$ and $\deg(2S, Y) = 15$, we have $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 3 + h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)$. Hence $\delta(2S, Y) \geq 3$.
We remark that by assumption $\pi_1(o) \notin \langle \pi_1(u), \pi_1(v)\rangle$, $\pi_2(u) \neq \pi_2(v)$ and $\pi_3(o) \neq \pi_3(u) = \pi_3(v)$.
To conclude this case we have to show that $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) > 0$ if and only if $\pi_2(o) \in \pi_2(S')$.

- Assume $\pi_2(o) \in \pi_2(S')$. Without loss of generality we may assume that $\pi_2(u) = \pi_2(o)$. Since $h^0(\mathcal{O}_Y(\varepsilon_2)) = 2$ then $|\mathcal{I}_o(\varepsilon_2)|$ is a singleton. Set $\{H\} := |\mathcal{I}_o(\varepsilon_2)|$. Since $H \cong \mathbb{P}^2 \times \mathbb{P}^1$ it is smooth, hence $(2S, Y) \cap H = (2\{o,u\}, H)$ scheme-theoretically and $\mathrm{Res}_H(2S, Y) = (2v, Y) \cup \{o, u\}$. Remark that $h^0\left(\mathcal{I}_{(2v,Y)\cup\{o,u\}}(1,0,1)\right) = h^0\left(Y_2, \mathcal{I}_{(2\eta_2(v),Y_2)\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right)$. We remark that $Y_2 \cong \mathbb{P}^2 \times \mathbb{P}^1$ and $h^0(\mathcal{O}_{Y_2}(1,1)) = 6 = \deg((2\eta_2(v), Y_2) \cup \{\eta_2(o), \eta_2(u)\})$. This last equality implies that

$$h^0\left(Y_2, \mathcal{I}_{(2\eta_2(v),Y_2)\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right) = h^1\left(Y_2, \mathcal{I}_{(2\eta_2(v),Y_2)\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right).$$

To show that $h^0\left(Y_2, \mathcal{I}_{(2\eta_2(v),Y_2)\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right) > 0$, we have to look at the hyperplanes of $\mathbb{P}^5 \supset \nu_2(Y_2)$ that contain both the tangent space $T_{\nu_2(\eta_2(v))}\nu_2(Y_2)$ and the points $\nu_2(\eta_2(\{o,u\}))$. Remark that $T_{\nu_2(\eta_2(v))}\nu_2(Y_2) \cap \nu_2(Y_2)$ is the

union of 2 linear spaces containing $\nu_2(\eta_2(v))$, one of dimension 2 and one of dimension 1, spanning the 3-dimensional projective space $T_{\nu_2(\eta_2(v))}\nu_2(Y_2)$. Since $\pi_3(u)=\pi_3(v)$, then $\nu_2(\eta_2(u))$ is a point of the 2-dimensional irreducible component of the tangent space $T_{\nu_2(\eta_2(v))}\nu_2(Y_2)\cap\nu_2(Y_2)$.
Thus $h^0\left(\mathcal{I}_{\mathrm{Res}_H(2S,Y)}(1,0,1)\right)>0$. Hence, by the long cohomology exact sequence induced by the exact sequence of the residue of S with respect to H, we get $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)>0$.

- Assume $\pi_2(o)\notin\pi_2(S')$. Since $h^0\left(\mathcal{O}_Y(\varepsilon_3)\right)=2$ then $|\mathcal{I}_u(\varepsilon_3)|$ is a singleton. Set $\{M\}:=|\mathcal{I}_u(\varepsilon_3)|$. Since M is smooth and $M\cap S=S'$, then $(2S,Y)\cap M=(2S',M)$ scheme-theoretically and $\mathrm{Res}_M(2S,Y)=S'\cup(2o,Y)$. We have $h^0\left(\mathcal{I}_{S'\cup(2o,Y)}(1,1,0)\right)=h^0\left(Y_3,\mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_3)}(1,1)\right)$. Obviously $h^0\left(Y_3,\mathcal{I}_{(2\eta_3(o),Y_3)}(1,1)\right)=2$. Since $\eta_3(S)$ is in the open orbit of $S(Y_3,3)$ for the action of $\mathrm{Aut}(\mathbb{P}^2)\times\mathrm{Aut}(\mathbb{P}^1)$, $h^0\left(Y_3,\mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_3)}(1,1)\right)=0$. The set S' is in the open orbit of $\mathrm{Aut}(M)$ for its action in $S(M,3)$. Since any 3×2 matrix has rank at most 2, we know that $\sigma_2(\nu(M))=\mathbb{P}^5$. Thus $h^0\left(H,\mathcal{I}_{(2S,Y)\cap H,H}(1,1,1)\right)=0$. The residual exact sequence of S with respect to M gives $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)=0$.

In summary if $k=3$ and $\{n_1,n_2\}=\{1,2\}$, then $S\in\mathbb{T}(Y,3)$ if and only if $\pi_2(o)\in\pi_2(S')$.

□

5. Main theorem. In this section we prove the main theorem of the present paper.

REMARK 5.1. Let $Y=(\mathbb{P}^1)^k$, for some $k\geq 2$. Given any two subsets $S,S'\in S(Y,3)$ such that $\#\pi_i(S)=\#\pi_i(S')=3$ for all i's, one can always find $f\in(\mathrm{Aut}(\mathbb{P}^1))^k$ such that $S=f(S')$. Since Y is the minimal multiprojective space containing both S and S', then $S\in\mathbb{T}(Y,3)$ if and only if $S'\in\mathbb{T}(Y,3)$.

LEMMA 5.1. *Let $Y=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ and let $S:=\{u,v,o\}\in S(Y,3)$ such that Y is the minimal multiprojective space containing S. Then $S\in\mathbb{T}(Y,3)$ if and only if there exist $h\in\{1,2,3\}$ and $i,j\in\{1,2,3\}\setminus\{h\}$, with $i<j$ such that*

$$\pi_h(u)=\pi_h(v)\neq\pi_h(o),\qquad \pi_i(o)=\pi_i(u)\neq\pi_i(v),\qquad \pi_j(o)=\pi_j(v)\neq\pi_j(u).$$

Proof. Up to a permutation of the index $h\in\{1,2,3\}$ we may assume $h=3$. Take $S\in S(Y,3)$ and let $X:=\nu(Y)$. If $\#\pi_i(S)=3$ for all i's, since $\dim\sigma_3(X)=7$ (see e.g. [29]), we have, by Remark 5.1, that $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)=0$. Hence $S\notin\mathbb{T}(Y,3)$. Now assume $\#\pi_i(S)\leq 2$ for some i. Remark that since Y is the minimal multiprojective space containing S then $\#\pi_i(S)=2$. We distinguish different cases depending on the number of indices $i\in\{1,2,3\}$ for which $\#\pi_i(S)=2$.

- If there exists only an index i such that $\#\pi_i(S)=2$ then S is as in Example 4.2 and by case (iv) of Proposition 4.2 we know that $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)=0$.
- If $\#\pi_i(S)=2$ for two indices, then S is as in Example 4.2 or as in Example 4.1. For both cases we have $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)=0$ (cf. case (iv) of Proposition 4.2 and Proposition 4.1).
- Finally, if $\#\pi_i(S)=2$ for all $i\in\{1,2,3\}$, then S is as in Example 4.2 and by case (iv) of Proposition 4.2 we get that $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right)=1$.

□

REMARK 5.2. Let $Y=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ and let $S\in S(Y,3)$ such that Y is the minimal multiprojective space containing S. We remark that the characterization of the elements $\mathbb{T}(Y,3)$ presented in Lemma 5.1 is the well-known description of the general element of the tangential variety $\tau(X)$, which is also called W-state in quantum information literature (cf. [24, 15]). Indeed let $S=\{u,v,o\}\in\mathbb{T}(Y,3)$ and without loss of generality take $\{i,j,h\}=\{1,2,3\}$. Then

$$u=(\alpha,b,\gamma),v=(a,\beta,\gamma),o=(\alpha,\beta,c),$$

for some distinct $\alpha,\beta,\gamma,a,b,c\in\mathbb{P}^1$. Now it is straightforward to see that the general $q\in\langle\nu(S)\rangle$ is actually an element of $T_{\nu(p)}X$ where $p=(\alpha,\beta,\gamma)$.

Remark 5.3. Fix $Y=(\mathbb{P}^1)^4$ and let $A\subset Y$ be a general subset of three distinct points. Since the 3-rd secant variety of $X:=\nu(Y)\subset\mathbb{P}^{15}$ is defective with defect 1 (cf. [27, 28]), then $\dim(\sigma_3(X))=13$ and $h^0\left(\mathcal{I}_{(2A,Y)}(1,1,1,1)\right)=2$. Hence, by Lemma 2.5 we get

$$h^1\left(\mathcal{I}_{(2A,Y)}(1,1,1,1)\right)=\deg(2A,Y)-\dim(\sigma_r(X))-1=1.$$

Moreover, by the semicontinuity theorem for cohomology (cf. [40, Ch. III §12]) we get $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1,1)\right)\geq 1$ and $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1,1)\right)\geq 2$ for all $S\in S(Y,3)$. We remark that Y is the minimal multiprojective space containing $S\in S(Y,3)$ if and only if $\pi_i(S)\geq 2$ for all $i=1,2,3,4$.

Thus $S(Y,3)\subset\mathbb{T}_1(Y,3)$ and the 3-rd Terracini locus $\mathbb{T}(Y,3)$ contains all subsets $S\in S(Y,3)$ such that $\pi_i(S)\geq 2$ for all i.

Remark 5.4. Let $Y=(\mathbb{P}^1)^k$ with $k\geq 5$. By [27, Theorem 2.3] we know that $\dim(\sigma_3(X))=3k+2$. Moreover, we recall that for a general S of cardinality 3

$$h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right)=\deg(2S,Y)-\dim(\sigma_r(X))-1=0.$$

So a general $S\subset Y$ with $\#S=3$ is not in the 3-rd Terracini locus $\mathbb{T}(Y,3)$. Thus by Remark 5.1, for all $S\in S(Y,3)$ such that $\#\pi_i(S)=3$ for all i, then $S\notin\mathbb{T}(Y,3)$.

Lemma 5.2. *Let $Y=(\mathbb{P}^1)^k$ with $k\geq 5$. Fix $S:=\{a,b,c\}\in S(Y,3)$ such that Y is the minimal multiprojective space containing S. Assume that there are at least $k-2$ indices i's for which $\pi_i(a)=\pi_i(b)$. Then S is either as in Example 4.2 or as in Example 4.1.*

Proof. Define $E:=\{i\in\{1,\dots,k\}\mid\pi_i(a)=\pi_i(b)\}$, by assumption $\#E\geq k-2$ and since $a\neq b$ then $\#E\leq k-1$. By permuting the factors of Y if necessary, one can always assume that E contains the last $k-2$ indices and that the index $1\notin E$. If $2\notin E$ then S is constructed as in Example 4.2 with $n_1=n_2=1$, else S is as in Example 4.1 where we took $m=1$. □

Lemma 5.3. *Let $Y=(\mathbb{P}^1)^k$ with $k\geq 5$. Fix $S\in S(Y,3)$ such that Y is the minimal multiprojective subspace containing S. If $S\in\mathbb{T}(Y,3)$ then S is either as in Example 4.2 or as in Example 4.1.*

Proof. Write $S:=\{u,v,z\}$. Since $S\in\mathbb{T}(Y,3)$, by Remark 5.4 we may assume that $\pi_i(u)=\pi_i(v)$ for at least one $i\in\{1,\dots,k\}$. With no loss of generality we may assume $i=1$. Since $h^0\left(\mathcal{O}_Y(\varepsilon_i)\right)=2$ for $i=1,2$, both $|\mathcal{I}_u(\varepsilon_1)|$ and $|\mathcal{I}_z(\varepsilon_2)|$ are singletons. Set $\{H\}:=|\mathcal{I}_u(\varepsilon_1)|$ and $\{M\}:=|\mathcal{I}_z(\varepsilon_2)|$. Since $v\in H$, then $S\subset H\cup M$. Moreover, since Y is the minimal multiprojective space containing S, then $z\notin H$ and $\#(S\cap M)\leq 2$.

Claim 5.3.1. *$h^1(\mathcal{I}_S(0,0,1,\dots,1)) = 0$ unless S is either as in Example 4.2 or as in Example 4.1.*

Proof. Call $\eta : Y \longrightarrow (\mathbb{P}^1)^{k-2}$ the projection onto the last $k-2$ factors of Y and set $Y' := (\mathbb{P}^1)^{k-2}$.
Assume $h^1(\mathcal{I}_S(0,0,1,\dots,1)) > 0$. Therefore either $\eta_{|S}$ is not injective or $\#\eta(S) = 3$ and $h^1\left(Y', \mathcal{I}_{\eta(S)}(1,\dots,1)\right) > 0$. In the first case S is either as in Example 4.2 or as in Example 4.1 by Lemma 5.2. In the second case by [17, Lemma 4.4] there is $i \in \{3,\dots,k\}$ such that $\#\pi_h(S) = 1$ for all $h \in \{3,\dots,k\} \setminus \{i\}$ contradicting the minimality of Y' for $\eta(S)$, which is a consequence of the minimality of Y for S. □

Assume by contradiction that S is neither as in Example 4.2 nor as in Example 4.1.

(a) Assume $\#(S \cap M) = 1$, i.e. $S \cap (H \cap M) = \emptyset$. So S is contained in the smooth part of $H \cup M$ and $\mathrm{Res}_{H\cup M}(2S) = S$. Since S is not as in one of the examples, by Claim 5.3.1 we get $h^1\left(\mathcal{I}_{\mathrm{Res}_{H\cup M}(2S,Y)}(0,0,1,\dots,1)\right) = h^1(\mathcal{I}_S(0,0,1,\dots,1)) = 0$. Moreover, by the restriction exact sequence of S (cf. Lemma 2.5), we get $h^0(\mathcal{I}_S(0,0,1,\dots,1)) = 2^{k-2} - 3$. Since by assumption $S \in \mathbb{T}(Y,3)$, then $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 0$ and more precisely $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 2^k - 3(k+1)$, where $k \geq 5$. Thus the residual exact sequence of $H \cup M$ gives $h^1\left(H \cup M, \mathcal{I}_{(2S,H\cup M),H\cup M}(1,\dots,1)\right) > 0$. Since by assumption $S \cap (H \cap M) = \emptyset$, $(2S, H \cup M)$ is equal to $(2u,H) \cup (2v,H) \cup (2z,M)$.
Denote by G the set of all $g \in (\mathrm{Aut}(\mathbb{P}^1))^k$ acting as the identity on the last $k-1$ factors of Y; we remark that the elements of G are 3-transitive on the first factor. Let G_u be the subgroup of G fixing also the first component $\pi_1(u)$ of $u \in S$. Hence, since we assumed $\pi_1(u) = \pi_1(v)$, any $g \in G_u$ fixes both u and v. Obviously $h^1\left(H \cup M, \mathcal{I}_{(2u,H)\cup(2v,H)\cup(2z,M),H\cup M}(1,\dots,1)\right) = h^1\left(H \cup M, \mathcal{I}_{(2u,H)\cup(2v,H)\cup(2g(z),M),H\cup M}(1,\dots,1)\right)$ for all $g \in G_u$. Thus it is sufficient to find a contradiction for a single $z' \in M \setminus H \cap M$ with $\pi_i(z') = \pi_i(z)$ for $i > 1$.
We may specialize z by considering a general $o \in H \cap M$. So it is sufficient to work on H rather than $H \cup M$. Denote by $Z := (2u,H) \cup (2v,H)$ and call A the union of Z and the double point $(2o, H \cap M)$. We want to use the Differential Horace Lemma with $H \cap M$ as a divisor of H (cf. Lemma 3.4). We remark that $Z \subset H$ satisfies the assumptions of the Differential Horace Lemma, i.e. both

$$h^1\left(H, \mathcal{I}_{\mathrm{Res}_{(H\cap M)}(Z)} \otimes \mathcal{L}(-H \cap M)\right) = 0$$

$$\text{and } h^1\left(H \cap M, \mathcal{I}_{Z\cap(H\cap M),H\cap M} \otimes \mathcal{L}_{|H\cap M}\right) = 0,$$

where $\mathcal{L} = \mathcal{O}(1,\dots,1)$. Indeed the latter is trivial since by assumption $\#S \cap M = 1$. The former is zero since H is the minimal multiprojective space containing Z and, by Proposition 2.9 and its proof, we know that $\mathbb{T}(H,2) = \emptyset$ and in particular that $\delta(2\{u,v\}, H) = 0$. Thus by Lemma 3.4, in order to show that $h^1(H, \mathcal{I}_A(1,\dots,1)) = 0$, it suffices to show that both

$$h^1\left(H \cap M, \mathcal{I}_{(Z\cap(H\cap M))\cup\{o\}}(1,\dots,1)\right) = 0$$

$$\text{and } h^1\left(H, \mathcal{I}_{\mathrm{Res}_{H\cap M}(A)}(1,0,1,\dots,1)\right) = 0.$$

Clearly since $(Z\cap(H\cap M))\cup\{o\}=\{o\}$ then $h^1\left(H\cap M,\mathcal{I}_{(Z\cap(H\cap M))\cup\{o\}}(1,\dots,1)\right)$ is trivially zero. The second equality follows from (3) of Remark 3.1 since we already pointed out that $\delta(2\{u,v\},H)=0$.

(b) Assume $\#(S\cap M)=2$. Taking $\{M_i\}=|\mathcal{I}_z(\varepsilon_i)|$ for $i=3,\dots,k$ and applying step (a) to $H\cup M_i$, we see that it is sufficient to handle the case with $\#\pi_i(S)=2$ for all i's.
Write $S=\{a,b,c\}$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 2-transitive, by composing with an element of $\mathrm{Aut}(\mathbb{P}^1)^k$, we may assume $\pi_i(S)=\{\alpha,\beta\}$ for all i's.
Without loss of generality we may also assume $\pi_i(a)=\alpha$ for all i's. Thus $\beta\in\{\pi_i(b),\pi_i(c)\}$ for all i. Moreover, since S is neither as in Example 4.2 nor as in Example 4.1, for all $A\subset S$ with $\#A=2$ then $\#\pi_i(A)=2$ for at least 3 indices i's.
We define the maximum number of common components that any two points of S can have as

$$t:=\max\{\#I\subset\{1,\dots,k\}\,|\,\exists A\subset S \text{ with } \#A=2 \text{ such that } \forall i\in I\ \pi_i(A)=1\}.$$

By relabeling if necessary, we may assume that $\{a,b\}$ is one of the subsets of S reaching such t. By assumption $t\le k-3$.
We distinguish different cases depending on the integer $k\ge 5$. In particular, for $k=5,6$, we will get to a contradiction with the assumption $\delta(2S,Y)>0$ by direct computation with Macaulay2 (cf. [39]).

(i) Assume $k=5$. So $t\le 2$ and since $\#\pi_i(S)=2$ for all i and $k>3$ then $t=2$. Permuting the factors of Y we may assume $\pi_i(b)=\alpha$ for $i=1,2$ and $\pi_i(b)=\beta$ for $i=3,4,5$. Since $\#\pi_i(S)=2$ for all i, then $\pi_1(c)=\pi_2(c)=\beta$. Since a and c have at most 2 common projections, then we may assume $\pi_i(c)=\alpha$ for $i=3,4$ and $\pi_5(c)=\beta$. Thus $S=\{a,b,c\}$ is such that

$$a=(\alpha,\alpha,\alpha,\alpha,\alpha),\ b=(\alpha,\alpha,\beta,\beta,\beta),\ c=(\beta,\beta,\alpha,\alpha,\beta)$$

and, up to a permutation of the elements of S and of the factors of Y, there is a unique such S.
By direct computation one can see that $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1,1,1)\right)=14$ and consequentially $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1,1,1)\right)=0$ contradicting the assumption.

(ii) Assume $k=6$. We have $t\le 3$. Moreover, since $\#\pi_i(S)=2$ for all i, then $t\ge 2$. We distinguish two different cases in dependence on the value $t\in\{2,3\}$.
Assume $t=3$. Permuting if necessary the factors of Y, we may assume $\pi_1(b)=\pi_2(b)=\pi_3(b)=\alpha$ and $\pi_4(b)=\pi_5(b)=\pi_6(b)=\beta$. Thus since $\#\pi_i(S)=2$ for all i's, then $\pi_1(c)=\pi_2(c)=\pi_3(c)=\beta$. Moreover c and a can have 2 or 3 common components. In the first case $S=\{a,b,c\}$ is such that

$$a=(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha),\ b=(\alpha,\alpha,\alpha,\beta,\beta,\beta),\ c=(\beta,\beta,\beta,\beta,\alpha,\alpha).$$

In the second case $S=\{a,b,c\}$ is such that

$$a=(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha),\ b=(\alpha,\alpha,\alpha,\beta,\beta,\beta),\ c=(\beta,\beta,\beta,\alpha,\alpha,\alpha).$$

We remark that up to permuting the factors of Y and relabeling the elements of S, these are the only cases for $t=3$. As before, by direct computation, one gets for both cases $\delta(2S,Y)=0$ contradicting the assumption.
Assume $t=2$. Permuting the factors of Y we may assume $\pi_1(b)=\pi_2(b)=\alpha$ (and hence $\pi_1(c)=\pi_2(c)=\beta$) and $\pi_3(b)=\pi_4(b)=\pi_5(b)=\pi_6(b)=\beta$. Since

$\#\{\pi_i(a),\pi_i(c)\} = \#\{\pi_i(b),\pi_i(c)\} = 1$ for at most 2 indices, among the set $\{3,4,5,6\}$ exactly 2 i's have $\pi_i(c) = \beta$, while the other ones have $\pi_i(c) = \alpha$. Thus $S = \{a, b, c\}$ is such that

$$a = (\alpha,\alpha,\alpha,\alpha,\alpha,\alpha),\ b = (\alpha,\alpha,\beta,\beta,\beta,\beta),\ c = (\beta,\beta,\beta,\beta,\alpha,\alpha).$$

Up to relabeling the points of S and a permutation of the factors of Y, there is a unique such S. By direct computation one gets $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1,1,1,1)\right) = 20$, so $\delta(2S, Y) = 0$ contradicting the assumption.

(iii) Now assume $k \geq 7$. Exchanging if necessary the names of the points of S we may assume $\pi_1(a) = \pi_1(b) = \alpha$ and hence $\pi_1(c) \neq \alpha$. For any $t \in \mathbb{P}^1$ set $S_t := \{a, b, c_t\}$, where $\pi_1(c_t) := t$ and $\pi_i(c_t) := \pi_i(c)$ for all $i > 1$. Since we are not as in Example 4.1 or Example 4.2, any two of the points of S differ in at least three coordinates, hence $\#S_t = 3$ for all t. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, for each $t \in \mathbb{P}^1 \setminus \{\pi_1(a)\}$ there is $g_t \in (\mathrm{Aut}(\mathbb{P}^1)^k) \subset \mathrm{Aut}(Y)$ such that $g_t(S_t) = S$. Thus $\delta(2S) = \delta(2S_t)$ for all $t \in \mathbb{P}^1 \setminus \{\pi_1(a)\}$. Denote $a_1 := \pi_1(a)$, by the semicontinuity theorem for cohomology it is sufficient to prove $\delta(2S_{a_1}, Y) = 0$. To show that $\delta(2S_{a_1}, Y) = 0$, we proceed by induction on the integer $n := k-7$. Assume $n = 0$, i.e. $k = 7$. Since $h^0(\mathcal{O}_Y(\varepsilon_1)) = 2$, $|\mathcal{I}_a(\varepsilon_1)|$ is a singleton. Set $\{H\} := |\mathcal{I}_a(\varepsilon_1)|$, so $H \supset S_{a_1}$ by definition. Since any two points of S differs in at least 3 coordinates, by [17, Lemma 4.4] we know that $h^1\left(H, \mathcal{I}_{S_{a_1}}(\hat{\varepsilon}_1)\right) = 0$. By case 3.2 of Lemma 3.2 we know that $\delta(2S_{a_1}, Y) = \delta(2S_{a_1}, H)$. In item (ii) we proved that for any subset $S \subset (\mathbb{P}^1)^6$ of three points such that any two of them have at least 3 distinct components, then $\delta(2S, (\mathbb{P}^1)^6) = 0$. Thus $\delta(2S_{a_1}, H) = 0$, and hence $\delta(2S_{a_1}, Y) = 0$.
Assume now $n > 0$, i.e. $k > 7$. As before, we set $\{H\} := |\mathcal{I}_a(\varepsilon_1)|$, so $H \supset S_{a_1}$ by definition. By the same argument we get $\delta(2S_{a_1}, Y) = \delta(2S_{a_1}, H)$. If c_{a_1} differs from a and from b in at least 3 coordinates, then the inductive assumption gives $\delta(2S_{a_1}, H) = 0$ and hence $\delta(2S, Y) = 0$. We conclude since $k > 7$ and $\#\pi_i(S) = 2$ for all i, so not all pairs of points of S may differ in only 3 coordinates.
Thus we proved that for all $k \geq 7$, then $\delta(S_{a_1}, Y) = 0$, so by the semicontinuity theorem for cohomology, for all $k \geq 7$ we get $\delta(S, Y) = 0$ contradicting the assumption $\delta(2S, Y) > 0$.

□

LEMMA 5.4. *Let $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Then each $S \in \mathbb{T}(Y, 3)$ is as in Example 4.2 for $k = 3$ and $n_1 = n_2 = 2$.*

Proof. Set $\mathcal{U} := \{S \in S(Y, 3) \,|\, Y$ is the minimal multiprojective space containing S$\}$. So any $S \in \mathcal{U}$ is such that $\#\pi_3(S) \geq 2$, $\pi_{1|S}$ and $\pi_{2|S}$ are injective and $\dim\langle\pi_1(S)\rangle = \dim\langle\pi_2(S)\rangle = 2$. The group $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathcal{U}$ with exactly 2 orbits:

(1) $\#\pi_3(S) = 3$;
(2) $\#\pi_3(S) = 2$.

Call O_1 the first orbit and O_2 the second one. Obviously $h^1(\mathcal{I}_{(2S,Y)}(1,1,1)) = h^1(\mathcal{I}_{(2S',Y)}(1,1,1))$ for all S, S' in the same orbit. Among the elements of O_1 there is the general subset of Y with cardinality 3. Since $\sigma_3(\nu(Y)) = \mathbb{P}^{17}$ (cf. [28]) $h^1\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 0$ for all $S \in O_1$. Note that the elements of O_2 are exactly the sets S described in Example 4.2 for $n_1 = n_2 = 2$ and $k = 3$. □

LEMMA 5.5. *Let $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. The 3-rd Terracini locus $\mathbb{T}(Y, 3)$ is empty.*

Proof. Let $S \in S(Y,3)$ be such that Y is the minimal multiprojective space that contains S, i.e. $\pi_{i|S}$ is injective and $\dim\langle \pi_i(S)\rangle = 2$ for all i. By the action of $\mathrm{Aut}(\mathbb{P}^2)^3$, we can reduce to work with a general set $S \in S(Y,3)$. Since $\sigma_3(X)$ is not defective (cf. [25, Example 4.1]) we know that $\dim(\sigma_3(X)) = 20$, so $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 6$. Hence, by Lemma 2.5, $\delta(2S,Y) = 0$. □

Lemma 5.6. *Let $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. If $S \in \mathbb{T}(Y,3)$ then S is either as in Example 4.1 or as in Example 4.2.*

Proof. Let $S \in S(Y,3)$ such that Y is the minimal multiprojective space containing S, i.e. $\pi_{1|S}$ is injective, $\dim\langle \pi_1(S)\rangle = \mathbb{P}^2$ and $\#\pi_i(S) \geq 2$ for all $i \in \{2,3\}$. We remark that S is as in Example 4.2 or as in Example 4.1 if and only if there exists an index $i \in \{2,3\}$ such that $\#\pi_i(S) = 2$.

Assume by contradiction that S is neither as in Example 4.2 nor as in Example 4.1, i.e. assume that $\#\pi_i(S) = 3$ for $i = 2,3$. Since $\mathrm{Aut}(\mathbb{P}^2)$ is transitive on the set of triples of linearly independent points of $\mathbb{P}^2$ and $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, S is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ on $S(Y,3)$. So we can deal with a general set $S \in S(Y,3)$. By [5, Theorem 4.5] we know that $\sigma_3(X)$ is non-defective, so $\sigma_3(X) = \mathbb{P}^{11}$ and hence $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1)\right) = 0$, contradicting the assumption. □

Lemma 5.7. *Let $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where $k \geq 5$ and $n_i \in \{1,2\}$ for all i's. If $S \in \mathbb{T}(Y,3)$ then S is either as in Example 4.1 or as in Example 4.2. In particular $\mathbb{T}(Y,3) = \emptyset$, unless $n_i = 1$ for at least $k-2$ indices i.*

Proof. We proceed by induction on the integer $t := \dim Y - k$.
The base case $t = 0$ corresponds to Lemma 5.3. Assume $t > 0$ and that the lemma is true for any multiprojective space Y of dimension at most $k + t - 1$. Since $t > 0$, there exists at least an index i such that $n_i = 2$, without loss of generality we may assume $i = 1$. Fix $S \in \mathbb{T}(Y,3)$. So we know that $\delta(2S,Y) > 0$ and Y is the minimal multiprojective space containing S. Thus $\pi_{1|S}$ is injective and $\langle \pi_1(S)\rangle = \mathbb{P}^2$. Fix a general $o \in \mathbb{P}^2 \setminus \pi_1(S)$. Choose a system of homogeneous coordinates $\{x_0,x_1,x_2\}$ of $\mathbb{P}^2$ such that $o = [1:0:0]$, the line $L := \{x_0 = 0\}$ contains no points of $\pi_1(S)$ and o is not contained in one of the three lines spanned by two of the points of $\pi_1(S)$. Let $\ell_o : \mathbb{P}^2 \setminus \{o\} \longrightarrow L$ denotes the linear projection from o, i.e. the rational map defined by $[a_0 : a_1 : a_2] \mapsto [0 : a_1 : a_2]$.

Write $Y = \mathbb{P}^2 \times Y'$ with $Y' = \prod_{i>1} \mathbb{P}^{n_i}$ and set $H := L \times Y' \in |\mathcal{O}_Y(\varepsilon_1)|$. The morphism ℓ_o extends to a morphism

$$f_o : \left(\mathbb{P}^2 \setminus \{o\}\right) \times Y' \longrightarrow H$$
$$(a,b) \mapsto (\ell_o(a), b),$$

We remark that $\#f_o(S) = 3$ and that H is the minimal multiprojective subspace of Y containing $f_o(S)$.

For each $\lambda \in \mathbb{K} \setminus \{0\}$ let $u_\lambda : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ denotes the automorphism of $\mathbb{P}^2$ defined by the formula $[a_0 : a_1 : a_2] \mapsto [\lambda a_0 : a_1 : a_2]$. Let $\mathbb{K}' \subseteq \mathbb{K} \setminus \{0\}$ be the set of all $\lambda \in \mathbb{K} \setminus \{0\}$ such that no line spanned by 2 of the points of $u_\lambda(\pi_1(S))$ contains o. For each $\lambda \in \mathbb{K}'$ we have $\#u_\lambda(\pi_1(S)) = 3$ and $u_\lambda(\pi_1(S))$ spans $\mathbb{P}^2$.
For each $\lambda \in \mathbb{K}'$ define

$$g_\lambda : Y \longrightarrow Y$$
$$(a,b) \mapsto (u_\lambda(a), b).$$

Composing f_o with the inclusion $j : H \subset Y$ we see that the rational map $j \circ f_o$ is a limit for λ going to 0 of the family $\{g_\lambda\}_{\lambda \in \mathbb{K}'}$ of automorphisms of Y. By the semicontinuity theorem for cohomology $\delta(2(j \circ f_o(S)), Y) \geq \delta(2S, Y) > 0$.

CLAIM 5.7.1. $\delta(2g_0(S), H) = \delta(2(j \circ f_o(S)), Y)$.

Proof. Since $\dim Y = \dim H + 1$, part (a) of Lemma 3.2 gives $\delta(2g_0(S), H) \leq \delta(2(j \circ f_o(S)), Y) \leq \delta(2g_0(S), H) + h^1(\mathcal{I}_S(\hat{\varepsilon}_1))$. To conclude the proof of Claim 5.7.1 it is sufficient to prove that $h^1(\mathcal{I}_S(\hat{\varepsilon}_1)) = 0$. Assume $h^1(\mathcal{I}_S(\hat{\varepsilon}_1)) > 0$. By [17, Lemma 4.4] either there are $u, v \in S$ such that $u \neq v$ and $\eta_1(u) = \eta_1(v)$ or there is $i \in \{2, \dots, k\}$ such that $\#\pi_h(S) = 1$ for all $h \in \{2, \dots, k\} \setminus \{i\}$. In the former case, i.e. if $\pi_i(u) = \pi_i(v)$ for all $i > 1$, S is as in Example 4.1. In the second case we are either in Example 4.2 or in Example 4.1. □

By Claim 5.7.1 and the inequality $h^0(\mathcal{O}_H(1, \dots, 1)) > 3 \dim H$ (true because $k \geq 5$) $f_o(S) \in \mathbb{T}(H, 3)$. By the inductive assumption $f_o(S)$ is as in one of the Examples 4.2 or 4.1 and in particular $n_h = 1$ for at least $k-2$ of the last $(k-1)$ indices h, say for $h \in \{3, \dots, k\}$. Moreover there is $A \subset f_o(S)$ such that $\#A = 2$ and $\#\pi_h(A) = 1$ for all $h > 2$. Since f_o acts as the identity on the last $(k-1)$ components of any $p \in Y \setminus H$, we get that S is described by the same example which describes $f_o(S)$. □

LEMMA 5.8. *Take $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} \times \mathbb{P}^{n_4}$ with $n_i \in \{1, 2\}$ for all i's and $n_1 + n_2 + n_3 + n_4 \geq 5$. If $S \in \mathbb{T}(Y, 3)$, then S is either as in Example 4.2 or as in Example 4.1.*

Proof. We will show the result by induction on the integer $t = n_1 + \cdots + n_4 - 5 \geq 0$.

First assume $t = 0$, i.e. $n_1 + n_2 + n_3 + n_4 = 5$. With no loss of generality we may assume $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since Y is the minimal multiprojective space containing S and $n_1 = 2$, the restriction $\pi_{1|S}$ is injective. Assume for the moment that $\pi_{i|S}$ is injective for $i = 2, 3, 4$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, S is in the same orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)^3$ of 3 general points of Y. We know that $\dim \sigma_3(\nu(Y)) = 17$ ([5, Theorem 4.5]), so $\delta(2S, Y) = 0$ contradicting the assumption. Thus we may assume $\#\pi_i(S) = 2$ for some $i \in \{2, 3, 4\}$. With no loss of generality we may assume that at least $\#\pi_3(S) = 2$. Since $\pi_{1|S}$ is injective, $\eta_{4|S}$ is injective. The set $\eta_4(S)$ is as in case (v) of Example 4.2. Using η_2 and η_3 instead of η_4 we see the existence of at least two indices $h \in \{2, 3, 4\}$ such that $\#\pi_h(S) = 2$. With no loss of generality we may assume $\#\pi_3(S) = \#\pi_4(S) = 2$, i.e. neither $\pi_{3|S}$ nor $\pi_{4|S}$ are injective. If there is $S' \subset S$ such that $\#S' = 2$ and $\#\pi_3(S') = \#\pi_4(S') = 1$, then we are in Example 4.2 or Example 4.1. The non-existence of such S' shows that we may name $S = \{a, b, c\}$ so that $\pi_4(a) = \pi_4(b)$, $\pi_3(a) = \pi_3(c)$. We distinguish two cases:

(i) $\#\pi_2(S) = 2$;

(ii) $\#\pi_2(S) = 3$.

Write $a = [a_1, a_2, a_3, a_4]$, $b = [b_1, b_2, b_3, b_4]$ and $c = [c_1, c_2, c_3, c_4]$. Since $\mathrm{Aut}(\mathbb{P}^2)$ is transitive on the set of all triples of linearly independent points, we may assume $a_1 = [1 : 0 : 0]$, $b_1 = [0 : 1 : 0]$ and $c_1 = [0 : 0 : 1]$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive we may assume $a_2 = a_3 = a_4 = \alpha$, $b_3 = \beta$, $b_4 = \alpha$, $c_3 = \alpha$ and $c_4 = \beta$, for some $\alpha \neq \beta \in \mathbb{P}^1$. Moreover, in case (i) we may assume $b_2 = c_2 = \beta$, while in case (ii) we may assume $b_2 = \beta$ and $c_2 = \gamma$, for some $\gamma \in \mathbb{P}^1$ with $\gamma \neq \alpha, \beta$. For both cases, by direct computation one gets $h^0\left(\mathcal{I}_{(2S,Y)}(1,1,1,1)\right) = 17$, so $\delta(2S, Y) = 0$ contradicting the assumption.

Now assume $t > 0$, i.e. $n_1 + n_2 + n_3 + n_4 \geq 6$. As in the proof of Lemma 5.7, one can use a linear projection from a general point of a 2-dimensional factor of Y to conclude by induction on the integer $n_1 + n_2 + n_3 + n_4$.□

REMARK 5.5. The case of $Y = (\mathbb{P}^2)^k$ with $k \geq 4$ is already contained in both Lemma 5.8 and Lemma 5.7 but it can be easily treated as follows. Let $S \in S(Y, 3)$ such that Y is the minimal multiprojective space containing S, i.e. $\pi_{i|S}$ is injective for all $i \leq k$. We can look at S as a general set of three distinct points by the action of $(\mathrm{Aut}(\mathbb{P}^2))^k$. By [5, Theorem 4.5] $\sigma_3(X)$ is never defective, therefore $h^1\left(\mathcal{I}_{(2S,Y)}(1,\ldots,1)\right) = 0$ and hence $\mathbb{T}(Y,3) = \emptyset$.

THEOREM 5.9. *Let Y be the minimal multiprojective space of $k \geq 1$ factors containing a set S of 3 points, where all $n_i \in \{1,2\}$. Then the following characterization of the 3-rd Terracini locus holds.*

$$\mathbb{T}(Y,3) \text{ is empty if and only if either } k = 1,2 \text{ or } Y = (\mathbb{P}^2)^k, \text{ for all } k \geq 3.$$

Moreover the non-empty $S \in \mathbb{T}(Y,3)$ can only be as follows.

- *(Y,S) either as in Example 4.1 or Example 4.2 with $k \geq 4$ factors.*
- *$Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ and $S \subset Y$ as in Example 4.2.*
- *$Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^1$, with $\{n_1, n_2\} = \{1,2\}$ and $S = \{u,v,o\} \subset Y$ as in Example 4.2 with the condition that $\pi_2(o) \in \pi_2(\{u,v\})$.*
- *$Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $S = \{u,v,o\} \subset Y$ as in Example 4.2 such that $\pi_i(u) = \pi_i(o)$ and $\pi_h(v) = \pi_h(o)$ for some $i,j \in \{1,2\}$.*
- *$Y = (\mathbb{P}^1)^4$, in this last case all $S \subset Y$ with $\#(S) = 3$ that have Y as minimal multiprojective space lie in $\mathbb{T}(Y,3)$.*

Proof. Let $S \in \mathbb{T}(Y,3)$ such that Y is the minimal multiprojective space containing S, so $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where all $n_i \in \{1,2\}$.

If $k = 1$ we always have $h^0\left(\mathcal{I}_{2S}(1)\right) = 0$, thus the case of $Y = \mathbb{P}^2$ is clear.

Assume $k = 2$. In this case $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $1 \leq n_1 \leq 2$ and $1 \leq n_1 \leq 2$. If $n_1 = n_2 = 1$, then obviously $h^0\left(\mathcal{I}_{2S}(1,1)\right) = 0$. If $n_i = 2$, then $\pi_{i|S}$ is injective and $\pi_i(S)$ is linearly independent.

Thus if $n_1 = n_2 = 2$, then S is in open orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2)$ of $S(Y,3)$. Since a general 3×3 matrix has rank 3 we get $\sigma_3(\nu(Y)) = \mathbb{P}^8$. Hence $h^0\left(\mathcal{I}_{(2S,Y)}(1,1)\right) = 0$, contradicting the assumption $S \in \mathbb{T}(Y,3)$.

Now assume $n_i = 1$ for exactly one i, say for $i = 1$. Since Y is the minimal multiprojective space containing containing Y, $\#\pi_1(S) \geq 2$ and $\#\pi_2(S) = 3$. Thus there is $S' \subset S$ such that $\#S' = \#\pi_1(S') = 2$. S' is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^2)$ on $S(Y,2)$. Since a general 2×3 matrix has rank 2, $\sigma_2(\nu(Y)) = \mathbb{P}^5$. Thus $h^0\left(\mathcal{I}_{(2S',Y)}(1,1)\right) = 0$. Hence $h^0\left(\mathcal{I}_{(2S,Y)}(1,1)\right) = 0$, contradicting the assumption $S \in \mathbb{T}(Y,3)$. This concludes the case of two factors.

The case of $k = 3$ is completely covered by Lemmas 5.1, 5.5, 5.6 and 5.4. In the case of $k = 4$ there is the defective 3-rd secant variety of the Segre embedding of $Y = (\mathbb{P}^1)^4$ (cf. Remark 5.3). For any other couple (S,Y) where $Y \not\cong (\mathbb{P}^1)^4$, Lemma 5.8 shows that S must be either as in Example 4.1 or as in Example 4.2. If $k \geq 5$ it is sufficient to use Lemma 5.7. □

6. Computing the maximal r-th Terracini defect. Fix any multiprojective space Y of dimension $n > 0$. For any $p \in Y$ the very ampleness of $\mathcal{O}_Y(1,\ldots,1)$ implies $h^1\left(\mathcal{I}_{(2p,Y)}(1,\ldots,1)\right) = 0$. For any integer $r \geq 2$ there are many $S \in S(Y,r)$

with $\delta(2S, Y) > 0$. In the following we compute the maximal value of all $\delta(2S, Y)$ for some multiprojective space Y of dimension n.

DEFINITION 6.1. For any integer $n > 0$, denote by $\mathcal{U}(n)$ the set of all multiprojective spaces of dimension n. For any integer $r \geq 2$, $n \geq 2$ define

$$\mathcal{E}(n, r) := \{(Y, S) \in \mathcal{U}(n) \times S(Y, r) \mid S \in \mathbb{T}_1(Y, r)\},$$

$$\mathbb{E}(n, r) := \{(Y, S) \in \mathcal{U}(n) \times S(Y, r) \mid S \in \mathbb{T}(Y, r)\}.$$

The set of all (n, r) such that $\mathcal{E}(n, r) \neq \emptyset$ is easily computed in Lemma 6.4 and we will show that $\mathcal{E}(n, r) \neq \emptyset$ if and only if $n \geq 3$ and $r \geq 2$.

NOTATION 6.2. Fix integers $n, r > 0$ and recall that given a finite $S \subset Y$, we defined $\delta(2S, Y) = h^1(\mathcal{I}_{(2S,Y)}(1, \dots, 1))$. Denote by

$$\delta_1(n, r) := \max\{\delta(2S, Y) \mid (Y, S) \in \mathcal{E}(n, r)\}.$$

We remark that given any $S \in S(Y, r)$ such that $h^0\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) > 0$, asking whether $S \in \mathbb{T}_1(Y, r)$ is equivalent to request that $\delta(2S, Y) > 0$. Similarly, if $S \in S(Y, r)$ is such that $\delta(2S, Y) > 0$, then to show that $S \in \mathbb{T}_1(Y, r)$ it suffices to prove $h^0\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) > 0$. In Proposition 6.5 we will show that

$$\delta_1(n, r) = (r - 1)(n + 1) - 1.$$

If we also prescribe that $(Y, S) \in \mathbb{E}(n, x)$, i.e. if we assume that Y is the minimal multiprojective space containing S, then we get the definition of the integer $\delta(n, x)$.

PROPOSITION 6.3. *Fix integers $n > 0$ and $r \geq 2$. Fix $Y \in \mathcal{U}(n)$ and $S \in S(Y, r)$. Then*

$$h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) \leq (r - 1)(n + 1).$$

The equality holds if and only if $Y = \mathbb{P}^n$.

Proof. Fix $Y \in \mathcal{U}(n)$, say $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $n_i > 0$ for all i's and $n_1 + \cdots + n_k = n$ and assume $k \geq 2$, i.e. assume $Y \ncong \mathbb{P}^n$. Fix $S \in S(Y, r)$ and take $o \in S$. Since $\mathcal{O}_Y(1, \dots, 1)$ is very ample, we have $h^1\left(\mathcal{I}_{(2o,Y)}(1, \dots, 1)\right) = 0$. Thus by (4) of Remark 3.1 we get $h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) \leq \deg(2(S \setminus \{o\}), Y)\big) = (r-1)(n+1)$, concluding the proof of the inequality.

Let us prove now that $h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) = (r-1)(n+1)$ if and only if $Y = \mathbb{P}^n$. The "if" part of the equality is clear, so we just need to prove the "only if" part and for this we will use induction on the integer n starting with the case $n = 2$.

Let $n = 2$ and assume by contradiction that $Y \ncong \mathbb{P}^2$, so $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Thus $h^0(\mathcal{O}_Y(1, 1)) = 4$. Since each tangent plane of $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ is tangent to a unique point of the smooth quadric $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ and $r \geq 2$, we have $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1)\right) = 0$ and hence $h^1\left(\mathcal{I}_{(2S,Y)}(1, 1)\right) = 3(r - 1) - 1 \neq 3(r - 1)$. Now assume $n > 2$ and remark that by assumption $h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) = (r-1)(n+1)$ and we have to prove that $Y = \mathbb{P}^n$. We distinguish two different cases depending on whether $r = 2$ or not.

(a) Assume $r = 2$. By assumption $h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) = n+1$. Since $\deg(2S) = 2(n + 1)$, we get $h^0(\mathcal{O}_Y(1, \dots, 1)) = n + 1$. Since $\dim Y = n$ and $\mathcal{O}_Y(1, ..., 1)$ is very ample, we get $Y \cong \mathbb{P}^n$.

(b) Assume $r > 2$. Write $S = A \cup B$ with $\#A = 2$ and $\#B = r-2$. By part (a) we have $h^1\left(\mathcal{I}_{(2A,Y)}(1,\dots,1)\right) \le n$. Thus by 4 of Remark 3.1 we get $h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) \le h^1\left(\mathcal{I}_{(2A,Y)}(1,\dots,1)\right) + \deg(2B,Y) \le n + (r-2)(n+1)$, which is absurd since by assumption $h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = (r-1)(n+1)$. □

Theorem 6.5 below shows that the following example is the only one which achieves the maximum $\delta_1(n,r)$.

Example 6.1. *Let $n \ge 3$, fix an integer $1 \le \mu \le n-1$ and let $r \ge \mu+1$, so $r > 1$ since $\mu \ge 1$. Let $L \subset \mathbb{P}^{n-1}$ be a μ-dimensional linear subspace and let $Y := \mathbb{P}^{n-1} \times \mathbb{P}^1$. Fix $o \in \mathbb{P}^1$ and and fix any finite set $S \subset L \times \{o\}$ with $\#S = r$ and such that $\langle \pi_1(S)\rangle = L$. The aim of this example is to show that $\delta(2S,Y) = (r-1)(n+1)-\mu$.*

Take $H := \pi_2^{-1}(o) \in |\mathcal{O}_Y(\varepsilon_2)|$. Note that $S \subset H$. Thus the residual exact sequence of $(2S,Y)$ with respect to H is

$$0 \longrightarrow \mathcal{I}_S(1,0) \longrightarrow \mathcal{I}_{(2S,Y)}(1,1) \longrightarrow \mathcal{I}_{(2S,H),H}(1,1) \longrightarrow 0. \tag{7}$$

We remark that $S \neq \emptyset$ and in particular $\#S \ge 2$. Moreover $H \cong \mathbb{P}^{n-1}$, so $h^0\left(H, \mathcal{I}_{(2S,H)}(1,1)\right) = 0$. Since by assumption $\langle \pi_1(S)\rangle = L$, where $\dim L = \mu$, we get $h^0\left(\mathcal{I}_S(1,0)\right) = n-1-\mu$. So by (7) we get $h^0\left(\mathcal{I}_{(2S,Y)}(1,1)\right) = n-1-\mu$. Thus $\delta(2S,Y) = r(n+1) - 2n + n - \mu - 1 = (r-1)(n+1) - \mu$.

In particular for $\mu = 1$, i.e. if L is a line, we obtain $\delta(2S,Y) = (r-1)(n+1)-1$. Since $h^0\left(\mathcal{O}_Y(1,1)\right) = 2n$ and $\deg(2S,Y) = r(n+1)$, when $\mu = 1$ we get $h^0\left(\mathcal{I}_{(2S,Y)}(1,1)\right) = 2n - r(n+1) + (r-1)(n+1) - 1 = n-2 > 0$. Thus if $\mu = 1$, $S \in \mathbb{T}_1(Y,r)$ and in particular $\delta_1(S,Y) = (r-1)(n+1)-1$.
Obviously also $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ gives an example, taking an L in the second factor of Y.

Lemma 6.4. *Fix integers $n \ge 2$ and $r \ge 2$. $\mathcal{E}(n,r) \neq \emptyset$ if and only if $n \ge 3$.*

Proof. For $n = 2$ we remark that $\mathcal{U}(2) = \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$. For both cases, by Proposition 6.3 we get $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = 0$. Viceversa, if $n \ge 3$ we may take $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$ and S as in Example 6.1. □

Remark 6.1. Let $n > 0$ and $r \ge 2$. By Proposition 6.3, for all $Y \in \mathcal{U}(n)$ and $S \in S(Y,r)$ the maximum value of $h^1\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right)$ is achieved when $Y = \mathbb{P}^n$. Clearly if $Y = \mathbb{P}^n$, $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) = 0$. Thus the couple $(Y,S) \in \mathcal{U}(n) \times S(Y,r)$ evincing $\delta_1(n,r)$ is such that Y is a multiprojective space with $k \ge 2$ factors.

Theorem 6.5. *Fix integers $n \ge 3$ and $r \ge 2$. Then $\delta_1(n,r) = (r-1)(n+1)-1$ and any (Y,S) with Y of $k \ge 2$ factors evincing $\delta_1(n,r)$ is as in Example 6.1 with $\mu = 1$.*

Proof. By Proposition 6.3, for all (Y,S)

$$\delta(2S,Y) \le (r-1)(n+1)-1.$$

The case $\mu = 1$ of Example 6.1 gives the inequality $\delta_1(n,r) \ge (r-1)(n+1)-1$. Thus it remains to prove that this is the only case.

Fix (Y,S) evincing $\delta_1(n,r)$. Thus $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where all $n_i > 0$ and are such that $n_1 + \cdots + n_k = n$. The finite set $S \in S(Y,r)$, is such that $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right) > 0$ and $h^1\left(\mathcal{I}_{(2S,Y)}(1,1)\right) \ge (r-1)(n+1)-1$.

We will show the result by induction on $n \ge 3$.

If $n = 3$ then $\mathcal{U}(3) = \{\mathbb{P}^3, \mathbb{P}^2 \times \mathbb{P}^1, (\mathbb{P}^1)^3\}$.

Clearly the case $Y = \mathbb{P}^3$ is excluded by Remark 6.1. If $Y = \mathbb{P}^2 \times \mathbb{P}^1$, it suffices to show that for any other r-uple of points $\hat{S} \in S(Y, r)$ that is not as in Example 6.1, we get $\delta(2\hat{S}, Y) < 4(r-1) - 1$. If $r = 2$ this is true since $\delta(2S, Y) = 2$ unless $S \in S(Y, 2)$ is as in Example 6.1. Moreover, since any 3×2 matrix has rank at most 2, $h^0(\mathcal{I}_{2S}(1, 1)) = 0$ if $r \geq 3$ and $\mathbb{P}^2 \times \mathbb{P}^1$ is the minimal multiprojective space containing S.

Let $Y = (\mathbb{P}^1)^3$. By Proposition 2.9 we exclude the case $r = 2$ since either $\delta(2S, Y)$ or $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right)$ is zero. Notice that if $h^0\left(\mathcal{I}_{2S}(1, 1, 1)\right) = 0$, then $\delta(2S, Y) = 4r - 8$. Now assume $h^0\left(\mathcal{I}_{2S}(1, 1, 1)\right) > 0$. If $r = 3$, Lemma 5.1 gives the only cases for which $S \in \mathbb{T}(Y, 3)$ and for such cases we already proved that $\delta(2S, Y) = 5 < \delta_1(3, 3)$ and $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) = 1$. Now assume $r \geq 4$ and write $S = S' \cup S''$ with $\#S' = 3$ and $S' \cap S'' = \emptyset$. We have

$$\delta(2S, Y) \leq 4(r-3) + \delta(2S', Y).$$

Thus for $r \geq 4$ we get $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1, 1)\right) = 0$ for all $S \in S(Y, r)$ that are not as in Example 6.1.

Assume that the proposition is true for all $n' < n$. We will prove the inductive step by induction on $r \geq 2$. Case (a) will be the base case and in case (b) we will show the inductive step.

(a) Assume $r = 2$ and let $L := \langle \nu(S) \rangle$.
First assume that Y has $k = 2$ factors, i.e. $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. With no loss of generality we may assume $n_1 \geq n_2$. To conclude this case it is sufficient to prove that $n_2 = 1$ and $\#\pi_2(S) = 1$ and we will do it by contradiction.
First assume $n_2 \geq 2$. Since $h^0\left(\mathcal{O}_Y(0, 1)\right) = n_2 + 1 > 2$, there is $M \in |\mathcal{I}_S(0, 1)|$. Thus $S \subset M$. If (S, M) is as in Example 6.1 there is nothing to prove, otherwise by the inductive step we get $h^1\left(M, \mathcal{I}_{(2S,M)}(1, 1)\right) \leq n - 2$. Since $\dim Y = \dim M + 1$, part 3.2 of Lemma 3.2 gives $h^1(\mathcal{I}_{(2S,Y)}(1, 1)) \leq n - 2 + 1 < n$ which is absurd since we took (Y, S) evincing $\delta_1(2, n) = n$.
Assume now that $\#\pi_2(S) = 2$. Again if $\#\pi_1(S) = 1$ then S is as in Example 6.1, so assume also $\#\pi_1(S) = 2$. Thus the minimal multiprojective space containing S is $Y = \mathbb{P}^1 \times \mathbb{P}^1$. So S is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^1)^2$ on $S(Y, 3)$. Hence $h^0\left(\mathcal{I}_{(2S,Y)}(1, 1)\right) = 0$ and consequently, since $\deg(2S, Y) = 15$ and $h^0\left(\mathcal{O}_Y(1, 1)\right) = 9$, we get $\delta(2S, Y) = 6 < \delta_1(4, 3)$.
Assume now Y has $k > 2$ factors. By Lemma 3.2 and the equality $\delta_1(n', 2) = (r-1)(n'+1) - 1$ for all $n' < n$, Y is the minimal multiprojective space containing S. Thus $Y = (\mathbb{P}^1)^k$. Fix $H \in |\mathcal{O}_Y(\varepsilon_k)|$ containing at least on point of S. Since $S \not\subseteq H$, $\#(S \cap H) = \#(S \setminus S \cap H) = 1$. Denote by $S := \{a, b\}$ and by relabeling if necessary, assume $S \cap H = \{a\}$ and $S \setminus S \cap H = \{b\}$. Consider the residual exact sequence of H:

$$0 \longrightarrow \mathcal{I}_{(2b,Y) \cup (a,Y)}(\hat{\varepsilon}_k) \longrightarrow \mathcal{I}_{(2S,Y)}(1, \dots, 1) \longrightarrow \mathcal{I}_{(2a,H),H}(1, \dots, 1) \longrightarrow 0. \quad (8)$$

Since $\#(S \cap H) = 1$ and $\mathcal{O}_H(1, \dots, 1)$ is very ample, $\delta(2a, H) = 0$. Since $\#(S \setminus S \cap H) = 1$, $\mathcal{O}_{Y_k}(1, \dots, 1)$ is very ample and $\dim Y - \dim Y_k = 1$, we have $h^1\left(\mathcal{I}_{(2b,Y)}(\hat{\varepsilon}_k)\right) = 0$. Since $\#(S \cap H) = 1$, $h^1\left(H, \mathcal{I}_{(2b,Y) \cup (a,Y)}(\hat{\varepsilon}_k)\right) \leq 1$. Thus (8) gives $h^1\left(\mathcal{I}_{(2S,Y)}(1, \dots, 1)\right) \leq 1 < n$, a contradiction.

(b) Assume now $r \geq 3$. Fix any $A \subset S$ such that $\#A = r - 1$. Since $\delta(2S, Y) \leq \delta(2A, Y) + n + 1$ (cf. Remark 3.1), the inductive assumption gives the pair (Y, A) is as in Example 6.1. Thus either $Y \cong \mathbb{P}^{n-1} \times \mathbb{P}^1$ or $Y \cong \mathbb{P}^1 \times$

$\mathbb{P}^{n-1}$. With no loss of generality we may assume $\mathbb{P}^{n-1}\times\mathbb{P}^1$. The inductive assumption gives the existence of a line $L_A\subset\mathbb{P}^{n-1}$ and a point $o_A\in\mathbb{P}^1$ such that $A\subset L\times\{o_A\}$. Since $r\geq 3$ there is $B\subset S$ with $\#B=r-1$, $B\cap A\neq\emptyset$ and $B\neq A$. We get $\{o_A\}=\pi_2(A)=\pi_2(B)=\{o_B\}$. Thus $\#\pi_2(S)=1$.
To conclude the proof it is sufficient to show that $\pi_1(S)$ spans a line and we will do it by induction on $r\geq 3$. Take for the moment $r=3$, assume that $\langle\pi_1(S)\rangle$ is a plane and set $M:=\langle\pi_1(S)\rangle\times\mathbb{P}^1$. By part 3.2 of Lemma 3.2 and the assumption $\delta(2S,Y)=2(\dim Y+1)-1$, we have $\delta(2S,M)\geq 2(\dim M+1)-1$. Moreover $\delta(2S,M)=2(\dim M+1)-1=7$, because M is not a projective space. However, by direct computation, one gets $\delta(2S,M)=3(\dim M+1)-6=6$.
Let $r\geq 4$. Take any 2 distinct subsets A, B of r with $\#A=\#B=r-1$. Since $\#A\cap B=r-2\geq 2$, the lines L_A and L_B have at least 2 common points. Thus $L_A=L_B$. Hence $\pi_1(S)$ spans a line.

□

Example 6.1 gives the following result, the last equality being true by Theorem 6.5.

THEOREM 6.6. *Fix integers $n>\mu\geq 2$ and $r\geq\mu+1$. Then there is $(Y,S)\in\mathcal{E}(n,r)$ such that $\delta(2S,Y)=(r-1)(n+1)-\mu=\delta_1(n,r)-\mu+1$.*

REMARK 6.2. If $S\in\mathcal{S}(Y,2)$ is such that Y is the minimal multiprojective space containing S, then $Y=(\mathbb{P}^1)^k$, for some $k\geq 1$. Proposition 2.9 gives $\mathbb{E}(n,2)=\emptyset$. By Theorem 5.9 we have that $\mathbb{E}(n,3)$ is given by all couples (Y,S) where $Y=\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\times(\mathbb{P}^1)^{n-n_1-n_2}$ with $1\leq n_2\leq n_1\leq 2$ and $n>n_1+n_2$, $S\subset Y$ is of cardinality 3 such that Y is the minimal multiprojective space containing S.

EXAMPLE 6.2. *Let $r,n\geq 3$ and let $Y:=(\mathbb{P}^1)^n$. Take $A\subset\mathbb{P}^1$ such that $\#A=r-1$ and define $S:=\{p_1,\dots,p_r\}\subset Y$ where*

$$p_i=(a_i,e_2,\dots,e_n)\ \text{for}\ i=1,\dots,r-1\ \text{with all}\ a_i\in A\ \text{and all}\ e_j\in\mathbb{P}^1$$
$$p_r=(o_1,\dots,o_r),\ \text{with}\ o_1\in\mathbb{P}^1\setminus A,\ o_k\in\mathbb{P}^1\ :\ o_k\neq e_k\ \text{for all}\ k=2,\dots,n$$

Set $S'=S\setminus\{p_r\}$ and let $Y':=\mathbb{P}^1\times\{e_2\}\times\cdots\times\{e_n\}$. Note that $Y'\cong\mathbb{P}^1$ is the minimal multiprojective subspace containing S' and that Y is the minimal multiprojective subspace containing S. From (4) of Remark 3.1 we know that $\delta(2S,Y)\geq\delta(2S',Y)\geq\delta(2S',Y')=\delta(2A,\mathbb{P}^1)=2(r-2)>0$. Take $H:=\pi_n^{-1}(e_n)\in|\mathcal{O}_Y(\varepsilon_n)|$. Since $S'\subset H$, the residual exact sequence of $2S'$ with respect to H gives

$$0\longrightarrow\mathcal{I}_{S'}(1,\dots,1,0)\longrightarrow\mathcal{I}_{(2S',Y)}(1,\dots,1)\longrightarrow\mathcal{I}_{(2S',H),H}(1,\dots,1)\longrightarrow 0 \tag{9}$$

Thus (9) gives $h^0\left(\mathcal{I}_{(2S',Y)}(1,\dots,1)\right)\geq h^0\left(\mathcal{I}_{S'}(1,\dots,1)\right)$. Since $\nu(S')$ spans a line, $h^0\left(\mathcal{I}_{S'}(1,\dots,1)\right)=2^n-2$. Since $n\geq 3$, $h^0\left(\mathcal{I}_{(2S',Y)}(1,\dots,1)\right)\geq n+2$. Thus $\delta(2S,Y)>0$ and $h^0\left(\mathcal{I}_{(2S,Y)}(1,\dots,1)\right)>0$.

The previous example gives us a bound on the quantity $\delta(n,r)$, introduced in Notation 6.2, namely

$$\delta(n,r)\geq 2(r-2).$$

We do not have a general formula about this quantity, but one can easily notice the following. Clearly $\delta(n,r)\leq\delta_1(n,r)=(r-1)(n+1)$. Moreover, if $r>n$, then

$\delta(n,r) = \delta_1(n,r)$, because r general points of $\mathbb{P}^n$ span $\mathbb{P}^n$. If $r = 2$ to compute $\delta(n,r)$ we only need to test $(\mathbb{P}^1)^n$ and we know that $\delta(n,2) = 0$ for $n > 2$, while $\delta(2,2) = 2$ since $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = 4$ and a general 2×2 matrix has rank 2.

PROPOSITION 6.7. *$\mathbb{E}(n,r) \neq \emptyset$ if and only if $n \geq 3$ and $r \geq 3$.*

Proof. If $n \geq 3$ and $r \geq 3$, Example 6.2 shows that $\mathbb{E}(n,r) \neq \emptyset$. The other implication follows from Lemma 6.4 since $\mathbb{E}(n,r) \subseteq \mathcal{E}(n,r)$ and $\mathbb{T}(Y,2) = \emptyset$ (cf. Proposition 2.9). □

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