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ORIGINAL PAPER

Holomorphic differential forms on moduli spaces of stable curves

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Abstract

We prove that the space of holomorphic p-forms on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points vanishes for $p = 14, 16, 18$ unconditionally and also for $p = 20$ under a natural assumption in the case $g = 3$. This result is consistent with the Langlands program and it is obtained by applying the Arbarello–Cornalba inductive approach to the cohomology of moduli spaces.

Keywords Moduli spaces · Stable curves · Holomorphic differential forms · Hodge numbers · Cohomology groups

Mathematics Subject Classification Primary 14H10 · Secondary 14C30

1 Introduction

The moduli space $\overline{\mathcal{M}}_{g,n}$ parameterizing stable curves of genus g with n marked points is a projective compactification with a beautiful geometric structure: all its boundary components are (products of) moduli spaces of the same kind but with smaller invariants. This remarkable property was employed by Enrico Arbarello and Maurizio Cornalba to perform an elegant inductive computation of the first few rational cohomology groups of $\overline{\mathcal{M}}_{g,n}$. In particular, in [1] they proved that $H^1(\overline{\mathcal{M}}_{g,n}) = H^3(\overline{\mathcal{M}}_{g,n}) = H^5(\overline{\mathcal{M}}_{g,n}) = 0$ and established an inductive approach to reduce the vanishing of odd cohomology (so long as it vanishes, since it is well known that $H^{11,0}(\overline{\mathcal{M}}_{1,11}) \neq 0$) to a finite number of explicit verifications in low genus.

A few years later, in [4] Gilberto Bini and the author pointed out that the same inductive procedure implies also the vanishing of the spaces of holomorphic p-forms $H^{p,0}(\overline{\mathcal{M}}_{g,n})$ for $0 < p < 11$. More recently, a renewed interest in the Arbarello-Cornalba method is witnessed by the papers [3] by Jonas Bergström, Carel Faber, and Sam Payne, where they compute that $H^7(\overline{\mathcal{M}}_{g,n}) = H^9(\overline{\mathcal{M}}_{g,n}) = 0$, and [5] by Samir Canning, Hannah Larson, and Sam Payne, where they prove inductively that the cohomology group $H^k(\overline{\mathcal{M}}_{g,n})$ is pure

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Hodge-Tate (hence, in particular, $H^{k,0}(\overline{\mathcal{M}}_{g,n}) = 0$) for any even $k \leq 12$. This is consistent with the Langlands program, predicting that $H^k(\overline{\mathcal{M}}_{g,n})$ should be pure Hodge-Tate for all even $k \leq 20$.

Here we move a small step forward along the same path by obtaining the following result:

Theorem 1 *We have*

$$H^{14,0}(\overline{\mathcal{M}}_{g,n}) = H^{16,0}(\overline{\mathcal{M}}_{g,n}) = H^{18,0}(\overline{\mathcal{M}}_{g,n}) = 0$$

for every g and n with $2g - 2 + n > 0$.
Furthermore, if $H^{20,0}(\overline{\mathcal{M}}_{3,15}) = H^{20,0}(\overline{\mathcal{M}}_{3,16}) = 0$ then $H^{20,0}(\overline{\mathcal{M}}_{g,n}) = 0$ for every g and n with $2g - 2 + n > 0$.

Once again, the crucial ingredient is a minor variant of the Arbarello-Cornalba inductive approach (see Lemma 1). Of course, the statement of Theorem 1 arises the following natural question:

Question 1 *Is $H^{20,0}(\overline{\mathcal{M}}_{3,15}) = H^{20,0}(\overline{\mathcal{M}}_{3,16}) = 0$?*

We work over the complex field $\mathbb{C}$.

2 The proofs

Lemma 1 *Let $0 < p \leq 21$ and assume $h^{p,0}(\overline{\mathcal{M}}_{g',n'}) = 0$ for every g', n' such that $p \geq 2g' - 2 + n' > 0$. Then $h^{p,0}(\overline{\mathcal{M}}_{g,n}) = 0$ for every g and n with $2g - 2 + n > 0$.*

Proof By double induction on g and n. Let $d(g, n) = 2g - 2 + n > 0$.

If $d(g, n) = 1$ we have either $g = 0$ and $n = 3$, or $g = 1$ and $n = 1$, and in both cases the claim is obvious.

Let now $d(g, n) > 1$. If $p \geq d(g, n)$ then the claim holds by assumption, hence let $p < d(g, n)$. In the long exact sequence of cohomology with compact supports:

$$\ldots \to H^k_c(\mathcal{M}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\overline{\mathcal{M}}_{g,n}) \to \ldots$$

we have $H^k_c(\mathcal{M}_{g,n}) = 0$ for $k < d(g, n)$ by [7]. Since the morphism

$$H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\overline{\mathcal{M}}_{g,n})$$

is compatible with the Hodge structures (see [1], p. 102), for $p < d(g, n)$ there is an injection

$$H^{p,0}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^{p,0}(\partial\overline{\mathcal{M}}_{g,n}). \tag{1}$$

Next we use the fact that each irreducible component of the boundary $\partial\overline{\mathcal{M}}_{g,n}$ is the image of a map from $\overline{\mathcal{M}}_{g-1,n+2}$ or $\overline{\mathcal{M}}_{h,m+1} \times \overline{\mathcal{M}}_{g-h,n-m+1}$, where $0 \leq h \leq g$ and both $2h - 2 + m + 1$ and $2(g - h) - 2 + n - m + 1$ are positive. By the analogue of Lemma (2.6) in [1] and the Hodge-Künneth formula, the map

$$\begin{aligned}
&H^{p,0}(\overline{\mathcal{M}}_{g,n}) \to H^{p,0}(\overline{\mathcal{M}}_{g-1,n+2}) \oplus \bigoplus_{h,m} H^{p,0}(\overline{\mathcal{M}}_{h,m+1} \times \overline{\mathcal{M}}_{g-h,n-m+1}) \\
&= H^{p,0}(\overline{\mathcal{M}}_{g-1,n+2}) \oplus \bigoplus_{h,m} (H^{0,0}(\overline{\mathcal{M}}_{h,m+1}) \otimes H^{p,0}(\overline{\mathcal{M}}_{g-h,n-m+1}) \oplus \\
&\bigoplus_{q \geq 1} H^{q,0}(\overline{\mathcal{M}}_{h,m+1}) \otimes H^{p-q,0}(\overline{\mathcal{M}}_{g-h,n-m+1}))
\end{aligned}$$

is injective whenever the map (1) is. The right hand side involves the terms $H^{p,0}(\overline{\mathcal{M}}_{g-1,n+2})$ and $H^{p,0}(\overline{\mathcal{M}}_{g-h,n-m+1})$ with either $h \geq 1$ or $h = 0$ and $m \geq 2$, hence vanishing by induction, and products of two terms which have $1 \leq q \leq 10$, since $p \leq 21$. Therefore by [4], Theorem 1, stating that $H^{q,0}(\overline{\mathcal{M}}_{g,n}) = 0$ for $0 < q < 11$, we obtain $H^{p,0}(\overline{\mathcal{M}}_{g,n}) = 0$. □

Remark 1 The assumption of Lemma 1 is not satisfied for every $11 \leq p \leq 21$: in particular, as it is well known $H^{11,0}(\overline{\mathcal{M}}_{1,11}) \neq 0$ (see for instance [6], Section 2.3) and also $H^{17,0}(\overline{\mathcal{M}}_{2,14}) \neq 0$ (see [6], Section 3.5).

Proof of Theorem 1 In order to apply Lemma 1 we have to fix an even integer p with $14 \leq p \leq 20$ and check that $H^{p,0}(\overline{\mathcal{M}}_{g',n'}) = 0$ for every g', n' such that $p \geq 2g' - 2 + n' > 0$.

If $g' = 0$ then all cohomology is tautological (hence algebraic) by [8].

If $g' = 1$ then all even cohomology is tautological by [10].

If $g' = 2$ then all even cohomology is tautological for $n' < 20$ by [11].

If $g' = 3$ then $\overline{\mathcal{M}}_{g',n'}$ is unirational (hence $H^{p,0}(\overline{\mathcal{M}}_{g',n'}) = 0$ for every $p > 0$) for $n' \leq 14$ by [9], Theorem 7.1 (notice that this range completely covers the case $p \leq 18$, while for $p = 20$ we need the additional assumption in the statement).

The same Theorem 7.1 in [9] yields the unirationality of $\overline{\mathcal{M}}_{g',n'}$ also for $g' = 4$ and $n' \leq 15$, $g' = 5$ and $n' \leq 12$, $g' = 6$ and $n' \leq 15$, $g' = 7$ and $n' \leq 11$, $g' = 9$ and $n' \leq 8$, $g' = 11$ and $n' \leq 10$.

Finally, by [2], Theorem B., $\overline{\mathcal{M}}_{g',n'}$ is unirational for $g' = 8$ and $n' \leq 11$ and $g' = 10$ and $n' \leq 3$, thus covering the last missing cases. □

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