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ABSTRACT

In this paper, we classify the orbits of the group $\mathrm{SL}(3,F)^3$ on the space $F^3 \otimes F^3 \otimes F^3$ for $F = \mathbb{C}$ and $F = \mathbb{R}$. This is known as the classification of complex and real three-qutrit states. We also give an overview of physical theories where these classifications are relevant.

I. INTRODUCTION

Elements of the vector space $(\mathbb{C}^3)^{\otimes n} = \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3$ (n factors $\mathbb{C}^3$) are called *n-qutrit states*. On this space acts the group $\mathrm{SL}(3,\mathbb{C})^n$, also called the SLOCC (Stochastic Local Quantum Operations assisted by Classical Communication) group. One of the basic problems in this context is to determine the orbits of this group or, in other words, to determine the classification of the n-qutrit states under SLOCC-equivalence. This has applications in quantum information theory, as well as in high energy particle physics; see Sec. II A.

For $n = 3$, the group and its representation can be constructed using a grading of the simple Lie algebra of type $\mathfrak{e}_6$ (see Sec. IV for the details). As a consequence, Vinberg's theory of θ-groups[1,2] can be used to classify the orbits. This has been carried out by Nurmiev in Ref. 3. In this paper, we revisit his classification, adding many details, such as descriptions of the stabilizers of the orbit representatives.

One of the main features of the construction of the representation using a grading is that the elements of $(\mathbb{C}^3)^{\otimes 3}$ are shown to have a Jordan decomposition. That is, each $v \in (\mathbb{C}^3)^{\otimes 3}$ can uniquely be written as $v = s + e$, where $s, e \in (\mathbb{C}^3)^{\otimes 3}$, s is *semisimple*, e is *nilpotent*, and $[s,e] = 0$ (the latter is the Lie product in the simple Lie algebra of type $\mathfrak{e}_6$). Here, the terms semisimple and nilpotent can be defined in the Lie algebra of type $\mathfrak{e}_6$. It is also possible to define them more geometrically as follows. An element $s \in (\mathbb{C}^3)^{\otimes 3}$ is semisimple if its orbit under $\mathrm{SL}(3,\mathbb{C})^3$ is closed (in the Zarisky topology). An element $e \in (\mathbb{C}^3)^{\otimes 3}$ is nilpotent if the closure of its orbit under $\mathrm{SL}(3,\mathbb{C})^3$ contains 0. An element is said to be *mixed* if it is neither semisimple nor nilpotent. In this paper, we show the following result.

Theorem 1.1. *There are 90 classes of elements of $(\mathbb{C}^3)^{\otimes 3}$ with the following properties. Each element of $(\mathbb{C}^3)^{\otimes 3}$ is $\mathrm{SL}(3,\mathbb{C})^3$-conjugate to an element of precisely one class. All elements of a fixed class have the same stabilizer in $\mathrm{SL}(3,\mathbb{C})^3$. We have 63 classes consisting of one nilpotent element (see Table I). There are four classes consisting of infinitely many semisimple elements and depending on, respectively, 3, 2, 1, and 1 parameters (see Sec. VII A). Elements of one of these classes are $\mathrm{SL}(3,\mathbb{C})^3$-conjugate if and only if they are conjugate under an explicitly given finite group of orders 648, 18, 6, and 3, respectively. Finally, there are in total 23 classes of mixed elements (see Tables III, IV, and V).*

TABLE I. Nilpotent complex three-qutrits.

N	3-qtrit	Char.	σ	$Z_{\widehat{G}}(h,e)^{\circ}$	K
1	$\lvert 012\rangle + \lvert 021\rangle + \lvert 102\rangle + \lvert 111\rangle + \lvert 120\rangle + \lvert 200\rangle$	6 12 6 6 6 6 6 6 6 6 6 12 6 6 6 12 6 6	id (12) (132)	id	$(\mathbb{Z}/3\mathbb{Z})^2$
2	$\lvert 012\rangle + \lvert 021\rangle + \lvert 102\rangle + \lvert 110\rangle + \lvert 111\rangle + \lvert 200\rangle$	6 6 0 6 6 6 6 6 6 6 0 6 0 6 6 6 6 6	id (23) (123)	id	$(\mathbb{Z}/3\mathbb{Z})^2$
3	$\lvert 002\rangle + \lvert 011\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 200\rangle$	0 6 6 0 0 6 0 6 0 6 6 0 6 0 0 6 0 6	id (23) (123)	id	$(\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$
4	$\lvert 002\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 110\rangle + \lvert 220\rangle$	6 0 6 6 6 0 6 0 6 0 6 6 6 6 6 0 6 0	id (23) (123)	T_1	$\mathbb{Z}/3\mathbb{Z}$
5	$\lvert 002\rangle + \lvert 020\rangle + \lvert 021\rangle + \lvert 110\rangle + \lvert 201\rangle$	3 3 0 6 3 3 3 3 3 3 0 6 0 6 3 3 3 3	id (23) (123)	T_1	$\mathbb{Z}/3\mathbb{Z}$
6	$\lvert 002\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 120\rangle + \lvert 210\rangle$	1 5 6 1 1 5 1 5 1 5 6 1 6 1 1 5 1 5	id (23) (123)	T_1	$\mathbb{Z}/3\mathbb{Z}$
7	$\lvert 002\rangle + \lvert 011\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 210\rangle$	3 0 3 3 3 3 3 3 3 3 3 0 3 3 3 0 3 3	id (12) (132)	T_1	$\mathbb{Z}/3\mathbb{Z}$
8	$\lvert 002\rangle + \lvert 020\rangle + \lvert 111\rangle + \lvert 200\rangle$	2 4 2 4 2 4	id	T_2	id
9	$\lvert 000\rangle + \lvert 011\rangle + \lvert 111\rangle + \lvert 122\rangle$	0 6 0 0 0 0 0 0 0 0 0 6 0 0 0 6 0 0	id (12) (132)	T_2	$S_3 \times \mathbb{Z}/3\mathbb{Z}$
10	$\lvert 002\rangle + \lvert 011\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 110\rangle + \lvert 200\rangle$	2 2 2 2 2 2	id	id	$(\mathbb{Z}/3\mathbb{Z})^3$
11	$\lvert 002\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 210\rangle$	2 0 4 2 4 2 4 2 4 2 2 0 4 2 2 0 4 2	id (12) (132)	T_2	id
12	$\lvert 002\rangle + \lvert 020\rangle + \lvert 100\rangle + \lvert 111\rangle$	1 5 0 1 0 1 0 1 0 1 1 5 0 1 1 5 0 1	id (12) (132)	T_2	$\mathbb{Z}/3\mathbb{Z}$
13	$\lvert 002\rangle + \lvert 011\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 110\rangle$	1 4 1 1 1 1 1 1 1 1 1 4 1 1 1 4 1 1	id (12) (132)	T_1	$(\mathbb{Z}/3\mathbb{Z})^2$
14	$\lvert 002\rangle + \lvert 010\rangle + \lvert 021\rangle + \lvert 100\rangle + \lvert 201\rangle$	3 0 0 3 3 0 3 0 3 0 0 3 0 3 3 0 3 0	id (23) (123)	$\mathrm{SL}(2,\mathbb{C})$	$(\mathbb{Z}/3\mathbb{Z})^2$
15	$\lvert 011\rangle + \lvert 022\rangle + \lvert 100\rangle$	2 4 1 0 1 0 1 0 1 0 2 4 1 0 2 4 1 0	id (12) (132)	$\mathrm{GL}(2,\mathbb{C}) \times T_1$	id

TABLE I. (*Continued*.)

N	3-qtrit	Char.	σ	$Z_{\widehat{G}}(h,e)^{\circ}$	K
16	$\lvert 002\rangle+\lvert 011\rangle+\lvert 020\rangle+\lvert 100\rangle$	2 2 1 1 1 1 1 1 1 1 2 2 1 1 2 2 1 1	id (12) (132)	T_2	$\mathbb{Z}/3\mathbb{Z}$
17	$\lvert 001\rangle+\lvert 010\rangle+\lvert 102\rangle+\lvert 120\rangle$	0 4 2 0 2 0 2 0 2 0 0 4 2 0 0 4 2 0	id (12) (132)	$\mathrm{GL}(2,\mathbb{C})$	$\mathbb{Z}/3\mathbb{Z}$
18	$\lvert 000\rangle+\lvert 011\rangle+\lvert 101\rangle+\lvert 112\rangle$	0 3 0 0 0 3 0 3 0 3 0 0 0 0 0 3 0 3	id (23) (123)	$\mathrm{GL}(2,\mathbb{C})$	$\mathbb{Z}/3\mathbb{Z}$
19	$\lvert 002\rangle+\lvert 010\rangle+\lvert 101\rangle$	1 2 0 1 1 2 1 2 1 2 0 1 0 1 1 2 1 2	id (23) (123)	T_3	id
20	$\lvert 000\rangle+\lvert 111\rangle$	0 2 0 2 0 2	id	T_4	$\mathbb{Z}/2\mathbb{Z}$
21	$\lvert 001\rangle+\lvert 010\rangle+\lvert 100\rangle$	1 1 1 1 1 1	id	T_3	id
22	$\lvert 000\rangle+\lvert 011\rangle+\lvert 022\rangle$	3 0 0 0 0 0 0 0 0 0 3 0 0 0 3 0 0 0	id (12) (132)	$\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C})$	$(\mathbb{Z}/3\mathbb{Z})^2$
23	$\lvert 000\rangle+\lvert 011\rangle$	2 0 0 1 0 1 0 1 0 1 2 0 0 1 2 0 0 1	id (12) (132)	$(\mathrm{GL}(2,\mathbb{C}))^2$	id
24	$\lvert 000\rangle$	0 1 2 0 0 1	id	$(\mathrm{GL}(2,\mathbb{C}))^2\times\mathrm{SL}(2,\mathbb{C})$	id

We also consider the real field. The elements of $(\mathbb{R}^3)^{\otimes n}=\mathbb{R}^3\otimes\cdots\otimes\mathbb{R}^3$ (n factors $\mathbb{R}^3$) are called *n-retrit states*. On this space, we consider the action of the group $\mathrm{SL}(3,\mathbb{R})^n$. This has applications in real quantum mechanics, as well as to a "black hole and black string/qutrit correspondence" in high-energy theoretical physics; see Sec. II B.

The main result of the present paper is the classification of the $\mathrm{SL}(3,\mathbb{R})^3$-orbits on the space $(\mathbb{R}^3)^3$. To obtain this classification, we use the classification of the complex orbits as well as Galois cohomological methods developed in Refs. 4 and 5. We summarize our findings for the real case in the following theorem.

Theorem 1.2. *There are 109 classes of elements of* $(\mathbb{R}^3)^{\otimes 3}$ *with the following properties. Each element of* $(\mathbb{R}^3)^{\otimes 3}$ *is* $\mathrm{SL}(3,\mathbb{R})^3$*-conjugate to an element of precisely one class. The elements of a fixed class all have the same stabilizer in* $\mathrm{SL}(3,\mathbb{R})^3$. *We have 70 classes consisting of one nilpotent element (the elements of Table I, along with Table II). There are six classes consisting of infinitely many semisimple elements (these are the families of Sec. VII A, as well as the two families found in Sec. VII B). Finally, there are in total 33 classes of mixed elements (the elements of Tables III, IV, and V, the elements given in (8.1), and those in Tables VI and VII).*

A. Organization of this paper

Section II is devoted to giving a general physical motivation for studying complex and real qutrits, while in Sec. III, we review the physical models involving three-qutrits.

In Secs. IV and V, we give a description of the mathematical techniques we used to obtain our computations. Using Vinberg theory of θ-groups, in Sec. IV, we describe how the $\mathrm{SL}(3,\mathbb{C})^3$-module $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ can be constructed starting from a suitable $\mathbb{Z}/3\mathbb{Z}$-grading on the Lie algebra $\mathfrak{e}_6$, induced by an order 3 automorphism. In Sec. V, we briefly recall how Galois cohomology can be used to achieve a description of real orbits, starting from a complete classification of the complex ones.

Finally, in Tables I–VII, we give the details of our computations that yield the classifications of the real and complex orbits, as well as the tables listing representatives of those orbits.

TABLE II. Nilpotent real three-qutrits.

N	Three-qutrit	σ
3	$-\lvert 002\rangle+\lvert 011\rangle+\lvert 020\rangle+\lvert 101\rangle+\lvert 112\rangle+\lvert 200\rangle$	id (23) (123)
9	$2\lvert 210\rangle+2\lvert 201\rangle-\frac{1}{4}\lvert 022\rangle-\lvert 011\rangle+\lvert 000\rangle$	id (12) (132)
20	$2\lvert 000\rangle-2\lvert 011\rangle-2\lvert 101\rangle-2\lvert 110\rangle$	id

TABLE III. Nilpotent part of mixed elements of the second family.

Nurmiev	Representative	$Z_{\widehat{G}}(s,h,e,f)^\circ$	Component group
1	$\lvert 021\rangle+\lvert 102\rangle$	id	$(\mathbb{Z}/3\mathbb{Z})^3$
2	$\lvert 021\rangle$	T_1	$(\mathbb{Z}/3\mathbb{Z})^2$

TABLE IV. Nilpotent part of mixed elements of the third family.

Nurmiev	Representative	π	$Z_{\widehat{G}}(s,h,e,f)^\circ$	Component group
1	$\lvert 012\rangle+\lvert 021\rangle+\lvert 102\rangle+\lvert 120\rangle$ $\lvert 102\rangle+\lvert 201\rangle+\lvert 012\rangle+\lvert 210\rangle$ $\lvert 210\rangle+\lvert 120\rangle+\lvert 201\rangle+\lvert 021\rangle$	id (1,2) (1,3)	id	$(\mathbb{Z}/3\mathbb{Z})^3$
2	$\lvert 012\rangle+\lvert 021\rangle+\lvert 102\rangle$ $\lvert 021\rangle+\lvert 012\rangle+\lvert 120\rangle$ $\lvert 102\rangle+\lvert 201\rangle+\lvert 012\rangle$ $\lvert 201\rangle+\lvert 102\rangle+\lvert 210\rangle$ $\lvert 120\rangle+\lvert 210\rangle+\lvert 021\rangle$ $\lvert 210\rangle+\lvert 120\rangle+\lvert 201\rangle$	id (2,3) (1,2) (1,2,3) (1,3,2) (1,3)	T_1	$(\mathbb{Z}/3\mathbb{Z})^2$
4	$\lvert 012\rangle+\lvert 021\rangle$ $\lvert 102\rangle+\lvert 201\rangle$ $\lvert 210\rangle+\lvert 120\rangle$	id (1,2) (1,3)	T_2	$\mathbb{Z}/3\mathbb{Z}$
5	$\lvert 012\rangle+\lvert 120\rangle$ $\lvert 102\rangle+\lvert 210\rangle$	id (1,2)	T_2	$\mathbb{Z}/3\mathbb{Z}$
7	$\lvert 012\rangle$ $\lvert 102\rangle$	id (1,2)	T_3	id

II. QUTRITS

A. Complex qutrits

A quantum trit (*qutrit*) is the superposition of three orthogonal basis states $\{\lvert 0\rangle, \lvert 1\rangle, \lvert 2\rangle\}$,[6] rather than the two that characterize a quantum bit (*qubit*), namely,

$$\lvert\psi\rangle=\alpha\lvert 0\rangle+\beta\lvert 1\rangle+\gamma\lvert 2\rangle, \text{ with } \alpha,\beta,\gamma\in\mathbb{C}:\lvert\alpha\rvert^2+\lvert\beta\rvert^2+\lvert\gamma\rvert^2=1. \tag{2.1}$$

Therefore, a qutrit is an element of $\mathbb{C}^3$ and the corresponding SLOCC group acting on it is $\mathrm{GL}(2,\mathbb{C})_3(\mathbb{C})$. The entanglement measures for qutrit systems are provided by relative invariants under the action of the multipartite local SLOCC group. For example, as the simplest relative

TABLE V. Nilpotent part of mixed elements of the fourth family.

Nurmiev	Representative	$Z_{\widehat{G}}(s,h,e,f)^\circ$	Component group
1	$\lvert 002\rangle + \lvert 020\rangle + \lvert 111\rangle + \lvert 200\rangle$	id	$(\mathbb{Z}/3\mathbb{Z})^2$
2	$\lvert 002\rangle + \lvert 011\rangle + \lvert 020\rangle + \lvert 101\rangle + \lvert 110\rangle + \lvert 200\rangle$	id	$(\mathbb{Z}/3\mathbb{Z})^2$
3	$\lvert 000\rangle + \lvert 111\rangle$	id	$(\mathbb{Z}/3\mathbb{Z})^3 \ltimes \mathbb{Z}/2\mathbb{Z}$.
4	$\lvert 001\rangle + \lvert 010\rangle + \lvert 100\rangle$	T_1	$\mathbb{Z}/3\mathbb{Z}$
5	$\lvert 000\rangle$	$\mathrm{SL}(2,\mathbb{C})$	$(\mathbb{Z}/3\mathbb{Z})^2$

TABLE VI. Nilpotent parts of mixed elements with semisimple part $b_1v_1 + b_2v_2$.

Nurmiev	representative
1	$2\lvert 220\rangle + 2\lvert 211\rangle + \lvert 021\rangle - \lvert 010\rangle$
2	$\lvert 102\rangle$ $-\lvert 102\rangle$

TABLE VII. Nilpotent parts of mixed elements with semisimple part av.

Nurmiev	Representative
1	$\lvert 120\rangle - \lvert 102\rangle - \lvert 021\rangle + \lvert 012\rangle$ $\lvert 210\rangle - \lvert 012\rangle - \lvert 201\rangle + \lvert 102\rangle$ $\lvert 021\rangle - \lvert 201\rangle - \lvert 120\rangle + \lvert 210\rangle$
4	$\lvert 210\rangle - \lvert 201\rangle$ $\lvert 120\rangle - \lvert 021\rangle$ $\lvert 012\rangle - \lvert 102\rangle$

SLOCC invariant for a two-qutrit state of the form $(A, B = 0, 1, 2)$,

$$\lvert\Psi\rangle = \Psi_{AB}\lvert A\rangle \otimes \lvert B\rangle, \tag{2.2}$$

one can take the determinant $\det(\Psi)$ of the 3×3 complex matrix Ψ_{AB}, which is invariant under $\mathrm{SL}(3,\mathbb{C}) \otimes \mathrm{SL}(3,\mathbb{C})$. The classification of two-qutrit states is thus very simple: different SLOCC classes are labeled by the rank of Ψ_{AB}. An operator representation of the qutrit density matrix has been developed, and qutrit entanglement has been studied; see, e.g., in Ref. 7. Furthermore, the generalized concurrence formula as a measure of two qutrit entanglement has also been introduced.[8]

At the present time, there is ongoing work to develop quantum computers using not only qubits but also qutrits with multiple states.[9] Indeed, qutrits exhibit several peculiar and interesting features when used for storing quantum information; for example, they are more robust to decoherence under certain environmental interactions;[10] however, in reality, their direct manipulation might be tricky, and it is thus often easier to use an entanglement with a qubit.[11] In Ref. 12, the maximum quantum violation of over 100 tight bipartite Bell inequalities on systems with up to four-dimensional Hilbert spaces was numerically determined, and it was found that there exist Bell inequalities that can be violated more with real qutrits than with complex qubits.

The physical implementation of a qutrit quantum computer in the context of trapped ions has been studied,[13] and quantum computer simulation packages for qutrits have been implemented.[14] New studies claim that qutrits offer a promising path toward extending the frontier of quantum computers.[15,16] In fact, accessing the third state in current quantum processors is primarily useful for researchers exploring the cutting edge of quantum computing, quantum physics,[17] and those interested in traditional, qubit-based algorithms alike. Promisingly, qutrits can not only increase the amount of information encoded in a single element, but they can also enable techniques that can dramatically decrease readout errors.[18] Recent progress in world-leading, cutting-edge quantum computer laboratories has shown how qutrit–qutrit gates can reduce the cost of decomposing three-qubit gates into basic two-qubit components.[19] This is, in part, due to the much larger state space accessible using qutrits: for instance, single qutrit operations pertain to a three-dimensional Hilbert space, while two-qutrit operations live in a nine-dimensional Hilbert space, which is more than twofold larger than the four-dimensional Hilbert space of the two-qubit case.

1. Neutrinos as qutrits

Recently, qutrits have found an intriguing application in high energy particle physics: indeed, neutrino flavor states, which are a superposition of three states, have been characterized in terms of qutrits. This can be done by mapping the density matrix for neutrinos to a *generalized Poincaré sphere*.[7,20]

The Poincaré sphere has its origin in optics and is a way of visualizing different types of polarized light using the mapping from SU_2 to S^3. A qubit represents a point on the Poincaré sphere of SU_2 defined as the complex projective line,

$$P^1\mathbb{C} = \frac{SU_2}{U_1}. \tag{2.3}$$

A generalization of the Poincaré sphere to SU_3 can be constructed,[21–23] yielding to the characterization of qutrits as points on the complex projective plane,[24,25]

$$P^2\mathbb{C} = \frac{SU_3}{U_2}. \tag{2.4}$$

An n-qutrit register can represent 3^n different states simultaneously, i.e., a superposition state vector in 3^n-dimensional Hilbert space.[7] Entanglement of neutrinos realized in terms of Poincaré sphere representation for two- and three-flavor neutrino states using SU_2 Pauli matrices and SU_3 Gell–Mann matrices has been investigated in Ref. 26.

B. Real qutrits: Retrits

Real quantum mechanics (that is, quantum mechanics defined over real vector spaces) dates back to the work of Stückelberg[27] and provides an interesting theory, whose study may shed some light to discriminate among the aspects of *quantum entanglement*, which are unique to standard quantum theory and those which are more generic over other physical theories endowed with this phenomenon.[28] Besides *rebits* (real qubits), in real quantum mechanics, there has been an interest in introducing *retrits*, i.e., qutrits with *real* coefficients for probability amplitudes of a three-level system, namely, a three-level quantum state that may be expressed as a *real* linear combination of the orthonormal basis states $|0\rangle$, $|1\rangle$, and $|2\rangle$; such a system may be in a superposition of three mutually orthonormal quantum states,[6] which span the Hilbert space $\mathcal{H}^3$,[25] with dim = 3 (over $\mathbb{C}$ for qutrits or over $\mathbb{R}$ for retrits). As for rebits, also for retrits, the density matrix of the processed quantum state ρ is real, i.e., at each point in the quantum computation, it holds that $\langle x|\rho|y\rangle \in \mathbb{R}$ for all $|x\rangle$, $|y\rangle$ in the computational basis.

1. Black hole/string charges as retrits

In recent years, the relevance of rebits in high-energy theoretical physics was highlighted by the determination of striking multiple relations between the entanglement of pure states of two and three qubits and the entropy of extremal black holes in $D = 4$ Maxwell–Einstein-scalar theories, which can be regarded as bosonic sectors of supergravity theories or equivalently as low-energy limits of string theory compactifications. In this framework, which has been subsequently dubbed "*black hole/qubit correspondence*" (see, e.g., Refs. 29–31 for reviews and lists of references), rebits acquire the physical meaning of the electric and magnetic charges of the extremal black hole, and they linearly transform under the generalized e.m. duality group $G(\mathbb{R})$ (named U-duality group in string theory; see further below) of the theory under consideration. In supergravity, the approximation of *real* (rather than integer) electric and magnetic charges of the black hole is often considered, thus disregarding the charge quantization.

Thus, a natural question arises out: *can one find entangled systems of real qutrits that can be related to other entropy formulas?*

The answer is positive, provided that one considers black hole and black string solutions in $D = 5$ space-time dimensions:[32] in this sense, the aforementioned $D = 4$ "*black hole/qubit correspondence*" can be lifted to a $D = 5$ "*black hole or black string/qutrit correspondence*."

It is here worth recalling that magic $\mathcal{N} = 2$, $D = 5$ supergravities[29,33,34] coupled to 5, 8, 14, and 26 vector multiplets, with $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})(3,\mathbb{C})_{\mathbb{R}}$, $SU^*(6) \simeq \mathrm{SL}(3,\mathbb{H})_{\mathbb{R}}$, and $E_{6(-26)} \simeq \mathrm{SL}(3,\mathbb{O})_{\mathbb{R}}$,[35] respectively, can be described by simple, rank-3 Jordan algebras $\mathfrak{J}_3^{\mathbb{A}}$ of 3×3 Hermitian matrices with entries taken from the four normed Hurwitz algebras:[36] the reals $\mathbb{R}$, complexes $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$, of total dimension $\dim_{\mathbb{R}} = 3q + 3$, with $q := \dim_{\mathbb{R}}\mathbb{A} = 1, 2, 4, 8$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, respectively.

Moreover, one can also replace in these Jordan algebras the division algebras by their split versions (i.e., split complexes $\mathbb{C}_s$, split quaternions $\mathbb{H}_s$, and split octonions $\mathbb{O}_s$), obtaining $\mathfrak{J}_3^{\mathbb{A}_s}$. For $\mathbb{C}_s$ and $\mathbb{H}_s$ having $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R}) \otimes \mathrm{SL}(3,\mathbb{R})$ and $G(\mathbb{R}) = \mathrm{SL}(6,\mathbb{R})$, one obtains non-supersymmetric (i.e., $\mathcal{N} = 0$) Maxwell–Einstein-scalar theories in $D = 5$, which have been investigated in Refs. 37 and 38, whereas for $\mathbb{O}_s$, one obtains the maximally supersymmetric, $\mathcal{N} = 8$, $D = 5$ supergravity[39] with 27 Abelian gauge fields transforming in the reflexive, real, defining module $\mathbf{27}$ of $G(\mathbb{R}) = E_{6(6)}$.

All these theories admit asymptotically flat, static, extremal black holes and black strings, whose Bekenstein–Hawking entropy can be expressed in terms of the (absolute value of the) homogeneous cubic polynomials $\mathbf{I}_{3,el}$ and $\mathbf{I}_{3,magn}$, which are invariant under the non-transitive action of $G(\mathbb{R})$ over the corresponding simple Jordan algebra $\mathfrak{J}_3$;[40] indeed, in all mentioned cases, G acting on $\mathfrak{J}_3$ is a θ-group in the sense of

Vinberg (see, e.g., also Ref. 41 and references therein). On the quantum information theory (QIT) side, such cubic polynomials describe the measure of a multipartite entanglement of a suitable system of qutrits; for a comprehensive account, see, e.g., Sec. 4 of Ref. 31.

III. FROM THREE QUTRITS TO THREE-CENTERED BLACK HOLES/STRINGS

A. Complex orbits: Entanglement of three qutrits

In Ref. 42, the following $\mathbb{Z}/3\mathbb{Z}$-grading of $\mathfrak{e}_6(\mathbb{C})$ was considered:

$$\begin{aligned}\mathfrak{e}_6(\mathbb{C}) &= \mathfrak{g}_{-1}(\mathbb{C}) \oplus \mathfrak{g}_0(\mathbb{C}) \oplus \mathfrak{g}_1(\mathbb{C}),\\ \mathfrak{g}_0(\mathbb{C}) &= \mathfrak{sl}_3(\mathbb{C})^{\oplus 3},\\ \mathfrak{g}_1(\mathbb{C}) &= \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \equiv \left(\mathbb{C}^3\right)^{\otimes 3},\\ \mathfrak{g}_{-1}(\mathbb{C}) &= \mathfrak{g}_1^*(\mathbb{C}),\end{aligned} \tag{3.1}$$

and the orbit action of the group $G_0(\mathbb{C}) = \mathrm{SL}(3,\mathbb{C})^{\times 3}$ acting on $\left(\mathbb{C}^3\right)^{\otimes 3}$ was completely classified. Note that $\mathfrak{g}_0(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{C})^{\oplus 3}$ is a maximal non-symmetric subalgebra of $\mathfrak{e}_6(\mathbb{C})$, defined as the fixed point algebra with respect to the action of a periodic automorphism of order 3, inducing a $\mathbb{Z}/3\mathbb{Z}$-grading on $\mathfrak{e}_6(\mathbb{C})$ itself. Moreover, $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ (non-transitively) acting on $\left(\mathbb{C}^3\right)^{\otimes 3}$ is a θ-group of type II in the sense of Vinberg;[1] indeed, it has a finite number of nilpotent orbits, and its ring of invariant polynomials has dimension 3, being finitely generated by the homogeneous polynomials of the integrity basis, of degrees 6, 9, and 12 (respectively, named $\mathbf{I}_6$, $\mathbf{I}_9$, and $\mathbf{I}_{12}$; see, e.g., Table III of Ref. 41 and references therein).

The above grading has application in Quantum Information Theory (QIT) because it concerns the classification of the entanglement of three pure multipartite qutrits, fitting the space $\left(\mathbb{C}^3\right)^{\otimes 3} \equiv (\mathbf{3},\mathbf{3},\mathbf{3})$ (in physicists' notation), under the SLOCC group $\mathrm{SL}(3,\mathbb{C})^{\times 3}$. For a recent account on the classification of the entanglement classes of three qutrits, see, e.g., Ref. 43 and references therein.

B. Real orbits: three-centered black holes/strings in $D = 5$

The maximally non-compact (i.e., split) real form of the $\mathbb{Z}/3\mathbb{Z}$-grading (3.1) reads

$$\begin{aligned}\mathfrak{e}_{6(6)} &= \mathfrak{g}_{-1}(\mathbb{R}) \oplus \mathfrak{g}_0(\mathbb{R}) \oplus \mathfrak{g}_1(\mathbb{R}),\\ \mathfrak{g}_0(\mathbb{R}) &= \mathfrak{sl}_3(\mathbb{R})^{\oplus 3},\\ \mathfrak{g}_1(\mathbb{R}) &= \left(\mathbb{R}^3\right)^{\otimes 3},\\ \mathfrak{g}_{-1}(\mathbb{R}) &= \mathfrak{g}_1'(\mathbb{R}).\end{aligned} \tag{3.2}$$

According to the classification carried out in Ref. 44, the split form $\mathfrak{e}_{6(6)}$ admits only two real forms of $\mathfrak{a}_2^{\oplus 3}$ as maximal (non-symmetric) subalgebras,

$$\mathfrak{e}_{6(6)} \supsetneq \begin{cases} i: & \mathfrak{sl}_3(\mathbb{R})^{\oplus 3},\\ ii: & \mathfrak{su}_{1,2} \oplus \mathfrak{sl}_3(\mathbb{C})_{\mathbb{R}}.\end{cases} \tag{3.3}$$

On the other hand, analogous embeddings exist for all other non-compact, real forms of $\mathfrak{e}_6(\mathbb{C})$, namely,[44]

$$\mathfrak{e}_{6(2)} \supsetneq \begin{cases} i: & \mathfrak{su}_{1,2}^{\oplus 3},\\ ii: & \mathfrak{su}_{1,2} \oplus \mathfrak{su}_3^{\oplus 2},\\ iii: & \mathfrak{sl}_3(\mathbb{R}) \oplus \mathfrak{sl}_3(\mathbb{C})_{\mathbb{R}},\end{cases} \tag{3.4}$$

$$\mathfrak{e}_{6(-14)} \supsetneq \mathfrak{su}_{1,2}^{\oplus 2} \oplus \mathfrak{su}_3, \tag{3.5}$$

$$\mathfrak{e}_{6(-26)} \supsetneq \mathfrak{su}_3 \oplus \mathfrak{sl}_3(\mathbb{C})_{\mathbb{R}}. \tag{3.6}$$

As embedding (3.3) *i* is related to the $\mathbb{Z}/3\mathbb{Z}$-grading (3.2) of $\mathfrak{e}_{6(6)}$, the question presents itself as to whether the other real embeddings above are related to suitable $\mathbb{Z}/3\mathbb{Z}$-gradings of the corresponding non-compact, real forms of $\mathfrak{e}_6$. Here, we will not go into this question.

Some observations are in order.

- In (not necessarily supersymmetric) Maxwell–Einstein-scalar theories, the embeddings (3.3) *i* and (3.4) *iii* are named *super-Ehlers*[45] (or *Jordan pairs*[46]) embeddings, and they express the fact that the Lie algebra of the $D = 3$ electric-magnetic (e.m.) duality group contains the Lie algebra of the $D \geqslant 4$ e.m. duality group as a proper subalgebra, along with a commuting summand $\mathfrak{sl}_{D-2}(\mathbb{R})$, specified for $D = 5$. By U-duality group we actually mean the "continuous limit/version" $G(\mathbb{R})$ of the U-duality group $G(\mathbb{Z})$ of superstring theory.[47]

- The embeddings (3.3) *i* and *ii* follow from consistent truncations of the corresponding Maxwell–Einstein-scalar theories in $D = 5$: the Lie algebra of the e.m. duality group of the ($\mathfrak{J}_3^{\mathbb{O}_s}$-based) *maximal* supergravity contains (as proper subalgebras) the Lie algebra of the e.m. duality group of $\mathcal{N} = 0$ $\mathfrak{J}_3^{\mathbb{C}_s}$-based Maxwell–Einstein-scalar theory, as well as of the $\mathcal{N} = 2$ $\mathfrak{J}_3^{\mathbb{C}}$-based magic supergravity, along with commuting summands $\mathfrak{sl}_3(\mathbb{R})$ and $\mathfrak{su}_{1,2}$, respectively. These truncations are consequences of division algebraic reductions (which are also supersymmetry reductions): $\mathbb{O}_s \to \mathbb{C}_s$ ($\mathcal{N} = 8 \to 0$) and $\mathbb{O}_s \to \mathbb{C}$ ($\mathcal{N} = 8 \to 2$), respectively.
- On the other hand, the embedding (3.6) follows from a consistent truncation of the corresponding supergravity theories: in $D = 5$, the Lie algebra of the e.m. duality group of the *exceptional* $\mathcal{N} = 2$, $J_3^{\mathbb{O}}$-based magic supergravity[33,34] contains (as proper subalgebra) the Lie algebra of the e.m. duality group of the $\mathcal{N} = 2$ $J_3^{\mathbb{C}}$-based magic supergravity, along with a commuting summand $\mathfrak{su}_3$. This truncation is a consequence of the division algebraic reduction $\mathbb{O} \to \mathbb{C}$ (which implies no reduction of supersymmetry).

In the present paper, we will study the orbit stratification of the non-transitive action on $(\mathbb{R}^3)^{\otimes 3}$ of the totally split real form $3\mathfrak{sl}_3(\mathbb{R})$. This corresponds to the embedding (3.3) *i*, in which one $\mathfrak{sl}_3(\mathbb{R}) \equiv \mathfrak{sl}_{3,h}(\mathbb{R})$ can be interpreted as the "horizontal" symmetry[48] of (asymptotically flat) three-centered extremal black hole/string solutions in the corresponding (ungauged) Maxwell–Einstein-scalar (super)gravity theory in $D = 5$.

In fact, *multi-centered* solutions (which are a natural generalization of single-centered solutions) to Maxwell–Einstein equations exist also in $D > 4$ (Lorentzian) space-time dimensions.[49] By considering $D = 5$, the asymptotically flat black objects are 0-branes (black *holes*) and 1-branes (black *strings*); in their extremal cases, their near-horizon geometries are $AdS_2 \otimes S^3$ and $AdS_3 \otimes S^2$. Asymptotically flat, extremal black holes in $D = 5$ are named Tangherlini black holes,[50,51] and they are a special instance of the Papapetrou–Majumdar black holes, whose multi-centered classes have been recently investigated in Ref. 49, in which the classification of asymptotically flat, extremal black hole solutions in Maxwell–Einstein theory in higher dimensions has been completed.

In $D = 5$, the fluxes of the two-form Abelian (electric) field-strengths and of their dual three-form Abelian (magnetic) field-strengths, which determine the *electric* charges of the extremal Tangherlini black hole and the *magnetic* charges of the extremal black string, fit into a representation $\mathcal{R}_{el}$ and into its dual/conjugate representation $\mathcal{R}_{magn} \equiv \mathcal{R}'_{el}$ of the $D = 5$ e.m. duality group $G(\mathbb{R})$. In the $D = 5$ Maxwell–Einstein-scalar theories pertaining to the real embeddings (3.3) *i* and (3.4) *iii* relevant to our treatment, the following group-theoretical data can be specified:

- $\mathcal{N} = 0$ Maxwell–Einstein-scalar theory based on $\mathfrak{J}_3^{\mathbb{C}_s}$: $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$, $\mathcal{R}_{el} = (\mathbf{3},\mathbf{3}')$, and $\mathcal{R}_{magn} = (\mathbf{3}',\mathbf{3})$, and each orbit of $\mathrm{SL}(3,\mathbb{R})^{\times 2}$ acting on $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ supports a unique class of single-centered (electric black hole and magnetic black string) solutions.[37,38]
- $\mathcal{N} = 2$ Maxwell–Einstein magic supergravity based on $\mathfrak{J}_3^{\mathbb{C}}$: $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}$, $\mathcal{R}_{el} = (\mathbf{3},\overline{\mathbf{3}})$, and $\mathcal{R}_{magn} = (\overline{\mathbf{3}},\mathbf{3})$, and each orbit of $\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}$ acting on $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ supports a unique class of single-centered (electric black hole and magnetic black string) solutions.[52]

From classical invariant theory, for both the aforementioned gravity theories in $D = 5$, the action of the e.m. duality group $G(\mathbb{R})$ on $\mathcal{R}_{el}$ or $\mathcal{R}_{magn}$ can be traced back, on $\mathbb{C}$, to the action of the θ-group of type I $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$ on its bi-fundamental representation $\mathbb{C}^3 \otimes \mathbb{C}^3$, characterized by the following properties (see, e.g., Ref. 41, and references therein):

1. The identity-connected component $(H_{m=1})_0$ of the stabilizer of the generic orbit $\mathcal{O}_{m=1} = \mathrm{SL}(3,\mathbb{C})^{\times 2}/H_{m=1}$ is given by $\mathrm{SL}(3,\mathbb{C})$.
2. The number of nilpotent orbits is finite.
3. The ring of invariant polynomials is one-dimensional, and it is finitely generated by a cubic homogeneous polynomial $\mathbf{I}_3$ (see, e.g., Table II of Ref. 41). Such a cubic polynomial is nothing but the cubic norm (determinant) of the rank-3 simple Jordan algebra $(J_3^{\mathbb{C}})_{\mathbb{C}} \simeq \mathbb{C}^3 \otimes \mathbb{C}^3$ of $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$, which can be regarded as the *reduced structure group* of $(J_3^{\mathbb{C}})_{\mathbb{C}}$ itself. The same will hold for the relevant forms on $\mathbb{R}$, *mutatis mutandis.*
   - **3.1.** In the $\mathcal{N} = 0$ Maxwell–Einstein-scalar theory based on $\mathfrak{J}_3^{\mathbb{C}_s}$:[37,38] $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ acting on $\mathcal{R}_{el} = (\mathbf{3},\mathbf{3}')$ and $\mathcal{R}_{magn} = (\mathbf{3}',\mathbf{3})$ admits a unique primitive invariant cubic homogeneous polynomial: $\mathbf{I}_{3,el}$ and $\mathbf{I}_{3,magn}$, which is nothing but the cubic norm (determinant) of the rank-3 simple Jordan algebra $J_3^{\mathbb{C}_s} \simeq \mathbb{R}^3 \otimes \mathbb{R}^3$ (respectively, its dual) of $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$, which can be regarded as the *reduced structure group* of $J_3^{\mathbb{C}_s}$ itself. Physically, $\mathbf{I}_{3,el}$ and $\mathbf{I}_{3,magn}$ determine the Bekenstein–Hawking entropy of the extremal, one-centered, static, asymptotically flat black hole and black string solutions of the theory.[37,38]
   - **3.2.** In the $\mathcal{N} = 2$ magic Maxwell–Einstein supergravity based on $\mathfrak{J}_3^{\mathbb{C}}$:[33,34] $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}$ acting on $\mathcal{R}_{el} = (\mathbf{3},\overline{\mathbf{3}})$ and $\mathcal{R}_{magn} = (\overline{\mathbf{3}},\mathbf{3})$ admits a unique primitive invariant cubic homogeneous polynomial: $\mathbf{I}_{3,el}$ and $\mathbf{I}_{3,magn}$, which is nothing but the cubic norm (determinant) of the rank-3 simple Jordan algebra $J_3^{\mathbb{C}} \simeq \mathbb{R}^3 \otimes \mathbb{R}^3$ (respectively, its dual) of $\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}$, which can be regarded as the *reduced structure group* of $J_3^{\mathbb{C}}$ itself. Physically, $\mathbf{I}_{3,el}$ and $\mathbf{I}_{3,magn}$ determine the Bekenstein–Hawking entropy of the extremal, one-centered, static, asymptotically flat black hole and black string solutions of the theory.[53,54]

As in $D = 4$, a crucial feature of multi-centered solutions is the existence of a global, "*horizontal*" symmetry group $\mathrm{SL}_{m,h}(\mathbb{R})$,[48] encoding the combinatoric structure of the m-centered solutions of the theory ($m \in \mathbb{N}$) and commuting with $G(\mathbb{R})$ itself. Actually, the "horizontal" symmetry group is $\mathrm{GL}(2,\mathbb{C})_m(\mathbb{R})$, where the additional scale symmetry $SO_{1,1}$ with respect to $\mathrm{SL}(m,\mathbb{R})$ is encoded by the homogeneity of the

$G(\mathbb{R})$-invariant polynomials in the black hole electric (or black string magnetic) charges; the subscript "h" stands for "horizontal" throughout. Thus, in the presence of an asymptotically flat $D = 5$ multi-centered black p-brane (with $p = 0$ and 1) solution with m centers, the dimension I_m of the ring of $(\mathrm{SL}_{m,h}(\mathbb{R}) \times G(\mathbb{R}))$-invariant homogeneous polynomials constructed with m distinct copies of the $G(\mathbb{R})$-representation $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ [in the defining irrepr. $\mathbf{m}$ of $\mathrm{SL}_{m,h}(\mathbb{R})$] is given by the general formula[48]

$$I_m = m \dim_{\mathbb{R}} \mathcal{R} - \dim_{\mathbb{R}} \mathcal{O}_m, \tag{3.7}$$

where

$$\mathcal{O}_m := \frac{\mathrm{SL}_{m,h}(\mathbb{R}) \times G(\mathbb{R})}{H_m(\mathbb{R})} \tag{3.8}$$

is a generally non-symmetric coset describing the generic, open $(\mathrm{SL}_{m,h}(\mathbb{R}) \times G(\mathbb{R}))$-orbit, spanned by the m copies of the representation $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$, each pertaining to one center of the multi-centered solution. We are not considering the intriguing possibility in which an m-centered solution in $D = 5$ is composed of m'. black holes and $m-m'$ black strings (or the other way around) since we are currently not knowledgeable whether this can be a well-defined solution of the equations of motion of the theories under consideration. By setting $m = 1$ and $m = 3$ in the aforementioned $D = 5$ Maxwell–Einstein-scalar theories, one obtains

$$\left.\begin{array}{ll} \underset{G(\mathbb{R})=\mathrm{SL}(3,\mathbb{R})^{\times 2}}{\mathcal{N}=0} & \mathfrak{J}_3^{\mathbb{C}_s} \\ \underset{G(\mathbb{R})=\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}}{\mathcal{N}=2} & \mathfrak{J}_3^{\mathbb{C}} \end{array}\right\} : \begin{array}{l} I_{m=1} = \dim_{\mathbb{R}}\left(\left(\mathbb{R}^3\right)^{\otimes 2}\right) - \dim_{\mathbb{R}}(\mathcal{O}_{m=1}) = 3\cdot 3 - 8 = \underset{\mathbf{I}_3}{1}, \\ I_{m=3} = 3\dim_{\mathbb{R}}\left(\left(\mathbb{R}^3\right)^{\otimes 2}\right) - \dim_{\mathbb{R}}(\mathcal{O}_{m=3}) = 3\cdot 9 - 3\cdot 8 = \underset{\mathbf{I}_6,\mathbf{I}_9,\mathbf{I}_{12}}{3} \end{array} \tag{3.9}$$

because

$$\begin{aligned} &\dim_{\mathbb{R}} \mathcal{R}_{el} = \dim_{\mathbb{R}} \mathcal{R}_{magn} = 9, \\ &(H_{m=1}(\mathbb{R}))_0 = \text{Lie group whose Lie algebra is a real form of } \mathfrak{a}_2 \Rightarrow \dim_{\mathbb{R}} \mathcal{O}_{m=1} = 8, \\ &(H_{m=3}(\mathbb{R}))_0 = \mathbb{I}(\text{identity}) \Rightarrow \mathcal{O}_{m=3} \simeq \mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R}) \Rightarrow \dim_{\mathbb{R}} \mathcal{O}_{m=3} = 24. \end{aligned}$$

The counting (3.9) for $m = 3$ implies that the ring of $(\mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R}))$-invariant homogeneous polynomials built out of three copies of $\mathcal{R}_{el}[= (\mathbf{3}, \mathbf{3}')$ and $(\mathbf{3}, \overline{\mathbf{3}})$ for $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})^{\times 2}$ and $\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}]$ or of $\mathcal{R}_{magn}[= (\mathbf{3}', \mathbf{3})$ and $(\overline{\mathbf{3}}, \mathbf{3})$ for $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})^{\times 2}$ and $\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}]$ has (real) dimension 3. By taking the complexification of all this, one retrieves the known result from classical invariant theory, namely, that $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ non-transitively acting on the tri-fundamental representation space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is a θ-group of type II in the sense of Vinberg,[1] and as anticipated in Sec. III A, it has the following properties:

(1) The identity-connected component $(H_{m=3})_0$ of the stabilizer of the generic three-centered orbit

$$\mathcal{O}_{m=3} := \frac{\mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R})}{H_{m=3}(\mathbb{R})} \tag{3.10}$$

is nothing but $\mathbb{I}$ (identity).
(2) The number of nilpotent orbits is finite.
(3) The ring of invariant polynomials is three-dimensional, and it is finitely generated by an integrity basis of homogeneous polynomials $\mathbf{I}_6$, $\mathbf{I}_9$, and $\mathbf{I}_{12}$ of degrees 6, 9, and 12 (see, e.g., Table III of Ref. 41 and references therein).

Some further observations are in order.

- For simplicity's sake, let us assume to work on $\mathbb{C}$. Since the "horizontal" Lie algebra $\mathfrak{a}_{2,h}$ stands on a different footing with respect to the Lie algebra $\mathfrak{a}_2 \oplus \mathfrak{a}_2$ of the e.m. duality group $G(\mathbb{C}) = \mathrm{SL}(3,\mathbb{C})^{\times 2}$, only invariance with respect to proper subgroup Sym_2 of permutation group Sym_3 of the three tensor factors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ should be taken into account, when considering three-centered black hole/string solutions in the $D = 5$ Maxwell–Einstein (super)gravity theories under consideration. In such a way, a classification invariant under
$$\mathrm{Sym}_2 \ltimes \left(\mathrm{SL}_{3,h}(\mathbb{C}) \times \left(\mathrm{SL}(3,\mathbb{C})^{\times 2}\right)\right) \tag{3.11}$$
(namely, $\mathrm{Sym}_2 \ltimes (\mathfrak{a}_{2,h} \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_2)$ at the Lie algebra level) should be considered. Therefore, besides the choice of the ground number field ($\mathbb{C}$ in QIT vs $\mathbb{R}$ in gravity), we are pointing out another important difference in the orbit classification procedure: while in QIT, one needs to classify the action of
$$\mathrm{Sym}_3 \ltimes \mathrm{SL}(3,\mathbb{C})^{\times 3} \text{ on } \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \equiv (\mathbf{3}, \mathbf{3}, \mathbf{3}), \tag{3.12}$$

in $D = 5$ Maxwell–Einstein (super)gravity, one needs to classify the action of

$$\mathrm{Sym}_2 \ltimes \left(SL_{3,h}(\mathbb{R}) \times \left(\mathrm{SL}(3,\mathbb{R})^{\times 2}\right)\right) \text{ on } \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \equiv \left(\mathbf{3},\mathbf{3},\mathbf{3}'\right), \tag{3.13}$$

and

$$\mathrm{Sym}_2 \ltimes \left(SL_{3,h}(\mathbb{R}) \times \mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}\right) \text{ on } \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \equiv \left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right) \tag{3.14}$$

in the $\mathcal{N} = 0$, $\mathfrak{J}_3^{\mathbb{C}_s}$-based theory[37,38] and the $\mathcal{N} = 2$, $\mathfrak{J}_3^{\mathbb{C}}$-based magic supergravity[33,34] in $D = 5$.

- In the $\mathcal{N} = 0$, $\mathfrak{J}_3^{\mathbb{C}_s}$-based Maxwell–Einstein theory and the $\mathcal{N} = 2$, $\mathfrak{J}_3^{\mathbb{C}}$-based magic supergravity, it holds that $\mathcal{R}_{el} = \left(\mathbf{3},\mathbf{3}'\right)$ and $\left(\mathbf{3},\overline{\mathbf{3}}\right)$ and $\mathcal{R}_{magn} = \left(\mathbf{3}',\mathbf{3}\right)$ and $\left(\overline{\mathbf{3}},\mathbf{3}\right)$. Therefore, as given above, *modulo the action of permutation groups*, the whole representation space on which $\mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R})$ (non-transitively) acts is $\mathbb{R}^3 \otimes \mathcal{R}_{el} \equiv (\mathbf{3}, \mathcal{R}_{el})$ or $\mathbb{R}^3 \otimes \mathcal{R}_{magn} \equiv (\mathbf{3}, \mathcal{R}_{magn})$. The complexification of all this yields an interpretation of the representation space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ as $\left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right)$ or $\left(\mathbf{3},\overline{\mathbf{3}},\mathbf{3}\right)$ of $\mathrm{SL}(3,\mathbb{C})^{\times 3}$, which is different from the interpretation of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ holding in QIT, namely, as $(\mathbf{3},\mathbf{3},\mathbf{3})$ of the SLOCC group $\mathrm{SL}(3,\mathbb{C})^{\times 3}$. However, one can realize that the classification of the orbit action of $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ over $(\mathbf{3},\mathbf{3},\mathbf{3})$ is the same as the one over $\left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right)$; see Sec. IV B. By switching from $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ to the split real form $\mathrm{SL}(3,\mathbb{R})^{\times 3}$, this directly implies that the classification of the orbit action of $\mathrm{SL}(3,\mathbb{R})^{\times 3}$ over $(\mathbf{3},\mathbf{3},\mathbf{3})$ is the same as the one over $\left(\mathbf{3},\mathbf{3},\mathbf{3}'\right)$.
- One may also simply consider the $G(\mathbb{R})$-invariant (electric and magnetic) polynomials constructed with m copies of $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ and conceive the "horizontal" symmetry $\mathrm{SL}_{m,h}(\mathbb{R})$ merely as a multiplet-organizing (i.e., spectrum-generating) symmetry. Setting $m = 3$, the counting, rather than the second of (3.9), now goes this way,

$$\left.\begin{array}{cc} \underset{G(\mathbb{R})=\mathrm{SL}(3,\mathbb{R})^{\times 2}}{\mathcal{N}=0} & \mathfrak{J}_3^{\mathbb{C}_s} \\ \underset{G(\mathbb{R})=\mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}}{\mathcal{N}=2} & \mathfrak{J}_3^{\mathbb{C}} \end{array}\right\} : \tilde{I}_{m=3} = 3\dim_{\mathbb{R}}\left(\left(\mathbb{R}^3\right)^{\otimes 2}\right) - \dim_{\mathbb{R}}(G(\mathbb{R})) = 3\cdot 9 - 2\cdot 8 = 11, \tag{3.15}$$

where $\tilde{I}_{m=3}$ denotes the number of $G(\mathbb{R})$-invariant (electric and magnetic) polynomials constructed with three copies of $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$, and we have used that the generic $G(\mathbb{R})$-orbit $\mathbf{O}_{m=3}$ spanned by three copies of $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ has a trivial identity connected component of the stabilizer,

$$\mathbf{O}_{m=3} \simeq G(\mathbb{R}) \Rightarrow \dim_{\mathbb{R}}(\mathbf{O}_{m=3}) = \dim_{\mathbb{R}}(G(\mathbb{R})) = 2\cdot 8 = 16. \tag{3.16}$$

By taking $Q^a \in \mathcal{R}_{el}$, $Q_a \in \mathcal{R}_{magn}$, $Q^{a\alpha} \in \mathbb{R}^3 \otimes \mathcal{R}_{el}$, and $Q_a^{\alpha} \in \mathbb{R}^3 \otimes \mathcal{R}_{magn}$, one can construct the $G(\mathbb{R})$-invariants (electric and magnetic singlets),

$$\mathbf{I}_{3,el} := \frac{1}{3!} d_{abc} Q^a Q^b Q^c \equiv \mathbf{1}_{el} \text{ of } G(\mathbb{R}), \tag{3.17}$$

$$\mathbf{I}_{3,magn} := \frac{1}{3!} d^{abc} Q_a Q_b Q_c \equiv \mathbf{1}_{magn} \text{ of } G(\mathbb{R}), \tag{3.18}$$

as well as the $(\mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R}))$-covariants,

$$\mathbf{I}_{el}^{\alpha\beta\gamma} := \frac{1}{3!} d_{abc} Q^{a\alpha} Q^{b\beta} Q^{c\gamma} = \mathbf{I}_{el}^{(\alpha\beta\gamma)} \equiv S^3\mathbb{R}^3 \equiv (\mathbf{10}, \mathbf{1}_{el}) \text{ of } \mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R}), \tag{3.19}$$

$$\mathbf{I}_{magn}^{\alpha\beta\gamma} := \frac{1}{3!} d^{abc} Q_a^{\alpha} Q_b^{\beta} Q_c^{\gamma} = \mathbf{I}_{magn}^{(\alpha\beta\gamma)} \equiv S^3\mathbb{R}^3 \equiv (\mathbf{10}, \mathbf{1}_{magn}) \text{ of } \mathrm{SL}_{3,h}(\mathbb{R}) \times G(\mathbb{R}), \tag{3.20}$$

where $a, b, c = 1, \ldots, 9$ and $\alpha, \beta, \gamma = 1, 2, 3$. From the explicit structure of $\mathcal{R}_{el}$ (respectively, $\mathcal{R}_{magn}$), the Latin lowercase indices are actually realized as pairs of contravariant-covariant (respectively, covariant-contravariant) fundamental (Greek lowercase) indices: $A^a \equiv A^{\alpha}{}_{\beta}$ (respectively, $A_a \equiv A_{\alpha}{}^{\beta}$); see, e.g., Refs. 40 and 55. Repeated indices are summed over (Einstein's convention). It is thus easy to realize that the $\tilde{I}_{m=3} = 11$ $G(\mathbb{R})$-invariant (electric and magnetic) polynomials constructed with three copies of $\mathcal{R}_{el}$ and $\mathcal{R}_{magn}$ organize as $\mathbf{10} \oplus \mathbf{1}$ in terms of $\mathrm{SL}_{3,h}(\mathbb{R})$-representations.

In conclusion, the classification of the orbit structure of the non-transitive action of

$$\mathrm{Sym}_2 \ltimes \left(\mathrm{SL}_{3,h}(\mathbb{R}) \times \left(\mathrm{SL}(3,\mathbb{R})^{\times 2}\right)\right) \text{ on } \left(\mathbf{3},\mathbf{3},\mathbf{3}'\right) \text{ and } \left(\mathbf{3},\mathbf{3}',\mathbf{3}\right) \tag{3.21}$$

concerns the classification of the three-centered extremal black holes and black strings in the $\mathcal{N} = 0$, $\mathfrak{J}_3^{\mathbb{C}_s}$-based Maxwell–Einstein-scalar theory[37,38] in $D = 5$. Analogously, the orbit structure of the non-transitive action of

$$\mathrm{Sym}_2 \ltimes (\mathrm{SL}_{3,h}(\mathbb{R}) \times \mathrm{SL}(3,\mathbb{C})_{\mathbb{R}}) \quad \text{on} \quad \left(\mathbf{3},\mathbf{3},\overline{\mathbf{3}}\right) \quad \text{and} \quad \left(\mathbf{3},\overline{\mathbf{3}},\mathbf{3}\right) \tag{3.22}$$

concerns the classification in the three-centered extremal black holes and black strings of the $\mathcal{N} = 2$, $\mathfrak{J}_3^{\mathbb{C}}$-based magic Maxwell–Einstein supergravity[33,34] in $D = 5$.

The exploitation of the classification of such orbits (which is the object of this paper) for the study of three-centered extremal black holes/strings in the aforementioned Maxwell–Einstein (super)gravity theories in $D = 5$ (Lorentzian) space-time dimensions goes beyond the scope of the present investigation, and we leave it for further future work.

IV. CONSTRUCTION OF THE REPRESENTATIONS

In this section, we show how the $\mathrm{SL}(3,\mathbb{C})^3$-module $(\mathbb{C}^3)^{\otimes 3}$ can be constructed using a $\mathbb{Z}/3\mathbb{Z}$-grading of the simple Lie algebra of type $\mathfrak{e}_6$. This construction pertains to Vinberg's theory of θ-groups.

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $m \geq 2$ be an integer, and fix a primitive mth root of unity ζ in $\mathbb{C}$. Then, the $\mathbb{Z}/m\mathbb{Z}$-gradings

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$$

(where $\mathfrak{g}_i$ are subspaces such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$) bijectively correspond to the automorphisms of $\mathfrak{g}$ of order m. Indeed, if a grading as above is given, then by defining $\theta(x) = \zeta^i x$ for $x \in \mathfrak{g}_i$ defines an automorphism of order m. Conversely, if θ is an automorphism of order m, then letting $\mathfrak{g}_i$ be the eigenspace of θ with eigenvalue ω^i yields a $\mathbb{Z}/m\mathbb{Z}$-grading. We also remark that given a grading as above, the subalgebra $\mathfrak{g}_0$ is reductive [see Ref. 56, Lemma 8.1(c)] and that $\mathfrak{g}_1$ is naturally a $\mathfrak{g}_0$-module.

The finite-order automorphisms have been classified, up to conjugacy in $\mathrm{Aut}(\mathfrak{g})$, by several authors. One classification is due to Kac; we refer to Ref. 57, Chap. X, Sec. 5 for a thorough treatment. The inner automorphisms in this classification correspond to labelings by non-negative integers of the affine Dynkin diagram. Here, we just consider the case where all labels are 0 or 1. Let Φ be the root system of $\mathfrak{g}$ with simple system Δ. Let $\tilde{\alpha}$ be the highest root of Φ with respect to the dominance order induced by Δ. Then, the nodes of the affine Dynkin diagram correspond to the elements of $\widetilde{\Delta} = \Delta \cup \{-\tilde{\alpha}\}$. The weight of the nodes $-\tilde{\alpha}$ is 1, whereas the weights of the other nodes are the coefficients of $\tilde{\alpha}$ with respect to Δ. Let $\Pi_0, \Pi_1 \subset \widetilde{\Delta}$ be the subsets consisting of the roots with labels 0 and 1, respectively. Then, we have the following (see Ref. 58, Chap. 3, Sec. 3.7):

- the automorphism θ associated with the labeling has order equal to the sum of the weights of the nodes corresponding to Π_0,
- Π_0 generates the root system of the semisimple part of $\mathfrak{g}_0$, and
- the module $\mathfrak{g}_1$ has $|\Pi_1|$ irreducible components.

Now, let $G = \mathrm{Aut}(\mathfrak{g})^\circ$ be the identity component of the automorphism group of $\mathfrak{g}$. The Lie algebra of G is isomorphic to $\mathfrak{g}$ (to be precise, it is equal to $\mathrm{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$). Let G_0 be the connected subgroup of G whose Lie algebra is isomorphic to $\mathfrak{g}_0$ (or equal to $\mathrm{ad}\mathfrak{g}_0$). Then, G_0 naturally acts on $\mathfrak{g}_1$. The representation of G_0 in $\mathrm{GL}(\mathfrak{g}_1)$ is called a θ-representation. They were introduced and studied in detail by Vinberg.[1,2] Among other things, he developed methods for classifying the orbits of G_0 in $\mathfrak{g}_1$.

A first observation here is that $\mathfrak{g}_1$ is closed under Jordan decomposition. Indeed, if $x = x_s + x_n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, then $\theta(x) = \theta(x_s) + \theta(x_n)$ is the Jordan decomposition of $\theta(x)$. However, also $\theta(x) = \zeta x = \zeta x_s + \zeta x_n$ is the Jordan decomposition of $\theta(x)$. Hence, $x_s, x_n \in \mathfrak{g}_1$. This immediately divides the orbits into three groups: nilpotent, semisimple, and mixed orbits [that consists, respectively, of nilpotent, semisimple, and mixed (that is, neither nilpotent nor semisimple) elements]. We will briefly indicate the methods for classifying these orbits in Secs. VI A, VII A, and VIII A.

Our main example is the automorphism of the simple Lie algebra $\mathfrak{g}$ of type $\mathfrak{e}_6$ with the Kac diagram displayed in Fig. 1.

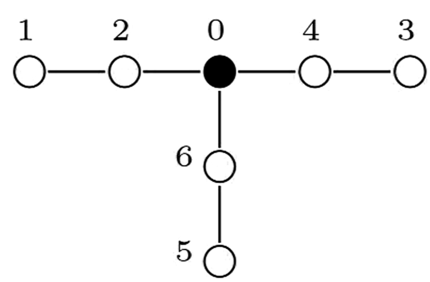

FIG. 1. Kac diagram of an automorphism of $\mathfrak{e}_6$.

Here, the black node has label 1 and the other nodes have label 0. From what is said above, it follows that θ has order 3, $\mathfrak{g}_0$ is semisimple of type $3\mathfrak{a}_2$, and the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$. Then, $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the corresponding root system of $\mathfrak{g}$. For a root $\alpha \in \Phi$, we denote the corresponding root space by $\mathfrak{g}_\alpha$. Let $\Delta = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$ be a set of simple roots of Φ such that in the corresponding Dynkin diagram α_i corresponds to node i as in Fig. 1 and such that $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}_0$ for $i = 1, 2, 3, 4, 5, 6$ and $\mathfrak{g}_{\alpha_0} \subset \mathfrak{g}_1$. Accordingly, we let α_5 denote minus the highest root of Φ. For $0 \le i \le 6$, we let $x_i \in \mathfrak{g}_{\alpha_i}$, $h_i \in \mathfrak{h}$, and $y_i \in \mathfrak{g}_{-\alpha_i}$ be "canonical generators," that is, letting C denote the Cartan matrix of the extended Dynkin diagram in Fig. 1, we have

$$[h_i, x_j] = C(j,i)x_j, [h_i, y_j] = -C(j,i)y_j, [x_i, y_j] = \delta_{ij}h_i.$$

By Ref. 1, Sec. 8, x_0 is a lowest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$. We have $[h_i, x_0] = 0$ for $i = 1, 3, 5$ and $[h_i, x_0] = -x_0$ for $i = 2, 4, 6$. Let $v_0 \in \mathfrak{g}_1$ be a highest weight vector. Then, it follows that $[h_i, v_0] = v_0$ for $i = 1, 3, 5$ and $[h_i, v_0] = 0$ for $i = 2, 4, 6$.

Now, we let $\hat{\mathfrak{g}} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$, where each $\mathfrak{a}_i$ is a copy of $\mathfrak{sl}(3,\mathbb{C})$. Let $\hat{x}_{i,j}, \hat{h}_{i,j}, \hat{y}_{i,j}$ for $j = 1, 2$ denote the canonical generators of $\mathfrak{a}_i$. Concretely, this means the following. Denote the 3×3-matrix with a 1 on position (i, j) and zeros elsewhere by e_{ij}. Then,

$$\hat{x}_{i,1} = e_{12}, \hat{x}_{i,2} = e_{23}, \hat{h}_{i,1} = e_{11} - e_{22}, \hat{h}_{i,2} = e_{22} - e_{33}, \hat{y}_{i,1} = e_{21}, \hat{y}_{i,2} = e_{32}.$$

Let V^i for $1 \le i \le 3$ be a copy of $\mathbb{C}^3$. In the sequel, we denote the standard basis of $\mathbb{C}^3$ by e_0, e_1, e_2. Then, $\hat{\mathfrak{g}}$ acts on $\mathcal{V} = V^1 \otimes V^2 \otimes V^3$ by

$$(z_1 + z_2 + z_3) \cdot (v_1 \otimes v_2 \otimes v_3) = z_1v_1 \otimes v_2 \otimes v_3 + v_1 \otimes z_2v_2 \otimes v_3 + v_1 \otimes v_2 \otimes z_3v_3.$$

Then, $\mathcal{V}$ is an irreducible $\hat{\mathfrak{g}}$-module with highest weight vector $u_0 = e_0 \otimes e_0 \otimes e_0$. We have $\hat{h}_{i,1} \cdot u_0 = u_0$ and $\hat{h}_{i,2} \cdot u_0 = 0$.

Let $\psi : \hat{\mathfrak{g}} \to \mathfrak{g}_0$ be the isomorphism mapping $\hat{a}_{i,1} \mapsto a_{2i-1}$, $\hat{a}_{i,2} \mapsto a_{2i}$, where $a \in \{x, h, y\}$, $1 \le i \le 3$. Then, $\mathfrak{g}_1$ becomes a $\hat{\mathfrak{g}}$-module by $\hat{z} \cdot x = [\psi(\hat{z}), x]$. Comparing highest weights, we see that the $\hat{\mathfrak{g}}$-modules $\mathfrak{g}_1$ and $\mathcal{V}$ are isomorphic and that there is a unique $\hat{\mathfrak{g}}$-module isomorphism $\varphi : \mathfrak{g}_1 \to \mathcal{V}$ mapping $v_0 \mapsto u_0$.

Consider the group $\widehat{G} = \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$. Then, $\mathrm{Lie}(\widehat{G}) = \hat{\mathfrak{g}}$. Let G, G_0 be as above, that is, $G = \mathrm{Aut}(\mathfrak{g})^\circ$ and G_0 is the connected subgroup corresponding to the subalgebra $\mathfrak{g}_0$. Since $\widehat{G}$ is simply connected, we have a surjective homomorphism $\Psi : \widehat{G} \to G_0$ whose differential is ψ. This defines a $\widehat{G}$-action on $\mathfrak{g}_1$ by $\hat{g} \cdot x = \Psi(\hat{g}) \cdot x$. In addition, $\mathcal{V}$ is a $\widehat{G}$-module given by

$$(g_1 \times g_2 \times g_3) \cdot (v_1 \otimes v_2 \otimes v_3) = g_1v_1 \otimes g_2v_2 \otimes g_3v_3.$$

Then, φ is also an isomorphism of $\widehat{G}$-modules. It follows that classifying the $\widehat{G}$-orbits in $\mathcal{V}$ is equivalent to classifying the G_0-orbits in $\mathfrak{g}_1$.

A. Permuting the tensor factors

Consider the vector space $\mathcal{V}$. Let $\pi \in S_3$ be a permutation of $\{1, 2, 3\}$. To π, we associate a linear map (denoted by the same symbol) $\pi : \mathcal{V} \to \mathcal{V}$, $\pi(v_1 \otimes v_2 \otimes v_3) = v_{\pi(1)} \otimes v_{\pi(2)} \otimes v_{\pi(3)}$. Furthermore, we also define the map $\pi : \widehat{G} \to \widehat{G}$, $\pi(g_1 \times g_2 \times g_3) = g_{\pi(1)} \times g_{\pi(2)} \times g_{\pi(3)}$. Then, obviously, we have

$$\pi(g) \cdot \pi(v) = \pi(g \cdot v) \quad \text{for } g \in \widehat{G}, v \in V.$$

It follows that $\widehat{G} \cdot \pi(v) = \pi(\widehat{G} \cdot v)$. Furthermore, denoting the stabilizer of $v \in \mathcal{V}$ in $\widehat{G}$ by $Z_{\widehat{G}}(v)$, we have $Z_{\widehat{G}}(\pi(v)) = \pi(Z_{\widehat{G}}(v))$.

As seen in Sec. III B, it is of interest to classify the orbits of $\widehat{G}$ in V up to permutation of the tensor factors or up to permutation of two of the tensor factors. In other words, instead of the group $\widehat{G}$, we consider the groups $\mathrm{Sym}_3 \ltimes \widehat{G}$ and $\mathrm{Sym}_2 \ltimes \widehat{G}$ (where Sym_2 permutes the second and third factors only). For the nilpotent elements, the permutation action is given in detail in Table I. For the semisimple and mixed elements, see Remarks 7.5 and 8.2, respectively.

B. The action on $V^1 \otimes V^2 \otimes (V^3)^*$

We can, of course, also consider the action of $\widehat{G}$ on the module $V^1 \otimes V^2 \otimes (V^3)^*$ (where the last factor is the dual module of V^3). Here, we argue that we get exactly the same orbits.

Let $\sigma_3 : \widehat{G} \to \widehat{G}$ be defined by $\sigma_3(g_1, g_2, g_3) = (g_1, g_2, g_3^{-T})$. Then, σ_3 is an involution of $\widehat{G}$. Define the action $\circ$ of $\widehat{G}$ on $\mathcal{V}$ by $g \circ v = \sigma_3(g) \cdot v$. The $\widehat{G}$-module with this action is isomorphic to $V^1 \otimes V^2 \otimes (V^3)^*$. We have that $g \cdot v = \sigma_3(g) \circ v$. Hence, the orbits of $\widehat{G}$ on $\mathcal{V}$ with respect to the $\cdot$ action and with respect to the $\circ$ action are the same. Hence, in this paper, we just consider the $\cdot$ action.

C. Notation

Throughout this paper, we use the *bra and ket notation* (cf. Ref. 59, Sec. 5.4.2.1) for the elements of $\mathcal{V}$, denoting the elementary tensor $e_{j_1} \otimes e_{j_2} \otimes e_{j_3}$ by the symbol $|j_1 j_2 j_3\rangle$.

By ζ, we will denote a fixed third primitive root of unity in $\mathbb{C}$.

In the sequel, we will freely use the notation introduced in this section. In particular, we will use the groups $\widehat{G}$, G, and G_0 and the Lie algebras $\mathfrak{g}$, $\mathfrak{g}_0$.

V. GALOIS COHOMOLOGY AND REAL ORBITS

One of the main goals of this paper is to achieve a classification of the orbits of $\widehat{G}(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ on the real space spanned by the elementary tensors $|j_1 j_2 j_3\rangle$ in $\mathcal{V}$. Our methods are based on Galois cohomology, to which we here give a brief introduction.

Let $\mathcal{G}$ be a subgroup of $\mathrm{GL}(n,\mathbb{C})$, stable under complex conjugation $g \mapsto \bar{g}$ (given by the complex conjugation of the matrix entries). By $\mathcal{G}(\mathbb{R})$, we denote the group of real points,

$$\mathcal{G}(\mathbb{R}) = \{g \in \mathcal{G} \mid \bar{g} = g\}.$$

We have the following definitions:

- $g \in \mathcal{G}$ is a *cocycle* if $\bar{g}g = 1$,
- two cocycles g_1, g_2 are *equivalent* if there is $h \in \mathcal{G}$ such that $h^{-1} g_1 \bar{h} = g_2$, and
- the set of equivalence classes of cocycles is the *first Galois cohomology set* $\mathrm{H}^1\mathcal{G}$ of $\mathcal{G}$.

We note that these definitions are an *ad hoc* version of the classical definition of Galois cohomology of a group, to the special case where the Galois group has order 2. We refer to the book by Serre[60] for a more general account.

The crucial result for our computations is the following theorem; see Ref. 60, Sec. I.5.4.

Theorem 5.1. *Suppose that $\mathrm{H}^1\mathcal{G} = 1$. Let $\mathcal{G}$ act on the set U. Suppose that U has an involution $u \mapsto \bar{u}$ such that $\overline{g \cdot u} = \bar{g} \cdot \bar{u}$ for $g \in \mathcal{G}$, $u \in U$. Let $u_0 \in U$ be real (that is, $\bar{u}_0 = u_0$), and consider the set of real points of the orbit of u_0,*

$$\mathcal{O}(\mathbb{R}) = \{v \in \mathcal{G} \cdot u_0 \mid \bar{v} = v\}.$$

Let $Z_{\mathcal{G}}(u_0) = \{g \in \mathcal{G} \mid g \cdot u_0 = u_0\}$ be the stabilizer of u_0 in $\mathcal{G}$. Then, there exists a bijection between $\mathrm{H}^1 Z_{\mathcal{G}}(u_0)$ and the set of $\mathcal{G}(\mathbb{R})$-orbits in $\mathcal{O}(\mathbb{R})$. This bijection is described explicitly as follows: Let $[g] \in \mathrm{H}^1 Z_{\mathcal{G}}(u_0)$; then, there exists $h \in \mathcal{G}$ such that $h^{-1}\bar{h} = g$ (as $\mathrm{H}^1\mathcal{G} = 1$). Then, the class $[g]$ corresponds to the $\mathcal{G}(\mathbb{R})$-orbit of $(h \cdot u_0)$.

In order to use this, we need to be able to compute the first Galois cohomology set of matrix groups. This can be done by computer. However, we also use some criteria to derive the cohomology sets directly.

Criterion 5.2. We have that $\mathrm{H}^1\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{H}^1\mathrm{SL}(n,\mathbb{C})$ are both trivial. Hence, in particular, $\mathrm{H}^1 T$ is trivial, where $T \subset \mathrm{GL}(n,\mathbb{C})$ is a torus.

The next criterion is Ref. 4, Corollary 3.2.2.

Criterion 5.3. If $\mathcal{G}$ is finite, Abelian, and of odd order, then $\mathrm{H}^1\mathcal{G}$ is trivial.

Criterion 5.4. Let $\mathcal{G}$ be a group of order p^m, where p is an odd prime. Then, $\mathrm{H}^1\mathcal{G}$ is trivial.

This follows from Ref. 4, Lemma 3.2.3. The next criterion is Ref. 4, Lemma 3.2.6.

Criterion 5.5. If $\mathcal{G}$ is finite of order $2p^m$, where p is an odd prime, then $\mathrm{H}^1\mathcal{G} = \{1, [c]\}$, where c is a real element of order 2.

The next criterion is Ref. 4, Proposition 3.3.16.

Criterion 5.6. Let $\mathcal{G}$ be a linear algebraic group with identity component $\mathcal{G}^\circ$. Suppose that $\mathcal{G}/\mathcal{G}^\circ$ is of order p^n for some odd prime p and $n \geq 0$ and $|\mathrm{H}^1\mathcal{G}^\circ| < p$. Then, the canonical map $\mathrm{H}^1\mathcal{G}^\circ \to \mathrm{H}^1\mathcal{G}$ is bijective.

VI. NILPOTENT ORBITS

In this section, we comment on the methods for classifying complex and real nilpotent orbits. We present the complex classification of Nurmiev,[3] to which we add some data (see below). We also classify the real nilpotent orbits.

From Ref. 4, Sec. 4.3, we recall some general results. Let $\mathfrak{a} = \oplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{a}_i$ be a $\mathbb{Z}/m\mathbb{Z}$-graded semisimple Lie algebra over a field of characteristic 0. Let H be a group of automorphisms of $\mathfrak{a}$ such that each element preserves the spaces $\mathfrak{a}_i$ for $i \in \mathbb{Z}/m\mathbb{Z}$ and such that H contains $\exp \mathrm{ad} x$ for all nilpotent $x \in \mathfrak{a}_0$. Then, we have the following.

- A nilpotent $e \in \mathfrak{a}_1$ lies in a *homogeneous* $\mathfrak{sl}_2$-triple (h, e, f). This means that $h \in \mathfrak{a}_0$, $f \in \mathfrak{a}_{-1}$, and

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

- Two nilpotent elements $e, e' \in \mathfrak{a}_0$, lying in homogeneous $\mathfrak{sl}_2$-triples (h, e, f), (h', e', f'), are H-conjugate if and only if there is $\sigma \in H$ with $\sigma(h) = h'$, $\sigma(e) = e'$, and $\sigma(f) = f'$.

A. The complex nilpotent orbits

Let $\mathfrak{a} = \oplus_{i\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{a}_i$ be a $\mathbb{Z}/m\mathbb{Z}$-graded semisimple Lie algebra over $\mathbb{C}$. Let A_0 be the connected subgroup of $\mathrm{Aut}(\mathfrak{a})$ with Lie algebra $\mathrm{ad}_{\mathfrak{a}}\mathfrak{a}_0$. Then, A_0 satisfies the requirements on H above. Moreover, as the base field is algebraically closed, it can be shown that if we have two homogeneous $\mathfrak{sl}_2$-triples (h,e,f), (h',e',f'), then e, e' are A_0-conjugate if and only if h, h' are A_0-conjugate (cf. Ref. 61, Theorem 8.3.6). Let $\mathfrak{h}_0$ be a fixed Cartan subalgebra of $\mathfrak{a}_0$. It follows that each nilpotent orbit has a representative e lying in a homogeneous $\mathfrak{sl}_2$-triple (h,e,f) such that $h\in\mathfrak{h}_0$. We also have that $h,h'\in\mathfrak{h}_0$ are A_0-conjugate if and only if they are conjugate under the Weyl group $N_{A_0}(\mathfrak{h}_0)/Z_{A_0}(\mathfrak{h}_0)$. This Weyl group is canonically isomorphic to the Weyl group of the root system of $\mathfrak{a}_0$ with respect to $\mathfrak{h}_0$. These facts can be used to devise an algorithm to classify the nilpotent A_0-orbits in $\mathfrak{a}_0$ (Ref. 61, Sec. 8.4.1). An alternative is Vinberg's support method[2] (Ref. 61, Sec. 8.4.2).

In our case, that is, with $\mathfrak{g}$, $\mathfrak{g}_0$, G, G_0 as in Sec. IV, there are 63 nonzero nilpotent orbits. Nurmiev[3] lists them up to permutation of the tensor factors (Sec. IV A). We have used the above indicated algorithms, and their implementation in the SLA[62] package of GAP4[63], to check the correctness of his list. We reproduce it here in Table I, where we also add the permutations that can be used to determine representatives of the orbits that are not in Nurmiev's list. The second column of Table I has the same representatives as Nurmiev, except that we use the qutrit notation. In the third column, we indicate a characteristic of the orbit by a list of integers. If e is a representative of a nilpotent orbit, lying in the homogeneous $\mathfrak{sl}_2$-triple (h,e,f), then the element h is called a characteristic of the orbit. The integers in the third column are to be interpreted as follows. Let $\alpha_1,\dots,\alpha_6$ be a fixed set of simple roots of $\mathfrak{g}_0$ with respect to the fixed Cartan subalgebra $\mathfrak{h}_0$. Then, by the above considerations, it follows that each nilpotent orbit has a representative e lying in a homogeneous $\mathfrak{sl}_2$-triple (h,e,f) with $\alpha_i(h)\geq 0$ for $1\leq i\leq 6$. The values $\alpha_i(h)$ are given in the third column. We remark that Nurmiev uses the action of the symmetric group S_3: if two orbits are carried to each other via a permutation of the tensor factors, then only one of them is listed. We do the same, except that in the fourth column, we list the permutations that have to be applied to the listed orbits to obtain the other orbits. Thus, the total number of nilpotent orbits is 62 (excluding 0). The fifth and sixth columns are devoted to the stabilizer

$$Z_{\widehat{G}}(h,e,f) = \{g\in\widehat{G} \mid g\cdot h = h, g\cdot e = e, g\cdot f = f\}$$

of a homogeneous $\mathfrak{sl}_2$-triple (h,e,f) containing a representative e of the given nilpotent orbit. The fifth column has a description of the identity component, and the sixth column has a description of the component group. These stabilizers have been determined by computing explicit sets of polynomial equations describing them, together with computational methods based on Gröbner bases. For this, we have used the computer algebra systems GAP4 and SINGULAR.[64]

B. The real nilpotent orbits

Let T denote the set of complex homogeneous $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Then, $\widehat{G}$ acts on T. Furthermore, T is closed under complex conjugation, and the triples fixed under complex conjugation form the set $T^{\mathbb{R}}$ of real homogeneous $\mathfrak{sl}_2$-triples. The group $\widehat{G}(\mathbb{R})$ acts on $T^{\mathbb{R}}$, and as seen at the beginning of this section, the nilpotent $\widehat{G}(\mathbb{R})$-orbits are in bijection with the $\widehat{G}(\mathbb{R})$-orbits in $T^{\mathbb{R}}$. Since $\mathrm{H}^1\widehat{G} = 1$, Theorem 5.1 implies the following.

Theorem 6.1. *Let $e\in\mathfrak{g}_1$ be a nilpotent element and (h,e,f) be an associated homogeneous $\mathfrak{sl}_2$-triple. Let $\mathcal{O} = \widehat{G}\cdot e$ and $\mathcal{O}(\mathbb{R}) = \{x\in\mathcal{O} \mid \bar{x} = x\}$. There is a bijection between $\widehat{G}(\mathbb{R})$-orbits in $\mathcal{O}(\mathbb{R})$ and $\mathrm{H}^1(Z_{\widehat{G}}(h,e,f))$.*

Among the centralizers in Table I, only three cases have non-trivial Galois cohomology. In order to compute generators of first cohomology set, we need a more explicit description of their stabilizers, which in all cases we denote as Z.

3 $|002\rangle + |011\rangle + |020\rangle + |101\rangle + |112\rangle + |200\rangle$: the stabilizer is isomorphic to $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and consists of the elements

$$\mathrm{diag}(\delta^2\zeta^2, \epsilon\delta^2\zeta^2, \epsilon\delta^2\zeta^2)\times\mathrm{diag}(\delta,\epsilon\delta,\epsilon\delta)\times\mathrm{diag}(\epsilon\zeta,\epsilon\zeta,\zeta),$$

where $\zeta^3 = \delta^3 = 1$ and $\epsilon^2 = 1$. By Criterion 5.5, it follows that H^1Z consists of Ref. 21 and $[c]$, where

$$c = \mathrm{diag}(1,-1,-1)\times\mathrm{diag}(1,-1,-1)\times\mathrm{diag}(-1,-1,1).$$

An element $g\in\mathrm{SL}_3(\mathbb{C})^3$ with $g^{-1}\bar{g} = c$ is

$$g = \mathrm{diag}(-1,i,i)\times\mathrm{diag}(-1,i,i)\times\mathrm{diag}(i,i,-1).$$

The corresponding orbit representative is

$$-|002\rangle + |011\rangle + |020\rangle + |101\rangle + |112\rangle + |200\rangle.$$

9 $|000\rangle + |011\rangle + |111\rangle + |122\rangle$: the identity component of the stabilizer is a two-dimensional torus, and the component group is isomorphic to $S_3 \times \mathbb{Z}/3\mathbb{Z}$. The identity component T_2 consists of the elements $\mathrm{diag}(1,1,1) \times \mathrm{diag}(s^{-1}, t^{-1}, st) \times \mathrm{diag}(s, t, s^{-1}t^{-1})$. The component group is generated by $\mathrm{diag}(\zeta^2, \zeta^2, \zeta^2) \times \mathrm{diag}(\zeta, \zeta, \zeta) \times \mathrm{diag}(1,1,1)$ and

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have $g_1^2 = 1$, $g_2^3 = 1$, and $g_1 g_2 g_1 = g_2^2 \mod T_2$. Hence, g_1, g_2 generate a subgroup of the component group isomorphic to S_3. Using Ref. 4, Proposition 3.3.17, it can be shown that $\mathrm{H}^1 Z = \{[1], [g_1]\}$. An element $g \in \mathrm{SL}_3(\mathbb{C})^3$ with $g^{-1}\bar{g} = g_1$ is

$$\begin{pmatrix} \frac{1}{2} & -1 & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & -\frac{1}{2}i \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \frac{1}{2}i \end{pmatrix}.$$

The corresponding orbit representative is

$$2|210\rangle + 2|201\rangle - \frac{1}{4}|022\rangle - |011\rangle + |000\rangle.$$

20 $|000\rangle + |111\rangle$: the stabilizer is isomorphic to $T_4 \rtimes \mathbb{Z}/2\mathbb{Z}$. The identity component T_4 consists of

$$\mathrm{diag}(s^{-1}u^{-1}, t^{-1}v^{-1}, stuv) \times \mathrm{diag}(s, t, s^{-1}t^{-1}) \times \mathrm{diag}(u, v, u^{-1}v^{-1}).$$

The component group is generated by

$$g_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have $\mathrm{H}^1 Z = \{[1], [g_0]\}$. An element $g \in \mathrm{SL}_3(\mathbb{C})^3$ with $g^{-1}\bar{g} = g_0$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \frac{1}{2}i \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \frac{1}{2}i \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \frac{1}{2}i \end{pmatrix}.$$

The corresponding orbit representative is

$$2|000\rangle - 2|011\rangle - 2|101\rangle - 2|110\rangle.$$

Summarizing, apart from the real nilpotent orbits with representatives given in Table I, we get the real nilpotent orbits with representatives given in Table II.

VII. SEMISIMPLE ELEMENTS

In this section, we consider the orbits of semisimple elements of $\mathfrak{g}_1$. We first describe the complex classification where we follow Nurmiev[3] adding many details on the proof of the correctness of the classification. These are then also needed for the classification in the real case.

A. Semisimple orbits: The complex case

Much of the theory needed to classify the semisimple orbits in $\mathfrak{g}_1$ is due to Vinberg.[1] Here, we give a short overview of some of the main concepts and results.

A Cartan subspace $\mathfrak{C} \subset \mathcal{V}$ is by definition a maximal subspace consisting of commuting semisimple elements. Vinberg[1] showed that any two Cartan subspaces are conjugated with respect to G_0. As a consequence, every semisimple orbit has a representative in any given Cartan subspace.

Definition 7.1. The Weyl group of a Cartan subspace $\mathfrak{C}$ *is* $W_{\mathfrak{C}} = N_{G_0}(\mathfrak{C})/Z_{G_0}(\mathfrak{C})$.

The group $W_{\mathfrak{C}}$ is a finite complex reflection group (Ref. 1, Theorem 8). Moreover, two elements from $\mathfrak{C}$ are G_0-conjugate if and only if they are $W_{\mathfrak{C}}$-conjugate (Ref. 1, Theorem 2). Hence, the classification of the semisimple orbits reduces to the classification of the $W_{\mathfrak{C}}$-orbits in $\mathfrak{C}$ where the latter is any fixed Cartan subspace. Those orbits are infinitely many, and it does not seem to be possible to present them in an irredundant list. However, it is possible to define several special subsets of $\mathfrak{C}$ such that each semisimple orbit has a point in exactly one of them. Second, two elements of the same subset are G_0-conjugate if and only if they are conjugate under an explicitly given finite group. Here, we show how that is done in our case.

Consider the following elements:

$$u_1 = |000\rangle + |111\rangle + |222\rangle,$$
$$u_2 = |012\rangle + |120\rangle + |201\rangle,$$
$$u_3 = |021\rangle + |210\rangle + |102\rangle.$$

By Ref. 3, see also Ref. 59, Sec. 5.4.4, Exercise 2; these span a Cartan subspace $\mathfrak{C}$ of $\mathfrak{g}_1$. We also checked this by computer. In the sequel, we fix this space.

Now, we briefly review the construction of complex reflections in the Weyl group given in Ref. 65, Sec. 3.3, adapted to our situation. Let $\mathfrak{h}$ be the centralizer of $\mathfrak{C}$ in the Lie algebra $\mathfrak{g}$. Then, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Consider the automorphism θ of order 3 (see Sec. IV). This automorphism stabilizes $\mathfrak{h}$. It induces a map θ_* of the dual space $\mathfrak{h}^*$ by $\theta_*(\mu)(h) = \mu(\theta(h))$ for $h \in \mathfrak{h}$. For a root α of $\mathfrak{h}^*$, we have that the elements $\pm\alpha, \pm\theta_*(\alpha), \pm\theta_*^2(\alpha)$ span a root subsystem of type A_2. By $U(\alpha)$, we denote the corresponding subalgebra of $\mathfrak{g}$. Let $\mathfrak{C}(\alpha) = \mathfrak{C} \cap U(\alpha)$; then, $\mathfrak{C}(\alpha)$ is of dimension 1. Let $\mathfrak{C}_0(\alpha) = \{p \in \mathfrak{C} \mid \alpha(p) = 0\}$. Then, $\mathfrak{C} = \mathfrak{C}_0(\alpha) \oplus \mathfrak{C}(\alpha)$. Moreover, defining the linear map $w_\alpha : \mathfrak{C} \to \mathfrak{C}$ by $w_\alpha(p) = p$ for $p \in \mathfrak{C}_0(\alpha)$, $w_\alpha(p) = \zeta p$ for $p \in \mathfrak{C}(\alpha)$, we obtain a complex reflection of $\mathfrak{C}$. The arguments in Ref. 65, Sec. 3.3 can also be used here to show that w_α is induced by an element of $N_{G_0}(\mathfrak{C})$. Hence, $w_\alpha \in W_{\mathfrak{C}}$.

The root system Φ is partitioned into twelve root subsystems of type A_2 as constructed above. This yields 12 reflections in $W_{\mathfrak{C}}$. In fact, they form a single conjugacy class. The reflections of the form w_α^2 also form a conjugacy class, and these are the two conjugacy classes in $W_{\mathfrak{C}}$ consisting of complex reflections.

In this case, the Weyl group $W_{\mathfrak{C}}$ has order 648. By the table in Ref. 1, Sec. 9, it is isomorphic to the group G_{25} in the Shephard–Todd classification and is also denoted as $W(\mathcal{L}_3)$ in Ref. 66. Using the above techniques, we can find all reflections in $W_{\mathfrak{C}}$ and show by a straightforward GAP calculation that $W_{\mathfrak{C}}$ is generated by complex reflections,

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2+\zeta & -1-2\zeta & 2+\zeta \\ 2+\zeta & 2+\zeta & -1-2\zeta \\ -1-2\zeta & 2+\zeta & 2+\zeta \end{pmatrix}.$$

Now, we describe how to obtain the special subsets of $\mathfrak{C}$ mentioned above. By W_p, we denote the stabilizer of $p \in \mathfrak{C}$ in $W_{\mathfrak{C}}$. It is known that W_p is also generated by complex reflections (Ref. 67, Theorem 1.5). For $p \in \mathfrak{C}$, set

$$\mathfrak{C}_p = \{h \in \mathfrak{C} \mid w\,h = h \quad \text{for all } w \in W_p\}$$
$$\mathring{\mathfrak{C}}_p = \{q \in \mathfrak{C}_p \mid W_q = W_p\}.$$

Remark 7.2. The set $\mathring{\mathfrak{C}}_p$ is Zariski open in $\mathfrak{C}_p$. It is determined by the inequalities $w\,q \neq q$ for all $w \in W_{\mathfrak{C}} \backslash W_p$. Because also the stabilizer W_q is generated by complex reflections, we have that $q \in \mathfrak{C}_p$ lies in $\mathring{\mathfrak{C}}_p$ if and only if $w\,q \neq q$ for all complex reflections in $W_{\mathfrak{C}}$ that do not lie in W_p. Let w be such a reflection. Then, $w = w_\alpha$ or $w = w_\alpha^2$ for $\alpha \in \Phi$. From the description of w_α, it follows that $w(q) \neq q$ is equivalent to $\alpha(q) \neq 0$. We can compute a linear polynomial $\psi_\alpha = c_1x_1 + c_2x_2 + c_3x_3$ such that $\alpha(a_1u_1 + a_2u_2 + a_3u_3) = \psi_\alpha(a_1, a_2, a_3)$. Hence, for $q = a_1u_1 + a_2u_2 + a_3u_3$, we have that $w(q) \neq q$ if and only if $\psi_\alpha(a_1, a_2, a_3) \neq 0$. Hence, by taking the product of all ψ_α, where α is such that $w_\alpha \notin W_p$, we can compute a polynomial inequality defining the set $\mathring{\mathfrak{C}}_p$.

Now, for $v \in W_{\mathfrak{C}}$ and $p, q \in \mathfrak{C}$, it is immediate that

$$v \cdot \mathring{\mathfrak{C}}_p = \mathfrak{C}_q{}^{\circ} \text{ if and only if } W_q = vW_pv^{-1}. \tag{7.1}$$

Let R denote the collection of all reflection subgroups of $W_{\mathfrak{C}}$ that are equal to a W_p for $p \in \mathfrak{C}$. Then, it follows that R is closed under conjugacy by $W_{\mathfrak{C}}$. Let $p_1, \dots, p_m \in \mathfrak{C}$ be such that $W_{p_1}, \dots, W_{p_m}$ are representatives of the different conjugacy classes in R. Then, (7.1) implies that each semisimple orbit has a point in unique $\mathfrak{C}_{p_i}{}^{\circ}$.

Let p be one of p_i. By (7.1), we see that the group of elements of $W_{\mathfrak{C}}$ mapping $\mathring{\mathfrak{C}}_p$ to itself is exactly the normalizer $N_{W_{\mathfrak{C}}}(W_p)$. By definition of $\mathring{\mathfrak{C}}_p$, it follows that the group of elements fixing each element of $\mathring{\mathfrak{C}}_p$ is exactly W_p. Hence, the group $\Gamma_p = N_{W_{\mathfrak{C}}}(W_p)/W_p$ acts naturally on $\mathring{\mathfrak{C}}_p$. Again, from (7.1), we immediately get the following result.

Theorem 7.3. *Two elements of $\mathring{\mathfrak{C}}_p$ are G_0-conjugate if and only if they are Γ_p-conjugate.*

We also have the following lemma, which is of fundamental importance for the complex and for the real classification.

Lemma 7.4. Let $p \in \mathfrak{C}$ and $q_1, q_2 \in \mathring{\mathfrak{C}}_p$. Let $Z_{G_0}(q_i) = \{g \in G_0 \mid g \cdot q_i = q_i\}$ be their stabilizers in G_0. Then, $Z_{G_0}(q_1) = Z_{G_0}(q_2)$.

Proof. Let $w \in W_{\mathfrak{C}}$ be a complex reflection. Then, in the above notation, we have $w = w_\alpha$ or $w = w_\alpha^2$ for $\alpha \in \Phi$. By the construction of w_α, it follows that $w_\alpha(q) = q$ if and only if $\alpha(q) = 0$. Set $\Phi_i = \{\alpha \in \Phi \mid \alpha(q_i) = 0\}$. Then, as $W_{q_1} = W_{q_2}$, it follows that $\Phi_1 = \Phi_2$. Let $\mathfrak{z}_i = \{x \in \mathfrak{g} \mid [x, q_i] = 0\}$. Then,

$$\mathfrak{z}_i = \mathfrak{h} \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space in $\mathfrak{g}$ corresponding to the root α. It follows that $\mathfrak{z}_1 = \mathfrak{z}_2$. By Ref. 68, Corollary 3.11, the stabilizers $Z_G(q_i)$ are connected. The Lie algebra of $Z_G(q_i)$ is $\mathfrak{z}_i$. Hence, $Z_G(q_1) = Z_G(q_2)$. However, $Z_{G_0}(q_i) = G_0 \cap Z_G(q_i)$. □

As seen above, each semisimple element is conjugate to one of the form $u = a_1u_1 + a_2u_2 + a_3u_3$. In Ref. 3, the representatives for semisimple orbits are classified in terms of the coefficients $\{a_i\}$ in five families, corresponding to the subsets of the form $\mathring{\mathfrak{C}}_p$. Each family corresponds to a conjugacy class of reflection subgroups of $W_{\mathfrak{C}}$. It is known that $W_{\mathfrak{C}}$ has six conjugacy classes of reflection subgroups, see, e.g., Ref. 69, Table 3, where the subgroups apart from the trivial subgroup and $W_{\mathfrak{C}}$ itself are denoted as $l_1, 2l_1, l_2$, and $3l_1$. The group $3l_1$ is not the stabilizer of a point, as it is of rank 3 and just stabilizes 0. Hence, when we include the trivial subgroup and $W_{\mathfrak{C}}$ itself, we get five families of the form $\mathring{\mathfrak{C}}_p$. Here, we give the description of these families following.[3] The fifth family is omitted because it is constituted only by the null vector. This family corresponds to the subgroup, which is $W_{\mathfrak{C}}$ itself. By Lemma 7.4, the elements of a fixed family all have the same stabilizer in G_0 and hence in $\widehat{G}$. We also describe the stabilizers in $\widehat{G}$. They have been determined using computational techniques based on Gröbner bases (as described in Ref. 4). We also give an explicit description of the groups Γ_p.

First Family: This family corresponds to the trivial subgroup. The parameters a_i satisfy the conditions

$$a_1a_2a_3 \neq 0,$$
$$(a_1^3 + a_2^3 + a_3^3)^3 - (3a_1a_2a_3)^3 \neq 0.$$

The stabilizer of the semisimple elements in the first family is finite of order 81 and generated by

$$\operatorname{diag}(\zeta, \zeta^2, 1, \zeta, \zeta^2, 1, \zeta, \zeta^2, 1)$$
$$\operatorname{diag}(\zeta^2, \zeta^2, \zeta^2, 1, 1, 1, \zeta, \zeta, \zeta)$$
$$\begin{pmatrix} 0 & 0 & \zeta^2 \\ 1 & 0 & 0 \\ 0 & \zeta & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & \zeta \\ \zeta^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \end{pmatrix}.$$

Here, the group Γ_p is $W_{\mathfrak{C}}$.

Second Family: This family corresponds to the reflection subgroup of order 3 generated by $\operatorname{diag}(1, 1, \zeta)$. It consists of the elements $a_1u_1 + a_2u_2$. The parameters a_1, a_2 satisfy the open condition

$$a_1a_2(a_1^3 + a_2^3) \neq 0.$$

The stabilizer of the elements of the second family is of the form $C \ltimes T_2$, where T_2 is a two-dimensional torus consisting of elements of the form

$$T_2(t_1, t_2) = \operatorname{diag}(t_1^{-1}t_2^{-1}, t_1, t_2, t_1, t_2, t_1^{-1}t_2^{-1}, t_2, t_1^{-1}t_2^{-1}, t_1),$$

and C is a group of order 9 generated by

$$\mathrm{diag}(\zeta,\zeta,\zeta,\zeta^2,\zeta^2,\zeta^2,1,1,1)$$
$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\times\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\times\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

The group Γ_p is of order 18 and generated (with respect to the basis u_1, u_2) by

$$\begin{pmatrix}0&-1\\-1&0\end{pmatrix},\begin{pmatrix}1&0\\0&\zeta\end{pmatrix}.$$

Third Family: This family corresponds to the reflection subgroup of order 9 generated by $\mathrm{diag}(1,1,\zeta)$ and $\mathrm{diag}(1,\zeta,1)$. It consists of the elements a_1u_1 with $a_1 \neq 0$. The stabilizer of the elements of the third family is of the form $C \ltimes T_4$, where T_4 is a four-dimensional torus consisting of elements of the form

$$T_4(t_1,t_2,t_3,t_4) = \mathrm{diag}(t_1^{-1}t_3^{-1}, t_2^{-1}t_4^{-1}, (t_1t_2t_3t_4)^{-1}, t_1, t_2, (t_1t_2)^{-1}, t_3, t_4, (t_3t_4)^{-1}),$$

and C is a group of order 3 generated by

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\times\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\times\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

The group Γ_p is of order 6 and generated by $-\zeta$.

Fourth Family: This family corresponds to the reflection subgroup of order 24 generated by $\mathrm{diag}(\zeta,1,1)$ and

$$\frac{1}{3}\begin{pmatrix}2+\zeta&2+\zeta&2+\zeta\\-1-2\zeta&2+\zeta&-1+\zeta\\-1-2\zeta&-1+\zeta&2+\zeta\end{pmatrix}.$$

It consists of the elements $a(u_2 - u_3)$. The stabilizer of the elements of the fourth family is of the form $F \ltimes C^\circ$, where C° consists of all matrices $A \times A \times A$, with $A \in \mathrm{SL}(3,\mathbb{C})$ and F is a group of order 3 generated by $\mathrm{diag}(\zeta,\zeta,\zeta,\zeta^2,\zeta^2,\zeta^2,1,1,1)$.

The group Γ_p is generated by ζ.

Remark 7.5. Here, we comment on the classification up to permutation of the tensor factors (see Sec. IV A). The elements of families 1 and 3 are stable under all permutations. The element $a(u_2 - u_3)$ of the fourth family is by a permutation of order 2 mapped to $-a(u_2 - u_3)$, whereas it is stable under permutations of order 3. Hence, if we consider the action of Sym_2, Sym_3, then $a(u_2 - u_3)$ is conjugate to $-a(u_2 - u_3)$. Therefore, here, we do not consider the group Γ_p of order 3, but rather a group of order 6. The elements $a_1u_1 + a_2u_2$ of the second family are mapped under a permutation of order 2 to $a_1u_1 + a_2u_3$. The Weyl group $W_{\mathfrak{C}}$ contains the transformation

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

mapping $a_1u_1 + a_2u_3$ to $a_2u_1 + a_1u_2$. Hence, if we also consider the action of the symmetric group, $a_1u_1 + a_2u_2$ is conjugate to $a_2u_1 + a_1u_2$. Therefore, in this case, the elements of the second family are conjugate under $\mathrm{Sym}_k \ltimes \widehat{G}$ $(k = 2, 3)$ if and only if they are conjugate under a group of order 36.

B. Semisimple orbits: The real case

Now, we recall some constructions from Ref. 5. The proofs in the latter paper were relative to the special case considered in that paper. However, because of Lemma 7.4, all proofs go through also in our case. Therefore, we omit them.

Let $p \in \mathfrak{C}$, and write $\mathcal{F} = \mathring{\mathfrak{C}}_p$. Recall that $\Gamma_p = N_{W_\mathfrak{C}}(W_p)/W_p$. Let

$$N_{\widehat{G}}(\mathcal{F}) = \{g \in \widehat{G} \mid g \cdot q \in \mathcal{F} \quad \text{for all } q \in \mathcal{F}\},$$
$$Z_{\widehat{G}}(\mathcal{F}) = \{g \in \widehat{G} \mid g \cdot q = q \quad \text{for all } q \in \mathcal{F}\}.$$

Define a map $\varphi : N_{\widehat{G}}(\mathcal{F}) \to \Gamma_p$ in the following way. Let $g \in N_{\widehat{G}}(\mathcal{F})$; then, $g \cdot p \in \mathcal{F}$, and hence, there is $w \in N_{W_\mathfrak{C}}(W_p)$ with $g \cdot p = w \cdot p$, and we set $\varphi(g) = wW_p$. Then, φ is well-defined and a surjective group homomorphism with kernel $Z_{\widehat{G}}(\mathcal{F})$.

We have the following theorem. For a proof, see Ref. 5, Proposition 5.2.4 or Ref. 70, Theorem 5.3.

Theorem 7.6. *Let $\mathcal{O} = \widehat{G} \cdot p$ be the orbit of p. Write $H^1(\Gamma_p) = \{[\gamma_1], \ldots, [\gamma_s]\}$. Suppose that for each γ_i, there is $n_i \in Z^1(N_{\widehat{G}}(\mathcal{F}))$ with $\varphi(n_i) = \gamma_i$. Then, $\mathcal{O}$ has a real point if and only if there exist $q \in \mathcal{O} \cap \mathcal{F}$ and $i \in \{1, \ldots, s\}$ with $\bar{q} = \gamma_i^{-1} q$. If the latter holds, then gq is a real point of $\mathcal{O}$, where $g \in \widehat{G}$ is such that $g^{-1}\bar{g} = n_i$.*

Using this theorem, we classify the real semisimple elements according to the complex family they lie in.

First family: A brute force computer calculation shows that $H^1 W_\mathfrak{C}$ is trivial. Hence, by Theorem 7.6, it follows that the orbits in this family having real points are exactly the orbits of $a_1u_1 + a_2u_2 + a_3u_3$ with all a_i real. Let Z denote the stabilizer of such a point in $\widehat{G}$. Since Z is a group of order 3^4, it follows that $\mathrm{H}^1 Z$ is trivial (Criterion 5.4). Therefore, the complex orbit of $a_1u_1 + a_2u_2 + a_3u_3$ with all a_i real corresponds to exactly one real orbit.

Second family: Here, Γ_p has 18 elements. By Criterion 5.5, $H^1\Gamma_p$ consists of the trivial class and the class of

$$\gamma = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, in this family, there are two groups of orbits having real points. The first consists of the orbits of $a_1u_1 + a_2u_2$ with a_i real. Let Z denote the stabilizer of such a point, which is given explicitly in Sec. VII A. By Criterion 5.6, we have $H^1 Z = 1$. Therefore, each complex orbit of $a_1u_1 + a_2u_2$ with a_i being real corresponds to exactly one real orbit.

The second group consists of the orbits of p with $\bar{p} = \gamma p$. These are $p = a(u_1 - u_2) + ib(u_1 + u_2)$ with $a, b \in \mathbb{R}$. We have that γ is induced by

$$n = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{7.2}$$

which is a cocycle. Setting

$$g = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & i & 0 \\ i & 0 & i \end{pmatrix} \times \begin{pmatrix} -\frac{1}{2}i & 0 & 0 \\ 0 & 1 & -1 \\ 0 & i & i \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & -\frac{1}{2}i \end{pmatrix}, \tag{7.3}$$

we have $g^{-1}\bar{g} = n$. Let $v_1 = g \cdot (u_1 - u_2)$ and $v_2 = g \cdot i(u_1 + u_2)$. Then,

$$\begin{aligned} v_1 &= -|212\rangle + |200\rangle + 2|120\rangle - 2|111\rangle + \frac{1}{2}|022\rangle - \frac{1}{2}|001\rangle, \\ v_2 &= -|222\rangle - |201\rangle + 2|121\rangle + 2|110\rangle - \frac{1}{2}|012\rangle - \frac{1}{2}|000\rangle. \end{aligned} \tag{7.4}$$

Hence, real representatives of these orbits are $a_1v_1 + a_2v_2$, and the polynomial conditions translate to $a_1^2 + a_2^2 \neq 0$ and $a_2(a_2^2 - 3a_1^2) \neq 0$. Let Z denote the stabilizer of such an element. A computer calculation shows that $\mathrm{H}^1 Z^\circ = 1$. Since the component group has order 9, Criterion 5.6 shows that $\mathrm{H}^1 Z = 1$. Hence, the complex orbits with these representatives correspond to one real orbit.

Third family: Here, Γ_p is generated by $-\zeta$. Hence, $H^1\Gamma_p = \{[1], [-1]\}$. Therefore, also here, the orbits with real representatives come in two groups. The first group has representatives au_1 with $a \in \mathbb{R}$. Let Z be the stabilizer of such an element in G. Then, $H^1 Z = 1$; hence, each of these orbits corresponds to one real orbit. The second group has representatives iau_1 with $a \in \mathbb{R}$. We have that -1 is induced by the cocycle $n = A \times A \times A$ with

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.5}$$

Setting

$$B = \begin{pmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & -\frac{1}{2}i \end{pmatrix} \tag{7.6}$$

and $g = B \times B \times B$, we have that $g^{-1}\bar{g} = n$. We have that $g \cdot iu_1 = v$ with

$$v = -2|001\rangle - 2|010\rangle - 2|100\rangle + 2|111\rangle - \frac{1}{8}|222\rangle. \tag{7.7}$$

Let Z be the stabilizer of v in $\widehat{G}$. By computing the Lie algebra of Z°, a further computer calculation shows that $\mathrm{H}^1 Z^\circ = 1$. Therefore, again, Criterion 5.6 shows that $\mathrm{H}^1 Z = 1$. Hence, each complex orbit with representative av, $a \in \mathbb{R}$, corresponds to one real orbit with the same representative.

Fourth family: Here, Γ_p is generated by ζ. Hence, $H^1\Gamma_p = 1$. It follows that the orbits with real representatives are the orbits of $a(u_2 - u_3)$ with $a \in \mathbb{R}$. Let Z be the stabilizer of such an element in $\widehat{G}$. Criterion 5.6 shows that $\mathrm{H}^1 Z = 1$. Hence, the complex orbit of $a(u_2 - u_1)$, with $a \in \mathbb{R}$, corresponds to one real orbit.

VIII. MIXED ELEMENTS

$x \in \mathfrak{g}_1$ is called *mixed* if it has Jordan decomposition $x = s + n$, where s and n are two non-zero semisimple and nilpotent elements, respectively, such that $[s, n] = 0$.

A. Mixed elements: The complex case

Each mixed element is conjugate under G_0 to an element whose semisimple part is one of the semisimple orbit representatives described in Sec. VII. Hence, fix a semisimple part s in one of the sets $\mathfrak{C}_p^\circ$. Then, consider its centralizer $\mathfrak{a} = \mathfrak{z}_\mathfrak{g}(s)$. Since $s \in \mathfrak{g}_1$, this algebra inherits the grading of $\mathfrak{g}$: $\mathfrak{a} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1$, where $\mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{g}_i$. The possible nilpotent parts of mixed elements with semisimple part s lie in $\mathfrak{a}_1$. The subalgebra $\mathfrak{a}_0$ is the Lie algebra of $Z_{G_0}(s)$. Hence, $Z_{G_0}(s)$ acts on $\mathfrak{a}_1$. It is clear that two mixed elements $s + e_1$, $s + e_2$ are G_0-conjugate if and only if they are $Z_{G_0}(s)$-conjugate. It follows that we can classify the possible nilpotent parts of the mixed elements with a semisimple part equal to s by classifying the nilpotent elements in $\mathfrak{a}_1$ under the action of $Z_{G_0}(s)$. The nilpotent orbits in $\mathfrak{a}_1$ under the action of the identity component $Z_{G_0}(s)^\circ$ can be classified with generic algorithms for the classification of the nilpotent orbits of a θ-group. Subsequently, it has to be seen which orbits are identified under by a component group. Also note that semisimple elements of a fixed $\mathfrak{C}_p^\circ$ have the same centralizer $\mathfrak{g}$ and the same stabilizer in G_0 by Lemma 7.4. In other words, they do not depend on the semisimple element s, just on the set $\mathfrak{C}_p^\circ$.

We now give the classification of the possible nilpotent parts of mixed elements. For each family of semisimple elements, we have such a classification. Here, we omit the first and fifth families: in the first case, the only possible nilpotent part is zero; in the latter, the semisimple part is zero.

We have computed the orbit classifications with the help of GAP. For the second and fourth families, we obtained a result equivalent to the tables in Ref. 3. Therefore, in these cases, we have taken the same representatives. For the third family, our computation gave quite different results, showing that Table 2 in Ref. 3 is erroneous.

Let s be a semisimple element of the ith family. Let $\mathfrak{a}$ be as above, and let $e \in \mathfrak{a}_1$ be nilpotent. Then, e lies in a homogeneous $\mathfrak{sl}_2$-triple (h, e, f) with $h \in \mathfrak{a}_0$, $f \in \mathfrak{a}_{-1}$. For each nilpotent element e in the classification, we also give a (not always very explicit) description of the stabilizer $Z_{\widehat{G}}(s, h, e, f) = \{g \in \widehat{G} \mid g \cdot s = s, g \cdot h = h, g \cdot e = e, g \cdot f = f\}$ in terms of the identity component and its component group.

The possible nilpotent parts of a mixed element with a semisimple part from the second family are given in Table III.

In the third family, we have $s = a(|000\rangle + |111\rangle + |222\rangle)$, where a is a nonzero scalar. Hence, s is stable under permutation of tensor factors. Therefore, if we let $\mathfrak{a} = \mathfrak{z}_\mathfrak{g}(s)$, then also $\mathfrak{a}_1$ is stable under permutations of the tensor factors. As seen in Sec. IV A, orbits that are obtained from each other by permutation share many properties, such as the structure of the stabilizer. In this case, there are 16 orbits of possible nilpotent parts. In Table IV, we give the representatives of these orbits in groups: different members of the same group are related by a permutation and hence have isomorphic centralizers. Therefore, we only describe the centralizer for the first element in a group. The numbering is according to the table for the same orbits in Nurmiev's paper (Ref. 3, Table 2).

Remark 8.1. Reference 3, Table 2 contains eight orbits (apart from the zero orbit, which we do not list here). Nurmiev's representative number 3 is conjugate to the second in our list under 2. Nurmiev's representative number 6 is conjugate to the second in our list under 5. Nurmiev's representative 8 is the second in our list under 7.

For later use, we explicitly give the element of order 2 in the stabilizer corresponding to $e = |000\rangle + |111\rangle$. It is $A \times A \times A$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark 8.2. Here, we comment on the classification up to permutation of the tensor factors (see Sec. IV A). The elements of Tables III and V are not conjugate to each other if we also consider permutations of the tensor factors. Hence, for the mixed elements of these families, there just remain the extra conjugacies coming from the semisimple parts (Remark 7.5). For the nilpotent parts of elements whose semisimple part is of the third family, the permutation action on the tensor factors is given in detail in Table IV.

B. Mixed elements: The real case

The real semisimple elements are divided into two types. There are the real semisimple elements in one of the families described in Sec. VII; we say that those are canonical semisimple elements. Second, we have the non-canonical semisimple elements: they are $\widehat{G}$-conjugate, but not $\widehat{G}(\mathbb{R})$-conjugate to elements of one of the families. From Sec. VII B, we see that they occur relative to the second and third families. Correspondingly, we have canonical and non-canonical mixed elements, whose semisimple parts are canonical and non-canonical, respectively.

Each complex orbit of a canonical mixed element has a real representative given in Tables II, IV, and V. Let $u = s + e$ be such an element. Then, $e \in \mathfrak{a}_1$, where $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}}(s)$. Let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple in $\mathfrak{a}$ containing e. Consider

$$Z_{\widehat{G}}(s, h, e, f) = \{g \in \widehat{G} \mid g \cdot s = s,\ g \cdot h = h,\ g \cdot e = e,\ g \cdot f = f\}.$$

Then, the $\widehat{G}(\mathbb{R})$-orbits contained in the complex orbit $\widehat{G} \cdot u$ correspond bijectively to $\mathrm{H}^1 Z_{\widehat{G}}(s, h, e, f)$. From the descriptions of these groups in the tables of Sec. VIII A, we see that this cohomology set is mostly trivial. In those cases, the complex orbit $\widehat{G} \cdot (s + e)$ contains one real orbit $\widehat{G}(\mathbb{R}) \cdot (s + e)$. The one exception is the third element in Table V. In this case, $|\mathrm{H}^1 Z_{\widehat{G}}(s, h, e, f)| = 2$ and the nontrivial cocycle is its element of order 2, which is $n = A \times A \times A$. Let $g = B \times B \times B$, where

$$B = \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \frac{1}{2} i \end{pmatrix}.$$

Then, $n = g^{-1}\bar{g}$ and $g \in Z_G(s)$. We have

$$g \cdot (|000\rangle + |111\rangle) = 2(|000\rangle - |011\rangle - |101\rangle - |110\rangle).$$

Therefore, here, we get the extra real mixed orbits with representatives

$$a(u_2 - u_3) + 2(|000\rangle - |011\rangle - |101\rangle - |110\rangle),\ a \in \mathbb{R}. \tag{8.1}$$

If s is not canonical, then s is of the form $s = g \cdot s_0$, where s_0 is a non-real element lying in one of the families. Furthermore, $g^{-1}\bar{g} = n$, where n is a cocycle in $Z_{\widehat{G}}(s_0)$ with $\bar{s}_0 = n^{-1} s_0$. If e_0 is a nilpotent element commuting with s_0 from one of the tables of the previous section, then $s + g \cdot e_0$ is a mixed element. However, in general, $g \cdot e_0$ is not real. Hence, the first thing we have to do is to see whether the orbit $Z_{\widehat{G}}(s) \cdot (g \cdot e_0)$ has real elements and find one in the affirmative case. On the other hand, if there are no real points in this orbit, then this e_0 will not lead to mixed elements with a semisimple part equal to s and can, therefore, be discarded.

For a semisimple element p, set $\mathfrak{u}_p = \mathfrak{z}_{\mathfrak{g}}(p) \cap \mathfrak{g}_1$. Then, $x \mapsto g \cdot x$ is a bijection $\mathfrak{u}_{s_0} \to \mathfrak{u}_s$. It also maps $Z_{\widehat{G}}(s_0)$-orbits to $Z_{\widehat{G}}(s)$-orbits. In addition, for $x \in \mathfrak{u}_{s_0}$, we have that $n\bar{x} \in \mathfrak{u}_{s_0}$ and $g \cdot x$ is real if and only if $n\bar{x} = x$. Hence, we define a map $\mu : \mathfrak{u}_{s_0} \to \mathfrak{u}_{s_0}$ by $\mu(x) = n\bar{x}$. Now, the authors of Ref. 5, Lemma 5.3.1 stated the following.

Lemma 8.3. Let $e \in \mathfrak{u}_{s_0}$ be nilpotent, and let $Y = Z_{\widehat{G}}(s_0) \cdot e$ be its orbit. Let $y_0 \in Y$. Then, $\mu(Y) = Y$ if and only if $\mu(y_0) \in Y$.

We use this lemma for classifying the non-canonical mixed orbits. From Sec. VII B, we see that this concerns only the second and third family.

Second family: Here, n, g are given in (7.2) and (7.3), respectively. We have $s_0 = a_1u_1 + a_2u_2$ where $a_2 = -\bar{a}_1$. Furthermore, $s = b_1v_1 + b_2v_2$ with $b_1, b_2 \in \mathbb{R}$ and v_1, v_2 given by (7.4). From Table III, we see that there are two nilpotent orbits in $\mathfrak{u}_{s_0}$.

For the first orbit, we have that $|021\rangle - |210\rangle$ is a representative fixed under μ. Furthermore,

$$g \cdot (|021\rangle - |210\rangle) = 2|220\rangle + 2|211\rangle + |021\rangle - |010\rangle.$$

Let e denote the latter element. Let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple containing e; then, the stabilizer $Z_{\widehat{G}}(s, h, e, f)$ has order 27 and, hence, trivial Galois cohomology by Criterion 5.4. Therefore, in this case, we get one real orbit.

For the second orbit, we have that $4i|102\rangle$ is a representative fixed under μ. Furthermore, $g \cdot 4i|102\rangle = |102\rangle$. Let (h, e, f) be as before, and set $Z = Z_{\widehat{G}}(s, h, e, f)$. By computing the Lie algebra of Z and a small computer calculation, it is seen that H^1Z° consists of two elements. Since the component group is of order 9, Criterion 5.6 implies that $H^1Z = H^1Z^\circ$. The nontrivial cocycle is

$$c = \mathrm{diag}(-1, 1, -1) \times \mathrm{diag}(1, -1, -1) \times \mathrm{diag}(-1, -1, 1).$$

Therefore, with

$$a = \mathrm{diag}(i, -1, i) \times \mathrm{diag}(-1, i, i) \times \mathrm{diag}(i, i, -1),$$

we get $a^{-1}\bar{a} = c$. Furthermore, $a \cdot |102\rangle = -|102\rangle$. Hence, here, we get two non-conjugate real nilpotent parts: $|102\rangle$ and $-|102\rangle$.

In conclusion, we have three nilpotent parts of a mixed element with a semisimple part equal to $g \cdot s_0$:

Third family: Here, $n = A \times A \times A$ with A as in (7.5) and $g = B \times B \times B$ with B as in (7.6). We have $s_0 = aiu_1$ where $a \in \mathbb{R}$ and $s = av$ with v given by (7.7). From Table IV, we see that there are 16 nilpotent parts of mixed elements with semisimple part s_0. They are divided into five groups with numbers 1, 2, 4, 5, and 7. We deal with them in that order.

We have that $|120\rangle - |102\rangle - |021\rangle + |012\rangle$ is a representative of orbit 1, fixed under μ. It is left invariant by g. The stabilizer in this case is of order 27; hence, the Galois cohomology is trivial (Criterion 5.4).

Let $e = |012\rangle + |021\rangle + |102\rangle$. Then, e is a representative of the first orbit in the second group. We have that $\mu(e)$ lies in the orbit of $|210\rangle + |120\rangle + |201\rangle$, which is a representative of the sixth orbit of the second group. Hence, by Lemma 8.3, the image of this orbit in $\mathfrak{u}_s$ has no real points. This then necessarily holds for all orbits of the second group.

Let $e = |210\rangle - |201\rangle$. Then, e is a representative of the first orbit in group number 4, fixed under μ. Furthermore, $g \cdot e = e$. Let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple containing e. Let Z denote the stabilizer of the quadruple (s, h, e, f) in $\widehat{G}$. Then, by a small computer calculation, it can be shown that H^1Z° is trivial. As the component group has order 3, it follows that $H^1Z = 1$. Therefore, there is one real orbit in this case (Criterion 5.6).

Let $e = |012\rangle + |120\rangle$. Then, e is a representative of the first orbit in group number 5. Here, $\mu(e)$ lies in the second orbit of group 5. Therefore, by Lemma 8.3, the image of this orbit in $\mathfrak{u}_s$ has no real points. The same holds for the second orbit of group number 5.

We have that $n \cdot |012\rangle$ lies in the orbit of $|021\rangle$. Hence, again, by Lemma 8.3, we see that the image of the first orbit of group number 7 has no real points.

Summarizing, we get the following real mixed orbits.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Sabino Di Trani: Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).
Willem A. de Graaf: Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).
Alessio Marrani: Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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