CHARACTERIZATION OF RECTIFIABILITY VIA LUSIN TYPE APPROXIMATION

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Abstract. We prove that a Radon measure μ on $\mathbb{R}^n$ can be written as $\mu = \sum_{i=0}^n \mu_i$, where each of the μ_i is an i-dimensional rectifiable measure if and only if for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a function g of class C^1 such that $\mu(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon$.

Keywords: Lipschitz functions, differentiability, Lusin type approximation.

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1 INTRODUCTION

A fundamental yet simple consequence of Rademacher's theorem and Whitney's theorem is the fact that Lipschitz functions on the Euclidean space admit a Lusin type approximation with C^1-functions, namely for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that

$$\mathscr{L}^n(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon,$$

where $\mathscr{L}^n$ denotes the Lebesgue measure, see [13, Theorem 5.3]. This fact has a central role in many pivotal results in Geometric Measure Theory, including the existence of the approximate tangent space to a rectifiable set, see [13, Lemma 11.1], and the validity of area and coarea formulas, see [13, §12].

On the one hand, this approximation property does not only hold for the Lebesgue measure: for instance it holds trivially for a Dirac delta. It is not difficult to see that the same property holds for any rectifiable measure and clearly the class of Radon measures for which the property holds is closed under finite sums.

On the other hand, it is known that there are measures μ for which Lipschitz functions do not admit a Lusin type approximation with respect to μ with functions of class C^1, see [9]. In this note we completely classify those measures, proving that the validity of such approximation property characterizes rectifiable measures, in the following sense.

Theorem 1.1. *Let μ be a positive Radon measure on $\mathbb{R}^n$. The measure μ can be written as $\mu = \sum_{i=0}^n \mu_i$, where each of the μ_i is an i-dimensional rectifiable measure if and only if for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a function g of class C^1 such that*

$$\mu(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon. \tag{1}$$

The proof of the "only if" part of Theorem 1.1 is a simple application of Whitney's theorem. The proof of the "if" part exploits some tools introduced in [1], including the notion of *decomposability bundle* of a measure μ, see [1, §2.6]: a map $x \mapsto V(\mu, x)$ which detects the maximal subspaces along which Lipschitz functions are differentiable μ-almost everywhere. For the purposes of this paper, we need to refine the result [1, Theorem 1.1 (ii)] on the existence of Lipschitz functions which are non-differentiable along directions which do not belong to the decomposability

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bundle. In [1], such non-differentiability is proved by finding a Lipschitz function f and for μ-almost every point x a sequence of points $y_i := x + t_i v \in \mathbb{R}^n$ converging to x along a direction $v \notin V(\mu, x)$, such that the corresponding incremental ratios $(f(y_i) - f(x))/t_i$ do not converge. Here we need to find a function f such that there exist points y_i as above, with the additional requirement that $y_i \in \operatorname{supp}(\mu)$, see Proposition 3.1. For a non-rectifiable measure μ, the existence of a μ-positive set of points x for which there are points $y_i \in \operatorname{supp}(\mu)$ approaching x along a direction $v \notin V(\mu, x)$ is guaranteed by Lemma 2.1.

We plan to investigate similar questions in Carnot groups, exploiting tools and techniques introduced in [3]. In this setting, similar questions have already attracted some interest. For instance, in [7] the authors proved a suitable extension of Lusin's approximation-type theorem for the surface measure of 1-codimensional $C^1_{\mathbb{H}}$-rectifiable surfaces in the Heisenberg groups $\mathbb{H}^n$, $n \geq 2$ and where the regular approximation of Lipschitz functions are found in the class of $C^1_{\mathbb{H}}$-regular functions. The authors also prove that in $\mathbb{H}^1$ there is a regular surface and a Lipschitz function that cannot be approximated by $C^1_{\mathbb{H}}$-regular functions. This different behaviour is connected to the algebraic structure of the tangents to 1-codimensional regular surfaces in the Heisenberg groups $\mathbb{H}^n$ when $n = 1$ or $n \geq 2$.

2 NOTATION AND PRELIMINARIES

We denote by $U(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r and by $B(x, r)$ the closed ball. In addition, for a Borel set E and a $\delta > 0$, we denote $B(E, \delta) := \bigcup_{y \in E} B(y, \delta)$. The unit sphere is denoted $\mathbb{S}^{n-1}$.

Given a Radon measure μ and a (possibly vector-valued) function f, we denote by $f\mu$ the measure

$$f\mu(A) := \int_A f \, d\mu, \quad \text{for every Borel set } A.$$

For a measure μ and a Borel set E we denote by $\mu \llcorner E$ the restriction of μ to E, namely the measure defined by

$$\mu \llcorner E(A) := \mu(A \cap E), \quad \text{for every Borel set } A.$$

The support of a positive Radon measure μ, denoted $\operatorname{supp}(\mu)$, is the intersection of all closed sets C such that $\mu(\mathbb{R}^n \setminus C) = 0$. For $0 \leq k \leq n$, the symbol $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^n$.

Definition 2.1 (Rectifiable sets and measures)**.** For $0 \leq k \leq n$, a set $E \subset \mathbb{R}^n$ is *k-rectifiable* if there are sets E_i $(i = 1, 2, \ldots)$ such that

(i) E_i is a Lipschitz image of $\mathbb{R}^k$ for every i;

(ii) $\mathscr{H}^k(E \setminus \bigcup_{i \geq 1} E_i) = 0$.

A Radon measure is said to be *k-rectifiable* if it is absolutely continuous with respect to $\mathscr{H}^k \llcorner E$, for some k-rectifiable set E.

As usual, the symbol $\mathrm{Gr}(k, n)$ denotes the Grassmannian of k-planes in $\mathbb{R}^n$, and we define $\mathrm{Gr} := \bigcup_{0 \leq k \leq n} \mathrm{Gr}(k, n)$. We endow Gr with the topology induced by the distance

$$d(V, W) := d_{\mathscr{H}}(V \cap U(0,1), W \cap U(0,1)),$$

where $d_{\mathscr{H}}$ is the Hausdorff distance. We recall the following definition, see [1, §2.6, §6.1 and Theorem 6.4].

Definition 2.2 (Decomposability bundle)**.** Given a positive Radon measure μ on $\mathbb{R}^n$ its *decomposability bundle* is a map $V(\mu, \cdot)$ taking values in the set Gr defined as follows. A vector $v \in \mathbb{R}^n$ belongs to $V(\mu, x)$ if and only if there exists a vector-valued measure T with $\operatorname{div} T = 0$ such that

$$\lim_{r \to 0} \frac{\mathbb{M}((T - v\mu) \llcorner B(x, r))}{\mu(B(x, r))} = 0,$$

where $\mathbb{M}((T - v\mu) \llcorner B(x, r))$ denotes the total variation of the vector-valued measure $(T - v\mu) \llcorner B(x, r)$.

Definition 2.3 (Tangent measures)**.** We define the map $T_{x,r}(y) = \frac{y-x}{r}$, and we denote by $T_{x,r}\mu$ the pushforward of μ under $T_{x,r}$, namely $T_{x,r}\mu(A) := \mu(x + rA)$ for every Borel set A. Given a measure μ and a point x, the family of *tangent measures* $\mathrm{Tan}(\mu, x)$, introduced in [12], consists of all the possible non-zero limits (with respect to the weak* convergence of measures) of $c_i T_{x,r_i}\mu$, for some sequence of positive real numbers c_i and some sequence of radii $r_i \to 0$. We know thanks to [12, Theorem 2.5] that $\mathrm{Tan}(\mu, x)$ is non-empty μ-almost everywhere.

Definition 2.4 (Cone over a k-plane)**.** For any $k \in \{1, \dots, n-1\}$, $0 < \vartheta < 1$, $x \in \mathbb{R}^n$ and $V \in \mathrm{Gr}(k, n)$ we let:

$$X(x, V, \vartheta) := x + \{v \in \mathbb{R}^n : |p_V(v)| \geq \vartheta |v|\},$$

where p_V denotes the orthogonal projection onto V. For notation convenience, for $k = 0$ and for every $0 < \vartheta < 1$, we define $X(x, 0, \vartheta) := \{x\}$.

Definition 2.5 (F_K distance between measures)**.** Given ϕ and ψ two Radon measures on $\mathbb{R}^n$, and given $K \subseteq \mathbb{R}^n$ a compact set, we define

$$F_K(\phi, \psi) := \sup \left\{ \left| \int f d\phi - \int f d\psi \right| : f \in \mathrm{Lip}_1^+(K) \right\}, \tag{2}$$

where $\mathrm{Lip}_1^+(K)$ denotes the class of 1-Lipschitz nonnegative functions with support contained in K. We also write $F_{x,r}$ for $F_{B(x,r)}$.

Lemma 2.1. *Let μ be a Radon measure on $\mathbb{R}^n$ with $\dim(V(\mu, x)) = k < n$, for μ-almost every x. Assume that $\mu(R) = 0$ for every k-rectifiable set R. Then for every $0 < \vartheta < 1$ and for every $\varepsilon > 0$*

$$\mathrm{supp}(\mu) \cap B(x, \varepsilon) \setminus X(x, V(\mu, x), \vartheta) \neq \emptyset, \tag{3}$$

for μ-almost every x.

Proof. Assume by contradiction that there exists a Borel set E with $\mu(E) > 0$ such that for every $x \in E$ there exists $\varepsilon > 0$ such that (3) fails. We claim that this implies that for μ-almost every $x \in E$ every tangent measure $\nu \in \mathrm{Tan}(\mu, x)$ satisfies

$$\mathrm{supp}(\nu) \subset X(0, V(\mu, x), \vartheta). \tag{4}$$

In order to prove (4), fix $x \in E$ such that $\mathrm{Tan}(\mu, x)$ is non-empty and consider any open ball $U(y, \rho) \subset \mathbb{R}^n \setminus X(0, V(\mu, x), \vartheta)$ and notice that since (3) fails, we have $T_{x,r}\mu(U(y, \rho)) = \mu(U(x + ry, r\rho)) = 0$ for every $r < \varepsilon/(|y| + \rho)$ which concludes in view of [2, Proposition 2.7]. Thanks to [5, Proposition 2.9] we infer that $\mathrm{supp}(\nu) \subset V(\mu, x)$ and in particular $\nu = c\mathscr{H}^k \llcorner V(\mu, x)$ for some $c > 0$. For every $W \in \mathrm{Gr}(k, n)$ denote

$$E_W := \{x \in \mathbb{R}^n : (k+1)F_{0,1}(\mathscr{H}^k \llcorner V(\mu, x), \mathscr{H}^k \llcorner W) < 20^{-k-4}\}.$$

By the compactness of the Grassmannian, there exists $W \in \mathrm{Gr}(k, n)$ such that $\mu(E_W) > 0$. On the other hand, by [12, §4.4(5)] and by the locality of tangent measures, see [12, §2.3(4)], we conclude that $\mu \llcorner E_W$ is supported on a k-rectifiable set. This however contradicts the assumption that $\mu(R) = 0$ for every k-rectifiable set R. □

Definition 2.6 (Cone-null sets)**.** For any $e \in \mathbb{S}^{n-1}$ and $\theta \in (0, 1)$ we let the *one-sided cone of axis e and amplitude θ* be the set

$$C(e, \theta) := \{v \in \mathbb{R}^n : \langle v, e \rangle \geq \theta |v|\}.$$

In the following we denote by $\Gamma(e, \theta)$ the family of Lipschitz curves $\gamma : E \subseteq \mathbb{R} \to \mathbb{R}^n$ such that $\gamma'(t) \in C(e, \theta)$ for $\mathcal{L}^1$-almost every $t \in E$. Finally, a Borel set B is said to be $C(e, \theta)$-null if $\mathscr{H}^1(\mathrm{im}(\gamma) \cap B) = 0$ for any $\gamma \in \Gamma(e, \theta)$.

Proposition 2.2. *Let E be a compact set in $\mathbb{R}^n$. Let $W \in \mathrm{Gr}(k, n)$, with $k < n$ and suppose that there exists $\theta_0 \in (0, 1)$ such that for any $e \in W^\perp$ the set E is $C(e, \theta_0)$-null. Then, for any $\theta_0 \leq \theta < 1$ and $\varepsilon > 0$ there exists $\delta_0 > 0$ such that*

$$\mathscr{H}^1(\mathrm{im}(\gamma) \cap B(E, \delta_0)) \leq \varepsilon,$$

for any $\gamma \in \Gamma(e, \theta)$. For any $\theta_0 \leq \theta < 1$, $0 < \delta < \delta_0$ and any $e \in W^\perp$, consider the function

$$\omega_{e,\theta,\delta}(x) := \sup_{\substack{\gamma \in \Gamma(e,\theta) \\ \gamma(b) = x + \lambda e}} \mathscr{H}^1(B(E, \delta) \cap \mathrm{im}(\gamma)) - \lambda |e|. \tag{5}$$

Then the following properties hold

(i) $0 \leq \omega_{e,\theta,\delta}(x) \leq \varepsilon$ *for any* $x \in \mathbb{R}^n$*,*

(ii) $\omega_{e,\theta,\delta}(x) \leq \omega_{e,\theta,\delta}(x+se) \leq \omega_{e,\theta,\delta}(x) + s|e|$ *for every* $s > 0$ *and any* $x \in \mathbb{R}^n$*. Moreover, if the segment* $[x, x+se]$ *is contained in* $B(E,\delta)$*, then* $\omega_{e,\theta,\delta}(x+se) = \omega_{e,\theta,\delta}(x) + s|e|$*,*

(iii) $|\omega_{e,\theta,\delta}(x+v) - \omega_{e,\theta,\delta}(x)| \leq \theta(1-\theta^2)^{-1/2}|v|$ *for every* $v \in V := e^\perp$*,*

(iv) $\omega_{e,\theta,\delta}$ *is* $(1+(n-1)\theta(1-\theta^2)^{-1/2})$*-Lipschitz.*

Proof. The first part of the proposition is an immediate consequence of *Step 1* in the proof of [1, Lemma 4.12]. On the other hand, the construction of the function $\omega_{e,\theta,\delta}$ was performed in the second step of the proof of [1, Lemma 4.12]. □

3 CONSTRUCTION OF NON-DIFFERENTIABLE FUNCTIONS

In this section we prove the existence of some suitable Lipschitz functions which are non-differentiable along directions that are quantitatively far away from the decomposability bundle. Given a measure μ as in Lemma 2.1, we prove that there are *many* functions which are non-differentiable on a set of positive μ-measure with the additional property that the non-differentiability is "detected" by the points in the support of μ, see Proposition 3.1.

Throughout this section we fix $k \in \{0, \dots, n-1\}$ and let μ be a Radon measure such that $\dim(V(\mu,x)) = k$ for μ-almost every $x \in \mathbb{R}^n$ and that $\mu(R) = 0$ for any k-rectifiable set R. Thanks to the strong locality principle, see [1, Proposition 2.9 (i)], and Lusin's Theorem we can assume, up to restriction to a compact subset $\tilde{K} \subset \operatorname{supp}(\mu)$ of positive μ-measure, that $V(\mu,x)$ is uniformly continuous on $\tilde{K}$. Up to restricting to a subset where the oscillation of V is small, we can assume that there are n continuous vector fields $e_1, \dots, e_n : \mathbb{R}^n \to \mathbb{S}^{n-1}$ such that

$$V(\mu,x) = \operatorname{span}\{e_1(x), \dots, e_k(x)\} \qquad \text{and} \qquad V(\mu,x)^\perp = \operatorname{span}\{e_{k+1}(x), \dots, e_n(x)\} \qquad \text{for every } x \in \tilde{K}.$$

The aim of this section is to prove the following

Proposition 3.1. *Let* μ *and* $\tilde{K}$ *be as above. There exists a Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *and a Borel set* $E \subseteq \tilde{K}$ *of positive* μ*-measure such that for* μ*-almost every* $x \in E$ *there exists a direction* $v \notin V(\mu,x)$ *and a sequence of points* $y_i = y_i(x) \in \tilde{K}$ *such that*

$$\frac{y_i - x}{|y_i - x|} \to v \qquad \text{and} \qquad \limsup_{i\to\infty} \frac{f(y_i) - f(x)}{|y_i - x|} - \liminf_{i\to\infty} \frac{f(y_i) - f(x)}{|y_i - x|} > 0.$$

Denoting $\alpha = 1/\sqrt{n}$, we apply Lemma 2.1 with the choice $\vartheta = \sqrt{1-\alpha^2}$ to find a compact subset K_α of $\tilde{K}$ with positive measure where

$$\operatorname{supp}(\mu) \cap B(x,r) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2}) \neq \emptyset \qquad \text{for any } r > 0 \text{ and every } x \in K_\alpha. \tag{6}$$

Lemma 3.2. *Let* μ *and* K_α *be as above. Then, we can find a compact set* $K \subseteq K_\alpha$ *of positive* μ*-measure and a continuous vector field* $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ *such that* $e(x)$ *is orthogonal to* $V(\mu,x)$ *at* μ*-almost every* $x \in \mathbb{R}^n$ *and such that:*

$$\operatorname{supp}(\mu) \cap B(x,r) \cap C(e(x), (n-k)^{-1}\alpha) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2}) \neq \emptyset \qquad \textit{for any } r > 0 \textit{ and for every } x \in K. \tag{7}$$

Proof. By the choice of α, the cones

$$C(e_{k+1}(x), (n-k)^{-1}\alpha), \dots, C(e_n(x), (n-k)^{-1}\alpha), C(-e_{k+1}(x), (n-k)^{-1}\alpha), \dots, C(-e_n(x), (n-k)^{-1}\alpha),$$

cover $\mathbb{R}^n \setminus X(0, V(\mu,x), \sqrt{1-\alpha^2})$ for every $x \in K_\alpha$. Hence there exists one vector field, which we denote e, among the $e_{k+1}, \dots, e_n, -e_{k+1}, \dots, -e_n$ for which the set of those $x \in K_\alpha$ where (7) holds has positive μ-measure. □

Definition 3.1. Throughout the rest of this section we will let α_0 be as in (6) and we fix $0 < \alpha < \alpha_0$. We also fix the compact set K and the continuous vector field $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ yielded by Lemma 3.2. We let $e_1, \dots, e_k : \mathbb{R}^n \to \mathbb{S}^{n-1}$ be continuous orthonormal vector fields spanning $V(\mu,x)$ at every $x \in K$ and we complete $\{e_1, \dots, e_k, e\}$ to a basis of $\mathbb{R}^n$ of orthonormal continuous vector fields that we denote by $\{e_1, \dots, e_k, e, e_{k+1}, \dots, e_{n-1}\}$.

Fix a ball $B(0,r)$ such that $K \subset B(0,r-1)$ and for any $\beta \in (0,1)$ we denote by X_β the family of Lipschitz functions $f : B(0,r) \to \mathbb{R}$ such that

$$|D_e f(x)| \le 1 \qquad \text{and} \qquad |D_{e_j} f(x)| \le \beta \quad \text{for any } j = 1, \dots, n-1, \tag{8}$$

for $\mathscr{L}^n$-almost every $x \in \mathbb{R}^n$. We metrize X_β with the supremum norm and we note that this make X_β a complete and separable metric space. Note also that X_β is non-trivial as it contains all the β-Lipschitz functions.

In the following definition we introduce some quantities which measure the incremental ratios "detected" by points in the support of μ, at fixed scales and along directions which are outside a cone whose axis is the decomposability bundle.

Definition 3.2. For any $\beta > 0$ and any $0 \le \sigma' < \sigma < 1$ we can define on X_β the maps

$$T^+_{\sigma',\sigma} f : x \mapsto \max\Big\{ \sup\Big\{ \frac{f(x+v) - f(x)}{|v|} : \sigma' < |v| \le \sigma \text{ and } x + v \in \mathrm{supp}(\mu) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2}) \Big\}, -n \Big\},$$

$$T^-_{\sigma',\sigma} f : x \mapsto \min\Big\{ \inf\Big\{ \frac{f(x+v) - f(x)}{|v|} : \sigma' < |v| \le \sigma \text{ and } x + v \in \mathrm{supp}(\mu) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2}) \Big\}, n \Big\}.$$

Proposition 3.3. *For any $0 \le \sigma' < \sigma < 1$ the functionals*

$$U^\pm_{\sigma',\sigma} f := \int_K T^\pm_{\sigma',\sigma} f(z)\, d\mu(z)$$

are Baire class 1 on X_β.

Proof. As a first step we show that the $T^+_{\sigma',\sigma} : X_\beta \to L^1(\mu \llcorner K)$ are continuous whenever $0 < \sigma' < \sigma < 1$. The functions $T^+_{\sigma',\sigma} f$ belong to $L^1(\mu \llcorner K)$ since K has finite measure and $|T^+_{\sigma',\sigma} f| \le \mathrm{Lip}(f) + n$. In addition, it is immediate to see that:

$$|T^+_{\sigma',\sigma} f(x) - T^+_{\sigma',\sigma} g(x)| \le \frac{2\|f-g\|_\infty}{\sigma'} \qquad \text{for } \mu\text{-almost every } x \in \mathbb{R}^n,$$

thanks to the fact that if at some $x \in \mathbb{R}^n$ we have $(B(x,\sigma) \setminus B(x,\sigma')) \cap (\mathrm{supp}(\mu) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2})) = \emptyset$, then $T_{\sigma',\sigma} f(x) = -n$ for any $f \in X_\beta$. Integrating with respect to μ, we infer that:

$$\|T^+_{\sigma',\sigma} f(x) - T^+_{\sigma',\sigma} g(x)\|_{L^1(\mu \llcorner K)} \le \frac{2\mu(K)}{\sigma'} \|f-g\|_\infty.$$

This implies in particular that $U^+_{\sigma',\sigma}$ is a continuous functional on X_β. Following verbatim the argument above, one can also prove the continuity of the functionals $T^-_{\sigma',\sigma}$ and of $U^-_{\sigma',\sigma}$.

In order to prove that $U^\pm_{0,\sigma}$ is of Baire class 1, thanks to [8, §24.B] we just need to show that for any $f \in X_\beta$ we have:

$$\lim_{j \to \infty} U^\pm_{j^{-1},\sigma} f = U^\pm_{0,\sigma} f. \tag{9}$$

This is an immediate consequence of the dominated convergence theorem since the sequence $(T^\pm_{j^{-1},\sigma} f)_j$ converges pointwise to $T^\pm_{0,\sigma} f$ and is dominated by the function constantly equal to n. □

We are now ready to prove the main result of the section, namely the fact that X_β contains *plenty* of Lipschitz functions whose non-differentiability at some points of K is "detected" by points in the support of μ.

Proposition 3.4. *Let $\beta < (8n^2)^{-1}\alpha$. Then for every $\sigma > 0$ the continuity points of $U^\pm_{0,\sigma}$ are contained in the set*

$$\mathcal{L}_\pm(\sigma) := \Big\{ f \in X_\beta : \pm U^\pm_{0,\sigma} f \ge \frac{\alpha}{16n} \mu(K) \Big\}.$$

In particular both $\mathcal{L}_+(\sigma)$ and $\mathcal{L}_-(\sigma)$ are residual in X_β.

Let us briefly explain here the idea of the proof. In our reduction, for every point $x \in K$ at any small scale there is a point $y \in \mathrm{supp}(\mu)$ such that $y - x$ is far away from $V(\mu, x)$, see Lemma 3.2. Hence the point y is not reached by Lipschitz curves passing through x and lying inside $\mathrm{supp}(\mu)$. By Proposition 2.2, we can find a Lipschitz function ω with small supremum norm which "jumps" with high derivative along the segment $[x, y]$, for any such point y. Assuming by contradiction that at a continuity point $g \in X_\beta$ the value of $U^+_{0,\sigma}$ is below a certain threshold, we reach a contradiction perturbing g by adding ω so that the value of $U^+_{0,\sigma}$ increases significantly.

Proof. We prove the result just for $U^+_{0,\sigma}$. The argument to prove the analogous statement for $U^-_{0,\sigma}$ can be obtained following verbatim that for $U^+_{0,\sigma}$ while making suitable changes of sign.

Assume by contradiction that g is a continuity point for $U^+_{0,\sigma}$ contained in $X_\beta \setminus \mathcal{L}_+(\sigma)$. It is easy to see by convolution that smooth functions are dense in X_β. Since g is a continuity point for $U^+_{0,\sigma}$, for any $\ell \in \mathbb{N}$ we can find a smooth function $h_\ell \in X_\beta$ such that $\|g - h_\ell\|_\infty \le 2^{-\ell}$ and $U^+_{0,\sigma} h_\ell \le \alpha\mu(K)/8n$ and for *any* $x \in \mathbb{R}^n$ we have

$$|D_e h_\ell(x)| \le 1 \qquad \text{and} \qquad |D_{e_j} h_\ell(x)| \le \beta \quad \text{for any } j = 1, \dots, n-1.$$

Let

$$A := \{y \in K : T^+_{0,\sigma} h_\ell(y) \le \alpha/8n\}.$$

Thanks to Besicovitch's covering theorem and [1, Lemma 7.5] we can cover μ-almost all A with countably many closed and disjoint balls $\{B(y_j, r_j)\}_{j\in\mathbb{N}}$ such that, for $0 < \eta, \chi < (n2^{10\ell})^{-1}\beta^2$

(i) $r_j \le 2^{-\ell}$, $\mu(A \cap B(y_j, r_j)) \ge (1-\eta)\mu(B(y_j, r_j))$ and $\mu(\partial B(y_j, r_j)) = 0$,

(ii) for any $z \in B(y_j, r_j)$

$$|e(z) - e(y_j)| + |\nabla h_\ell(y_j) - \nabla h_\ell(z)| + \Big|\frac{h_\ell(z) - h_\ell(y_j)}{|z - y_j|} - \nabla h_\ell(z)\Big[\frac{z - y_j}{|z - y_j|}\Big]\Big| \le \chi^4,$$

(iii) for any $j \in \mathbb{N}$ we can find $0 < \rho_j < (n2^\ell)^{-1}\beta^2$ and a compact subset $\tilde{A}_j$ of $A \cap B(y_j, (1-2\rho_j)r_j)$ such that $\mu(\tilde{A}_j) \ge (1-2\eta)\mu(B(y_j, r_j))$ and $\tilde{A}_j$ is $C(e(y_j), 2^{-10\ell}\chi^2)$-null.

For any $j \in \mathbb{N}$ we let ϕ_j be a smooth $2(\rho_j r_j)^{-1}$-Lipschitz function such that $0 \le \phi_j \le 1$, $\phi_j = 1$ on $B(y_j, (1-\rho_j)r_j)$ and it is supported on $B(y_j, r_j)$. Now fix $0 < \varepsilon < \beta\chi^2$. Thanks to Proposition 2.2 we can find $\delta_j \le 2^{-j}\rho_j r_j$ and a function ω_j such that:

1. $0 \le \omega_j(x) \le \varepsilon\beta\rho_j r_j$ for any $x \in \mathbb{R}^n$,
2. $\omega_j(x) \le \omega_j(x + se(y_j)) \le \omega_j(x) + s$, for every $s > 0$ and any $x \in \mathbb{R}^n$. Moreover, if the segment $[x, x + se(y_j)]$ is contained in $B(\tilde{A}_j, \delta_j)$, then $\omega_j(x + se(y_j)) = \omega_j(x) + s$,
3. $|\omega_j(x + v) - \omega_j(x)| \le 2^{-9\ell}\chi^2|v|$, for every $v \in e(y_j)^\perp$,
4. ω_j is $1 + 2^{-9\ell}\chi^2$-Lipschitz.

We thus define the function g_ℓ as

$$g_\ell := (1-2\chi)\Big(h_\ell + \sum_{j\in\mathbb{N}} [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j\omega_j\Big). \tag{10}$$

First we estimate the supremum distance

$$\begin{aligned} \|g - g_\ell\|_\infty \le& \|g - h_\ell\|_\infty + 2\chi\|h_\ell\|_\infty + (1-2\chi)\|h_\ell - (1-2\chi)^{-1}g_\ell\|_\infty \\ \le& 2^{-\ell} + \chi(\|g\|_\infty + 2^{-\ell}) + (1-2\chi)\Big\|\sum_{j\in\mathbb{N}}(1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle)\Big\|_\infty \\ \le& 2^{-\ell}(2 + \|g\|_\infty + (1 + (n-1)\beta^2)^{1/2}) \le 2^{-\ell}(4 + \|g\|_\infty), \end{aligned} \tag{11}$$

where the last inequality follows from the choice of β. The above computation shows that the sequence g_ℓ converges in the supremum distance.

Let us now prove that $g_\ell \in X_\beta$. If $z \notin \cup_j B(y_j, r_j)$ then the functions h_ℓ and g_ℓ and their gradients coincide at z and hence g_ℓ satisfies (8) on $(\cup_j B(y_j, r_j))^c$. If on the other hand $z \in \cup_j B(y_j, r_j)$, there exists a unique $j \in \mathbb{N}$ such that $z \in B(y_j, r_j)$. In particular, differentiating (10) we get

$$\nabla g_\ell(z) = (1-2\chi)\Big[\nabla h_\ell(z) + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\nabla\phi_j(z)\omega_j(z) + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j(z)\nabla\omega_j(z)\Big].$$

So that, for $\mathscr{L}^n$-almost every $x \in \mathbb{R}^n$ we have

$$|\langle \nabla g_\ell(z), e(z)\rangle| \le (1-2\chi)\Big|\langle \nabla h_\ell(z), e(z)\rangle + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle\Big| + 4\varepsilon\beta,$$

where in the estimate above we have used the fact that $|-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1| \le 2$, $\|\nabla\phi\|_{L^\infty(\mathscr{L}^n)} \le 2(\rho_j r_j)^{-1}$ and $\|\omega_j\|_\infty \le \varepsilon\beta\rho_j r_j$. Now we replace z with y_j in the first addendum, by means of the estimate (ii), obtaining

$$|\langle \nabla g_\ell(z), e(z)\rangle| \le 3(1-2\chi)\chi^2 + (1-2\chi)\Big|\langle \nabla h_\ell(y_j), e(y_j)\rangle\big(1 - \phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle\big) + \phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle\Big| + 2\varepsilon\beta.$$

Finally, substituting z with y_j in the argument of the vector field e we deduce thanks to (ii) that

$$\begin{aligned}|\langle \nabla g_\ell(z), e(z)\rangle| \le& 3(1-2\chi)\chi^2 + 2\varepsilon\beta + 6(1-2\chi)(1+2^{-9\ell}\chi)\chi^2 \\ &+ (1-2\chi)\big|\langle \nabla h_\ell(y_j), e(y_j)\rangle\big(1 - \phi_j(z)\langle \nabla\omega_j(z), e(y_j)\rangle\big) + \phi_j(z)\langle \nabla\omega_j(z), e(y_j)\rangle\big| \\ \le& 3(1-2\chi)\chi^2 + 2\varepsilon\beta + 6(1-2\chi)(1+2^{-9\ell}\chi)\chi^2 + (1-2\chi) \le 1,\end{aligned}$$

where the the last inequality follows from the choice of χ, β, ε. Furthermore, for any $q = 1, \dots, n-1$ we infer similarly that:

$$\begin{aligned}|g_\ell(z + te_q(z)) - g_\ell(z)| \le& (1-2\chi)\big|h_\ell(z + t\,e_q(z)) - h_\ell(z)\big| \\ &+ (1-2\chi)|[1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle](\phi_j(z + t\,e_q(z)) - \phi_j(z))\omega_j(z)| \\ &+ (1-2\chi)|[1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle]\phi_j(z)(\omega_j(z + t\,e_q(y_j)) - \omega_j(z))| \\ &+ (1-2\chi)|[1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle]\phi_j(z)(\omega_j(z + t\,e_q(z)) - \omega_j(z + t\,e_q(y_j)))| + o(|t|) \\ \le& (1-2\chi)\beta|t| + 4(1-2\chi)(\beta\varepsilon\rho_j r_j)(\rho_j r_j)^{-1}|t| + 3\cdot 2^{-9\ell}(1-2\chi)\chi^2|t| + 3(1-2\chi)(1+2^{-9\ell}\chi)\chi^4|t| + o(|t|) \\ \le& (1-2\chi)(\beta + 4\beta\varepsilon + 4\cdot 2^{-9\ell}\chi^2 + 4(1+2^{-9\ell}\chi)\chi^4)|t| \le (1-2\chi)(1+10\chi^2)\beta|t| + o(|t|) < \beta|t|,\end{aligned}$$

provided $|t|$ is chosen sufficiently small (depending on z) and where the second to last inequality holds thanks to the choice of χ, ε and for ℓ sufficiently big, in such a way that $2^{-\ell} \le \beta$. The above bound implies that in particular

$$|\langle \nabla g_\ell(z), e_q(z)\rangle| \le \beta \text{ for } \mathscr{L}^n\text{-almost every } x \in \mathbb{R}^n. \tag{12}$$

This concludes the proof that for ℓ sufficiently big we have $g_\ell \in X_\beta$.

The next step in the proof is to show that the functions g_ℓ satisfy the inequality $U^+_{0,\sigma}g_\ell \ge \alpha\mu(K)/8n$ for ℓ sufficiently big, and this contradicts the continuity of $U^+_{0,\sigma}$ at g (recall that we supposed $U^+_{0,\sigma}g \ge \alpha\mu(K)/16n$). In order to see this, we first estimate from below the partial derivative of g_ℓ along e on the points of $\tilde{A}_j$ for any j. So, let us fix for any $j \in \mathbb{N}$ a point $z \in \tilde{A}_j$. Then, let $0 < \lambda_0 < \delta_j$ be so small that $\phi_j(z + \lambda e(z)) = 1$ for any $0 < \lambda < \lambda_0$ and note that

$$\begin{aligned}\langle g_\ell(z + \lambda e(z)) - g_\ell(z), e(z)\rangle \ge& (1-2\chi)\Big[(h_\ell(z + \lambda e(z)) - h_\ell(z)) + [1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle](\omega_j(z + \lambda e(z)) - \omega_j(z))\Big] \\ \ge& (1-2\chi)\big[-\chi^2\lambda + \lambda\langle \nabla h_\ell(z), e(z)\rangle + [1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle]\lambda\big] \ge \lambda(1-2\chi)(1-4\chi^2) \ge (1-6\chi)\lambda.\end{aligned}$$

This implies in particular that for any unit vector $v \in C(e(z), (n-k)^{-1}\alpha)$, for any $\lambda > 0$ we have

$$\begin{aligned}g_\ell(z + \lambda v) - g_\ell(z) \ge& g_\ell(z + \lambda v) - g_\ell(z + \lambda\langle e(z), v\rangle e(z)) + g_\ell(z + \lambda\langle e(z), v\rangle e(z)) - g_\ell(z) \\ \ge& \alpha(n-k)^{-1}(1-6\chi)\lambda - \beta\sqrt{n-1}\lambda \ge (\alpha/2(n-k))\lambda - \beta n\lambda > \alpha\lambda/4(n-k),\end{aligned} \tag{13}$$

where the last inequality follows from the choice of β. However, thanks to choice of K, see (7), we infer that

$$T^+_{0,\sigma}g_\ell(z) \geq \alpha/4(n-k) \qquad \text{for any } z \in \cup_j \tilde{A}_j.$$

This allows us to infer that

$$\begin{aligned}
U^+_{0,\sigma}g_\ell &= \int_A T^+_{0,\sigma}g_\ell d\mu + \int_{K\setminus A} T^+_{0,\sigma}g_\ell d\mu \geq \int_A T^+_{0,\sigma}g_\ell d\mu + \alpha\mu(K\setminus A)\\
&= \int_{A\setminus\cup_j\tilde{A}_j} T^+_{0,\sigma}g_\ell d\mu + \sum_{j\in\mathbb{N}}\int_{A_j} T^+_{0,\sigma}g_\ell d\mu + \alpha\mu(K\setminus A)\\
&\geq -\mu(A\setminus\bigcup_{j\in\mathbb{N}}A_j)\mathrm{Lip}(g_\ell) + \frac{\alpha}{4(n-k)}\mu(\bigcup_{j\in\mathbb{N}}A_j) + \alpha\mu(K\setminus A)\\
&\geq -2\mu(A\setminus\bigcup_{j\in\mathbb{N}}A_j) + \frac{\alpha}{4(n-k)}\mu\Big((K\setminus A)\cup\bigcup_{j\in\mathbb{N}}A_j\Big)\\
&\geq -4\eta\mu(K) + \frac{\alpha}{4(n-k)}(1-2\eta)\mu(K) \geq \frac{\alpha}{8n}\mu(K).
\end{aligned}$$

for ℓ sufficiently big.

Since the functional $U^+_{0,\sigma}$ is of Baire class 1, thanks to [11, Chapter 7] we know that the set of the continuity points of $U^+_{0,\sigma}$ is residual. However, since thanks to the above argument $\mathcal{L}_+(\sigma)$ contains the continuity points of $U^+_{0,\sigma}$, we conclude that $\mathcal{L}_+(\sigma)$ is residual in X_β. □

Proof of Proposition 3.1. Let $\beta := (16n^2)^{-1}\alpha$ and let $\mathfrak{c}(\alpha) := \alpha/16n$ note that since the countable intersection of residual sets is residual, we can find a Lipschitz function f in X_β such that $f \in \cap_{\sigma\in\mathbb{Q}\cap(0,1)}(\mathcal{L}_+(\sigma)\cap\mathcal{L}_-(\sigma))$. In particular, for any $\sigma > 0$ we have

$$U^-_{0,\sigma}f \leq -\mathfrak{c}(\alpha)\mu(K) < \mathfrak{c}(\alpha)\mu(K) \leq U^+_{0,\sigma}f.$$

Letting $\Delta T_\sigma f(z) := T^+_{0,\sigma}f(z) - T^-_{0,\sigma}f(z)$ and $C_\sigma := \{z \in K : \Delta T_\sigma(z) > \mathfrak{c}(\alpha)\}$, there holds

$$2\mathfrak{c}(\alpha)\mu(K) \leq \int_K \Delta T_\sigma(z) d\mu \llcorner K(z) \leq \mu(K\setminus C_\sigma)\mathfrak{c}(\alpha) + 2\mathrm{Lip}(f)\mu(C_\sigma).$$

Thanks to the above computation we infer in particular that $\mu(C_\sigma) \geq \mathfrak{c}(\alpha)\mu(K)/2\mathrm{Lip}(f)$ for any $\sigma > 0$. Thus, defined $E := \bigcap_{j\in\mathbb{N}}\bigcup_{l\geq j} C_{1/l}$, Fatou's Lemma implies that:

$$\frac{\mathfrak{c}(\alpha)\mu(K)}{2\mathrm{Lip}(f)} \leq \limsup_{p\to\infty}\mu(C_{1/p}) \leq \int \limsup_{p\to\infty}\mathbb{1}_{C_{1/p}}d\mu = \mu(E),$$

where $\mathbb{1}_{C_{1/p}}$ denotes the indicator function of the set $C_{1/p}$. Therefore, E is a Borel set of positive μ-measure such that for μ-almost every $z \in E$ there exists a sequence of natural numbers (depending on z) such that $p \to \infty$ and $\Delta T_{1/p} > \mathfrak{c}(\alpha)$. In particular, for μ-almost every $z \in E$ we have:

$$\mathfrak{c}(\alpha) < \liminf_{p\to\infty}(T^+_{0,1/p}f(z) - T^-_{0,1/p}f(z)) = \lim_{p\to\infty}(T^+_{0,1/p}f(z) - T^-_{0,1/p}f(z)), \tag{14}$$

where the last identity comes from the fact that $p \mapsto T^+_{0,1/p}f(z)$ is decreasing and $p \mapsto T^-_{0,1/p}f(z)$ is increasing for any z. However, thanks to the definition of $T^+_{0,1/p}f$ and $T^-_{0,1/p}f$ it is immediate to see that for μ-almost every $z \in E$ we can find a sequence $y_i = y_i(z) \in \mathrm{supp}(\mu)\cap B(z, i^{-1})\setminus X(0, V(\mu, x), \sqrt{1-\alpha^2})$ such that

$$\frac{y_i - z}{|y_i - z|} \to v \qquad \text{and} \qquad \limsup_{i\to\infty}\frac{f(y_i)-f(z)}{|y_i-z|} - \liminf_{i\to\infty}\frac{f(y_i)-f(z)}{|y_i-z|} > \frac{\mathfrak{c}(\alpha)}{2}.$$

□

4 PROOF OF THEOREM 1.1

Without loss of generality we can restrict our attention to finite measures. Assume that μ is a finite sum of rectifiable measures. For every $\varepsilon > 0$ there exist finitely many disjoint, compact submanifolds S_j for $(j = 1, \dots, N)$ of class C^1 (of any dimension between 0 and n) such that denoting $K := \bigcup_{j=1}^N S_j$ it holds $\mu(\mathbb{R}^n \setminus K) < \varepsilon/2$. Consider now any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$. By [1, Theorem 1.1 (i)] and Lusin's theorem, we can find a closed subset $C \subset K$ such that $\mu(K \setminus C) < \varepsilon/2$ and for every $x \in C$ the differential $d_{V(\mu,x)} f(x)$, see [1, §2.1], exists and is continuous. Let $d : C \to \mathbb{R}^n$ be obtained extending $d_{V(\mu,\cdot)} f$ to be zero in the directions orthogonal to $V(\mu, \cdot)$. By [1, Proposition 2.9 (iii)] and since the S_j's have positive mutual distances, we can apply Whitney's extension theorem, see [6, Theorem 6.10], deducing that there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $g = f$ and $dg = d$ on C. Hence Lipschitz functions admit a Lusin type approximation with respect to μ with functions of class C^1.

Assume now that μ is not a finite sum of rectifiable measures and write $\mu = \sum_{k=0}^n \mu \llcorner E_k$, where $E_k := \{x \in \mathbb{R}^n : \dim(V(\mu, x)) = k\}$. Then there exists $k \in \{0, \dots, n-1\}$ such that $\mu \llcorner E_k$ is not a k-rectifiable measure: the case $k = n$ can be excluded by combining [1, Theorem 1.1(i)] and [4, Theorem 1.14], so as to ensure that a measure on $\mathbb{R}^n$ whose decomposability bundle has dimension n is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^n$. Let ν be the supremum of all k-rectifiable measures $\sigma \le \mu \llcorner E_k$ and let E be any Borel set such that $\nu = \mu \llcorner (\mathbb{R}^n \setminus E)$. We claim that $\mu \llcorner E$ satisfies the assumptions of Lemma 2.1.

To prove the claim, consider a k-dimensional surface S that is the graph of some function $h : W \to W^\perp$ of class C^1, where $W \in \mathrm{Gr}(k, n)$. Assume by contradiction that $\eta := \mu \llcorner (E \cap S)$ is non-zero. If $G = \{\mu_t := \mathscr{H}^1 \llcorner E_t\}_{t \in I} \in \mathscr{F}_\eta$ is a family as in [1, Proposition 2.8 (ii)], then $\mathrm{supp}(\mu_t) \subset S$ for almost every $t \in I$. Since both $V(\eta, x)$ and $\mathrm{Tan}(S, x)$ are k-dimensional, this implies that $V(\eta, x) = \mathrm{Tan}(S, x)$ for η-almost every x. Fix now a point $y \in \mathrm{supp}(\eta)$ and observe that the family $\{\mathscr{H}^1 \llcorner p_W(E_t)\}_{t \in I}$ belongs to $\mathscr{F}_{(p_W)_\sharp \eta}$ (as $(p_W)_\sharp \mu_t$ is absolutely continuous with respect to $\mathscr{H}^1 \llcorner p_W(E_t)$ for any t) and that $V((p_W)_\sharp \eta, \cdot)$ is k-dimensional $(p_W)_\sharp \eta$-almost everywhere. By [4, Corollary 1.12], we infer that $(p_W)_\sharp \eta$ is absolutely continuous with respect to $\mathscr{H}^k \llcorner W$. Finally, Since p_W is locally bi-Lipschitz from S to W, this implies that η is absolutely continuous with respect to $\mathscr{H}^k \llcorner S$, which contradicts the maximality of σ. Hence, $\mu \llcorner E$ satisfies the assumptions of Lemma 2.1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be the Lipschitz function obtained from Proposition 3.1. Clearly there exists no function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 which coincides with f on a set of positive $\mu \llcorner E$ measure, hence Lipschitz functions do not admit a Lusin type approximation with respect to μ with functions of class C^1.

Remark 4.1. It is evident from the last lines in the proof of Theorem 1.1 that the condition that g is of class C^1 can be replaced by the condition that g is differentiable everywhere.

Remark 4.2. In Theorem 1.1 the condition (1) can be strengthened to

$$\mu(\{x \in \mathbb{R}^n : g(x) \neq f(x) \text{ or } d_V g(x) \neq d_V f(x)\}) < \varepsilon, \tag{15}$$

where d_V denotes the "tangential differential" defined in [1, Theorem 1.1]. This follows immediately from [3, Proposition 6.2], see also [7, Theorem B]. On the other hand one cannot replace (1) with the condition

$$\mu(\{x \in \mathbb{R}^n : d_V g(x) \neq d_V f(x)\}) < \varepsilon, \tag{16}$$

since the latter does not force any geometric structure on μ. More precisely, for every Radon measure μ and every Lipschitz function f, for every $\varepsilon > 0$ one can find a function g of class C^1 such that (16) holds, see [10, Theorem 2.1].

REFERENCES

[1] Giovanni Alberti and Andrea Marchese. "On the differentiability of Lipschitz functions with respect to measures in the Euclidean space". In: *Geom. Funct. Anal.* 26.1 (2016), pp. 1–66. ISSN: 1016-443X. DOI: 10.1007/s00039-016-0354-y.

[2] Camillo De Lellis. *Rectifiable sets, densities and tangent measures*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. vi+127. ISBN: 978-3-03719-044-9. DOI: 10.4171/044.

[3] Guido De Philippis, Andrea Marchese, Andrea Merlo, Andrea Pinamonti, and Filip Rindler. *On the converse of Pansu's Theorem*. 2022. DOI: 10.48550/ARXIV.2211.06081.

[4] Guido De Philippis and Filip Rindler. "On the structure of $\mathcal{A}$-free measures and applications". In: *Ann. of Math. (2)* 184.3 (2016), pp. 1017–1039. ISSN: 0003-486X. DOI: 10.4007/annals.2016.184.3.10.

[5] Giacomo Del Nin and Andrea Merlo. "Endpoint Fourier restriction and unrectifiability". In: *Proc. Amer. Math. Soc.* 150.5 (2022), pp. 2137–2144. ISSN: 0002-9939. DOI: 10.1090/proc/15857.

[6] Ronald F. Evans Lawrence C.; Gariepy. *Measure theory and fine properties of functions. Revised edition.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299. ISBN: 978-1-4822-4238-6 28-01.

[7] Antoine Julia, Sebastiano Nicolussi Golo, and Davide Vittone. "Lipschitz Functions on Submanifolds of Heisenberg Groups". In: *International Mathematics Research Notices* (Mar. 2022). ISSN: 1073-7928. DOI: 10.1093/imrn/rnac066.

[8] Alexander S. Kechris. *Classical descriptive set theory*. Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. ISBN: 0-387-94374-9. DOI: 10.1007/978-1-4612-4190-4.

[9] Andrea Marchese. "Lusin type theorems for Radon measures". In: *Rend. Semin. Mat. Univ. Padova* 138 (2017), pp. 193–207. ISSN: 0041-8994. DOI: 10.4171/RSMUP/138-9.

[10] Andrea Marchese and Andrea Schioppa. "Lipschitz functions with prescribed blowups at many points". In: *Calc. Var. Partial Differential Equations* 58.3 (2019), Paper No. 112, 33. ISSN: 0944-2669. DOI: 10.1007/s00526-019-1559-3.

[11] John C. Oxtoby. *Measure and category. A survey of the analogies between topological and measure spaces*. Graduate Texts in Mathematics, Vol. 2. Springer-Verlag, New York-Berlin, 1971, pp. viii+95.

[12] David Preiss. "Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities". In: *Ann. of Math. (2)* 125.3 (1987), pp. 537–643. ISSN: 0003-486X. DOI: 10.2307/1971410.

[13] Leon Simon. *Lectures on geometric measure theory*. Vol. 3. Proceedings of the Centre for Mathematical Analysis, Australian National University. Australian National University, Centre for Mathematical Analysis, Canberra, 1983, pp. vii+272. ISBN: 0-86784-429-9.

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