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The Quantization of Proca Fields on Globally Hyperbolic Spacetimes: Hadamard States and Møller Operators

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Abstract. This paper deals with several issues concerning the algebraic quantization of the real Proca field in a globally hyperbolic spacetime and the definition and existence of Hadamard states for that field. In particular, extending previous work, we construct the so-called Møller $*$-isomorphism between the algebras of Proca observables on paracausally related spacetimes, proving that the pullback of these isomorphisms preserves the Hadamard property of corresponding quasifree states defined on the two spacetimes. Then, we pull back a natural Hadamard state constructed on ultrastatic spacetimes of bounded geometry, along this $*$-isomorphism, to obtain an Hadamard state on a general globally hyperbolic spacetime. We conclude the paper, by comparing the definition of an Hadamard state, here given in terms of wavefront set, with the one proposed by Fewster and Pfenning, which makes use of a supplementary Klein–Gordon Hadamard form. We establish an (almost) complete equivalence of the two definitions.

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Contents

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1. Introduction

The (algebraic) quantization of a quantum field propagating in a globally hyperbolic curved spacetime (M, g) [6,54] and the definition of meaningful quantum states have been and continue to be at the forefront of scientific research. Linearized theories are the first step of all perturbative procedures, so the definition of physically meaningful states for linearized field equations is an important task.

Gaussian, also known as quasifree, states $\omega : \mathcal{A} \to \mathbb{C}$ on the relevant CCR or CAR unital $*$-algebra $\mathcal{A}$ of observables of a given quantum field are an important family of (algebraic) states [38]. They are completely defined by assigning the two-point function, a bidistribution $\omega_2(x, y)$ on the sections used to smear the field operator.

A crucial physical requirement on ω is the so-called *Hadamard condition*, which is needed, in particular, for defining locally covariant renormalization procedures of Wick polynomials [16,38] and for the mathematical formulation of locally covariant perturbative renormalization in quantum field theory [50].

1.1. Generalized Klein–Gordon Vector Fields

All the notations and conventions used in this section to briefly summarize our results will be defined precisely later. For a charged (i.e., complex) Klein–Gordon field A, possibly vector-valued, the construction of Hadamard states amounts to finding distributional bisolutions $\Lambda_2^{\pm}(x,y)$ of the Klein–Gordon equation $\mathsf{N}A = 0$ describing the two-point functions[1]

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}^*(\mathfrak{f}')) = \int_{\mathsf{M}\times\mathsf{M}} \Lambda_2^{+}(x,y)_{cd}\gamma^{ca}(x)\gamma^{db}(y)\overline{\mathfrak{f}_a(x)}\mathfrak{f}'_b(y)\mathrm{vol}_g \otimes \mathrm{vol}_g =: \Lambda_2^{+}(\bar{\mathfrak{f}},\mathfrak{f}')\,,$$

and

$$\omega(\hat{\mathfrak{a}}^*(\mathfrak{f}')\hat{\mathfrak{a}}(\mathfrak{f})) = \int_{\mathsf{M}\times\mathsf{M}} \Lambda_2^{-}(x,y)_{cd}\gamma^{ca}(x)\gamma^{db}(y)\overline{\mathfrak{f}'_a(x)}\mathfrak{f}_b(y)\mathrm{vol}_g \otimes \mathrm{vol}_g =: \Lambda_2^{-}(\mathfrak{f},\bar{\mathfrak{f}}')\,.$$

Above, the generators of the CCR $*$-algebra of the Proca field $\hat{\mathfrak{a}}(\mathfrak{f})$ and $\hat{\mathfrak{a}}^*(\mathfrak{f}') = \hat{\mathfrak{a}}(\mathfrak{f})^*$ are the (algebraic) field operators smeared with smooth compactly supported complex sections $\mathfrak{f},\mathfrak{f}'$ of the relevant *complex* vector bundle $\mathsf{E} \to \mathsf{M}$. That bundle is equipped with a non-degenerate Hermitian[2] fiberwise scalar product (not necessarily positive) γ. In case of the standard complex vector Klein–Gordon field over (M, g) constructed out the 1-form Hodge D'Alembertian or the Levi-Civita vector D'Alembertian, the vector bundle E is the one of smooth 1-forms $\mathsf{T}^*\mathsf{M}_{\mathbb{C}} := \mathsf{T}^*\mathsf{M} + i\mathsf{T}^*\mathsf{M}$ and the Hermitian scalar product γ is the indefinite one induced by the metric g in $\mathsf{T}^*\mathsf{M}_{\mathbb{C}}$, i.e., $\gamma = g^\sharp$. In the general case, a *Klein–Gordon operator* N is by definition a second-order operator on the smooth sections of E which is *normally hyperbolic* [2,3]: its principal symbol σ_{N} satisfies

$$\sigma_{\mathsf{N}}(\xi) = -g^\sharp(\xi,\xi)\,\mathrm{Id}_{\mathsf{E}} \quad \text{for all } \xi \in \mathsf{T}^*\mathsf{M}, \text{where } \mathrm{Id}_{\mathsf{E}} \text{ is the identity automorphism of } \mathsf{E}.$$

N is also required to be *formally self-adjoint* with respect to the Hermitian scalar product (generally non-positive!) induced on the space of complex sections $\mathfrak{f}$ by γ and the volume form vol_g,

$$(\mathfrak{f}|\mathfrak{g}) := \int_{\mathsf{M}} \overline{\mathfrak{f}_a}(x)\gamma^{ab}(x)\mathfrak{g}_b(x)\mathrm{vol}_g(x)\,.$$

The scalar complex Klein–Gordon field is encompassed by simply taking $\mathbb{C}$ as canonical fiber of E and using the trivial *positive* scalar product.

The requirements on the bidistributions $\Lambda_2^{\pm}$ are, where G_{N} is the *causal propagator* of N,

(1) $\mathsf{N}_x\Lambda_2^{\pm}(x,y) = \Lambda_2^{\pm}(x,y)\mathsf{N}_y = 0$ and $\Lambda_2^{+} - \Lambda_2^{-} = -i\mathsf{G}_{\mathsf{N}}$;

(2) $\Lambda_2^{\pm}(\bar{\mathfrak{f}},\mathfrak{f}) \geqslant 0\,,\quad$ where $\Lambda^{\pm}(\bar{\mathfrak{f}},\mathfrak{f}) = 0$

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[1]We use throughout the convention of summation over the repeated indices.

[2]In this work to be a Hermitian or real scalar product does not include the positivity condition, though it is always assumed to be non-degenerate.

implies $\mathfrak{f} = \mathsf{N}\mathfrak{g}$ for a compactly supported section $\mathfrak{g}$;

(3) $WF(\Lambda_2^{\pm}) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \triangleright 0\}$.

The second part of (1) corresponds to the *canonical commutations relations*, the first part is the "on-shell" condition, while condition (2) is the *positivity* requirement on two functions. Then, the *Gelfand–Naimark–Segal construction* gives rise to a $*$-representation of $\mathcal{A}_g$ in terms of densely defined operators in a Hilbert space which, as a consequence of the above requirements (1) and (2) and the Wick rule, is a Fock space. Here ω is the expectation value referred to vacuum state and the action of the image of the representation on the vacuum state produces the dense invariant domain of the representation itself. Requirement (3) is the celebrated *Hadamard condition* (also known as the *microlocal spectrum condition*) which ensures the correct short-distance behavior of the n-point functions of the field. This condition has a long history which can be traced back to [20], passing to [39] and [48,49] (see [38] for a review). It plays a crucial role in various contexts of quantum field theory in curved spacetime. In particular, but not only, in perturbative renormalization and semiclassical quantum gravity. More recently, Gérard and Wrochna in [23,25], proved that condition (1)–(3) can be controlled at the same time by using methods of pseudodifferential calculus in spacetimes *of bounded geometry* (see also the subsequent papers [26–30]).

When dealing with *real* quantum fields, as in this work, for instance the Klein–Gordon real vector field A, a single bidistribution $\omega_2(x, y)$ is sufficient to define a quasifree state ω:

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{f}')) = \int_{\mathsf{M}\times\mathsf{M}} \omega_2(x, y)_{cd}\gamma^{ca}(x)\gamma^{db}(y)\mathfrak{f}_a(x)\mathfrak{f}'_b(y)\mathrm{vol}_g \otimes \mathrm{vol}_g$$

where $\hat{\mathfrak{a}}(\mathfrak{f}) = \hat{\mathfrak{a}}(\mathfrak{f})^*$ is the (algebraic) field operators smeared with smooth *real* compactly supported sections $\mathfrak{f}$ of a relevant *real* vector bundle $\mathsf{E} \to \mathsf{M}$, equipped with a fiberwise real symmetric non-degenerate (but not necessarily positive) scalar product γ. As before, a *Klein–Gordon operator* N is by definition a second-order differential operator on the smooth sections of E which is normally hyperbolic (same definition as for the complex case) and formally self-adjoint with respect to the real symmetric scalar product (not necessarily positive)

$$(\mathfrak{f}|\mathfrak{g}) := \int_{\mathsf{M}} \mathfrak{f}_a(x)\gamma^{ab}(x)\mathfrak{g}_b(x)\mathrm{vol}_g(x) .$$

In the case of the standard real vector Klein–Gordon field (constructed out of the Hodge D'Alembertian or the Levi-Civita D'Alembertian), the bundle is exactly $\mathsf{T}^*\mathsf{M}$, equipped with a real symmetric non-degenerate but indefinite fiberwise scalar product induced by the metric g on $\mathsf{T}^*\mathsf{M}$, namely $\gamma = g^\sharp$. The theory of the scalar real Klein–Gordon field is encompassed simply by taking $\mathbb{R}$ as canonical fiber of E with trivial positive scalar product.

In the real case, defining the symmetric bilinear form $\mu(\mathfrak{f},\mathfrak{f}') := \frac{1}{2}(\omega_2(\mathfrak{f},\mathfrak{f}')+\omega_2(\mathfrak{f}',\mathfrak{f}))$, conditions (1)–(3) are replaced by

(1)′ $\mathsf{N}_x\omega_2(x,y) = \omega_2(x,y)\mathsf{N}_y = 0$ and $\omega_2(\mathfrak{f},\mathfrak{f}') - \omega_2(\mathfrak{f}',\mathfrak{f}) = i\mathsf{G}_\mathsf{N}(\mathfrak{f},\mathfrak{f}')$;

(2)′ $\mu_2(\mathfrak{f},\mathfrak{f}) \geqslant 0$ where $\mu(\mathfrak{f},\mathfrak{f}) = 0$
implies $\mathfrak{f} = \mathsf{N}\mathfrak{g}$ for a compactly supported section $\mathfrak{g}$;

(3)′ $|\mathsf{G}_\mathsf{N}(\mathfrak{f},\mathfrak{f}')|^2 \leqslant 4\mu(\mathfrak{f},\mathfrak{f})\,\mu(\mathfrak{f}',\mathfrak{f}')$;

(4)′ $WF(\omega_2^\pm) = \{(x,k_x;y,-k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x,k_x) \sim_\parallel (y,k_y), k_x \triangleright 0\}$.

The apparently new continuity condition (3)′ for the real case is actually embodied in the positivity condition (2) for the complex case [21]. As a matter of fact, (2)′ and (3)′ together give rise to positivity of the whole state ω on $\mathcal{A}_g$ induced by ω_2 in the real case. Once again, the GNS construction gives rise to a representation of the (complex) unital $*$-algebra $\mathcal{A}_g$ generated by the field operators $\hat{\mathfrak{a}}(\mathfrak{f})$ exactly as in the complex case.

Since the Klein–Gordon equations are normally hyperbolic, not only they are *Green hyperbolic* so that the Green operators $\mathsf{G}^\pm_\mathsf{P}$ and the causal propagator $\mathsf{G}_P = \mathsf{G}^+_P - \mathsf{G}^-_P$ can be therefore defined, but *the Cauchy problem* is also *automatically* well posed [2,3]. An important implication of this fact is that the two-point function of a quasifree state can be defined as a Hermitian or real bilinear form—in the complex and real case, respectively—on the Cauchy data of solutions of the Klein–Gordon equation (e.g., see [46]). We follow this route in the present paper and, to this end, we will translate (1)–(3) and (1)′–(4)′ in the language of Cauchy data.

1.2. Issues with the Quantization of the Proca Field

Most of the quantum theories are described by *Green hyperbolic operators* [2,3], as Klein–Gordon operators N discussed above or the *Proca operator* [14,53], studied in this work,

$$\mathsf{P} = \delta d + m^2$$

acting on smooth 1-forms $A \in \Omega^1(\mathsf{M})$ and where $m^2 > 0$ is a constant. These operators are usually formally self-adjoint w.r.t. a (Hermitian or real symmetric) scalar product induced by the analog γ on the fibers of the relevant vector bundle. In general γ is *not positive definite*. Very common and physical examples are: the *standard vector* Klein–Gordon field, the Proca field, the Maxwell field, more generally, the Yang–Mills field and also the linearized gravity. Referring to the Proca, and in general all 1-form fields, we have that $\gamma = g^\sharp$ is the inverse (indefinite!) Lorentzian metric of the spacetime (M, g).

Unfortunately, in those situations, the Hadamard condition (4) and (5)′ are in conflict with the positivity of states, respectively, (3) and (2)′–(3)′. It is known that for a vectorial Klein–Gordon operator that is formally self-adjoint w.r.t. an *indefinite Hermitian/real symmetric scalar product*, the existence of quasifree Hadamard states is forbidden (see the comment after [51, Proposition 5.6] and [27, Section 6.3]).

The case of a (real) Proca field seems to be even more complicated at first glance. In fact, on the one hand differently from the Klein–Gordon operator, the Proca operator is not even *normally hyperbolic* and this makes more difficult (but not impossible) the proof of the well-posedness of the Cauchy problem, in particular. On the other hand, similarly to the case of the vectorial Klein–Gordon theory, the Proca theory deals with an indefinite fiberwise scalar product. Actually, as we shall see in the rest of the work, *these two apparent drawbacks cooperate to permit the existence of quasifree Hadamard states*. Positivity of the two-point function ω_2 is restored when dealing with a *constrained* space of Cauchy conditions that make well-posed the Cauchy problem.

In the present paper, we study the existence of quasifree Hadamard states for the real Proca field on a general globally hyperbolic spacetime. A definition of Hadamard states for the Proca field was introduced by Fewster and Pfenning in [14], to study *quantum energy inequalities*, with a definition more involved than the one based on conditions (3) and $(4)'$ above. They also managed to prove that such states exist in globally hyperbolic spacetimes whose Cauchy surfaces are compact.

Differently from Fewster–Pfenning's definition, here we adopt a definition of Hadamard state which directly relies on conditions (3) and $(4)'$ above and we consider a generic globally hyperbolic spacetime. At the end of the work, we actually prove that the two definitions of Hadamard states are substantially equivalent.

Before establishing that equivalence, using the technology of the Møller operators we introduced in [46] for normally hyperbolic operators, and here extended to the Proca field, we prove the existence of quasifree Hadamard states in every globally hyperbolic spacetime, also in the case in which their Cauchy hypersurfaces are not compact.

As a matter of fact, it is enough to focus our attention on *ultrastatic* spacetimes of *bounded geometry*. In this class of spacetimes, we directly work at the level of initial data for the Proca equation and we establish the following, also by taking advantage of some technical results of spectral theory applied to *elliptic Hilbert complexes* [4].

1. The initial data of the Proca equations are a subspace C_Σ of the initial data of *a couple* of Klein–Gordon equations, one scalar and the other vectorial, however, both defined on bundles with fiberwise *positive* real symmetric scalar product;
2. The difference of a pair of certain Hadamard two-point functions for two above-mentioned Klein–Gordon fields becomes positive once that its arguments are restricted to C_Σ. There, it defines a two-point function ω_2 for a quasifree state ω of the Proca field;
3. ω is also Hadamard since it is the difference of two two-point functions of Klein–Gordon fields which are Hadamard. They are Hadamard in view of known results of microlocal analysis of pseudodifferential operators

on Cauchy surfaces of bounded geometry, for more details the interested reader can refer to [21].

Every field theory defined on a globally hyperbolic spacetime (M, g) is connected to one defined on an ultrastatic spacetime of bounded geometry $(\mathbb{R} \times \Sigma, -dt^2 + h)$ through a Møller operator: the associated Møller $*$-isomorphism between the algebras of Proca observables preserves the Hadamard condition. We therefore conclude that every globally hyperbolic spacetime (M, g) admits a Hadamard state for the Proca field. This state is nothing but the Hadamard state on $(\mathbb{R} \times \Sigma, -dt^2 + h)$ pulled back to (M, g) by the Møller $*$-isomorphism.

One novelty of this paper is in particular a direct control of the positivity of the two-point functions, obtained by spectral calculus of elliptic Hilbert complexes. Some microlocal property of the Møller operators then guarantees the validity of the Hadamard condition without exploiting the classical so-called *deformation argument*, or better, by re-formulating it into a new form in terms of Møller operators.

1.3. Main Results

We explicitly state here the principal results established in this paper referring, for the former, to the notions introduced in the previous section. Below, $\mathsf{G}^{\pm}_{\mathsf{P}}$ denote the retarded and advanced Green operators of the Proca equation (2.3), we shall discuss in Sect. 3. The symbol $\kappa_{g'g}$ denotes a linear fiber-preserving isometry from the spaces of smooth sections $\Gamma(\mathsf{V}_g)$ to $\Gamma(\mathsf{V}_{g'})$ constructed in Section 3. Here, V_g indicates the vector bundle of real 1-forms over the spacetime (M, g) whose sections are the argument of the Proca operator P.

Theorem 1 (Theorems 3.2 and 3.7). *Let (M, g) and (M, g') be globally hyperbolic spacetimes, with associated real Proca bundles V_g and $\mathsf{V}_{g'}$ and Proca operators P, P'.*
If the metric is paracausally related $g \simeq g'$, then there exists a $\mathbb{R}$-vector space isomorphism $\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$, called **Møller operator** *of g, g' (with this order), such that the following facts are true.*

(1) *The restriction, called* **Møller map**
$$\mathsf{S}^0 := \mathsf{R}|_{\mathsf{Ker}_{sc}(\mathsf{P})} : \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathsf{Ker}_{sc}(\mathsf{P}')$$
is well-defined vector space isomorphism with inverse given by
$$(\mathsf{S}^0)^{-1} := \mathsf{R}^{-1}|_{\mathsf{Ker}_{sc}(\mathsf{P}')} : \mathsf{Ker}_{sc}(\mathsf{P}') \to \mathsf{Ker}_{sc}(\mathsf{P})\,.$$
(2) *It holds $\kappa_{gg'}\mathsf{P}'\mathsf{R} = \mathsf{P}$.*
(3) *The causal propagators $\mathsf{G}_{\mathsf{P}} := \mathsf{G}^+_{\mathsf{P}} - \mathsf{G}^-_{\mathsf{P}}$ and $\mathsf{G}_{\mathsf{P}'} := \mathsf{G}^+_{\mathsf{P}'} - \mathsf{G}^-_{\mathsf{P}'}$, respectively, of P and P', satisfy $\mathsf{R}\mathsf{G}_{\mathsf{P}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{P}'}$.*
(4) *It holds $\mathsf{R}^{\dagger_{gg'}}\mathsf{P}'\kappa_{g'g}|_{\Gamma_c(\mathsf{V}_g)} = \mathsf{P}|_{\Gamma_c(\mathsf{V}_g)}$, where the adjoint $^{\dagger_{gg'}}$ is defined in Definition 3.3.*

The next result (Theorem 2) permits us to promote R to a $*$-isomorphism $\mathcal{R}$ of the algebras of field operators $\mathcal{A}$, $\mathcal{A}'$, respectively, associated with the paracausally related metrics g and g', with the associated P, P', and generated by respective Hermitian field operators $\mathfrak{a}(\mathfrak{f})$ and $\mathfrak{a}'(\mathfrak{f}')$ with $\mathfrak{f}, \mathfrak{f}'$ compactly supported smooth real sections of V. We will introduce these notions in Sect. 4. These field operators satisfy respective CCRs

$$[\mathfrak{a}(\mathfrak{f}),\mathfrak{a}(\mathfrak{h})] = i\mathsf{G}_{\mathsf{P}}(\mathfrak{f},\mathfrak{h})\mathbb{I}\,, \quad [\mathfrak{a}'(\mathfrak{f}'),\mathfrak{a}'(\mathfrak{h}')] = i\mathsf{G}_{\mathsf{P}'}(\mathfrak{f}',\mathfrak{h}')\mathbb{I}'$$

and the said unital $*$-algebra isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ is uniquely determined by the requirement

$$\mathcal{R}(\mathfrak{a}'(\mathfrak{f})) = \mathfrak{a}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f})\,, \quad \mathfrak{f} \in \Gamma_c(\mathsf{V}_{g'})\,.$$

The final important result regards the properties of $\mathcal{R}$ for the algebras of a pair of paracausally related metrics g, g' when it acts on the states $\omega : \mathcal{A} \to \mathbb{C}$, $\omega' : \mathcal{A}' \to \mathbb{C}$ of the algebras in terms of pullback.

$$\omega' = \omega \circ \mathcal{R}\,.$$

As is known, the most relevant (quasifree) states in algebraic QFT are *Hadamard states* characterized by the *microlocal spectrum condition* valid for the wavefront set of their two-point functions or, equivalently, an universal short-distance structure of the distribution defining the two-point function. A definition of Hadamard state for the Proca field was first stated by Fewster and Pfenning in [14] and corresponds to Definition 6.1 in this paper. That definition requires the existence of a bisolution of the *Klein Gordon* field satisfying the microlocal spectrum condition. This bisolution is next used to construct the two-point function of the Proca field. Differently, in this work we adopt a direct definition (Definition 4.5) which only requires the validity of the microlocal spectrum condition directly for the two-point function of the Proca two-point function. We also prove that our definition, exactly as it happens for Fewster and Pfenning's definition, satisfies some physically relevant properties. In addition to these general results, we also prove that *the Hadamard property is preserved by the Møller operators* as one of main results of this work.

Theorem 2 (Theorem 4.9). *Let g, g' be paracausally related metric and consider the corresponding Proca operators P, P'. Finally, refer to the associated on-shell CCR-algebras $\mathcal{A}$ and $\mathcal{A}'$.*
Let $\omega : \mathcal{A} \to \mathbb{C}$ be a quasifree Hadamard state. The pullback state $\omega' : \mathcal{A}' \to \mathbb{C}$ by $\omega' = \omega \circ \mathcal{R}$ satisfies,

1. *ω' is a well-defined state;*
2. *ω' is quasifree;*
3. *ω' is a Hadamard state.*

Attention is next focused on the *existence problem* of quasifree Hadamard states for the real Proca field in a *generic* globally hyperbolic spacetime. The technology of Møller operators allows us to reduce the construction of Hadamard states for the real Proca field to the special case of an *ultrastatic spacetime* $(\mathbb{R}\times\Sigma, -dt^2+h)$. In this class of spacetimes, if assuming the further geometric hypothesis of *bounded geometry*, we provide a direct construction of a Hadamard state just working on the space of initial data C_Σ for the Proca equation $\mathsf{P}A = 0$ where $A \in \Gamma(\mathsf{V}_g)$ has compact Cauchy data. Here, A decomposes as $A = A^{(0)}dt + A^{(1)}$, where $A^{(0)}$ and $A^{(1)}$ and are, respectively, a

0-form and a 1-form on $\{t\} \times \Sigma$. As we shall prove, this space of initial data is actually *constrained* in order to satisfy the existence and uniqueness theorem for the Cauchy problem:

$$C_\Sigma := \Big\{ (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in \Omega^0_c(\Sigma)^2 \times \Omega^1_c(\Sigma)^2 \;\Big|\; \pi^{(0)} = -\delta^{(1)}_h a^{(1)} \,, \\ (\Delta^{(0)}_h + m^2) a^{(0)} = \delta^{(1)}_h \pi^{(1)} \Big\} \,,$$

where $(a^{(0)}, \pi^{(0)}) := (A^{(0)}, \partial_t A^{(0)})|_{t=0}$ and $(a^{(1)}, \pi^{(1)}) := (A^{(1)}, \partial_t A^{(1)})|_{t=0}$.

Theorem 3 (Propositions 5.7 and 5.9). *Consider the $*$-algebra $\mathcal{A}_g$ of the real Proca field on the ultrastatic spacetime $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, with (Σ, h) a smooth complete Riemannian manifold. Let $\eta_0 := -1$, $\eta_1 := 1$ and $h^\sharp_{(j)}$ denote the standard inner product of j-forms on Σ induced by h. Then, the two-point function*

$$\omega_\mu(\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{f}')) = \omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') := \mu(A, A') + \frac{i}{2}\sigma^{(P)}(A, A')$$

defines a quasifree state ω_μ on $\mathcal{A}_g$ where $\mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathsf{V}_g)$. Above,

$$A = \mathsf{G_P}\mathfrak{f} \,, \qquad A' = \mathsf{G_P}\mathfrak{f}' \,, \qquad \sigma^{(P)}(A, A') = \int_{\mathsf{M}} g^\sharp(\mathfrak{f}, \mathsf{G_P}\mathfrak{f}') \operatorname{vol}_g$$

$$\mu(A, A') := \sum_{j=0}^{1} \frac{\eta_j}{2} \int_\Sigma h^\sharp_{(j)}(\pi^{(j)}, (\overline{\Delta^{(j)} + m^2})^{-1/2} \pi^{(j)\prime}) \\ + h^\sharp_{(j)}(a^{(j)}, (\overline{\Delta^{(j)} + m^2})^{1/2} a^{(j)\prime}) \operatorname{vol}_h$$

where $\Delta^{(j)}$ is the Hodge Laplacian for compactly supported real smooth j-forms on (Σ, h).
Finally, ω_μ is Hadamard if (Σ, h) is of bounded geometry.

Above, the bar denotes the closure of the considered operators defined in suitable L^2-spaces of forms according to the theory of elliptic Hilbert complexes.

Using the fact that every globally hyperbolic spacetime is paracausally related to an ultrastatic spacetime with bounded geometry and combining the two previous theorems, we can conclude that Proca fields can be quantized in any globally hyperbolic spacetime and admit Hadamard states.

Theorem 4 *Let (M, g). be a globally hyperbolic spacetime and refer to the associated CCR-algebras $\mathcal{A}_g$ of the real Proca field. Then there exists a quasifree Hadamard state on $\mathcal{A}_g$.*

Eventually, coming back to the alternative definition of Hadamard states proposed by Fewster and Pfenning in [14], we prove an almost complete equivalence theorem, which is the last main achievement of this work.

Theorem 5 (Theorem 6.6). *Consider the globally hyperbolic spacetime (M, g) and a quasifree state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g) with two-point function $\omega \in \Gamma'_c(\mathsf{V}_g \otimes \mathsf{V}_g)$. The following facts are true.*

(a) *If ω is Hadamard according to Fewster and Pfenning, then it is also Hadamard according to Definition 4.5.*
(b) *If (M, g) admits a Hadamard state according to Fewster and Pfenning and ω is Hadamard according to Definition 4.5, then ω is Hadamard in the sense of Fewster–Pfenning's definition.*

The existence of Hadamard states according to Fewster–Pfenning's definition was proved in [14] for spacetimes with compact Cauchy surfaces. For these spacetimes, the equivalence of the two definitions is complete.

1.4. Structure of the Paper

The paper is structured as follows. In Sect. 3, we will provide a detailed analysis of the Møller maps and the Møller operator for classical Proca fields. In particular, we will analyze the relation between the Møller operators and the causal propagators of Proca operators on paracausally related globally hyperbolic spacetimes. In Sect. 4, we will extend the action of the Møller operators to a $*$-isomorphism of the CCR-algebras of free Proca fields. This will allow us to pullback quasifree Hadamard states preserving the microlocal spectrum condition. In this section, we also analyze the general properties of Hadamard states including their existence. The explicit construction of Hadamard states in an ultrastatic spacetime is performed in Sect. 5. In Sect. 6, we show that the microlocal spectrum condition is essentially equivalent to the definition of Hadamard states proposed by Fewster and Pfenning. Finally, we conclude our paper with Sect. 7, where open issues and future prospects are presented.

2. Mathematical Setup

2.1. Conventions and Notation of Geometric Tools in Spacetimes

Throughout all the paper, the symbols $\subset$ and $\supset$ allow the case $=$.
We explicitly adopt the signature $(-, +, \cdots, +)$ for Lorentzian metrics.

Throughout, (M, g) denotes a **spacetime**, i.e., a paracompact, connected, oriented, time-oriented, smooth, Lorentzian manifold M, whose Lorentzian metric is g. As in [46], the Lorentzian metrics g of spacetimes are hereafter supposed to be equipped with their own temporal orientation.

All considered spacetimes (M, g) are also **globally hyperbolic**. In other words, a (smooth) **Cauchy temporal function** $t : \mathsf{M} \to \mathbb{R}$ exists. By definition dt is timelike, past directed and

$$(\mathsf{M}, g) \quad \text{is isometric to} \quad (\mathbb{R} \times \Sigma, g') \quad \text{with metric} \quad g' = -\beta dt \otimes dt + h_t\,,$$

where $\beta : \mathbb{R} \times \Sigma \to \mathbb{R}$ is a smooth positive function, h_t is a Riemannian metric on each slice $\Sigma_t := \{t\} \times \Sigma$ varying smoothly with t, and these slices are smooth **spacelike Cauchy hypersurfaces**. By definition, they are achronal sets intersected exactly once by every inextensible timelike curve (see [45] for a recent up-to-date survey on the subject).

According to [46], given two globally hyperbolic metrics g and g' on M , $g \preceq g'$ means that $V_p^{g+} \subset V_p^{g'+}$ *for all* $p \in \mathsf{M}$, where $V_p^{g+} \subset \mathsf{T}_p\mathsf{M}$ is the open cone of future directed timelike vectors at p in (M, g).

Two globally hyperbolic metrics g and g' on M are **paracausally related**, written $g \simeq g'$, if there exists a finite sequence of globally hyperbolic metrics $g_1 = g, g_2 \dots, g_n = g'$ on M such that for each pair of consecutive metrics either

$$g_k \preceq g_{k+1} \qquad \text{or} \qquad g_{k+1} \preceq g_k \,.$$

For a discussion on this notion, its properties, and examples, we refer to [46, Section 2].

We henceforth denote by $\Gamma(\mathsf{E})$ the real vector space of smooth sections of any real vector bundle $\mathsf{E} \to \mathsf{M}$. More precisely, as in [46], we denote with $\Gamma_c(\mathsf{E}), \Gamma_{fc}(\mathsf{E}), \Gamma_{pc}(\mathsf{E}), \Gamma_{sc}(\mathsf{E})$ the space of sections, respectively, with **compact support**, **future-compact** (i.e., whose support stays before a smooth spacelike Cauchy surface), **past-compact** (i.e., whose support stays after a smooth spacelike Cauchy surface), and **spatially compact support** (i.e., whose support on every smooth spacelike Cauchy surface is compact). If $\mathsf{E} \to \mathsf{M}$ and $\mathsf{E}' \to \mathsf{M}'$ are two vector bundles, $\mathsf{E} \boxtimes \mathsf{E}'$ denotes the external tensor product of the vector bundles. This vector bundle has base $\mathsf{M} \times \mathsf{M}'$ and fiber at (p, p') given by the tensor products of the respective fibers at $p \in \mathsf{M}$ and $p' \in \mathsf{M}'$ respectively. If $\mathfrak{f} \in \Gamma(\mathsf{E})$ and $\mathfrak{f}' \in \Gamma(\mathsf{E}')$, the section $\mathfrak{f} \otimes \mathfrak{f}' \in \Gamma(\mathsf{E} \boxtimes \mathsf{E}')$ is defined by $\mathfrak{f} \otimes \mathfrak{f}'(p, p') := \mathfrak{f}(p) \otimes \mathfrak{f}'(p')$. The tensor product of linear operators acting on sections of an external product bundle are denoted by $\otimes$.

2.2. Smooth Forms, Hodge Operators, and the Proca Equation

In this work, we frequently deal with real smooth k-forms $\mathfrak{f}, \mathfrak{g} \in \Omega^k(\mathsf{M})$, where $k = 0, \dots, n = \dim \mathsf{M}$ (and one usually adds $\Omega^{n+1}(\mathsf{M}) = \Omega^{-1}(\mathsf{M}) = \{0\}$). The **Hodge** *real* **inner product** can be computed by integrating the fiberwise product with respect to the volume form induced by g:

$$(\mathfrak{f}|\mathfrak{g})_{g,k} := \int_{\mathsf{M}} \mathfrak{f} \wedge *\mathfrak{g} = \int_{\mathsf{M}} g^\sharp_{(k)}(\mathfrak{f}, \mathfrak{g}) \, \mathrm{vol}_g \,,$$

where at least one of the two forms has compact support and $g^\sharp_{(k)}$ is the natural inner product of k-forms induced by g. This symmetric real scalar product $(\cdot|\cdot)_{g,k}$ is always non-degenerate but it is not positive when g is Lorentzian as in the considered case. It is positive when g is Riemannian. If $k = 1$, we simply write

$$(\mathfrak{f}|\mathfrak{g})_g = \int_{\mathsf{M}} g^\sharp(\mathfrak{f}, \mathfrak{g}) \, \mathrm{vol}_g \,. \tag{2.1}$$

In this context, $d^{(k)} : \Omega^k(\mathsf{M}) \to \Omega^{k+1}(\mathsf{M})$ is the exterior derivative and $\delta_g^{(k)} : \Omega^k(\mathsf{M}) \to \Omega^{k-1}(\mathsf{M})$ is the codifferential operator acting on the relevant spaces of smooth k-forms $\Omega^k(\mathsf{M})$ on M depending on the metric g on M. $d^{(k)}$ and $\delta_g^{(k+1)}$ are the **formal adjoint** of one another with respect to the Hodge product (2.1) in the sense that

$$(d^{(k)}\mathfrak{f}|\mathfrak{g})_{g,k+1} = (\mathfrak{f}|\delta_g^{(k+1)}\mathfrak{g})_{g,k}\,, \quad \forall \mathfrak{f} \in \Omega^k(\mathsf{M})\,,$$
$$\forall \mathfrak{g} \in \Omega^{k+1}(\mathsf{M}) \quad \text{if } \mathfrak{f} \text{ or } \mathfrak{g} \text{ is compactly supported.}$$

In the rest of the paper, we will often omit the indices$_{g,k}$ and $^{(k)}$ referring to the metric and the order of the used forms, when the choice of the used metric and order will be obvious from the context.

If (M, g) is globally hyperbolic, we call **Proca bundle** the real vector bundle $\mathsf{V}_g := (\mathsf{T}^*\mathsf{M}, g^\sharp)$ obtained by endowing the cotangent bundle with the fiber metric given by the dual metric $g^\sharp$ (also appearing in (2.1)) defined by

$$g^\sharp(\omega_p, \omega'_p) := g(\sharp\omega_p, \sharp\omega'_p) \quad \text{for every } \omega, \omega' \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M},$$

where $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{TM})$ is the standard musical isomorphism.

By construction, $\Gamma(\mathsf{V}_g) = \Omega^1(\mathsf{M})$ and $\Gamma_c(\mathsf{V}_g) = \Omega^1_c(\mathsf{M})$. Here and henceforth, $\Omega^k_c(\mathsf{M}) \subset \Omega^k(\mathsf{M})$ is the subspace of compactly supported real smooth k-forms on M.

The formally self-adjoint **Proca operator** P on (M, g) is defined by choosing a (mass) constant $m > 0$, *the same for all globally hyperbolic metrics we will consider on* M em in this work,

$$\mathsf{P} = \delta d + m^2 : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g), \tag{2.2}$$

where $d := d^{(1)}$, $\delta := \delta_g^{(2)}$. Actually P depends also on g, but we shall not indicate those dependencies in the notation for the sake of shortness.

The **Proca equation** we shall consider in this paper reads

$$\mathsf{P}A = 0 \quad \text{for } A \in \Gamma_{sc}(\mathsf{V}_g)\,, \tag{2.3}$$

where, as said above, $\Gamma_{sc}(\mathsf{V}_g)$ is the space of real smooth 1-forms which have compact support on the Cauchy surfaces of the globally hyperbolic spacetime (M, g).

3. Møller Maps and Møller Operators

The construction of the so-called Møller operator for hyperbolic PDEs (coming from the realm of quantum field theories on curved spacetimes) has been studied extensively in various contexts in quantum field theory, see, e.g., [8–10,12,46,47]. The key idea was to inspired by the scattering theory: Starting with two "free theories" described by the space of solutions of *normally hyperbolic operators* (see (3.3)) N_0 and N_1 in corresponding spacetimes (M, g_0) and (M, g_1), respectively, we connected them through an "interaction spacetime" (M, g_χ) with a "temporally localized" interaction defined by interpolating the two metrics by means of a smoothing function χ. Here we need two Møller maps: Ω_+ connecting (M, g_0) and (M, g_χ)—which reduces to the identity in the past when χ is switched off—and a second Møller map connecting (M, g_χ) to (M, g_1)—which reduces to the identity in the future when χ constantly takes the value 1. The "S-matrix" given by the composition $\mathsf{S} := \Omega_-\Omega_+$ will be the Møller map connecting N_0 and N_1.

As remarked in [46, Section 6], all the results concern vector-valued normally hyperbolic operators acting on real vector bundles whose fiber metric does not depend on the globally hyperbolic metrics g chosen on M. These operators are also assumed to be formally self-adjoint with respect to the associated real symmetric scalar product on the sections of the bundle. As already pointed out in the Introduction, to quantize the theory defining quantum states on an associated $*$-algebra of observables, the fiberwise metric on E should be assumed to be *positive*.

This section aims to extend the construction of the Møller operators to Proca fields. The main difficulties we have to face with respect to the case of the Klein–Gordon equation are the following:

- the fiber metric of the Proca bundle depends on the underlying globally hyperbolic metrics g chosen on M (and it is not positive definite);
- Proca operators are not normally hyperbolic.

The next two sections are devoted to tackle these technical issues before starting with the construction of the Møller maps.

3.1. Linear Fiber-Preserving Isometry

As said above, to construct Møller maps for the Proca field we should be able to compare different fiberwise metrics on $\mathsf{T}^*\mathsf{M}$ when we change the metric g on M. This will be done by defining suitable fiber-preserving isometries.

If g and g' are globally hyperbolic on M and $g \simeq g'$, it is possible to define a linear fiber-preserving isometry from $\Gamma(\mathsf{V}_g)$ to $\Gamma(\mathsf{V}_{g'})$ we denote with $\kappa_{g'g}$ and we shall take advantage of it very frequently in the rest of this work. In other words, if $\mathfrak{f} \in \Gamma(\mathsf{V}_g)$, then $\kappa_{g'g}\mathfrak{f} \in \Gamma(\mathsf{V}_{g'})$, the map $\kappa_{g'g} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ is $\mathbb{R}$ linear, and

$$g'^{\sharp}((\kappa_{g'g}\mathfrak{f})(p), (\kappa_{g'g}\mathfrak{g})(p)) = g^{\sharp}(\mathfrak{f}(p), \mathfrak{g}(p)) \quad \forall p \in \mathsf{M}\,.$$

Let us describe the (highly non-unique) construction of $\kappa_{gg'}$. If $\chi \in C^\infty(\mathsf{M};[0,1])$ and $g_0 \preceq g_1$, then

$$g_\chi := (1-\chi)g_0 + \chi g_1 \tag{3.1}$$

is a Lorentzian metric globally hyperbolic on M (see [46, Section 2] for details) and satisfies

$$g_0 \preceq g_\chi \preceq g_1\,.$$

Now consider the product manifold $\mathsf{N} := \mathbb{R} \times \mathsf{M}$, equipped with the indefinite non-degenerate metric

$$h := -dt \otimes dt + g_t\,,$$

where $g_t = (1 - f(t))g_0 + f(t)g_\chi$ and $f : \mathbb{R} \to [0, 1]$ is smooth and $f(t) = 0$ for $t \leqslant 0$, $f(t) = 1$ for $t \geqslant 1$. Notice that g_t is Lorentzian according to [46] because $g_0 \preceq g_\chi$ and h is indefinite non-degenerate by construction. At this point $\tilde{\kappa}_{\chi 0} : \mathsf{TM} \to \mathsf{TM}$ is the fiber-preserving diffeomorphism such that $\tilde{\kappa}_{\chi 0}(x, v)$ is the parallel transport form $(0, x)$ to $(1, x)$ of $v \in \mathsf{T}_x\mathsf{M} \subset \mathsf{T}_{(0,x)}\mathsf{N}$ along the complete h-geodesic $\mathbb{R} \ni t \mapsto (t, x) \in \mathsf{N}$. Standard theorems on joint smoothness of the flow of ODEs depending on parameters assure that

$\tilde{\kappa}_{\chi 0} : \mathsf{TM} \to \mathsf{TM}$ is smooth. Notice that $\tilde{\kappa}_{\chi 0}|_{\mathsf{T}_x\mathsf{M}} : \mathsf{T}_x\mathsf{M} \to \mathsf{T}_x\mathsf{M}$ is also a h-isometry from known properties of the parallel transport and thus it is a g_0, g_χ-isometry by construction because $h_{(t,x)}(v,v) = g_t(v,v)$ if $v \in \mathsf{T}_x\mathsf{M} \subset \mathsf{T}_{(t,x)}\mathsf{N}$. Taking advantage of the musical isomorphisms, $\tilde{\kappa}_{\chi 0}$ induces a fiber bundle map $\kappa_{\chi 0} : T^*\mathsf{M} \to T^*\mathsf{M}$ which can be seen as a map on the sections of $\Gamma(\mathsf{V}_{g_0})$ and producing sections of $\Gamma(\mathsf{V}_{g_\chi})$, preserving the metrics $g_0^\sharp$, $g_\chi^\sharp$. Then the required Proca bundle isomorphism $\kappa_{g'g} = \kappa_{g_1 g_0}$ is defined by composition:

$$\kappa_{1,0} = \kappa_{1\chi}\kappa_{\chi 0}.$$

where $\kappa_{1\chi}$ from $\Gamma(\mathsf{V}_{g_\chi})$ to $\Gamma(\mathsf{V}_{g_1})$ is defined analogously to $\kappa_{\chi 0}$. The general case $g \simeq g'$ can be defined by composing the fiber preserving linear isometries $\kappa_{g_{k+1}g_k}$ or $\kappa_{g_k,g_{k+1}}$.

3.2. Klein–Gordon Operator Associated with a Proca Operator and Green Operators

We pass to tackle the issue of normal hyperbolicity of P. As we shall see here, it is not really necessary to construct the Møller maps, and the weaker requirement of *Green hyperbolicity* is sufficient.

Let N be the **Klein–Gordon operator** associated with the Proca operator P (2.2) acting on 1-forms

$$\mathsf{N} := \delta d + d\delta + m^2 : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)\,. \tag{3.2}$$

Notice that this operator is **normally hyperbolic**: its principal symbol σ_{N} satisfies

$$\sigma_{\mathsf{N}}(\xi) = -g^\sharp(\xi,\xi)\,\mathrm{Id}_{\mathsf{V}_g} \text{for all } \xi \in \mathsf{T}^*\mathsf{M},\ \text{where } \mathrm{Id}_{\mathsf{V}_g} \text{ is the identity automorphism of } \mathsf{V}_g. \tag{3.3}$$

Therefore, the Cauchy problem for N is well posed [2,3]. Both N and P are formally self-adjoint with respect to the Hodge scalar product (2.1) on $\Omega^1_c(\mathsf{M}) = \Gamma_c(\mathsf{V}_g)$.

Since $m^2 > 0$ and $\delta_g^{(1)}\delta_g^{(2)} = 0$, it is easy to prove that the Proca equation (2.3) is *equivalent* to the pair of equations

$$\mathsf{N}A = 0\,, \quad \text{for } A \in \Gamma_{sc}(\mathsf{V}_g)\,, \tag{3.4}$$

$$\delta A = 0\,. \tag{3.5}$$

As already noticed, differently from N, the Proca operator is not normally hyperbolic. However, it is **Green hyperbolic** [2,3,5] as N, in particular, there exist linear maps, dubbed **advanced Green operator** $\mathsf{G}^+_{\mathsf{P}} : \Gamma_{pc}(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ and **retarded Green operator** $\mathsf{G}^-_{\mathsf{P}} : \Gamma_{fc}(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ uniquely defined by the requirements

(i.a) $\mathsf{G}^+_{\mathsf{P}} \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^+_{\mathsf{P}}\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$,
(ii.a) $\mathrm{supp}\,(\mathsf{G}^+_{\mathsf{P}}\mathfrak{f}) \subset J^+(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$;
(i.b) $\mathsf{G}^-_{\mathsf{P}} \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^-_{\mathsf{P}}\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{V}_g)$,
(ii.b) $\mathrm{supp}\,(\mathsf{G}^-_{\mathsf{P}}\mathfrak{f}) \subset J^-(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{V}_g)$;

The **causal propagator** of P is defined as

$$\mathsf{G}_{\mathsf{P}} := \mathsf{G}^+_{\mathsf{P}} - \mathsf{G}^-_{\mathsf{P}} : \Gamma_c(\mathsf{V}_g) \to \Gamma_{sc}(\mathsf{V}_g)\,. \tag{3.6}$$

All these maps are also continuous with respect to the natural topologies of the definition spaces [5]. As a matter of fact (see [5, Proposition 3.19] and also [3]), the advanced and retarded Green operator $\mathsf{G}^{\pm}_{\mathsf{P}} : \Gamma_{pc/fc}(\mathsf{V}_g) \to \Gamma_{pc/fc}(\mathsf{V}_g)$ can be written as

$$\mathsf{G}^{\pm}_{\mathsf{P}} := \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{G}^{\pm}_{\mathsf{N}} = \mathsf{G}^{\pm}_{\mathsf{N}}\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)$$

where $\mathsf{G}^{\pm}_{\mathsf{N}}$ are the analogous Green operators for the Klein–Gordon operator N. Therefore,

$$\mathsf{G}_{\mathsf{P}} := \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{G}_{\mathsf{N}} = \mathsf{G}_{\mathsf{N}}\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\,. \tag{3.7}$$

The fact that P is normally hyperbolic can be proved just by checking that the operators above satisfy the requirements which define the Green operators as stated above, using the analogous properties for $\mathsf{G}^{\pm}_{\mathsf{N}}$.

Eq. (3.7) and the analogous properties for G_{N} entail

$$\mathsf{G}_{\mathsf{P}}(\Gamma_c(\mathsf{V}_g)) = \{A \in \Gamma_{sc}(\mathsf{V}_g) \mid \mathsf{P}A = 0\}\,. \tag{3.8}$$

Indeed, if $\mathsf{P}A = 0$ then $\mathsf{N}A = 0$ and $\delta A = 0$. If $A \in \Gamma_{sc}(\mathsf{V}_g)$, [46, Theorem 3.8] implies $A = \mathsf{G}_{\mathsf{N}}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, so that $A = \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)A = \mathsf{G}_{\mathsf{P}}\mathfrak{f}$ as said. Furthermore,

$$\mathrm{Ker}\mathsf{G}_{\mathsf{P}} = \{\mathsf{P}\mathfrak{g} \mid \mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\}\,. \tag{3.9}$$

Indeed, if $\mathsf{P}A = 0$ then $m^2\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{P}A = \mathsf{N}A = 0$. If $A \in \Gamma_{sc}(\mathsf{V}_g)$, again [46, Theorem 3.8] implies that $A = \mathsf{N}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$. Since we also know that $\delta A = 0$, the form (3.3) of N yields $A = \mathsf{P}\mathfrak{f}$. On the other hand, if $A = \mathsf{P}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, then $\mathsf{G}_{\mathsf{P}}A = \mathsf{G}^+_{\mathsf{P}}\mathfrak{f} - \mathsf{G}^-_{\mathsf{P}}\mathfrak{f} = \mathfrak{f} - \mathfrak{f} = 0$.

On account of [46, Proposition 3.6], for any smooth function $\rho : \mathsf{M} \to (0, +\infty)$ also $\rho\mathsf{P}$ is Green hyperbolic and $\mathsf{G}^{\pm}_{\rho\mathsf{P}} = \mathsf{G}^{\pm}_{\mathsf{P}}\rho^{-1}$.

3.3. Proca Møller Maps

A **smooth Cauchy time function** in a globally hyperbolic spacetime (M, g) relaxes the notion of temporal Cauchy function, it is a smooth map $t : \mathsf{M} \to \mathbb{R}$ such that dt is everywhere timelike and past directed, the level surfaces of t are smooth spacelike Cauchy surfaces and (M, g) is isometric to $(\mathbb{R} \times \Sigma, h)$. Here, t identifies with the natural coordinate on $\mathbb{R}$ and the Cauchy surfaces of (M, g) identify with the sets $\{t\} \times \Sigma$.

From now on we indicate by N_0, N_1, N_χ the Klein–Gordon operators (3.2) on M constructed out of g_0, g_1 and g_χ, respectively, where the globally hyperbolic metric g_χ is defined as in (3.1) (and thus $g_0 \preceq g_\chi \preceq g_1$ [46, Theorem 2.18]) and depends on the choice of a function $\chi \in C^\infty_0(\mathsf{M}, [0, 1])$. Similarly, P_0, P_1, P_χ denote the Proca operators (2.2) on M constructed out of g_0, g_1 and g_χ, respectively.

We can state the first technical result.

Proposition 3.1. *Let g_0, g_1 be globally hyperbolic metrics satisfying $g_0 \preceq g_1$ and let be $\chi \in C^\infty(\mathsf{M}; [0,1])$. Choose*

(a) *a smooth Cauchy time g_1-function $t : \mathsf{M} \to \mathbb{R}$ and $\chi \in C^\infty(\mathsf{M}; [0,1])$ such that $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$;*
(b) *a pair of smooth functions $\rho, \rho' : \mathsf{M} \to (0, +\infty)$ such that $\rho(p) = 1$ for $t(p) < t_0$ and $\rho'(p) = \rho(p) = 1$ if $t(p) > t_1$. (Notice that $\rho = \rho' = 1$ constantly is allowed.)*

Then the following facts are true where g_χ is defined as in (3.1).

(1) *The operators*

$$\mathsf{R}_+ : \Gamma(\mathsf{V}_{g_0}) \to \Gamma(\mathsf{V}_{g_\chi}) \qquad \mathsf{R}_+ := \kappa_{\chi 0} - \mathsf{G}^+_{\rho \mathsf{P}_\chi}\left(\rho \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0} \mathsf{P}_0\right),$$
$$\mathsf{R}_- : \Gamma(\mathsf{V}_{g_\chi}) \to \Gamma(\mathsf{V}_{g_1}) \qquad \mathsf{R}_- := \kappa_{1\chi} - \mathsf{G}^-_{\rho \mathsf{P}_1}\left(\rho' \mathsf{P}_1 \kappa_{1\chi} - \rho \kappa_{1\chi} \mathsf{P}_\chi\right)$$

are linear space isomorphisms, whose inverses are given by

$$\mathsf{R}_+^{-1} : \Gamma(\mathsf{V}_{g_\chi}) \to \Gamma(\mathsf{V}_{g_0}) \qquad \mathsf{R}_+^{-1} = \kappa_{0\chi} + \mathsf{G}^+_{\mathsf{P}_0}(\rho \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \rho \kappa_{0\chi}),$$
$$\mathsf{R}_-^{-1} : \Gamma(\mathsf{V}_{g_1}) \to \Gamma(\mathsf{V}_{g_\chi}) \qquad \mathsf{R}_-^{-1} := \kappa_{\chi 1} + \mathsf{G}^-_{\rho \mathsf{P}_\chi}\left(\rho' \kappa_{\chi 1} \mathsf{P}_1 - \rho \kappa_{1\chi} \mathsf{P}_\chi\right).$$

By composition we define the ***Møller operator****:*

$$\mathsf{R} : \Gamma(\mathsf{V}_{g_0}) \to \Gamma(\mathsf{V}_{g_1}) \qquad \mathsf{R} := \mathsf{R}_- \circ \mathsf{R}_+,$$

which is also a linear space isomorphism.

(2) *It holds*

$$\rho \kappa_{0\chi} \mathsf{P}_\chi \mathsf{R}_+ = \mathsf{P}_0 \qquad \text{and} \qquad \rho' \kappa_{\chi 1} \mathsf{P}_1 \mathsf{R}_- = \rho \mathsf{P}_\chi .$$

and also

$$\rho \kappa_{0\chi} \mathsf{P}_\chi = \mathsf{P}_0 \mathsf{R}_+^{-1} \qquad \text{and} \qquad \rho' \kappa_{\chi 1} \mathsf{P}_1 = \mathsf{P}_\chi \mathsf{R}_-^{-1} .$$

(3) *If $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_0})$ or $\Gamma(\mathsf{V}_{g_\chi})$, respectively, then*

$$(\mathsf{R}_+ \mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) < t_0, \tag{3.10}$$
$$(\mathsf{R}_- \mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) > t_1 . \tag{3.11}$$

Proof. First of all, we notice that the operator R_+ is well defined on the whole space $\Gamma(\mathsf{V}_{g_0})$ since for all sections $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_0})$ we have that $(\mathsf{P}_\chi \frac{\kappa_{\chi 0}}{\rho} - \frac{\kappa_{\chi 0}}{\rho} \mathsf{P}_0)\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_{g_1})$: indeed by definition, there exists a $t_0 \in \mathbb{R}$ such that on $t^{-1}(-\infty, t_0)$ and we have that $\mathsf{P}_\chi = \mathsf{P}_0$, $\kappa_{\chi 0} = \mathrm{Id}$ and t is a smooth g_1-Cauchy time function. Moreover, since $g_\chi \preceq g_1$ it follows that $\Gamma_{pc}(\mathsf{V}_{g_1}) \subset \Gamma_{pc}(\mathsf{V}_{g_\chi}) = \mathsf{Dom}(G_{\mathsf{P}_\chi})$.

To prove (1), we can first notice that

$$\begin{aligned} \mathsf{R}_+^{-1} \circ \mathsf{R}_+ &= \left(\kappa_{0\chi} + \mathsf{G}^+_{\mathsf{P}_0}(\rho \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \kappa_{0\chi})\right) \circ \Big(\kappa_{\chi 0} - \mathsf{G}^+_{\rho \mathsf{P}_\chi}(\rho \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0} \mathsf{P}_0)\Big) \\ &= \mathrm{Id} - \kappa_{0\chi} \mathsf{G}^+_{\rho \mathsf{P}_\chi}\left(\rho \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0} \mathsf{P}_0\right) + \mathsf{G}^+_{\mathsf{P}_0}(\rho \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \kappa_{0\chi}) \kappa_{\chi 0} \\ &\quad - \mathsf{G}^+_{\mathsf{P}_0}(\rho \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \rho \kappa_{0\chi}) \mathsf{G}^+_{\rho \mathsf{P}_\chi}\left(\rho \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0} \mathsf{P}_0\right). \end{aligned}$$

To conclude it is enough to show that everything cancels out except the identity operator, but that just follows by using basic properties of Green operators and straightforward algebraic steps. We easily see that the last addend can be recast as:

$$\begin{aligned}&\mathsf{G}^+_{\mathsf{P}_0}\left(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi}\right)\mathsf{G}^+_{\rho\mathsf{P}_\chi}\left(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right)\\ &= \mathsf{G}^+_{\mathsf{P}_0}\rho\kappa_{0\chi}\mathsf{P}_\chi\mathsf{G}^+_{\rho\mathsf{P}_\chi}\left(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right) - \mathsf{G}^+_{\mathsf{P}_0}\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\\ &= \mathsf{G}^+_{\mathsf{P}_0}\kappa_{0\chi}\left(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right) - \kappa_{0\chi}\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0),\end{aligned}$$

which fulfills its purpose.
A specular computation proves that R_+^{-1} is also a right inverse. Almost identical reasonings prove that R_-^{-1} is a two sided inverse of R_- which is also well defined, then bijectivity of R is obvious.

(2) follows by the following direct computation

$$\begin{aligned}\rho\kappa_{0\chi}\mathsf{P}_\chi\mathsf{R}_+ &= \rho\kappa_{0\chi}\mathsf{P}_\chi\left(\kappa_{\chi 0} - \mathsf{G}^+_{\rho\mathsf{P}_\chi}\left(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right)\right)\\ &= \kappa_{0\chi}\kappa_{\chi 0}\mathsf{P}_0 = \mathsf{P}_0.\end{aligned}$$

(3) Let us prove (3.10). In the following P^* denotes the formal dual operator of P acting on the sections of the dual bundle $\Gamma_c(V_g^*)$. If $\mathfrak{f}' \in \Gamma_c(\mathsf{V}_g^*)$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$ or $\mathfrak{f} \in \Gamma_{fc}(V_g)$, respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P^*}\mathfrak{f}', \mathfrak{f}\rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^+_P\mathfrak{f}\rangle \,\mathrm{vol}_g\ , \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P^*}\mathfrak{f}', \mathfrak{f}\rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^-_P\mathfrak{f}\rangle \,\mathrm{vol}_g\ , \tag{3.12}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P^*}$ indicate the Green operators of P^*. Consider now a compactly supported smooth section $\mathfrak{h}$ whose support is included in the set $t^{-1}((-\infty, t_0))$. Taking advantage of Equation (3.12), we obtain

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = \int_{\mathsf{M}} \langle \mathsf{G}^-_{(\rho\mathsf{P}_\chi)^*}\mathfrak{h}, (\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

since $\mathrm{supp}(\mathsf{G}^-_{(\rho\mathsf{P}_\chi)^*}\mathfrak{h}) \subset J_-^{g_\chi}(\mathrm{supp}(\mathfrak{h}))$ and thus that support does not meet $\mathrm{supp}((\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f})$ because $((\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f})(p)$ vanishes if $t(p) < t_0$. As $\mathfrak{h}$ is an arbitrary smooth section compactly supported in $t^{-1}((-\infty, t_0))$,

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

entails that $\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f} = 0$ on $t^{-1}((-\infty, t_0))$. The proof of (3.11) is strictly analogous, so we leave it to the reader. □

Using Proposition 3.1, we can pass to the generic case $g \simeq g'$.

Theorem 3.2. *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes, with associated Proca bundles* V_g *and* $\mathsf{V}_{g'}$ *and Proca operators* P, P'.
If $g \simeq g'$*, then there exist (infinitely many) vector space isomorphisms,*

$$\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$$

such that

(1) *referring to the said domains,*

$$\mu\kappa_{gg'}\mathsf{P}'\mathsf{R} = \mathsf{P}$$

for some smooth $\mu : \mathsf{M} \to (0,+\infty)$ *(which can always be chosen* $\mu = 1$ *constantly in particular), and a smooth fiberwise isometry* $\kappa_{gg'} : \Gamma(\mathsf{V}_{g'}) \to \Gamma(\mathsf{V}_g)$.

(2) *The restriction, called* **Møller map**

$$\mathsf{S}^0 := \mathsf{R}|_{\mathsf{Ker}_{sc}(\mathsf{P})} : \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathsf{Ker}_{sc}(\mathsf{P}')$$

is well-defined vector space isomorphism with inverse given by

$$(\mathsf{S}^0)^{-1} := \mathsf{R}^{-1}|_{\mathsf{Ker}_{sc}(\mathsf{P}')} : \mathsf{Ker}_{sc}(\mathsf{P}') \to \mathsf{Ker}_{sc}(\mathsf{P})\,.$$

Proof. Since $g \simeq g'$, there exists a finite sequence of globally hyperbolic metrics $g_0 = g, g_1, .., g_N = g$ such that at each step $g_k \preceq g_{k+1}$ or $g_{k+1} \preceq g_k$. For all $k \in \{0, .., N\}$, we can associate with the metric a Proca operator P_k.
At each step, the hypotheses of Proposition 3.1 are verified; in fact, we can choose functions ρ_k and ρ'_k and the Møller map is given by $\mathsf{R}_k = \mathsf{R}_{k-} \circ \mathsf{R}_{k+}$. The general map is then built straightforwardly by composing the N maps constructed step by step:

$$\mathsf{R} = \mathsf{R}_N \circ ... \circ \mathsf{R}_1.$$

Regarding (1), by direct calculation we get that $\mu = \prod_{k=1}^N \rho'_k$, while $\kappa_{gg'} = \kappa_{g_0 g_1} \circ ... \circ \kappa_{g_{N-1} g_N}$. The proof of (2) is trivial. □

3.4. Causal Propagator and Møller Operator

The rest of this section is devoted to study the relation between Møller maps and the causal propagator of Proca operators. To this end, we use a natural extension of the notion of *adjoint operator* introduced in [46, Section 4.5].

Let g and g' (possibly $g \neq g'$) globally hyperbolic metric and let V_g and $\mathsf{V}_{g'}$ be a Proca bundle on the manifold M. Consider a $\mathbb{R}$-linear operator

$$\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})\,,$$

where $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{V}_g)$ is a $\mathbb{R}$-linear subspace and $\mathsf{Dom}(\mathsf{T}) \supset \Gamma_c(\mathsf{V}_g)$.

Definition 3.3. An operator

$$\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{V}_{g'}) \to \Gamma_c(\mathsf{V}_g)$$

is said to be the **adjoint of** T **with respect to** g, g' (with the said order) if it satisfies

$$\int_{\mathsf{M}} g'^{\sharp}\left(\mathfrak{h}, \mathsf{T}\mathfrak{f}\right)(x)\,\mathrm{vol}_{g'}(x) = \int_{\mathsf{M}} g^{\sharp}\left(\mathsf{T}^{\dagger_{gg'}}\mathfrak{h}, \mathfrak{f}\right)(x)\,\mathrm{vol}_{g}(x) \quad \forall \mathfrak{f} \in \mathsf{Dom}(\mathsf{T})\,,\ \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}).$$

When $g = g'$, we use the simplified notation $\mathsf{T}^{\dagger} := \mathsf{T}^{\dagger_{gg}}$.

As in [46], the adjoint operator satisfies a lot of useful properties which we summarize in the following proposition. Since the proof is analogous to the one of [46, Proposition 4.11], we leave it to the reader. Though the rest of this paper deal with the real case only, we state the theorem encompassing the case where the sections are complex and the fiber scalar product is made Hermitian by adding a complex conjugation of the left entry in the usual fiberwise real $g^\sharp$ inner product, which becomes $g^\sharp(\bar{\mathfrak{f}}, \mathfrak{g})$, where the bar denotes the complex conjugation. Definition 3.3 extends accordingly. For this reason $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$, and the complex conjugate $\bar{c}$ reduces to c itself when $\mathbb{K} = \mathbb{R}$. We keep the notation V_g for indicating either the real or complex vector bundle $\mathsf{T}^*\mathsf{M}$ or $\mathsf{T}^*\mathsf{M} + i\mathsf{T}^*\mathsf{M}$ corresponding to two possible choices of $\mathbb{K}$.

Proposition 3.4. *Referring to the notion of adjoint in Definition 3.3, the following facts are valid.*

(1) *If the adjoint $\mathsf{T}^{\dagger_{gg'}}$ of T exists, then it is unique.*
(2) *If $\mathsf{T} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ is a differential operator and $g = g'$, then $\mathsf{T}^{\dagger_{gg}}$ exists and is the restriction of the formal adjoint to $\Gamma_c(\mathsf{E})$. (In turn, the formal adjoint of T is the unique extension to $\Gamma(\mathsf{E})$ of the differential operator $\mathsf{T}^\dagger$ as a differential operator.)*
(3) *Consider a pair of $\mathbb{K}$-linear operators $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})$, $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{V}_{g'})$ with $\mathsf{Dom}(\mathsf{T}), \mathsf{Dom}(\mathsf{T}') \subset \Gamma(\mathsf{V}_g)$ and $a, b \in \mathbb{K}$. Then*
$$(a\mathsf{T} + b\mathsf{T}')^{\dagger_{gg'}} = \bar{a}\mathsf{T}^{\dagger_{gg'}} + \bar{b}\mathsf{T}'^{\dagger_{gg'}}$$
provided $\mathsf{T}^{\dagger_{gg'}}$ and $\mathsf{T}'^{\dagger_{gg'}}$ exist.
(4) *Consider a pair of $\mathbb{K}$-linear operators $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})$, $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{V}_{g''})$ with $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{V}_g)$ and $\mathsf{Dom}(\mathsf{T}') \subset \Gamma(\mathsf{V}_{g'})$ such that*
   (i) *$\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) \supset \Gamma_c(\mathsf{V}_g)$,*
   (ii) *$\mathsf{T}^{\dagger_{gg'}}$ and $\mathsf{T}'^{\dagger_{g'g''}}$ exist,*
   then $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ exists and
$$(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}} = \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} .$$
(5) *If $\mathsf{T}^{\dagger_{gg'}}$ exists, then $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}} = \mathsf{T}|_{\Gamma_c(\mathsf{V}_g)}$.*
(6) *If $\mathsf{T} : \mathsf{Dom}(T) = \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ is bijective, admits $\mathsf{T}^{\dagger_{gg'}}$, and T^{-1} admits $(\mathsf{T}^{-1})^{\dagger_{g'g}}$, then $\mathsf{T}^{\dagger_{gg'}}$ is bijective and $(\mathsf{T}^{-1})^{\dagger_{g'g}} = (\mathsf{T}^{\dagger_{gg'}})^{-1}$.*

Now we are ready to prove that the operators R admit adjoints and we explicitly compute them.

Proposition 3.5. *Let g_0, g_1 be globally hyperbolic metrics satisfying $g_0 \preceq g_1$. Let R_+, R_- and R be the operators defined in Proposition 3.1 and fix, once and for all, $\rho = c_0^\chi$ and $\rho' = c_0^1$ where c_0^χ, c_0^1 are the unique smooth functions on M such that:*
$$\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0} \qquad \mathrm{vol}_{g_1} = c_0^1 \mathrm{vol}_{g_0}. \tag{3.13}$$
Then we have:

(1) *$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} : \Gamma_c(\mathsf{V}_{g_\chi}) \to \Gamma_c(\mathsf{V}_{g_0})$ satisfies:*
$$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \left(c_0^\chi \kappa_{0\chi} - (c_0^\chi \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \kappa_{0\chi})\, \mathsf{G}^-_{\mathsf{P}_\chi}\right)|_{\Gamma_c(\mathsf{V}_\chi)}$$
and can be recast in the form
$$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \mathsf{P}_0 \kappa_{0\chi} \mathsf{G}^-_{\mathsf{P}_\chi}|_{\Gamma_c(\mathsf{V}_\chi)}.$$

(2) $\mathsf{R}_-^{\dagger g_\chi g_1} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_\chi})$ *satisfies*

$$\mathsf{R}_-^{\dagger g_\chi g_1} = \left(c_1^\chi \kappa_{\chi 1} - (c_1^\chi \kappa_{\chi 1}\mathsf{P}_1 - \mathsf{P}_\chi \kappa_{\chi 1})\, \mathsf{G}^+_{\mathsf{P}_1}\right)|_{\Gamma_c(\mathsf{V}_1)},$$

and can be recast in the form

$$\mathsf{R}_-^{\dagger g_\chi g_1} = P_\chi \kappa_{\chi 1} \mathsf{G}^+_{\mathsf{P}_1}|_{\Gamma_c(\mathsf{V}_1)}.$$

(3) *The map* $\mathsf{R}^{\dagger g_0 g_1} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_0})$ *defined by* $\mathsf{R}^{\dagger g_0 g_1} := \mathsf{R}_+^{\dagger g_0 g_\chi} \circ \mathsf{R}_-^{\dagger g_\chi g_1}$ *is invertible and*

$$(\mathsf{R}^{\dagger g_0 g_1})^{-1} = (\mathsf{R}^{-1})^{\dagger g_1 g_0} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_0})\,.$$

We call it ***adjoint Møller operator****.*
Moreover $\mathsf{R}^{\dagger g_0 g_1}$ *is a homeomorphism with respect to the natural (test section) topologies of the domain and of the co-domain.*

Proof. We start by proving points (1) and (2). For any $\mathfrak{f} \in \mathsf{Dom}(\mathsf{R}_+) = \Gamma(\mathsf{V}_{g_0})$ and $\mathfrak{h} \in \Gamma_c(\mathsf{V}_{g_\chi})$ we have

$$\int_\mathsf{M} g_\chi^\sharp(\mathfrak{h}, \mathsf{R}_+\mathfrak{f})\, \mathrm{vol}_{g_\chi} = \int_\mathsf{M} g_\chi^\sharp\left(\mathfrak{h}, \left(\kappa_{\chi 0} - \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\left(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right)\right)\mathfrak{f}\right) \mathrm{vol}_{g_\chi}$$
$$= \int_\mathsf{M} g_\chi^\sharp(\mathfrak{h}, \kappa_{\chi 0}\mathfrak{f})\, \mathrm{vol}_{g_\chi} - \int_\mathsf{M} g_\chi^\sharp\left(\mathfrak{h}, \left(\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\left(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right)\right)\mathfrak{f}\right) \mathrm{vol}_{g_\chi}.$$

We now split the problem and compute the adjoint of the two summands separately.
The adjoint of the first one follows immediately by exploiting the properties of the existing isometry and Eqs. (3.13)

$$\int_\mathsf{M} g_\chi^\sharp(\mathfrak{h}, \kappa_{\chi 0}\mathfrak{f})\, \mathrm{vol}_{g_\chi} = \int_\mathsf{M} g_0^\sharp(c_0^\chi \kappa_{0\chi}\mathfrak{h}, \mathfrak{f})\, \mathrm{vol}_{g_0}.$$

For the second summand the situation is trickier and we cannot split the calculation in two more summands since it is crucial that the whole difference $(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)$ acts on a general $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_\chi})$ before we apply the Green operator whose domain is $\Gamma_{pc}(\mathsf{V}_{g_\chi})$.
So we first rewrite $\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi} = \mathsf{G}^+_{\mathsf{P}_\chi}\frac{1}{c_0^\chi}$ use the properties of standard adjoints of Green operators for formally self-adjoint Green hyperbolic differential operators to get

$$\int_\mathsf{M} g_\chi^\sharp\left(\mathfrak{h}, \left(\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\left(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right)\right)\mathfrak{f}\right) \mathrm{vol}_{g_\chi}$$
$$= \int_\mathsf{M} g_\chi^\sharp\left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \left(\mathsf{P}_\chi \kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\right)\mathfrak{f}\right) \mathrm{vol}_{g_\chi}.$$

Now we are tempted to exploit the linearity of the integral and of the fiber product, but first, to ensure that the two integrals individually converge, we need to introduce a cutoff function:

- We notice again that there is a Cauchy surface of the foliation Σ_{t_0} such that for all leaves with $t < t_0$ the operator $\left(\mathsf{P}_\chi \kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\right) = 0$;

- So take a $t' < t_0$ and define a cutoff smooth function $s : \mathsf{M} \to [0,1]$ such that $s = 0$ on all leaves with $t < t'$.

In this way, we are allowed to rewrite our last integral and split it in two convergent summands without modifying its numerical value.

$$\begin{aligned}
&\int_{\mathsf{M}} g^\sharp_\chi\Big(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, (\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0)s\mathfrak{f}\Big)\mathrm{vol}_{g_\chi} \\
&\qquad = \int_{\mathsf{M}} g^\sharp_\chi\left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \mathsf{P}_\chi\kappa_{\chi 0}s\mathfrak{f}\right)\mathrm{vol}_{g_\chi} - \int_{\mathsf{M}} g^\sharp_\chi\left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0 s\mathfrak{f}\right)\mathrm{vol}_{g_\chi} \\
&\qquad = \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left(c_0^\chi\kappa_{0\chi}\mathsf{P}_\chi\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f}\right)\mathrm{vol}_{g_0} - \int_{\mathsf{M}} \mathsf{g}^\sharp_0(\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f})\mathrm{vol}_{g_0} \\
&\qquad = \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left((c_0^\chi\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f}\right)\mathrm{vol}_{g_0} \\
&\qquad = \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left((c_0^\chi\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \mathfrak{f}\right)\mathrm{vol}_{g_0}.
\end{aligned}$$

where in the last identities we have used properties of the standard adjoints of the formally self-adjoint operators, of the isometries and of the cutoff function. Since the domain of the operator is just made up of compactly supported sections, we may exploit the inverse property of the Green operators to immediately obtain that

$$c_0^\chi\kappa_{0\chi} - (c_0^\chi\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\,\mathsf{G}^-_{\mathsf{P}_\chi}|_{\Gamma_c(\mathsf{V}_\chi)} = \mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}|_{\Gamma_c(\mathsf{V}_\chi)}.$$

To see that the image of the operators is indeed compactly supported we can focus on $\mathsf{R}^{\dagger_{g_0 g_\chi}}$, the rest follows straightforwardly. The first summand $c_0^\chi\kappa_{0\chi}$ does not modify the support of the sections, whereas the second does. Let us fix $\mathfrak{f} \in \Gamma_c(\mathsf{V}_{g_\chi})$, then $\mathrm{supp}\,(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f}) \subset J^-_{g_\chi}(\mathrm{supp}\,\mathfrak{f})$ which means that $\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f} \in \Gamma_{sfc}$, i.e., it is spacelike and future compact. The thesis follows by again observing that there is a Cauchy surface such that in its past $\left(\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\right)\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f} = 0$. The computation of the adjoint of R_- is almost identical to the one just performed.

The first part of (3) is an immediate consequence of property (4) in Proposition 3.4, while the invertibility of the adjoint can be proved by explicitly showing that the operator

$$(\mathsf{R}_+^{\dagger_{g_0 g_\chi}})^{-1} = \left(\frac{\kappa_{\chi 0}}{c_0^\chi} + \left(\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{G}^-_{\mathsf{P}_0}\right)\right)\Big|_{\Gamma_c(\mathsf{V}_{g_0})}$$

serves as a left and right inverse of $\mathsf{R}_+^{\dagger_{g_0 g_\chi}}$. An analogous argument can be used for $\mathsf{R}_-^{\dagger_{g_\chi g_1}}$.

The continuity of both the adjoint and its inverse comes by the same arguments used in the proof of [46, Theorem 4.12] (with the only immaterial difference that this time the smooth isometry $\kappa_{\chi 0}$ is included in the definition of the Møller operator.) □

Remark 3.6. An interesting fact to remark is that having defined the adjoints over compactly supported sections makes the dependence on the auxiliary volume fixing functions disappear.

We conclude the section, by proving the second part of Theorem 1.

Theorem 3.7. *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes, with associated Proca bundles* V_g *and* $\mathsf{V}_{g'}$ *and Proca operators* P, P'.
If $g \simeq g'$, *it is possible to specialize the* $\mathbb{R}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ *of Proposition 3.2 such that the following further facts are true.*

(1) *The causal propagators* G_{P} *and* $\mathsf{G}_{\mathsf{P}'}$ (3.6), *respectively, of* P *and* P', *satisfy*

$$\mathsf{R}\mathsf{G}_{\mathsf{P}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{P}'} \,.$$

(2) *It holds*

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{P}'\kappa_{g'g}|_{\Gamma_c(\mathsf{V}_g)} = \mathsf{P}|_{\Gamma_c(\mathsf{V}_g)} \,.$$

R *as above is called* **Møller operator** *of* g, g' *(with this order).*

Proof. Since $g \simeq g'$ and the Møller map is defined as the composition $\mathsf{R} = \mathsf{R}_N \circ ... \circ \mathsf{R}_1$, we can use properties (4) in Proposition 3.4 and reduce to the case where $g = g_0 \preceq g_1 = g'$. With this assumption, (2) can be obtained following the proof of Proposition 3.1 and (3) is identical to [46, Theorem 4.12 (5)]. So we leave it to the reader.

It remains to prove (1). Decomposing R as above, we define the maps $\mathsf{R}_{\pm}^{g_0 g_\chi}$, $\mathsf{R}_{\pm}^{g_\chi g_1}$ by choosing the various arbitrary functions as in Proposition 3.5. We first notice

$$\mathsf{R}_+\mathsf{G}^+_{\mathsf{P}_0}\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \left(\kappa_{\chi 0} - \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\,(c_0^\chi \mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\right)\mathsf{G}^+_{\mathsf{P}_0}\left(\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}\right)|_{\Gamma_c(\mathsf{V}_\chi)}$$
$$= \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\kappa_{\chi 0}\left(\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}\right)|_{\Gamma_c(\mathsf{V}_\chi)} = \mathsf{G}^+_{\mathsf{P}_\chi} - \mathsf{G}^+_{\mathsf{P}_\chi}\left(\mathsf{P}_\chi - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\kappa_{0\chi}\right)\mathsf{G}^-_{\mathsf{P}_\chi}.$$

where the first equality follows by definition, in the second one we have used the properties of Green operators, while in the third one we have just equated the two expressions for the adjoint operator according to (1) in Proposition 3.5 and performed some trivial algebraic manipulations.
Another analogous computation can be performed for the retarded Green operator yielding

$$\mathsf{R}_+\mathsf{G}^+_{\mathsf{P}_0}\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \mathsf{G}^-_{\mathsf{P}_\chi} - \mathsf{G}^+_{\mathsf{P}_\chi}\left(\mathsf{P}_\chi - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\kappa_{0\chi}\right)\mathsf{G}^-_{\mathsf{P}_\chi}.$$

Therefore, subtracting the two terms we get

$$\mathsf{R}_+\mathsf{G}_{\mathsf{P}_0}\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \mathsf{R}_+(\mathsf{G}^+_{\mathsf{P}_0} - \mathsf{G}^-_{\mathsf{P}_0})\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \mathsf{G}_{\mathsf{P}_\chi}.$$

Applying now R_- and its adjoint we get the claimed result. □

4. Møller ∗-Isomorphisms and Hadamard States

4.1. The CCR Algebra of Observables of the Proca Field

We now pass to introduce the algebraic formalism to quantize the Proca field [14, 53].

Let (M, g) be a globally hyperbolic spacetime, V_g be a Proca bundle and denote by $\mathsf{P} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ the Proca operator. Following [38], we call **on-shell Proca** CCR **$*$-algebra**, the $*$-algebra defined as

$$\mathcal{A}_g = \mathfrak{A}_g / \mathfrak{I}_g$$

where:

- $\mathfrak{A}_g$ is the free complex unital algebra finitely generated by the set of abstract elements $\mathbb{I}$ (the unit element), $\mathfrak{a}(\mathfrak{f})$ and $\mathfrak{a}(\mathfrak{f})^*$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, and endowed with the unique (antilinear) $*$-involution which associates $\mathfrak{a}(\mathfrak{f})$ to $\mathfrak{a}(\mathfrak{f})^*$ and satisfies $\mathbb{I}^* = \mathbb{I}$ and $(ab)^* = b^* a^*$.
- $\mathfrak{I}_g$ is the two-sided $*$-ideal generated by the following elements of $\mathfrak{A}_f$:

1. $\mathfrak{a}(a\mathfrak{f} + b\mathfrak{h}) - a\mathfrak{a}(\mathfrak{f}) - b\mathfrak{a}(\mathfrak{h})\,, \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$;
2. $\mathfrak{a}(\mathfrak{f})^* - \mathfrak{a}(\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$;
3. $\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{h}) - \mathfrak{a}(\mathfrak{h})\mathfrak{a}(\mathfrak{f}) - i\mathsf{G}_{\mathsf{P}}(\mathfrak{f}, \mathfrak{h})\mathbb{I}\,, \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$;
4. $\mathfrak{a}(\mathsf{P}\mathfrak{f}), \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$.

The four entries of the list, respectively, implement linearity, hermiticity of the generators, canonical commutation relations and the equations of motion for the quantum field.

Remark 4.1. As in [14], we adopt in this paper the interpretation of $\mathfrak{a}(\mathfrak{f})$ is $(\mathfrak{a}|\mathfrak{f})$, where the pairing is the Hodge inner product of 1-forms (2.1).
An equivalence class in $\mathcal{A}_g$ is denoted by $[\mathfrak{a}(\mathfrak{f})] = \hat{\mathfrak{a}}(\mathfrak{f})$, the equivalence class corresponding to the identity is denoted by $[\mathbb{I}] = \mathrm{Id}$. The Hermitian elements of the algebra $\mathcal{A}_g$ are called **observables**.

Remark 4.2. Requirement 4, when we pass to the quotient algebra corresponds to the distributional relation $\mathsf{P}\hat{\mathfrak{a}} = 0$, when we take Remark 4.1 into account and the fact that P is formally self-adjoint. Since every solution of the Proca equation is a co-closed solution of the Klein–Gordon equation and *vice versa*, we conclude that $\delta\hat{\mathfrak{a}} = 0$, i.e., $\hat{\mathfrak{a}}(d\mathfrak{f}) = 0$ for every $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, must be valid.
If, moreover, we deprive the ideal $\mathfrak{I}_g$ of the generators in 4, the quotient algebra is said to be **off-shell**; however, it would still be convenient to assume the closedness constraint when defining the off-shell algebra. That is when defining the relevant ideal of the free off-shell algebra, we should keep 1–3, we should drop 4, and we should replace it with the weaker condition

4'. $\hat{\mathfrak{a}}(d\mathfrak{f}), \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$.

This work, however, deals with the on-shell algebra only, we shall indicate by $\mathcal{A}_g$ throughout. A study of the off-shell algebra, which may result in some relevance in perturbative renormalization procedure will be done elsewhere.

4.2. Møller $*$-Isomorphism and Hadamard States

From now on, let X be a topological vector space, we indicate by X' its topological dual. For example, $\Gamma'_c(V_g)$ represents the space of distributions acting on compactly supported test functions and shall not be confused with the space of compactly supported distributions.

Having built the CCR-algebra, the subsequent step in quantization consists in finding a way to associate numbers with the abstract operators in $\mathcal{A}_g$

by identifying a distinguished state. For the sake of completeness, let us recall that a **state** over the Proca algebra $\mathcal{A}_g$ a $\mathbb{C}$-linear functional $\omega : \mathcal{A}_g \to \mathbb{C}$ which is

(i) **Positive:** $\omega(a^* a) \geqslant 0 \quad \forall a \in \mathcal{A}_g$,
(ii) **Normalized:** $\omega(\mathbb{I}) = 1$

A generic element of the CCR-algebras $\mathcal{A}_g$ of a quantum field can be written as a finite polynomial of the generators $\hat{\mathfrak{a}}(f)$, where the zero grade term is proportional to $\mathbb{I}$. To specify the action of a state, it is sufficient to know its action on the monomials, i.e., its **n-point functions**:

$$\omega_n(\mathfrak{f}_1, \dots, \mathfrak{f}_n) := \omega(\hat{\mathfrak{a}}(\mathfrak{f}_1) \dots \hat{\mathfrak{a}}(\mathfrak{f}_n))$$

with $\mathfrak{f}_1, \dots, \mathfrak{f}_n \in \Gamma_c(\mathsf{V}_g)$.

If we impose continuity with respect to the usual topology on the space of compactly supported test sections we can uniquely extend the n-point functions to distributions in $\Gamma'_c(\mathsf{V}_g^{n\boxtimes})$ we shall hereafter indicate by the symbol $\tilde{\omega}_n$.

Among all possible states the physical ones are the so-called *quasifree* (or *Gaussian*) *Hadamard* states. Quasifree means that the n-point distributions of the state have a structure resembling the one of a free theory, i.e., they all can be recovered just by knowing the two-point distribution.

Definition 4.3. Consider the globally hyperbolic spacetime (M, g) and a state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g). ω is called **quasifree**, if for all choices of $\mathfrak{f}_i \in \Gamma_c(V_g)$

(i) $\omega_n(\mathfrak{f}_1, \dots, \mathfrak{f}_n) = 0$, if $n \in \mathbb{N}$ is odd,
(ii) $\omega_{2n}(\mathfrak{f}_1, \dots, \mathfrak{f}_{2n}) = \sum_\Pi \omega_2(f_{i_1}, f_{i_2}) \cdots \omega_2(f_{i_{n-1}}, f_{i_n})$, if $n \in \mathbb{N}$ is even,

where Π refers to the class of all possible decompositions of the set $\{1, 2, \dots, 2n\}$ into n pairwise disjoint subsets of 2 elements $\{i_1, i_2\}$, $\{i_3, i_4\}$, ..., $\{i_n - 1, i_n\}$ with $i_{2k-1} < i_{2k}$ for $k = 1, 2, \dots, n$.

Regarding the notion of Hadamard state for the Proca field, which is a vector field, we adopt the notions of microlocal analysis for vector-valued distributions introduced in [51].

Remark 4.4. The interpretation of the action of a distribution on test sections is formalized in the sense of the Hodge product (2.1). This interpretation is necessary in order to agree with the interpretation of the symbol $\hat{\mathfrak{a}}(\mathfrak{f})$ stated in Remark 4.1, since some of the distributions we shall consider in the rest of the paper arise from field operators, as the two-point functions $\omega_2(\mathfrak{f}, \mathfrak{g}) := \omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{g}))$. If

$$\Gamma_c(\mathsf{V}_g) \ni \mathfrak{g} \mapsto \omega_2(\cdot, \mathfrak{g}) \in \Gamma'_c(\mathsf{V}_g)$$

is well defined and continuous, ω_2 actually defines a distribution of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ and *vice versa*, as a consequence of the *Schwartz kernel theorem* as clarified below.

From now on, if $F \in \Gamma'_c(\mathsf{V}_g)$ and $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, the action of the former on the latter is therefore interpreted as the Hodge product (2.1)

$$F(\mathfrak{f}) = (F|\mathfrak{f}) = (\mathfrak{f}|F) = \int_{\mathsf{M}} \mathsf{g}^\sharp(F, \mathfrak{f})\mathrm{vol}_g \,.$$

With a straightforward extension of the Definition 3.3, operators working on a generic space of k test forms $\mathsf{T} : \Omega^k_c(\mathsf{M}) \to \Omega^k_c(\mathsf{M})$ can be extended to the topological duals, i.e., the associated distributions, in terms of the action $\mathsf{T}^\dagger$ on the argument of the distribution:

$$(\mathsf{T}F)(\mathfrak{f}) := F(\mathsf{T}^\dagger\mathfrak{f}) \,.$$

For instance, if $F \in {\Omega^2}'_c(\mathsf{M})$ and $H \in {\Omega^0}'_c(\mathsf{M})$,

$$(\delta F)(\mathfrak{f}) := F(d\mathfrak{f}) \,, \quad (dH)(\mathfrak{f}) := H(\delta\mathfrak{f}) \,, \quad \mathfrak{f} \in \Omega^1_c(\mathsf{M}) \,.$$

If $\mathsf{S} : \Gamma_c(\mathsf{V}_g) \to \Gamma'_c(\mathsf{V}_g)$ is continuous (in particular if $\mathsf{S} : \Gamma_c(\mathsf{V}_g) \to \Gamma_c(\mathsf{V}_g)$ is continuous), the standard Schwartz kernel theorem permits to introduce the distribution indicated with the same symbol $\mathsf{S} \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$, which is the unique distribution such that

$$\mathsf{S}(\mathfrak{f} \otimes \mathfrak{g}) := \mathsf{S}(\mathfrak{f}, \mathfrak{g}) := (\mathsf{S}\mathfrak{g})(\mathfrak{f}) \;\text{“} = (\mathfrak{f}|\mathsf{S}\mathfrak{g})'' \,.$$

Conversely, a distribution of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ defines a unique map $\Gamma_c(\mathsf{V}_g) \to \Gamma'_c(\mathsf{V}_g)$ that fulfills the identity above. In the rest of the work we shall take advantage of these facts and notations above. Furthermore, we adopt the notion of *wavefront set* of a distribution on test sections of a vector bundle on M as defined in [51].

Definition 4.5. Consider the globally hyperbolic spacetime (M, g) and a state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g). ω is called **Hadamard** if it is quasifree and its two-point function $\omega_2 \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ satisfies the **microlocal spectrum condition**,[3] i.e.,

$$WF(\omega_2) = \mathcal{H} := \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_\parallel (y, k_y), k_x \triangleright 0\}. \tag{4.1}$$

Above, $(x, k_x) \sim_\parallel (y, k_y)$ means that x and y are connected by a lightlike geodesic and k_y is the co-parallel transport of k_x from x to y along said geodesic, whereas $k_x \triangleright 0$ means that the covector k_x is future pointing.

As for Klein–Gordon scalar field theory, Hadamard states for Proca fields have two important properties which were also established in [14] for the notion of Hadamard state adopted there. We present here independent proofs only based on Definition 4.5. Indeed, [14] uses a definition of Hadamard states which is apparently different from our definition. A comparison of the two definitions and an equivalence result appear in Section 6.

[3] The notion of wavefront set refers to distributions acting on *complex* valued test sections in view of the pervasive use of the Fourier transform. For this reason, when dealing with these notions we consider the natural complex extension of the involved distributions, by imposing that they are also $\mathbb{C}$-linear.

The first property of Hadamard states is the fact that the difference between the two-point functions of a pair of Hadamard states is a smooth function. This fact plays a crucial role in the point-splitting renormalization procedure (for instance of Wick polynomials and time-ordered polynomials [34–37] and of the stress-energy tensor [33,43,54]) and is, in fact, one of the reasons for assuming that Hadamard states are the physically relevant ones.

Proposition 4.6. *Let $\omega, \omega' \in \Gamma'_c(V_g \boxtimes V_g)$ be a pair of Hadamard states on the algebra $\mathcal{A}_g$ of the Proca field according to Definition 4.5. Then, $\omega - \omega' \in \Gamma(V_g \boxtimes V_g)$, i.e., $\omega - \omega'$ is smooth.*
More generally, $\omega - \omega'$ is smooth if ω, ω' are distributions satisfying (4.1) such that their antisymmetric parts coincide mod. C^∞.

Proof. Let us first prove the second statement. Let us define $\omega_2^+(\mathfrak{f}, \mathfrak{g}) := \omega_2(\mathfrak{f}, \mathfrak{g})$ and $\omega_2^-(\mathfrak{f}, \mathfrak{g}) := \omega_2(\mathfrak{g}, \mathfrak{f})$,

$$N^+ := \{(x,k) \in T^*\mathsf{M}\backslash\{0\} \mid k_a k^a = 0\,,\ k \triangleright 0\}\,,$$
$$N^- := \{(x,k) \in T^*\mathsf{M}\backslash\{0\} \mid k_a k^a = 0\,,\ k \triangleleft 0\}\,,$$

$$\Gamma' := \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x; y, k_y) \in \Gamma\}\,. \tag{4.2}$$

for every $\Gamma \subset \mathsf{T}^*\mathsf{M}^2 \setminus \{0\}$. If both distributions satisfy (4.1), then

$$WF(\omega_2^\pm)' \subset N^\pm \times N^\pm\,. \tag{4.3}$$

With the hypotheses of the proposition define $R^\pm := \omega_2^\pm - \omega_2'^\pm$. Since $\omega_2^+ - \omega_2^- = \omega_2'^+ - \omega_2'^- + F$ where F is a smooth function, we have that $R^+ = -R^-$ mod. C^∞. At this juncture, (4.3) yields $WF(R^+)' \cap WF(R^-)' = \emptyset$ because $N^+ \cap N^- = \emptyset$. Since $WF(R^+) = WF(-R^- + F) = WF(-R^-) = WF(R^-)$, we conclude that the wavefront set of the distributions $R^\pm$ is empty and thus they are smooth functions. This is the thesis of the second statement. The latter statement implies the former because, since both ω and ω' are states on the Proca $*$-algebra, their antisymmetric part must be identical and it amounts to $i\mathsf{G}_P$; furthermore, ω and ω' satisfy (4.1) in view of Definition 4.5. □

The second property regards the so-called propagation property of the Hadamard singularity or also the local–global feature of Hadamard states. It has a long history which can be traced back to [20] passing through [39], [48,49] and [51] (and the recent [42]) at least.

Proposition 4.7. *Consider a globally hyperbolic spacetime (M, g) and a globally hyperbolic neighborhood $\mathcal{N}$ of a smooth spacelike Cauchy surface Σ of (M, g). Finally, let $\omega_\mathcal{N}$ be a quasifree state for the on-shell algebra of the Proca field in $(\mathcal{N}, g|_\mathcal{N})$. The following facts are valid.*

(a) *There exists a unique quasifree state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca field on the whole (M, g) which restricts to $\omega_\mathcal{N}$ on the Proca algebra on $\mathcal{N}$.*
(b) *If $\omega_\mathcal{N}$ is Hadamard according to Definition 4.5, then ω is.*

Proof. (a) According to (3.9), $\mathsf{G_P}\mathfrak{f} = 0$ for $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$ if and only if $\mathfrak{f} = \mathsf{P}\mathfrak{g}$ for some $\mathfrak{g} \in \Gamma_c(\mathsf{V}_g)$. We will use this fact to construct ω out of $\omega_{\mathcal{N}}$. Consider two other smooth spacelike surfaces (for both M and $\mathcal{N}$) Σ_+ in the future of Σ and Σ_- in the past of Σ. Let $\chi^+, \chi^- : \mathsf{M} \to [0,1]$ be smooth maps such that $\chi^+(p) = 0$ if p stays in the past of Σ_- and $\chi^+(p) = 1$ if p stays in the future of Σ_+ and $\chi^+ + \chi^- = 1$. Then, defining

$$\mathsf{T}\mathfrak{f} := \mathsf{P}\chi^+\mathsf{G_P}\mathfrak{f}\,, \quad \mathfrak{f} \in \Gamma_c(\mathsf{V}_g) \tag{4.4}$$

we have that $\mathsf{T}\mathfrak{f} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})$ (more precisely *supp*$(\mathsf{T}\mathfrak{f})$ stays between Σ_- and Σ_+), and

$$\mathsf{T}\mathfrak{f} - \mathfrak{f} = \mathsf{P}\mathfrak{g} \quad \text{for some } \mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\,, \tag{4.5}$$

because by standard properties of Green operators:

$$\begin{aligned}\mathsf{G_P}\mathsf{T}\mathfrak{f} &= \mathsf{G_P^+}\mathsf{T}\mathfrak{f} - \mathsf{G_P^-}\mathsf{T}\mathfrak{f} = \left(\mathsf{G_P^+}\mathsf{P}\right)\chi^+\mathsf{G_P}\mathfrak{f} - \mathsf{G_P^-}\mathsf{P}(1-\chi^-)\mathsf{G_P}\mathfrak{f}\\ &= \chi^+\mathsf{G_P}\mathfrak{f} - \mathsf{G_P^-}\left(\mathsf{P}\mathsf{G_P}\mathfrak{f}\right) + \mathsf{G_P^-}\mathsf{P}\chi^-\mathsf{G_P}\mathfrak{f} = \chi^+\mathsf{G_P}\mathfrak{f} + \chi^-\mathsf{G_P}\mathfrak{f} = \mathsf{G_P}\mathfrak{f}.\end{aligned}$$

Therefore, we can apply (3.9) obtaining (4.5).
With these results, let us define

$$\omega_2(\mathfrak{f},\mathfrak{g}) := \omega_{\mathcal{N}2}(\mathsf{T}\mathfrak{f},\mathsf{T}\mathfrak{g})\,, \quad \mathfrak{f},\mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\,. \tag{4.6}$$

Taking the continuity properties of $\mathsf{G_P}$ into account, we leave to the reader the elementary proof of the fact that there is a unique distribution $\Gamma_c'(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ such that its value on $\mathfrak{f}\otimes\mathfrak{g}$ coincides with[4] $\omega_2(\mathfrak{f},\mathfrak{g})$. (We will indicate that distribution by ω_2 with the usual misuse of language.) Furthermore, in view of the definition of T, it is obvious that ω_2 is also a bisolution of the Proca equation, since $\mathsf{G_P}\mathsf{P} = \mathsf{P}\mathsf{G_P} = 0$. Using Definition 4.3 to construct a candidate quasifree state ω on $\mathcal{A}_g$ out of its two-point function ω_2, it is clear that the positivity requirement is guaranteed because $\omega_{\mathcal{N}}$ satisfies it. We conclude that there is a quasifree state ω on $\mathcal{A}_g$, whose two-point function is (4.6), and this two-point function is a distribution which is also bisolution of the Proca equation. Finally, observe that ω extends to the whole $\mathcal{A}_g$ the state $\omega_{\mathcal{N}}$ since the states are quasifree and the two-point function of the former extends the two-point function of the latter. Indeed,

$$\omega_2(\mathfrak{f},\mathfrak{g}) = \omega_{\mathcal{N}2}(\mathsf{T}\mathfrak{f},\mathsf{T}\mathfrak{g}) = \omega_{\mathcal{N}2}(\mathfrak{f},\mathfrak{g}) \quad \text{if } \mathfrak{f},\mathfrak{g} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})\,.$$

This is because, specializing (3.9) and (4.4)–(4.5) to the globally hyperbolic spacetime $(\mathcal{N}, g|_{\mathcal{N}})$ since $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})$, we have that $\mathsf{T}\mathfrak{f} - \mathfrak{f} = \mathsf{P}\mathfrak{g}$ with $\mathfrak{g} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})$ and $\omega_{\mathcal{N}2}$ vanishes when one argument has the form $\mathsf{P}\mathfrak{g}$, because it is a bisolution of the Proca equation in $\mathcal{N}$.
There is only one such quasifree state which is an extension of $\omega_{\mathcal{N}}$ to the whole algebra $\mathcal{A}_g$, and such that its two-point function is a bisolution of the Proca equation. In fact, another such extension ω' would satisfy

$$\omega_2'(\mathfrak{f},\mathfrak{g}) = \omega_2'(\mathsf{T}\mathfrak{f},\mathsf{T}\mathfrak{g}) = \omega_{\mathcal{N}}(\mathsf{T}\mathfrak{f},\mathsf{T}\mathfrak{g}) = \omega_2(\mathsf{T}\mathfrak{f},\mathsf{T}\mathfrak{g}) = \omega_2(\mathfrak{f},\mathfrak{g})\,, \quad \text{for all } \mathfrak{f},\mathfrak{g} \in \Gamma_c(\mathsf{V}_g).$$

[4]If ω_2 indicates the distribution associated with the two-point function: $\omega_2 = \omega_{\mathcal{N}2} \circ \mathsf{T}\otimes\mathsf{T}$.

(b) We pass to the proof that ω is Hadamard if $\omega_{\mathcal{N}}$ is. We have to prove that (4.1) is valid if it is valid for $\omega_{\mathcal{N}}$ in $(\mathcal{N}, g|_{\mathcal{N}})$. Interpreting the two-point functions as distributions of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$,

$$\omega_2 = \omega_{\mathcal{N}2} \circ \mathsf{P}\chi^+\mathsf{G}_\mathsf{P} \otimes \mathsf{P}\chi^+\mathsf{G}_\mathsf{P} \,. \tag{4.7}$$

The wavefront sets of G_P and $\mathsf{P}\chi^+\mathsf{G}_\mathsf{P}$ can be computed as follows. First of all, from (3.7),

$$\mathsf{G}_\mathsf{P} = \mathsf{Q}\mathsf{G}_\mathsf{N} = \mathsf{G}_\mathsf{N}\mathsf{Q} \tag{4.8}$$

where $\mathsf{Q} = I + m^{-2} d\delta_g$. It is known that

$$WF(\mathsf{G}_\mathsf{N}) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \setminus \{0\} \mid (x, k_x) \sim_\parallel (y, k_y)\}$$

Notice that, in particular $k_x \neq 0$ and $k_y \neq 0$ nor simultaneously by definition, nor separately since they are connected by a co-parallel transport.
So, since Q is a differential operator we immediately deduce by 4.8 that $WF(\mathsf{G}_\mathsf{P}) \subset WF(\mathsf{G}_N)$. Then, we associate with the two operator their distributional kernels $\mathsf{G}_\mathsf{P}(x,y)$ and $\mathsf{G}_\mathsf{N}(x,y)$ and recast equation 4.8 in the form:

$$\mathsf{G}_\mathsf{P}(x,y) = (\mathrm{Id}_x \otimes \mathsf{Q}_y)\, \mathsf{G}_\mathsf{N}(x,y),$$

which, by standard microlocal analysis results, implies that

$$WF(\mathsf{G}_\mathsf{N}) \subset Char(\mathrm{Id}_x \otimes \mathsf{Q}_y) \cup WF(\mathsf{G}_\mathsf{P}).$$

However, explicit computations give that $Char(\mathrm{Id}_x \otimes \mathsf{Q}_y) = \{(x, k_x; y, 0) | (x, k_x) \in \mathsf{T}^*\mathsf{M}, y \in \mathsf{M}\}$ which does not intersect $WF(\mathsf{G}_\mathsf{N})$ at any point, implying

$$WF(\mathsf{G}_\mathsf{N}) \subset WF(\mathsf{G}_\mathsf{P}) \subset WF(\mathsf{G}_\mathsf{N})$$

.
So G_P and G_Q have the same wavefront set. Therefore, since $\mathsf{P}\chi^+$ is a smooth differential operator,

$$WF(\mathsf{P}\chi\mathsf{G}_\mathsf{N}) \subset \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \setminus \{0\} \mid (x, k_x) \sim_\parallel (y, k_y)\}$$

A this point, a standard estimate of composition of wavefront sets in (4.7) yields (see, e.g., [38])

$$WF(\omega_2) \subset \mathcal{H}$$

where the Hadamard wavefront set $\mathcal{H}$ is the one in (4.1). To conclude the proof, it is sufficient to establish the converse inclusion. To this end, observe that, since the antisymmetric part of ω_2 is $\omega_2^+ - \omega_2^- = i\mathsf{G}_\mathsf{P}$,

$$WF(\mathsf{G}_\mathsf{P}) \subset WF(\omega_2^+) \cup WF(\omega_2^-) \,,$$

where we adopted the same notation as at the beginning of the proof of Proposition 4.6: $\omega_2^+ = \omega_2$, $\omega_2^-(\mathfrak{f}, \mathfrak{g}) = \omega_2(\mathfrak{g}, \mathfrak{f})$. If, according to that notation, the prime applied to wavefront sets is defined as in (4.2), the above inclusion can be re-phrased to

$$\begin{aligned} &\{(x, k_x; y, k_y) \in T^*\mathsf{M}^2 \setminus \{0\} \mid (x, k_x) \sim_\parallel (y, k_y)\} \\ &\qquad = WF(\mathsf{G}_\mathsf{P})' \subset WF(\omega_2^+)' \cup WF(\omega_2^-)' \end{aligned} \tag{4.9}$$

Above

$$WF(\omega_2^+)' \subset \mathcal{H}' = \{(x, k_x; y, k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_\parallel (y, k_y), k_x \triangleright 0\}$$

and, with a trivial computation,

$$WF(\omega_2^-)' \subset \{(x, -k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_\parallel (y, k_y), k_y \triangleright 0\} ,$$

Now suppose that $(x, k_x; y, k_y) \in \mathcal{H}'$ does not belong to $WF(\omega_2^+)'$. According to (4.9), $(x, k_x; y, k_y) \notin WF(\mathsf{G_P})'$ (notice that $\mathcal{H}' \ni (x, k_x; y, k_y) \notin WF(\omega_2^-)'$ since the two sets are disjoint). This is impossible because every $(x, k_x; y, k_y) \in \mathcal{H}'$ belongs to $WF(\mathsf{G_P})'$ as it immediately arises by comparing the explicit expressions of these two sets written above. In summary, $\mathcal{H}' \subset WF(\omega_2)'$, that is $\mathcal{H} \subset WF(\omega_2)$, concluding the proof. □

Hadamard states turned also out to be relevant in the study of quantum energy conditions [14,15,17] and in black hole physics [11,22,40,44,52] (see references in [42] for a summary)

We are finally ready to extend the Møller operator to the quantum algebras, proving that they are indeed isomorphic. To this end, for any paracausally related metric $g \simeq g'$, we define an isomorphism of the free algebras $\mathcal{R}_{gg'} : \mathfrak{A}_{g'} \to \mathfrak{A}_g$ as the unique unital $*$-algebra isomorphism between the said free unital $*$-algebras such that

$$\mathcal{R}_{gg'}(\mathfrak{a}'(\mathfrak{f})) = \mathfrak{a}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_{g'}) ,$$

where R is a Møller operator of g, g' and the adjoint $\mathsf{R}^{\dagger_{gg'}}$ is defined as in Proposition 3.5.

4.3. Møller $*$-Isomorphism and the Pullback of Hadamard States

When we pass to the quotient algebras, the preservation of the causal propagators discussed in the previous sections, immediately implies that the induced map on the quotient algebras is an isomorphism, that we call **Møller $*$-isomorphism.**

Proposition 4.8. *Let now $\mathcal{R}_{gg'} : \mathcal{A}_{g'} = \mathfrak{A}_{g'}/\mathfrak{I}_{g'} \to \mathcal{A}_g = \mathfrak{A}_g/\mathfrak{I}_g$ be the quotient morphism constructed out of $\mathcal{R}_{gg'}$. Then $\mathcal{R}$ is well defined and is indeed a $*$-algebra isomorphism.*

Proof. The proof of this statement is identical to the one performed in [46, Proposition 5.6]. Indeed, it just relies on the preservation of the causal propagators proved in Theorem 3.7, which implies that the associated CCR-ideals are $*$-isomorphic. □

The final step in our construction is to define a pullback of the Møller $*$-isomorphism to the states and then to prove that the Hadamard condition is preserved, as done in [46, Theorem 5.14] for normally hyperbolic field theories.

Theorem 4.9. *Let $\mathcal{R}_{gg'}$ be the Møller *-isomorphism and let $\omega : \mathcal{A}_g \to \mathbb{C}$ be a quasifree Hadamard state, we define the pullback state $\omega' : \mathcal{A}_{g'} \to \mathbb{C}$ by $\omega' = \omega \circ \mathcal{R}_{gg'}$. The following facts are true:*

1. *ω' is a well-defined state;*

2. ω' *is quasifree;*
3. ω' *is a Hadamard state.*

Proof. The proof of 1–2 is trivial and discussed in [46, Proposition 5.11]. The proof of 3 follows from the Hadamard propagation property stated in Proposition 4.7. To prove the statement we can just focus on the case in which the Møller operator is constructed out of two spacetimes such that $g \preceq g'$, the reasoning can then be iterated at each step of the paracausal chain.
The two-point function of the pullback state can be written as

$$\begin{aligned}\omega_2'(\mathfrak{f},\mathfrak{h}) &= \omega'(\hat{\mathfrak{a}}'(\mathfrak{f})\hat{\mathfrak{a}}'(\mathfrak{h})) = \omega(\mathcal{R}_{gg'}(\hat{\mathfrak{a}}'(\mathfrak{f})\hat{\mathfrak{a}}'(\mathfrak{h}))) = \omega(\hat{\mathfrak{a}}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f})\hat{\mathfrak{a}}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}))\\ &= \omega_2(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f},\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}).\end{aligned}$$

We recall that the operator is the composition of two pieces $\mathsf{R}^{\dagger_{gg'}} = \mathsf{R}_+^{\dagger_{gg_\chi}} \circ \mathsf{R}_-^{\dagger_{g_\chi g'}}$ and split the proof in two steps.
First we focus on the bidistribution $\omega_2^\chi(\mathfrak{f},\mathfrak{h}) := \omega_2(\mathsf{R}_+^{\dagger_{gg_\chi}}\mathfrak{f},\mathsf{R}_+^{\dagger_{gg_\chi}}\mathfrak{h})$ on (M, g_χ) defining a quasifree state therein. By the property 3.10, in the region in which $g_\chi = g$, there is a t_0 a Cauchy surface Σ_{t_0} in common for g and g_χ, a common globally hyperbolic neighborhood $\mathcal{N}$ of that Cauchy surface such that $\omega_2^\chi(\mathfrak{f},\mathfrak{h}) = \omega_2(\mathfrak{f},\mathfrak{h})$ when the supports of $\mathfrak{f}$ and $\mathfrak{g}$ are chosen in $\mathcal{N}$ and thus the corresponding state is Hadamard in $(\mathcal{N}, g_\chi)$. Now Proposition 6.3 implies that ω_2^χ is Hadamard in the whole (M, g_χ). Secondly, the same argument can be used once again for the operator $\mathsf{R}_-^{\dagger_{g_\chi g'}}$ on the Hadamard state ω^χ on (M, g_χ), proving that the state induced by $\omega_2(\mathsf{R}^{\dagger_{gg'}}\cdot,\mathsf{R}^{\dagger_{gg'}}\cdot) = \omega_2^\chi(\mathsf{R}_-^{\dagger_{g_\chi g'}}\cdot,\mathsf{R}_-^{\dagger_{g_\chi g'}}\cdot)$ is Hadamard as well on (M, g'). In other words, the full Møller operator preserves the Hadamard form. □

5. Existence of Proca Hadamard States in Globally Hyperbolic Spacetimes

This section is devoted to the construction of Hadamard states for the real Proca field in a generic globally hyperbolic spacetime. Actually, the technology of Møller operators, in particular Theorem 4.9, allows us to reduce the construction of Hadamard states for the Proca equation to the special case of an ultrastatic spacetime with Cauchy hypersurfaces of bounded geometry. Indeed, as shown in [46, Proposition 2.23], for any globally hyperbolic spacetime (M, g), there exists a paracausally related globally hyperbolic spacetime (M, g_0) which is ultrastatic. In other words, first of all (M, g_0) is isometric to $\mathbb{R}\times\Sigma$ where (Σ, h_0) is a t-independent complete Riemannian manifold and $g_0 = -dt\otimes dt + h_0$, where t is the natural coordinate on $\mathbb{R}$ and dt is past directed. We also denote by ∂_t the tangent vector to the submanifold $\mathbb{R}$ of $\mathbb{R}\times\Sigma$. In view of the completeness of h, these spacetimes are globally hyperbolic (see, e.g., [18]) and Σ is a Cauchy surface of this spacetime. In turn, it is possible to change the metric on Σ in order that the final metric, indicated by h is both complete and *of bounded geometry* [32]. By construction, the final ultrastatic spacetime $(\mathsf{M}, -dt \otimes dt + h)$ is paracausally related to (M, g_0) because the intersection of the corresponding open cones is non-empty as it always contains ∂_t. By transitivity (M, g) is paracausally related with $(\mathbb{R} \times \Sigma, -dt \otimes dt + h)$.

Hence, we assume without loss of generalities, that $(\mathsf{M}, g) = (\mathbb{R}\times\Sigma, -dt\otimes dt + h)$ is a globally hyperbolic ultrastatic spacetime, with dt past directed, whose spatial metric h is complete. When dealing with the construction of Hadamard states we also assume that the spatial manifold (Σ, h) is also of bounded geometry. In the final part of the section, we will come back to consider a generic globally hyperbolic spacetime (M, g)

5.1. The Cauchy Problem in Ultrastatic Spacetimes

We study here the Cauchy problem for the Proca (real and complex) field in ultrastatic spacetimes $(\mathsf{M}, g) = (\mathbb{R}\times\Sigma, -dt\otimes dt+h)$, where (Σ, h) is complete. A more general treatise appears in [53] where the Cauchy problem is studied, also in the presence of a source of the Proca field, in a generic globally hyperbolic spacetime and the continuity of the solutions with respect to the initial data is focused.

Let us consider the Proca equation (2.3) (where $m^2 > 0$) on the above ultrastatic spacetime. As observed in [14], every smooth 1-form $A \in \Omega^1(\mathsf{M})$ naturally uniquely decomposes as

$$A(t, p) = A^{(0)}(t, p)dt + A^{(1)}(t, p) \tag{5.1}$$

where $A^{(i)}(t, \cdot) \in \Omega^i(\Sigma)$ for $i = 0, 1$ and $t \in \mathbb{R}$. By direct inspection and taking the equivalence of (2.3) and (3.4)–(3.5) into account, one sees that Eq. (2.3) is equivalent to the constrained double Klein–Gordon system

$$\partial_t^2 A^{(0)} = -(\Delta_h^{(0)} + m^2)A^{(0)}\,, \tag{5.2}$$

$$\partial_t^2 A^{(1)} = -(\Delta_h^{(1)} + m^2)A^{(1)}\,, \tag{5.3}$$

$$\partial_t A^{(0)} = -\delta_h^{(1)} A^{(1)}\,. \tag{5.4}$$

Above, $\Delta_h^{(k)} := \delta_h^{(k+1)} d^{(k)} + d^{(k-1)}\delta_h^{(k)}$ is the Hodge Laplacian on (Σ, h) for k-forms and the last condition (5.4) is nothing but the constraint $\delta_g^{(1)} A = 0$ arising from (2.3).

The theory for the fields $A^{(1)}$ and $A^{(0)}$ is a special case of the theory of *normally hyperbolic equations on corresponding vector bundles with positive inner product* over a globally hyperbolic spacetime [2,3]. In our case,

(1) there is a real vector bundle $\mathsf{V}_g^{(1)}$ with basis M, canonical fiber isomorphic to $T_q^*\Sigma$, and equipped with a fiberwise real symmetric scalar product induced by $h_q^\sharp$. $A^{(1)} \in \Gamma(V_g^{(1)})$;

(2) there is another real vector bundle $\mathsf{V}_g^{(0)}$ with basis M, canonical fiber isomorphic to $\mathbb{R}$, and equipped with a positive fiberwise real symmetric scalar product given by the natural product in $\mathbb{R}$. $A^{(0)} \in \Gamma(V_g^{(0)})$.

Evidently

$$\mathsf{V}_g = \mathsf{V}_g^{(0)} \oplus \mathsf{V}_g^{(1)}\,. \tag{5.5}$$

Equations (5.2) and (5.3) admit existence and uniqueness theorems for smooth compactly supported Cauchy data and corresponding smooth spacelike compact solutions in $\Gamma_{sc}(\mathsf{V}_g^{(0)})$ and $\Gamma_{sc}(\mathsf{V}_g^{(1)})$, respectively, as a consequence of very well-known results in the theory of normally hyperbolic equations [2,3,31]. However, when viewing $A^{(0)}$ and $A^{(1)}$ as parts of the Proca field A, we have also to deal with the additional constraint (5.4). Notice that (5.4) imposes two constraints on the Cauchy data of $A^{(0)}$ and $A^{(1)}$ on Σ:

$$\partial_t A^{(0)}(0,p) = -\delta_h^{(1)} A^{(1)}(0,p) \qquad \partial_t^2 A^{(0)}(0,p) = -\delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

The second constraint is only apparently of the second order. Indeed, taking (5.2) into account, it can be rewritten as a condition of the Cauchy data

$$(\Delta_h^{(0)} + m^2) A^{(0)}(0,p) = \delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

At this juncture we observe that, with some elementary computation (use $\Delta_h^{(0)}\delta_h^{(1)} = \delta_h^{(1)}\Delta_h^{(1)}$), Eqs. (5.2) and (5.3) imply also the crucial condition

$$(\partial_t^2 + \Delta_h^{(0)} - m^2)(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)}) = 0$$

which, in turn, implies Eq. (5.4) , if the initial condition of that scalar Klein–Gordon equation for $(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)})$ are the zero initial conditions. This exactly amounts to

$$\partial_t A^{(0)}(0,p) = -\delta_h^{(1)} A^{(1)}(0,p) \quad \text{and} \quad (\Delta_h^{(0)} + m^2) A^{(0)} = \delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

In summary, we are naturally led to focus on this Cauchy problem

$$\partial_t^2 A^{(0)} + (\Delta_h^{(0)} + m^2) A^{(0)} = 0\,, \tag{5.6}$$
$$\partial_t^2 A^{(1)} + (\Delta_h^{(1)} + m^2) A^{(1)} = 0\,, \tag{5.7}$$
$$(\partial_t^2 + \Delta_h^{(0)} - m^2)(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)}) = 0\,, \tag{5.8}$$

with initial data

$$A^{(0)}(0,\cdot) = a^{(0)}(\cdot), \quad \partial_t A^{(0)}(0,\cdot) = \pi^{(0)}(\cdot), \quad A^{(1)}(0,\cdot) = a^{(1)}(\cdot),$$
$$\partial_t A^{(1)}(0,\cdot) = \pi^{(1)}(\cdot) \tag{5.9}$$

where $a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}$ are pairs of smooth compactly supported, respectively, 0 and 1, forms on Σ, and the constraints are valid

$$\pi^{(0)} = -\delta_h^{(1)} a^{(1)}\,, \quad (\Delta_h^{(0)} + m^2) a^{(0)} = \delta_h^{(1)} \pi^{(1)}\,. \tag{5.10}$$

If A is a spacelike compact solution of the Proca equation (2.3), then it satisfies (5.2)–(5.4) and its Cauchy data (5.9) satisfy the constraints (5.10). On the other hand, if we have smooth compactly supported Cauchy data (5.9), then the two Klein–Gordon equations (5.2) and (5.3) admit unique spacelike compact smooth solutions which also satisfies (5.8) as a consequence. If the said Cauchy data satisfy the constraint (5.10), then also (5.4) is satisfied, because it is equivalent to the unique solution of (5.8) with zero Cauchy data. In that case, the two solutions $A^{(0)}$ and $A^{(1)}$ define a unique solution of the Proca equation with the said Cauchy data.

We have established the following result completely extracted from the theory of normally hyperbolic equations.

Proposition 5.1. *Let* $(\mathsf{M}, g) = (\Sigma, -dt \otimes dt + h)$ *be a smooth globally hyperbolic ultrastatic spacetime with* dt *past directed, where* h *is a smooth complete Riemannian metric on* Σ. *Consider the Cauchy problem on* (M, g) *for the smooth* 1*-form* A *satisfying the Proca equation* (2.3) *for* $m^2 > 0$, *with smooth compactly supported Cauchy data* (5.9) *on* Σ *viewed as the* $t = 0$ *time slice.*
The Proca Cauchy problem for A *with constraints* (5.10) *is equivalent, regarding existence and uniqueness of spacelike compact smooth solutions , to the double normally hyperbolic Klein–Gordon constrained Cauchy problem* (5.2)–(5.4), *for the fields* $A^{(0)} \in \Gamma_{sc}(\mathsf{V}_g^{(0)})$ *and* $A^{(1)} \in \Gamma_{sc}(\mathsf{V}_g^{(1)})$, *with the same initial data* (5.9) *and constraints* (5.10). *As a consequence,*

1. *every smooth spacelike compact solution of the Proca equation* $A \in \Gamma_{sc}(\mathsf{V}_g)$ (2.3) *defines compactly supported smooth Cauchy data on* Σ *which satisfy the constraints* (5.10);
2. *if the Cauchy data are smooth, compactly supported and satisfy* (5.10), *then there is a unique smooth spacelike compact solution of the Proca equation* $A \in \Gamma_{sc}(\mathsf{V}_g)$ (2.3) *associated with them;*
3. *the support of a solution* $A \in \Gamma_{sc}(\mathsf{V}_g)$ *with smooth compactly supported initial data satisfies* $supp(A) \subset J^+(S) \cup J^-(S)$, *where* $S \subset \Sigma$ *is the union of the supports of the Cauchy data.*

Remark 5.2. (1) All the discussion above, and Proposition 5.1 in particular, extends to the case of a *complex* Proca field and corresponding associated complex Klein Gordon fields. The stated results can be extended easily to the case of the non-homogeneous Proca equation and also considering continuity properties of the solutions with respect to the source and the initial data referring to natural topologies. (See [53] for a general discussion.)

(2) A naive idea may be that we can freely fix smooth compactly supported Cauchy data for $A^{(1)}$ and then define associated Cauchy conditions for $A^{(0)}$ by solving the constraints (5.10). In this case the true degrees of freedom of the Proca field would be the vector part $A^{(1)}$, whereas $A^{(0)}$ would be a constrained degree of freedom. This viewpoint is incorrect, if we decide to deal with spacelike compact solutions, because the second constraint in Equation (5.10) in general does not produce a compactly supported function $a^{(0)}$ when the source $\delta_h^{(1)}\pi^{(1)}$ is smooth compactly supported (the smoothness of $a^{(0)}$ is, however, guaranteed by elliptic regularity from the smoothness of $\delta_h^{(1)}\pi^{(1)}$). $a^{(0)}$ is compactly supported only for some smooth compactly supported initial conditions $\pi^{(1)}$. Therefore, the linear subspace of initial data (5.9) compatible with the constraints (5.10) does not include *all* possible compactly supported initial conditions $\pi^{(1)}$ which, therefore, cannot be freely chosen.

(3) However, this space of constrained Cauchy data is non-trivial, i.e., it does not contain only zero initial conditions and in particular there are couples $(a^{(0)}, \pi^{(1)})$ such that both elements do not vanish. This is because, for every smooth compactly supported 1-form $f^{(1)}$ (with $\delta^{(1)} f^{(1)} \neq 0$ in particular) and for every smooth compactly supported 2-form $f^{(2)}$,

$$a^{(0)} := \delta_h^{(1)} f^{(1)} \qquad \pi^{(1)} := \left(\Delta_h^{(1)} + m^2\right) f^{(1)} + \delta_h^{(2)} f^{(2)}$$

are smooth, and compactly supported, they solve the non-trivial constraint in (5.10) $\delta_h^{(1)} \pi^{(1)} = (\Delta_{(0)} + m^2) a^{(0)}$ and $f^{(1)}, f^{(2)}$ can be chosen in order that neither of $a^{(0)}$ and $\pi^{(1)}$ vanishes. The easier constraint $\pi^{(0)} = -\delta_h^{(1)} a^{(1)}$ is solved by every smooth compactly supported 1-form $a^{(1)}$ by defining the smooth compactly supported 0-form $\pi^{(0)}$ correspondingly.

5.2. The Proca Symplectic Form in Ultrastatic Spacetimes

Consider two solutions $A, A' \in \Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{Ker P}$ of the Proca equation in our ultrastatic spacetime, choose $t \in \mathbb{R}$ and consider the bilinear form

$$\sigma_t^{(\mathsf{P})}(A, A') := \int_\Sigma h^\sharp(a_t^{(1)}, \pi_t^{(1)'} - da_t^{(0)'}) - h^\sharp(a_t^{(1)'}, \pi_t^{(1)} - da_t^{(0)}) \, \mathrm{vol}_h \,, \tag{5.11}$$

where we are referring to the Cauchy data on Σ of the smooth spacelike compact solutions of the Proca equation. Σ is viewed as the time slice at time t. As is well known, it is possible to define a natural symplectic form for the Proca field in general globally hyperbolic specetimes [5] with properties analogous to the ones we are going to discuss here. In this section, we, however, stick to the ultrastatic spacetime case which is enough for our ends.

According to [5] (with an argument very similar to the proof of Propositions 3.12 and 3.13 in [46]) we have immediately that

$$\sigma_t^{(\mathsf{P})}(A, A') = \sigma_{t'}^{(\mathsf{P})}(A, A') \quad \forall t, t' \in \mathbb{R} \,,$$

and, omitting the index t as the symplectic form is independent of it,

$$\sigma^{(\mathsf{P})}(A, A') = \int_{\mathsf{M}} g^\sharp \left(\mathfrak{f}, \mathsf{G_P} \mathfrak{f}'\right) \, \mathrm{vol}_g \tag{5.12}$$

where A, f (resp. A, f') are related by $A := \mathsf{G_P}\mathfrak{f}$ (resp. $A' := \mathsf{G_P}\mathfrak{f}'$).

Remark 5.3. The important identity (5.12) is also valid in a generic globally hyperbolic spacetime when $\sigma^{(P)}$ is interpreted as the general symplectic form of the Proca field according to [5].

Let us suppose to deal with the Cauchy data of the real vector space $C_\Sigma \subset \Omega_c^0(\Sigma)^2 \times \Omega_c^1(\Sigma)^2$ of smooth compactly supported Cauchy data (a_0, π_0, a_1, π_1) subjected to the linear constraints (5.10),

$$C_\Sigma := \Big\{ (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in \Omega_c^0(\Sigma)^2 \times \Omega_c^1(\Sigma)^2 \; \Big| \; \pi^{(0)} = -\delta_h^{(1)} a^{(1)} \,,$$

$$(\Delta_h^{(0)} + m^2)a^{(0)} = \delta_h^{(1)}\pi^{(1)} \Big\} \,. \tag{5.13}$$

Not only the Cauchy problem is well behaved in that space as a consequence of Proposition 5.1, but we also have the following result which, in particular, implies that the Weyl algebra of the real Proca field has trivial center.

Proposition 5.4. *The bilinear antisymmetric map $\sigma^{(P)} : C_\Sigma \times C_\Sigma \to \mathbb{R}$ defined in (5.11) is non-degenerate, and therefore, it is a symplectic form on C_Σ.*

Proof. Taking (3.8) into account, suppose that $\Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{KerP} \ni A' = \mathsf{G_P}\mathfrak{f}$ whose Cauchy data are $(a^{(0)'}, \pi^{(0)'}, a^{(1)'}, \pi^{(1)'}) \in C_\Sigma$ is such that $\sigma^{(\mathsf{P})}(A, A') = 0$ for all $A = \mathsf{G_P}\mathfrak{f} \in \Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{KerP} \equiv C_\Sigma$, we want to prove that $A' = 0$ namely, its initial conditions are $(0, 0, 0, 0)$. From (5.12), using the fact that $g^\sharp$ is non-degenerate, we have that $A' = \mathsf{G_P}\mathfrak{f}' = 0$ so that its Cauchy data are the zero data in view of the well-posedness of the Cauchy problem Proposition 5.1. □

To conclude this section we prove that, when using Cauchy data in C_Σ, the expression of $\sigma^{(\mathsf{P})}$ can be rearranged in order to make contact with the analogous symplectic forms of the two Klein–Gordon fields $A^{(0)}$ and $A^{(1)}$ the solution A is made of, as discussed in Section 5.1. Indeed, remembering the constraint $\pi^{(0)} = -\delta_h^{(1)} a^{(1)}$, and using the duality of δ and d, part of the integral in the right-hand side of (5.11) can be rearranged to

$$\begin{aligned}
&\int_\Sigma h^\sharp(a_t^{(1)}, da_t^{(0)'}) - h^\sharp(a_t^{(1)'}, da_t^{(0)}) \,\mathrm{vol}_h \\
&\quad= \int_\Sigma h^\sharp(\delta_h^{(1)} a_t^{(1)}, a_t^{(0)'}) - h^\sharp(\delta_h^{(1)} a_t^{(1)'}, a_t^{(0)}) \,\mathrm{vol}_h \\
&\quad= -\int_\Sigma h^\sharp(\pi_t^{(0)}, a_t^{(0)'}) - h^\sharp(\pi_t^{(0)'}, a_t^{(0)}) \,\mathrm{vol}_h \,.
\end{aligned}$$

As a consequence, if $\eta_i = 0$ for $i = 1$ and $\eta_i = -1$ for $i = 0$ and $h^\sharp_{(i)}$ is $h^\sharp$ for $i = 1$ and the pointwise product for $i = 0$,

$$\sigma^{(\mathsf{P})}(A, A') = \sum_{i=0}^{1} \eta_i \int_{\Sigma_t} h^\sharp_{(i)}(a_t^{(i)}, \pi_t^{(i)'}) - h^\sharp_{(i)}(a_t^{(i)'}, \pi_t^{(i)}) \,\mathrm{vol}_h \,. \tag{5.14}$$

In other words, referring to the (Klein–Gordon) symplectic forms introduced in [46] for normally hyperbolic equations (5.2) and (5.3)

$$\sigma^{(\mathsf{P})}(A, A') = \sigma^{(1)}(A^{(1)}, A^{(1)'}) - \sigma^{(0)}(A^{(0)}, A^{(0)'})$$

where $\sigma^{(k)}$ is the symplectic form for a normally hyperbolic field operator on a real vector bundle defined, e.g., [46, Proposition 3.12].

A similar result is valid for the causal propagators. Decomposing $\mathfrak{f} = \mathfrak{f}^{(0)} dt + \mathfrak{f}^{(1)} \in \Gamma_c(\mathsf{V}_g)$ where $\mathfrak{f}^{(0)} \in \Gamma_c(\mathsf{V}_g^{(0)})$ and $\mathfrak{f}^{(1)} \in \Gamma_c(\mathsf{V}_g^{(1)})$, (5.12), the analogs for scalar and vector Klein Gordon fields [46] and (5.14) imply

$$\int_{\mathsf{M}} g^\sharp(\mathfrak{f}, \mathsf{G_P}\mathfrak{f}')\mathrm{vol}_g = \int_{\mathsf{M}} h^\sharp(\mathfrak{f}^{(1)}, \mathsf{G}^{(1)}\mathfrak{f}^{(1)'})\mathrm{vol}_g - \int_{\mathsf{M}} \mathfrak{f}^{(0)}\mathsf{G}^{(0)}\mathfrak{f}^{(0)'}\mathrm{vol}_g$$

where $\mathsf{G}^{(i)}$, $i = 0, 1$ are the causal propagators for the normally hyperbolic operators

$$\mathsf{N}^{(i)} := \partial_t^2 + \Delta_h^{(i)} + m^2 I : \Gamma_{sc}(\mathsf{V}_g^{(i)}) \to \Gamma_{sc}(\mathsf{V}_g^{(i)}) \quad i = 0, 1$$

according to the theory of [46]. Here $\Delta_h^{(0)}$ coincides with the standard Laplace-Beltrami operator for scalar fields on Σ.

Remark 5.5. With the same argument, the found results immediately generalize to the case of *complex* k-forms. More precisely, if the Cauchy data belong to $C_\Sigma + iC_\Sigma$,

$$\sigma^{(\mathsf{P})}(\overline{A}, A') = \sigma^{(1)}(\overline{A^{(1)}}, A^{(1)'}) - \sigma^{(0)}(\overline{A^{(0)}}, A^{(0)'}) \,,$$

where the left-hand side is again (5.11) evaluated for complex Proca fields, i.e., complex Cauchy data. Above, the bar denotes the complex conjugation and the Cauchy data of the considered complex Proca fields satisfy the constraints (5.10). Furthermore,

$$\int_{\mathsf{M}} g^\sharp(\bar{\mathfrak{f}}, \mathsf{G}_\mathsf{P}\mathfrak{f}')\mathrm{vol}_g = \int_{\mathsf{M}} h^\sharp(\overline{\mathfrak{f}^{(1)}}, \mathsf{G}^{(1)}\mathfrak{f}^{(1)'})\mathrm{vol}_g - \int_{\mathsf{M}} \overline{\mathfrak{f}^{(0)}}\mathsf{G}^{(0)}\mathfrak{f}^{(0)'}\mathrm{vol}_g$$

where the smooth compactly supported sections are complex. We have used the same symbols as for the real case for the causal propagators since the associated operators commute with the complex conjugation. As a consequence, a standard argument about the uniqueness of Green operators implies that the causal propagators for the real case are nothing but the restriction of the causal propagator of the complex case which, in turn, are the trivial complexification of the real ones.

5.3. The Proca Energy Density in Ultrastatic Spacetimes

Starting from the Proca Lagrangian in every curved spacetime (see, e.g., [13])

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu \quad \text{with} \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

and referring to local coordinates $(x^0, \ldots, x^{n-1})$ adapted to the split $\mathsf{M} = \mathbb{R} \times \Sigma$ of our ultrastatic spacetime, where $x^0 = t$ runs along the whole $\mathbb{R}$ and $x^1, \ldots, x^{n-1}$ are local coordinates on Σ, the energy density reads in terms of initial conditions on Σ of the considered Proca field

$$\begin{aligned} T_{00} = \frac{1}{2} h^\sharp(\pi^{(1)} - da^{(0)}, \pi^{(1)} - da^{(0)}) + \frac{1}{2} h^\sharp_{(2)}(da^{(1)}, da^{(1)}) \\ + \frac{m^2}{2}\left(h^\sharp(a^{(1)}, a^{(1)}) + a^{(0)}a^{(0)}\right) \geqslant 0 \,. \end{aligned} \tag{5.15}$$

Above $h^\sharp_{(2)}$ is the natural scalar product for the 2-forms on Σ induced by the metric tensor. It is evident that the energy density is nonnegative since the metric h and its inverse $h^\sharp$ are positive by hypothesis. The total energy at time t is the integral of T_{00} on Σ, using the natural volume form, when replacing

$A^{(0)}$ and $A^{(1)}$ for the respective Cauchy data. As ∂_t is a Killing vector and the solution is spacelike compact, the total energy is finite and constant in time.

$$E^{(P)} = \frac{1}{2}\int_\Sigma \Big(h^\sharp(\pi^{(1)} - da^{(0)}, \pi^{(1)} - da^{(0)}) + h^\sharp_{(2)}(da^{(1)}, da^{(1)}) \\ + m^2\big(h^\sharp(a^{(1)}, a^{(1)}) + a^{(0)}a^{(0)}\big)\Big)\mathrm{vol}_h\,. \tag{5.16}$$

Using Hodge duality of d and δ and the definition of the Hodge Laplacian, the expression of the total energy can be rearranged to

$$E^{(P)} = \frac{1}{2}\int_\Sigma \Big(h^\sharp(\pi^{(1)}, \pi^{(1)}) + h^\sharp(da^{(0)}, da^{(0)}) - 2h^\sharp(\pi^{(1)}, da^{(0)}) - \delta_h^{(1)}a^{(1)}\delta_h^{(1)}a^{(1)} \\ + h^\sharp(a^{(1)}, \Delta_h^{(1)}a^{(1)}) + m^2(a^{(0)}a^{(0)} + h^\sharp(a^{(1)}, a^{(1)}))\Big)\mathrm{vol}_h\,.$$

Using again the Hodge duality of d and δ the third term in the integral can be rearranged to

$$-\int_\Sigma h^\sharp(\pi^{(1)}, da^{(0)})\mathrm{vol}_h = -\int_\Sigma a^{(0)}\delta_h^{(1)}\pi^{(1)}\mathrm{vol}_h\,.$$

The term $\delta^{(1)}\pi^{(1)}$ above and the term $\delta_h^{(1)}a^{(1)}\delta_h^{(1)}a^{(1)}$ appearing in the expression for the total energy can be worked out exploiting the constraints (5.10). Inserting the results in the found formula for the total energy, we finally find, with the notation already used for the symplectic form,

$$E^{(P)} = \sum_{i=0}^{1}\eta_i\frac{1}{2}\int_\Sigma h^\sharp_{(i)}(\pi^{(i)}, \pi^{(i)}) + h^\sharp_{(i)}(a^{(i)}, (\Delta_h^{(i)} + m^2 I)a^{(i)})\,\mathrm{vol}_h\,, \tag{5.17}$$

when the used Cauchy data belong to the constrained space C_Σ. It is now clear that the total energy of the Proca field is the difference between the total energies of the two Klein–Gordon fields composing it exactly as it happened for the symplectic form. This difference is, however, positive when working on smooth compactly supported initial conditions satisfying the constraints (5.10), because the found expression of the energy is the same as the one computed with the density (5.15).

Remark 5.6. With the same argument, the found result immediately generalizes to the case of *complex* k-forms and one finds

$$\sum_{i=0}^{1}\eta_i\frac{1}{2}\int_\Sigma h^\sharp_{(i)}(\overline{\pi^{(i)}}, \pi^{(i)}) + h^\sharp_{(i)}(\overline{a^{(i)}}, (\Delta_h^{(i)} + m^2 I)a^{(i)})\mathrm{vol}_h \\ = \frac{1}{2}\int_\Sigma \Big(h^\sharp(\overline{\pi^{(1)} - da^{(0)}}, \pi^{(1)} - da^{(0)}) + h^\sharp_{(2)}(\overline{da^{(1)}}, da^{(1)}) \\ + m^2\big(h^\sharp(\overline{a^{(1)}}, a^{(1)}) + \overline{a^{(0)}}a^{(0)}\big)\mathrm{vol}_h\Big) \geqslant 0 \tag{5.18}$$

where the bar over the forms denotes the complex conjugation and $(a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)})$ are *complex* forms of $C_\Sigma + iC_\Sigma$.

5.4. Elliptic Hilbert Complexes and Proca Quantum States in Ultrastatic Spacetimes

We can proceed to the construction of quasifree states. As we shall see shortly, this construction for the Proca field uses some consequences of the spectral theory applied to the theory of *elliptic Hilbert complexes* [4] defined in terms of the closure of Hodge operators in natural L^2 spaces of forms.

Some of the following ideas were inspired by [14]. However, we now work in the space of Cauchy data instead of in the space of smooth supported compacted forms and/or modes. Furthermore, we do not assume restrictions on the topology of the Cauchy surfaces used in [14] to impose a pure point spectrum to the Hodge Laplacians.

To define quasifree states for the Proca field we observe that, as P is Green hyperbolic, the CCR algebra $\mathcal{A}_g$ is isomorphic to the analogous unital $*$-algebra $\mathcal{A}_g^{(symp)}$ generated by the **solution-smeared field operators** $\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A)$, for $A \in \mathsf{Ker}_{sc}(P)$, which are $\mathbb{R}$-linear in A, Hermitian, and satisfy the commutation relations[5]

$$\left[\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A), \sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A')\right] = i\sigma^{(\mathsf{P})}(A, A')I\,. \tag{5.19}$$

The said unital $*$-algebra isomorphism $F : \mathcal{A}_g \to \mathcal{A}_g^{(symp)}$ is completely defined as the unique homomorphism of unital $*$-algebras that satisfies

$$F : \hat{\mathfrak{a}}(\mathfrak{f}) \mapsto \sigma^{(\mathsf{P})}\left(\hat{\mathfrak{a}}, \mathsf{G}_{\mathsf{P}}\mathfrak{f}\right) \quad \text{with } A = \mathsf{G}_{\mathsf{P}}\mathfrak{f}, \quad \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)\,.$$

The definition is well posed in view of (5.12), (3.8), (3.9), and the definition of $\mathcal{A}_g$. Within this framework, the two-point function ω_2 is interpreted as the integral kernel of

$$\omega\left(\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A)\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A')\right)\,.$$

In particular, its antisymmetric part is universally given by $\frac{i}{2}\sigma^{(P)}(A, A')$ due to (5.19). The specific part of the two-point function is therefore completely embodied in its symmetric part $\mu(A, A')$.

According to this observation, a general recipe for real (bosonic) CCR in generic globally hyperbolic spacetimes to define a quasifree state on the *-algebra $\mathcal{A}_g$ (e.g., see [38,39,54] for the scalar case and [21, Chapter 4, Proposition 4.9] for the generic case of real bosonic CCRs) is to assign a real scalar product on the space of spacelike compact solutions

$$\mu : \mathsf{Ker}_{sc}(\mathsf{P}) \times \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathbb{R}$$

satisfying

(a) the strict positivity requirement $\mu(A, A) \geqslant 0$ where $\mu(A, A) = 0$ implies $A = 0$;

(b) the continuity requirement with respect to the relevant symplectic form $\sigma^{(P)}$ (see, e.g., [21, Proposition 4.9]),

$$\sigma^{(\mathsf{P})}(A, A')^2 \leqslant 4\mu(A, A)\mu(A', A')\,. \tag{5.20}$$

[5] Notice that, as $\sigma^{(\mathsf{P})}(A, A')$ is non-degenerate, we have that $\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A) = 0$ only if $A = 0$.

The continuity requirement directly arises form the fact that the quasifree state induced by μ on the whole $*$-algebra $\mathcal{A}_g \equiv \mathcal{A}_g^{symp}$ according to Definition 4.3 is a positive functional. The converse implication, though true, is less trivial [21, 39]. The two mentioned requirements are nothing but the direct translation of $(2)'$ and $(3)'$ stated in the Introduction. (Regarding the latter, observe that $\sigma^{(P)}$ corresponds to the causal propagator at the level of solutions—Eq. (5.12) in our case—as discussed in Section 5.2.) At this point, it should be clear that the quasifree state defined by μ has two-point function, viewed as bilinear map on $\Gamma_c(\mathsf{V}_g) \times \Gamma_c(\mathsf{V}_g)$,

$$\omega_\mu(\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{f}')) = \omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') := \mu(\mathsf{G_P}\mathfrak{f}, \mathsf{G_P}\mathfrak{f}') + \frac{i}{2}\sigma^{(P)}(\mathsf{G_P}\mathfrak{f}, \mathsf{G_P}\mathfrak{f}')\,.$$

However, since the Cauchy problem is well posed on the time slices Σ of an ultrastatic spacetime $(\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, as proved in Proposition 5.1, we can directly define μ (and $\sigma^{(\mathsf{P})}$) in the space of Cauchy data C_Σ on Σ, for smooth spacelike compact solutions, viewed as the time slice at $t = 0$,

$$\mu : C_\Sigma \times C_\Sigma \to \mathbb{R}\,.$$

In view of the peculiarity of the Cauchy problem for the Proca field as discussed in Sect. 5.1, the real vector space of the Cauchy data C_Σ is *constrained*. We underline that working at the level of constrained initial data does not affect the construction of quasifree states. Indeed, it is sufficient that the space of constrained initial conditions is a real (or complex) vector space and that the constrained Cauchy problem is well posed. With this in mind, referring to the canonical decomposition $A = A^{(0)}dt + A^{(1)}$ of a real smooth spacelike compact solution A of the Proca equation, we remember that

$$C_\Sigma := \left\{ (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in \Omega^0_c(\Sigma)^2 \times \Omega^1_c(\Sigma)^2 \;\middle|\; \pi^{(0)} = -\delta^{(1)}_h a^{(1)}\,, \right.$$
$$\left. (\Delta^{(0)}_h + m^2)a^{(0)} = \delta^{(1)}_h \pi^{(1)} \right\}\,.$$

Above, $(a^{(0)}, \pi^{(0)}) := (A^{(0)}, \partial_t A^{(0)})|_{t=0}$ and $(a^{(1)}, \pi^{(1)}) := (A^{(1)}, \partial_t A^{(1)})|_{t=0}$.

With the said definitions and *where A denotes both a solution of Proca equation and its Cauchy data on Σ*, we have the first result.

Proposition 5.7. *Consider the $*$-algebra $\mathcal{A}_g$ of the real Proca field on the ultrastatic spacetime $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, with dt past directed and (Σ, h) a smooth complete Riemannian manifold. Let $\eta_0 := -1$, $\eta_1 := 1$ and $h^\sharp_{(j)}$ denote the standard inner product of j-forms on Σ induced by h. The bilinear map on the space C_Σ of real smooth compactly supported Cauchy data (5.13)*

$$\mu(A, A') := \sum_{j=0}^{1} \frac{\eta_j}{2} \int_\Sigma h^\sharp_{(j)}(\pi^{(j)}, (\overline{\Delta^{(j)} + m^2})^{-1/2}\pi^{(j)\prime})$$
$$+ h^\sharp_{(j)}(a^{(j)}, (\overline{\Delta^{(j)} + m^2})^{1/2} a^{(j)\prime}) \,\mathrm{vol}_h \qquad (5.21)$$

is a well-defined symmetric positive inner product which satisfies (5.20) and thus it defines a quasifree state ω_μ on $\mathcal{A}_g$ completely defined by its two-point *function*

$$\omega_\mu\left(\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{f}')\right) = \omega_{\mu 2}(\mathfrak{f},\mathfrak{f}') := \mu\left(\mathsf{G_P}\mathfrak{f},\mathsf{G_P}\mathfrak{f}'\right) + \frac{i}{2}\sigma^{(P)}\left(\mathsf{G_P}\mathfrak{f},\mathsf{G_P}\mathfrak{f}'\right) \tag{5.22}$$

where $\mathfrak{f},\mathfrak{f}' \in \Gamma_c(\mathsf{V}_g)$ *satisfy*

$$\sigma^{(P)}\left(\mathsf{G_P}\mathfrak{f},\mathsf{G_P}\mathfrak{f}'\right) = \int_{\mathsf{M}} g^\sharp(\mathfrak{f},\mathsf{G_P}\mathfrak{f}')\,\mathrm{vol}_g\,.$$

The bar over the operators in (5.21) denotes the closure in suitable Hilbert spaces of the operators originally defined on domains of compactly supported smooth functions. To explain this formalism, before starting with the proof we have to permit some technical facts about the properties of the Hodge operators at the level of L^2 spaces. Given the complete Riemannian manifold (Σ,h), with $n := \dim(\Sigma)$ consider the Hilbert space $\mathcal{H}_h := \bigoplus_{k=0}^n L^2_k(\Sigma,\mathrm{vol}_h)$, where the sum is orthogonal and $L^2_k(\Sigma,\mathrm{vol}_h)$ is the complex Hilbert space of the square-integrable k-forms with respect to the relevant *Hermitian* Hodge inner product:

$$(a|b)_k := \int_\Sigma h^\sharp_{(k)}(\overline{a},b)\,\mathrm{vol}_h\,, \quad a,b \in L^2_k(\Sigma,\mathrm{vol}_h)\,,$$

where $\overline{a}$ denotes the pointwise complex conjugation of the complex form a. The overall inner product on $\mathcal{H}_h$ will be indicated by $(\cdot|\cdot)$ and the Hilbert space adjoint of a densely defined operator $A : D(A) \to \mathcal{H}_h$, with $D(A) \subset \mathcal{H}_h$, will be denoted by $A^* : D(A^*) \to \mathcal{H}_h$. The closure of A will be denoted by the bar: $\overline{A} : D(\overline{A}) \to \mathcal{H}_h$.

If $\Omega_c(\Sigma)_\mathbb{C} := \bigoplus_{k=0}^n \Omega^k_c(\Sigma)_\mathbb{C}$ denotes the dense subspace of complex compactly supported smooth forms $\Omega^k_c(\Sigma)_\mathbb{C} := \Omega^k_c(\Sigma) + i\Omega^k_c(\Sigma)$, define the two operators (we omit the index h for shortness)

$$d := \oplus_{k=0}^n d^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C}\,, \quad \delta := \oplus_{k=0}^n \delta^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C}$$

with $d^{(n)} := 0$ and $\delta^{(0)} := 0$. Finally, introduce the Hodge Laplacian as

$$\Delta := \sum_{k=0}^n \Delta^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C} \quad \text{with } \Delta^{(k)} := \delta^{(k+1)}d^{(k)} + d^{(k-1)}\delta^{(k)}.$$

Since (Σ,h) is complete, Δ can be proved to be essentially self-adjoint, for instance exploiting the well-known argument by Chernoff [7] (or directly referring to [1]). Since Δ is essentially self-adjoint, if $c \in \mathbb{R}$, also $\Delta + cI$ is essentially self-adjoint. In particular, its unique self-adjoint extension is its closure $\overline{\Delta + cI}$.

Referring to the theory of *elliptic Hilbert complexes* developed in [4, Section 3] and focusing in particular on [4, Lemma 3.3] based on previous achievements established in [1], we can conclude that the following couple of facts are true. (The compositions of operators are henceforth defined with their natural domains: $D(A+B) := D(A) \cap D(B)$, $D(AB) = \{x \in D(B) \mid Bx \in D(A)\}$, $D(aA) := D(A)$ for $a \neq 0$, $D(0A) := \mathcal{H}_h$, and $A \subset B$ means $D(A) \subset D(B)$ with $B|_{D(A)} = A$.)

(a) The identities hold

$$\overline{d}^* = \overline{\delta}\,, \quad \overline{\delta}^* = \overline{d} \tag{5.23}$$

where * denotes the adjoint in the Hilbert space $\mathcal{H}_h$.

(b) The unique self-adjoint extension $\overline{\Delta}$ of Δ satisfies

$$\overline{\Delta} = \overline{d}\,\overline{\delta} + \overline{\delta}\overline{d} = \sum_{k=0}^{n} \overline{\Delta^{(k)}} \quad \text{with } \overline{\Delta^{(k)}} := \overline{\delta^{(k+1)}}\,\overline{d^{(k)}} + \overline{d^{k-1}}\,\overline{\delta^{(k)}}. \tag{5.24}$$

A trivial generalization of the decomposition as in (5.24) holds for $\overline{\Delta + cI} = \overline{\Delta} + cI$ with $c \in \mathbb{R}$.

We are now prompt to prove a preparatory technical lemma—necessary to establish Proposition 5.7—that will be fundamental for showing that the bilinear map μ is positive on the space C_Σ.

Lemma 5.8. *For every given $k = 0, 1, \ldots, n$, $c > 0$, and $\alpha \in \mathbb{R}$, the identities hold*

$$\begin{aligned}
(\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}} x &= \overline{d^{(k)}} (\overline{\Delta^{(k)} + cI})^\alpha x\,, \\
&\qquad \forall x \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}}) \\
(\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}} y &= \overline{\delta^{(k-1)}} (\overline{\Delta^{(k)} + cI})^\alpha y\,, \\
&\qquad \forall y \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}})\,.
\end{aligned}$$

Proof. Since $dd = 0$ and $\delta\delta = 0$, from (5.23), we also have $\overline{d}\,\overline{d}x = 0$ if $x \in D(\overline{d})$ and $\overline{\delta}\,\overline{\delta}y = 0$ if $y \in D(\overline{\delta})$, and thus (5.24) yields[6]

$$\overline{d}\,\overline{\Delta} \supset \overline{d}\,\overline{\delta}\,\overline{d} = \overline{\Delta}\,\overline{d}\,.$$

However, if $D(\overline{d}\,\overline{\Delta}) \supsetneq D(\overline{d}\,\overline{\delta}\,\overline{d})$, we would have $x \in D(\overline{\Delta}) = D(\overline{\delta}\,\overline{d}) \cap D(\overline{d}\,\overline{\delta})$ such that $\overline{\Delta}x = \overline{\delta}\,\overline{d}x + \overline{d}\,\overline{\delta}x \in D(\overline{d})$, but $x \notin D(\overline{d}\overline{\delta}\overline{d})$, namely $\overline{\delta}\,\overline{d}x \notin D(\overline{d})$. This is impossible since $\overline{\delta}\,\overline{d}x + \overline{d}\,\overline{\delta}x \in D(\overline{d})$, $D(\overline{d})$ is a subspace and $\overline{d}\,\overline{\delta}x \in D(\overline{d})$ (and more precisely $\overline{d}\,\overline{d}\,\overline{\delta}x = 0$ as stated above). Therefore,

$$\overline{d}\,\overline{\Delta} = \overline{d}\,\overline{\delta}\,\overline{d} = \overline{\Delta}\,\overline{d}$$

and the same result is valid with δ in place of d. Evidently, in both cases $\overline{\Delta}$ can be replaced by the self-adjoint operator $\overline{\Delta} + cI = \overline{\Delta + cI}$ for every $c \in \mathbb{R}$:

$$\overline{d}\,\overline{\Delta + cI} = \overline{\Delta + cI}\,\overline{d}\,, \quad \overline{\delta}\,\overline{\Delta + cI} = \overline{\Delta + cI}\,\overline{\delta}\,. \tag{5.25}$$

We henceforth assume $c > 0$. In that case, as Δ is already positive on its domain, the spectrum of the self-adjoint operator $\overline{\Delta + cI}$ is strictly positive and thus $\overline{\Delta + cI}^{-1} : \mathcal{H}_h \to D(\overline{\Delta + cI})$ is well defined, self-adjoint and bounded. The former identity in (5.25) also implies that $D(\overline{d}\,\overline{\Delta + cI}) = D(\overline{\Delta + cI}\,\overline{d})$, so that

$$\overline{\Delta + cI}^{-1} \overline{d}\,\overline{\Delta + cI}|_{D(\overline{d}\,\overline{\Delta + cI})} x = \overline{d}|_{D(\overline{d}\,\overline{\Delta + cI})} x\,.$$

[6]It holds $(B + C)A = BC + BA$, but $AB + AC \subset A(B + C)$.

By construction, we can choose $x = \overline{\Delta + cI}^{-1} y$ with $y \in D(\overline{d})$ in view of the definition of the natural domain of the composition $\overline{d}\,\overline{\Delta + cI})$. In summary,

$$\overline{\Delta + cI}^{-1}\overline{d}y = \overline{d}\,\overline{\Delta + cI}^{-1} y\,, \quad \forall y \in D(\overline{d})\,.$$

Since the argument is also valid for δ, we have established that

$$\overline{\Delta + cI}^{-1}\overline{d} \subset \overline{d}\,\overline{\Delta + cI}^{-1}\,, \quad \overline{\Delta + cI}^{-1}\overline{\delta} \subset \overline{\delta}\,\overline{\Delta + cI}^{-1}$$

Iterating the argument, for every $n = 0, 1, \ldots$,

$$(\overline{\Delta + cI}^{-1})^n\overline{d} \subset \overline{d}\,(\overline{\Delta + cI}^{-1})^n\,, \quad (\overline{\Delta + cI}^{-1})^n\overline{\delta} \subset \overline{\delta}\,(\overline{\Delta + cI}^{-1})^n\,.$$

This result extends to complex polynomials of $\overline{\Delta + cI}^{-1}$ in place of powers by linearity. Using the spectral calculus (see, e.g., [41]) where $\mu_{xy}(E) = (x|P_E y)$ and P is the projector-valued spectral measure of $\overline{\Delta + cI}^{-1}$, the found result for $\overline{d}$ can be written

$$\int_{[0,b]} p(\lambda) d\mu_{x,\overline{d}y}(\lambda) = \int_{[0,b]} p(\lambda) d\mu_{\overline{\delta}x,y}(\lambda) \tag{5.26}$$

for every complex polynomial p, where $[0, b]$ is a sufficiently large interval to include the (bounded positive) spectrum of $\overline{\Delta + cI}^{-1}$, $x \in D(\overline{\delta})$, $y \in D(\overline{d})$, and where we have used $\overline{\delta} = \overline{d}^*$. Since the considered regular Borel complex measures are finite and supported on the compact $[0, b]$, we can pass in (5.26) from polynomials p to generic continuous functions f in view of the Stone–Weierstrass theorem. At this juncture, $P_E^* = P_E$ and the uniqueness part of Riesz' representation theorem for regular complex Borel measures, implies that

$$(P_E\overline{\delta}y|x) = (P_E y|\overline{d}x) \quad \text{for all } x \in D(\overline{\delta}), y \in D(\overline{d}), \text{ and every Borel set } E \subset \mathbb{R}.$$

which means $P_E\overline{\delta} \subset \overline{d}^* P_E$, namely $P_E\overline{\delta} \subset \overline{\delta}P_E$. Analogously, we also have $P_E\overline{d} \subset \overline{d}P_E$.

If $f : \mathbb{R} \to \mathbb{C}$ is measurable and *bounded*, the standard spectral calculus and (5.23), with a procedure similar to the one used to prove $P_E\overline{\delta} \subset \overline{\delta}P_E$ and taking into account the fact that $D(f(\overline{\Delta + cI}^{-1})) = \mathcal{H}_h$, yields

$$f(\overline{\Delta + cI}^{-1})\overline{\delta} \subset \overline{\delta} f(\overline{\Delta + cI}^{-1})\,, \quad f(\overline{\Delta + cI}^{-1})\overline{d} \subset \overline{d} f(\overline{\Delta + cI}^{-1}) \tag{5.27}$$

If f is unbounded, we can choose a sequence of bounded measurable functions f_n such that $f_n \to f$ pointwise. It is easy to prove that (see, e.g., [41]) $x \in D(\int_{\mathbb{R}} f dP)$ entails $\int_{\mathbb{R}} f_n dPx \to \int_{\mathbb{R}} f dPx$. This is the case for instance for $f(\lambda) = \lambda^\beta$ with $\beta < 0$ restricted to $[0, b]$. Referring to this function and the pointed out result for some sequence of bounded functions with $f_n \to f$ pointwise, the latter of (5.27) implies that[7] ,

$$(\overline{\Delta + cI})^\alpha\overline{d}x = \overline{d}(\overline{\Delta + cI})^\alpha x \quad \text{if } x \in D((\overline{\Delta + cI})^\alpha) \cap D(\overline{d}) \text{ and } \overline{d}x \in D((\overline{\Delta + cI})^\alpha),$$

[7]Below, $\alpha > 0$ otherwise $(\overline{\Delta + cI})^\alpha$ is bounded in view of its spectral properties and (5.27) is enough to conclude the proof.

where we used also the fact that $\overline{d}$ is closed. The case of δ can be worked out similarly. Summing up, we have proved that, if $\alpha \in \mathbb{R}$,

$$(\overline{\Delta + cI})^\alpha \overline{d} x = \overline{d}(\overline{\Delta + cI})^\alpha x\,, \quad \forall x \in D((\overline{\Delta + cI})^\alpha) \cap D((\overline{\Delta + cI})^\alpha \overline{d})$$
$$(\overline{\Delta + cI})^\alpha \overline{\delta} y = \overline{\delta}(\overline{\Delta + cI})^\alpha y\,, \quad \forall y \in D((\overline{\Delta + cI})^\alpha) \cap D((\overline{\Delta + cI})^\alpha \overline{\delta})\,.$$

Let us remark that for $\alpha \leqslant 0$ it is sufficient to choose $x \in D(\overline{d})$ and $y \in D(\overline{\delta})$. For every given $k = 0, 1, \ldots, n$, $c > 0$, and $\alpha \in \mathbb{R}$, taking the decomposition of $\mathcal{H}_h$ into account the above formulae imply

$$(\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}} x = \overline{d^{(k)}}(\overline{\Delta^{(k)} + cI})^\alpha x\,,$$
$$\forall x \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}})$$
$$(\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}} y = \overline{\delta^{(k-1)}}(\overline{\Delta^{(k)} + cI})^\alpha y\,,$$
$$\forall y \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}})\,.$$

That is the thesis. □

We are now prompted to prove that the bilinear map defined by Equation (5.21) defines a quasifree state defined by the two-point function given by (5.22) establishing the thesis of Proposition 5.7.

Proof of Proposition 5.7. To continue with the proof of the proposition, we now demonstrate that μ is well defined and positive. That bilinear form is well defined because $\Omega_c^{(j)}(\Sigma) \subset D(\overline{\Delta^{(j)} + m^2 I}^\alpha)$ for $\alpha \leqslant 1$ as one immediately proves from spectral calculus. Furthermore, the integrand in the right-hand side of Equation (5.21) is the linear combination of products of L^2 functions (of which one of the two has also compact support). Let us pass to the positivity issue. Our strategy is to rewrite $\mu(A, A)$, where $A = (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in C_\Sigma$, as the quadratic form of the energy $\mu(A, A) = E^{(P)}(A_o)$, where the right-hand side is defined in Equation (5.16), for a new set of initial data A_o which are not necessarily smooth and compactly supported but such that $E^{(P)}(A_o)$ is well defined. If $A \in C_\Sigma$, define for $j = 0, 1$

$$\begin{aligned} A_o &= (a_o^{(0)}, \pi_o^{(0)}, a_o^{(1)}, \pi_o^{(1)}) \\ a_o^{(j)} &:= (\overline{\Delta^{(j)} + m^2 I})^{-1/4} a^{(j)} \\ \pi_o^{(j)} &:= (\overline{\Delta^{(j)} + m^2 I})^{-1/4} \pi^{(j)} \end{aligned} \tag{5.28}$$

Notice that the definition is well posed and the forms $a_o^{(j)}$ and $\pi_o^{(j)}$ belong to the respective Hilbert spaces of j-forms, because $\Omega_c^{(j)}(\Sigma) \subset D(\overline{\Delta^{(j)} + m^2 I}^\alpha)$ for $\alpha \leqslant 1$ as said above. Furthermore, the new forms are real since the initial ones are real and $\overline{\Delta^{(j)} + m^2 I}^\alpha$ commutes with the complex conjugation.[8] At this juncture, we have from (5.21)

$$\mu(A, A) = \sum_{j=0}^{1} \eta_j \int_\Sigma h^\sharp_{(j)}(\pi_o^{(j)}, \pi_o^{(j)}) + h^\sharp_{(j)}(a_o^{(j)}, (\overline{\Delta^{(j)} + m^2 I}) a_o^{(j)}) \mathrm{vol}_h \tag{5.29}$$

[8]It easily arises from spectral calculus using the fact that the complex conjugation is bijective from $\mathcal{H}_h$ to $\mathcal{H}_h$, continuous, and commutes with $\overline{\Delta^{(j)} + m^2 I}$.

Furthermore, though the new Cauchy data stay outside C_Σ in general, they, however, satisfy the natural generalization of the constraints defining C_Σ in view of Lemma 5.8:

$$\pi_o^{(0)} = -\overline{\delta_h^{(1)}} a_o^{(1)}\,, \quad \overline{(\Delta_h^{(0)} + m^2)} a_o^{(0)} = \overline{\delta_h^{(1)}} \pi_o^{(1)}\,. \tag{5.30}$$

These identities arise immediately from Definitions (5.28), the constraints (5.10), and by applying Lemma 5.8 and paying attention to the fact that $\Omega_c^{(j)}(\Sigma) \subset D((\overline{\Delta^{(j-1)} + cI})^\alpha \overline{\delta^{(j)}})$ for every $\alpha \leqslant 1$ and also using $(\overline{\Delta^{(j)} + m^2 I})$ $(\overline{\Delta^{(j)} + m^2 I})^{-1/4} = (\overline{\Delta^{(j)} + m^2 I})^{-1/4} \overline{\Delta^{(j)} + m^2 I}$ (e.g., [41, (f) in Proposition 3.60]). Using (5.23) and (5.30) in the right-hand side of (5.29), we can proceed backwardly as in the proof that (5.16) is equivalent to (5.17). Indeed, the only ingredients we used in that proof were the constraint equations which are valid also for A_o and the duality of δ and d with respect to the Hodge inner product, which extends to $\overline{\delta}$ and $\overline{d}$. In summary,

$$\mu(A, A) = \frac{1}{2}\int_\Sigma \Big(h^\sharp_{(1)}(\pi_o^{(1)} - \overline{d^{(0)}} a_o^{(0)}, \pi_o^{(1)} - \overline{d^{(0)}} a_o^{(0)}) + h^\sharp_{(2)}(\overline{d^{(1)}} a_o^{(1)}, \overline{d^{(1)}} a_o^{(1)})$$
$$+ m^2 \big(h^\sharp_{(1)}(a_o^{(1)}, a_o^{(1)}) + a_o^{(0)} a_o^{(0)} \big) \Big) \mathrm{vol}_h\,.$$

From that identity, it is clear that $\mu(A, A) \geqslant 0$ and $\mu(A, A) = 0$ implies $A_o = 0$, which in turn yields $A = 0$ because the operators $\overline{\Delta^{(j)} + m^2 I}^{1/4}$ are injective. We have established that $\mu : C_\Sigma \times C_\Sigma \to \mathbb{R}$ is a positive real symmetric inner product.

Let us pass to prove (5.20). First of all, we change the notation concerning the scalar product μ making explicit the decomposition of A, and we work with *complex* valued forms. We use

$$A = (a, \pi) = (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)})\,, \quad a := (a^{(0)}, a^{(1)})\,, \quad \pi := (\pi^{(0)}, \pi^{(1)})$$

so that, if $(a, \pi), (a', \pi') \in (L^2_0(\Sigma, \mathrm{vol}_h) \oplus L^2_1(\Sigma, \mathrm{vol}_h)) \times (L^2_0(\Sigma, \mathrm{vol}_h) \oplus L^2_1$ $(\Sigma, \mathrm{vol}_h))$ are such that the right-hand side below is defined, we can write

$$\mu((\overline{a}, \overline{\pi}), (a', \pi')) := \sum_{j=0}^{1} \frac{\eta_j}{2} \int_\Sigma h^\sharp_{(j)}(\overline{\pi^{(j)}}, H^{-1}_{(j)} \pi^{(j)\prime}) + h^\sharp_{(j)}(\overline{a^{(j)}}, H_{(j)} a^{(j)\prime}) \mathrm{vol}_h$$

where $H_{(j)} := \overline{\Delta^{(j)} + m^2 I}^{1/2}$, and the bar on forms denotes the complex conjugation. Finally, for $\alpha = \pm 1$, we defined

$$H^\alpha a := (H^\alpha_{(0)} a^{(0)}, H^\alpha_{(1)} a^{(1)})\,, \quad H^\alpha \pi := (H^\alpha_{(0)} \pi^{(0)}, H^\alpha_{(1)} \pi^{(1)})\,.$$

By direct inspection, one sees that, if $(a, \pi), (a', \pi') \in C_\Sigma + iC_\Sigma$, then the right-hand side of the first identity below is well defined and

$$\Lambda((a, \pi), (a', \pi')) := \frac{1}{2}\mu\left((\overline{\pi} + iH^{-1}\overline{a}, \overline{a} - iH\overline{\pi}), (\pi' - iH^{-1}a', a' + iH\pi')\right)$$
$$= \mu((\overline{a}, \overline{\pi}), (a', \pi')) + \frac{i}{2}\sigma^{(\mathsf{P})}((\overline{a}, \overline{\pi}), (a', \pi'))$$

where $\sigma^{(P)}$ is the right-hand side of (5.14), which, however, coincides with the original symplectic form (5.11) evaluated on complex Cauchy data because

$(a,\pi),(a',\pi') \in C_\Sigma + iC_\Sigma$ and Remark 5.5 holds. Finally notice that if $(a,\pi) \in C_\Sigma + iC_\Sigma$ then $a_o := \pi - iHa$ and $\pi_o := a + iH^{-1}\pi$ satisfy the constraints (though they do not belong to $C_\Sigma + iC_\Sigma$ in general)

$$\pi_o^{(0)} = -\overline{\delta_h^{(1)}} a_o^{(1)} \,, \quad H_{(0)} a_o^{(0)} = \overline{\delta_h^{(1)}} \pi_o^{(1)} \,.$$

The proof is direct, using Lemma 5.8 once more. As a consequence, exploiting the same argument to prove (5.18) and observing that H^α commutes with the complex conjugation—so that it holds $\overline{\pi} - iH^{-1}\overline{a} = \overline{\pi + iH^{-1}a}$ for instance—we have that

$$\begin{aligned} 2\Lambda((a,\pi),(a'\pi')) &= \mu\left((\overline{\pi} + iH^{-1}\overline{a}, \overline{a} - iH\overline{\pi}), (\pi - iH^{-1}a, a + iH\pi)\right) \\ &= \mu\left(\overline{(\pi - iH^{-1}a, a + iH\pi)}, (\pi - iH^{-1}a, a + iH\pi)\right) \geqslant 0 \,. \end{aligned}$$

The final inequality is due to the fact that μ is (the complexification of) a *real* positive bilinear symmetric form. All that means in particular that the *Hermitian* form Λ on $(C_\Sigma + iC_\Sigma) \times (C_\Sigma + iC_\Sigma)$ is (semi)positively defined and thus it satisfies the Cauchy–Schwartz inequality. In particular,

$$(Im\Lambda((a,\pi),(a',\pi')))^2 \leqslant |\Lambda((a,\pi),(a',\pi'))|^2 \leqslant \Lambda((a,\pi),(a,\pi))\, \Lambda((a',\pi'),(a',\pi')) \,.$$

If choosing $(a,\pi),(a',\pi') \in C_\Sigma$ (thus *real* forms), the above inequality specializes to

$$\sigma^{(\mathsf{P})}((a,\pi),(a',\pi'))^2 \leqslant 4\mu((a,\pi),(a,\pi))\, \mu((a',\pi'),(a',\pi'))$$

which is the inequality (5.20) we wanted to prove. □

5.5. Hadamard States in Ultrastatic and Generic Globally Hyperbolic Spacetimes

With the next proposition, we show that the quasifree states defined in Proposition 5.7 is a Hadamard state when (Σ, h) is of bounded geometry. To prove the assertion, we will take advantage of the general formalism developed in [21] and [24]. An alternative proof, which does not assume that the manifold is of bounded geometry (however, we here take advantage of [32]), could be constructed along the procedure developed in [19] and extending it to the vectorial Klein–Gordon field.

Proposition 5.9. *If the metric h on the time slice Σ is of bounded geometry, then the quasifree state $\omega_\mu : \mathcal{A}_g \to \mathbb{C}$ defined in Proposition 5.7 is Hadamard according to Definition 4.5.*

Proof. Consider a pair of *complex* Klein–Gordon fields $A^{(0)}$ and $A^{(1)}$ in the ultrastatic spacetime $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, with (Σ, h) a smooth complete Riemannian manifold of bounded geometry obeying the normally hyperbolic equations (5.2) and (5.3) in the respective vector bundles on M, according to Section 5.1. We stress that we now assume that the two fields are complex. Referring to [21, Chapter 4], we define the *covariances*, for $j = 0, 1$

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)'}) := \frac{1}{2} \int_\Sigma h^\sharp_{(j)}(\overline{\pi^{(j)}}, H^{-1}_{(j)} \pi^{(j)'}) + h^\sharp_{(j)}(\overline{a^{(j)}}, H_{(j)} a^{(j)'}) \,\mathrm{vol}_h$$

$$+ \frac{i}{2}\sigma^{(j)}(\overline{A^{(j)}}, A^{(j)'}) \tag{5.31}$$

$$\lambda^-_{(j)}(A^{(j)}, A^{(j)'}) := \frac{1}{2}\int_\Sigma h^\sharp_{(j)}(\pi^{(j)'}, H^{-1}_{(j)}\overline{\pi^{(j)}}) + h^\sharp_{(j)}(a^{(j)'}, H_{(j)}\overline{a^{(j)}}) \operatorname{vol}_h$$
$$+ \frac{i}{2}\sigma^{(j)}(A^{(j)'}, \overline{A^{(j)}}) \tag{5.32}$$

where $H_{(j)} := \overline{\Delta^{(j)} + m^2}^{1/2}$, $\sigma^{(j)}$ are the symplectic forms of the corresponding Klein–Gordon fields taking place in the right-hand side of (5.14), now evaluated on complex fields. Above, $a^{(j)}, \pi^{(j)} \in \Omega^j_c(\Sigma)_{\mathbb{C}}$ are the Cauchy data on Σ of $A^{(j)}$, respectively, and $a^{(j)'}, \pi^{(j)'} \in \Omega^j_c(\Sigma)_{\mathbb{C}}$ are the Cauchy data on Σ of $A^{(j)'}$, respectively. Notice that we are not imposing constraints on these initial data since we are dealing with independent Klein–Gordon fields. $\lambda^\pm_{(j)}$ are evidently positive because, if all involved forms in the right-hand side are smooth and compactly supported, then the right-hand side of the identity above is well defined and

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)'}) := \frac{1}{2}\int_\Sigma h^\sharp_{(j)}(\overline{H^{1/2}a^{(j)} + iH^{-1/2}\pi^{(j)}}, H^{1/2}_{(j)}a^{(j)'}$$
$$+iH^{-1/2}\pi^{(j)'}) \operatorname{vol}_h .$$

The case of $\lambda^-_{(j)}$ is strictly analogous. Furthermore,

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)'}) - \lambda^-_{(j)}(A^{(j)}, A^{(j)'}) = i\sigma^{(j)}(\overline{A^{(j)}}, A^{(j)'}) .$$

Therefore, $\lambda^\pm_{(j)}$ satisfy the hypotheses of [21, Proposition 4.14][9] so that they define a pair, for $j = 0, 1$, of gauge-invariant quasifree states for the complex Klein–Gordon fields, respectively, associated with Equations (5.2) and (5.3). We pass to prove that both states are Hadamard exploiting the fact that (Σ, h) is of bounded geometry. By rewriting the covariances $\lambda^\pm_{(j)}$ as $\lambda^\pm_{(j)} = \pm qc^\pm_{(j)}$ ($q = i\sigma^{(j)}$), a quick computation shows that

$$c^\pm_{(j)} = \frac{1}{2}\begin{bmatrix} I & \pm H^{-1}_{(j)} \\ \pm H_{(j)} & I \end{bmatrix} .$$

We can immediately realize that the operator $c^\pm_{(j)}$ is the same Hadamard projector obtained in [24, Section 5.2][10]—see also [21, Section 11] for a more introductory explanation for the scalar case. This operator belongs to the necessary class of pseudodifferential operators $C^\infty_b(\mathbb{R}; \Psi^1_b(\Sigma))$ because (Σ, h) is of bounded geometry. Therefore, on account of [24, Proposition 5.4], the two quasifree states associated with $\lambda^+_{(j)}$, for $A^{(j)}$ and $j = 0, 1$, are Hadamard.

[9] The reader should pay attention to the fact that the Cauchy data used in [21], *in the complex case*, are defined as $(f_0, f_1) := (a, -i\pi)$ *instead of our* (a, π)! This is evident by comparing (2.4) and (2.20) in [21]. With the choice of [21], $i\overline{(f_0, f_1)}^t \cdot q(f'_0, f'_1) = \int \overline{f_0} f'_1 + \overline{f_1} f'_0 \operatorname{vol}_h = i\sigma(\overline{(a, \pi)}, (a', \pi'))$, where $\cdot q \equiv \sigma_1$ (the Pauli matrix) according to [21].

[10] It follows immediately since $b^+(t) = -b^-(t) = H := \overline{\Delta^{(j)} + m^2 I}^{1/2}$.

In other words, the Schwartz kernels provided by the two-point functions $\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)$, viewed as distributions of $\Gamma(\mathsf{V}_g^{(j)} \boxtimes \mathsf{V}_g^{(j)})'$, satisfy

$$WF(\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = \mathcal{H}\,,$$

where $\mathcal{H}$ is defined in (4.1) and $\mathsf{G}^{(i)}$, $i = 0, 1$ are the causal propagators for the normally hyperbolic operators

$$\mathsf{N}^{(i)} := \partial_t^2 + \Delta_h^{(i)} + m^2 I : \Gamma_{sc}(\mathsf{V}_g^{(i)}) \to \Gamma_{sc}(\mathsf{V}_g^{(i)}) \quad i = 0, 1\,.$$

Above and from now on, we use the same notation to indicate a bidistribution and the associated Schwartz kernel. Notice that we have used the same symbol $\mathsf{G}^{(j)}$ of the causal propagator we used for the real vector field case. This is because the causal propagators for the complex fields are the direct complexification of the scalar case (see Remark 5.5). We pass now to focus on the expression of $\omega_{\mu 2}$ provided in (5.22) taking the usual decomposition $\Omega^1_c(\mathsf{M})_{\mathbb{C}} \ni \mathfrak{f} = \mathfrak{f}^{(0)} dt + \mathfrak{f}^{(1)}$ into account. It can be written

$$\omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') = \omega^{(1)}_{\mu 2}(\mathfrak{f}^{(1)}, \mathfrak{f}^{(1)'}) - \omega^{(0)}_{\mu 2}(\mathfrak{f}^{(0)}, \mathfrak{f}^{(0)'})$$

where, comparing (5.21) and (5.22) with (5.31) for *real* arguments $\mathfrak{f}, \mathfrak{f}' \in \Gamma(\mathsf{V}_g)$, we find

$$\omega^{(j)}_{\mu 2}(\mathfrak{f}^{(j)}, \mathfrak{f}^{(j)'}) = \lambda^+_{(j)}(\mathsf{G}^{(j)}\mathfrak{f}^{(j)}, \mathsf{G}^{(j)}\mathfrak{f}^{(j)'})\,.$$

We have

$$WF(\pm\omega^{(j)}_{\mu 2}) = WF(\pm\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = WF(\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = \mathcal{H} \quad \text{for } j = 0, 1.$$

Taking (5.5) into account, we now observe that $\omega_{\mu 2} \in \Gamma(\mathsf{V}_g \boxtimes \mathsf{V}_g)' = \Gamma((\mathsf{V}_g^{(0)} \oplus V_g^{(1)}) \boxtimes (\mathsf{V}_g^{(0)} \oplus V_g^{(1)}))'$. As a matter of fact, however, $\omega_{\mu 2}$ does not have mixed components acting on sections of $V_g^{(1)} \boxtimes \mathsf{V}_g^{(0)}$ and $V_g^{(0)} \boxtimes \mathsf{V}_g^{(1)}$ and the only components of that distribution are those which work on sections of $V_g^{(0)} \boxtimes \mathsf{V}_g^{(0)}$ and $V_g^{(1)} \boxtimes \mathsf{V}_g^{(1)}$. These are, respectively, represented by $-\omega^{(0)}_{\mu 2}$ and $\omega^{(1)}_{\mu 2}$ whose wavefront set is $\mathcal{H}$ in both cases. The remaining two components have empty wavefront set since they are the zero distributions. Applying the definition of wavefront set of a vector-valued distribution [51], we conclude that

$$WF(\omega_{\mu 2}) = WF(-\omega^{(0)}_{\mu 2}) \cup WF(\omega^{(1)}_{\mu 2}) \cup \emptyset \cup \emptyset = \mathcal{H} \cup \mathcal{H} \cup \emptyset \cup \emptyset = \mathcal{H}\,,$$

concluding the proof. □

Combining the results obtained so far, we get the main result of this paper.

Theorem 5.10. *Let* (M, g) *be a globally hyperbolic spacetime and refer to the* CCR*-algebra* $\mathcal{A}_g$ *of the real Proca field. Then there exists a quasifree Hadamard state on* $\mathcal{A}_g$.

Proof of Theorem 4. As already explained in the beginning of Sect. 5, for any globally hyperbolic spacetime (M, g), there exists a paracausally related globally hyperbolic spacetime (M, g_0) which is ultrastatic and whose spatial metric is of bounded geometry. In particular, in this class of spacetimes, the quasifree states defined in Proposition 5.7 satisfy the microlocal spectrum condition, as proved in Proposition 5.9. Therefore, since the pullback along a Møller $*$-isomorphism preserves the Hadamard condition on account of Theorem 4.9, we can conclude. □

6. Comparison with Fewster–Pfenning's Definition of Hadamard States

Though the paper [14] by Fewster and Pfenning concerns *quantum energy inequalities*, it also offers a general theoretical discussion about the algebraic quantization of the Proca and the Maxwell fields in curved spacetime. In particular, the authors propose a definition of a Hadamard state which appears to be technically different from ours at first glance, even if it shares a number of important features with ours. This section is devoted to a comparison of the two definitions for the Proca field.

6.1. Proca Hadamard States According to Fewster and Pfenning

The definition of Hadamard state stated in [14, Equation (35)] is formulated in terms of causal normal neighborhoods of smooth spacelike Cauchy surfaces (see also below) and the *global Hadamard parametrix* for distributions which are bisolutions of the vectorial Klein–Gordon equation. Our final goal is to prove an equivalence theorem of our definition of Hadamard state Definition 4.5 and the one adopted in [14].

As a first step, we translate the original Fewster–Pfenning's definition of a Hadamard state into an equivalent form which will turn out to be more useful for our comparison. The equivalence of the version stated below of Fewster–Pfenning's definition and the original one was established in [14, Section III C] (see also the comments under Definition 6.1).

Definition 6.1 (**Fewster–Pfenning's definition of Proca Hadamard state**). Consider the globally hyperbolic spacetime (M, g) and a state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g). ω is called **Hadamard** if it is quasifree and its two-point function has the form

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{h})) = W_g(\mathfrak{f}, Q\mathfrak{h}) \tag{6.1}$$

$\forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$, where $Q : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ in the differential operator $Q = \mathrm{Id} + m^{-2}(d\delta_g)$. Above, $W_g \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ is a Klein–Gordon distributional bisolution such that

$$W_g(\mathfrak{f}, \mathfrak{g}) - W_g(\mathfrak{g}, \mathfrak{f}) = i\mathsf{G}_{\mathsf{N}}(\mathfrak{f}, \mathfrak{g}) \mod C^\infty , \tag{6.2}$$

G_{N} being the causal propagator of the Klein–Gordon operator (3.2) and which satisfies the microlocal spectrum condition

$$WF(W_g) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \setminus \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \triangleright 0\} .$$

(6.3)

Remark 6.2. The equivalence of Definition 6.1 and the original one stated in [14] relies on Sahmann–Verch's [51] generalization to vector (and spinor) fields of some classic Radzikowski results originally formulated for scalar fields. In practice, (a) if a distribution which is a bisolution of the vectorial Klein–Gordon equation and it is of Hadamard form in a normal causal neighborhoods of a smooth spacelike Cauchy surface, then it necessarily has the wavefront set of the form (6.3) ((a) [51, Theorem 5.8]) and its antisymmetric part satisfies (6.2) directly from the definition of parametrix; (b) if a distribution which is a bisolution of the vectorial Klein–Gordon equation satisfies (6.3) and (6.2), then it is of Hadamard form in some normal causal neighborhoods of a smooth spacelike Cauchy surface (see [51, Remark 5.9. (i)]).

For the Proca fields, it has been established in [14] the property of propagation of the Hadamard condition stated in the next proposition. That result was already established for the Hadamard states of scalar and vector (including spinor) fields in [20,39,51] (see [38,42] for a general recap for the KG scalar field). The pivotal tool is the already mentioned notion of *causal normal neighborhood $\mathcal{N}$ of a smooth spacelike Cauchy surface* Σ in a globally hyperbolic spacetime $(\mathsf{M}; g)$. The notion introduced in [39] has been recently improved (closing a gap in the geometric definition of Hadamard states) in [42].[11] The propagation results established in [39,51] and [14] are valid with the improved notion of causal normal neighborhoods and Hadamard states of [42].

Proposition 6.3. *Let $\omega : \mathcal{A}_g \to \mathbb{C}$ be a quasifree state for the Proca field in the globally hyperbolic spacetime (M, g). Let $\mathcal{N}$ be causal normal neighborhood of a Cauchy surface Σ of (M, g). Suppose that the restriction of ω to $(\mathcal{N}, g|_{\mathcal{N}})$ is Hadamard according to Definition 6.1. Then ω is Hadamard in (M, g) according to the same definition.*

Remark 6.4. In order to compare Proposition 4.7 and Proposition 6.3, we stress that the requirement that the neighborhood $\mathcal{N}$ of a Cauchy surface is causal normal can be relaxed also in Proposition 6.3 to make contact with our Proposition 4.7. One may only assume that $(\mathcal{N}, g|_{\mathcal{N}})$ is globally hyperbolic also therein. That is a consequence of the following facts.

(a) Every causal normal neighborhood $\mathcal{N} \subset \mathsf{M}$ of a Cauchy surface Σ of (M, g) is, by definition [39,42], a globally hyperbolic spacetime with respect to the restriction of the metric and Σ is also a Cauchy surface in $(\mathcal{N}, g|_{\mathcal{N}})$.
(b) Every smooth spacelike Cauchy surface admits a causal normal neighborhood [39,42].
(c) According to the proof of [39, Lemma 2.2] whose validity extends to [42], every neighborhood of a smooth spacelike Cauchy surface includes a causal normal neighborhood of that Cauchy surface.[12]

[11] Where these open sets are named normal neighborhoods of smooth spacelike Cauchy surfaces, omitting "causal."
[12] Essentially because convex normal neighborhoods of points form a topological basis of any spacetime and in view of [42, Proposition 9].

The smoothness property corresponding to our Proposition 4.6 also holds for Hadamard bisolutions in the sense of Fewster–Pfenning. In [14], it is an immediate consequence of (6.1) and the analogous feature of Klein–Gordon bisolutions (see the discussion on p. 4488 in [14]).

Proposition 6.5. *Let $\omega, \omega' \in \Gamma'_c(V_g \boxtimes V_g)$ be a pair of bisolutions of the Proca equation satisfying the Hadamard condition (6.1) for corresponding Klein–Gordon bisolutions which, in turn, satisfy (6.2). Then, the differences between the two bisolutions are smooth: $\omega - \omega' \in \Gamma(V_g \boxtimes V_g)$.*

Finally, [14] also contains a proof of the existence of Hadamard states for the Proca (and the Maxwell) field in globally hyperbolic spacetimes with compact Cauchy surfaces (whose first homology group is trivial when treating the Maxwell field). This proof establishes first the existence in ultrastatic spacetimes and next it exploits a standard deformation argument [54].

6.2. An (Almost) Equivalence Theorem

We are in a position to state and prove our equivalence result.

Theorem 6.6. *Consider the globally hyperbolic spacetime (M, g) and a quasifree state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the $*$-algebra of observables on (M, g) of the real Proca field. Let $\omega_2 \in \Gamma'_c(V_g \boxtimes V_g)$ be the two-point function of ω. The following facts are true.*

- (a) *If ω is Hadamard according to Definition 6.1, then it is also Hadamard according to Definition 4.5.*
- (b) *If (M, g) admits a Proca quasifree Hadamard state according to Definition 6.1 and ω is Hadamard according to Definition 4.5, then ω is Hadamard in the sense of Definition 6.1.*

Proof. The following argument is identical to the one used in 4.7 to prove $WF(\mathsf{G_P}) = WF(\mathsf{G_N})$, but we repeat it here to keep this section self-contained. First of all notice that, since $\omega_2(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g})$, then viewing ω_2 and W_g as bidistributions, we have $\omega(x,y) = (Id_x \otimes Q_y) W(x,y)$ (where we have used the fact that Q is formally self-adjoint) taking Remark 4.4 into account).
Now suppose that ω is Hadamard according to Definition 6.1. Since W_g satisfies the microlocal spectrum condition and the differential operator $I \otimes Q$ is smooth, we have

$$WF(\omega_2) \subset WF(W_g) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \triangleright 0\}\,.$$

Notice that, in particular, k_x and k_y cannot vanish (simultaneously or separately) if they take part of $WF(W_g)$. Let us prove the converse inclusion to complete the proof of (a). Again from known results, from $\omega_2(x,y) = (Id_x \otimes Q_y) W_g(x,y)$, we have

$$WF(W_g) \subset Char(I \otimes Q) \cup WF(\omega_2)\,.$$

However, by direct inspection, one sees that

$$Char(I \otimes Q) = \{(x, k_x; y, 0) \mid (x, k_x) \in \mathsf{T}^*\mathsf{M}\,, y \in \mathsf{M}\}\,,$$

so that

$$WF(\omega_2) \subset WF(W_g) \subset WF(\omega_2) \cup \{(x, k_x; y, 0) \mid (x, k_x) \in \mathsf{T}^*\mathsf{M}\,, y \in \mathsf{M}\} \tag{6.4}$$

However, $WF(W_g) \cap \{(x, k_x; y, 0) \mid (x, k_x) \in \mathsf{T}^*\mathsf{M}\,, y \in \mathsf{M}\} = \emptyset$ and thus we can rewrite the chain of inclusions (6.4) as

$$WF(\omega_2) \subset WF(W_g) \subset WF(\omega_2) \quad \text{so that} \quad WF(\omega_2) = WF(W_g)\,.$$

This is the thesis of (a) because we have established that Definition 4.5 is satisfied by ω.

To prove (b), let us assume that ω satisfies Definition 4.5. By hypotheses, the antisymmetric part of ω_2 is $-iG_{\mathsf{P}}$. Let Ω be another quasifree state of the Proca field which satisfies Definition 6.1. Also the antisymmetric part of Ω_2 is $-iG_{\mathsf{P}}$.

Due to Proposition 4.6,

$$F(x, y) := \omega_2(x, y) - \Omega_2(x, y)\,.$$

is a smooth function. Furthermore, it is a symmetric bisolution of the Proca equation. In particular, it therefore satisfies[13] $F(\mathfrak{f}, d\mathfrak{h}^{(0)}) = 0$, where $\mathfrak{h}^{(0)} \in \Omega^0_c(\mathsf{M})$, so that

$$F(\mathfrak{f}, Q\mathfrak{g}) = F(\mathfrak{f}, \mathfrak{g}) + \frac{1}{m^2} F(\mathfrak{f}, d(\delta_g \mathfrak{g})) = F(\mathfrak{f}, \mathfrak{g})\,.$$

Collecting everything together, we can assert that, for some distributional bisolution of the Klein–Gordon equation W_g which satisfies (6.2), (6.3) and is associated with the Hadamard state Ω, it holds

$$\omega_2(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, Q\mathfrak{g})\,.$$

If we re-absorb F in the definition of W_g,

$$W'_g(\mathfrak{f}, \mathsf{Q}\mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, Q\mathfrak{g})\,.$$

the new W'_g is again a distributional bisolution of the Klein–Gordon equation which satisfies (6.2), (6.3) and

$$\omega_2(f, g) = W'_g(\mathfrak{f}, Q\mathfrak{g})\,.$$

In other words, the Hadamard state ω according to Definition 4.5 is also Hadamard in the sense of Definition 6.1 concluding the proof of (b). □

Remark 6.7. Regarding (b), the existence of Hadamard states in the sense of Definition 6.1 has been established in [14] for globally hyperbolic spacetimes whose Cauchy surfaces are compact: in those types of spacetimes at least, the two definitions are completely equivalent. We expect that actually the equivalence is complete, even dropping the compactness hypothesis (see the conclusion section). This issue will be investigated elsewhere.

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[13]We are grateful to C. Fewster for this observation.

7. Conclusion and Future Outlook

We conclude this paper by discussing some open issues which are raised in this paper and we leave for future works.

On an ultrastatic spacetime $\mathsf{M} = \mathbb{R} \times \Sigma$, the one-parameter group of isometries given by time translations has an associated action on $\mathcal{A}_g$ in terms of $*$-algebras isomorphisms α_u completely induced by

$$\alpha_u(\hat{\mathfrak{a}}(\mathfrak{f})) := \hat{\mathfrak{a}}(\mathfrak{f}_u)$$

for every $\mathfrak{f} \in \Gamma_c(\mathsf{M})$, where $\mathfrak{f}_u(t,p) := \mathfrak{f}(t-u,p)$ for every $t, u \in \mathbb{R}$ and $p \in \Sigma$. It shall not be difficult to prove that the Hadamard state constructed in Theorem 3 is invariant under the action of α_u

$$\omega_\mu(\alpha_u(a)) = \omega_\mu(a) \quad \forall u \in \mathbb{R} \quad \forall a \in \mathcal{A}_g$$

It should be also true that the map

$$\mathbb{R} \ni u \mapsto \omega_\mu(b\alpha_u(a)) \in \mathbb{C}$$

is continuous for every $a, b \in \mathcal{A}_g$ which would assure (see, e.g., [41]) that $\alpha := \{\alpha_h\}_{h\in\mathbb{R}}$ is unitarily implementable by a strongly continuous unitary representation of $\mathbb{R}$ in the GNS representation of ω_μ and that the vacuum vector of the Fock-GNS representation is left invariant under the said unitary representation. We expect that the self-adjoint generator of that unitary group has a positive spectrum where, necessarily, the vacuum state is an eigenvector with eigenvalue 0. In other words, ω_μ should be a *ground state* of α. We finally expect that ω_μ is *pure* (on the Weyl algebra associated with the symplectic space $((\mathsf{Ker P}) \cap \Gamma_{sc}(\mathsf{M}), \sigma^{(\mathsf{P})})$ and it is the *unique quasifree algebraic state which is invariant under* α. We can summarize the previous discussion in the following question.

Question 7.1. Is the Hadamard state defined in Theorem 3 a *ground state* for the time translation? More precisely, is it the unique, pure, quasifree algebraic state which is invariant under action of α?

Last, but not least, we have seen in Sect. 6 that if a globally hyperbolic manifold admits a Proca quasifree Hadamard state according to the definition of Fewster–Pfenning, then Definition 4.5 and 6.1 are equivalent. This is the case, for example, for globally hyperbolic spacetimes whose Cauchy surfaces are compact. We do expect to extend this result for the whole class of globally hyperbolic spacetime.

Conjecture 7.2. Definition 4.5 and 6.1 are equivalent on any globally hyperbolic spacetime.

As is evident from our quasi equivalence theorem, a complete equivalence would take place if a Hadamard state according to [14] is proved to exist for every globally hyperbolic spacetime. As a matter of fact, we expect that every globally hyperbolic spacetime (M, g) admits a quasifree Proca Hadamard state ω according to Fewster and Pfenning. This state should exist in every paracausally related ultrastatic spacetime $(\mathbb{R} \times \Sigma, -dt^2 + h)$ with complete

Cauchy surfaces of bounded geometry. With the same argument used for our existence proof of Hadamard states or the deformation argument exploited in [14], it should be possible to export this state to the original space (M, g). We expect that the Hadamard Klein–Gordon bisolution for the real Proca field on $(\mathbb{R} \times \Sigma, -dt^2 + h)$ used to define ω according to (6.1) in Definition 6.1 should have this form.

$$W_g(\mathfrak{f}, \mathfrak{f}') := \mu(\mathsf{G_N}\mathfrak{f}, \mathsf{G_N}\mathfrak{f}') + \frac{i}{2}\sigma^{(N)}(\mathsf{G_N}\mathfrak{f}, \mathsf{G_N}\mathfrak{f}') , \quad \mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathbb{R} \times \Sigma) ,$$

where N is the Klein–Gordon operator (3.2) associated with P and $\mathsf{G_N}$ its causal propagator. The bilinear symmetric form $\mu : \big((\Omega^0_c(\Sigma))^2 \times (\Omega^1_c(\Sigma))^2\big) \times \big((\Omega^0_c(\Sigma))^2 \times (\Omega^1_c(\Sigma))^2\big) \to \mathbb{R}$ is defined as in (5.21), but with the crucial difference that here its arguments are not restricted to $C_\Sigma \times C_\Sigma$.

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