UNIVERSITA' DEGLI STUDI DI TRENTO - DIPARTIMENTO DI ECONOMIA

A PRIMER ON THE TOOLS AND CONCEPTS OF COMPUTABLE ECONOMICS

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A Primer on the Tools and Concepts of Computable Economics

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Abstract

Computability theory came into being as a result of Hilbert's attempts to meet Brouwer's challenges, from an intuitionistc and constructive standpoint, to formalism as a foundation for mathematical practice. Viewed this way, constructive mathematics should be one vision of computability theory. However, there are fundamental differences between computability theory and constructive mathematics: the Church-Turing thesis is a disciplining criterion in the former and not in the latter; and classical logic - particularly, the law of the excluded middle - is not accepted in the latter but freely invoked in the former, especially in proving universal negative propositions. In *Computable Economics* an eclectic approach is adopted where the main criterion is numerical content for economic entities. In this sense both the computable and the constructive traditions are freely and indiscriminately invoked and utilised in the formalization of economic entities. Some of the mathematical methods and concepts of computable economics are surveyed in a pedagogical mode. The context is that of a digital economy embedded in an information society.

Key Words:Computable Economics, Computability, Digital Economy, Constructivity

JEL Classification Codes:B41,C60,C63,C65

1 Basics and Motivation[1]

> "Classical mathematics fails to observe meaningful distinctions having to do with integers. .... Classical mathematicians do concern themselves sporadically with whether numbers can be 'computed', but only on an ad hoc basis. The distinction is not observed in the systematic development of classical mathematics, nor would the tools available to the classicist permit him to observe the distinction systematically even if he were so inclined."
>
> [5], p.7

The above seemingly simple but, in fact, rather profound observation by Errett Bishop, who revived and put constructive mathematics on an applicable footing, captures the essential weakness of classical real analysis. How can a mathematics that cannot maintain and develop a 'meaningful distinction' between numbers that can and cannot be computed be the basis of formalization in a subject such as economics which is quintessentially a quantitative subject, especially in its many policy oriented concerns? A digital economy, almost by definition, is quantified in terms of integers or rational numbers. If so, an economic theory that relies on a mathematics that cannot meaningfully distinguish the computable numbers from those that cannot be computed cannot, by definition, be quantitative in numerical modes. How can economists continue to maintain the fiction that their subject is numerically meaningful and use their formal propositions, derived by using a non-numerical mathematics, to claim applicable policy prescriptions of significance to the daily lives of people, societies and nations?

Over the years many economists, both within and without the citadel that may be called, for want of a better name, orthodoxy, have tried to find ways to infuse, systematically, an element of systematic numerical content in the precise sense mentioned above by Errett Bishop. Kenneth Arrow, Robert Clower, Alain Lewis, Maury Osborne, Herbert Simon and others come most immediately to mind as pioneers who made serious efforts to try to modify orthodox economic theory in directions that could have made it depend on numbers and processes that operate on them in economically meaningful ways. Despite their prestige and their efforts, the citadel remains founded on classical mathematics.

There is a great deal of discussion, in academic and other professional circles, about the predominance of the digital economy in modern times, even to the extent of claiming that such an economy characterizes a post-modern information society. However, an economic theory in the digital mode, for an information society, is not readily available for pedagogical purposes. To be sure, there are many kinds of books on the information society; with increasing frequency, even books and professional articles on *e-commerce* appear and give the impression that there is a seamless adaptation of various kinds of orthodox economic the-

[1]I am deeply indebted to Tom Boylan, Duncan Foley, John McCall and Joe McCauley. Each, in his own way, has stamped an imprint on the contents of this paper. Alas, it is I who must bear all responsibilities for the remaining infelicities.

ory readily available for facilitating rigorous analysis. In this contribution I take exception to this view and try to indicate the kind of economics that may be relevant for the analysis of a digital economy embedded in an information society. The guiding principles for the approach taken here is given by Kolmogorov:

> *"Quite probably, with the development of the modern computing technique it will be clear that in very many cases it is reasonable to conduct the study of real phenomena avoiding the intermediary stage of stylizing them in the spirit of the ideas of mathematics of the infinite and the continuous, and passing directly to discrete models. This applies particularly to the study of systems with a complicated organization capable of processing information. In the most developed such systems the tendency to discrete work was due to reasons that are by now sufficiently clarified. It is a paradox requiring an explanation that while the human brain of a mathematician works essentially according to a discrete principle, nevertheless to the mathematician the intuitive grasp, say, of the properties of geodesics on smooth surfaces is much more accessible than that of properties of combinatorial schemes capable of approximating them.*
>
> *Using the brain, as given by the Lord, a mathematician may not be interested in the combinatorial basis of his work. But the artificial intellect of machines must be created by man, and man has to plunge into the indispensable combinatorial mathematics. For the time being it would still be premature to draw final conclusions about the implications for the general architecture of the mathematics of the future."*
>
> Andrei N Kolmogorov: 'Combinatorial Foundations of Information Theory and the Calculus of Probabilities" ([24], pp. 30-1)

The kind of economic fundamentals that underpin the mathematical formalisms of a computable economics for a digital economy seeks a return to classical building blocks and classical frameworks in their most elementary forms. In the rest of this section I shall try to indicate what I mean by this in the most basic, elementary, way. But before that I should like to substantiate my claim that I seek a 'return to classical building blocks and classical frameworks' by invoking the pertinent observations by two eminent theorists who have contributed much to the foundations of orthodox economic analysis in the *non-digital* domain: Tjalling Koopmans and Steve Smale. In his classic and thoughtful *Three Essays on the State of Economic Science*, Koopmans ([25], p.60; italics added) observed, almost *sotto voce*:

> "Before turning to [the] discussion [of the model of competitive equilibrium] it is worth pointing out that in this particular study our authors [Arrow and Debreu] have *abandoned demand and supply functions as tools of analysis*, even as applied to individuals. The emphasis is entirely on the *existence of some set of compatible optimising choices* ... . The problem is *no longer* conceived as

that of *proving that a certain set of equations has a solution*. It has been reformulated as one of proving that a certain number of maximizations of individual goals under independent restraints can be simultaneously carried out"

The new emphasis brought with it a new formalism and a mathematics to encapsulate it that was entirely divorced from numerical meaning and digital significance. The continuous in its real number versions came to be the vehicle of analysis and digital implementations required approximations which were, correspondingly, divorced from theory. It is not as if it was necessary to recast the fundamental economic problem of finding equilibrium solutions between supply and demand, 'even as applied to individuals', as one of finding a proof of the existence a solution to 'maximizations of individual goals under independent restraints'. As Smale perceptively remarked:

> "We return to the subject of equilibrium theory. The existence theory of the static approach is deeply rooted to the use of the mathematics of fixed point theory. Thus *one step in the liberation from the static point of view would be to use a mathematics of a different kind*. Furthermore, proofs of fixed point theorems traditionally use difficult ideas of algebraic topology, and this has obscured the economic phenomena underlying the existence of equilibria. Also the economic equilibrium problem presents itself most directly and *with the most tradition* not as a fixed point problem, but *as an equation, supply equals demand*. Mathematical economists have translated the problem of solving this equation into a fixed point problem."
>
> .....
>
> "I think it is fair to say that for the main existence problems in the theory of economic equilibrium, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and *even mathematics with dynamic and algorithmic overtones*."
>
> [46], p.290; italics added.

These, then, are the economic precepts, against the analytical backdrop provided by Komogorov's above reflections on formalisms for information theory and the combinatorial and discrete underpinnings of a digital machine with which to pursue quantitative analysis, that will circumscribe the world of computable economics. The 'mathematics of a different kind' that Smale refers to will be constructive analysis and computability theory (including computable analysis). Given the algorithmic foundations of both constructive analysis and computability theory[2] and the intrinsic dynamic form and content of algorithms,

[2] However, computability theory is disciplined by the Church-Turing Thesis; the algorithmic underpinnings of constructive analysis are not so constrained. On the other hand, the former freely invokes classical logical principles in proof exercises and the latter does not accept some of the key precepts of classical logic. I shall be opportunisitc in my reliance on either of these algorithmic mathematics to formulate, analyse and prove economic propositions.

it is clear that this will be a 'mathematics with dynamic and algorithmic overtones'. This means, thus, that computable economics is a case of a new kind of mathematics in old economic bottles. The 'new kind of mathematics' implies new questions, new frameworks, new proof techniques - all of them with algorithmic and dynamic content for digital domains and ranges.

Some of the key formal concepts of computable economics are, therefore: *solvability & Diophantine decision problems*, *decidability* & *undecidability*, *computability & uncomputability*, *satisfiability*, *completeness* & *incompleteness*, *recursivity* and *recursive enumerability*, *degrees of solvability* (*Turing degrees*), *universality* & the *Universal Turing Machine* and *Computational*, *algorithmic* and *stochastic complexity*. The proof techniques of computable economics, as a result of the new formalisms, will be, typically, invoking methods of: *Diagonalization*, *The Halting Problem for Turing Machines*, *Rice's Theorem*, *Incompressibility theorems*, *Specker's Theorem*, *Recursion Theorems.* For example, the *recursion theorems* will replace the use of traditional, non-constructive and uncomputable, topological fix point theorems, routinely used in orthodox mathematical analysis. The other theorems have no counterpart in non-algorithmic mathematics.

In the spirit of pouring new mathematical wines into old economic bottles, the kind of economic problems of a digital economy that computable economics is immediately able to grant a new, digital, lease of life are the classic ones of: computable and constructive existence and learning of rational expectations equilibria, computable learning and complexity of learning, computable and bounded rationality, computability, constructivity and complexity of general equilibrium models, undecidability, self-reproduction and self-reconstruction of models of economic dynamics (growth & cycles), uncomputability and incompletenes in (finite and infinite)game theory and of Nash Equilibria,decidability (playability) of arithmetical games, the intractability (computational complexity) of optimization operators; etc. Naturally, I shall not be able to go through all or even any of these examples in this contribution; but suitable hints and relevant references will be supplied.

I must warn any sympathetic, but critical, reader of one possible and serious misapprehension. Considering a digital economy has nothing to do with any kind of finitism. However, such a caveat must be balanced by the fact that appeal to finitism will not obviate the paradoxes of the countable infinite that are ubiquitous in the mathematics of the discrete. It is easy and almost routinely simple to construct perfectly ordinary example from microeconomics and game theory where the fundamentals are formalized axiomatically and described finitely but still to be able generate undecidable properties in them. For example, considering a game with a finite number of players and strategies also finite in number does not mean that simple counting arguments can locate Nash equilibria. Equally, it is not true, as some have suggested that the non-computability of preference orderings might be circumvented by rendering the commodities and their power sets *finite*. There are many different ways to demonstrate the existence of undecidabiliteis in finite structures, but this is not a direction I wish to pursue in this essay especially because it is elegantly and rigorously discussed

in [9] (but cf. also [52], for detailed discussions).

This essay, therefore, is organised as follows. The next section considers elementary examples to illustrate some of the above concepts and three examples of relevant references from the standard literature to make a case for the discrete, the digital and the ubiquity of the discontinuous. Section three is a discursive and general presentation of the elements that should necessarily form a theory of an information society to embed in a digital economy. Section four outlines the elements of the 'new mathematics' that underpins computable economics for a digital economy. The essay ends with some reflections of omitted issues, possible futures and some further methodological and technical observations.

2 Notes on the Discrete, Digital and the Discontinuous in Economics

> "It is difficult to test theories of dynamic, history-dependent systems. The saturation with content - with diverse meaningful symbolic structures - only makes matters worse. There is not even a well-behaved Euclidean space of numerical measurements in which to plot and compare human behavior with theory."
>
> Newell and Simon ([36], p.13)

Suppose we are given a simple, linear, equation, perhaps an excess demand equation, and attempt to 'solve' it:

$$\alpha x + \beta y = \theta \tag{1}$$

α, β, θ: *integer* (or *rational*) valued parameters;
x, y: *integer* (or *rational*) valued variables;

What is so unusual or strange - at least to an economist - about what one is trying to do, i.e., 'attempting to solve the equation'? An immediate reaction by a conventionally educated economist – and many others – would be to dismiss the attempt as futile since we are given one equation in two variables and asked to 'solve' it! Traditionally, one would expect the concept of solution to carry with it the associated idea that there should be as many equations as variables. But suppose, instead, one means by 'solution' a search for the *integer* (or *rational*) values of the variables x and y given *integer* (or *rational*) valued parameters α, β, and θ for which the equation is satisfied as given above – then the scenario is wholly different.

In the alternative perspective of the new mathematics, the relevant questions associated with the

concept of 'solution' or 'satisfiability' would be:

- Is there a uniform way, *a priori*, of telling whether *integer* (or *rational*) values of x and y exist such that the equation is *satisfied* – i.e., can be '*solved*';

- Is there a general method of finding (in addition to 'proving existence') all such values of x and y that satisfy the equation;

In other words, how can the problem of the '*satisfiability*' and '*solvability*' of such equations be studied and what methods are available to systematise and routinise their use? The paradoxical answer to both of these questions is that the problems of 'solvability' and 'satisfiability' are intractable and their systematic and routinised study is almost impossible. To get a handle on this remark, let me generalise just a little and ask:

Problem 1 *Given a polynomial equation with integer coefficients, decide whether or not the equation has any integer solutions. (In the jargon of computability theory or number theory, we are testing Diophantine equations for solutions).*

For example, is there an integer x satisfying:

$$3x^8 + 7x^5 - 18x^2 - 427x + 10 = 0 \tag{2}$$

If the equation holds for some x, then:

$$10 = x\left(-3x^7 - 7x^4 + 18x + 427\right) \tag{3}$$

Thus relevant x is one of the eight numbers ± 1, ± 2, ± 5, ± 10. To see whether or not the equation has a solution, just plug in each of those

eight numbers for x. If any of the eight works, we have a solution; while if none of them work, there is no solution. Generalizing, we have a nice algorithm for testing the polynomial equation

$$p(x) = 0 \tag{4}$$

for solutions: If c, the constant term of p, equals 0, then $x = 0$ is a solution. If $c \neq 0$, find all of its divisors. Then plug each divisor of c into p. The equation $p(x) = 0$ has a solution iff one of these plug-ins yields 0 as an answer. This algorithm can easily be written up as a program – the divisors of c can be found either by brute force, checking all integers from $\{-|c|\}$ through $\{|c|\}$, or (preferably) by a more efficient search.

Now consider the equation:

$$15x + 33y = 28 \tag{5}$$

Does it have any *integer* solutions? No, for a quite simple reason: Regardless of x and y , the left-hand side must be a multiple of 3, so it cannot equal 28. In general, the linear equation with integer coefficients:

$$ax + by = c \tag{6}$$

cannot be satisfied for integer values of the variables unless c is a multiple of the greatest common divisor of a and b [denoted $\gcd(a, b)$]. Conversely, if c is a multiple of $\gcd(a, b)$, then standard elementary number theory implies:

$$\exists(x,y), s.t \;\; ax + by = c \tag{7}$$

is *satisfiable*. So again, a simple *procedure* based on divisibility lets us test $ax + by = c$ for solutions, solvability and satisfiability:

Claim 2 *There exists a solution iff gcd(a,b) is a divisor of c.*

Of course, that still leaves the question of how to crank out the computation of gcd(a,b). But that can be done easily. One could, by brute force, check numbers from 2 through $|a|$ to find which of them divide both a and b. Or, more efficiently, one could use the *Euclidean algorithm*, a 2500-year-old "program" devised for this purpose. Generalize the above examples to: find a *procedure* that, given a polynomial equation with *integer* coefficients, *determines* whether or not the equation has an *integer solution*. Is this problem *decidable*? To answer this rigorously, it is first necessary to be precise about the meaning of *procedure*. This term was made precise as a result of the development of computability theory via the notion of effective calculability by a *Turing Machine* or by *partial recursive functions* or by $\lambda-$*definable functions* or by *Register Machines* or by other formalisms, all of them equal to one another by the Church-Turing Thesis. The above claim can then be properly generalised as the following problem, known as *Hilbert's 10th Problem* (to which I return more formally later on, below):

Problem 3 Determination of the solvability of a Diophantine equation*: Given a Diophantine equation with any (finite) number of unknown quantities and with rational integral numerical coefficients; to devise (i.e., construct) a process according to which it can be determined, by a finite number of operations, whether the equation is solvable in rational integers.*

Remark 4 *Hilbert did not ask for an algorithm to find a solution in case one exists, because that presents 'no problem' in principle, provided time and patience are freely available. Brute force suffices again – just systematically try out the possible combinations of values for the variables until you find one that works. What brute force will not do is to inform you when the equation has no solutions. In this latter case, as brute force testing progresses, you will suspect more and more strongly that there is no solution, but without ever knowing for sure. Hilbert asked for some other algorithm that would definitively settle each case.*

It is this subtle distinction between an affirmation and a refutation that is captured by the formal differences underpinning the definitions of *recursive & recursively enumerable sets*. In computable economics one way to impose structure on observable variables will be by way of supposing their membership in one or the other of the above sets - rather like the way one imagines, in orthodox mathematical economics that variable defined over compact sets.

I claim that, by definition, variables, parameters and coefficients defining basic behavioural functions in economics (particularly the economics of a digital economy) - supply, demand, production, etc., - should be formalised as *Diophantine equations* and their solutions sought as equilibrium market values along the above lines. In one sense, *every* economic problem for a digital economy, approached by computable economics, can be formulated as a *Diophantine decision problem*. I shall have a little more to say about these things below. How does such a formalism relate to the traditional optimization exercises of orthodox theory? Let me illustrate with simple examples, again, in such a way that concepts of computable economics relevant for analyzing a digital economy (embedded in an information society) can also be highlighted.

Consider the following three-variable Boolean formula:

$$\neg x_3 \cap (x_1 \cup \neg x_2 \cup x_3) \tag{8}$$

Just as in the case of equations with integer (or rational) values, given a truth assignment $t(x_i) = 1$ or 0 for each of the variables x_i $(i = 1, ..3)$, the above Boolean formula can be evaluated to be true or false, globally. For example the following assignments gives it the value *true*: $t(x_1) = 1; t(x_2) = 1; t(x_3) = 0$. Boolean formulas which can be made true by some truth assignments are said to be *satisfiable*.

Now consider the Boolean formula:

$$(x_1 \cup x_2 \cup x_3) \cap (x_1 \cup \{\neg x_2\}) \cap (x_2 \cup \{\neg x_3\}) \cap (x_3 \cup \{\neg x_1\}) \cap (\{\neg x_1 \cup \{\neg x_2\} \cup \{\neg x_3\}) \tag{9}$$

Remark 5 *Each subformula within parenthesis is called a clause; The variables and their negations that constitute clauses are called literals; It is 'easy' to 'see' that for the truth value of the above Boolean formula to be true all the subformulas within each of the parenthesis will have to be true. It is equally 'easy' to see that no truth assignments whatsoever can satisfy the formula such that its global value is true. This Boolean formula is unsatisfiable.*

Problem 6 *The Satisfiability Problem*

Given m clauses, $C_i (i = 1, \ldots, m)$, containing the literals (of) $x_j (j = 1, \ldots, n)$, *determine* if the formula $C_1 \cap C_2 \cap \ldots\ldots \cap C_m$ is satisfiable.

Determine means 'find an (efficient) algorithm'. To date it is not known whether there is an efficient algorithm to solve the satisfiability problem – i.e., to determine the truth value of a Boolean formula. This is 'more-or-less' equivalent to the above mentioned unsolvability of Hilbert's 10th Problem. Now to go from here to an optimization framework is a purely mechanical affair. Denoting the union operator as ordinary addition and the negation operator related to arithmetic operators as: $\neg x = (1 - x)$ and noting that it is necessary, for each clause C, there should, at least, be one true literal, we have, for any formula:

$$\sum_{x \in C} x + \sum_{x \in C} (1 - x) \geq 1 \tag{10}$$

With these conventions, the previous Boolean formula becomes the following *integer linear programming* (*ILP*) problem:

$$x_1 + x_2 + x_3 \geq 1 \tag{11}$$

$$x_1 + (1 - x_2) \geq 1 \tag{12}$$

$$x_2 + (1 - x_3) \geq 1 \tag{13}$$

$$x_3 + (1 - x_1) \geq 1 \tag{14}$$

$$(1 - x_1) + (1 - x_2) + (1 - x_3) \geq 1 \tag{15}$$

$$0 \leq x_1, x_2, x_3 \leq 1, \text{ and integer} \tag{16}$$

Definition 7 *A Boolean formula consisting of many clauses connected by conjunction (i.e., $\cap$) is said to be in Conjunctive Normal Form (CNF).*

Remark 8 *A CNF is satisfiable iff the equivalent ILP has a feasible point.*

Clearly, the above system of equations and inequalities do not, as yet, represent an ILP since there is no 'optimisation'. However, it can be turned into a complete ILP in the ordinary sense by, for example, replacing the first of the above inequalities into:

$$Max\ y,\ s.t: x_1 + x_2 + x_3 \geq y \tag{17}$$

Remark 9 *The formula is satisfiable iff the optimal value of y, say $\hat{y}$ exists and satisfies $\hat{y} \geq 1$.*

Finally a warning on *ad hoc* 'approximate' solutions: an *ILP* cannot, except for flukes, be solved just by rounding the solution to the 'corresponding' linear programming (*LP*) problem. Since the constraints corresponding to the satisfiability problem for any formula in *CNF*, with at least two literals in each clause is always satisfied by the fractional values: $x_j = \frac{1}{2}$, $\forall x_j$, a feasible solution to the *LP* is always trivially available. But, rounding a feasible LP solution to an acceptable ILP – i.e., *deciding* whether it can be rounded – is as hard as the original satisfiability problem. As for: 'how hard?', 'what does hard mean?' and 'what is easy?', are questions that are for computational complexity theory.

These, then, are some of the elementary transformations and alternate formulations that the mathematics of computable economics entails, if a digital

economy is to be seriously formalized and analyzed in the same way orthodox theory has handled simple, real-valued domain, optimizations. Satisfiability and decidability replace blind optimizations; this is why Simon got it right from the outset when he sought to underpin classical behavioural economics with *satisficing* behaviour.

At another, more basic economic theoretic level of even the first principles course, Edward Chamberlain's 1948 report on class room experiments to test the validity of the competitive equilibrium hypothesis is an acknowledged classic of modern experimental economics[3] ([8]). The key notion underlying the experiment was the fact that the *process* of demand and supply adjustments, whether towards an equilibrium or not, implied demand and supply *step functions* and, hence, of course, an excess demand function of that form, as well (cf. [8], op.cit, Figure 1; cf. also [48], the two diagrams on p.118)[4].

Most kinds of experimental economic setups, where market *processes* need to be simulated to test the validity of theoretically predicted equilibria, whether under classical competitive or non-competitive conditions, whether in game-like situations or not, implies - whether convergent or nor - the use of *step functions*. This, coupled to the fact that there is no theory of an algorithmic[5]

[3] Although Irving Fisher's earlier *analogue* experiments should be considered the true fountainhead for the field of experimental economics:

> "The [hydraulic] mechanism just described is the physical analogue of the ideal economic market. The elements which contribute to the determination of prices are represented each with its appropriate rôle and open to the scrutiny of the eye. We are thus enabled not only to obtain a clear and analytical *picture* of the interdependence of the many elements in the causation of prices, but also to employ the mechanism as an instrument of investigation and by it, study some complicated variations which could scarcely be successfully followed without its aid." ([14], p.44, italics in original)

I shall not pursue doctrine-historical priorities in this paper.

[4] I cannot resist mentioning that William Thornton pointed out, at the very dawn of supply-demand analysis of equilibria, that the process of an English auction would, in general and for a single commodity, converge to a different value from the one to which a Dutch auction would converge. Even that acknowledged last standard bearer of the 'classical' flag, John E Cairnes acknowledged that this observation would cast decisive doubt on the 'classical' equilibrium solution in terms of demand and supply:

> "...[W]hat is the explanation of market prices?. This question, after having been discussed by economists from Turgôt and Adam Smith to Mill, was at length supposed to have received its definitive solution in the chapters on 'Demand and Supply' in the Principles of Political Economy by the latter authority. That solution, however, has lately been challenged by Mr. Thornton, I must own it seems to me, so far as the negative portion of his criticism is concerned with success." ([7], p.97; italics added)

I was unable to resist the temptation to quote Cairnes in view of the fact that I am, after all, the holder of the John E Cairnes Chair of Economics! Of course, experimental economists are aware of such ambiguities and indeterminacies and device procedures that are consistent with orthodox microeconomic postulates and guarantee convergence to determinate equilibria (cf, for example, [47], in particular, p.944, ff. My point, however, is that *none* of these procedures can be underpinned by rationality postulates that are algorithmic; nor are they, themselves, formally algorithmic and cannot be shown to converge to computable equilibria.

[5] By '*algorithmic*' I mean, of course, either *computable* in the strict recursion theoretic

process that guarantees convergence to any of the theoretical equilibria predicted by orthodox theory - whether of the competitive[6], non-competitive or game theoretic variety - makes it obviously difficult 'to plot and compare human behavior with theory', as Newell and Simon, masters *par excellence* of empirical and behavioral economics, pointed out years ago.

Another acknowledged classic. albeit in the altogether different area of the dynamics of the stock market, was Louis Bachelier's astonishing doctoral dissertation of 1900 which heralded the advent of the formal study of the random character of stock market price dynamics ([3]). Bachelier's heuristic solution to the integral equation:

$$P_{x,t_1+t_2} = \int_{-\infty}^{\infty} P_{z-x,t_1} P_{x,t_2} dx \tag{18}$$

is the diffusion process:

$$P_{x,t} = \frac{1}{2\pi k\sqrt{t}} \exp\left(-\frac{z^2}{4\pi k^2 t}\right) \tag{19}$$

where x and z are prices and t is time.

More than half a century after Bachelier's path-breaking paper, Maury Osborne ([37]), in reviving a study of the underpinnings of the seeming randomness of price and returns behaviour in the stock market, pointed out some of the mathematical infelicities in that classic. One of the crucial points in cleaning up some of the mathematical infelicities was the question of the existence of the first and second partial derivatives w.r.t z in the postulated solution. In an attempt to remove the infelicity, Osborne's own reformulated model was, in its turn, underpinned by three crucial empirical assumptions:

- Prices move in *discrete* units of 1/8 of a dollar[7];
- In each pre-defined unit of time only a *finite*, *integral* number of transactions can be made on any commodity or entity traded in the stock market;

sense or constructive in any of the standard senses - i.e., Bishop-style constructivism, Russian constructivism, etc.

[6] The claims of computable general equilibrium theory to have devised constructive algorithms to show the effective computation of competitive equilibria is demonstrably false. This can be shown in view of a decisive invoking of an undecidable disjunction via the use of the Bolzano-Weierstrass theorem at a crucial step in the construction of the 'algorithm'.

[7] Anyone working with one of the readily available symbolic mathematical softwares, for example *Matlab* or *Mathematica*, will have experiences with the kind of heavily truncated approximations of money market rates that are part of the built-in functions in them. In *Matlab* the command **format bank** is a built-in instruction for working with only two decimal places for currency. The requirements of economic theory cannot be met by even the most precise digital computer. To this obvious difficulty must be added the added complication of conversion of real numbers expressed as decimal fractions into dyadic fractions. The simple fact that the binary fraction for 0.1 is non-terminating and can lead to unexpected catastrophes when truncated arbitrarily should be warning enough to any somnambulant economic theorist. But I know of no economic theorist who has ever shown any kind of awareness of even this simple fact.

- The validity of the Weber-Fechner law[8];

Interestingly, Osborne's assumptions and empirical framework (and results) underpinned and implied the use of *step functions*. (cf, for persuasive and colourful arguments on this point, his fascinating book [38], in particular Figures 2.4-1, 2.4-2 and 2.4-3, pp.23-6). The kind of high frequency money market and commodity market data, daily reported even in the popular financial press, that underpin the first two assumptions are readily available for empirical analysis in any number of routinely accessible data banks.

If, as is routinely done in almost every variety of experimental or empirical economic research, it is assumed that this kind of economically relevant data is generated by a probability space in which rational decision makers reside, then it appears to be necessary for the investigator to work with the formalism of *empirical distribution functions*[9], if also the assumption of *step functions* is to be maintained.[10] But this is not normal practice in economic theory or even in the theory of finance (even of the computational variety).

On the other hand, when market data, whether of the stock market variety or of the more traditional goods and money market varieties, are to be experimentally or empirically analyzed on the basis of standard economic theory, there is a conundrum that has no straightforward formal solution. All of orthodox, and even non-orthodox, mathematically formalized economic theory pre-supposes the domain of real numbers (and, occasionally, even the non-standard numbers). Moreover, the transition is made, from the price ($\mathbf{p}$) dynamics of excess demand via primitive *tatônnement* discussions:

$$\dot{\mathbf{p}} = \mathbf{z}(\mathbf{p}, t); \qquad \mathbf{p}_{t_0} = \bar{\mathbf{p}} \tag{20}$$

to a stochastic differential equation:

$$\dot{\mathbf{p}} = \mathbf{z}(\mathbf{p}, t, \eta_t); \qquad \mathbf{p}_{t_0} = \tilde{\mathbf{p}} \tag{21}$$

where, now, η_t is a random function and $\tilde{\mathbf{p}}$ is a random initial value.

If (21) had very little or no justification in economic theory, there would be even more reason to accuse the interpretation of market price fluctuations in

[8] The subjective probability basis of orthodox utility theory and its tenuous underpinnings in psychophysics was, in the early literature, given a tenuous formal foundation via the Weber-Fechner law (cf., for example, [32], p.380).

[9] The *empirical distribution function*, F_n of n points $a_1, a_2, ..., a_n$ on the (Euclidean) line is the *step function* with *jumps* $1/n$ at $a_1, a_2, ..., a_n$. Thus, $nF_n(x)$ equals the number of points $a_k \in (-\infty, x]$ and F_n is a distribution.(cf. [11], Chapter I, §12, p.36,ff)

[10] By using a *Cantor-type distribution* (cf.[11], op.cit, pp34-6) it is possible to generate the famous *Devil's Staircase*, a case of a continuous distribution function without a density. Nonlinearly coupled market dynamics can lead to the *Devil's Staircase*.

I have often conjectured that the persistence of 'far from equilibrium' configurations of typical e-commerce markets, for example, are generated by the *nonlinearly coupled dynamics* of supply chains and demand dispersions – the one giving rise to hierarchical structures and the other to asynchronous interactions of heterogeneous economic entities. The former amenable to combinatorial, graph theoretic, formalisms and analysis; the latter, to dynamical systems analysis via, say, encapsulation as cellular automata.

terms of (22) to be pure *ad hockery*. However, in a somewhat perverse way, a powerful theorem in mathematical economics, the so-called *Sonnenschein-Debreu-Mantel theorem on excess demand functions* can, in fact, be invoked to justify the assumption of arbitrary *aggregate* functional specifications for price dynamics. If we are prepared to analyze *market* price dynamics then the practice of going from (4) to the more specific form of the stochastic differential equation:

$$d\mathbf{p} = \alpha(\mathbf{p}, t)dt + \beta(\mathbf{p}, t)d\Theta(t) \tag{22}$$

(where,$\Theta(t)$: is a Wiener process; and $\alpha(\mathbf{p}, t)$: a drift coefficient) could, indeed, be made to make rigorous economic theoretical sense.

However, to go from (22) and (23), via a basis in economic theory or by way of the kind of kind of restrictions assumed by Osborne on the domain over which prices are defined, and the constraints on the frequency of decision making, to difference schemes requires an approximation theory that no amount of *ad hockery, black magic* or *alchemy* in numerical analysis and complexity theory will alleviate. Still, with princely unconcern for mathematical rigour and economic theoretical consistencies, market data is analyzed in the *digital* domain, with *digital* means, using, for example, the Ito Stochastic calculus, underpinned by one or the other of the rational theories of decision making in measure-theoretically justified probabilistic spaces with little or no *digital* basis.

In this connection it must also be pointed out that even within the citadel of macroeconomic dynamics, growth and business cycle theory, the traditional assumption of postulating that empirical data has been generated from an underlying probability space in which rational agents reside and economic institutions are located has been questioned with increasing vigour by orthodox theorists, even of the dominant school (cf. [27]).

Many years ago, in one of the most fundamentally innovative discussions on a core area of monetary economics, Clower and Howitt remarked that a realistic analysis of the *Transactions Theory of the Demand for Money* implied the use of proof techniques involving 'the use of number theory - a branch of mathematics unfamiliar to most economists' ([**?**], p.452, footnote 3). The simple reason was that they imposed simple, realistic, constraints on the domain of analysis - discrete units. Thirty years later[11], in the ferment of the digital economy, not an iota of progress has been made on familiarising economists with number theory or any other kind of discrete mathematics, in spite of lip service to computational economics.

Clower and Howitt began by supposing that individual traders produce, sell and purchase only discrete units of one stock-flow good. Using 'the generic symbols S, D and M' to denote *measurable quantities* of production-supply, demand and money prices and y, the constant level of production per unit of time of S, the time-paths of the inventories of the variables had the following patterns (in the special case $S = 3, D = 2, y = 1, m = 0$ and $M_0 = 1$; Fig.1, p.451 in [**?**]):

[11] The original version of the paper appeared as a UCLA Economics Discussion Paper in 1974.

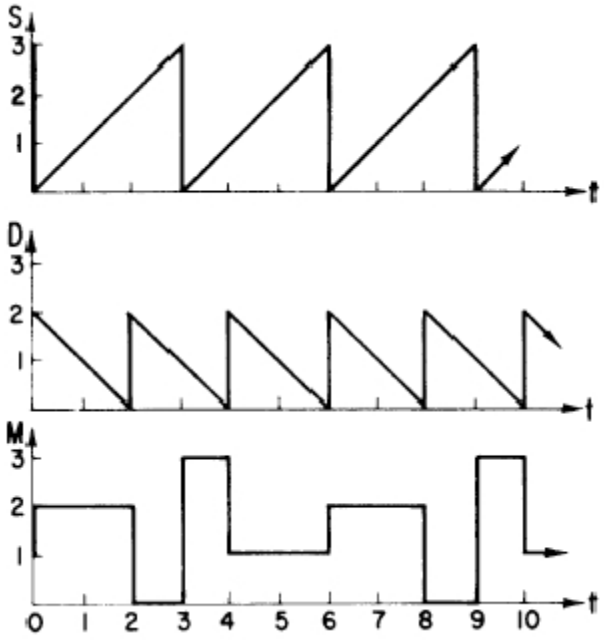

Figure 1: Time Paths of Inventories

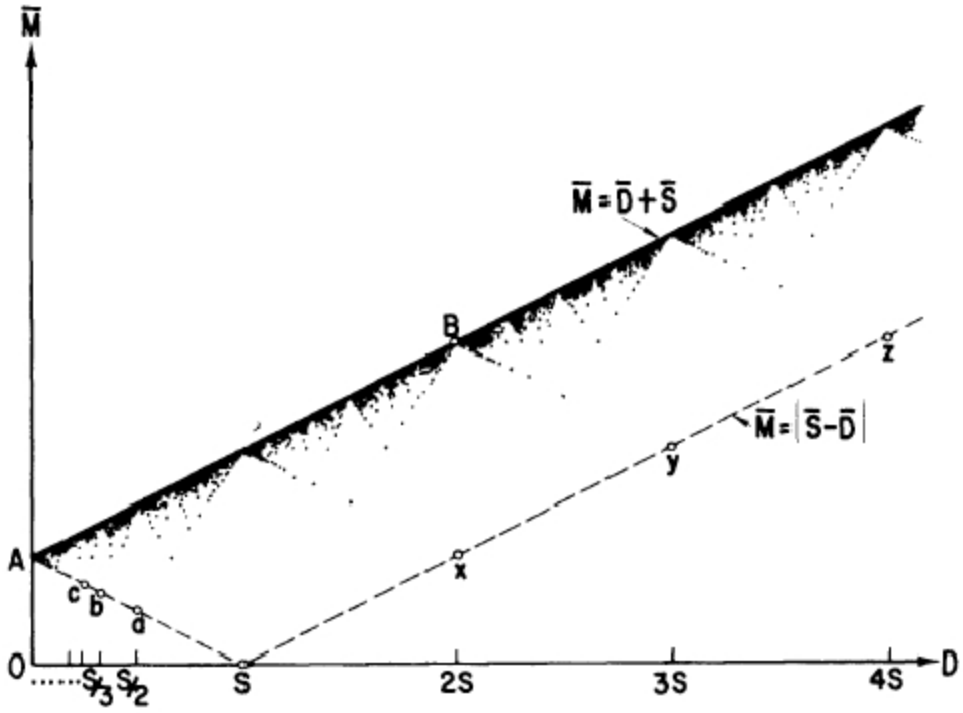

Figure 2: The Finance Function

Corresponding to the initial assumptions, the average money holdings, $\bar{M}$, denoted as the *finance function*, $F(S, D)$, is given by:

$$F(S, D) = \bar{M} = \bar{S} + \bar{D} - G(S, D) \tag{23}$$

where the 'barred variables' denote average holdings and $G(S, D)$, the *divisor function* is given by:

$$G(S, D) = \begin{cases} GCD \text{ of } S \text{ and } D \text{ if } S/D \text{ is rational} \\ 0, \text{ otherwise} \end{cases} \tag{24}$$

The graph of this perfectly natural and realistic *finance function,* $F(S, D)$,reproduced from [**?**] (Fig.2, p.452), is shown below.

The simple economics of the seemingly bizarre behaviour depicted in the diagram is as follows[12]:

[12] The points, $x, y, z, ...$, along the longer lower boundary and those on the shorter lower

"[The graph's] most notable characteristic is that for any given value of S the finance function contains *a jump discontinuity* at every value of D for which S/D is rational. These discontinuities have a straightforward economic interpretation. The trader can economize on money balances most effectively by so *coordinating* purchases and sales that sales often or always occur simultaneously with purchases, in which case S and D will have a common divisor that differs significantly from zero. If, instead, most or all sales are *poorly coordinated* with purchases, most purchases will have to be financed with money carried over from previous sales, and S and D will have a common divisor that is close to zero."

ibid, p.452; italics added

In other words, whenever $S = D$, then, since $G(S, D) = S$, the trader need not hold money; however, if demand is reduced *by even a fraction*, then the divisor function will drop to the neighbourhood of zero and money balances will rise *discontinuously*. This prediction is not based on any kind of *ad hoc* assumptions about risk and uncertainty or placing agents and institutions in probabilistic environments, various *Phelpsian islands* and so on.

Chamberlin's experimental domain is permeated with step functions; Osborne, in the sphere of computational finance, seeks to constrain the domain of variables to the discrete; Clower and Howitt do constrain the domain of elementary supply-demand analysis of a classic problem in monetary theory to rational valued variables and invoke the methods of number theory and dynamical systems theory to solve their intractable problem. What such innovative criticisms and doubts - almost all of them rigorously formulated - imply, most obviously, are a series of irrelevancies at the core, foundational, levels of economic and methodological basics. For example, the irrelevancy of basing theory on the domain of real numbers, the excessive *informational* burdens on the rationality postulates of individual behaviour that make ordinary decision making impossible, the *uncomputability* of almost any equilibrium predicted by theory, and so on.

If it was simply inertia and tradition that were the root causes of lapses in rigour or neglect of the importance of considering the proper domain for relevant economic variables and respect for the modern digital tools of analysis then one would have expected a change in practice and expertise to have taken place at least by osmosis, if not hard work and effort toward learning the mathematics of the discrete and the digital. Since none of these has happened and to avoid unnecessary compromises with theoretical and empirical rigour, whilst respecting the discrete/digital nature of observational data, all of the economic theoretical formalizations are achieved via the mathematics of the digital com-

boundary, $a, b, c, ...$, are attained when D is an exact multiple or divisor of S, where, therefore, $\bar{M} = \left|\bar{S} - \bar{D}\right|$. The values on the upper boundary, along the line AB, are attained when $\frac{S}{D}$ is irrational, when, therefore: $\bar{M} = \bar{S} + \bar{D}$.In the remaining cases, when $\frac{S}{D}$ is *rational* but neither it nor its reciprocal is an *integer*, yield values in the region bounded by the above lines.

puter and algorithms: *viz.*, *computability theory* and *constructive mathematics*. Therefore, the underlying economic theory is *Computable Economics*. However, this should not lull the reader into thinking that all the formal structures in Computable Economics presuppose *computability* by agents or institutions; in fact, such a formalism is replete with undecidabilities and uncomputabilites and call forth a wholly new ontology for applied and theoretical economics. In particular, working within a probability space as the generator of empirical data and a space in which rational agents decide optimally with the calculus of expected utility maximization mediated by subjective probabilities will have to be given up. But the alternatives are quite rich and even philosophically and epistemologically exciting and challenging.

3 Information, Computation and Emergent Complexities in a Digital Economy

> "*Discrete* forms of storing and processing *information* are fundamental. They are at the base of the very measure of the 'quantity of information' expressed in 'bits', numbers of binary symbols. ...[T]he discrete part of *information theory* is to some extent destined to play a leading organizing part in the development of *combinatorial* finite mathematics. ... [I]t is not clear why *information theory* should be based so essentially on *probability theory*, as the majority of textbooks would have it. It is my task to show that *this dependence on previously created probability theory is not, in fact, inevitable.*"
>
> [24], p.31; italics added

The theoretical foundations of a digital economy are based on digitally underpinned information, computation and communication structures for agents and institutions that are conceived as emergent, complex, evolving entities. In this essay the digital economy will be considered to be embedded in a formal information society - just as, orthodox economic theory is the theory of modern industrial societies and not of primitive economies. Hence an outline of the characteristics of what a formal information society entails will have to be discussed. To this must be added an outline of how agents and institutions can formally be situated in a digital economy that is embedded in an information society and theorised economic theoretically?

Information problems have become of central concern in orthodox economic theory of every hue and variety. However, the nature of the formal information theory that has been harnessed in the service of orthodoxy, whether in economic theory or in econometrics, has been based on equally orthodox (subjective or measure theoretic) probability theory. Not a single consideration of an information problem in any kind of formalism of economic theory or econometrics has de-linked it from either subjective probabilistic underpinnings or a measure theoretic basis. *A fortiori*, there is, so far as I know, no formal

information-based problem in economics - whether microeconomic, macroeconomic, IO, Game Theory or econometrics - that is demonstrably computable (or shown to be formally uncomputable). Moreover, the formalism of communications, in the same collection of economic and econometric disciplines mentioned above, by and between agents and institutions, has never been based on a formal information theory that is independent of probability theory and, hence, on discrete or digital mechanisms; viz., on computable mechanisms.

On the other hand, so far as I know, no one has proposed a *theory of the information society*, particularly one that is compatible with any of the standard economic theories - let alone of the dominant or orthodox economic theory - of modern industrial economies. *Any* theory of the information society will have to be underpinned by and built on the foundations that were devised by Alan Turing, Claude Shannon, Norbert Wiener, John von Neumann, Andrei Komogorov and Herbert Simon. Turing's notion of *computability*; Shannon's measure and formalization of *coding* for *communication* and *sampling* to transform the analog - in which, as pointed out above, much theorising resides - to the digital within a framework of information theory; Wiener's epistemological and methodological contributions to the mathematics of the stochastic processes that underpin the noise contaminating the information that is communicated between social agents and institutions and the latter's attempt to coordinate and control their smooth functioning according to specified criteria; von Neumann's experimental design criteria, utilizing the formalization of cellular automata, for self-reproduction and self-reconstruction in addition to his contributions to the development of the stored-program digital computer; Kolmogorov's creative genius abandoning his own early creation of a measure theoretic basis for probability theory and developing a combinatorial and computable basis for it and resurrecting the once discredited frequency theory of probability to provide a digital basis for information theory; and, closer to home, so to speak, Herbert Simon's almost single-handed development of modern cognitive science on the basis of a theory of computation and information processing by realistic cognitive mechanisms. On these sterling building blocks are to be found a theory of the digital information society.

To the above core theoretical bases of an information society, particularly where experimental self-reproduction and self-reconstruction theories of the digital economy are at issue, have, moreover, to be supplemented with a vision of the *emergent complexities* of new market forms, innovations and societal transformations that are brought about by new technologies, not just of a scientific or engineering nature.

If these are the foundations on which a theory of a digital information society is founded, any subsidiary theory of agents or institutions, such as economic theory, cannot but aim at a consistency with the basics, as suggested above, that came out of the various defining themes and theories of Turing, Shannon, Wiener, von Neumann, Simon and Kolmogorov. Paradoxically, economic theory as taught and harnessed for practice today has nothing to do with any of these underlying foundations of information theory, the theory of computation, communication theory, cognitive science, theories of self-reproduction and

self-reconstruction based on computability and the need to tackle, theoretically and empirically, emergent complexities of the digital foundations of individual and institutional behaviour and evolution . How, then, can the economics and decision processes of a digital economy be analysed?

To answer this question it will be necessary to identify, clearly and formally, those foundations of the economic theory of individual behaviour, institutional evolution, market complexities and technological innovations that depend on theories of information, computation, communication and emergence. It is possible that an orthodox economic theorist would claim, as hinted in the opening paragraph of this section, that the citadel is well aware of informational, computational and emergent factors of the foundations of economic theory. Such an orthodox theorist could easily point to work by Nobel Prize winning economists and their work on asymmetric information. Perhaps, even more fundamentally, a hand may be waved in the direction of the information revolution that brought with it, in one fell swoop, also the microfoundations movement in macroeconomics heralded by the celebrated *Phelps volume* ([41])[13]; a volume that, in almost every conceivable way, made it possible for the newclassical revolution to sweep all before it. And, then, such a theorist would or could triumphantly point out to the 'signal processing' basis of newclassical economics; go even further and point out to the fact that rational agents in such economies are, essentially, signal processors and if such things are not based on information, communication and computation theories then what else can be? Indeed, such theorists, the dominant ones at the moment, could even point out to a wholly new trend in macrodynamics whereby the theory is now felt to have been solidly and firmly based in recursive mathematical structures: Markov processes, dynamic programming and Wald's sequential decision making (as claimed, for example, in [30]). The residual complaint that such theories are incapable of encapsulating emergent structures[14] may well be side-tracked by the believable claim that it is next on the research agenda and initial steps are in the process of being taken or, at least, been considered seriously.

It is trivial to show that the formally rigorous core of orthodox neoclassical economic theory, *General Equilibrium Theory*, with or without informational and communication constraints, whether in its alleged 'computable' incarnation or not, has no computable, viz., digital, numerical content whatsoever. In fact *General Equilibrium Theory*, in its present form and contrary to various explicit

[13] Information problems in macroeconomics were centrally placed by the pioneers of the subject, particularly by Lindahl and Keynes, from the outset. But they, too, neglected the link with computation and relied on either subjective or logical probability theory for a justification of an information theoretic consideration of individual or institutional decision process. The same can be said of Stigler's later revival of Hayekian information and knowledge themes, in the early 60s.

[14] Occasionally, vague references to Hayek's work on *The Sensory Order* might be invoked, especially by so-called members of one variety of *Austrian economics*. But this is not something to which orthodoxy can turn to; moreover, the claims of the Austrians are formally thin and digitally meaningless. There are earlier approaches to emergent evolution of agents and institutions, more solid and better based on digital structures, going all the way back to John Stuart Mill, to which we can turn. That, however, is not a story that I shall dwell upon here.

assertions, *cannot* formally be made numerically meaningful in any algorithmic formalism. Hence there is no meaningful way to encapsulate computable self-reproduction or self-reconstruction processes as general equilibrium models. This is not the place to demonstrate this negative assertion in any formal way. The point, however, is that this theory, as presently formalized, can provide no basis for an empirical or experimental analysis of a digital economy.

Above all, nothing in formal economic theory has given content to formalizing agents, whether rational or not, with numerically meaningful cognitive content. Not a single formal learning scheme developed and implemented within formal economic theory has ever been built on either recursion theoretic or constructive bases. Some rare instances of learning or inference schemes can, at best, be classed as numerically meaningful except for the fact that they are implemented by numerically and algorithmically unformalizable rational agents seeking uncomputable fix points or entities. There are, to be sure, pseudo-formal claims to the contrary; but it is easy to show that such claims are false. All of the above reservations and comments apply, *pari passu*, to all formal theories of economic dynamics, whether of theories of the business cycle or growth and development.

The issue of financial markets - asset markets, for example - in a digital economy, and the question of the theory of computational finance in general, is a more complicated story. There is a healthy, vigorous and rigorous attempt to place this field of research on a behavioural basis (eg., [51], although not quite in the sense of classical behavioural economics) and to consider computable formalizations and discuss the complexities of emergent financial markets in algorithmic frameworks (eg., [34]). If these embryonic formal tendencies crystallise in numerically meaningfully ways, then it is clear that financial markets might well be underpinned by agents and institutions that are algorithmically rich and formally rigorous in *information, computation and emergent complexities.*

In sum, there is no extant economic theory for a digital economy, with the exception of *behavioural economics* as developed by Herbert Simon[15] and *Computable Economics*, that has ever linked information combinatorially with computation and communication in the context of computationally meaningful procedural agents in algorithmically evolutionary institutions. Classical behavioural economics is not based on recursion theory. Computable economics is squarely based on recursion theory. Either of the theories could form the basis for an analysis of *information, computation and emergent complexities in a digital economy.* I choose the latter, *pro tempore*, partly because I have come to believe that the former is, largely, a 'subset' of the latter. I am aware that Simon would disagree with such an assertion; alas, he is no longer among us to refute my statement, as he, no doubt, would have with conviction and force.

However, I shall retain the fundamental insight that agents and institutions in any decision theoretic context should most fruitfully be considered problem solvers. Hence any formalization of such entities has as its starting point, even in

[15] I have come to refer to this kind of behavioural economics as 'classical', in contrast to the newer varieties based on non-numerical microfoundations.

computable economics, the algorithmic formalization of *problems*, an algorithmic characterization of agents and institutions and the crucial questions asked are always about solvability of problems - and if solvable, how hard or easy. Naturally, this also means a recognition that there are unsolvable problems and, in computable economics, under the assumption of the Church-Turing Thesis and the Kolmogorov-Chaitin-Solomonoff Thesis, these are also *absolutely unsolvable*. This will not be the case in constructive mathematics. This will be an inherent feature of a digital economy characterized by computable economics; but a characterization of such an economy in terms of orthodox economic theory would be unable to determine its limits w.r.t problem solvability.

4 Computable Economics for a Digital Economy

> "From the point of view of the mathematician the property of being digital should be of greater interest than that of being electronic. ... That the machine is digital ... means firstly that numbers are represented by sequences of digits which can be as long as one wishes. One can therefore work to any desired degree of accuracy. ....This is in sharp contrast with analogue machines, and continuous variable machines .... . A second advantage of digital computing machines is that they are not restricted in their applications to any particular type of problem."
>
> Alan Turing: *Lecture to the London Mathematical Society on 20 February 1947* ([53], p.106)

In this section a broad brush picture of the nature and scope of computable economics, from the point of view providing the economic theory of a digital economy as characterised in the previous section, is discussed. On a larger canvas, it would be natural to include *Constructive Economics* on an equal footing with computable economics under the one umbrella of *Algorithmic Economics*. However, for the specific purposes of an economic theory of a digital economy, especially as characterised in the previous section, it is more useful to confine attention to an eclectic computable economics which 'opportunistically' invokes constructive principles whenever necessary in the manner of relative computation and appeals to oracles. There are many technical reasons for this, not least my own mathematical incompetence in constructive measure theory.

First of all, the overall nature of the overall modelling strategy in computable economics is disciplined by what may be called the following 'credo':

1. · The triple $\{assumption, proof, conclusion\}$ will always be understood in terms of $\{input\ data, a\lg orithm, output\ data\}$.

2. · Mathematics is best regarded as a very high level programming language.

3. · Every proof is an algorithm in the strict recursion theoretic sense. This, of course, means that classical logic is freely and almost uncritically

invoked. There will be no qualms about invoking the law of the excluded middle. However, that does not mean that there will by undecidable disjunctions in any algorithm (see the next criterion).

4. · To understand a *theorem* of computable economics, in algorithmic terms, represent the *assumptions* as *input data* and the *conclusions* as *output data*. Then try to convert the proof into an algorithm which will take in the input and produce the desired output. If you are unable to do this, it is probably because the proof relies essentially on the *law of excluded middle*. This step will identify any inadvertent infusion of undecidable disjunctions in existential statements.

5. · If we take algorithms and data structures to be fundamental, then it is natural to define and understand functions in these terms. If a function does not correspond to an algorithm, what can it be? Hence, take the stand that functions are, by definition, computable.

6. · Given a putative function f, we do not ask "Is it computable?", or "Is it constructive?", but rather "What are the data types of the domain and of the range?" This question will often have more than one natural answer, and we will need to consider both restricted and expanded domain/range pairs. Distinguishing between these pairs will require that we reject undecidable propositions. If you attempt to pair an expanded domain for f with a restricted range, you will come to the conclusion that $f$is non-computable.

Secondly, the optimization paradigm of orthodox economic analysis and, indeed, of almost every kind of formal decision theory except classical behavioural economics and some parts of the cognitive sciences, is replaced by the more general paradigm of *Diophantine decision problems*. This is specifically to acknowledge the fact that the domain of discourse in computable economics are the computable numbers, in general, and the rational numbers (or the integers), in particular. Thus, one does not arbitrarily force the domain of discourse to be the real numbers simply because the economic theorist is only competent in real analysis. The available, natural, domain of analysis in economic and financial markets, in a digital economy, will be rational, integer or, perhaps, the algebraic numbers. This will be adequately respected in the assumptions made about the analytical and decision theoretic framework.

Thirdly, the implementation of a *Diophantine decision problem* will be in the form of asking for the solvability or not of a *Diophantine representation* of market equilibrium in a digital economy. This will lead, seamlessly, to an exploitation of the connection with the *Halting Problem for Turing Machines* and the powerful methods developed for showing the unsolvability of *Hilbert's 10th Problem*. In a very general way, the connections come about as follows:

A relation of the form

$$D\left(a_1, a_2, ..., a_n, x_1, x_2, ..., x_m\right) = 0$$

where:

Definition 10 ***Definition 11*** *D is a polynomial with integer coefficients with respect to all the variables $a_1, a_2, ..., a_n, x_1, x_2, ..., x_m$ (also integer or rational valued), separated into **parameters** $a_1, a_2, ..., a_n$ and **unknowns** $x_1, x_2, ..., x_m$, is called a **parametric Diophantine equation**.*

Definition 12 *D in Definition 10 defines a set F of the parameters for which there are values of the unknowns such that:*

$$\langle a_1, a_2, ..., a_n \rangle \in \mathcal{F} \iff \exists x_1, x_2, ..., x_m \left[D\left(a_1, a_2, ..., a_n, x_1, x_2, ..., x_m\right) = 0 \right]$$

Loosely speaking, the relations denoted in the above two definitions can be called *Diophantine representations.* Then sets, such as F, having a Diophantine representation, are called simply *Diophantine.* With this much terminology at hand, it is possible to state the fundamental problem of Diophantine equations as follows:

Problem 13 *A set, say $\langle a_1, a_2, ..., a_n \rangle \in \mathcal{F}$, is given. Determine if this set is Diophantine. If it is, find a Diophantine representation for it.*

Of course, the set F may be so structured as to possess equivalence classes of properties, P and relations, R.Then it is possible also to talk, analogously, about a *Diophantine representation of a Property P* or a *Diophantine representation of a Relation R.* For example, in the latter case we have:

$$R\left(a_1, a_2, ..., a_n\right) \iff \exists x_1, x_2, ..., x_m \left[D\left(a_1, a_2, ..., a_n, x_1, x_2, ..., x_m\right) = 0 \right]$$

Hence, given, say partially ordered preference relations, it is possible to ask whether it is Diophantine and, if so, search for a Diophantine representation for it. Next, how can we talk about the solvability of a Diophantine representation? This is where undecidability (and uncomputability) will enter this family of 'inviting flora of rare equations' - through a remarkable connection with recursion theory, summarized in the next Proposition:

Proposition 14 *Given any parametric Diophantine equation, D, it is possible to construct a Turing Machine, M, such that M will eventually **Halt**, beginning with a representation of the parametric n-tuple, $\langle a_1, a_2, ..., a_n \rangle$, **iff** D in Definition 9 is solvable for the unknowns, $x_1, x_2, ..., x_m$.*

But, then, given the famous result on the *Unsolvability of the Halting problem for Turing Machines*, we are forced to come to terms with the unsolvability of Diophantine equations[16].

[16] It must, of course, be remembered that all this is predicated upon an acceptance of the *Church-Turing Thesis.*

Several remarks are in order here. The perceptive reader would have realised that the computable economy is a *Diophantine economy* and, hence, so is the digital economy. Therefore, algorithmic agents and algorithmic institutions will routinely face formally unsolvable Diophantine decision problems. This is the context in which *boundedly rational agents* could be defined and their behaviour experimentally and empirically analysed. Next, it is precisely the existence of formally unsovlable decision problems in a Diophantine economy that calls forth *satisficing behaviour*. These two remarks are made to dispel the conventional misconception that a boundedly rational agent is simply orthodoxy's omnipotent substantively rational agent cognitively constrained in various *ad hoc* ways; and that *satisficing* is simply a 'second best' optimization outcome. If anything, the truth is exactly the opposite - but I shall not pursue further discussions to clarify these remarks in any great detail in this essay.

Fourthly, following a tradition that regrettably was stillborn, market interactions will be modelled in terms of *coupled dynamical systems* taking as the fountainhead for such an approach the path breaking paper by Richard Goodwin in 1947 ([16]). For over a century markets have been modelled with various metaphors in mind: from Walras and Pareto invoking analogue calculation machines and iterated dynamical systems as metaphors in the late 19th century to markets as evolutionary mechanisms in more recent years and even all the way to markets as institutions and institutions as algorithms in the modern sense by the more imaginative modern economists such as Simon and Scarf. There is no doubt that markets have evolutionary aspects that have to be encapsulated in any modelling formalism; there is also no question that the fertile idea of markets as analogue or digital calculating devices may suggest interesting formalisms. However, the need to model the dynamic interaction of markets as the source of emerging complexities and evolving novelties seems best handled in terms of explicit dynamics. For this nothing, surely, is more obvious than coupled dynamical systems and since there is a tradition, albeit comprehensively neglected, that can be invoked, I shall do so.[17] In particular, this exercise will be squarely within the Turing tradition of enabling novelty and complexity to evolve via the coupling of *simple* dynamical systems. The general idea is as follows[18].

[17] As a pupil of Richard Goodwin I was privileged to be shown by him how two Phillips Hydraulic Machines could be coupled *linearly* to study trade between two national economies and to demonstrate the emergence of complex dynamics as two parameters were systematically 'tuned'. In his 'Reminiscences' on the occasion of the efforts at the LSE to repair and refurbish the Phillips Machine, about a decade and half ago, Goodwin wrote as follows:

> "..I was very excited to find that Phillips had two of his magical machines in London, so I could reproduce what I had analyzed back in 1947 in my dynamical coupling paper. If I remember correctly, Phillips did not believe we could produce erratic behaviour by coupling his machines - but we did"
>
> Goodwin, undated manuscript, probably Summer, 1991.

[18] I am simply reproducing the model in [54]. The reasons are as follows: first of all, to familiarize the reader with the strategy adopted by Turing; secondly, to hint that by suitable adaptations and re-interpretations, a similar or more general model can be devised for coupled markets; thirdly, to indicate that much can be done within the 'linear' fold; finally, to show

Begin, in the simplest case, with two *linear differential equations*, encapsulating the dynamics of demand-supply in two *linearly* coupled markets[19]:

$$\frac{dx_r}{dt} = ax_r + by_r \tag{25}$$

$$\frac{dy_r}{dt} = cx_r + dy_r \tag{26}$$

Transform them into standard variational equations via 'diffusion' coefficients μ and ν:

$$\frac{dx_r}{dt} = (ax_r + by_r) + \mu(x_{r+1} - 2x_r + x_{r-1}) \tag{27}$$

$$\frac{dy_r}{dt} = (cx_r + dy_r) + \nu(y_{r+1} - 2y_r + y_{r-1}) \tag{28}$$

Assuming stable or neutrally stable dynamics for market interactions when 'diffusive' interaction is absent means the following conditions between the trace and the determinant of the characteristic equations will have to be satisfied for the constant of the system (25)-(26):

$$\alpha = a + d \leq 0 \tag{29}$$

$$\beta = ad - bc > 0 \tag{30}$$

The general linear system of equations can be orthogonalised by introducing the coordinate transformations, ξ_i, η_i $(\forall i = 0 \dots N)$, for x and y, respectively, and using the relationship[20]:

$$\sum_{r=1}^{N} \exp[\frac{2\pi irs}{N}\ ] = \begin{cases} 0 & if \quad 0 < s < N \\ N & if \quad s = 0 \text{ or } s = N \end{cases}$$

Then, it is easy to show that the orthogonalised system of 2N decoupled equations in the new coordinate system are:

$$\frac{d\xi_s}{dt} = (a - 4\mu \sin^2 \frac{\pi s}{N})\xi_s + b\eta_s \tag{31}$$

and:

$$\frac{d\eta_s}{dt} = c\xi_s + (d - 4\nu \sin^2 \frac{\pi s}{N})\eta_s \tag{32}$$

Denoting by $\sigma, a_\sigma$and d_σ, the following:

how to link bifurcation analysis with the search for emergent complexities in a manageable way. Unfortunately, due to space considerations I will have to leave much in the form of bare hints and skeletal discussions and leave it to the good reader to infer adequately.

[19] If it might help the reader to follow the rest of the analysis in this part, s/he could think of x and y as excess demand *levels* in two coupled markets of two trading economies.

[20] In the first case, since the l.h.s is a geometric progression and 0 < s < N, the result is immediate; in the second case, in either of the alternatives, s=0 or s=N, all terms are equal to 1, hence the l.h.s sums to N (cf. [54], p.39).

$$\sigma = \sin^2(\frac{\pi s}{N}) \quad (0 \leq \sigma \leq 1) \tag{33}$$

$$a_\sigma = a - 4\mu\sigma \tag{34}$$

and:

$$d_\sigma = d - 4\nu\sigma \tag{35}$$

Then, the trace and the determinant of the characteristic equation for the orthogonalised system will be:

$$\alpha^* \equiv a_\sigma + d_\sigma \tag{36}$$

and:

$$\beta^* \equiv a_\sigma d_\sigma - bc \tag{37}$$

It is easy to calculate α^* decreases monotonically with increases in μ, ν and σ. On the other hand, since β^* depends on a_σ and d_σ multiplicatively, there is no unambiguous way of relating changes in β^* to changes in the relevant parameters,μ, ν and σ. However, it is clear that as the parameters are varied or change autonomously, and given that α^* decreases (or increases) monotonically, the dynamics of the system loses stability as β^* becomes negative. The loss of stability, from a stable node or focus, to a saddle-point, is called a *Turing Bifurcation*.[21]

It is this loss of stability that Turing exploited to provide a beautiful, simple but counter-intuitive example of how, from a formless initial structure, form - i.e., 'order' - can be generated. It is counter-intuitive in that one expects a diffusive mechanism to iron out - i.e., smooth out - inhomogeneities. Instead, starting with a minor inhomogeneity in a linear, coupled, dynamical system, the Turing Bifurcation leads to a growth of form - i.e., inhomogeneities giving rise to *morphogenesis*.

At this point it is apposite to state that my conjecture here is to interpret 'diffusion' and 'diffusive' interaction in terms of 'competition' in coupled market dynamics. The full implication of this analogy must be left for a different exercise.

Several modelling remarks are in order at this point. Given the various, scattered, strictures that I have earlier made against modelling in the domain of the continuous, real, variable, it will not do to work with *differential equations* without care and caveats. Either one has to work with difference equations or explicit, equivalent discretizations must accompany any such formalization in terms of differential or any other continuous time dynamical system. It is for this reason I have circumscribed this discussion in terms of *linear* differential equations; they can routinely be discretized to equivalent difference systems. Such is not the case with *nonlinear* dynamical systems. Some clarifying remarks on this issue will be made in the concluding section.

[21] This can be contrasted and compared with the more familiar *Hopf Bifurcation*, where the parameter variation and loss of stability involves foci and limit cycles.

However, an alternative strategy, that which is advocated in the computable economic research program for a digital economy, is to go directly from the differential equation to its equivalent *Turing Machine*. Again, this is a routinisable strategy. By extension, coupled dynamical systems can be modelled as coupled Turing machines - or, as has been stated many times in the recent 'complexity' literature, the economy as a massively parallel system of interacting markets. In this sense, a discrete dynamical system or coupled Turing machines can easily be re-interpreted in terms of a *cellular automata* system activated on a grid. It is, then, immediate, to reinterpret the *cellular automaton* as a *finite automaton*. If this automaton is not capable of *universal computation,* then it will not be capable of either self-reproduction or self-reconstruction. Thus, such markets even if they are capable of evolving complex novelties are not capable of survival.

The themes that Turing broached, and the kind of analysis he developed, provides a fertile source for those interested in a dynamical approach to complex adaptive systems analysis. This is amply illustrated by works such as those by Kelso ([21])and Levin ([29]. In particular, Kelso's work integrates pattern detection with a dynamical systems and simulation perspective at the forefront, eschewing all the paraphernalia of 'statistics and probability' and showing the virtues of metastability - more particularly, but misleadingly, sometimes referred to as 'life at the edge of chaos'. This would be equivalent to simulating and locating coupled Turing machine configurations, when studied as cellular automata on a grid, with initial conditions compatible with universal computation. The question whether an economic system is capable of self-reproduction, i.e., growth, and innovations will, then, be answered by investigating whether it, modelled as coupled dynamical systems or one of its above equivalents can be shown to be capable of interesting bifurcations and universal computations. More can be said but in the absence of simulations to illustrate some of the claims, it is better left for a different exercise.

In this chapter, where the main theme is the study of digital economies, the framework of the Turing bifurcation enables an encapsulation also of positive feedback between interacting markets, whilst simultaneously, the propagation of inhomogeneities, via varieties of competitive mechanisms and couplings, leads to complex novelties in a self-organizing order. Both agents and markets - or, more generally, institutions - become so-called *complex adaptive systems* and hence *evolution*, too, is encapsulated in the formalism. None of this requires the agents, the markets or any other institution to be embedded in a probability space or to act as expected utility maximizers. Indeed, agents, markets and institutions will be interpreted either as nonlinear oscillators or as Turing Machines; the former as *universal dynamical systems*; the latter as Universal Turing Machines capable of *universal computation*. I don't see anything strange about this; after all the Lucasians view agents as *signal processors*; it is not necessary to stretch one's imagination too far to make a transition from agents as *signal processors* to agents as *nonlinear oscillators* or Turing machines.

The final modelling precept is the place of induction or abduction in the computable economic modelling of a digital economy embedded in an information society: the need for agents and institutions to *learn* and *infer* over time. This

is similar, perhaps even equivalent to, *pattern recognition* by *complex adaptive systems*. Although the frontiers of current research advocates such an approach, here the aim would be to use the Turing Machine formalism of agents, markets and institutions to re-interpret learning and inference as induction or abduction by such machines from data sequences. This means induction and abduction are given formalisms in terms of computations on data sequences without either assuming that the data sequences are emanating from probability spaces or are embedded in them. The idea will be to divorce both induction and abduction, formally, from any reliance on probability considerations. The theoretical framework for such an exercise is *algorithmic complexity theory*.

The above *modelling strategies* form the core and the key of the computable economic approach to modelling the dynamics of the digital economy as embedded in an information society. These strategies are not meant to be implemented in any particular sequence. For example, it may be useful - or not, as the case may be - to move from a dynamical system formulation to its Turing Machine equivalence and, then, to a Diophantine formalism to investigate solvability - or the other way about.

5 Mathematical Methods of Computable Economics

> Computer science ... is not actually a science. It does not study natural objects. Neither is it, as you might think, mathematics; although it does use mathematical reasoning pretty extensively. Rather, computer science is like engineering - it is all about getting something to do something, rather than just dealing with abstractions ... .
>
> But this is not to say that computer science is all practical, down to earth bridge-building. Far from it. Computer science touches on a variety of deep issues. ... . It naturally encourages us to ask questions about *the limits of computability*, *about what we can and cannot know about the world around us.*"
>
> Richard Feynman: **Feynman Lectures on Computation** ([13], p.xiii; italics added)

As always, the versatility and audacity of Feynman's intellect has captured the essence of a vast subject of immense complexity with a few well chosen phrases of subtle depth. Dentists do not deal with abstractions; that is why an equally versatile and audacious intellect such as Keynes, long years ago, hoped that economists might some day become as humble as dentists. Instead, economic theory has concentrated, for about the past half a century or so, on 'just dealing with abstractions'. Moreover, borrowing another famous Keynesian phrase, these abstractions have been dealt with using the 'pretty polite techniques' of irrelevant, non-numerical, mathematics. No wonder, then, that economic theory has not been able to 'get something to do something'. Instead,

computable economics is 'like engineering - it is all about getting something to do something'. At the same time, using the tools and the mathematics of computer science and allied disciplines, it is also 'about what we can and cannot know about the world around us' - i.e., epistemology.

To give quantitative, numerically meaningful, content to the general statements and modelling strategies discussed in previous sections, it is necessary to formalize the economic entities that characterise computable economics, agents, institutions, technologies, etc., in relevant ways. In this section, therefore, a menu of the kind of mathematics that is necessary to do so will be outlined. It should remind the reader of what s/he would have found in appendices and 'cookbooks' of mathematical economics or econometrics in years gone by - before all economists and econometricians were expected to have become minor or major experts of real analysis, measure theory, smooth dynamical systems and stochastic processes. No economic theorist worth his salt would dream of trying to tackle or study business cycles, growth or development without first mastering numerical analysis, stochastic processes, filtering theory, dynamical systems theory and real analysis and much else. My generation had to learn topology from excellent mathematical economics texts such as the ones by Nikaido before even beginning to understand the meaning of the excess demand function and equilibrium in the market for carrots, apples and tractors, all formalized in vector spaces and indexed in the continuum. Compared to all this paraphernalia, the mathematics of computable economics is refreshingly algorithmic for the modern generation brought up not with pencil and paper but with a monitor and the keyboard.

Suppose the starting point of the computable economist who is studying a digital economy is the following:

Conjecture 15 *Observable variables are sequences that are generated from recursively enumerable but not recursive sets, if rational agents underpin their generation.*

An aside: In 1974 Georg Kreisel posed the following problem:

> "We consider theories, ... and ask if every sequence of natural numbers or every real number which is well defined (*observable*) *according to the theory* must be recursive or, more generally, *recursive in the data*. ..... Equivalently, we may ask whether any such sequence of numbers, etc., can also be generated by an ideal computing or Turing Machine if the data are used as input. The question is certainly not empty because most objects considered in a ... theory are not computers in the sense defined by Turing. ....."
>
> [26], p.11

The above conjecture has been formulated after years of pondering on Kreisel's typically thought-provoking question and the feelings expressed in the earlier sections. More recently, a reading - belated and undigested though it may well

be, for the moment - of Osborne's stimulating book ([38]), was also a source of inspiration in the formulation of the conjecture as an empirical disciplining criterion for computable economics.

The conjecture is also is akin to the orthodox economic theorist and her handmaiden, the econometrician, assuming that all observable data emanate from a structured probability space and the problem of inference is simply to determine, by statistical or other means the parameters that characterise their probability distributions. Whilst Einstein in his epistemology may not believe that the Good Lord is malicious enough to be playing roulette with us, mortals, the economic theorist and the econometricians seem to think they know better. I prefer Einstein's epistemology and, hence, begin with the above conjecture. If the computable economist's starting point is the above conjecture then it follows that:

Theorem 16 *Only dynamical systems capable of computation universality can generate sequences that are members of sets that are recursively enumerable but not recursive.*

Theorem 17 *Only dynamical systems capable of universal computation can extract patterns inherent in arbitrary, digitally generated, data, without assuming their generation by an underlying probability model*

Corollary 18 *Asymptotically stable dynamical systems are not capable of computation universality.*

Proposition 19 *Only dynamical systems capable of computation universality are consistent with the* ***no arbitrage*** *hypothesis.*

Theorem 20 *Rational economic agents in the sense of economic theory are equivalent to suitably indexed Turing Machines; i.e, decision processes implemented by rational economic agents - viz., choice behaviour - is equivalent to the computing behaviour of a suitable indexed Turing Machine.*

Put another way, this theorem states that the process of rational choice by an economic agent is equivalent to the computing activity of a suitably programmed Turing Machine.

Conjecture 21 *Dynamical systems capable of computation universality can persist in disequilibrium configurations for long time periods.*

Theorems 16 and 17, Corollary 18, Proposition 19 and Conjecture 21 have been the basis of my work in [55] and in chapter 4 of [56]. Ideas underpinned by related results are also discussed in the important and earlier work by Peter Albin ([2], chapter 7).

Theorem 22 ***(Rabin, 1957)*** *T]here are games in which the player who* ***in theory*** *can always win cannot do so* ***in practice*** *because it is impossible to supply him with* ***effective instructions*** *regarding how he should play in order to win.*

The next item has been mentioned twice already in this essay; but I restate it here just for completion.

Problem 23 ***Hilbert's 10th Problem: Determination of the solvability of a Diophantine equation***

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients; to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

Theorem 24 ***Undecidability of Hilbert's tenth problem***

There is no algorithm which, for a given arbitrary Diophantine equation, would tell whether the equation has a solution or not.

Theorem 25 ***Halting Problem for Turing Machines***

Suppose we are given a Turing Machine computable function $f_n(m)$. Then there is no general algorithm for determining, for arbitrary $n \geq 0$ and $m \geq 0$, whether $f_n(m)$ is defined.

Theorem 26 ***Rice's Theorem****: Let C be a class of partial recursive functions. Then C is not recursive unless it is the empty set, or the set of all partial recursive functions.*

Claim 27 *Validity of the Church-Turing Thesis on Effective Calculability*

Claim 28 *Validity of the Kolmogorov-Chaitin-Solomonoff-Martin Löf Thesis on Randomness*

Anyone who is able to formalize these theorems, corollaries and conjectures and work with them as disciplining economic theoretical criteria would have mastered all the necessary mathematics of computable economics. The rest of this section is a guide for those who are perplexed by the above theorems, corollaries, conjectures and propositions.

Theorem 29 ***Specker's Theorem*** *in Computable Analysis ([49], pp. 145-58)*
A sequence exists with an upper bound but without a least upper bound.

Theorem 30 *The Pour-El/Richards Theorem*

There exists an Ordinary Differential Equation (ODE) s.t : $\varphi'(t) = F[t, \varphi(t)]$ with $\varphi(0) = 0$, s.t $F(x, y)$ is computable on the rectangle $[0 \leq x \leq 1, -1 \leq y \leq 1]$, but no solution of the ODE is computable on any interval $[0, \delta], \delta \geq 0$.

Theorem 31 ***Fix Point Theorem***

Suppose that $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is a recursive operator (or a recursive program P). Then there is a partial function f_ϕ that is the least fixed point of Φ :

Theorem 32 $\Phi(f_\phi) = f_\phi$;
If $\Phi(g) = g$, *then* $f_\phi \sqsubseteq g$.

Remark 33 *If, in addition to being partial,* f_ϕ *is also total, then it is the* ***unique least fixed point.***

Definition 34 *Computational 3D Sperner ([39], p.510)*

Given an integer n (in binary) and a polynomial-time algorithm computing for each point of the $n \times n \times n$ subdivision of the cube a legal color, find a tetrachromatic cubelet (one that has four colours).

Theorem 35 *For any* $k \geq 2$, $k - D$ *Sperner is inefficient (i.e., belongs to the class PPAD - Polynomial Parity Argument in a Directed Graph;[39], p.510)*

Proposition 36 *There exists a probability measure* $m(\cdot)$ *that is* ***universal*** *in the sense of being invariant except for an inessential additive constant such that:*

$$\log_2 m(\cdot) \approx K(\cdot) \tag{38}$$

where $K(\cdot)$ is the Kolmogorov-Chaitin *algorithmic complexity.*
The statement above is phrased with induction and abduction in mind. More conventionally, in terms of the terminology of recursion theory, the proposition is usually stated as the **Invariance Theorem** (due to Kolmogorov, Solomonoff and Chaitin)*:*

Theorem 37 $\exists$ *a* ***universal*** *partial recursive function* f_0, *such that, for any other partial recursive function* f, *there is a constant* c_f *such that for all (binary) strings* $x, y : K_{f_0}(x \mid y) \leq K_f(x \mid y) + c_f$.

Where K, as above is the *Kolmogorov(-Chaitin algorithmic) complexity and:*

$$K_S(x) = \min\{\mid p \mid : S(p) = n(x)\} \text{ and } K_S(x) = \infty \text{ if there are no such } p \tag{39}$$

S : a method of programming; p : a program with $\mid p \mid$ denoting the smallest program that gives rise to $x \in D$, a domain of combinatorially defined objects, the elements of which are given a standard enumeration by numbers $n(x)$.In other words, $K_S(x)$ is the smallest program that gives rise to the object $x \in D$ and is its complexity w.r.t the specifying method S.

More concretely stated, the core implication of the *invariance theorem* is, for a given reference Turing Machine UTM, the length of the shortest program to compute x is: $\min\{l(p) : UTM(p) = x\}$, where $l(p)$ is the finite binary string identified with p.Therefore:

Definition 38 $K(x) = \min\{l(UTM) + l(p) : UTM(p) = x\} \pm 1$

where, $l(T)$: length of the *self-delimiting* encoding for a Turing Machine UTM.

Since, for each n there are 2^n binary strings of length n, but only $\sum_{i=o}^{n} 2^i = 2^n - 1$ possible *shorter* 'descriptions' we can find at least one binary string x of length n such that $K(x) \geq n$. Such strings are referred to as *incompressible*.

Theorem 39 ***The Incompressibility Theorem***

Let k be a positive integer. For each fixed y, every finite set A of cardinality m has at least $m(1 - 2^{-c}) + 1$ elements x with $K(x \mid y) \geq \log m - c$, where $K(x \mid y)$ is the complexity of the object x given y.

Finally, related to invariance theorems in the domain of algorithmic complexity theory and the fix point theorem (Theorem 30, above) of classical recursion theory, we have the *recursion theorem*, essential for understanding self-reproduction and self-reconstruction:

Theorem 40 ***Recursion Theorem*** *Let T be a Turing Machine that computes a function:*

$$t : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^* \tag{40}$$

Then, there is a Turing Machine R that computes a function:

$$r : \Sigma^* \longrightarrow \Sigma^* \tag{41}$$

such that, $\forall \omega$:

$$r(\omega) = t\left(\langle R \rangle, \omega\right) \tag{42}$$

where,

$\langle R \rangle$: denotes the encoding of the Turing Machine into its standard representation as a bit string;

and the $*(star)$ operator denotes its standard role as a *unary operator* defined as: $A^* = \{x_1, x_2, ..., x_k \mid k \geq 0, \forall x_i \in A\}$

The idea behind the *recursion theorem* is to formalize the activity of a Turing Machine that can obtain its own description and, then, compute with it. All malicious 'hackers', perhaps with no knowledge of this theorem, are invoking this theorem every time they generate viruses! More seriously, this theorem is essential, too, for formalizing, recursion theoretically, a model of *growth* in a digital economy and to determine and learn, computably and constructively, *rational expectations equilibria* (cf. [58]). The *fix point theorem* and the *recursion theorem* are also indispensable in the computable formalization of *policy ineffectiveness* postulates (cf. [33] for a pioneering vision of this approach), *time inconsistency* and *credibility* in the theory of macroeconomic policy. Even more than in microeconomics, where topological fix point theorems have been indispensable in the formalizations underpinning existence proofs, the role of the above *fix point theorem* and the related *recursion theorem* are absolutely fundamental in what I come to call *Computable Macroeconomics*.

6 Ways Ahead

> "Mathematize as he will, and seek algorithms as he will, the empirical scientist is not going to aspire to an algorithm or proof procedure for the whole of his science; he would not want it if he could have it. He will want rather to keep a large class of his sentences open to the contingencies of future observation. It is only thus that his theory can claim empirical import."
>
> [43], p.155

If the ubiquity of the digital computer and the popular references to a digital economy embedded in an information society leads to new convictions in the mathematical economic community and the mathematical economist begins to reconsider the mathematical underpinnings of economic theory, along some of the lines indicated in the previous sections, then there are at least three more directions of research that could be explored with much hope for an enriched and truly numerically rigorous foundations for the subject: *interval analysis*, *computable & constructive analysis* and *numerical analysis & recursion theory*.

First, the relevance of *interval analysis* as a basis for enhancing and making sure that digital computers are used with care and the theorist does not demand more precision than could be delivered by such machines, even if the theory has been based on computable or constructive mathematics. Recently Brian Hayes, in one of his fascinating regular columns on *Computing Science* in the **American Scientist** ([19]) reminded us - at least those of us concerned with respecting the discrete and finite precision nature of digital computers - of the dangers of arbitrary approximations and routinised truncations of standard computations:

> "On February 25, 1991, a Patriot missile battery assigned to protect a military installation at Dahrahn, Saudi Arabia, failed to intercept a Scud missile, and the malfunction was blamed on *an error in computer arithmetic*. ... In combination with other peculiarities of the control software, the inaccuracy caused *a miscalculation of almost 700 meters* in the predicted position of the incoming missile. *Twenty-eight soldiers died*."
>
> ibid, p.484; italics added.

What was this tragic 'error in computer arithmetic'? It is simply due to the fact the binary fraction for the decimal fraction $10^{-1} = 0.1$ is *not terminating*:

$$10^{-1} = (0.1)_{10} = (.0001100110011...)_2 = (0\ 0011\ 0011\ 0011\ ....)_2 \qquad (43)$$

In other words, the decimal fraction, in its binary notation, cycles and is non-terminating and will have to be truncated with unpredictable consequences, unless a serious approximation analysis is included in the software which truncates automatically for some predetermined instruction. But there is another

alternative, similar to the Clower-Howitt admonishment to the economist to master number theory (or some mathematics of the discrete); in the case of pitfalls due to the discrete and finite nature of the digital computer and its arithmetic, the alternative would be to use *Interval Analysis*, where an '*interval of real numbers is treated as a new kind of number, represented by a pair of real numbers, namely its right and left end points'* ([35], p.vii; italics added). This is to go part of the way towards the *analytical* part of computable economics - i.e, *computable or constructive analysis*. Had such numbers been used in the software that was built into the operation of the control software referred to above, the error would have been eliminated. Maury Osborne's warning to traders in the stock market, not to approximate by the continuous that which is intrinsically discrete, made over a quarter of a century ago, rings a similar tone:

> As for the question of replacing rows of closely spaced dots by solid lines, you can do that too if you want to, and the governors of the exchange and the community of brokers and dealers who make markets will bless you. If *you* think in terms of solid lines while the *practice is in terms of dots and little steps up and down*, this *misbelief* on your part is worth, I would say conservatively, to the governors of the exchange, *at least eighty million dollars per year.*"
>
> [38], p.34; italics added.

Today, I would put the conservative estimate at more than several multiples of that figure of 27 years ago, even adjusting for inflation. The reason, once again, a reliance on an illegitimate domain of analysis, unrealistic assumptions and the wrong mathematics for analyzing digital data, by digital agents in a digital medium using a digital machine for computing discrete numbers.

Is there, then, no way to justify the use of real numbers as the relevant domain of analysis for a digital economy and, hence, to rely on orthodox theory and its well developed mathematical structures and practice to analyse it? Paradoxically, in spite of my various and even, at times, virulent, negative remarks on the irrelevancies of orthodox mathematical economics for the purposes of analyzing a digital economy embedded in an information society, there *is* a way out - in fact, there are several ways out. It is possible to redo orthodox mathematical economics using recursive analysis or computable analysis. This chapter is already disproportionately long; hence the case for underpinning orthodox economic theory with the mathematics of recursive analysis or computable analysis cannot be fully substantiated here. However, some useful indications might show that the distance between computable economics and orthodox mathematical economics need not be as vast as I have made out in the main part of this work.

The assumptions underlying Conjecture 14 and Theorem 15 are stepping stones towards a compromise approach to the study of a digital economy with the tools of computable economics. The reason, concisely summarised, is as follows. Consider the typical space in which ordinary mathematical economic

exercises are conducted - a *Banach Space*. Such a space is characterised by three fundamental classical mathematical properties: *linearity*, *limit* and *norm*. Thus, a topology on a Banach space can be characterised by a sequence of elements of the space. On the other hand, the sequence of elements of a recursively enumerable set can be ordered as a computable sequence. Therefore, it is quite easy to endow a Banach space with relative computability properties. If, in addition, the notion of a computable, continuous, function of a real variable, as defined, say, by Grzegorczyk or Lacombe, in the 50s, is also harnessed, we are almost fully equipped to continue using the full paraphernalia of orthodox concepts in a slightly altered, but perfectly meaningful in algorithmic and dynamic senses. However, there are other ways to define computable, continuous, functions - some of which I prefer to the standard way using the notions and methods of Grzegorczyk or Lacombe, but I shall not elaborate on the alternatives here[22].

Next, we need to characterise the computable reals in an effective way. This can be done in many alternative ways: for example, following the tradition of classical mathematical analysis, by effectivizing Cauchy sequences; or, by sticking to the tradition of mathematical logic and effectivizing the Dedekind cuts. I prefer the latter mode simply because recursion theory is, by now, a defining sub-discipline of mathematical logic and much of the motivating forces of constructive mathematics are still underpinned by a puritanism about logical principles. But either way, the effectivizing is straightforward. Anyone familiar with the standard way the real number system is built up from the natural and rational numbers via one of the above two methods, should have no difficulties mastering the way the computable reals are defined.

But lest the enthusiastic classical mathematical economist gets carried away with the promise of 'business as usual', some warning signposts must be placed. It is partly with this purpose that theorems 25, 28 and 29 above have been included as part of the *Mathematical Methods of Computable Economics*. From theorem 25[23] it is easy to show that equality between two computable real numbers is, in general, undecidable. From theorem 28 it is clear that a much beloved theorem of elementary classical analysis, where a bounded monotone sequence converges, is not valid in computable analysis - i.e., a sequence exists with an upper bound but without a least upper bound. On the other hand, it will be possible to eliminate the reliance on the classical fix point theorems of Brouwer, Kakutani, etc., which are non-constructive and uncomputable, by using theorems 30 and 39, with some careful reformulations.

An exactly similar path can be carved with constructive analysis, but I will have to leave it for another exercise. *Uniform continuity. located points*, etc., will replace classical starting points and duality theorems, for example, will not

[22] My preferable method is to rely on Goodstein's *Uniform Calculus* ([17]), where the whole of elementary analysis is developed from the starting point of *uniform continuity*. This starting point, together with the fact that the *Uniform Calculus* does not rely on any dubious logical principle invoking proof by *reductio ad absurdum,* places it sensitively between computable calculus and constructive analysis.

[23] The contribution by da Costa and Doria to this volume develops a more general version of this theorem, valid in classical analysis.

have the unambiguity that is routinely invoked by those who rely on classical mathematical analysis. I think I have given enough hints to show how one can remain within the analytical fold, but not lose numerical and algorithmic meaning.

Algorithms and dynamics are most easily recognized, without any need to understand the mathematical underpinnings of the former, in dynamical systems theory. In the spectacular developments achieved in dynamical systems theory in the second half of the 20th century, the digital computer played a decisive part. However, there is a close connection between algorithms and dynamical systems via numerical analysis. The use of the digital computer to study continuous dynamical systems requires the analyst or the experimenter to first discretise the system to be studied. The discretisation processes for nonlinear dynamical systems are often intractable and undecidable. On the other hand, paradoxically, until very recently the mathematical foundation for numerical analysis was not developed in a way that was consistent with the mathematical foundation of the digital computer - i.e., computability theory. As a result we have, in economics, a plethora of attempts and claims about *computational economics* that are not well founded on recursion theoretic, constructive or numerical analytic foundations. Bishop's observation, quoted in the very first lines of this chapter, is fully applicable to the *ad hockery* practised by *computational economists*, fully similar to the activities of the classical mathematical economist.

Now, there are, to the best of my knowledge, two ways out of the dilemma faced by the *computational economist*. Either be rigorous about the theory of approximations and numerical analysis in discretising the continuous; or, look for a mathematical foundation for numerical analysis taking heed of the following observations remarks by Smale, et.al:

> "There is a substantial conflict between theoretical computer science and numerical analysis. These two subjects with common goals have grown apart. For example, computer scientists are uneasy with calculus, whereas numerical analysis thrives on it. On the other hand numerical analysts see *no use for the Turing machine.*
>
> The conflict has at its roots another age-old *conflict*, that *between the continuous and the discrete.* Computer science is oriented by the digital nature of machines and by its discrete foundations given by Turing machines. For numerical analysis, systems of equations and differential equations are central and this discipline depends heavily on the continuous nature of the real numbers. ...
>
> Use of Turing machines yields a unifying concept of the algorithm well formalized. ....
>
> The situation in numerical analysis is quite the opposite. Algorithms are primarily a means to solve practical problems. *There is not even a formal definition of algorithm in the subject.* ....
>
> A major obstacle to reconciling scientific computation and computer science is the present view of the machine, that is, the digital

computer. As long as the computer is seen simply as a finite or discrete subject, it will be difficult to systematize numerical analysis. We believe that the Turing machine as a foundation for real number algorithms can only obscure concepts.

Towards resolving the problem we have posed, we are led to expanding the theoretical model of the machine to allow real numbers as inputs."

[6], p.23; italics added.

This is a strategy that is a compromise between using an analog computer and a digital one, on the one hand, and, on the other, between accepting either constructive or computable analysis and classical real analysis. The model of computation developed with great ingenuity by Smale and his co-workers may well be the best way to retain much of classical mathematical economics while still being able to pose and answer meaningfully questions about decidability, computability and computational complexity - and to retain numerical meaning in the whole framework.

As for my remark about being rigorous about the theory of approximations and numerical analysis in discretising the continuous, particularly when working within the framework of dynamical systems theory and using the digital computer[24], what I mean is the need to avoid the paradoxes posed by *phantom solutions* (cf. [42] and [45]). The exact sense in which this problem confounds the connection between the continuous and the discrete can be described with a simple example. Consider the Verhulst equation, almost the simplest conceivable, empirically relevant, nonlinear differential equation:

$$\dot{x} = x\,(1 - x) \tag{44}$$

To simulate and study this system using a digital computer it is necessary to find an *equivalent* nonlinear difference equation representation for it. By *equivalent* I mean one with the same long-run, steady state, phase portrait - basins of attraction, limit points, etc. Now, mercifully, this system is exactly integrable and therefore the analyst knows what s/he is looking for, so far as implementing 'equivalence'. The solution for the Verhulst equation is the logistic curve; so, the difference scheme approximation to it should also generate the logistic curve. But often, particularly in economics, the following seemingly obvious nonlinear difference equation system is chosen as the approximation for digital implementation:

$$\Delta x_n = x_{n+1} - x_n = x_n\,(1 - x_n)\,\Delta t_n \tag{45}$$

Now, this seemingly simple and straightforward difference scheme *cannot* be solved in closed form and, hence, it is not clear that it is an 'equivalent' approximation to the original Verhulst equation. However, the following approximation can be solved in closed form (for small h):

[24] These remarks are irrelevant should it be possible to use an analogue computer (cf.[57]).

$$x_{n+1} - x_n = (x_{n+1} - x_n x_{n+1})\, h \tag{46}$$

with closed form solution given by:

$$x_n = [1 - (1 - x_0^{-1})(1 - h)^n]^{-1} \tag{47}$$

which, for *small* h, approaches the logistic curve:

$$x(t) = [1 - (1 - x_0^{-1})e^{-t}]^{-1} \tag{48}$$

However, for $h \geqslant 1$,spurious solutions - i.e., *phantom solutions* - are generated.

How can an economist, wedded to modelling in the continuous domain, avoid the dilemma of phantom solutions? This is a pertinent question for at least two reasons: the ubiquity of the digital computer and the easy access to simulation software such as *Matlab* and *Mathematica.* My own analytical answer to this dilemma is to study numerical analysis as dynamical systems in their own right (cf. [50], especially chapter 4), in the first instance; then, as a second step, transform the dynamical system to an equivalent Turing machine. This way a solid mathematical foundation for numerical analysis is at hand in a very direct way.

Let me end with less rigorously grounded speculations about the nature of the discrete and its relevance for economic modelling. The digital economy may well be an artefact of the fact that the digital computer is ubiquitous. I can imagine, counterfactually, a world of dominated by the analogue computer, as it once nearly did, and it would be inconceivable that the economic world would then not have been interpreted, modelled and named the analogue economy. But, increasingly, there are tendencies in many of the neuro-, physical- and other natural sciences, and in some of the pure sciences, to acknowledge the fact that 'reality' may well be discrete. Not very recently, Richard Feynman ([12], p.467) wondered:

> "Can physics be simulated by a universal computer?

Feynman, in his characteristically penetrating way, then asked three obviously pertinent questions to make the above query meaningful:

- What kind of physics are we going to *imitate*?
- What kind of *simulation* do we mean?
- Is there a way of simulating rather than imitating physics?

Before providing absolutely fundamental, but tentative, answers to the above queries, he adds a penetrating caveat (ibid, p.468; italics in original):

> "I want to talk about the possibility that there is to be an *exact* simulation, that the computer will do *exactly* the same as nature."

The nature of my discussion above is also about 'what kind of economics are going to simulate?' - not imitate; and about 'exact simulation' of a relevant economic theory. Feynman's answer to part of the first question was that the kind of physics we should simulate are 'quantum mechanical phenomena', because (ibid, p. 486):

> "...I'm not happy with all the analyses that go with just the classical theory, because nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."

But he was careful to point out, also, that there was a crucial mathematical difference between 'quantizing' and 'discretizing' (ibid, p. 488; italics added):

> "*Discretizing is the right word.* Quantizing is a different kind of mathematics. If we talk about *discretizing* ... of course I pointed out that we're going to *have to change the laws of physics.* Because the laws of physics as written now have, in the classical limit, a continuous variable everywhere ... ."

It is this that I mean in my vision, outlined in the earlier sections, for economics, if we want to be seriously rigorous about using the digital computer to study the real world around us, in its economic manifestations. *Ad hoc* patching up, unrigorous approximations, arbitrary discretizations of theoretical entities, modelled with continuous variables that have no correspondence in economic reality have no place in a rigorous digital world

He was not the only giant in the natural sciences who wondered thus: Einstein, Schrödinger, Hamming, Toffoli, Fredkin and most recently, Penrose, too, have had speculative thoughts along similar lines. Einstein, in perhaps his last published work, seems to suggest that a future physics may well be in terms of the discrete:

> "One can give good reasons why reality cannot be represented as a continuous field. ...."
> [10], p.166

Roger Penrose, in his recently published, massive, vision of *The Road to Reality*, was even more explicit:

> [W]e may still ask *whether the real-number system is really 'correct' for the description of physical reality at its deepest level.* When quantum mechanical ideas were beginning to be introduced early in the 20th century, there was the feeling that perhaps we were now beginning to witness *a discrete or granular nature to the physical world* at its smallest scales.... Accordingly, various physicists attempted to build up an alternative picture of the world in which discrete processes governed all action at the tiniest levels. ....

> In the late 1950s, I myself tried this sort of thing, coming up with a scheme that I referred to as the theory of 'spin networks', in which the discrete nature of quantum-mechanical *spin* is taken as the fundamental building block for a *combinatorial* (i.e. a discrete rather than real-number-based) approach to physics."
>
> [40], pp.61-2; italics in the second paragraph as in original.

These speculations on the granular structure of 'reality' at some deep level arose out of purely theoretical developments in the subject, but in continuous interaction with the epistemology of measurement in well-designed and sound experimental environments. Whether or not the economy is digital or not is not something we can theorize about in the same way - or, perhaps, have not done so till now.

In one of the great classical textbooks of mathematical logic, [44], with painstaking detail and meticulous fidelity, extracted and presented, in prose of unsurpassed exactitude, the logic that was inherent in the normal practice of mathematics. I have often wondered what kind of logic for mathematicians would be the result if such an exercise were carried out *exclusively* on the textbooks by Goodstein ([17]), Lorenzen ([31]), Bishop ([4]), Weihrauch ([59]), Aberth ([1]), Landau ([28]), Hardy ([18]) and Keisler ([20]). The first one eschews *reductio ad absurdum* but retains other standard undecidable disjunctions in proof procedures; the next two are elementary and advanced texts, respectively, on constructive analysis, although it is the second that is consistently free of any reliance on the use of the *law of the excluded middle*; the books by Weihrauch and Aberth are explicitly on computable analysis, hence relying on the Church-Turing Thesis but accepting the strictures of classical logic; next, Landau and Hardy are introductory texts on classical analysis; finally, Keisler is an introductory textbook on non-standard analysis. In Rosser, the latter three were treated with great care and much sympathy and the nature of the implicit logic used in them were extracted with finesse - but Keisler required an appendix for its treatment ([44], Appendix D). In more recent times Knuth ([22]; [23]) has attempted a related exercise by investigating standard textbooks to try to understand whether there is such a thing as a 'mathematical way of thinking' which is distinct from an 'algorithmic way of thinking'. I believe a good case can be made for the identity between the two modes of thinking had Knuth looked only at books such as those by Goodstein, Lorenzen, Bishop, Weihrauch and Aberth; contrariwise, had he looked at all of the above and a majority coming out of the Bourbakian stables, then he may have been led to believe that there was no clear cut answer. Similarly, it is my conviction that Rosser's admirable exercise would have resulted in at least three volumes on *Logic for Mathematicians* (or, at least, three new appendices) - depending on the mathematician's philosophical stance. It seems to me that the way of thinking of the mathematical economist is confined to those that are predominantly present in the classical mathematical analyst. As a result the emergence of new forms of economic societies, such as the digital one embedded in an information society, receives an inappropriate economic interpretation and a one-dimensional

formalization which emasculates the digital underpinnings of modern economic transactions and institutions..

Over the past few years Duncan Foley has been giving such issues some considerable thought and has also pondered over the kinds of questions I have raised, from the point of view of computable economics. His perceptive criticisms and imaginative interpretations of the kind of exercise I have reported above are worth quoting in full, to conclude this essay[25]:

> "Why raise these issues of abstraction in relation to your critique of real-analysis-based mathematical economics and your ingenious suggestions about correcting or improving this field by using constructive or computable mathematical methods? One reason is to raise a warning that the shift from real to computable numbers, logically salutary as it might be, does not in itself address the more fundamental question of the fact that theoretical economic quantity and price are themselves complex abstractions from actually observable transactions, which might have limits as tools for analysis. The second reason perhaps might give some insight into the limits your critique has encountered among the mathematical economists so far. The abstraction from concrete transactions to "quantity" and "price" on which traditional mathematical economics rests is deeply tied up with the project of applying calculus and a fortiori real analysis to economic interactions. (The Classical political economists had a much weaker and more robust notion of the regularity of economic interactions in their method of "long-period" averages.) When you ask mathematical economists to re-think their ideas of proof and computability from the point of view of meta-mathematical developments of the twentieth century you are also implicitly asking them to re-think the abstractions of quantity and price altogether. I suspect this is at the root of the question that I think lies behind most mathematical economists incomprehension of your logical critique, which is, what mistake are we making in adhering to real analysis?"
>
> [15]

Perhaps the force of circumstances and the realities of a digital economy, embedded in an information society, will force the classical orthodoxies to begin to think in novel ways and use numerically relevant, algorithmic and dynamic, mathematics - even if logical criticisms may fail to entice them to enlightened paths.

[25] I am afraid I will be doing some violation to the broad and deep points Foley was making by only quoting a part of his letter to me. I shall be happy to make available a copy of the full letter to any reader who makes an explicit request for it!

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