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Strict Deformation Quantization and Local Spin Interactions

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Abstract: We define a strict deformation quantization which is compatible with any Hamiltonian with local spin interaction (e.g. the Heisenberg Hamiltonian) for a spin chain. This is a generalization of previous results known for mean-field theories. The main idea is to study the asymptotic properties of a suitably defined algebra of sequences invariant under the group generated by a cyclic permutation. Our point of view is similar to the one adopted by Landsman, Moretti and van de Ven (Rev Math Phys 32(10):2050031, 2020, https://doi.org/10.1142/S0129055X20500312), who considered a strict deformation quantization for the case of mean-field theories. However, the methods for a local spin interaction are considerably more involved, due to the presence of a strictly smaller symmetry group.

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1. Introduction

In this paper we provide a rigorous C^*-algebraic framework for the study of the semi-classical properties of any Hamiltonian with local spin interaction for a spin chain. This

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covers, for example, the Heisenberg Hamiltonian. This result is achieved by means of a suitable strict deformation quantization, whose construction is the main result of this paper—cf. Theorem 24.

Strict deformation quantization originates with Berezin [3] and Bayen et al. [1,2] and it is based on the idea of "deforming" a given commutative Poisson algebra representing a classical system into a given non-commutative algebra modelling the associated quantized system. In Rieffel's approach [17] the deformed algebras are C^*-algebras.

Notably, the "classical-to-quantum" interpretation of a strict deformation quantization is not the unique point of view which can be taken. In Landsman's approach [10,11] the starting point of a strict deformation quantization is often taken to be a continuous field of C^*-algebras. The latter models an increasingly larger sequence of quantum physical systems, whose limit defines a macroscopic classical theory. The advantage of this point of view is that it leads to a rigorous notion of the *classical limit* of quantum theories [11]. This in turn yields a mathematically sound description of several physically interesting *emergent* phenomena, e.g. symmetry breaking [14,18,19]. This paper is considering this "micro-to-macro" point of view on strict deformation quantization.

From a technical point of view a **strict deformation quantization** is defined by the following data:

1. A commutative Poisson C^*-algebra $\mathcal{A}_\infty$, namely a commutative C^*-algebra $\mathcal{A}_\infty$ equipped with a Poisson structure $\{\ ,\ \}\colon \widetilde{\mathcal{A}}_\infty \times \widetilde{\mathcal{A}}_\infty \to \widetilde{\mathcal{A}}_\infty$ defined on a dense $*$-subalgebra $\widetilde{\mathcal{A}}_\infty \subseteq \mathcal{A}_\infty$—cf. Sect. 3.3.
2. A continuous bundle of C^*-algebras [7] $\prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$, where $\overline{\mathbb{N}} = \mathbb{N}\cup\{\infty\}$; (Thorough the whole paper we will stick to the case of continuous bundle of C^*-algebras over $\overline{\mathbb{N}}$, see [11] for the generic case.)
3. A family of linear maps, called **quantization maps**, $Q_N\colon \widetilde{\mathcal{A}}_\infty \to \mathcal{A}_N$, $N \in \overline{\mathbb{N}}$, such that
   (a) $Q_\infty = \mathrm{Id}_{\widetilde{\mathcal{A}}_\infty}$ and $Q_N(a)^* = Q_N(a^*)$ for all $a \in \widetilde{\mathcal{A}}_\infty$. Moreover, the assignment
   $$\overline{\mathbb{N}} \ni N \mapsto Q_N(a) \in \mathcal{A}_N,$$
   defines a continuous section of the bundle $\prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$.
   (b) For all $a, a' \in \widetilde{\mathcal{A}}_\infty$ it holds
   $$\lim_{N\to\infty} \|Q_N(\{a,a'\}) - iN[Q_N(a), Q_N(a')]\|_{\mathcal{A}_N} = 0\,. \tag{1}$$
   (c) For all $N \in \mathbb{N}$, $Q_N(\widetilde{\mathcal{A}}_\infty)$ is a dense $*$-subalgebra of $\mathcal{A}_N$.

The algebra $\mathcal{A}_\infty$ represents the classical (macroscopic) observables of the physical system. Likewise, the fibers $\mathcal{A}_N$, $N \in \mathbb{N}$, of the bundle $\prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$ recollect the quantum observables of the (increasingly larger) quantum system.

A relevant example is the strict deformation quantization described in [12,16] for the C^*-algebra $[B]_\pi^\infty$ of (equivalence classes of) symmetric sequences—cf. Sect. 2 for further details. In this scenario the role of the commutative C^*-algebra $\mathcal{A}_\infty$ is played by $[B]_\pi^\infty = C(S(B))$—here $B = M_\kappa(\mathbb{C})$, $\kappa \in \mathbb{N}$, while $S(B)$ denotes the states space over B. The continuous bundle of C^*-algebras $\prod_{N\in\mathbb{N}} B_\pi^N$ is such that, for $N \in \mathbb{N}$, $B_\pi^N \subseteq B^N$ is the N-th symmetric tensor product of B.

From a physical point of view the ensuing quantization maps Q_N are of particular interest as they relate to mean-field theories like the Curie-Weiss model [9, §2], for which the interaction between N spin sites is described by

$$H_{\mathrm{CW},N} := -\frac{J}{2N} \sum_{j_1+j_2=N-2}^{N} I^{j_1} \otimes \sigma_3 \otimes I^{j_2} \otimes \sigma_3 - h \sum_{j=1}^{N-1} I^{N-1-j} \otimes \sigma_1 \otimes I^j, \quad (2)$$

where $\sigma_3, \sigma_1 \in M_2(\mathbb{C})$ denote the Pauli's matrices while $I \in M_2(\mathbb{C})$ is the identity matrix and $I^j := I^{\otimes j}$. Here $J \in \mathbb{R}$ represents the strength of the spin interaction whereas $h \in \mathbb{R}$ models an external magnetic field acting on the system. As observed in [12] one may recognize that

$$H_{\mathrm{CW},N}/N = Q_N(h_{\mathrm{CW}}) + R_N,$$

where $h_{\mathrm{CW}} \in C(S(B))$ while $R_N \in B_\pi^N$ is such that $\|R_N\|_N = O(1/N)$.

The physical interpretation is that $C(S(B))$ is the algebra containing macroscopic observables, i.e. observables of an infinite quantum system describing classical thermodynamics as a limit of quantum statistical mechanics. This has furthermore led to a significant contribution in the study of the classical limit of ground states [12–14, 18]. More precisely, in such works a mathematically rigorous description of the limit of ground states ω_N of $H_{\mathrm{CW},N}$ in the regime of large particles $N \to \infty$ is given. In particular, a classical counterpart ω_∞ (i.e. a probability measure) of the quantum ground state $\omega_N \in S(B_\pi^N)$ is constructed with the property that $\omega_\infty(a) := \lim_{N\to\infty} \omega_N(Q_N(a))$ for all $a \in C(S(B))$. Additionally, this algebraic approach has revealed the existence of several physical *emergent phenomena*, see [19] for an overview. These results are consistent with the point of view of [11]—which is also the one considered in this paper—for which a quantum theory is pre-existing and the classical limit is computed in a second step, not vice versa.

As characteristic for mean-field models, the Curie-Weiss Hamiltonian describes the energy of a system of N spin sites under the assumption that the interaction is *non-local*, namely that every spin site interacts with all other spin site. This leads to interesting results, but it is ultimately an approximation as one would rather expect each spin site to interact with finitely many neighbouring spin sites. An exemplary model based on such a local interaction is the celebrated quantum Heisenberg Hamiltonian (for a spin chain) [6, §6.2]

$$H_{\mathrm{HE},N} := -\sum_{j=0}^{N-1} I^{N-2-j} \otimes \sum_{p,q=1}^{3} J^{pq} \sigma_p \otimes \sigma_q \otimes I^j - \sum_{j=0}^{N-1} I^{N-1-j} \otimes \sum_{p=1}^{3} h^p \sigma_p \otimes I^j, \quad (3)$$

where J^{pq} is the symmetric matrix describing the spin interaction while h^p are the components of an external magnetic field—here for $j = N-1$ the contribution in the first sum reads $\sum_{p,q=1}^{3} J^{pq} \sigma_q \otimes I^{N-2} \otimes \sigma_p$. For this model the interaction is restricted to two neighbouring sites.

Similarly to what happens with mean-field models one may wonder whether there exists a strict deformation quantization of a suitable C^*-algebra such that

$$H_{\mathrm{HE},N}/N = Q_N(h_{\mathrm{HE}}) + O(1/N).$$

The purpose of this paper is to prove that this is in fact the case, cf. Theorem 24. In [15] a different (though similar in spirit) point of view is taken, and a strict deformation is considered such that $H_{\mathrm{HE},N} = Q_\kappa(h_{\mathrm{HE},N}) + O(1/\kappa)$ where the semi-classical parameter

κ corresponds to the increasing dimension of the single site algebra $B = M_\kappa(\mathbb{C})$ for a *fixed* number N of lattice sites. In contrast, this paper deals with an arbitrary but fixed dimension $\kappa \in \mathbb{N}$ considering instead the increasing number N of spin sites as the semi-classical parameter.

Our result is particularly relevant because it provides an excellent basis for studying the classical limit of local quantum spin systems. Similarly to the case of mean-field theories [12,14,18,19], one may now consider a rigorous C^*-algebraic formalization of the limit of ground states or Gibbs states [8,9]. The latter can be used for the study of spontaneous symmetry breaking and phase transitions in realistic models such as the Heisenberg model.

From a technical point of view, the methods of this paper profit of those of [12,16] for mean-field models. Nevertheless the results obtained therein do not apply straight away to our case. As a matter of fact the strict deformation quantization for mean-field models (like the Curie-Weiss Hamiltonian) profits of:

1. A large symmetry group, that is, mean-field models are symmetric under the permutation group $\mathfrak{S}_N$ of all N spin sites. This leads to a high symmetry property which can be exploited in several steps of the construction, cf. [12,16].
2. A fairly explicit description of the classical algebra $[B]_\pi^\infty = C(S(B))$. One may define $[B]_\pi^\infty$ in terms of equivalence classes of "symmetric sequences"—cf. Remark 3— but the description in terms of $C(S(B))$ simplifies the discussion, e.g. it allows to identify a Poisson structure in a rather direct way.

Contrary to this case, local quantum spin Hamiltonians (e.g. the Heisenberg model defined in (3)) are invariant under the strictly smaller subgroup generated by a fixed cyclic permutation of N objects. This spoils the possibility of applying the arguments of [12,16]. The latter have to be reconsidered to take into account the smaller symmetry group. Moreover, the classical algebra $[B]_\gamma^\infty$ for such models does not have a "simple" explicit description. As a matter of fact, $[B]_\gamma^\infty$ is defined as the C^*-algebra generated by (equivalence classes of) "γ-sequences"—cf. Definition 5. Nevertheless it is still possible to prove all properties of $[B]_\gamma^\infty$ relevant for the discussion of its strict deformation quantization.

The paper is structured as follows. In Sect. 2 we introduce the notion of "γ-sequences"—cf. Definition 2—and discuss their properties. The main result in this section is the proof that the C^*-algebra $[B]_\gamma^\infty$ generated by (equivalence classes of) γ-sequences is a commutative C^*-algebra. The latter will play the role of the classical algebra $\mathcal{A}_\infty$ for which we will present a strict deformation quantization.

In Sect. 3 we state and prove the main theorem of this paper, which provides a strict deformation quantization of the commutative C^*-algebra $[B]_\gamma^\infty$. To this avail, Sect. 3.1 is devoted to prove Proposition 12 which provides the continuous bundle of C^*-algebra $[B]_\gamma$ needed in the formulation of Theorem 24. The main technical hurdle of this section is to prove that, given a γ-sequence $(a_N)_N$, the sequence of the norms $(\|a_N\|_N)_N$ is convergent. While this is straightforward for symmetric sequences (i.e. those used when dealing with mean-field models) for γ-sequences this is non-trivial and has to be discussed carefully. Sections 3.2–3.3 discuss further relevant properties of (equivalence classes of) γ-sequences as well as the Poisson structure on the C^*-algebra $[B]_\gamma^\infty$. Eventually Theorem 24 is proved by recollecting all results from the previous sections.

For the sake of clarity the following theorem recollects in a concise fashion the content of the main Theorem 24 together with the other relevant results of the paper.

Theorem 1 (main results). *The algebra* $[B]^\infty_\gamma := \overline{[\dot{B}]^\infty_\gamma}$ *of equivalence classes of* γ*-sequences—cf. Definitions 2–5—is a commutative* C^**-algebra which is also endowed with a Poisson structure* $\{\,,\,\}_\gamma$*—cf. Propositions 6–22.*

Moreover, the data $[B]^\infty_\gamma$ *and* $B^N_\gamma := \overline{\gamma}_N(B^N)$*—cf. Eq. (7)—define a continuous bundle* $[B]_\gamma$ *of* C^**-algebras—cf. Proposition 12.*

Finally, there exists a family of quantization maps $Q_N\colon [\dot{B}]^\infty_\gamma \to [B]^N_\gamma$, $N \in \overline{\mathbb{N}}$ *such that the data* $[B]^\infty_\gamma, [B]_\gamma, \{Q_N\}_{N\in\overline{\mathbb{N}}}$ *define a strict deformation quantization—cf. Theorem 24.*

2. The Algebra of γ-Sequences

2.1. Definition of γ*-sequences.* In this section we will introduce γ-sequences and discuss their properties.

To fix some notations, let $\kappa \in \mathbb{N}$ and set $B := M_\kappa(\mathbb{C})$. For the sake of simplicity we shall denote by $B^N := B^{\otimes N}$, where $N \in \mathbb{N}$, with the convention that $B^0 = \mathbb{C}$. The state space over B^N will be denoted by $S(B^N)$: Given $\eta \in S(B)$ we set $\eta^N := \eta^{\otimes N} \in S(B^N)$. Whenever needed we will denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Following [12] we denote by $I, b_1, \ldots, b_{\kappa^2-1}$, $I \in B$ being the identity matrix, a basis of B (as a $\mathbb{R}$-vector space) abiding by the requirements

$$\operatorname{tr}(b_j) = 0, \qquad b_j^* = b_j, \qquad [b_j, b_\ell] = i c^m_{j\ell} b_m, \qquad \forall j, \ell = 1, \ldots, \kappa^2 - 1\,. \tag{4}$$

where $c^m_{j\ell}$ denotes the structure constants of $\mathfrak{su}(\kappa)$. In the particular case $\kappa = 2$ we may choose $b_j = \sigma_j/2$ while $c^m_{j\ell} = \varepsilon_{j\ell s}\delta^{sm}$, ε_{ijk} being the Levi-Civita symbol. We will denote by $\widetilde{B}$ the vector space generated by $\{b_j\}_{j=1}^{\kappa^2-1}$. The latter corresponds to the $\ker\tau$, being $\tau\colon B \to \mathbb{C}$ the normalized trace defined by $\tau(a) := \operatorname{tr}(a)/\kappa$.

We then consider the linear operator (left-shift operator) $\gamma_N\colon B^N \to B^N$ uniquely defined by continuous and linear extension of the following map defined on elementary tensors[1]

$$\gamma_N(a_{(1)} \otimes \cdots \otimes a_{(N)}) := a_{(2)} \otimes \cdots \otimes a_{(N)} \otimes a_{(1)} \qquad a_{(1)}, \ldots, a_{(N)} \in B\,. \tag{5}$$

The operator γ_N is an algebra endomorphism, moreover, $\gamma_N^N = \mathrm{Id}_B$, $\mathrm{Id}_B\colon B \to B$ being the identity operator. We denote by $\overline{\gamma}_N\colon B^N \to B^N$ the **averaged** γ_N **operator**, defined by

$$\overline{\gamma}_N := \frac{1}{N}\sum_{j=0}^{N-1} \gamma_N^j\,. \tag{6}$$

Clearly $\gamma_N \circ \overline{\gamma}_N = \overline{\gamma}_N = \overline{\gamma}_N \circ \gamma_N$: We denote by

$$B^N_\gamma := \overline{\gamma}_N(B^N), \tag{7}$$

the C^*-subalgebra of B^N made by γ_N-invariant elements.

Through this paper we will mostly consider sequences $(a_N)_N = (a_N)_{N\in\mathbb{N}}$ with $a_N \in B^N$ for all $N \in \mathbb{N}$. A sequence $(a_N)_{N\geq K}$ will be implicitly extended to $(a'_N)_{N\in\mathbb{N}}$ where $a'_N = a_N$ for $N \geq K$ and $a'_N = 0$ for $N < K$.

[1] In the forthcoming discussion we will use the notation a_N to denote an element $a_N \in B^N$. When we will need to use a subindex without necessarily stating the degree of the element we will use the notation $a_{(k)}$ so that $a_{(k)} \in B^{M(k)}$, $M(k) \in \mathbb{N}$.

Definition 2. A sequence $(a_N)_N$ is called γ**-sequence** if there exists $M \in \mathbb{N}$ and $a_M \in B^M$ such that

$$a_N = \overline{\gamma}_N^M a_M := \begin{cases} \overline{\gamma}_N(I^{N-M} \otimes a_M) & N \geq M \\ 0 & N < M \end{cases} . \tag{8}$$

where $I \in B$ denotes the identity of B and $I^N = I^{\otimes N}$.

Remark 3.

(i) For fixed $N, M \in \mathbb{N}$, $N \geq M$, $\overline{\gamma}_N^M : B^M \to B_\gamma^N$ is a linear operator with operator norm smaller than 1. This implies that, $(\overline{\gamma}_N^M a_M)_N$ is bounded with

$$\|(\overline{\gamma}_N^M a_M)_N\|_\infty := \sup_{N \in \mathbb{N}} \|\overline{\gamma}_N^M a_M\|_N \leq \|a_M\|_M,$$

where $\| \ \|_M$ denotes the norm on B^M.

(ii) It is worth comparing our construction with the one presented in the literature [11,12,16], based on symmetric sequences. We stress that the latter are exploited to deal with the Curie-Weiss Hamiltonian—or more generally with mean-field theories [11, §10]— which prescribe a non-local interaction between spin sites. On the other hand we are interested in models compatible with Hamiltonian describing a local interaction between spin sites—e.g. the Heisenberg Hamiltonian, cf. Remark 4. To describe the non-local interaction algebraically one considers the symmetrization operator $S_N : B^N \to B^N$ defined by continuous and linear extension of

$$S_N(a_{(1)} \otimes \cdots \otimes a_{(N)}) := \frac{1}{N!} \sum_{\varsigma \in \mathfrak{S}_N} a_{(\varsigma(1))} \otimes \cdots \otimes a_{(\varsigma(N))} \qquad a_{(1)}, \ldots, a_{(N)} \in B,$$

where $\mathfrak{S}_N$ is the set of permutation of N objects [11,16]. Considering the C^*-subalgebra $B_\pi^N := S_N B^N \subset B^N$ one then defines a **symmetric-sequence** (shortly, π-sequence) to be a sequence $(a_N)_N$ such that there exists $M \in \mathbb{N}$ and $a_M \in B_\pi^M$ fulfilling

$$(a_N)_N = (\pi_N^M a_M)_N := (S_N(I^{N-M} \otimes a_M))_{N \geq M} .$$

One immediately sees the relation with Definition 2: Actually a γ-sequence is defined in a way similar to π-sequences but averaging over a strictly smaller subgroup of $\mathfrak{S}_N$. In fact γ-sequences and π-sequences share many similar properties, although π-sequences are generally speaking better behaved.

2.2. Asymptotic properties of γ-sequences. In what follows we will be mainly interested in the asymptotic behaviour as $N \to \infty$ of the sequences under investigations. For this reason, following [16], we introduce the $\sim$**-equivalence relation**

$$(a_N)_N \sim (b_N)_N \iff \lim_{N \to \infty} \|a_N - b_N\|_N = 0 . \tag{9}$$

For a given sequence $(a_N)_N$ we will denoted by $[a_N]_N := [(a_N)_N]$ the corresponding equivalence class with respect to (9). The $\sim$-equivalence relation (9) has a nice interplay with the **full C^*-product** $\prod_{N\in\mathbb{N}} B^N$ defined by

$$\prod_{N\in\mathbb{N}} B^N := \{(a_N)_N \mid (\|a_N\|_N)_N \in \ell^\infty(\mathbb{N})\}\,. \tag{10}$$

As it is well-known [4] $\prod_{N\in\mathbb{N}} B^N$ is a C^*-algebra with respect to sup norm $\|(a_N)_N\|_\infty := \sup_{N\in\mathbb{N}} \|a_N\|_N$. Moreover, the **direct C^*-sum**

$$\bigoplus_{N\in\mathbb{N}} B^N := \{(a_N)_N \in \prod_{N\in\mathbb{N}} B^N \mid \lim_{N\to\infty} \|a_N\|_N = 0\}, \tag{11}$$

is a closed two-sided ideal in $\prod_{N\in\mathbb{N}} B^N$ and thus we may consider the quotient

$$[B]_\sim := \prod_{N\in\mathbb{N}} B^N / \bigoplus_{N\in\mathbb{N}} B^N, \tag{12}$$

which is nothing but the space of $\sim$-equivalence classes $[a_N]_N$ for bounded sequences $(a_N)_N$. Importantly, $[B]_\sim$ is a C^*-algebra with norm

$$\|[a_N]_N\|_{[B]_\sim} = \limsup_{N\to\infty} \|a_N\|_N\,. \tag{13}$$

Remark 4. (i) Since both γ- and π-sequences are bounded—*cf.* Remark 3-(i)—they lead to well-defined elements $[\overline{\gamma}_N^M a_M]_N, [\pi_N^M a_M]_N \in [B]_\sim$. One may wonder whether $[\overline{\gamma}_N^M a_M]_N = [0]_N$ for a non-zero $a_M \in B^M$. This is in fact possible, but we postpone this discussion to Sect. 3.2 where we will prove that, for a given equivalence class $[\overline{\gamma}_N^M a_M]_N$ it is possible to extract a "canonical representative"—cf. Definition 16—with the property that $[\overline{\gamma}_N^M a_M]_N = [0]_N$ if and only if the canonical representative vanishes—cf. Proposition 18.

(ii) With reference to Eq. (3) we have (considering $\kappa = 2$)

$$\frac{1}{N} H_{\mathrm{HE},N} = \overline{\gamma}_N^2 \Big(\sum_{p,q=1}^3 J^{pq} \sigma_p \otimes \sigma_q \Big) + \overline{\gamma}_N^1 \Big(\sum_{p=1}^3 h^p \sigma_p \Big),$$

showing the relation between γ-sequences and the Heisenberg Hamiltonian. Similarly, as discussed in [12], Eq. (2) leads to

$$\frac{1}{N} H_{\mathrm{CW},N} = -\pi_N^2 (J \sigma_3 \otimes \sigma_3) + \pi_N^1 (h\sigma_1) + O(1/N),$$

showing that $(H_{\mathrm{CW},N}/N)_{N\geq 1}$ is equivalent to a π-sequence. At this stage it is worth observing that γ-sequences model an arbitrary Hamiltonian with local spin interaction. We say that $H_N \in B_\gamma^N$ is a (translation invariant) **Hamiltonian with local spin interaction** if and only if

$$H_N = \sum_{|i-j|\leq \ell} \sum_{p,q=1}^3 J^{pq} \sigma_p(i) \sigma_q(j) + \sum_{i=1}^N \sum_{p=1}^3 h^p \sigma_p(i), \tag{14}$$

were $\sigma_p(i)\sigma_q(j)$ is a short notation for $I^{i-1} \otimes \sigma_p \otimes I^{j-i-1} \otimes \sigma_q \otimes I^{N-i-j}$ and similarly $\sigma_p(i) = I^{i-1} \otimes \sigma_p \otimes I^{N-i}$. The parameter $\ell \in \mathbb{N}$ determines the number of spin sites which interact with a fixed spin site i—e.g. for the Heisenberg Hamiltonian $\ell = 1$. The strength of the interaction and of the external magnetic field is determined by J^{pq}, h^p. Notice that the latter do not depend on the spin site: This entails that we are considering translation invariant local spin interactions. Any Hamiltonian H_N as per Eq. (14) leads to a γ-sequences as per Definition (2). Indeed, we have

$$H_N/N = \sum_{m=0}^{\ell-1} \overline{\gamma}_N^{2+m}\Big(\sum_{p,q=1}^{3} J^{pq}\sigma_p \otimes I^m \otimes \sigma_q\Big) + \overline{\gamma}_N^1\Big(\sum_{p=1}^{3} h^p \sigma_p\Big).$$

(iii) The $\sim$-equivalence relation (9) provides a first example showing the different behaviour of γ-sequences with respect to π-sequences. To this avail, let $a_M \in B_\pi^M$ and let us consider the π-sequence $(\pi_N^M a_M)_N$. By direct inspection one immediately sees that, for all $N' \geq N \geq M$

$$\pi_{N'}^N \pi_N^M a_M = S_{N'}\Big[I^{N'-N} \otimes S_N(I^{N-M} \otimes a_M)\Big] = \pi_{N'}^M a_M,$$

which shows that the family of maps $\pi_N^M : B_\pi^M \to B_\pi^N$ is "consistent", namely $\pi_{N'}^N \circ \pi_N^M = \pi_{N'}^M$. The same property does not apply for γ-sequences, but it holds only asymptotically. Indeed for $a_M \in B^M$ one has, for $N \geq M$,

$$\begin{aligned}\overline{\gamma}_N^M a_M &= \frac{1}{N}\sum_{j=0}^{N-M} \gamma_N^j (I^{N-M} \otimes a_M) + R_N \\ &= \frac{1}{N}\sum_{j=0}^{N-M} I^{N-M-j} \otimes a_M \otimes I^j + R_N, \qquad \|R_N\|_N \leq \frac{M-1}{N}\|a_M\|_M .\end{aligned}$$

It then follows that

$$\overline{\gamma}_{N'}^N \overline{\gamma}_N^M a_M = \overline{\gamma}_{N'}^N\Big[\frac{1}{N}\sum_{j=0}^{N-M} I^{N-M-j} \otimes a_M \otimes I^j\Big] + \overline{\gamma}_{N'}^N R_N = \overline{\gamma}_{N'}^M a_M + R'_{N'},$$

where we used the γ-invariance while

$$\|R'_{N'}\|_{N'} = \Big\|\overline{\gamma}_{N'}^N R_N - \frac{M-1}{N}\overline{\gamma}_{N'}^M a_M\Big\|_{N'} \leq \|R_N\|_N + \frac{M-1}{N}\|a_M\|_M = O(1/N).$$

This shows that, although $\overline{\gamma}_{N'}^N \circ \overline{\gamma}_N^M \neq \overline{\gamma}_{N'}^M$ one still has

$$\lim_{N\to\infty} \|[\overline{\gamma}_{N'}^N \overline{\gamma}_N^M a_M]_{N'} - [\overline{\gamma}_{N'}^M a_M]_{N'}\|_{B_\sim} = 0. \tag{15}$$

As we shall see, naively speaking most the results obtained for π-sequences holds true also for γ-sequences but only asymptotically—in the sense of relation (9).

2.3. The algebra generated by γ-sequences. In what follows we will consider the $*$-algebra $\dot{B}^\infty_\gamma \subset \prod_{N\in\mathbb{N}} B^N$ generated by γ-sequences together with its projection $[\dot{B}]^\infty_\gamma \subset [B]_\sim$. As we will see, the latter algebra enjoys remarkable properties, in particular, it can be completed to a commutative C^*-algebra $[B]^\infty_\gamma$.

Definition 5. Let $\dot{B}^\infty_\gamma$ be the $*$-algebra generated by γ-sequences—cf. Definition 2. We denote by $[\dot{B}]^\infty_\gamma \subset [B]_\sim$ the projection of $\dot{B}^\infty_\gamma$ in $[B]_\sim$, that is, $[\dot{B}]^\infty_\gamma$ is the $*$-algebra generated by equivalence classes of γ-sequences. Thus, $[a_N]_N \in [\dot{B}]^\infty_\gamma$ if and only if

$$[a_N]_N = \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell}[\overline{\gamma}_N^{M(k_1)}(a_{(k_1)})\ldots\overline{\gamma}_N^{M(k_\ell)}(a_{(k_\ell)})]_N,$$

where $a_{(k_j)} \in B^{M(k_j)}$ while the sum over $\ell, k_1, \ldots, k_\ell$ is finite. We denote by $[B]^\infty_\gamma := \overline{[\dot{B}]^\infty_\gamma}$ the closure of $[\dot{B}]^\infty_\gamma$ in $[B]_\sim$, that is, the C^*-algebra generated by equivalence classes of γ-sequences. To wit, an equivalence class $[a_N]_N$ belongs to $[B]^\infty_\gamma$ if and only if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and $[a'_N]_N \in [\dot{B}]^\infty_\gamma$ such that $\|a_N - a'_N\|_N < \varepsilon$ for all $N \geq N_\varepsilon$.

Proposition 6. *Let $a_{M_1}, \ldots, a_{M_\ell}$, $\ell \in \mathbb{N}$, be such that $a_{M_j} \in B^{M_j}$, $j = 1, \ldots, \ell$. Then:*

$$[\overline{\gamma}_N^{M_1}(a_{M_1})\ldots\overline{\gamma}_N^{M_\ell}(a_{M_\ell})]_N = \left[\overline{\gamma}_N\left(\frac{1}{N^{\ell-1}}\sum_{|j|_\ell=N-|M|_\ell} I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_\ell}\otimes a}_{}{}_{M_\ell}\right)\right]_N, \tag{16}$$

where $|j|_\ell := j_1 + \cdots + j_\ell$, $|M|_\ell = M_1 + \cdots + M_\ell$ is a short notation while

$$I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_\ell}\otimes a}_{}{}_{M_\ell} := \frac{1}{\ell}\sum_{\varsigma\in\mathfrak{S}_\ell} I^{j_1}\otimes a_{M_{\varsigma(1)}}\otimes\cdots\otimes I^{j_\ell}\otimes a_{M_{\varsigma(\ell)}}, \tag{17}$$

denotes "total weighted symmetrization" over the factor $a_{M_1}, \ldots, a_{M_\ell}$[2]*. In particular $[B]^\infty_\gamma$ is a commutative C^*-subalgebra of $[B]_\sim$.*

Proof. Let $a_{M_1} \in B^{M_1}, \ldots, a_{M_\ell} \in B^{M_\ell}$, $\ell \in \mathbb{N}$. We will prove that, for N large enough,

$$\overline{\gamma}_N^{M_1}(a_{M_1})\ldots\overline{\gamma}_N^{M_\ell}(a_{M_\ell}) = \overline{\gamma}_N\left(\frac{1}{N^{\ell-1}}\sum_{|j|_\ell=N-|M|_\ell} I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_\ell}\otimes a}_{}{}_{M_\ell}\right) + R_N, \tag{18}$$

where $R_N \in B^N$ is such that $\|R_N\|_N = O(1/N)$. On account of (9) this implies Eq. (16).

We proceed by induction over $\ell \in \mathbb{N}$. For $\ell = 1$ the right-hand side of Eq. (18) reduces to $\overline{\gamma}_N^M a_M + R_N$ so that we may choose $R_N = 0$. For $\ell = 2$ we find, for N large enough (say, $N \geq 2(M_1 + M_2)$),

[2] For example $I^{j_1}\otimes \overbrace{a_{M_1}\otimes I^{j_2}\otimes a}_{}{}_{M_2} = (I^{j_1}\otimes a_{M_1}\otimes I^{j_2}\otimes a_{M_2} + I^{j_1}\otimes a_{M_2}\otimes I^{j_2}\otimes a_{M_1})/2.$

$$
\begin{aligned}
\overline{\gamma}_N^{M_1}(a_{M_1})\overline{\gamma}_N^{M_2}(a_{M_2}) &= \overline{\gamma}_N\bigg(\Big(I^{N-M_1}\otimes a_{M_1}\Big)\,\overline{\gamma}_N(I^{N-M_2}\otimes a_{M_2})\bigg)\\
&= \overline{\gamma}_N\bigg(\Big(I^{N-M_1}\otimes a_{M_1}\Big)\frac{1}{N}\sum_{j=0}^{N-1}\gamma_N^j(I^{N-M_2}\otimes a_{M_2})\bigg)\\
&= \overline{\gamma}_N\bigg(\Big(I^{N-M_1}\otimes a_{M_1}\Big)\frac{1}{N}\sum_{j=M_1}^{N-M_2-1}\gamma_N^j(I^{N-M_2}\otimes a_{M_2})\bigg)+R_N\\
&= \overline{\gamma}_N\bigg(\frac{1}{N}\sum_{j_1+j_2=N-M_1-M_2} I^{j_2}\otimes a_{M_2}\otimes I^{j_1}\otimes a_{M_1}\bigg)+R_N,\\
&= \overline{\gamma}_N\bigg(\frac{1}{N}\sum_{j_1+j_2=N-M_1-M_2} I^{j_2}\otimes \overbrace{a_{M_2}\otimes I^{j_1}\otimes a}_{}{}_{M_1}\bigg)+R_N,
\end{aligned}
$$

where in the last line we used the symmetry of j_1, j_2 as well as the γ_N-invariance of the whole term. The remainder R_N coincides with

$$
\|R_N\|_N = \left\|\overline{\gamma}_N\bigg((I^{N-M_1}\otimes a_{M_1})\frac{1}{N}\sum_{\substack{j\in\{0,\dots,M_1-1\}\\ \cup\{N-M_2,\dots,N-1\}}}\gamma_N^j(I^{N-M_2}\otimes a_{M_2})\bigg)\right\|_N \le \frac{C_{M_1,M_2}}{N},
$$

where $C_{M_1,M_2} > 0$ is a constant depending on a_{M_1}, a_{M_2}. Roughly speaking, we removed the values of j for which a_{M_1} and a_{M_2} have "overlapping positions". This happens in $M_1 + M_2$ cases, whose fraction vanishes as $N \to \infty$.

This proves Eq. (18) for $\ell = 2$. Proceeding by induction on ℓ, we now assume that Eq. (18) holds for all $\ell' < \ell$ and prove it for ℓ. To this avail we consider, for $N \ge 2|M|_\ell$,

$$
\begin{aligned}
&\overline{\gamma}_N^{M_1}(a_{M_1})\dots\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\\
&\quad= \overline{\gamma}_N\bigg(\frac{1}{N^{\ell-2}}\sum_{|j|_{\ell-1}=N-|M|_{\ell-1}} I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{}{}_{M_{\ell-1}}\bigg)\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\\
&\qquad+R_N\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\\
&\quad= \overline{\gamma}_N\bigg(\frac{1}{N^{\ell-2}}\sum_{|j|_{\ell-1}=N-|M|_{\ell-1}} (I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{}{}_{M_{\ell-1}})\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\bigg)+R'_N,
\end{aligned}
$$

where $\|R'_N\|_N \le \|R_N\|_N\|a_{M_\ell}\| = O(1/N)$. Thus, we focus on

$$
\begin{aligned}
&\overline{\gamma}_N\bigg(\frac{1}{N^{\ell-2}}(I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{}{}_{M_{\ell-1}})\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\bigg)\\
&\quad= \overline{\gamma}_N\bigg(\frac{1}{N^{\ell-1}}(I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{}{}_{M_{\ell-1}})\sum_{j_\ell=0}^{N-1}\gamma_N^{j_\ell}(I^{N-M_\ell}\otimes a_{M_\ell})\bigg),
\end{aligned}
$$

where $j_1,\dots,j_{\ell-1}$ are such that $|j|_{\ell-1} = N - |M|_{\ell-1}$. We now proceed as in the case $\ell = 2$ by considering only those values j_ℓ for which the position of a_{M_ℓ} “overlaps” with the ones of $I^{j_1},\dots,I^{j_\ell}$ and not with those of $a_{M_1},\dots,a_{M_{\ell-1}}$. Notice that, in focusing only on these j_ℓ's we are neglecting a contribution R''_N with $\|R''_N\|_N = O(1/N)$. We obtain

$$\overline{\gamma}_N\left(\frac{1}{N^{\ell-1}}(I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{M_{\ell-1}})\sum_{j_\ell=0}^{N-1}\gamma_N^{j_\ell}(I^{N-M_\ell}\otimes a_{M_\ell})\right)$$
$$=\overline{\gamma}_N\left(\frac{1}{N^{\ell-1}}\sum_{h_1=0}^{j_1-M_\ell} I^{h_1}\otimes a_{M_\ell}\otimes I^{j_1-M_\ell-h_1}\otimes\overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{M_{\ell-1}}\right)+\cdots+\overline{\gamma}_N$$
$$\left(\frac{1}{N^{\ell-1}}I^{j_1}\otimes \overbrace{a_{M_1}\otimes\cdots\otimes\sum_{0\leq h_{\ell-1}\leq j_\ell-M_\ell} I^{h_{\ell-1}}\otimes a_{M_\ell}\otimes I^{j_{\ell-1}-M_\ell-h_{\ell-1}}\otimes a}_{M_{\ell-1}}\right)+R''_N, \tag{19}$$

where $\|R''_N\|_N = O(1/N)$ while the sum over the h_p is empty if $j_p < M_\ell$—notice that at least one of these sums is not empty if N is large enough. Notice that each of the $\ell-1$ sets of ℓ indexes

$$\{h_1, j_1-M_\ell-h_1, j_2,\ldots,j_{\ell-1}\},\ \{j_1,h_2,j_2-M_\ell-h_2,j_3,\ldots,j_{\ell-1}\},$$
$$\ldots\ \{j_1,\ldots,j_{\ell-2},h_{\ell-1},j_{\ell-1}+M_\ell-h_{\ell-1}\}.$$

is such that its elements sum to $N-|M|_\ell$. Considering now the summation over $j_1,\ldots,j_{\ell-1}$ and using the γ_N-invariance each subset of indexes provides the same contribution. We are lead to

$$\overline{\gamma}_N\left(\frac{1}{N^{\ell-2}}\sum_{|j|_{\ell-1}=N-|M|_{\ell-1}}(I^{j_1}\otimes\overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{M_{\ell-1}})\overline{\gamma}_N^{M_\ell}(a_{M_\ell})\right)$$
$$=\overline{\gamma}_N\left(\frac{\ell-1}{N^{\ell-1}}\sum_{|j|_\ell=N-|M|_\ell}I^{j_1}\otimes\overbrace{a_{M_1}\otimes\cdots\otimes I^{j_{\ell-1}}\otimes a}_{M_{\ell-1}}\otimes I^{j_\ell}\otimes a_{M_\ell}\right)+R''_N,$$
$$=\overline{\gamma}_N\left(\frac{1}{N^{\ell-1}}\sum_{|j|_\ell=N-|M|_\ell}I^{j_1}\otimes\overbrace{a_{M_1}\otimes\cdots\otimes I^{j_\ell}\otimes a}_{M_\ell}\right)+R''_N,$$

where in the last line we used that for all $\varsigma\in\mathfrak{S}_\ell$ there are ℓ permutations which are equivalent to ς up to a cyclic permutation. Indeed, for any permutation of $a_{M_1},\ldots,a_{M_\ell}$ we may use the γ_N-invariance to write the corresponding contribution fixing the position of the factor a_{M_ℓ}. This boils down to a permutation of $a_{M_1},\ldots,a_{M_{\ell-1}}$ which is repeated ℓ times.

By induction this proves Eq. (18) for all $\ell\in\mathbb{N}$ and thus Eq. (16). □

Remark 7. (i) The appearance of the total weighted symmetrization (17) ensures that, when $a_{M_j} = I^{M_j}$ for all $j\in\{1,\ldots,\ell\}$, the right-hand side of Eq. (16) coincides with $[I^N]_N$. This is related to the fact that $\frac{1}{N^{\ell-1}}\sum_{|j|_\ell=N-|M|_\ell}=(\ell-1)!+O(1/N)$.
(ii) A closer inspection to the remainder term R_N of Eq. (18) reveals that

$$\left\|[R_N,\overline{\gamma}_N^{M'_1}(a_{M'_1})\ldots\text{-}\gamma_N^{M'_{\ell'}}(a_{M'_{\ell'}})]\right\|_N = O(1/N^2), \tag{20}$$

for all $\ell', M'_1,\ldots,M'_{\ell'}\in\mathbb{N}$, and $a_M\in B^M$. Roughly speaking, the reason for this is due to the estimate $\|R_N\|_N = O(1/N)$ together with the fact that both $(R_N)_N$ and

$(\overline{\gamma}_N^{M_1'} a_{M_1'} \dots \overline{\gamma}^{M_{\ell'}'} a_{M_{\ell'}'})_N$ are sequences with "an increasing number of identities". In more details, Eq. (18) implies

$$\overline{\gamma}_N^{M_1'}(a_{M_1'}) \dots \overline{\gamma}^{M_{\ell'}'}(a_{M_{\ell'}'}) = \frac{1}{N^{\ell'-1}} \sum_{|j|_{\ell'}=N-|M'|_{\ell'}} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{M_1'} \otimes \cdots \otimes I^{j_{\ell'}} \otimes a}_{M_{\ell'}'}) + R_N',$$

where $\|R_N'\|_N = O(1/N)$. This implies

$$[R_N, \overline{\gamma}_N^{M_1'}(a_{M_1'}) \dots \overline{\gamma}_N^{M_{\ell'}'}(a_{M_{\ell'}'})] = [R_N, R_N'] + \left[R_N, \frac{1}{N^{\ell'-1}} \sum_{|j|_{\ell'}=N-|M'|_{\ell'}} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{M_1'} \otimes \cdots \otimes I^{j_{\ell'}} \otimes a}_{M_{\ell'}'})\right].$$

The first contribution is estimated by $\|[R_N, R_N']\|_N = O(1/N^2)$ while for the second contribution we have

$$\begin{aligned} &\left\|\left[R_N, \frac{1}{N^{\ell'-1}} \sum_{|j|_{\ell'}=N-|M'|_{\ell'}} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{M_1'} \otimes \cdots \otimes I^{j_{\ell'}} \otimes a}_{M_{\ell'}'})\right]\right\| \\ &\le \frac{1}{N} \sum_{p=0}^{N-1} \left\|\left[R_N, \frac{1}{N^{\ell'-1}} \sum_{|j|_{\ell'}=N-|M'|_{\ell'}} \gamma_N^p(I^{j_1} \otimes \overline{a_{M_1'} \otimes \cdots \otimes I^{j_{\ell'}} \otimes a}_{M_{\ell'}'})\right]\right\| \\ &\le \frac{L}{N^2} C_{M_1,\dots,M_{\ell'}}. \end{aligned}$$

where $C_{M_1,\dots,M_{\ell'}} > 0$ does not depend on N. In the last inequality we used the estimate $\|R_N\|_N = O(1/N)$ and that, on account of the structure of R_N—cf. the proof of Proposition 6—and of $I^{j_1} \otimes a_{M_1'} \otimes \cdots \otimes I^{j_{\ell'}} \otimes a_{M_{\ell'}'}$, the sum over p is non-vanishing for finitely many values, say L, where L is N-independent. This proves Eq. (20).

(iii) In complete analogy with Definition 5 one may introduce the C^*-algebra $[B]_\pi^\infty \subset [B]_\sim$ generated by equivalence classes of π-sequences [16, Def. II.1]. Moreover, as shown in [11,12,16], for any $a_M \in B_\pi^M$ and $a_{M'} \in B_\pi^{M'}$ one finds

$$[\pi_N^M(a_M)\pi_N^{M'}(a_{M'})]_N = [\pi_N^{M+M'}(S_{M+M'}(a_M \otimes a_{M'}))]_N, \tag{21}$$

which shows that also $[B]_\pi^\infty$ is a commutative C^*-algebra. In fact, the product of π-sequences is (asymptotically as $N \to \infty$) a π-sequence. Additionally, one may prove that the system $\{B_\pi^N\}_{N\in\mathbb{N}}, \{\pi_N^M\}_{N\ge M}$ is a **generalized inductive system** [4,5]. This streamlines the identification of a bundle of C^*-algebras $\prod_{N\in\overline{\mathbb{N}}} B_\pi^N$ out of which a strict deformation quantization can be constructed [12,16]. The situation for γ-sequences is slightly different. Indeed, Eq. (16) shows that the product of γ-sequences is not a γ-sequence, even if its $\sim$-equivalence class is considered. Nevertheless, Eq. (16) shows that the product of equivalence classes of γ-sequences is commutative. As we shall see in Sect. 3.1 this will be enough to identify a continuous bundle of C^*-algebras $\prod_{N\in\overline{\mathbb{N}}} B_\gamma^N$ out of which a strict deformation quantization is obtained. Finally it is worth observing that, for all $a \in B$, one finds $\overline{\gamma}_N^1 a = \pi_N^1 a$ so that, given the results of [12,16], $[B]_\pi^\infty \subseteq [B]_\gamma^\infty$.

(iv) By standard Gelfand duality [4, § II.2] we find

$$[B]_\gamma^\infty \simeq C(K([B]_\gamma^\infty)),$$

where $K([B]_\gamma^\infty)$ denotes the character space over $[B]_\gamma^\infty$. An element $\varphi \in K([B]_\gamma^\infty)$ is completely characterized by

$$\varphi_M(a_M) := \varphi([\overline{\gamma}_N^M a_M]_N), \qquad a_M \in B^M,$$

which identifies a sequence $\{\varphi_M\}_{M\in\mathbb{N}}$ of states with $\varphi_M \in S(B^M)$. These states are "asymptotically equivalent" because of Eq. (15). Indeed considering $\overline{\gamma}_N^M : B^M \to B_\gamma^N$, $\overline{\gamma}_N^M a_M = \overline{\gamma}_N(I^{N-M} \otimes a_M)$, we find

$$\lim_{N\to\infty}(\varphi_N \circ \overline{\gamma}_N^M)(a_M) = \lim_{N\to\infty} \varphi\left([\overline{\gamma}_{N'}^N \overline{\gamma}_N^M a_M]_{N'}\right) = \varphi\left([\overline{\gamma}_{N'}^M a_M]_{N'}\right) = \varphi_M(a_M)\,.$$

A similar argument goes for $[B]_\pi^\infty$, where $K([B]_\pi^\infty)$ can be explicitly characterized. In particular $K([B]_\pi^\infty) \simeq S(B)$ [16, Lem. IV.4]. This identification may also be seen as a consequence of the prominent quantum De Finetti Theorem [11, Thm. 8.9]. As shown in [12], $S(B)$ is a stratified manifold which carries a Poisson structure.

3. Strict Deformation Quantization of γ-Sequences

The goal of this section is to construct a strict deformation quantization of the commutative algebra $[B]_\gamma^\infty$. To this avail in Sect. 3.1 we will identity a suitable continuous bundle of C^*-algebras $[B]_\gamma$ by means of a standard construction [10,11]. In Sect. 3.2 we will introduce the notion of "canonical representative" for an element $[a_N]_N \in [\dot{B}]_\gamma^\infty$—cf. Definitions 16–19. Eventually, in Sect. 3.3 will show that $[B]_\gamma^\infty$ carries a Poisson structure and we will prove Theorem 24, which provides the strict deformation quantization of $[B]_\gamma^\infty$.

3.1. The continuous bundle of C^*-algebras $[B]_\gamma$ associated with $[B]_\gamma^\infty$. In this section we will define a continuous bundle of C^*-algebras $[B]_\gamma$ over $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ whose fibers are $[B]_\gamma^N := B_\gamma^N$ for $N \in \mathbb{N}$ and $[B]_\gamma^\infty$, defined as per Definition 5, for $N = \infty$.

To this avail we briefly recall the main definitions and results we need—cf. [11, App. C.19], [4, §IV.1.6]. We denote by $C(\overline{\mathbb{N}})$ the space of $\mathbb{C}$-valued sequences $(\alpha_N)_{N\in\mathbb{N}}$ such that $\alpha_\infty := \lim_{N\to\infty} \alpha_N \in \mathbb{C}$ exists. A **continuous bundle (or field) of C^*-algebras** over $\overline{\mathbb{N}}$ is a triple $\mathcal{A}$, $\{\mathcal{A}_N\}_{N\in\overline{\mathbb{N}}}$, $\{\psi_N\}_{N\in\overline{\mathbb{N}}}$ made by C^*-algebras $\mathcal{A}$, $\mathcal{A}_N$, $N \in \overline{\mathbb{N}}$, and surjective homomorphisms $\psi_N : \mathcal{A} \to \mathcal{A}_N$ such that:

(i) The norm of $\mathcal{A}$ is given by $\|a\|_{\mathcal{A}} := \sup_{N\in\overline{\mathbb{N}}} \|\psi_N(a)\|_{\mathcal{A}_N}$;
(ii) For all $\alpha = (\alpha_N)_{N\in\overline{\mathbb{N}}} \in C(\overline{\mathbb{N}})$ and $a \in \mathcal{A}$ there exists a $\alpha a \in \mathcal{A}$ with the property that $\psi_N(\alpha a) = \alpha_N \psi_N(a)$.
(iii) For all $a \in \mathcal{A}$, $(\|\psi_N(a)\|_{\mathcal{A}_N})_{N\in\overline{\mathbb{N}}} \in C(\overline{\mathbb{N}})$.

A **continuous section** of $\mathcal{A}$ is an element $a \in \prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$ such that there exists $a' \in \mathcal{A}$ fulfilling $a_N = \psi_N(a')$ for all $N \in \overline{\mathbb{N}}$. Clearly $\mathcal{A}$ can be identified with its continuous sections, therefore, in the forthcoming discussion we shall always regard $a \in \mathcal{A}$ as an element $a \in \prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$. For this reason from now on we will implicitly identify ψ_N, $N \in \overline{\mathbb{N}}$, with the projection $\prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N \to \mathcal{A}_N$.

Remark 8. In applications, it is often difficult to identity a continuous bundle of C^*-algebras by assigning the triple $\mathcal{A}$, $\{\mathcal{A}_N\}_{N\in\overline{\mathbb{N}}}$, $\{\psi_N\}_{N\in\overline{\mathbb{N}}}$ directly. However, a useful result—cf. [10, Prop. 1.2.3], [11, Prop. C.124]—shows that it is in fact sufficient to identify a dense set of (a posteriori) continuous sections of $\mathcal{A}$. Actually, let $\widetilde{\mathcal{A}} \subseteq \prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N$ be such that:

1. For all $N \in \overline{\mathbb{N}}$ the set $\{a_N \mid a \in \widetilde{\mathcal{A}}\}$ is dense in $\mathcal{A}_N$;
2. $\widetilde{\mathcal{A}}$ is a $*$-algebra;
3. For all $\widetilde{a} \in \widetilde{\mathcal{A}}$, it holds $\lim_{N\to\infty} \|\widetilde{a}_N\|_{\mathcal{A}_N} = \|\widetilde{a}_\infty\|_{\mathcal{A}_\infty}$, i.e. $(\|\widetilde{a}_N\|_{\mathcal{A}_N})_{N\in\overline{\mathbb{N}}} \in C(\overline{\mathbb{N}})$.

Then defining $\mathcal{A}$ by

$$\mathcal{A} := \left\{ a \in \prod_{N\in\overline{\mathbb{N}}} \mathcal{A}_N \mid \forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N},\ \exists a' \in \widetilde{\mathcal{A}}\colon \|a_N - a'_N\|_{\mathcal{A}_N} < \varepsilon\ \forall N \geq N_\varepsilon \right\}, \tag{22}$$

one may prove that $\mathcal{A}$ is a continuous bundle of C^*-algebras over $\overline{\mathbb{N}}$ [10,11]. In fact, $\mathcal{A}$ is the smallest continuous bundle of C^*-algebras over $\overline{\mathbb{N}}$ which contains $\widetilde{\mathcal{A}}$.

We will now prove that $\prod_{N\in\overline{\mathbb{N}}}[B]^N_\gamma$ identifies a continuous bundle of C^*-algebras where $[B]^N_\gamma := B^N_\gamma$ for $N \in \mathbb{N}$ while $[B]^\infty_\gamma$ denotes the C^*-algebra introduced in Definition 5. To this avail we will identity a subset $\widetilde{\mathcal{A}} \subset \prod_{N\in\overline{\mathbb{N}}}[B]^N_\gamma$ fulfilling conditions 1–2–3 of Remark 8. From a technical point of view, condition 3 will require to prove that, for all $[a_N]_N \in [B]^\infty_\gamma$, the sequence $(\|a_N\|_N)_N$ has a limit as $N \to \infty$: This is proved in Proposition 10. To this avail, the following Lemma comes in handy.

Lemma 9. *Let $(\alpha_N)_{N\in\mathbb{N}}$ be a bounded sequence of real numbers such that*

$$\exists C_1, C_2 \in \mathbb{R},\ \exists N_0 \in \mathbb{N} :\ \alpha_N \geq \alpha_K + C_1\frac{1}{K} + C_2\frac{K}{N}\ \forall N \geq K \geq N_0\,. \tag{23}$$

Then $(\alpha_N)_N \in C(\overline{\mathbb{N}})$, i.e. $\alpha_\infty := \lim_{N\to\infty} \alpha_N \in \mathbb{R}$ exists.

Proof. Let $(\alpha_{N_j})_{j\in\mathbb{N}}$ be a convergent subsequence of $(\alpha_N)_{N\in\mathbb{N}}$. Then for all $K \geq N_0$ we find

$$\lim_{j\to\infty} \alpha_{N_j} \geq \lim_{j\to\infty}\left(\alpha_K + C_1\frac{1}{K} + C_2\frac{K}{N_j}\right) = \alpha_K + \frac{C_1}{K}\,.$$

Since this holds true for all convergent subsequences we conclude that

$$\liminf_{N\to\infty} \alpha_N \geq \alpha_K + \frac{C_1}{K} \qquad \forall K \geq N_0\,.$$

Considering again a convergent subsequence $(\alpha_{K_j})_{j\in\mathbb{N}}$ of $(\alpha_N)_{N\in\mathbb{N}}$ the above inequality implies

$$\lim_{j\to\infty} \alpha_{K_j} \le \lim_{j\to\infty} \left(\liminf_{N\to\infty} \alpha_N - \frac{C_1}{K_j} \right) = \liminf_{N\to\infty} \alpha_N .$$

Since this holds for all convergent subsequences we conclude that

$$\limsup_{N\to\infty} \alpha_N \le \liminf_{N\to\infty} \alpha_N ,$$

therefore, $\lim_{N\to\infty} \alpha_N = \liminf_{N\to\infty} \alpha_N = \limsup_{N\to\infty} \alpha_N$ exists and it is finite. □

Proposition 10. *For all $[a_N]_N \in [B]^\infty_\gamma$ the sequence $(\|a_N\|_N)_N$ is in $C(\overline{\mathbb{N}})$. In particular we have*

$$\|[a_N]_N\|_{[B]^\infty_\gamma} (= \|[a_N]_N\|_{[B]_\sim}) = \lim_{N\to\infty} \|a_N\|_N .$$

Proof. To begin with we prove the claim for $[a_N]_N = [\overline{\gamma}^M_N a_M]_N$. We will then move to $[a_N]_N \in [\dot{B}]^\infty_\gamma$ eventually discussing $[a_N]_N \in [B]^\infty_\gamma$.

$\boxed{\overline{\gamma}^M_N a_M}$ Let $a_M \in B^M$, $M \in \mathbb{N}$, and let us consider $[\overline{\gamma}^M_N a_M]_N$. Let $N, K \in \mathbb{N}$, $N \ge K \ge M$ and consider $\omega_K \in S(B^K)$. We decompose ω_K in a finite convex combination of product states

$$\omega_K = \sum_{p_1,\dots,p_K} \omega_K^{p_1\dots p_K} \eta_{p_1} \otimes \cdots \otimes \eta_{p_K},$$

where $\eta_{p_\ell} \in S(B)$ for all $\ell = 1, \dots, K$. We then consider

$$\omega_{K,N} := \sum_{p_1,\dots,p_K} \omega_K^{p_1\dots p_K} \tau^r \otimes (\eta_{p_1} \otimes \cdots \otimes \eta_{p_K})^q \in S(B^N),$$

where $N = r + qK$, $q \in \mathbb{N}$ and $r \in \{0, \dots, K-1\}$ while $\tau \in S(B)$ is normalized the trace state. We consider

$$\omega_{K,N}(\overline{\gamma}^M_N a_M) = \omega_{K,N}\left(\frac{1}{N} \sum_{j=0}^{N-1} I^{N-M-j} \otimes a_M \otimes I^j \right).$$

By direct inspection we have that, for all $\ell \in \{0, \dots, K-1\}$,

$$\frac{1}{N}\left[\sum_{p_1,\dots,p_K} \omega_K^{p_1\dots p_K} \tau^r \otimes (\eta_{p_1} \otimes \cdots \otimes \eta_{p_K})^q \right]\left(I^{N-M-\ell} \otimes a_M \otimes I^\ell \right)$$
$$= \frac{1}{N} \omega_K(\gamma_K^\ell (I^{K-M} \otimes a_M)).$$

The same contribution arises if $j \le N - r - M = qK - M$ and $j = \ell \mod K$. The number of such j's is roughly

$$q - M/K = N/K - r/K - M/K = N/K + O(1),$$

where the $O(1)$ contribution is bounded both in N and in K. The net result is

$$\begin{aligned}\omega_{K,N}(\overline{\gamma}_N^M a_M) &= \sum_{\ell=0}^{K-1}\left(\frac{N}{K}+O(1)\right)\frac{1}{N}\omega_K(\gamma_K^\ell(I^{K-M}\otimes a_M))\\ &+\frac{1}{N}\sum_{j=N-r-M+1}^{N-1}\omega_{K,N}\big(\overline{\gamma}_N^j(I^{N-M}\otimes a_M)\big)\\ &=\omega_K(\overline{\gamma}_K^M a_M)+O(K/N),\end{aligned}$$

where we observed that the sum over $j\in[N-r-M+1,N-1]$ contains at most $r+M-1=O(K)$ terms each of which is bounded by $\|a_M\|_M$. Overall we find

$$\|\overline{\gamma}_N^M a_M\|_N \geq |\omega_{K,N}(\overline{\gamma}_N^M a_M)| = \left|\omega_K(\overline{\gamma}_K^M a_M)+C\frac{K}{N}\right| \geq |\omega_K(\overline{\gamma}_K^M a_M)| - C\frac{K}{N},$$

where $C>0$ depends on M but not on N or K. The arbitrariness of $\omega_K\in S(B^K)$ leads to

$$\|\overline{\gamma}_N^M a_M\|_N \geq \|\overline{\gamma}_K^M a_M\|_K - C\frac{K}{N}.$$

Thus, Lemma 9 applies to the sequence $(\|\overline{\gamma}_N^M a_M\|_N)_N$ proving that $\lim_{N\to\infty}\|\overline{\gamma}_N^M a_M\|_N$ exists.

$[\dot{B}]_\gamma^\infty$ We now consider an arbitrary element $[a_N]_N$. Although our proof works for an arbitrary element of $[a_N]_N\in[\dot{B}]_\gamma^\infty$, for the sake of (notational) simplicity we restrict ourself to the case

$$[a_N]_N=\left[\sum_{k_1,k_2}c^{k_1k_2}\overline{\gamma}_N^{M(k_1)}(a_{(k_1)})\overline{\gamma}_N^{M(k_2)}(a_{(k_2)})\right]_N,$$

where the sum over k_1,k_2 is finite. To prove that $(\|a_N\|_N)_N$ has a limit as $N\to\infty$ we rely on Eq. (16) together with an argument similar in spirit to the one used for the case of a single γ-sequence. In fact, Proposition 6 implies that

$$\begin{aligned}&\left\|\sum_{k_1,k_2}c^{k_1k_2}\overline{\gamma}_N^{M(k_1)}(a_{(k_1)})\overline{\gamma}_N^{M(k_2)}(a_{(k_2)})\right\|_N\\ &=\left\|\sum_{k_1,k_2}c^{k_1k_2}\frac{1}{N}\sum_{j_1+j_2=N-M(k_1)-M(k_2)}\overline{\gamma}_N\left(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}\right)\right\|_N\\ &\quad+O(1/N),\end{aligned}$$

so that we may restrict to the first factor on the right-hand side. As for the case of single γ-sequence let $N,K\in\mathbb{N}$ be such that $N\geq K\geq\max_{k_1,k_2}\{2(M(k_1)+M(k_2))\}$ where the maximum is taken over all pairs $k_1,k_2\in\mathbb{N}$ appearing in the sum defining $[a_N]_N$. We consider $\omega_K\in S(B^K)$ and, as above, we set

$$\omega_{N,K}:=\sum_{p_1,\ldots,p_K}\omega_K^{p_1\cdots p_K}\tau^r\otimes(\eta_{p_1}\otimes\cdots\otimes\eta_{p_K})^q\in S(B^N),$$

where $N = r + qK$, $q \in \mathbb{N}$ and $r \in \{0, \ldots, K-1\}$ while $\omega_K = \sum_{p_1,\ldots,p_K} \omega_K^{p_1\cdots p_K}$ $\eta_{p_1} \otimes \cdots \otimes \eta_{p_K}$ is an arbitrary finite convex decomposition of ω_K into product states. We then evaluate

$$\omega_{K,N}\Bigg(\sum_{k_1,k_2} c^{k_1k_2}\frac{1}{N}\sum_{j_1+j_2=N-M(k_1)-M(k_2)} \overline{\gamma}_N\big(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}\big)\Bigg)$$
$$= \sum_{k_1,k_2} c^{k_1k_2}\frac{1}{N}\sum_{j_1+j_2=N-M(k_1)-M(k_2)} \omega_{K,N}\left(\overline{\gamma}_N\big(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}\big)\right).$$

To this avail we fix k_1, k_2 and split the sum over j_2 in two cases:

(a) Let consider the sum for $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$. For $0 \leq \ell \leq K - M(k_1) - M(k_2)$ we find, with the same argument used for a single γ-sequence,

$$\omega_{K,N}\left(\overline{\gamma}_N\big(I^{N-M(k_1)-M(k_2)-\ell}\otimes a_{(k_1)}\otimes I^{\ell}\otimes a_{(k_2)}\big)\right)$$
$$= \omega_K\left(\overline{\gamma}_K(I^{K-M(k_1)-M(k_2)-\ell}\otimes a_{(k_1)}\otimes I^{\ell}\otimes a_{(k_2)})\right) + O(K/N),$$

The number of j_2's such that $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$ and $j_2 = \ell \mod K$ is roughly $q = N/K + O(1)$, therefore, summing over such j_2's leads to a contribution of

$$\omega_{K,N}\Bigg(\frac{1}{N}\sum_{\substack{j_1+j_2=N-M(k_1)-M(k_2)\\ 0\leq j_2\leq N-r-M(k_1)-M(k_2)\\ j_2\leq K-M(k_1)-M(k_2) \mod K}} \overline{\gamma}_N\big(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}\big)\Bigg)$$
$$= \omega_K\Bigg(\frac{1}{K}\sum_{j_1+j_2=K-M(k_1)-M(k_2)} \overline{\gamma}_K(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)})\Bigg) + O(K/N).$$

It remains to discuss the sum over $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$ with $j_2 \in [K - M(k_1) - M(k_2), K-1] \mod K$: In this case we find

$$\Bigg|\omega_{K,N}\Bigg(\frac{1}{N}\sum_{\substack{j_1+j_2=N-M(k_1)-M(k_2)\\ 0\leq j_2\leq N-M(k_1)-M(k_2)-r\\ K-M(k_1)-M(k_2)\leq j_2\leq K-1 \mod K}} \overline{\gamma}_N\big(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}\big)\Bigg)\Bigg|$$
$$\leq \frac{1}{N}\sum_{\substack{j_1+j_2=N-M(k_1)-M(k_2)\\ 0\leq j_2\leq N-M(k_1)-M(k_2)-r\\ K-M(k_1)-M(k_2)\leq j_2\leq K-1 \mod K}} \|a_{(k_1)}\|_{M(k_1)}\|a_{(k_2)}\|_{M(k_2)}$$
$$\leq \frac{1}{N}\left(\frac{N}{K} + O(1)\right)(M(k_1) + M(k_2) - 1)\|a_{(k_1)}\|_{M(k_1)}\|a_{(k_2)}\|_{M(k_2)} = O(1/K),$$

where we observed that, for each of the $M(k_1) + M(k_2) - 1$ values of $\ell \in [K - M(k_1) - M(k_2), K-1]$, there are $q = N/K + O(1)$ values of $j_2 \leq N - M(k_1) - M(K_2) - r$ such that $j_2 = \ell \mod K$. Loosely speaking these contributions arise when j_2 is such that "$a_{(k_2)}$ overlaps with the (translated) position of $a_{(k_1)}$". This does not allow to reconstruct ω_K, therefore, these cases are estimated by $O(1/K)$.

(b) If $j_2 \in [N - M(k_1) - M(k_2) - r + 1, N - M(k_1) - M(k_2)]$—which is empty if $r = 0$—we have

$$\omega_{K,N}\Bigg(\frac{1}{N} \sum_{\substack{j_1+j_2=N-M(k_1)-M(k_2) \\ N-M(k_1)-M(k_2)-r \le j_2 \le N-M(k_1)-M(k_2)}} \overline{\gamma}_N\big(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}\big)\Bigg) = O(K/N).$$

Recollecting our result we have

$$\begin{aligned}
&\left|\omega_{K,N}\Bigg(\sum_{k_1,k_2} c^{k_1k_2}\frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \overline{\gamma}_N\big(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}\big)\Bigg)\right| \\
&= \Bigg|\omega_K\Bigg(\sum_{k_1,k_2} c^{k_1k_2}\frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \overline{\gamma}_K(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\Bigg) \\
&\quad +O(1/K) + O(K/N)\Bigg| \\
&= \left|\omega_K\Bigg(\sum_{k_1,k_2} c^{k_1k_2}\frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \overline{\gamma}_K(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\Bigg)\right| \\
&\quad -\frac{C_1}{K} - C_2\frac{K}{N}.
\end{aligned}$$

where $C_1, C_2 > 0$ do not depend neither on N nor on K. The arbitrariness of $\omega_K \in S(B^K)$ leads to

$$\begin{aligned}
&\left\|\sum_{k_1,k_2} \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \overline{\gamma}_N\Big(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}\Big)\right\|_N \\
&\geq \left\|\sum_{k_1,k_2} \frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \overline{\gamma}_K\Big(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}\Big)\right\|_K \\
&\quad -\frac{C_1}{K} - C_2\frac{K}{N}.
\end{aligned}$$

Thus, Lemma 9 applies and the limit

$$\lim_{N\to\infty} \|\overline{\gamma}_N^{M(k_1)}(a_{(k_1)})\overline{\gamma}_N^{M(k_2)}(a_{(k_2)})\|_N,$$

exists and it is finite.

$[B]^\infty_\gamma$ Finally, let $[a_N]_N \in [B]^\infty_\gamma$. Then, for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and $[a'_N]_N \in [\dot{B}]^\infty_\gamma$ such that

$$\|a_N - a'_N\|_N < \varepsilon \qquad \forall N \geq N_\varepsilon\,.$$

Moreover, since $(\|a'_N\|_N)_N$ is convergent, there exists $N'_\varepsilon \in \mathbb{N}$ such that

$$\big|\|a'_N\|_N - \|a'_M\|_M\big| < \varepsilon \qquad \forall N, M \geq N'_\varepsilon\,.$$

For $N, M \geq \max\{N_\varepsilon, N'_\varepsilon\}$ we then have

$$\begin{aligned} |\|a_N\|_N - \|a_M\|_M| &\leq \left|\|a_N\|_N - \|a'_N\|_N\right| + \left|\|a'_N\|_N - \|a'_M\|_M\right| + \left|\|a'_M\|_M - \|a_M\|_M\right| \\ &\leq \|a_N - a'_N\|_N + \left|\|a'_N\|_N - \|a'_M\|_M\right| + \|a'_M - a_M\|_M \leq 3\varepsilon, \end{aligned}$$

proving that $(\|a_N\|_N)_N$ is a Cauchy sequence. □

Remark 11. The result of Proposition 10 applies also for π-sequences. For this latter case the proof streamlines because

$$\|\pi_N^M a_M\|_N = \|\pi_N^K \pi_K^M a_M\|_N \leq \|\pi_K^M a_M\|_K,$$

so that $(\|\pi_N^M a_M\|)_N$ is decreasing. The difficulties in moving from $[B]_\pi^\infty$ to $[B]_\gamma^\infty$ is twofold. On the one hand, for γ-sequences $\|\overline{\gamma}_N^M a_M\|_N$ is not decreasing, although it fulfils a similar properties asymptotically. On the other hand, the product of γ-sequences is not a γ-sequence, even when equivalence classes are considered. This requires a different strategy to ensure the existence of the limit $\lim\limits_{N\to\infty} \|a_N\|_N$ for $[a_N]_N \in [B]_\gamma^\infty$.

The following proposition proves the existence of the continuous bundle of C^*-algebras of interest.

Proposition 12. *Let $\{B_\gamma^N\}_{N\in\mathbb{N}}$ be the family of C^*-algebras introduced in Eq. (7). Let $\{[B]_\gamma^N\}_{N\in\overline{\mathbb{N}}}$ be defined by $[B]_\gamma^N := B_\gamma^N$ for $N \in \mathbb{N}$ while $[B]_\gamma^\infty$ is the C^*-algebra generated by equivalence classes of γ-sequences, cf. Definition 5. Let $[\dot{B}]_\gamma \subset \prod_{N\in\overline{\mathbb{N}}}[B]_\gamma^N$ be the subset defined by*

$$[\dot{B}]_\gamma := \left\{(A_N)_{N\in\overline{\mathbb{N}}} \in \prod_{N\in\overline{\mathbb{N}}} [B]_\gamma^N \mid \exists (a_N)_N \in \dot{B}_\gamma^\infty : A_N = \begin{cases} a_N & N \in \mathbb{N} \\ [a_N]_N & N = \infty \end{cases}\right\}. \quad (24)$$

Then $[\dot{B}]_\gamma$ fulfils conditions 1.–2.–3. and thus it leads to a continuous bundle of C^*-algebras

$$[B]_\gamma := \left\{(A_N)_{N\in\overline{\mathbb{N}}} \in \prod_{N\in\overline{\mathbb{N}}} [B]_\gamma^N \mid \forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N},\ \exists A' \in [\dot{B}]_\gamma : \|A_N - A'_N\|_N < \varepsilon\ \forall N \geq N_\varepsilon\right\}. \quad (25)$$

Proof. We will prove conditions 1–2–3 of Remark 8. The space $[\dot{B}]_\gamma$ is a $*$-algebra, therefore, condition 2 is fulfilled. Concerning condition 1, we have to prove that

$$Z_M := \{A_M \in [B]_\gamma^M \mid (A_N)_{N\in\overline{\mathbb{N}}} \in [\dot{B}]_\gamma\} \subseteq [B]_\gamma^M,$$

is dense in $[B]_\gamma^M$ for all $M \in \overline{\mathbb{N}}$. For $M \in \mathbb{N}$ it is enough to observe that, for all $a_M \in [B]_\gamma^M = B_\gamma^M$, we may consider $(A_N)_{N\in\overline{\mathbb{N}}} \in [\dot{B}]_\gamma$ defined by

$$A_N = \begin{cases} \overline{\gamma}_N^M a_M & N \in \mathbb{N} \\ [\overline{\gamma}_N^M a_M]_N & N = \infty \end{cases},$$

which leads to $A_M = \overline{\gamma}_M a_M = a_M$, i.e. $Z_M = [B]_\gamma^M$. If $M = \infty$ we have $Z_\infty = [\dot{B}]_\gamma^\infty$ whose closure is per definition $[B]_\gamma^\infty$—cf. Definition 5.

Finally condition 3 is equivalent to

$$\lim_{N\to\infty} \|A_N\|_N = \lim_{N\to\infty} \|a_N\|_N = \|[a_N]_N\|_{[B]_\gamma^\infty} = \|A_\infty\|_{[B]_\gamma^\infty} \qquad \forall (A_N)_{N\in\overline{\mathbb{N}}} \in [\dot{B}]_\gamma,$$

where the existence of $\lim\limits_{N\to\infty} \|a_N\|_N$ is ensured by Proposition 10. □

3.2. Canonical representative of $[a_N]_N \in [\dot{B}]_\gamma^\infty$. To proceed further in the construction of the deformation quantization of $[B]_\gamma^\infty$ we have to discuss the possibility of identifying a canonical representative of an element $[a_N]_N \in [\dot{B}]_\gamma^\infty$. This is required for both endowing $[B]_\gamma^\infty$ with a Poisson structure as well as for defining the quantization maps $Q_N \colon [\dot{B}]_\gamma^\infty \to [B]_\gamma^N$—cf. Theorem 24.

To begin with, we address the following problem: Given $[\overline{\gamma}_N^M a_M]_N \in [\dot{B}]_\gamma^\infty$ does it hold

$$[\overline{\gamma}_N^M a_M]_N = [0]_N \Longleftrightarrow a_M = 0 \,?$$

A positive answer in this direction would imply that, given an equivalence class $[\overline{\gamma}_N^M a_M]_N$, one is able to determine uniquely the γ-sequence $(\overline{\gamma}_N^M a_M)_N$. Unfortunately, the answer to this question is negative because

$$[\overline{\gamma}_N^M a_M]_N = [\overline{\gamma}_N^{M+K}(I^K \otimes a_M)]_N = [\overline{\gamma}_N^{M+K}(a_M \otimes I^K)]_N ,$$

although the associated sequences are not the same. Indeed

$$\overline{\gamma}_M^{M+K}(I^K \otimes a_M) = 0 \neq \overline{\gamma}_M a_M = \overline{\gamma}_M^M a_M \,.$$

This counterexample suggests to focus on the C^*-subalgebra $\widetilde{B}^M$ where $\widetilde{B} = \ker \tau$, $\tau \in S(B)$ being the trace state—cf. Sect. 2. In fact, therein the situation is slightly better as shown by the following Lemma.

Lemma 13. *Let* $\widetilde{a}_M \in \widetilde{B}^M$ *be such that* $[\overline{\gamma}_N^M \widetilde{a}_M]_N = [0]_N$. *Then* $\widetilde{a}_M = 0$.

Proof. Per definition $[\overline{\gamma}_N^M \widetilde{a}_M]_N = [0]_N$ if and only if $\lim_{N\to\infty} \|\overline{\gamma}_N^M \widetilde{a}_M\|_N = 0$. Let $\omega_M \in S(B^M)$ and let $\tau \in S(B)$ be the normalized trace state $\tau(a) := \mathrm{tr}(a)/\kappa$. Let $N \geq M+1$, $q \in \mathbb{N}$ and $r \in \{0, \ldots, M\}$ be such that $N = r + q(M+1)$. We consider the state

$$\omega_{M,N} := \tau^r \otimes (\tau \otimes \omega_M)^q \in S(B^N) \,.$$

By direct inspection we find that

$$\begin{aligned}\omega_{M,N}(\overline{\gamma}_N^M \widetilde{a}_M) \\ = [\tau^r \otimes (\tau \otimes \omega_M)^q]\left(\frac{1}{N}\sum_{j=0}^{N-1} \gamma_N^j (I^{N-M} \otimes \widetilde{a}_M)\right) = \frac{1}{M+1}\omega_M(\widetilde{a}_M) + O(1/N),\end{aligned} \tag{26}$$

Indeed, for $j = 0$ the resulting contribution is $\omega_M(\widetilde{a}_M)/N$. The same contribution appears when $j = 0 \mod M+1$: Since $j \in \{0, \ldots, N-1\}$ this happens q times, moreover, $q = N/(M+1) + O(1)$ leading to the right-hand side of Eq. (26). Whenever $j \neq 0 \mod M+1$ the resulting contribution is 0, on account of the fact that τ vanishes on $\widetilde{B}$.

Equation (26) implies that, for all $\omega_M \in S(B^M)$,

$$0 = \lim_{N\to\infty} \|\overline{\gamma}_N^M \widetilde{a}_M\| \geq \frac{1}{M+1} |\omega_M(\widetilde{a}_M)| \,.$$

The arbitrariness of ω_M leads to $\|\widetilde{a}_M\|_M = 0$, that is, $\widetilde{a}_M = 0$. □

Thus, although the equivalence class $[\overline{\gamma}_N^M a_M]_N$ does not identify a unique sequence $(\overline{\gamma}_N^M a_M)_N$, Lemma 13 suggests that a (a posteriori unique) canonical representative may be extracted by working with the "$\widetilde{B}$-irreducible components" of the γ-sequence. To this avail, we introduce the notion of $\widetilde{B}$-irreducibility. This identifies those elements in B^M which cannot be written as $I \otimes a_{M-1}$ or $a_{M-1} \otimes I$ for some $a_{M-1} \in B^{M-1}$.

Definition 14. An element $a_M \in B^M$ is called $\widetilde{B}$**-irreducible**, and we write $a_M \in B^M_{\mathrm{IRR}}$, if either $M = 0$ or

$$(\tau \otimes \omega_{M-1})(a_M) = (\omega_{M-1} \otimes \tau)(a_M) = 0, \tag{27}$$

for all $\omega_{M-1} \in S(B^{M-1})$.

Remark 15. Notice that, per definition, $a_0 \in \mathbb{C}$ is $\widetilde{B}$-irreducible, moreover, $B^1_{\mathrm{IRR}} = \widetilde{B}$, $B^2_{\mathrm{IRR}} = \widetilde{B}^2$. For the sake of completeness, Appendix A provides a complete characterization of B^M_{IRR} for all $M \in \mathbb{N}$.

The notion of $\widetilde{B}$-irreducible elements leads to a proper definition of "canonical representative" for a γ-sequence—cf. Definition 16. Indeed, let consider an arbitrary $a_M \in B^M$. By considering a basis $I, b_1, \ldots, b_{\kappa^2-1}$ of B fulfilling (4) we may decompose a_M as

$$\begin{aligned}
a_M &= a_0 I^M + \sum_{\substack{j_1+j_2=M-1\\ k}} c^k_{j_1 j_2} I^{j_1} \otimes b_k \otimes I^{j_2} \\
&+ \sum_{\substack{j_1+j_2+j_3=M-2\\ k_1,k_2}} c^{k_1 k_2}_{j_1 j_2 j_3} I^{j_1} \otimes b_{k_1} \otimes I^{j_2} \otimes b_{k_2} \otimes I^{j_3} \\
&+ \cdots + \sum_{\substack{j_1+\cdots+j_{\ell+1}=M-\ell\\ k_1,\ldots,k_\ell}} c^{k_1\ldots k_\ell}_{j_1\cdots j_{\ell+1}} I^{j_1} \otimes b_{k_1} \otimes \cdots \otimes I^{j_\ell} \otimes b_{k_\ell} \otimes I^{j_{\ell+1}} \\
&+ \cdots + \sum_{k_1,\ldots,k_M} c^{k_1\ldots k_M} b_{k_1} \otimes \cdots \otimes b_{k_M},
\end{aligned} \tag{28}$$

where $a_0, c^{k_1\ldots k_\ell}_{j_1\cdots j_{\ell+1}} \in \mathbb{C}$ and the sum over $k_1, \ldots, k_\ell$ is finite. At this stage we observe that $(\overline{\gamma}_N^M a_M)_N = (\overline{\gamma}_N^M a'_M)_N$ where $a'_M \in B^M$ is defined by

$$\begin{aligned}
a'_M &= a_0 I^M + I^{M-1} \otimes \sum_{\substack{j_1+j_2=M-1\\ k}} c^k_{j_1 j_2} b_k + \sum_{\substack{j_1+j_2+j_3=M-2\\ k_1,k_2}} c^{k_1 k_2}_{j_1 j_2 j_3} I^{j_1+j_3} \otimes b_{k_1} \otimes I^{j_2} \otimes b_{k_2} \\
&+ \cdots + \sum_{\substack{j_1+\cdots+j_{\ell+1}=M-\ell\\ k_1,\ldots,k_\ell}} c^{k_1\ldots k_\ell}_{j_1\cdots j_{\ell+1}} I^{j_1+j_{\ell+1}} \otimes b_{k_1} \otimes \cdots \otimes I^{j_\ell} \otimes b_{k_\ell} \\
&+ \cdots + \sum_{k_1,\ldots,k_M} c^{k_1\ldots k_M} b_{k_1} \otimes \cdots \otimes b_{k_M} \\
&= \sum_{j=0}^{M} I^{M-j} \otimes a'_j \in \bigoplus_{j=0}^{M} I^{M-j} \otimes B^j_{\mathrm{IRR}}\,.
\end{aligned} \tag{29}$$

We stress that some of the a'_j's may vanish in the process. However, it is important to observe that, moving from a_M to a'_M, the γ-sequence (and thus its equivalence class) does not change. Notice that, if we replace a_M with $I^K \otimes a_M$ or $a_M \otimes I^K$, the $\widetilde{B}$-irreducible elements $\{a'_j\}_{j=0}^M$ do not change.

Definition 16. Let $(\overline{\gamma}_N^M a_M)_N$ be a γ-sequence and let $\sum_{j=0}^M I^{M-j} \otimes a'_j$ be the element defined as per Eq. (29), where $a'_j \in B_{\text{IRR}}^j$ for all $j \in \{0, \dots, M\}$. The sequence

$$\sum_{j=0}^{M} (\overline{\gamma}_N^j a'_j)_N \in \dot{B}_\gamma^\infty,$$

is called the **canonical representative** of $[\overline{\gamma}_N^M a_M]_N$.

Remark 17. (i) It is worth pointing out that, while $(\overline{\gamma}_N^M a_M)_N = (\overline{\gamma}_N^M a'_M)_N$ for $a'_M = \sum_{j=0}^M I^{M-j} \otimes a'_j$, for the canonical representative we only have equality of equivalence classes, i.e. $[\overline{\gamma}_N^M a_M]_N = \sum_{j=0}^M [\overline{\gamma}_N^j a'_j]_N$. In particular we have

$$(\overline{\gamma}_N^M a_M)_N = (\overline{\gamma}_N^M a'_M)_N = \sum_{j=0}^{M} (\overline{\gamma}_N^j a'_j)_N + R_N, \tag{30}$$

where $\|R_N\| = O(1/N^\infty)$. For example if $a_M = a_0 I^M$ then the canonical representative is the constant sequence $a_N = a_0 I^N$, $N \in \mathbb{N}$, which coincides with $(\overline{\gamma}_N^M a_M)_N$ only for $N \geq M$.

(ii) On account of the previous discussion we observe that the algebra generated by γ-sequences of the form $(\overline{\gamma}_N^M a_M)_N$ for $a_M \in B_{\text{IRR}}^M$, $M \in \mathbb{N}$, exhaust the whole space $\dot{B}_\gamma^\infty$.

The following proposition shows that the canonical representative introduced in Definition 16 is unique.

Proposition 18. *Let $M \in \mathbb{N}$ and $a_j \in B_{\text{IRR}}^j$ for all $j = 0, \dots, M$. Then*

$$\lim_{N\to\infty} \left\| \sum_{j=0}^{M} \overline{\gamma}_N^j a_j \right\|_N = 0 \Longleftrightarrow a_0 = 0, \dots, a_M = 0. \tag{31}$$

Proof. The proof is similar to the one of Lemma 13. By direct inspection we have

$$0 = \lim_{N\to\infty} \left\| \sum_{j=0}^{M} \overline{\gamma}_N^j a_j \right\|_N \geq \lim_{N\to\infty} \left| \tau^N \left(\sum_{j=0}^{M} \overline{\gamma}_N^j a_j \right) \right| = |a_0|.$$

Let now $\eta \in S(B)$ and let $\omega_{\eta,N} := \tau^r \otimes (\tau^{2M-1} \otimes \eta)^q \in S(B^N)$, where $N = r + 2Mq$, $q \in \mathbb{N}$ and $r \in \{0, \dots, 2M-1\}$. We have

$$0 = \lim_{N\to\infty} \left\| \sum_{j=0}^{M} \overline{\gamma}_N^j a_j \right\|_N \geq \lim_{N\to\infty} \left| \omega_{\eta,N} \left(\sum_{j=1}^{M} \overline{\gamma}_N^j a_j \right) \right| = \frac{1}{2M} |\eta(a_1)|,$$

which implies $a_1 = 0$ because of the arbitrariness of $\eta \in S(B)$. Notice that $\omega_{\eta,N}(\overline{\gamma}_N^j a_j) = 0$ for all $j \geq 2$ on account of the assumption $a_j \in B^j_{\mathrm{IRR}}$.

Proceeding by induction we may assume that $a_1 = \cdots = a_{\ell-1} = 0$ and prove that $a_\ell = 0$. To this avail let $\eta_1, \ldots, \eta_\ell \in S(B)$ and set

$$\omega_{\eta_1,\ldots,\eta_\ell,N} := \tau^r \otimes (\tau^{2M-\ell} \otimes \eta_1 \otimes \cdots \otimes \eta_\ell)^q \in S(B^N),$$

where $N = r + 2Mq$, $q \in \mathbb{N}$ and $r \in \{0, \ldots, 2M - 1\}$. Using the inductive hypothesis we find

$$0 = \lim_{N\to\infty} \left\| \sum_{j=\ell}^{M} \overline{\gamma}_N^j a_j \right\|_N \geq \lim_{N\to\infty} \left| \omega_{\eta_1,\ldots,\eta_\ell,N} \left(\sum_{j=\ell}^{M} \overline{\gamma}_N^j a_j \right) \right| = \frac{1}{2M} |(\eta_1 \otimes \cdots \otimes \eta_\ell)(a_\ell)|,$$

where, with the same argument as above, the contributions arising from a_j, $j \geq \ell + 1$, vanish. The arbitrariness of $\eta_1, \ldots, \eta_\ell \in S(B)$ implies $a_\ell = 0$. □

Summing up, every equivalence class $[\overline{\gamma}_N^M a_M]_N \in [\dot{B}]_\gamma^\infty$ has a unique canonical representative obtained by decomposing a_M into its $\widetilde{B}$-irreducible components.

We shall now discuss the notion of canonical representative for a generic element $[a_N]_N \in [\dot{B}]_\gamma^\infty$. Proposition 6 and Remark 17-(ii) lead to the following definition.

Definition 19. Let $[a_N]_N \in [\dot{B}]_\gamma^\infty$ be such that

$$\begin{aligned}[a_N]_N &= \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} [\overline{\gamma}_N^{M(k_1)}(a_{(k_1)}) \ldots \overline{\gamma}_N^{M(k_\ell)}(a_{(k_\ell)})]_N \\ &= \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} \frac{1}{N^{\ell-1}} \sum_{|j|_\ell = N - |M(k)|_\ell} [\overline{\gamma}_N(I^{j_1} \otimes \overline{a_{(k_1)} \otimes \cdots \otimes I^{j_\ell} \otimes a}_{(k_\ell)})]_N.\end{aligned}$$

where $a_{k_j} \in B_{\mathrm{IRR}}^{M(k_j)}$ for all k_j, while the sum over $\ell, k_1, \ldots, k_\ell$ is finite and $|M(k)|_\ell := M(k_1) + \cdots + M(k_\ell)$. The sequence

$$\sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} \left(\frac{1}{N^{\ell-1}} \sum_{|j|_\ell = N - |M(k)|_\ell} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{(k_1)} \otimes \cdots \otimes I^{j_\ell} \otimes a}_{(k_\ell)}) \right)_{N \geq |M(k)|_\ell}, \tag{32}$$

is called the **canonical representative** of $[a_N]_N$.

Similarly to Proposition 18 we have the following result, showing that the canonical representative introduced in Definition 19 is unique.

Proposition 20. *It holds*

$$\begin{aligned}&\lim_{N\to\infty} \left\| \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} \frac{1}{N^{\ell-1}} \sum_{|j|_\ell = N - |M(k)|_\ell} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{(k_1)} \otimes \cdots \otimes I^{j_\ell} \otimes a}_{(k_\ell)}) \right\|_N = 0 \\ &\iff \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} \frac{1}{N^{\ell-1}} \sum_{|j|_\ell = N - |M(k)|_\ell} \overline{\gamma}_N(I^{j_1} \otimes \overline{a_{(k_1)} \otimes \cdots \otimes I^{j_\ell} \otimes a}_{(k_\ell)}) \\ &= 0 \quad \forall N \in \mathbb{N},\end{aligned} \tag{33}$$

where the sum over $\ell, k_1, \ldots, k_\ell \in \mathbb{N}$ is finite and $a_{k_j} \in B_{\mathrm{IRR}}^{M(k_j)}$ for all k_j.

Proof. For the sake of clarity, we will discuss the proof for $\ell \leq 2$. This simplifies the construction without affecting the validity of the argument. We thus consider the sequence

$$a_N := \sum_{k_1,k_2} c^{k_1k_2}\frac{1}{N}\sum_{|j|=N-|M(k)|_2} \overline{\gamma}_N(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)}), \tag{34}$$

where the sum over k_1, k_2 is finite and $a_{(k)} \in B^{M(k)}_{\mathrm{IRR}}$ for all k. Notice that, whenever $M(k_1) = 0$ or $M(k_2) = 0$ the corresponding contribution reduces to a single γ-sequence up to a remainder $O(1/N)$. We have to prove that $\lim_{N\to\infty} \|a_N\|_N = 0$ implies $a_N = 0$ for all $N \in \mathbb{N}$.

We observe that $\|a_N\|_N \underset{N\to\infty}{\longrightarrow} 0$ entails

$$0 = \lim_{N\to\infty}\|a_N\|_N \geq |\tau^N(a_N)| = \sum_{\substack{k_1:\, M(k_1)=0\\ k_2:\, M(k_2)=0}} c^{k_1k_2}a_{(k_1)}a_{(k_2)},$$

so that we may assume $(M(k_1), M(k_2)) \neq (0, 0)$ in (34).

We now analyse (34) with the help of the following parameters:

$$\overline{M} := \max_{k_1,k_2}\max\{M(k_1), M(k_2)\},$$

$$\underline{M}_1 := \min_{k_1,k_2}\max\{M(k_1), M(k_2)\}, \qquad \underline{M}_2 := \min_{\substack{k_1:\, M(k_1)\leq \underline{M}_1\\ k_2:\, M(k_2)\leq \underline{M}_1}} \min\{M(k_1), M(k_2)\}. \tag{35}$$

Roughly speaking $\overline{M}$ is the maximal degree of the $a_{(k)}$'s appearing in (34). The parameter $\underline{M}_1 \leq \overline{M}$ is the minimal "bigger length" among all pairs (k_1, k_2) appearing in (34). Notice that $\underline{M}_1 > 0$ on account of the hypothesis $(M(k_1), M(k_2)) \neq (0, 0)$. Finally $\underline{M}_2 \leq \underline{M}_1$ is the minimal length of the $a_{(k)}$'s appearing when considering only those pairs (k_1, k_2) for which $\max\{M(k_1), M(k_2)\} \leq \underline{M}_1$—notice that this implies $M(k) = \underline{M}_1$ for at least one between $k \in \{k_1, k_2\}$.

Let $\omega_{\underline{M}_1} \in S(B^{\underline{M}_1})$, $\omega_{\underline{M}_2} \in S(B^{\underline{M}_2})$ and let $\omega_{\underline{M}_1,\underline{M}_2,N} \in S(B^N)$ be defined by

$$\omega_{\underline{M}_1,\underline{M}_2,N} := \tau^r \otimes (\tau^{\overline{M}} \otimes \omega_{\underline{M}_1} \otimes \tau^{\overline{M}} \otimes \omega_{\underline{M}_2})^q, \tag{36}$$

where $N = r + (2\overline{M} + \underline{M}_1 + \underline{M}_2)q$, $q \in \mathbb{N}$, $r \in \{0, \dots, 2\overline{M} + \underline{M}_1 + \underline{M}_2 - 1\}$. We consider

$$\begin{aligned}\omega_{\underline{M}_1,\underline{M}_2,N}(a_N) &= \sum_{k_1,k_2} c^{k_1k_2}\frac{1}{N}\sum_{|j|=N-|M(k)|_2}\omega_{\underline{M}_1,\underline{M}_2,N}\Big[\overline{\gamma}_N(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)})\Big]\\ &= \sum_{\substack{k_1:\, M(k_1)\leq \underline{M}_1\\ k_2:\, M(k_2)\leq \underline{M}_1}} c^{k_1k_2}\frac{1}{N}\sum_{|j|=N-|M(k)|_2}\omega_{\underline{M}_1,\underline{M}_2,N}\Big[\overline{\gamma}_N(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)})\Big],\end{aligned} \tag{37}$$

where in the second line we observed that

$$\omega_{\underline{M}_1,\underline{M}_2,N}\Big[\overline{\gamma}_N(I^{j_1}\otimes a_{(k_1)}\otimes I^{j_2}\otimes a_{(k_2)})\Big] = 0 \qquad \text{if } \min\{M(k_1), M(k_2)\} > \underline{M}_1\,;$$

This follows from the fact that, if $\underline{M}_1 < M_2 \leq \overline{M}$, for all $a_{M_2} \in B_{\text{IRR}}^{M_2}$ we have

$$\omega_{\underline{M}_1,\underline{M}_2,N}(\overline{\gamma}_N(a_{N-M_2} \otimes a_{M_2})) = 0,$$

no matter the choice of $a_{N-M_2} \in B^{N-M_2}$. In fact, for all $j \in \{0, \ldots, N-1\}$ one finds that

$$\omega_{\underline{M}_1,\underline{M}_2,N}\big[\gamma_N^j(a_{N-M_2} \otimes a_{M_2})\big] = \tau^r \otimes (\tau^{\overline{M}} \otimes \omega_{\underline{M}_1} \otimes \tau^{\overline{M}} \otimes \omega_{\underline{M}_2})^q\big[\gamma_N^j(a_{N-M_2} \otimes a_{M_2})\big],$$

is non vanishing only if the position of a_{M_2} "overlaps completely" with either $\omega_{\underline{M}_1}$ or with $\omega_{\underline{M}_2}$, however, this is not possible because $M_2 > \underline{M}_1 \geq \underline{M}_2$. Notice that overlapping with both states is impossible since each pair $\omega_{\underline{M}_1}, \omega_{\underline{M}_2}$ is separated by $\tau^{\overline{M}}$ and $M_2 \leq \overline{M}$.

We now analyse the remaining contributions (37) of $\omega_{\underline{M}_1,\underline{M}_2,N}(a_N)$. Notice that the condition $M(k_1) \leq \underline{M}_1$ and $M(k_2) \leq \underline{M}_1$ implies $M(k) = \underline{M}_1$ for at least one between k_1, k_2. In fact, we also have $M(k_1), M(k_2) \geq \underline{M}_2$ which implies $M(k_2) = \underline{M}_2$ or $M(k_1) = \underline{M}_2$ for at least one pair (k_1, k_2). Moreover, by direct inspection:

(a) If $j_2 = \overline{M} \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$ then

$$\begin{aligned}&\omega_{\underline{M}_1,\underline{M}_2,N}\big[\overline{\gamma}_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\big]\\&= \frac{1}{2\overline{M} + \underline{M}_1 + \underline{M}_2}\Big[\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_2)} \otimes a_{(k_2)})\\&\quad+\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)})\Big] + O(1/N),\end{aligned} \tag{38}$$

with the convention that the contribution vanishes if, say, $\underline{M}_2 < M(k_1)$—this may happen if $\underline{M}_2 < \underline{M}_1$ and $M(k_1) = \underline{M}_1$. This restrict the non-vanishing contributions to those pairs (k_1, k_2) such that $\{M(k_1), M(k_2)\} = \{\underline{M}_1, \underline{M}_2\}$. Notice that there exists at least one such pair on account of the definition of $\underline{M}_2$—cf. Eq. (35). To prove (38) it suffices to observe that for all $\ell \in \{0, \ldots, N-1\}$ we have

$$\omega_{\underline{M}_1,\underline{M}_2,N}\big[\gamma_N^\ell(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\big] = \begin{cases}\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_2)} \otimes a_{(k_2)}) \\ \text{if } \ell = 0 \mod \underline{M}_1 + \underline{M}_2 + 2\overline{M} \\ \omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}) \\ \text{if } \ell = \underline{M}_1 + \overline{M} \mod \underline{M}_1 + \underline{M}_2 + 2\overline{M} \\ 0 \quad \text{otherwise}\end{cases}$$

Since the number of $\ell \in \{0, \ldots, N-1\}$ such that $\ell = 0 \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$ (*resp.* $\ell = \underline{M}_1 + \overline{M} \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$) is roughly $N/(2\overline{M} + \underline{M}_1 + \underline{M}_2) + O(1)$ the formula for $\omega_{\underline{M}_1,\underline{M}_2,N}\big[\overline{\gamma}_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\big]$ follows.

(b) Similarly, if $j_2 = 2\overline{M} + \underline{M}_1 \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$ then

$$\begin{aligned}&\omega_{\underline{M}_1,\underline{M}_2,N}\big[\overline{\gamma}_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)})\big]\\&= \frac{1}{2\overline{M} + \underline{M}_1 + \underline{M}_2}\Big[\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_1}(I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)})\\&\quad+\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_1)} \otimes a_{(k_1)})\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k_2)} \otimes a_{(k_2)})\Big] + O(1/N),\end{aligned}$$

where again the contribution is non-vanishing if and only if $\{M(k_1), M(k_2)\} = \{\underline{M}_1, \underline{M}_2\}$.

(c) In all other cases the contribution vanishes.

The number of $j_2 \in \mathbb{N}$ such that $j_2 \leq N - |\underline{M}(k)|_2$ and $j_2 = \overline{M} \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$ (*resp.* $j_2 = 2\overline{M} + \underline{M}_1 \mod 2\overline{M} + \underline{M}_1 + \underline{M}_2$) is roughly $N/(2\overline{M} + \underline{M}_1 + \underline{M}_2) + O(1)$. Moreover, we have

$$\omega_{\underline{M}_2}(I^{\underline{M}_2 - M(k)} \otimes a_{(k)}) = (\omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2})(I^{\underline{M}_1 - M(k)} \otimes a_{(k)}),$$

where $\omega_{\underline{M}_1 - \underline{M}_2} \in S(B^{\underline{M}_1 - \underline{M}_2})$ is arbitrarily chosen.

Thus, combining cases (a)–(b) we find

$$\omega_{\underline{M}_1, \underline{M}_2, N}(a_N) = \sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} \frac{c^{k_1 k_2}}{(2\overline{M} + \underline{M}_1 + \underline{M}_2)^2}$$

$$\left(\frac{1}{2}\omega_{\underline{M}_1} \quad + \frac{1}{2}\omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2}\right)^2 \left[I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}\right] + O(1/N).$$

Since $\|a_N\|_N \geq |\omega_{\underline{M}_1, \underline{M}_2, N}(a_N)|$ and $\|a\|_N \underset{N\to\infty}{\longrightarrow} 0$ we find

$$\sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} \left(\frac{1}{2}\omega_{\underline{M}_1} + \frac{1}{2}\omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2}\right)^2 \left[I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}\right] = 0, \tag{39}$$

for all $\omega_{\underline{M}_1} \in S(B^{\underline{M}_1})$, $\omega_{\underline{M}_2} \in S(B^{\underline{M}_2})$ and $\omega_{\underline{M}_1 - \underline{M}_2} \in S(B^{\underline{M}_1 - \underline{M}_2})$. Choosing

$$\omega_{\underline{M}_1} = \omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2},$$

we have

$$\sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} (\omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2})^2 \left[I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}\right] = 0, \tag{40}$$

for all $\omega_{\underline{M}_2} \in S(B^{\underline{M}_2})$ and $\omega_{\underline{M}_1 - \underline{M}_2} \in S(B^{\underline{M}_1 - \underline{M}_2})$. This implies that in the general case, unfolding $(\omega_{\underline{M}_1} + \omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2})^2$ and using Eq. (40) we have

$$\sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} (\omega_{\underline{M}_1} \otimes \omega_{\underline{M}_1}) \left[I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}\right]$$
$$+ \sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} \left(\omega_{\underline{M}_1} \otimes \omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2}\right) \left[(I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)}) \otimes_\pi (I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)})\right] = 0, \tag{41}$$

for all $\omega_{\underline{M}_1} \in S(B^{\underline{M}_1})$, $\omega_{\underline{M}_2} \in S(B^{\underline{M}_2})$ and $\omega_{\underline{M}_1 - \underline{M}_2} \in S(B^{\underline{M}_1 - \underline{M}_2})$ while $a \otimes_\pi a' := a \otimes a' + a' \otimes a$. Equation (41) is now linear in $\omega_{\underline{M}_1 - \underline{M}_2} \otimes \omega_{\underline{M}_2}$. Since convex combinations of states in $S(B^{\underline{M}_1 - \underline{M}_2}) \otimes S(B^{\underline{M}_2})$ generate $S(B^{\underline{M}_1})$ we find that

$$\sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} (\omega_{\underline{M}_1} \otimes \omega_{\underline{M}_1}) \left[I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)}\right]$$
$$+ \sum_{\substack{(k_1,k_2): \\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1 k_2} \left(\omega_{\underline{M}_1} \otimes \omega'_{\underline{M}_1}\right) \left[(I^{\underline{M}_1 - M(k_1)} \otimes a_{(k_1)}) \otimes_\pi (I^{\underline{M}_1 - M(k_2)} \otimes a_{(k_2)})\right] = 0, \tag{42}$$

for all $\omega_{\underline{M}_1}, \omega'_{\underline{M}_1} \in S(B^{\underline{M}_1})$. Choosing $\omega_{\underline{M}_1} = \omega'_{\underline{M}_1}$ we find

$$\sum_{\substack{(k_1,k_2):\\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1k_2}(\omega_{\underline{M}_1} \otimes \omega_{\underline{M}_1})\Big[I^{\underline{M}_1-M(k_1)} \otimes a_{(k_1)} \otimes I^{\underline{M}_1-M(k_2)} \otimes a_{(k_2)}\Big] = 0,$$

out which Eq. (42) simplifies to

$$\sum_{\substack{(k_1,k_2):\\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1k_2}\big(\omega_{\underline{M}_1} \otimes \omega'_{\underline{M}_1}\big)\Big[(I^{\underline{M}_1-M(k_1)} \otimes a_{(k_1)}) \otimes_\pi (I^{\underline{M}_1-M(k_2)} \otimes a_{(k_2)})\Big] = 0, \tag{43}$$

for all $\omega_{\underline{M}_1}, \omega'_{\underline{M}_1} \in S(B^{\underline{M}_1})$.

The arbitrariness of $\omega_{\underline{M}_1}, \omega'_{\underline{M}_1} \in S(B^{\underline{M}_1})$ and the fact that any $\omega_{2\underline{M}_1} \in S(B^{2\underline{M}_1})$ can be written as a convex combination of product states in $\omega_{\underline{M}_1} \otimes \omega'_{\underline{M}_1}$ lead to

$$\sum_{\substack{(k_1,k_2):\\ \{M(k_1),M(k_2)\}=\{\underline{M}_1,\underline{M}_2\}}} c^{k_1k_2}(I^{\underline{M}_1-M(k_1)} \otimes a_{(k_1)}) \otimes_\pi (I^{\underline{M}_1-M(k_2)} \otimes a_{(k_2)}) = 0\,. \tag{44}$$

On account of the symmetry in k_1, k_2 in the sum, we may assume that $M(k_1) = \underline{M}_1$ and $M(k_2) = \underline{M}_2$ for all pairs (k_1, k_2). Equation (44) reduces to

$$\sum_{\substack{k_1:\ M(k_1)=\underline{M}_1\\ k_2:\ M(k_2)=\underline{M}_2}} c^{k_1k_2}a_{(k_1)} \otimes_\pi (I^{\underline{M}_1-\underline{M}_2} \otimes a_{(k_2)}) = 0\,. \tag{45}$$

Let $\omega_{\underline{M}_1} \in S(B^{\underline{M}_1})$ and $\omega_{\underline{M}_2} \in S(B^{\underline{M}_2})$ and let

$$\omega_{\underline{M}_1,\underline{M}_2} := \frac{1}{2}\omega_{\underline{M}_1} \otimes \tau^{\underline{M}_1-\underline{M}_2} \otimes \omega_{\underline{M}_2} + \frac{1}{2}\tau^{\underline{M}_1-\underline{M}_2} \otimes \omega_{\underline{M}_2} \otimes \omega_{\underline{M}_1}\,.$$

Then Eq. (45) leads to

$$\begin{aligned} 0 &= \omega_{\underline{M}_1,\underline{M}_2}\Bigg(\sum_{\substack{k_1:\ M(k_1)=\underline{M}_1\\ k_2:\ M(k_2)=\underline{M}_2}} c^{k_1k_2}a_{(k_1)} \otimes_\pi (I^{\underline{M}_1-\underline{M}_2} \otimes a_{(k_2)})\Bigg)\\ &= (\omega_{\underline{M}_1} \otimes \omega_{\underline{M}_2})\Bigg(\sum_{\substack{k_1:\ M(k_1)=\underline{M}_1\\ k_2:\ M(k_2)=\underline{M}_2}} c^{k_1k_2}a_{(k_1)} \otimes_\pi a_{(k_2)})\Bigg), \end{aligned}$$

with the convention that $\omega_{\underline{M}_\ell}(a_{(k)}) = 0$ if $M(k) \neq \underline{M}_\ell$. This shows that Eq. (45) implies

$$\sum_{\substack{k_1:\ M(k_1)=\underline{M}_1\\ k_2:\ M(k_2)=\underline{M}_2}} c^{k_1k_2}a_{(k_1)} \otimes_\pi \otimes a_{(k_2)} = 0\,. \tag{46}$$

We now observe that Eq. (46) is equivalent to

$$\sum_{\substack{k_1:\, M(k_1)=\underline{M}_1 \\ k_2:\, M(k_2)=\underline{M}_2}} c^{k_1k_2}\frac{1}{N}\sum_{j_1+j_2=N-\underline{M}_1-\underline{M}_2} \overline{\gamma}_N\big(I^{j_1}\otimes \overbrace{a_{(k_1)}\otimes I^{j_2}\otimes a}_{(k_2)}\big)=0,\quad \forall N\geq \underline{M}_1-\underline{M}_2\,. \tag{47}$$

Indeed, by direct inspection Eq. (46) implies that

$$\sum_{\substack{k_1:\, M(k_1)=\underline{M}_1 \\ k_2:\, M(k_2)=\underline{M}_2}} c^{k_1k_2} I^{j_1}\otimes \overbrace{a_{(k_1)}\otimes I^{j_2}\otimes a}_{(k_2)}=0,\qquad \forall j_1, j_2\in\mathbb{N}, \tag{48}$$

and thus it implies Eq. (47). Conversely, if Eq. (47) holds true then evaluation on the state $\tau^{\ell_1}\otimes\omega_{\underline{M}_1}\otimes\tau^{\ell_2}\otimes\omega_{\underline{M}_2}$ leads to

$$\begin{aligned}0&=(\tau^{\ell_1}\otimes\omega_{\underline{M}_1}\otimes\tau^{\ell_2}\otimes\omega_{\underline{M}_2})\Big(\textstyle\sum_{\substack{k_1:\, M(k_1)=\underline{M}_1 \\ k_2:\, M(k_2)=\underline{M}_2}} c^{k_1k_2}\frac{1}{N}\sum_{j_1+j_2=N-\underline{M}_1-\underline{M}_2}\overline{\gamma}_N\big(I^{j_1}\otimes \overbrace{a_{(k_1)}\otimes I^{j_2}\otimes a}_{(k_2)}\big)\Big)\\ &=\tfrac{1}{N}(\omega_{\underline{M}_1}\otimes\omega_{\underline{M}_2})\Big(\textstyle\sum_{\substack{k_1:\, M(k_1)=\underline{M}_1 \\ k_2:\, M(k_2)=\underline{M}_2}} c^{k_1k_2}\frac{1}{N}a_{(k_1)}\otimes_\pi a_{(k_2)}\Big),\end{aligned}$$

where ℓ_1, ℓ_2 are such that $\ell_1+\ell_2=N-\underline{M}_1-\underline{M}_2$ while $\omega_{\underline{M}_1}\in S(B^{\underline{M}_1})$ and $\omega_{\underline{M}_2}\in S(B^{\underline{M}_2})$ are arbitrary states. This implies Eq. (45).

By comparison with (34) we conclude that Eq. (47) is nothing but the sum of the terms in a_N whose pairs k_1, k_2 fulfils $\{M(k_1), M(k_2)\}=\{\underline{M}_1, \underline{M}_2\}$.

At this stage we may either argue that this is in contradiction with the definition of $\underline{M}_1, \underline{M}_2$—unless $a_N=0$—because $\min\limits_{\substack{k_1:\, M(k_1)\leq\underline{M}_1 \\ k_2:\, M(k_2)\leq\underline{M}_1}} \min\{M(k_1), M(k_2)\}>\underline{M}_2$. Alternatively we may consider the remaining contribution to a_N and argue again as above identifying new values $\underline{M}_1, \underline{M}_2, \overline{M}$. In either case we have $a_N=0$ for all $N\in\mathbb{N}$ as claimed. □

Remark 21. The notion of canonical representative applies also for symmetric sequences. Indeed, let $[\pi_N^M a_M]_N\in[B]_\pi^\infty$ where $a_M\in B_\pi^M$. As for γ-sequences one has $[\pi_N^M a_M]_N=[\pi_N^{M+K}(I^K\otimes a_M)]_N$ so that the π-sequence generating $[\pi_N^M a_M]_N$ is not uniquely determined. Nevertheless, since $a_M\in B_\pi^M$, one obtain the following unique decomposition:

$$\begin{aligned}a_M&=S_M(\widetilde{a}_0 I^M+I^{M-1}\otimes\widetilde{a}_1+\cdots+\widetilde{a}_M)\\ &=S_M\Big(\sum_{j=0}^M I^{M-j}\otimes\widetilde{a}_j\Big)\in S_M\Big(\bigoplus_{j=0}^M I^{M-j}\otimes\widetilde{B}_\pi^j\Big).\end{aligned}$$

With this decomposition at hand the canonical representative of $[\pi_N^M a_M]_N$ is defined by

$$\Big(\sum_{j=0}^M \pi_N^j\widetilde{a}_j\Big)_N.$$

This point of view is equivalent to the one adopted in [12].

3.3. The Poisson structure of $[B]_\gamma^\infty$. In this section we will endow $[B]_\gamma^\infty$ with a Poisson structure defined on $[\dot{B}]_\gamma^\infty$. Eventually we will discuss the deformation quantization of $[B]_\gamma$.

We recall that a **Poisson structure** over a C^*-algebra $\mathcal{A}$ is given by a bilinear map $\{\ ,\ \}\colon \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ defined on a dense $*$-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ which fulfils:

$$\{a, a'\} = -\{a', a\}, \qquad \{a, a'\}^* = \{a^*, a'^*\} \tag{49}$$

$$\{a, a'a''\} = \{a, a'\}a'' + a'\{a, a''\}, \tag{50}$$

$$\{a, \{a', a''\}\} = \{\{a, a'\}, a''\} + \{a', \{a, a''\}\}, \tag{51}$$

for all $a, a', a'' \in \mathcal{A}_0$.

Proposition 22. *Let* $\{\ ,\ \}_\gamma : [\dot{B}]_\gamma^\infty \times [\dot{B}]_\gamma^\infty \to [\dot{B}]_\gamma^\infty$ *be the bilinear map defined by*

$$\{[a_N]_N, [a'_N]_N\}_\gamma := [iN[a_N^{\mathrm{CAN}}, a_N'^{\mathrm{CAN}}]]_N, \tag{52}$$

where $(a_N^{\mathrm{CAN}})_N$ *denotes the canonical representative of* $[a_N]_N$*—cf. Definitions 16–19. Then* $\{,\ \}_\gamma$ *is a Poisson structure on* $[B]_\gamma^\infty$.

Proof. Notice that $\{,\ \}_\gamma$ fulfils condition (49) because so does the pointwise commmutator $i[,\ ]$.

The non-trivial part of the proof is to prove that $\{,\ \}_\gamma$ is well-defined, namely that $\{[a_N]_N, [a'_N]_N\}_\gamma$ is a well-defined element of $[\dot{B}]_\gamma^\infty$. Moreover, we also have to prove conditions (50)-(51): The latter do not follow from the properties of the commutator because Eq. (52) uses the canonical representative and in general $[a_N^{\mathrm{CAN}}, a_N^{\mathrm{CAN}}] \neq [a_N^{\mathrm{CAN}}, a_N^{\mathrm{CAN}}]^{\mathrm{CAN}}$. For these reasons we proceed in several steps:

$\boxed{\gamma}$ As a first step, we prove that $\{[a_N]_N, [a'_N]_N\}_\gamma \in [\dot{B}]_\gamma^\infty$ for the case of two equivalence classes of γ-sequences. Since the commutator is linear, on account of Remark 17-(ii) we may reduce to the case $[a_N]_N = [\overline{\gamma}_N^M a_M]_N$, $[a'_N]_N = [\overline{\gamma}_N^{M'} a_{M'}]_N$ for $a_M \in B_{\mathrm{IRR}}^M$ and $a_{M'} \in B_{\mathrm{IRR}}^{M'}$. In this latter case we find, for large enough N, say $N \geq 2(M + M')$,

$$\begin{aligned} iN[\overline{\gamma}_N^M a_M, \overline{\gamma}_N^{M'} a_{M'}] &= iN\gamma_N\left(\left[I^{N-M} \otimes a_M, \gamma_N(I^{N-M'} \otimes a_{M'})\right]\right) \\ &= i\gamma_N\left(\sum_{\substack{j\in\{0,\ldots,M-1\}\\ \cup\{N-M'-1,\ldots,N-1\}}} \left[I^{N-M} \otimes a_M, \gamma_N^j(I^{N-M'} \otimes a_{M'})\right]\right) \\ &= i\gamma_N\left(I^{N-M-2M'} \otimes \sum_{j=0}^{M+M'} \left[I^{M'} \otimes a_M \otimes I^{M'}, \gamma_{M+2M'}^j(I^{M'+M} \otimes a_{M'})\right]\right) \\ &=: \overline{\gamma}_N^{M+2M'} a_{M+2M'}, \end{aligned}$$

which implies

$$\{[\overline{\gamma}_N^M a_M]_N, [\overline{\gamma}_N^{M'} a_{M'}]_N\}_\gamma = [iN[\overline{\gamma}_N^M a_M, \overline{\gamma}_N^{M'} a_{M'}]]_N = [\overline{\gamma}_N^{M+2M'} a_{M+2M'}]_N \in [\dot{B}]_\gamma^\infty.$$

This proves that the Poisson bracket between $[\overline{\gamma}_N^M a_M]_N$ and $[\overline{\gamma}_N^{M'} a_{M'}]_N$ is an element of $[\dot{B}]_\gamma^\infty$.

$[\dot{B}]^{\infty}_{\gamma}$ We now consider the general case of $[a_N]_N, [a'_N]_N \in [\dot{B}]^{\infty}_{\gamma}$. Using again linearity and Remark 17-(ii) we may restrict to case

$$[a_N]_N = [\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}]_N, \qquad [a'_N]_N = [\overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}]_N,$$

where $a_M, a'_M \in B^M_{\mathrm{IRR}}$ for all M's and $\ell, \ell' \in \mathbb{N}$ are arbitrary but fixed. We observe that Eq. (18)—cf. Proposition 6—leads to

$$(a_N^{\mathrm{CAN}})_N = (\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell})_N + R_N, \quad (a_N'^{\mathrm{CAN}})_N = (\overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}})_N + R'_N,$$

where $\|R_N\|_N = O(1/N) = \|R'_N\|_N$. This implies

$$\begin{aligned}[a_N^{\mathrm{CAN}}, a_N'^{\mathrm{CAN}}] = {} & [\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}] \\ & + [R_N, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}] + [\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}, R'_N] \\ & + [R_N, R'_N]\,.\end{aligned}$$

At this stage we observe that

$$\big[iN[\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}]\big]_N \in [\dot{B}]^{\infty}_{\gamma}\,.$$

Indeed, this is due to the identity

$$[a_N, a'_N a''_N] = [a_N, a'_N] a''_N + a'_N [a_N, a''_N],$$

together with the fact that the result holds true for $\ell = \ell' = 1$. Moreover, we have $\|N[R_N, R'_N]\|_N = O(1/N)$ so that it remains to discuss the term

$$\big\|N[R_N, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}]\big\|_N = O(1/N),$$

where the latter estimate is due to Remark 7-(ii). Overall we have shown that

$$iN[a_N^{\mathrm{CAN}}, a_N'^{\mathrm{CAN}}] = iN[\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}] + R''_N,$$

where $\|R''_N\|_N = O(1/N)$ and by direct inspection fulfils Eq. (20). This implies in particular that

$$\big[iN[a_N^{\mathrm{CAN}}, a_N'^{\mathrm{CAN}}]\big]_N = \big[iN[\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}, \overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}]\big]_N \in [\dot{B}]^{\infty}_{\gamma}\,.$$

so that $\{,\ \}_\gamma$ is well-defined.

(50) − (51) By proceeding in a completely analogous way one also proves conditions (50)–(51). Indeed, considering without loss of generality

$$[a_N]_N = [\overline{\gamma}_N^{M_1} a_{M_1} ... \overline{\gamma}_N^{M_\ell} a_{M_\ell}]_N, \qquad [a'_N]_N = [\overline{\gamma}_N^{M'_1} a_{M'_1} ... \overline{\gamma}_N^{M'_{\ell'}} a_{M'_{\ell'}}]_N,$$

$$[a''_N]_N = [\overline{\gamma}_{N^M_{1''}} a_{M_{1''}} ... \overline{\gamma}_N^{M''_{\ell''}} a_{M''_{\ell''}}]_N,$$

we find

$$
\begin{aligned}
&(iN)^2[a_N^{\mathrm{CAN}}, [a_N'^{\mathrm{CAN}}, a_N''^{\mathrm{CAN}}]] \\
&= iN[a_N^{\mathrm{CAN}}, iN[\overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'}, \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}]] + iN[a_N^{\mathrm{CAN}}, R_N] \\
&= iN[\overline{\gamma}_N^{M_1} a_{M_1} \ldots \overline{\gamma}_N^{M_\ell} a_{M_\ell}, iN[\overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'}, \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}]] \\
&+ iN[R_N', iN[\overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'}, \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}]] + iN[a_N^{\mathrm{CAN}}, R_N].
\end{aligned}
$$

The first contribution fulfils (51). With an argument similar in spirit to Remark 7-(ii), the second contribution can be estimated by

$$
\|[R_N', iN[\overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'}, \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}]]\|_N = O(1/N^2)\,.
$$

Finally $\|N[a_N^{\mathrm{CAN}}, R_N]\|_N = O(1/N)$ because of Remark 7-(ii) so that

$$
\begin{aligned}
&(iN)^2[a_N^{\mathrm{CAN}}, [a_N'^{\mathrm{CAN}}, a_N''^{\mathrm{CAN}}]] \\
&= iN[\overline{\gamma}_N^{M_1} a_{M_1} \ldots \overline{\gamma}_N^{M_\ell} a_{M_\ell}, iN[\overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'}, \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}]] + R_N''',
\end{aligned}
$$

with $\|R_N'''\|_N = O(1/N)$. This proves condition 51 for $\{,\ \}_\gamma$.
By proceeding in a similar fashion we also have

$$
\begin{aligned}
&iN[a_N^{\mathrm{CAN}}, a_N'^{\mathrm{CAN}} a_N''^{\mathrm{CAN}}] \\
&= iN[\overline{\gamma}_N^{M_1} a_{M_1} \ldots \overline{\gamma}_N^{M_\ell} a_{M_\ell}, \overline{\gamma}_N^{M_1'} a_{M_1'} \ldots \overline{\gamma}_N^{M_{\ell'}'} a_{M_{\ell'}'} \overline{\gamma}_N^{M_1''} a_{M_1''} \ldots \overline{\gamma}_N^{M_{\ell''}''} a_{M_{\ell''}''}] + R_N''',
\end{aligned}
$$

where $\|R_N'''\|_N = O(1/N)$ while the first contribution fulfils (50). This proves condition (50) for $\{,\ \}_\gamma$. □

Remark 23. The proof of Proposition 22 shows that, if $a_M \in B_{\mathrm{IRR}}^M$ and $a_{M'} \in B_{\mathrm{IRR}}^{M'}$ then

$$
\{[\overline{\gamma}_N^M a_M]_N, [\overline{\gamma}_N^{M'} a_{M'}]_N\}_\gamma = [\overline{\gamma}_N^{M+2M'} a_{M+2M'}]_N,
$$

where $a_{M+2M'} \in B^{M+2M'}$ is not $\widetilde{B}$-irreducible in general.

At last, we can finally state and prove the main theorem of this paper.

Theorem 24. *Let $[B]_\gamma \subset \prod_{N\in\overline{\mathbb{N}}}[B]_\gamma^N$ be the continuous bundle of C^*-algebras defined as per Proposition 12. For $K \in \overline{\mathbb{N}}$ let $Q_K : [\dot{B}]_\gamma^\infty \to [B]_\gamma^K$ be the linear map defined by*

$$
Q_K([a_N]_N) := \begin{cases} a_K^{\mathrm{CAN}} & K \in \mathbb{N} \\ [a_N]_N & K = \infty \end{cases} \tag{53}
$$

where $(a_N^{\mathrm{CAN}})_N$ is the canonical representative of $[a_N]_N$ as per Definitions 16–19. Then the family of maps $\{Q_N\}_{N\in\overline{\mathbb{N}}}$ defines a strict deformation quantization of $[B]_\gamma^\infty$.

Proof. Notice that Q_N is well-defined for all $N \in \overline{\mathbb{N}}$ on account of the uniqueness of the canonical representative—cf. Propositions 18–20.

With reference to Sect. 1 we have

$$[B]_\gamma \leftrightarrow \prod_{N \in \overline{\mathbb{N}}} \mathcal{A}_N, \qquad [B]_\gamma^N \leftrightarrow \mathcal{A}_N, \qquad [\dot{B}]_\gamma^\infty \leftrightarrow \widetilde{\mathcal{A}}_\infty, \qquad [B]_\gamma^\infty \leftrightarrow \mathcal{A}_\infty .$$

We will now prove conditions 3(a)–3(b)–3(c).

3(a) Per definition we have $Q_\infty := \mathrm{Id}_{[\dot{B}]_\gamma^\infty}$ as well as $Q_N([a_N]_N)^* = Q_N([a_N^*]_N)$ for all $[a_N]_N \in [\dot{B}]_\gamma^\infty$. Moreover, Eq. (53) defines an element in the space $[\dot{B}]_\gamma$—cf. Eq. (24)—and thus a continuous section of $[B]_\gamma$ as per Proposition 12.

3(b) By direct inspection one has

$$[iK[Q_K([a_N]_N), Q_K([a'_N]_N)]]_K = [iK[a_K^{\mathrm{CAN}}, a_K'^{\mathrm{CAN}}]]_K = \{[a_N]_N, [a'_N]_N\}_\gamma,$$

which implies Eq. (1). Notice that, on account of Remark 23, in general

$$Q_K(\{[a_N]_N, [a'_N]_N\}_\gamma) \neq iK[a_K^{\mathrm{CAN}}, a_K'^{\mathrm{CAN}}],$$

despite the fact that the equivalence classes of the associated sequences are equal.

3(c) By direct inspection one finds that $Q_M([\dot{B}]_\gamma^\infty) = B_\gamma^M$ for all $M \in \mathbb{N}$. Indeed, by proceeding as in the proof of Proposition 12, let $a_M \in B_\gamma^M$, for $M \in \mathbb{N}$. Then we have $a_M = \overline{\gamma}_M(a_M) = \sum_{j=0}^M \overline{\gamma}_M^j a_j$ for $a_j \in B_{\mathrm{IRR}}^j$—cf. Eq. (29). This implies that

$$a_M = \overline{\gamma}_M(a_M) = \sum_{j=0}^M \overline{\gamma}_M^j a_j = Q_M\left(\left[\sum_{j=0}^M \overline{\gamma}_N^j a_j\right]_N\right),$$

thus proving that $Q_M([\dot{B}]_\gamma^\infty) = B_\gamma^M$. □

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Declarations

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A. Characterization of $\widetilde{B}$-Irreducible Elements

This section is devoted to characterize the set B^M_{IRR} of $\widetilde{B}$-irreducible elements in B^M—cf. Definition 14. To this avail, we introduce the following convenient family of linear maps.

Definition 25. Let $M, \ell \in \mathbb{N}$, $\ell \geq 2$, and $j_1, \ldots, j_{\ell-1} \in \mathbb{N}$ such that $j_1 + \cdots + j_{\ell-1} = M - \ell$. We denote by $\iota_M^{j_1 \ldots j_{\ell-1}} \colon \widetilde{B}^\ell \to B^M$ the linear map defined by

$$\iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{a}_\ell) := \sum_{k_1, \ldots, k_\ell} c^{k_1 \ldots k_\ell} b_{k_1} \otimes I^{j_1} \otimes \cdots \otimes I^{j_{\ell-1}} \otimes b_{k_\ell}, \tag{54}$$

where $I, b_1, \ldots, b_{\kappa^2-1}$ is a basis of B fulfilling (4) and $\widetilde{a}_\ell = \sum_{k_1, \ldots, k_\ell} c^{k_1 \ldots k_\ell} b_{k_1} \otimes \cdots \otimes b_{k_\ell}$, the sum over $k_1, \ldots, k_\ell$ being finite.

Remark 26. (i) If $\ell = M$ one has $\iota_M^{j_1 \ldots j_M}(\widetilde{a}_M) = \widetilde{a}_M$. Moreover, by direct inspection $\iota_M^{j_1 \ldots j_{\ell-1}}$ does not depend on the chosen basis $I, b_1, \ldots, b_{\kappa^2-1}$.

(ii) On account of Definition 14 we have $\iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{B}^\ell) \subseteq B^M_{\mathrm{IRR}}$. Moreover, $\iota_M^{j_1 \ldots j_{\ell-1}}$ is injective. Indeed, if $\iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{a}_\ell) = 0$ then for all $\eta_1, \ldots, \eta_\ell \in S(B)$ we have

$$0 = [\eta_1 \otimes \tau^{j_1} \otimes \cdots \otimes \tau^{j_{\ell-1}} \otimes \eta_\ell](\iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{a}_\ell)) = (\eta_1 \otimes \cdots \otimes \eta_\ell)(\widetilde{a}_\ell)\,.$$

The arbitrariness of $\eta_1, \ldots, \eta_\ell$ entails $\omega_\ell(\widetilde{a}_\ell) = 0$ for all $\omega_\ell \in S(B^\ell)$, therefore, $\widetilde{a}_\ell = 0$.

Let $I, b_1, \ldots, b_{\kappa^2-1}$ be a basis of B fulfilling (4) and let $a_M \in B^M_{\mathrm{IRR}}$, $M \geq 2$. By considering Eq. (28) for $a_M \in B^M_{\mathrm{IRR}}$ we find

$$\begin{aligned} a_M = \iota_M^{M-2}(\widetilde{a}_2) + \sum_{j_1+j_2=M-3} \iota_M^{j_1 j_2}(\widetilde{a}_{3|j_1 j_2}) \\ + \cdots + \sum_{j_1+\cdots+j_{\ell-1}=M-\ell} \iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{a}_{\ell|j_1 \ldots j_{\ell-1}}) + \cdots + \widetilde{a}_M, \end{aligned} \tag{55}$$

where $\widetilde{a}_{\ell|j_1 \ldots j_{\ell-1}} \in \widetilde{B}^\ell$ for all $\ell, j_1, \ldots, j_{\ell-1}$.

Equation (55) provides a description of B^M_{IRR} in terms of "$\widetilde{B}$-components". To this avail we consider the vector space

$$\widetilde{\boldsymbol{B}}^M := \widetilde{B}^2 \oplus \bigotimes_{j_1+j_2=M-3} \widetilde{B}^3 \oplus \cdots \oplus \bigotimes_{j_1+\cdots+j_{\ell-1}=M-\ell} \widetilde{B}^\ell \oplus \cdots \oplus \widetilde{B}^M, \tag{56}$$

where $\widetilde{\boldsymbol{B}}^0 = \mathbb{C}$ and $\widetilde{\boldsymbol{B}}^1 = \widetilde{B}$. We then define the linear map

$$\Phi_M \colon \widetilde{\boldsymbol{B}}^M \to B^M \; \Phi_M(\widetilde{\boldsymbol{a}}_M) := \begin{cases} a_0 & M = 0 \\ \widetilde{a}_1 & M = 1 \\ \iota_M^{M-2}(\widetilde{a}_2) + \cdots & \\ + \sum_{j_1+\cdots+j_{\ell-1}=M-\ell} \iota_M^{j_1 \ldots j_{\ell-1}}(\widetilde{a}_{\ell|j_1 \ldots j_{\ell-1}}) + \cdots + \widetilde{a}_M & M \geq 2 \end{cases}, \tag{57}$$

where $\widetilde{\boldsymbol{a}}_M \in \widetilde{\boldsymbol{B}}^M$ is given by

$$\widetilde{\boldsymbol{a}}_M = \begin{cases} a_0 & M = 0 \\ \widetilde{a}_1 & M = 1 \\ \widetilde{a}_2 \oplus \bigotimes_{j_1+j_2=M-3} \widetilde{a}_{3|j_1 j_2} \oplus \cdots \oplus \bigotimes_{j_1+\cdots+j_{\ell-1}=M-\ell} \widetilde{a}_{\ell|j_1+\cdots+j_{\ell-1}} \oplus \cdots \oplus \widetilde{a}_M & M \geq 2 \end{cases} . \tag{58}$$

Equation (55) can be rephrased by saying that for all $a_M \in B^M_{\text{IRR}}$ there exists $\widetilde{\boldsymbol{a}}_M \in \widetilde{\boldsymbol{B}}^M$ such that $\Phi_M(\widetilde{\boldsymbol{a}}_M) = a_M$. The following lemma shows that Φ_M is in fact an isomorphism, proving that $B^M_{\text{IRR}} \simeq \widetilde{\boldsymbol{B}}^M$.

Lemma 27. *For all $M \in \mathbb{N}$, the map $\Phi_M : \widetilde{\boldsymbol{B}}^M \to B^M_{\text{IRR}}$ is an isomorphism.*

Proof. There is nothing to prove for $M \in \{0, 1\}$, therefore, we assume $M \geq 2$.
From Eqs. (55), (57), we have that Φ_M is linear and surjective: Thus, it remains to prove that $\Phi(\widetilde{\boldsymbol{a}}_M) = 0$ implies $\widetilde{\boldsymbol{a}}_M = 0$. We now prove that, if $\Phi(\widetilde{\boldsymbol{a}}_M) = 0$, then all components of $\widetilde{\boldsymbol{a}}_M$ appearing in Eq. (58) vanish.
To this avail let $\eta_1, \eta_2 \in S(B)$. By direct inspection we have

$$0 = (\eta_1 \otimes \tau^{M-2} \otimes \eta_2)[\Phi(\widetilde{\boldsymbol{a}}_M)] = (\eta_1 \otimes \eta_2)(\widetilde{a}_2) .$$

Notice that no other term from $\Phi(\widetilde{\boldsymbol{a}}_M)$ provides a non-vanishing contribution because $\tau(\widetilde{B}) = \{0\}$. The arbitrariness of $\eta_1, \eta_2 \in S(B)$ leads to $\omega_2(\widetilde{a}_2) = 0$ for all $\omega_2 \in S(B^2)$ and thus $\widetilde{a}_2 = 0$.
We now proceed by proving that $\widetilde{a}_{3|j_1 j_2} = 0$ for all $j_1 + j_2 = M - 3$. To this avail let j_1, j_2 be such that $j_1 + j_2 = M - 3$ and let $\eta_1, \eta_2, \eta_3 \in S(B)$. Since we already proved that $\widetilde{a}_2 = 0$ it follows that

$$0 = (\eta_1 \otimes \tau^{j_1} \otimes \eta_2 \otimes \tau^{j_2} \otimes \eta_3)[\Phi(\widetilde{\boldsymbol{a}}_M)] = (\eta_1 \otimes \eta_2 \otimes \eta_3)(\widetilde{a}_{3|j_1 j_2}) .$$

Once again, the arbitrariness of $\eta_1, \eta_2, \eta_3 \in S(B)$ (as well as the one of j_1, j_2) leads to $\widetilde{a}_{3|j_1 j_2} = 0$ for all $j_1 + j_2 = M - 3$. Proceeding by induction we find $\widetilde{a}_{\ell|j_1 \ldots j_{\ell-1}} = 0$ for all $j_1 + \cdots + j_{\ell-1} = M - \ell$. Thus $\widetilde{\boldsymbol{a}}_M = 0$. □

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