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Component Groups of Stabilizers in Algebraic Groups: Computational Methods and Applications

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Introduction

This dissertation is submitted to conclude the doctoral programme in Mathematics at the University of Trento of the author, Emanuele Di Bella, supervised by professor Willem Adriaan de Graaf.

This thesis concerns topics intersecting different areas of Mathematics, such as Lie theory, linear algebra, differential geometry, algebraic geometry and related fields. Moreover, computational aspects play a fundamental role. Specifically, the core contributions of this work are computational in nature.
The dissertation is divided into four chapters. Chapter 1 is a collection of basic results that are well-known in the literature. The goal of this chapter is to provide the reader with the necessary background and to establish the notation.
In Chapters 2, 3 and 4 we report the three main works that we have conducted during the doctoral programme. In particular, Chapter 2 ([28]) consists of the computation of component groups of stabilizers of nilpotent orbit representatives in simple Lie algebras. Chapter 3 ([29]) and Chapter 4 concern two examples of θ-groups. In this case the goal is to classify the real orbits of the action of the θ-group on a suitable subspace of the underlying Lie algebra. For more details about technical background, known results and possible outcomes, we refer the reader to the introduction at the beginning of each chapter. Here, instead, we proceed to describe the recurring elements that are shared by each work so that it is clear to the reader what is the main path of the PhD program of the author. In particular, the computation of the component group of the stabilizer of an orbit representative is the most relevant element which plays a fundamental role in each chapter. Formally, this can be stated as follows: let $\mathfrak{g}$ be a simple Lie algebra and $G = \mathrm{Aut}(\mathfrak{g})^\circ$; let $\mathcal{O}$ be a nilpotent or semisimple orbit of the action of G on $\mathfrak{g}$ and let x be a representative of $\mathcal{O}$. We want to explicitly calculate representatives of the quotient $Z_H(x)/Z_H(x)^\circ$, where $H = G$ in Chapter 2, $H = G_0$ in Chapter 3 and $H = [G_0, G_0]$ in Chapter 4, G_0 being the θ-group. We give more precise details for component groups in Section 1.5 and for θ-groups in Section 1.6. In Chapter 2 it is clear from the title what is the role of the component groups, while in the other chapters they play a key role thanks to Theorem 1.8.1 which states a correspondence between the elements of the first Galois cohomology of the stabilizer of an orbit representative and the real orbits contained in its complex orbit.
As illustrated in the previous paragraph, the component group plays a central role in this dissertation. This topic has been tackled many times during the last century. One of the earliest works is due to Alekseevskiĭ ([2]), where some calculations on component groups for nilpotent elements in exceptional Lie algebras appear. Later, other sources showed new theoretical approaches to the topic, for example [15], [43] and [51]. However, a unified approach to determine the component groups for nilpotent elements in exceptional Lie algebras only appeared in 1998, due to Sommers ([60]). In all the cited sources, the isomorphism type of the component group is computed, but not the explicit generators. Because of this, it is hard to apply the obtained results. This weakness is clear in

the context of Galois cohomology. Indeed, the component group plays a fundamental role in the algorithm presented in [8]. Here, if representatives of the component groups are given, it is possible to deduce the classification of the real orbits of a θ-group, as introduced above. For this and similar applications, it is then required to compute the representatives of the component groups. However, in the literature, it is rare to find such calculations; often, these are for very specific situations and do not provide general methods that are immediately applicable (see, for example, [48]). In this thesis, the explicit computation of representatives of component groups is the innovative result that makes this work original and an advancement in the literature.

The use of computer algebra systems to approach the problem illustrated above is new. In this setting, it is clear that in all three works we needed to develop general and ad hoc methods to compute such component groups. Sometimes, the computational approach used for a specific component group serves as a blueprint for solving other cases in different works. Let the notation be as above. Then, for example, in Section 2.4 we aim to determine the component group $Z_G(e)/Z_G(e)^\circ$, e being a fixed nilpotent element in an exceptional Lie algebra $\mathfrak{g}$. For doing so, we want to construct explicit automorphisms of $\mathfrak{g}$. Jacobson-Morozov's theorem guarantees the existence of an $\mathfrak{sl}_2$-triple (h,e,f) (see Theorem 1.4.2). Then, we set $\mathfrak{c} = \mathfrak{z}_{\mathfrak{g}}(h,e,f)$ and work on the direct sum $\mathfrak{c} \oplus \mathfrak{c}^\perp$, where $\mathfrak{c}^\perp$ is the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{g}$ with respect to the Killing form. In particular, since it turns out $\mathfrak{c}^\perp$ is a $\mathfrak{c}$-module, we compute the corresponding decomposition into irreducible $\mathfrak{c}$-modules and the highest weight vectors. Then, we use the fact that an element of $Z_G(h,e,f)$ has to map such vectors in an appropriate manner. In Chapter 4 we work on a Lie algebra $\mathfrak{g}$ of type E_8 endowed with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$. Here, for any orbit representative e, we let $\mathfrak{c} = \mathfrak{z}_{\mathfrak{g}_0}(h,e,f)$ and observe that $\mathfrak{g}_1$ is a $[\mathfrak{c},\mathfrak{c}]$-module. Again, we use the decomposition of a module into irreducible ones. Even though we are working on a different setting, we proceed with techniques similar to what we developed in Chapter 2. Another notable recurring method can be found in Sections 3.6 and 4.6. In these situations, we look for elements in the Weyl group W of $\mathfrak{g}$ that lift to representatives of the component groups; here, we use a computational approach to verify theoretical requirements on the elements of W, leading to the construction of suitable subgroups of W corresponding to the desired component groups.

Clearly, the fact that we aim to compute explicitly generators of component groups also implies that we often need to use various algorithms; this appears to be another constant in the whole thesis. In particular we use Gröbner basis computation, introduced in Section 1.2. The implications of this strategy is clear in our results. On some occasions, this algorithm works very well and for this reason we do not need to use deep theoretical approaches, as brute force can be enough. This is not always the case, so that when the algorithm for computing a Gröbner basis takes too long to terminate or does not reduce significantly the equations, we need to use some different strategies. Observe that often these strategies have the main goal to reduce the complexity of the calculations, so that most of the time we still have to compute the Gröbner basis of some polynomials at the end. Clearly, there can be minor adjustments that reduce the complexity (see Example 1.5.2). However, this is not always sufficient. For example, in Chapter 2 and Chapter 4, we deal with the adjoint representation of Lie algebras of type E_8, hence we have to handle 248×248 matrices. In this case, usually minor adjustments do not reduce the complexity of the calculations significantly. In this and other situations we need to develop very specific algorithms to perform our computations. The most articulated can be found in Sections 2.4, 3.6 and 4.5.

Finally, due to the main role of computational aspects, in the next page we give an overview of software and packages that we mostly use.

Computational tools

Here we list software we use and briefly describe the role in our works.

- Magma ([10]): this software is very useful when computing Gröbner bases. In particular, using a suitable algorithm according to the strategy described in the corresponding handbook and based on the theoretical framework of [21],[33].
- GAP4 ([34]): we use the following packages:
  - SLA ([23]): this package contains main features that we need. First of all, it helps us dealing with basic tools concerning semisimple Lie algebras, like root systems, Weyl groups and so on. Moreover, it contains most of the functions corresponding to the theoretical setting of Section 1.4, like the ones for computing $\mathfrak{sl}_2$-triples, nilpotent orbits and weighted Dynkin diagrams. Finally, it also has some tools to work on θ-groups.
  - CoReLG ([31]): some features of this package overlap with SLA. It adds some tools to work with real forms and rational extensions.
  - GalCohom: this program explicitly computes the first Galois cohomology set $H^1(R, G)$ of a real linear algebraic subgroup G of $\mathrm{GL}(n, \mathbb{C})$. The group is given by a list of real matrices that form a basis of the Lie algebra of G and a list of complex matrices whose elements are representatives of the elements of the component group. A detailed description of the algorithm can be found in [8].
- Singular ([22]): this is a computer algebra system that we use inside GAP sessions (thanks to [19]) to perform computations with Gröbner basis. Even though we mostly use Magma for this purpose, sometimes we prefer to make some calculations using this software.

Chapter 1

Preliminaries

We remark that the material presented in this chapter is primarily based on the comprehensive treatments found in [17], [24] and [42]. We refer the reader to these texts for further details and complete proofs.
In this chapter we aim to introduce the necessary background. We mostly prefer to omit details that fall outside the scope of this work. However, on some occasions we also show some proofs. This is a deliberate choice, since in some important proofs there are crucial steps that have a strong impact on our original strategies and ideas contained in the subsequent chapters. For example the proofs of Theorems 1.2.7 and 1.2.9 give a precise idea of what we do when using Gröbner basis computations. Similarly, Proposition 1.5.1 plays a decisive role in our computational strategies and the proof familiarizes the reader with the computation of representatives of component groups. Another goal of Chapter 1 is to provide the reader with a deeper understanding of our computational strategies. In the whole dissertation many examples arise, however we also deemed it necessary to include basic examples in Chapter 1 (see Examples 1.2.10 and 1.5.2) to show explicitly how to perform basic algorithms that we use recurrently.
In this chapter, we assume the reader to be familiar with the standard theory of Lie algebras and algebraic groups (as presented, for example, in [41] and [42]). In particular, we take for granted the notions of ideal, solvability, semisimplicity and basic facts from representation theory. The next section aims to fix the notation and recall specific characterizations, notably for reductive algebras, that will be of frequent use in the following.

1.1 Basic concepts

When not otherwise specified, $\mathfrak{g}$ denotes a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. We denote by $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (or simply ad, if it is clear which Lie algebra we are referring to) the adjoint representation of $\mathfrak{g}$.
For any Lie algebra $\mathfrak{g}$, we denote by $\kappa_{\mathfrak{g}}(x, y)$ the *Killing form*, for $x, y \in \mathfrak{g}$. We will also simply denote it by κ when the underlying Lie algebra is clear from the context.
Since reductive Lie algebras play a central role in this work, we recall some basic facts about them.

Definition 1.1.1. *A Lie algebra $\mathfrak{g}$ is called reductive if its adjoint representation is completely reducible. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is said to be reductive in $\mathfrak{g}$ if the $\mathfrak{a}$-module $\mathfrak{g}$ is completely reducible.*

Proposition 1.1.2. *The following statements are equivalent:*

1. $\mathfrak{g}$ *is reductive;*
2. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$*;*
3. $\mathfrak{g}$ *has a finite-dimensional representation such that the associated trace form is non-degenerate;*
4. $\mathfrak{g}$ *has a faithful completely reducible representation.*

Reductive Lie algebras play an important role in all the upcoming chapters. While the second characterization in Proposition 1.1.2 is the most frequently used in this dissertation, the other equivalent statements will also be relevant.

Next, we recall the definitions of nilpotent and semisimple elements:

Definition 1.1.3. *An element $x \in \mathfrak{g}$ is called nilpotent (resp. semisimple) if* $\operatorname{ad}x$ *is a nilpotent (resp. semisimple) endomorphism.*

We also recall that in any semisimple Lie algebra $\mathfrak{g}$, for any $x \in \mathfrak{g}$, there exists a unique pair of elements $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Moreover, if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is any finite-dimensional representation of $\mathfrak{g}$, then $\rho(x) = \rho(x_s) + \rho(x_n)$ coincides with the usual Jordan decomposition of the endomorphism $\rho(x)$ ([42, Corollary 6.4]).
A fundamental role will be played by $G = \operatorname{Aut}(\mathfrak{g})^\circ$, the identity component of the group of automorphisms of $\mathfrak{g}$. Observe that G is a connected algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra is equal to $\operatorname{ad}\mathfrak{g} = \{\operatorname{ad}y \mid y \in \mathfrak{g}\}$ ([47, Proposition 1.120]). We let $\exp : \operatorname{ad}\mathfrak{g} \to G$ denote the exponential map. The group G is generated by the elements $e^{\operatorname{ad}x}$ for all nilpotent $x \in \mathfrak{g}$, and it is known as the *inner automorphism group* of $\mathfrak{g}$ ([47, Corollary 0.20]).
Let $\hat{G}$ be a connected group having $\mathfrak{g}$ as its Lie algebra. Then, we denote by $\operatorname{Ad}_{\hat{G}} : \hat{G} \to \operatorname{Aut}(\mathfrak{g})$ the adjoint representation of $\hat{G}$. We recall that the map $\operatorname{ad}_{\mathfrak{g}}$ also arises as the differential at the identity of the adjoint representation of $\hat{G}$. For any map f, we will denote its differential at the identity by $(df)_e$. Hence, we write $d(\operatorname{Ad}_{\hat{G}})_e = \operatorname{ad}_{\mathfrak{g}}$. Observe that since $\hat{G}$ is connected, the image of $\operatorname{Ad}_{\hat{G}}$ coincides with G. In particular, since $\ker(\operatorname{Ad}_{\hat{G}}) = Z(\hat{G})$, we have $\hat{G}/Z(\hat{G}) \cong G$.

It is clear that G acts on its Lie algebra. We are interested in the orbits of this action.

Definition 1.1.4. *The orbit of a nilpotent (resp. semisimple) element is called a nilpotent (resp. semisimple) orbit.*

The motivation behind the above definition lies in the following elementary observation: for any nilpotent (resp. semisimple) element $x \in \mathfrak{g}$ and any $\sigma \in G$, we have

$$\operatorname{ad}(\sigma(x)) = \sigma \operatorname{ad}x \, \sigma^{-1}.$$

Hence, $\sigma(x)$ is nilpotent (resp. semisimple) as well. Therefore, the orbit of a nilpotent (resp. semisimple) element consists entirely of nilpotent (resp. semisimple) elements. We say that two elements x, x' lie in the same orbit if $x \in G \cdot x'$. Equivalently, we say that x and x' are G-conjugate, or that they are equivalent under the action of G.

1.2 Gröbner basis

In this section, we introduce the basic theory of Gröbner bases, emphasizing the properties that will play a fundamental role in the sequel.
Throughout, k will denote a field, and $k[x_1, \dots, x_n]$ will be the polynomial ring in n indeterminates over k. Moreover, let us fix the following notation. Here, $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{N}$ and $x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n}$. Such an element is called a monomial. On the set of monomials we can define a relation in the following way.

Definition 1.2.1. *An order $\leq$ on the set $\{x^\alpha : \alpha \in \mathbb{N}^n\}$ of the monomials of $k[x_1, \dots, x_n]$ is called a monomial order if*

- *$\leq$ is total, i.e. for all $\alpha, \beta \in \mathbb{N}^n$ we have $x^\alpha \leq x^\beta$ or $x^\beta \leq x^\alpha$;*
- *$\leq$ is multiplicative, i.e. if $x^\alpha \leq x^\beta$, then $x^\alpha x^\gamma \leq x^\beta x^\gamma$, for all $\gamma \in \mathbb{N}^n$;*
- *There are no infinite descending chains $x^{\alpha(1)} > x^{\alpha(2)} > \dots$.*

Example 1.2.2. A notable example of monomial order is the lexicographic order. Let $\alpha, \beta \in \mathbb{N}^n$. Then, suppose that i is the smallest index such that $\alpha_i \neq \beta_i$. In this case we define $\alpha <_{\text{lex}} \beta$ if $\alpha_i < \beta_i$. For example, let $x_1^2 x_2^3 x_3^4, x_1^2 x_2^2 x_3^9 \in k[x_1, x_2, x_3]$. Then $x_1^2 x_2^3 x_3^4 >_{\text{lex}} x_1^2 x_2^2 x_3^9$.
Although there are different monomial orders, the lexicographic one will play an important role for us, hence we will tacitly use it unless specified otherwise.

Now, we fix a monomial order $\leq$ on $\{x^\alpha : \alpha \in \mathbb{N}^n\}$ and let $g \in k[x_1, \dots, x_n]$. Let α be maximal with respect to $\leq$ and such that the coefficient of x^α in g is nonzero. Then, such a coefficient is said to be the leading coefficient of g and denoted by $\mathrm{LC}(g)$. Also, x^α is said to be the leading monomial of g, denoted by $\mathrm{LM}(g)$. Then, we have the following algorithm to perform polynomial division with remainder. In particular, for fixed polynomials $f_1, \dots, f_s$, we compute $r \in k[x_1, \dots, x_n]$ such that $g = h_1 f_1 + \dots + h_s f_s + r$ where $h_i \in k[x_1, \dots, x_n]$ and no monomial of r is divisible by an $\mathrm{LM}(f_i)$. The division algorithm proceeds as follows:

- set $g' := g, r := 0$;
- if $\mathrm{LM}(f_i)$ divides $\mathrm{LM}(g')$ for some i, then we set $g' := g' - (\mathrm{LC}(g')/\mathrm{LC}(f_i))x^\beta f_i$, where x^β is defined as the monomial such that $x^\beta \mathrm{LM}(f_i) = \mathrm{LM}(g')$. If such i does not exist, we set $g' := g' - \mathrm{LC}(g')\mathrm{LM}(g')$ and $r := r + \mathrm{LC}(g')\mathrm{LM}(g')$.
- if $g' = 0$, the algorithm is concluded; otherwise, repeat the previous step.

At this point, we can introduce the definition of Gröbner bases:

Definition 1.2.3. *Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Then, $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(f) \mid f \in I \setminus \{0\} \rangle$ is called the initial ideal of I.*

Definition 1.2.4. *Let $I \subset k[x_1, \dots, x_n]$ be an ideal and $B \subset I$ such that $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(g) \mid g \in B \setminus \{0\} \rangle$. Then B is called a Gröbner basis of I.*

Remark 1.2.5. Observe that the division algorithm strongly depends on the choice of the polynomial f_i with the property that $\mathrm{LM}(f_i)$ divides $\mathrm{LM}(g')$ (it is obvious that such an index is not unique in general). In particular, different remainders r can arise. However, it is known that such a remainder r is unique modulo B. This is a crucial fact in the computation of Gröbner bases. Even though we do not go into details of this fact, as this goes in a different direction with respect to what we intend to describe in this paragraph, it feels necessary to keep this in mind in the following, for the sake of correctness.

Next, we present some well-known facts that answer natural questions arising when dealing with Gröbner bases:

Lemma 1.2.6 (Dickson)**.** *Let $A \subset \mathbb{N}^n$ and $I = \langle x^\alpha \mid \alpha \in A \rangle \subset k[x_1, \dots, x_n]$. Then, there is a finite $A' \subseteq A$ such that $I = \langle x^\alpha \mid \alpha \in A' \rangle$.*

In particular, Lemma 1.2.6 ensures that every ideal has a finite Gröbner basis. Moreover, we state and prove the following theorem:

Theorem 1.2.7 (Hilbert)**.** *Every ideal of $k[x_1, \dots, x_n]$ is generated by a finite number of elements.*

Proof. Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Lemma 1.2.6 implies that there are $g_1, \dots, g_s \in I$ such that $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(g_i) \mid 1 \le i \le s \rangle$. Moreover, let $f \in I$, then using the above algorithm we find $r \in k[x_1, \dots, x_n]$ such that there are $h_1, \dots, h_s \in k[x_1, \dots, x_n]$ with $f = h_1 g_1 + \dots + h_s g_s + r$ and no monomial of r is divisible by an $\mathrm{LM}(g_i)$. But $r = f - (h_1 g_1 + \dots + h_s g_s)$, hence $\mathrm{LM}(r)$ lies in $\langle \mathrm{LM}(I) \rangle$. Hence there is an i such that $\mathrm{LM}(g_i)$ divides $\mathrm{LM}(r)$, which is only possible when $r = 0$. In other words, the elements $g_1, \dots, g_s$ generate I. □

An immediate consequence of Theorem 1.2.7, that can be deduced from the proof, is that a Gröbner basis of an ideal I also generates I. Then, consider the following simple lemma:

Lemma 1.2.8. *Let I be the ideal generated by $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, and let $a = (a_1, \dots, a_n) \in k^n$. Then, $f_1(a) = \dots = f_s(a) = 0$ if and only if $f(a) = 0$ for all $f \in I$.*

Then, it is clear that given an ideal I and two different generating sets of I, say $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_r\}$, solving the system of equations $f_1(a) = \dots = f_s(a) = 0$ is equivalent to solving $g_1(a) = \dots = g_s(a) = 0$.

At this point, it is worth commenting on a computational pattern that frequently occurs in this thesis. Given a system of polynomial equations defined by $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, we can compute a Gröbner basis $B = \{g_1, \dots, g_r\}$ for the ideal $I := \langle f_1, \dots, f_s \rangle$. As stated above, the zero locus of the original system is precisely the zero locus of B. While we omit the details of the algorithms computing B (referring the reader to [10],[21],[33]), we stress that their complexity heavily depends on the number of equations, their degrees, and the number of indeterminates. When the algorithms terminate in a reasonable time, we can replace the initial system with B. This considerably simplifies the problem of finding solutions, this being a consequence of the following result:

Theorem 1.2.9 (Elimination Theorem)**.** *Let $I \subset k[x_1, \dots, x_n]$ be an ideal and B a Gröbner basis of I with respect to $<_{\mathrm{lex}}$. Then $B \cap k[x_{l+1}, \dots, x_n]$ is a Gröbner basis of $I \cap k[x_{l+1}, \dots, x_n]$, for any integer l such that $1 \le l \le n-1$.*

Proof. Set $B_l = B \cap k[x_{l+1}, \dots, x_n]$ and $I_l = I \cap k[x_{l+1}, \dots, x_n]$. We need to show that $\langle \mathrm{LM}(I_l) \rangle = \langle \mathrm{LM}(g) \mid g \in B_l \rangle$. One inclusion is obvious as $B_l \subset I_l$. For the other one, we proceed as follows. Let $f \in I_l$. Then $f \in I$ and since B is a Gröbner basis of I, there exists a $g \in B$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(f)$. But $f \in k[x_{l+1}, \dots, x_n]$ so also $\mathrm{LM}(g) \in k[x_{l+1}, \dots, x_n]$. Then, let x^α be a monomial of g. If $\alpha_i > 0$ for some $i \le l$, then $x^\alpha >_{\mathrm{lex}} \mathrm{LM}(g)$, which is a contradiction. Hence $x^\alpha \in k[x_{l+1}, \dots, x_n]$ and also $g \in k[x_{l+1}, \dots, x_n]$. Therefore, $g \in B_l$ and $\mathrm{LM}(f) \in \langle \mathrm{LM}(g) \mid g \in B_l \rangle$. □

In other words, we deduce that a Gröbner basis with respect to $<_{\mathrm{lex}}$ has a triangular structure, making the problem of solving the corresponding system of equations less challenging. In order to clarify how we deal with the theoretical setting described in this section, we show the following straightforward example:

Example 1.2.10. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}), G = \mathrm{Aut}(\mathfrak{sl}(3, \mathbb{C}))^\circ$ and $\hat{G} = \mathrm{SL}(3, \mathbb{C})$. Here we show how to compute the stabilizer in G of the nilpotent element

$$x = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

with $x \in \mathfrak{g}$. For the computation of the component group, we refer the reader to Section 2.2, in which we developed a general approach for classical simple Lie algebras. In this case, we show how to compute the stabilizer by direct calculations. While the approach developed in Section 2.2 provides a more efficient method for these specific calculations, the direct computation presented here perfectly illustrates the general computational strategy occurring in various situations throughout this thesis.
First of all, let

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then, requiring $M \in Z_{\hat{G}}(x)$ is equivalent to $\det(M) = 1$ and $Mx - xM = 0$, the entries of the matrix $Mx - xM$ being polynomials in $k[a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}]$. Explicitly, we need to solve the following system of equations:

$$\begin{gathered} a_{21} + a_{31} = 0, \quad a_{11} - a_{22} - a_{32} = 0, \quad a_{11} + a_{12} - a_{23} - a_{33} = 0, \\ a_{31} = 0, \quad a_{21} - a_{32} = 0, \quad a_{21} + a_{22} - a_{33} = 0, \\ a_{31} = 0, \quad a_{31} + a_{32} = 0, \\ a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - 1 = 0. \end{gathered}$$

Then we proceed using the software Magma, with the following input:

```
P<a11,a12,a13,a21,a22,a23,a31,a32,a33>:=PolynomialRing(Rationals(),9);
F<i>:= CyclotomicField(4);
R:= ChangeRing(P,F);
r:=[a21+a31, a11-a22-a32, a11+a12-a23-a33, a31, a21-a32,a21+a22-a33,
a31,a31+a32,a11*a22*a33-a11*a23*a32-a12*a21*a33+a12*a23*a31+a13*a21*a32-
a13*a22*a31-1];
gb:=GroebnerBasis(r);
```

The theoretical setting of this section allows us to proceed by computing the solutions of the system of equations given by the output gb, which is

$$\begin{gathered} a_{11} - a_{33} = 0, \quad a_{12} - a_{23} = 0, \quad a_{21} = 0, \\ a_{22} - a_{33} = 0, \quad a_{31} = 0, \quad a_{32} = 0, \\ a_{33}^3 - 1 = 0. \end{gathered}$$

We observe that this system of equations is much easier to solve. At this point we usually proceed with some calculations by hand. Here, it is immediate to deduce the space of solutions of the above system, which implies that

$$Z_{\hat{G}}(x) = \left\{ \begin{pmatrix} \omega & a & b \\ 0 & \omega & a \\ 0 & 0 & \omega \end{pmatrix} \;\middle|\; a, b, \omega \in \mathbb{C}, \omega^3 = 1 \right\}.$$

Then, since $\mathrm{Ker}(\mathrm{Ad}\hat{G}) = \{I, \omega I, \omega^2 I\}$, we deduce that

$$Z_G(x) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}.$$

Observe that this is connected as it is homeomorphic to $\mathbb{C}^2$. As we are clarifying in Section 1.5, this means that its component group is trivial.

1.3 Automorphisms of a semisimple Lie algebra

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. By $\ell = \dim \mathfrak{h}$ we denote the rank of $\mathfrak{g}$. For a root $\alpha \in \Phi$ we denote the corresponding root space by $\mathfrak{g}_\alpha$, that is

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi$ be a basis of simple roots. For $\alpha, \beta \in \Phi$ we write $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. The matrix $(\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^{\ell}$ is then the Cartan matrix of Φ. We also let W be the Weyl group of $\mathfrak{g}$. We recall a few basic properties of W. It is a finite group generated by the reflections s_α, associated with the roots $\alpha \in \Phi$, acting on $\mathfrak{h}^*$ by $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$. Notably, W is faithfully represented as a subgroup of the permutations of Φ and it is generated by the set of simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$. Furthermore, the action of the Weyl group preserves the inner product induced by the Killing form on $\mathfrak{h}^*$.

Proposition 1.3.1. *Let the notation be as above. There are nonzero $h_i \in \mathfrak{h}$, $x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, $x_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ for $1 \le i \le \ell$, such that for all $1 \le i, j \le \ell$:*

$$[h_i, h_j] = 0, \quad [x_{\alpha_i}, x_{-\alpha_j}] = \delta_{ij} h_i, \quad [h_j, x_{\pm\alpha_i}] = \pm\langle \alpha_i, \alpha_j^\vee \rangle x_{\pm\alpha_i}. \tag{1.3.1}$$

Moreover, the elements $\{h_i, x_{\pm\alpha_i} \mid 1 \le i \le \ell\}$ span $\mathfrak{g}$.

Definition 1.3.2. *Let $\{h_i, x_{\pm\alpha_i} \mid 1 \le i \le \ell\}$ be a generating set of $\mathfrak{g}$, satisfying relations* (1.3.1)*. Then, such a set is called a* canonical generating set *of $\mathfrak{g}$. Sometimes it can be convenient to use the notation with y_{α_i} in place of $x_{-\alpha_i}$.*

The following result is crucial in many steps of some algorithm that we developed (see, for example, Section 2.4). For all the details, see [44, Chapter IV].

Theorem 1.3.3. *Let $\{h_i^l, x_{\pm\alpha_i}^k \mid 1 \le i \le \ell\}$ for $k = 1, 2$ be two canonical generating sets. Then mapping $h_i^1 \mapsto h_i^2$, $x_{\pm\alpha_i}^1 \mapsto x_{\pm\alpha_i}^2$ extends to a unique automorphism of $\mathfrak{g}$.*

Sketch of proof. Let β be a positive root. We write $\beta = \alpha_{j_1} + \dots + \alpha_{j_s}$, α_{j_d} being a positive root for all $1 \le d \le s$, so that $\alpha_{j_1} + \dots + \alpha_{j_m}$, with $m \le s$, is still a positive root (this is possible thanks to [42, Corollary 10.2]). Then, we can consider the two bases of $\mathfrak{g}$ composed of the elements

$$\{h_i^k, \mathrm{ad}(x_{\pm\alpha_{j_1}}^k) \dots \mathrm{ad}(x_{\pm\alpha_{j_{s-1}}}^k) x_{\pm\alpha_{j_s}}^k \mid 1 \le i \le \ell, \beta = \alpha_{j_1} + \dots + \alpha_{j_s} \in \Phi^+\}$$

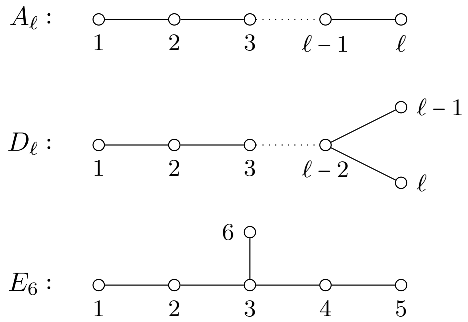
for $k = 1, 2$. It is possible to show that the two associated multiplication tables are identical (in particular, it holds that the coefficients of the multiplication tables are rational and only depend on the Cartan matrix; see [44, Chapter IV, Theorem 2] for all the details). Then, the linear mapping associating the corresponding elements $h_i^1 \mapsto h_i^2$ and $\mathrm{ad}(x_{\pm\alpha_{j_1}}^1) \dots \mathrm{ad}(x_{\pm\alpha_{j_{s-1}}}^1) x_{\pm\alpha_{j_s}}^1 \mapsto \mathrm{ad}(x_{\pm\alpha j_1}^2) \dots \mathrm{ad}(x_{\pm\alpha_{j_{s-1}}}^2) x_{\pm\alpha_{j_s}}^2$ is the desired automorphism. It is obviously unique as we are working with generating sets. $\square$

Corollary 1.3.4. *Let $\{h_i, x_{\pm\alpha_i} \mid 1 \le i \le \ell\}$ be a fixed canonical generating set. Let π be a permutation of $\{1, \dots, \ell\}$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_{\pi(i)}, \alpha_{\pi(j)}^\vee \rangle$ for all i, j. Then it follows that there is a unique automorphism σ_π of $\mathfrak{g}$ such that $\sigma_\pi(h_i) = h_{\pi(i)}$, $\sigma_\pi(x_{\pm\alpha_i}) = x_{\pm\alpha_{\pi(i)}}$.*

Proof. Since the permutation π preserves the Cartan matrix, we can proceed exactly as in the proof of Theorem 1.3.3 □

Definition 1.3.5. *An automorphism σ_π constructed as above is called a* diagram automorphism. *It is clear that the set of diagram automorphisms is a group (as $\sigma_{\pi_1\pi_2} = \sigma_{\pi_1}\sigma_{\pi_2}$). It will be denoted Γ in the following.*

Since we are dealing with diagram automorphisms throughout the whole thesis, it could be convenient to discuss this in more detail. First, it is clear that since a diagram automorphism has to permute the simple roots, it can be visualized immediately when looking at Dynkin diagrams (also, this justifies the nomenclature). Then, one has to keep in mind that we require the preservation of the Cartan matrix.



For example, for $A_\ell (\ell > 1)$, it is clear that the only permutation which preserves the Cartan matrix is the one mapping $j \mapsto \ell - j + 1$; for $D_\ell (\ell > 4)$ we have the permutation $j \mapsto j$ for $j \le \ell - 2$, $\ell \mapsto \ell - 1$, $\ell - 1 \mapsto \ell$; for D_4, we consider all the permutations of $1, 3, 4$, hence $\Gamma(D_4) \cong S_3$; for E_6, we let $1 \mapsto 5$, $2 \mapsto 4$ and get the only nontrivial permutation which returns a nontrivial diagram automorphism; for all the other simple Lie algebras, Γ is trivial. Diagram automorphisms play a key role in the following well-known result, holding for any semisimple Lie algebra $\mathfrak{g}$ ([44, §IX.4], [53, §4, Theorem 1]):

$$\mathrm{Aut}(\mathfrak{g}) = G \rtimes \Gamma. \tag{1.3.2}$$

The above result has notable implications in our work. Indeed, it is not easy a priori to describe $\mathrm{Aut}(\mathfrak{g})$. However, the formula (1.3.2) implies that for this purpose we only need to deal with the finite group Γ when $\mathfrak{g}$ is given. We will show a simple application of this result in Example 1.5.2.

1.4 Nilpotent orbits

In this thesis we frequently deal with nilpotent orbits. We have already given basic definitions in Section 1.1. Here we present the most relevant results in the literature that we will use throughout the next chapters. We begin with some results on $\mathfrak{sl}_2$-triples and then give some characterizations of nilpotent orbits of simple Lie algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Φ its root system. We denote by Φ^+ and Φ^- the sets of positive and negative roots, respectively. By an $\mathfrak{sl}_2$-triple we mean a set of three elements (h, e, f) where $h \in \mathfrak{h}, e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$, for some $\alpha \in \Phi^+$ and its corresponding root space $\mathfrak{g}_\alpha$, such that the Lie algebra generated by this set is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Then, the following relations hold:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Sometimes we refer to e as the nilpositive element of the triple and to f as the nilnegative one. Here we show some basic results about $\mathfrak{sl}_2$-triples (see [25, Section 2.13] for more details):

Proposition 1.4.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. Let $e, h \in \mathfrak{g}$ be such that $[h, e] = 2e$ and $h \in [e, \mathfrak{g}]$. Then there is an $f \in \mathfrak{g}$ such that (h, e, f) is an $\mathfrak{sl}_2$-triple.*

Theorem 1.4.2 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. Let $e \in \mathfrak{g}$ be a nonzero nilpotent element. Then, there are $h, f \in \mathfrak{g}$ such that (h, e, f) is an $\mathfrak{sl}_2$-triple.*

We will often work with an $\mathfrak{sl}_2$-triple of a nonzero nilpotent element. In general, for us this is convenient when performing explicit calculations (we will give some more details about this in Section 1.5). Clearly, the above results guarantee the existence of the triple. Moreover, we can deduce a straightforward algorithm to find it explicitly. Indeed, if $\mathfrak{g}$ is a given Lie algebra, then we can compute a basis of the space $[e, \mathfrak{g}]$ and solve some linear equations to find an element $h \in [e, \mathfrak{g}]$ satisfying $[h, e] = 2e$. Theorem 1.4.2 guarantees that the equations have solutions. Similarly, we find the space of elements of $\mathfrak{g}$ satisfying $[h, y] = -2y$ and in there find an f with $[e, f] = h$. The equations have solutions by Proposition 1.4.1. In particular, such a solution is unique. Indeed, we have the following result:

Proposition 1.4.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. Let $(h, e, f), (h, e, f')$ be two $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Then $f = f'$.*

Summarizing, when we declare that we pick an $\mathfrak{sl}_2$-triple containing a nilpotent element e, we are in fact using this algorithm. In GAP4, this is performed by the command SL2Triple(L, e), for a semisimple Lie algebra L and a nonzero nilpotent element $e \in L$.
We conclude the introductory comments about $\mathfrak{sl}_2$-triples by giving some details on their conjugacy relations. Observe that G acts on the set of triples. Indeed, $g \cdot (h, e, f) := (g \cdot h, g \cdot e, g \cdot f)$. Then, we have the following important theorem ([25, Theorem 8.1.4]):

Theorem 1.4.4. *Let $(h, e, f), (h', e', f')$ be two $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Then the following are equivalent:*

- *e, e' are G-conjugate;*
- *$(h, e, f), (h', e', f')$ are G-conjugate;*
- *h, h' are G-conjugate.*

The above theorem is in fact saying that if $e = \sigma(e')$ for some $\sigma \in G$, then there is a $\sigma' \in G$ such that $e = \sigma'(e'), h = \sigma'(h'), f = \sigma'(f')$. Finding such a σ' is not a trivial task. We developed an algorithm to perform this efficiently, as described in Section 2.3.

In the second part of this section, we provide some tools that are useful to describe nilpotent orbits in simple Lie algebras. This topic is strictly related to what we do in Chapter 2. However, the basic results that we are citing here are useful throughout the

whole dissertation.
We recall that given an integer n, a partition is defined as a tuple $[d_1, \dots, d_k]$ of positive integers such that $d_1 \geq d_2 \geq \dots \geq d_k$ and $\sum_{i=1}^{k} d_i = n$. We denote by $\mathcal{P}(n)$ the set of partitions of n. Then we have the following results ([17, Theorems 5.1.1, 5.1.2, 5.1.3, 5.1.4]):

Theorem 1.4.5. *There is a one-to-one correspondence between nilpotent orbits of classical Lie algebras and subsets of $\mathcal{P}(n)$:*

- *For type A_n, the orbits are in one-to-one correspondence with $\mathcal{P}(n+1)$;*
- *For type B_n, the orbits are in one-to-one correspondence with the subset of $\mathcal{P}(2n+1)$ consisting of partitions in which even parts occur with even multiplicity;*
- *For type C_n, the orbits are in one-to-one correspondence with the subset of $\mathcal{P}(2n)$ consisting of partitions in which odd parts occur with even multiplicity;*
- *For type D_n, the orbits are in one-to-one correspondence with the subset of $\mathcal{P}(2n)$ consisting of partitions in which even parts occur with even multiplicity, except that the partitions totally composed of even parts correspond to two orbits.*

We are not going into the details of the proof. However, we remark that the partitions represent the lengths of the Jordan blocks of the Jordan normal form of each nilpotent element of the corresponding orbit. In Example 1.2.10, the Jordan normal form corresponding to the nilpotent element $x \in \mathfrak{sl}(3, \mathbb{C})$ has a unique block of size 3×3. Hence, it corresponds to the partition $[3]$.

There is a similar characterization for exceptional Lie algebras, which involves special subalgebras instead of partitions. First, we need to recall some definitions. We use the notation of Section 1.3.

Definition 1.4.6. *Given a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is said to be a Borel subalgebra.*

Definition 1.4.7. *Given a semisimple Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is said to be parabolic if it contains a Borel subalgebra.*

In other words, for fixed basis $\Delta \subset \Phi$ and Cartan subalgebra $\mathfrak{h}$, we have that a parabolic subalgebra containing $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is of the form $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+ \cup \Psi} \mathfrak{g}_\alpha$, $\Psi \subset \Phi$ being a root subsystem with basis $\Delta' \subset \Delta$. For this reason we also denote by $\mathfrak{p}_{\Delta'}$ a parabolic subalgebra constructed this way.

Definition 1.4.8. *Let the notation be as above. We have a direct sum decomposition $\mathfrak{p}_{\Delta'} = \mathfrak{l}_{\Delta'} \oplus \mathfrak{n}_{\Delta'}$, where the reductive subalgebra $\mathfrak{l}_{\Delta'} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ is said to be a Levi subalgebra and $\mathfrak{n}_{\Delta'} := \bigoplus_{\alpha \in \Phi^+ \setminus \Psi} \mathfrak{g}_\alpha$ is the nilradical of $\mathfrak{p}_{\Delta'}$.*

Definition 1.4.9. *A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is said to be distinguished if* $\dim \mathfrak{l} = \dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

Now we have enough tools to state the following fundamental theorem ([17, Theorem 8.2.12]):

Theorem 1.4.10. *There is a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}$ and G-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$.*

Thanks to the above theorem, we can use a very intuitive notation (given for the first time by Bala and Carter, see [15]). In particular, a nilpotent orbit (up to G-conjugacy) is denoted $K(a_i)$, where K is the semisimple type of $[\mathfrak{l}, \mathfrak{l}]$, while i is the number of simple roots in any Levi subalgebra of $\mathfrak{p}_{\mathfrak{l}}$. If $i = 0$, we omit a_0 and only write K. It happens that isomorphic Levi subalgebras are sometimes not conjugate; in this case we put a tilde on the simple part involving short roots (one may check that this is enough to avoid ambiguity). Moreover, for E_7 it happens that there are isomorphic non-conjugate Levi subalgebras; we distinguish those by using prime and double prime notations.

Finally, we come to a very important tool which uniquely defines nilpotent orbits. First, we give some remarks on the Weyl group W. From what we said in Section 1.3, we know that W acts on $\mathfrak{h}^*$. It also acts on $\mathfrak{h}$ itself. Indeed, for $\alpha \in \Phi$ there exists a unique $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $[h_\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$. Define $s_\alpha^\vee : \mathfrak{h} \to \mathfrak{h}$ by $s_\alpha^\vee(h) = h - \alpha(h)h_\alpha$. Then the group generated by the reflections $s_\alpha^\vee$ is isomorphic to W ([25, Remark 2.9.9]). Moreover, we state the following important result.

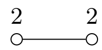
Proposition 1.4.11. *Two elements of $\mathfrak{h}$ are W-conjugate if and only if they are G-conjugate.*

Then, observe that for any nonzero nilpotent element e, we can always pick an $\mathfrak{sl}_2$-triple containing e and a semisimple element h with the property $\alpha(h) \geq 0$ for any $\alpha \in \Delta$. This is a consequence of Theorems 1.4.2, 1.4.4 and the fact that $D_\Delta := \{h \in \mathfrak{h}_{\mathbb{Q}} \mid \alpha(h) \geq 0 \text{ for all } \alpha \in \Delta\}$ is a fundamental domain for the action of W on $\mathfrak{h}$ (see [25, Chapter 8]) together with Proposition 1.4.11. However, the following result states an even stricter property.

Proposition 1.4.12. *Let h be as above, then $\alpha(h) \in \{0, 1, 2\}$ for any $\alpha \in \Delta$.*

In particular, thanks to Propositions 1.4.11, 1.4.12 and the fact that D_Δ is a fundamental domain, we can uniquely associate to a nilpotent orbit a Dynkin diagram in which the node enumerated i is labeled with $\alpha_i(h)$. We call it the *weighted Dynkin diagram* of the orbit. Clearly, the only requirement to correctly detect a nilpotent orbit is to know the exact enumeration of the simple roots on the diagram. It is not rare to find different enumerations when dealing with different sources and software, hence we will be clear on this when it is necessary.

Example 1.4.13. Let x be as in Example 1.2.10. We want to compute the weighted Dynkin diagram of the nilpotent orbit of x under the action of $\hat{G} = \mathrm{SL}(3, \mathbb{C})$. Let $e_{ij} \in \hat{G}$ denote the matrix whose only nonzero entry is a 1 in the (i, j)-th position. We have already noted that x is conjugate to $x' := x - e_{13}$. It is convenient to use x' for the calculations. We can compute a triple for x, using the algorithm described above, and then conjugate the triple to another one with $h \in D_\Delta$. Set $h_1 := \mathrm{diag}(1, -1, 0), h_2 := \mathrm{diag}(0, 1, -1)$. It is clear that $h_1, h_2 \in [x', \mathfrak{g}]$, since $[x', e_{21}] = h_1, [x', e_{32}] = h_2$. Hence, we can look for the semisimple element in the subspace of $[x', \mathfrak{g}]$ generated by h_1, h_2, so that we get a semisimple element in the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of trace zero (which is often convenient for the calculations). Then, we set $h := ah_1 + bh_2$ and we look for values of a, b such that $[h, x'] = 2x'$. Thus, we get the simple system of equations given by $2a - b - 2 = 0$ and $-a + 2b - 2 = 0$, which yields $a = b = 2$. Therefore, $h = \mathrm{diag}(2, 0, -2)$ is the desired semisimple element. Thanks to the algorithm, we have in fact also constructed the third element, which is $f := 2e_{21} + 2e_{32}$. Moreover, for $i = 1, 2$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$, where $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ maps an element of $\mathfrak{h}$ into its i-th diagonal entry. Then, $\alpha_i(h) = 2$, for $i = 1, 2$. This means that h already lies in D_Δ. Hence, we conclude that the following weighted Dynkin diagram is associated to the orbit of x':



1.5 Component groups

A crucial role is played by the component group of the stabilizer of a nilpotent or semisimple element. We will need to compute it for different reasons and in many different ways. Indeed, sometimes the computation can be very challenging and we may need to develop different methods that fit well for certain specific situations (but less so for others). Here we give a formal description of what we mean by such component groups.
Let $e \in \mathfrak{g}$ and consider the stabilizer

$$Z_G(e) = \{\sigma \in G \mid \sigma(e) = e\}.$$

Then $Z_G(e)$ is an algebraic subgroup of G. Let $Z_G(e)^\circ$ be the connected component of the identity. There are $g_1, \dots, g_r \in G$ with $g_1 = 1$ such that $Z_G(e)$ is the disjoint union of the cosets $g_i Z_G(e)^\circ$. By determining the components of $Z_G(e)$ we mean determining such $g_1, \dots, g_r$. The identity component $Z_G(e)^\circ$ is determined by its Lie algebra $\mathfrak{z}_\mathfrak{g}(e) = \{x \in \mathfrak{g} \mid x \cdot e = 0\}$. This Lie algebra can be readily determined by linear algebra techniques. The *component group* is the quotient $Z_G(e)/Z_G(e)^\circ$; we see that the g_i are representatives of the different cosets in the component group. We will discuss below how exactly we determine the components of a stabilizer. For the purpose of this section, we recall that an element $u \in G$ is said to be *unipotent* if $u - \mathrm{id}_\mathfrak{g}$ is nilpotent; moreover, a subgroup $U \subset G$ is said to be *unipotent* if each element $u \in U$ is unipotent.
Now, set $Z_G(h,e,f) = \{\sigma \in G \mid \sigma(h) = h, \sigma(e) = e, \sigma(f) = f\}$. Often, when e is a nilpotent element of $\mathfrak{g}$, we will need to compute $Z_G(h,e,f)/Z_G(h,e,f)^\circ$, for some $\mathfrak{sl}_2$-triple containing e. This is usually convenient from a computational point of view, as there are more equations deriving from the requirements $\sigma(h) = h, \sigma(f) = f$ and this usually implies that the methods of Section 1.2 are more efficient. Moreover, by Theorem 1.4.2, the existence of a triple is always guaranteed. This is particularly convenient due to the following result:

Proposition 1.5.1. *Let e be a nonzero nilpotent element and (h,e,f) an $\mathfrak{sl}_2$-triple. Then,*

$$Z_G(e) = Z_G(h,e,f) \ltimes U$$

where U is a unipotent group.

Proof. For simplicity, we let $t = (h,e,f)$. Moreover, we let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into $\mathrm{ad}h$ eigenspaces. Since $\mathfrak{z}_\mathfrak{g}(e)$ is stable under $\mathrm{ad}h$ and by representation theory of $\mathfrak{sl}_2$, we have that $\mathfrak{z}_\mathfrak{g}(e) = \bigoplus_{i\geq 0} \mathfrak{z}_{\mathfrak{g}_i}(e)$ and $\mathfrak{z}_{\mathfrak{g}_i}(e) = \mathfrak{z}_\mathfrak{g}(e) \cap \mathfrak{g}_i$. We let $\mathfrak{u} \coloneqq \bigoplus_{i>0} \mathfrak{z}_{\mathfrak{g}_i}(e)$. Then, $\mathfrak{u}$ is a nilpotent ideal of $\mathfrak{z}_\mathfrak{g}(e)$. In particular, it holds $\mathfrak{z}_\mathfrak{g}(e) = \mathfrak{u} \oplus \mathfrak{z}_\mathfrak{g}(t)$. Let $U \subseteq Z_G(e)$ be the connected normal subgroup having $\mathfrak{u}$ as Lie algebra. Since $\mathfrak{u} \cap \mathfrak{z}_\mathfrak{g}(t) = \{0\}$, then $U \cap Z_G(h,e,f)$ has to be a finite group. Therefore, it has to be trivial as U is unipotent. Then, let $x \in Z_G(e)$. Then t and xt are two $\mathfrak{sl}_2$-triples with the same nilpositive element. Hence, by ([17, Theorem 3.4.10]), there is a $y \in U$ such that $(yx)t = t$, i.e. $yx \in Z_G(t)$. The fact that U is normal gives the desired semidirect product decomposition. □

The above proposition shows, in particular, that representatives of the component group $Z_G(h,e,f)/Z_G(h,e,f)^\circ$ are also representatives of the component group $Z_G(e)/Z_G(e)^\circ$ (this follows immediately from the connectedness of U).
We can use Proposition 1.5.1 to show an application of (1.3.2) that is recurrent in our computations.

Example 1.5.2. In this example we show how using diagram automorphisms help in reducing the complexity of some calculations. Here we let $\mathfrak{g} = \mathfrak{sl}(5,\mathbb{C})$, $\hat{G} = \mathrm{SL}(5,\mathbb{C})$ and we want to compute the component group of the nilpotent orbit representative

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can use the following $\mathfrak{sl}_2$-triple containing e:

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For simplicity, we let t be the $\mathfrak{sl}_2$-triple (h, e, f). We want to compute $Z_{\hat{G}}(t)/Z_{\hat{G}}(t)^\circ$ first. To get some info about the structure of the stabilizer, we can proceed as follows on GAP4. First of all, we construct $\mathfrak{sl}(5,\mathbb{C})$ explicitly by generators:

```
gap> x:=[ ]; y:= [ ];;
gap> for i in [1..4] do
>     x1:= NullMat(5,5); x1[i][i+1]:= 1;
>     y1:= NullMat(5,5); y1[i+1][i]:= 1;
>     Add( x, LieObject(x1) ); Add( y, LieObject(y1) );
> od;
gap> z:= List( [1..4], i -> x[i]*y[i] );;
gap> gl5:= FullMatrixLieAlgebra( CF(120), 5 );;
gap> sl5:= Subalgebra( gl5, Concatenation( x, y, z ) );;
gap> t:=[ LieObject( [ [ 0, 0, 0, 0, 0 ], [ 1, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ] ] ),
LieObject( [ [ 1, 0, 0, 0, 0 ], [ 0, -1, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ] ] ),
LieObject( [ [ 0, 1, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ] ] ) ];;
```

At this point, we may proceed as in Example 1.2.10, with t in place of x and with $M = (a_{ij})$ for $1 \le i, j \le 5$. In particular, we get a system of equations requiring

$$Mh - hM = 0, \quad Me - eM = 0, \quad Mf - fM = 0, \quad \det(M) = 1.$$

Computing the corresponding Gröbner basis, we reduce to the problem of solving the following system of equations:

$$\begin{gathered} a_{11} - a_{22} = 0, \quad a_{12} = 0, \quad a_{13} = 0, \quad a_{14} = 0, \quad a_{15} = 0, \quad a_{21} = 0, \quad a_{23} = 0, \quad a_{24} = 0, \\ a_{25} = 0, \quad a_{31} = 0, \quad a_{32} = 0, \quad a_{41} = 0, \quad a_{42} = 0, \quad a_{51} = 0, \quad a_{52} = 0, \\ a_{22}^2(a_{33}a_{44}a_{55} - a_{33}a_{45}a_{54} - a_{34}a_{43}a_{55} + a_{34}a_{45}a_{53} + a_{35}a_{43}a_{54} - a_{35}a_{44}a_{53}) - 1 = 0. \end{gathered}$$

Even though the procedure of Example 1.2.10 helped in reducing the initial system of equations (that we do not show here as it is particularly intricate), it is still a challenge to determine the components of the resulting set of solutions. For this reason, we can go through a slightly different procedure, as follows. First of all, we compute $\mathfrak{z}_{\mathfrak{g}}(t)$:

```
gap> C:=LieCentralizer(sl5,Subalgebra(sl5,t));
<Lie algebra of dimension 9 over CF(120)>
```

```
gap> DC:= LieDerivedSubalgebra( C );
<Lie algebra of dimension 8 over CF(120)>
gap> SemiSimpleType(DC);
A2
```

In this case we observe that $\mathfrak{z}_{\mathfrak{g}}(t)$ is reductive, i.e. $\mathfrak{z}_{\mathfrak{g}}(t) = \mathfrak{s} \oplus \mathfrak{t}$, where $\mathfrak{s} = [\mathfrak{z}_{\mathfrak{g}}(t), \mathfrak{z}_{\mathfrak{g}}(t)]$ is simple of type A_2 and $\mathfrak{t}$ is a 1-dimensional center. It is clear that for any $g \in Z_{\hat{G}}(t)$, $g(x) = \lambda x$, for any $x \in \mathfrak{t}$, $\lambda \in \mathbb{C}^\times$. Moreover, since $Z_{\hat{G}}(t)$ acts on $\mathfrak{s}$ by conjugation, we obtain a natural homomorphism from $Z_{\hat{G}}(t)$ into $\mathrm{Aut}(\mathfrak{s})$. By (1.3.2), we know that $\mathrm{Aut}(\mathfrak{s}) = \mathrm{Aut}(\mathfrak{s})^\circ \rtimes \Gamma_s$, where $\Gamma_s \cong S_2$ is the group of diagram automorphisms of $\mathfrak{s}$. In particular, we get that for any $s \in \mathrm{Aut}(\mathfrak{s})$, $s = (g, ())$ or $s = (g, (1,2))$, for some $g \in \mathrm{Aut}(\mathfrak{s})^\circ$. In other words, up to an inner automorphism, any element of $Z_{\hat{G}}(t)$ acts on the Cartan subalgebra of $\mathfrak{s}$ by either fixing the semisimple canonical generators or swapping them. Then, we compute the semisimple elements of a fixed set of canonical generators:

```
gap> h1:= CanonicalGenerators( RootSystem(DC) )[3][1];
LieObject( [ [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0 ],
[ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, -1 ] ] )
gap> h2:= CanonicalGenerators( RootSystem(DC) )[3][2];
LieObject( [ [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0 ],
[ 0, 0, 0, -1, 0 ], [ 0, 0, 0, 0, 1 ] ] )
```

Thanks to the above remarks, we can solve the two systems of equations:

$$\begin{gathered} Mh - hM = 0, \quad Me - eM = 0, \quad Mf - fM = 0, \\ \det(M) = 1, \quad Mh_1 - h_1 M = 0, \quad Mh_2 - h_2 M = 0; \end{gathered}$$

$$\begin{gathered} Mh - hM = 0, \quad Me - eM = 0, \quad Mf - fM = 0, \\ \det(M) = 1, \quad Mh_1 - h_2 M = 0, \quad Mh_2 - h_1 M = 0. \end{gathered}$$

It turns out that the ideal generated by the polynomials of the second one has a trivial Gröbner basis, hence there are no solutions. The first one reduces to solving:

$$\begin{gathered} a_{11} - a_{22} = 0, \quad a_{12} = 0, \quad a_{13} = 0, \quad a_{14} = 0, \quad a_{15} = 0, \quad a_{21} = 0, \quad a_{22}^2 a_{33} a_{44} a_{55} - 1 = 0, \\ a_{23} = 0, \quad a_{24} = 0, \quad a_{25} = 0, \quad a_{31} = 0, \quad a_{32} = 0, \quad a_{34} = 0, \quad a_{35} = 0, \quad a_{41} = 0, \\ a_{42} = 0, \quad a_{43} = 0, \quad a_{45} = 0, \quad a_{51} = 0, \quad a_{52} = 0, \quad a_{53} = 0, \quad a_{54} = 0. \end{gathered}$$

It is then immediate to deduce the set of solutions, which is

$$\left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{C}^\times,\ a^2 bcd = 1 \right\} \subset \hat{G}.$$

It is clearly connected (it is homeomorphic to $(\mathbb{C}^\times)^3$). Combining the solutions of both systems, we conclude that the component group $Z_{\hat{G}}(t)/Z_{\hat{G}}(t)^\circ$ is trivial, hence $Z_G(t)/Z_G(t)^\circ$ is trivial as well. Finally, by Proposition 1.5.1, we conclude that the group $Z_G(e)/Z_G(e)^\circ$ is trivial.

We conclude this section with some remarks on some computational aspects that we deal with when computing component groups.

The simple results obtained in Examples 1.2.10 and 1.5.2 are coherent with the well-known fact that $Z_G(x)/Z_G(x)^\circ$ is trivial for any $x \in \mathfrak{g}$, with $\mathfrak{g}$ of type $A_l (l \geq 1)$ (see for example [17, §6.1]). For further results on the computation of component groups of stabilizers of nilpotent elements in classical simple Lie algebras, we refer the reader to Section 2.2. Even though these examples are quite elementary, the approach and the techniques used are very common in most of the explicit computations that we perform in the whole thesis. However, there is a recurring trouble that mostly appears when dealing with Lie algebras of higher dimension. Indeed, it is not always immediate to calculate the exact number of connected components from the equations. Often this is not an issue. Indeed, we do not need to compute the component group directly from the equations, but it is enough to compute a finite group containing it. This is our goal most of the times and is a much easier task. Then, it is enough to check directly if two elements of such a finite group belong to the same connected component. Here we show two ways of doing so. The following result is known in the literature but we could not find a direct reference, hence we report our own adaptation.

Proposition 1.5.3. *Let σ be an automorphism of finite order of $\mathfrak{g}$. Let $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid \sigma(x) = x\}$. Then σ is inner if and only if $\mathfrak{g}_0$ has a Cartan subalgebra that is also a Cartan subalgebra of $\mathfrak{g}$.*

Proof. ($\Rightarrow$) At first, observe that σ being of finite order implies it being semisimple. Assuming that σ is inner means that $\sigma \in G$. Hence $\sigma \in T$, for some maximal torus $T \subset G$. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra such that $\mathrm{Lie}(T) = \mathrm{ad}\mathfrak{t}$ is a Cartan subalgebra of $\mathrm{ad}\mathfrak{g}$. For any $t \in \mathfrak{t}, e^{\mathrm{ad}t} \in T$, hence $\sigma e^{\mathrm{ad}t} = e^{\mathrm{ad}t}\sigma$. This implies that $\sigma \mathrm{ad} t \sigma^{-1} = \mathrm{ad}t$, which is equivalent to $\mathrm{ad}\sigma(t) = \mathrm{ad}t$. Since $\mathfrak{g}$ is semisimple, this implies $\sigma(t) = t$, for any $t \in \mathfrak{t}$. Therefore, $\mathfrak{t} \subset \mathfrak{g}_0$ is the requested Cartan subalgebra.
($\Leftarrow$) Conversely, suppose that σ fixes a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Consider the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The connected algebraic subgroup T of G whose Lie algebra is $\mathrm{ad}\mathfrak{t}$ consists of all maps that are the identity on $\mathfrak{t}$ and act on each root space $\mathfrak{g}_\alpha$ as multiplication by a scalar λ_α where $\lambda_{-\alpha} = \lambda_\alpha^{-1}$ and $\lambda_\gamma = \lambda_\alpha\lambda_\beta$ if $\alpha + \beta = \gamma$. Since σ fixes $\mathfrak{t}$ pointwise it has to lie in T and therefore in G. □

Thanks to Proposition 1.5.3, we see that for finite order automorphisms we have a straightforward algorithm for deciding whether they are inner or not. Indeed: we compute the subalgebra $\mathfrak{g}_0$, a Cartan subalgebra $\mathfrak{h}$ of it, and check whether $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.
There is a more general algorithm to check whether a given element of $\mathrm{GL}(n, \mathbb{C})$ lies in a connected algebraic subgroup H of $\mathrm{GL}(n, \mathbb{C})$, using only the Lie algebra of H. This method is outlined in [8, Remark 5.8] and this is the method that we use most. In the present dissertation we call this algorithm IsEltOf. The input to this algorithm is a basis $\mathcal{B}$ of the Lie algebra of a connected algebraic subgroup A of $\mathrm{GL}(n, \mathbb{C})$ and an element $g \in \mathrm{GL}(n, \mathbb{C})$. The output of IsEltOf($\mathcal{B}$, g) is true if $g \in A$ and false otherwise. For our purposes, on most occasions, the Lie algebra of the relevant algebraic subgroup is the centralizer of an orbit representative. We need this algorithm because some of our constructions may give too many elements of the component group, and we need a method to test whether two elements are equal modulo the identity component.

1.6 Vinberg's θ-groups

In [66] Vinberg introduced and studied a class of representations of reductive algebraic groups that since then have become known as θ-groups in the literature. Here we present results contained in that reference. Their construction starts by considering a semisimple complex Lie algebra $\mathfrak{g}$ and a $\mathbb{Z}/m\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i \text{ with } [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \text{ for } i, j \in \mathbb{Z}/m\mathbb{Z}. \tag{1.6.1}$$

Such gradings are closely related to automorphisms of $\mathfrak{g}$ of order m. Starting with a grading (1.6.1) and a primitive m-th root of unity $\omega \in \mathbb{C}$, we define $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\theta(x) = \omega^i x$ for $x \in \mathfrak{g}_i$, and extend θ to $\mathfrak{g}$ by linearity. Then θ is an automorphism of order m. Conversely, if θ is an automorphism of order m then we let $\mathfrak{g}_i$ be the eigenspace of θ with eigenvalue ω^i and we obtain a grading of the form (1.6.1). We can proceed analogously when $\mathfrak{g}$ is $\mathbb{Z}$-graded. In this case, the grading corresponds to a subgroup $\{\theta_t \mid t \in \mathbb{C}^\times\} \subset \mathrm{Aut}(\mathfrak{g})$ where $\theta_t(x) = t^i x$, for $x \in \mathfrak{g}_i$. We let the grading correspond to an automorphism $\theta = \theta_t$ where $t \in \{x \in \mathbb{C}^\times \mid |x| \neq 1\}$. Clearly, this implies that θ is not uniquely defined, but for our purposes that has no relevant consequences.
Notable results about θ-groups can be found in the main works of Vinberg ([66, 67]) and Elashvili ([68]). Here we cite the basic results that have relevant consequences in the sequel. First of all, we have the following theorem:

Theorem 1.6.1. *Consider a grading of the form* (1.6.1)*. Let κ denote the Killing form of $\mathfrak{g}$. Then:*

1. *For each i, $\mathfrak{g}_i$ contains the semisimple and nilpotent parts of its elements;*
2. *the restriction of κ to $\mathfrak{g}_0$ is non-degenerate;*
3. *let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. Then the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. In particular, $\mathfrak{g}_0 \neq 0$;*
4. *$\mathfrak{g}_0$ is reductive in $\mathfrak{g}$.*

We skip the proof of Theorem 1.6.1, as it is a bit technical and it goes beyond our aims (for all the details we refer to [25, Theorem 8.3.1]). However, we spend some words on the first assertion which is easy to prove but also important for our purposes. Indeed, θ being an automorphism implies that it maps nilpotent elements of $\mathfrak{g}$ to nilpotent elements of $\mathfrak{g}$. Analogously, it maps semisimple elements of $\mathfrak{g}$ into semisimple elements of $\mathfrak{g}$. Now, if $x \in \mathfrak{g}_i$ and $x = x_s + x_n$ is its Jordan decomposition, then $\theta(x) = \theta(x_s) + \theta(x_n)$. Also, $\theta(x) = \lambda^i x$, for some $\lambda \in \mathbb{C}$. However, we have the Jordan decomposition $\lambda^i x = \lambda^i x_s + \lambda^i x_n$, hence, by uniqueness, it has to be $\theta(x_s) = \lambda^i x_s, \theta(x_n) = \lambda^i x_n$, both lying in $\mathfrak{g}_i$. This is as simple as it is significant and the reason is explained below.
Let $G = \mathrm{Aut}(\mathfrak{g})^\circ$, as usual. This time, we let $\mathfrak{g}$ be graded as in (1.6.1). We let $G^\theta := \{g \in G \mid \theta g = g\theta\}$. It follows that $\mathrm{Lie}(G^\theta) = \{y \in \mathrm{ad}(\mathfrak{g}) \mid \theta y = y\theta\}$ (see, for example, [25, Lemma 3.6.8]). Let G_0 be the identity component of G^θ. Then $\mathrm{Lie}(G_0) = \mathrm{Lie}(G^\theta)$ which coincides with $\{\mathrm{ad}x \mid x \in \mathfrak{g}, \theta(\mathrm{ad}x)\theta^{-1} = \mathrm{ad}x\} = \{\mathrm{ad}x \mid x \in \mathfrak{g}, \mathrm{ad}\theta(x) = \mathrm{ad}x\}$. Hence, G_0 is the unique connected subgroup of G with Lie algebra $\mathrm{ad}\mathfrak{g}_0$. Also, the grading implies that $\mathrm{ad}\mathfrak{g}_0(\mathfrak{g}_1) \subset \mathfrak{g}_1$. Then, by [25, Corollary 4.2.10], $\mathfrak{g}_1$ is stable under the action of G_0. Now, the remark of the previous paragraph allows us to study nilpotent and semisimple orbits of this action. This is relevant both in Chapter 3 and Chapter 4. Moreover, we can define $\Theta : G \to G$, such that $\Theta(g) = \theta g \theta^{-1}$. In our case, we easily get $\mathrm{d}\Theta = \theta$. We remark that $G^\theta = G^\Theta$.
We also show three results that are often used when dealing with θ-groups ([25, Lemma 8.3.5, Theorem 8.3.6, Corollary 8.3.8]. The first two are similar to what has been shown in Section 1.4:

Lemma 1.6.2. *Let $e \in \mathfrak{g}_1$ be a nonzero nilpotent element. Then there exist $f \in \mathfrak{g}_{-1}$ and $h \in \mathfrak{g}_0$ such that (h, e, f) is an $\mathfrak{sl}_2$-triple.*

We say that an $\mathfrak{sl}_2$-triple (h, e, f) as in Lemma 1.6.2 is *homogeneous*.

Theorem 1.6.3. *Let $(h,e,f),(h',e',f')$ be two $\mathfrak{sl}_2$-triples as in Lemma 1.6.2. Then the following are equivalent:*

- *e, e' are G_0-conjugate;*
- *$(h,e,f),(h',e',f')$ are G_0-conjugate;*
- *h, h' are G_0-conjugate.*

Proposition 1.6.4. *The number of nilpotent orbits of the action of G_0 on $\mathfrak{g}_1$ is finite.*

We conclude this section with the analogue of Proposition 1.5.1 for θ-groups.

Proposition 1.6.5. *Let e be a nonzero nilpotent element and (h,e,f) an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0, e \in \mathfrak{g}_1, f \in \mathfrak{g}_{-1}$. Let $S := [G_0, G_0]$. Then,*

$$Z_S(e) = Z_S(h,e,f) \ltimes U_0$$

where U_0 is a unipotent group.

Proof. Thanks to Proposition 1.5.1, we know that

$$Z_G(e) = Z_G(h,e,f) \ltimes U$$

where U and $\mathfrak{u}$ are as in Section 1.5. Observe that $\theta(\mathfrak{u}) = \mathfrak{u}$, hence $\Theta(U) = U$. Also, if $g \in Z_G(h,e,f)$, then $\Theta(g)(e) = \theta g \theta^{-1}(e) = e$, similarly $\Theta(g)(h) = h$ and $\Theta(g)(f) = f$; hence, $\Theta(Z_G(h,e,f)) = Z_G(h,e,f)$. Therefore, $Z_G(e)^{\Theta} = Z_G(h,e,f)^{\Theta} \ltimes U^{\Theta}$. Since G_0 is the identity component of G^{Θ}, $U^{\Theta} \subset G_0$ (since U^{Θ} is connected). In particular, we get

$$Z_{G_0}(e) = Z_{G_0}(h,e,f) \ltimes U^{\Theta}.$$

Moreover, observe that $U^{\Theta} \cap Z(G_0)$ is trivial, as U^{Θ} is unipotent. In other words, $U^{\Theta} \subset S$. Hence, letting $U_0 := U^{\Theta}$, the proof is complete. □

1.7 Irreducible representations of reductive Lie algebras

Here we recall some well-known facts on irreducible representations of reductive Lie algebras. Specifically, we refer to the structure theory of irreducible representations of semisimple Lie algebras (see, e.g., [42, §20]) that generalizes directly to the case of reductive Lie algebras. We do not show all the details of this generalization, but we focus on the most relevant consequences for finite-dimensional representations. Such a theory is based on the role of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and it is proven that every irreducible representation arises as a quotient of $\mathcal{U}(\mathfrak{g})$. Since it is a bit technical to describe all the steps of this construction, we only focus on its consequences (for reductive Lie algebras).

Now, let $\mathfrak{c}$ be a reductive Lie algebra, i.e., $\mathfrak{c} = \mathfrak{c}' \oplus \mathfrak{z}(\mathfrak{c})$ where $\mathfrak{c}' = [\mathfrak{c},\mathfrak{c}]$ is semisimple and $\mathfrak{z}(\mathfrak{c})$ is the center of $\mathfrak{c}$. Let $\rho : \mathfrak{c} \to \mathfrak{gl}(V)$ be an irreducible finite-dimensional representation such that $\rho(x)$ is semisimple for all $x \in \mathfrak{z}(\mathfrak{c})$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{c}$. Then $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{z}(\mathfrak{c})$, where $\mathfrak{a}'$ is a Cartan subalgebra of $\mathfrak{c}'$. We consider the root system of $\mathfrak{c}'$ with respect to $\mathfrak{a}'$ and let $h_1,\dots,h_r$, $x_1,\dots,x_r$, $y_1,\dots,y_r$ be a canonical generating set (here $h_i \in \mathfrak{a}'$, x_i, y_i are root vectors corresponding to the simple roots, respectively the negative simple roots). Let $h_{r+1},\dots,h_\ell$ be a basis of $\mathfrak{z}(\mathfrak{c})$, so that $h_1,\dots,h_\ell$ is a basis of $\mathfrak{a}$. For $\mu\in\mathfrak{a}^*$, we set

$$V_\mu = \{v\in V \mid \rho(h)v = \mu(h)v \text{ for all } h\in\mathfrak{a}\}.$$

If $V_\mu \neq 0$ then we say that μ is a *weight* of V and V_μ is the corresponding *weight space*. Any nonzero element of V_μ is called a *weight vector* of weight μ. A weight λ of V is said to be a *highest weight* if $\rho(x_i)V_\lambda = 0$ for $1\le i\le r$. Because ρ is irreducible, V has a unique highest weight λ and $\dim V_\lambda = 1$. A nonzero element $v_0\in V_\lambda$ is called a *highest weight vector* of V. Fixing such a vector v_0 we have that the elements

$$\rho(y_{i_1})\cdots\rho(y_{i_k})v_0 \text{ for } k\ge 0,\ 1\le i_j\le r$$

span V.
Let now ρ be as above, except we assume it is not necessarily irreducible. We recall the following proposition ([44, Chapter III, Theorem 10]):

Proposition 1.7.1. *Let $\mathfrak{l}$ be a finite-dimensional linear Lie algebra over a field K of characteristic zero. Then, $\mathfrak{l}$ is completely reducible over K if and only if $\mathfrak{l} = \mathfrak{l}'\oplus\mathfrak{z}(\mathfrak{l})$, where $\mathfrak{l}' = [\mathfrak{l},\mathfrak{l}]$, and the elements of $\mathfrak{z}(\mathfrak{l})$ are semisimple.*

In particular, thanks to the above result with $\rho(\mathfrak{c})$ in place of $\mathfrak{l}$, we get that V is completely reducible. Moreover, the above description of a spanning set of an irreducible module gives a straightforward method to find a decomposition of V as a direct sum of irreducible modules. This works as follows. First we compute the space $V^+ = \{v\in V \mid \rho(x_i)v = 0 \text{ for } 1\le i\le r\}$. Secondly, we decompose V^+ as a direct sum of weight spaces. Thirdly, for each weight space we fix a basis. Those basis vectors are exactly the highest weight vectors of the different irreducible modules in the decomposition.
We say that ρ (or the $\mathfrak{c}$-module V) is *multiplicity free* if the dimension of each weight space in V^+ is 1. This is the same as saying that V decomposes as a sum of pairwise non-isomorphic irreducible modules. In that case this decomposition is uniquely determined.

1.8 Galois cohomology

A significant tool that we will use throughout the next sections is Galois cohomology. Here we summarize the main results that we use for determining the real orbits within a complex one. We remark that the theoretical framework of this section is inspired by [8] and this source also contains efficient algorithms to perform explicit calculations concerning Galois cohomology.
In this section we let $\mathcal{G}$ be any group with a fixed automorphism σ of order 2. An element $g\in\mathcal{G}$ is called a *cocycle* if $g\sigma(g)=1$. The set of cocycles is denoted by $Z^1(\mathcal{G},\sigma)$ (we omit σ when it is clear which automorphism is being used). We can define an equivalence relation on $Z^1(\mathcal{G},\sigma)$: $g,g'\in\mathcal{G}$ are equivalent if $g' = hg\sigma(h)^{-1}$ for some $h\in\mathcal{G}$. The set of equivalence classes is denoted by $H^1(\mathcal{G},\sigma)$, or also by $H^1\mathcal{G}$ if it is clear which automorphism σ is used. It is called the first Galois cohomology set of $\mathcal{G}$. Suppose $\mathcal{G}$ acts on a set X which also has a map $\sigma: X\to X$ such that $\sigma(\sigma(x)) = x$ for each $x\in X$. We suppose that the action of $\mathcal{G}$ is compatible with σ, that is, $\sigma(g)\sigma(x) = \sigma(gx)$ for all $g\in\mathcal{G}, x\in X$. Let $x_0\in X$ be invariant under σ, i.e., $\sigma(x_0)=x_0$. Let $Z_\mathcal{G}(x_0) = \{g\in\mathcal{G}\mid gx_0 = x_0\}$ be its stabilizer. We have a map $i_*: H^1(Z_\mathcal{G}(x_0),\sigma)\to H^1(\mathcal{G},\sigma)$ by $i_*([g]) = [g]$, where the latter is the class of the cocyle g in $H^1(\mathcal{G},\sigma)$. The kernel of this map is $\ker(i_*) = \{[g]\in H^1(Z_\mathcal{G}(x_0),\sigma)\mid g \text{ is equivalent to 1 in } Z^1(\mathcal{G},\sigma)\}$. By $\mathcal{G}^\sigma, X^\sigma$ we denote

the elements of $\mathcal{G}, X$ that are fixed under σ. We are now ready to state a crucial result (see [59, Section I.5.4, Corollary 1 of Proposition 36] for a proof):

Theorem 1.8.1. *Let $x_0 \in X^\sigma$ and let $Y = \mathcal{G}{\cdot}x_0$ be its $\mathcal{G}$-orbit. Let $Y^\sigma = \{y \in Y \mid \sigma(y) = y\}$. Then the $\mathcal{G}^\sigma$-orbits in Y^σ are in bijection with $\ker(i_*)$. The bijection is given as follows. Let $[g] \in \ker(i_*)$, then there is an $h \in \mathcal{G}$ such that $g = h^{-1}\sigma(h)$. The class $[g]$ corresponds to the $\mathcal{G}^\sigma$-orbit of $h \cdot x_0$.*

Now we outline how to determine the first Galois cohomology set of a group, assuming we can do it for a normal subgroup and the quotient by it. We sketch the method as it is also contained in other sources (for example [8]). It will be used in Section 3.7.
Let $\mathcal{G}$ be a group with involution σ. Let $\mathcal{N}$ be a normal subgroup that is stable under σ. Then σ induces an involution on the quotient $\mathcal{C} = \mathcal{G}/\mathcal{N}$ which we also denote by σ. Let $\pi : \mathcal{G} \to \mathcal{C}$ be the projection map. Hence we have an exact sequence

$$1 \longrightarrow \mathcal{N} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{C} \longrightarrow 1$$

where i is the inclusion map. The map i yields a map $i_* : H^1(\mathcal{N}, \sigma) \to H^1(\mathcal{G}, \sigma)$, $i_*([n]) = [i(n)]$; similarly we have a map $\pi_* : H^1(\mathcal{G}, \sigma) \to H^1(\mathcal{C}, \sigma)$, $\pi_*([g]) = [\pi(g)]$.

Lemma 1.8.2. *We have* $\mathrm{im}(i_*) = \ker(\pi_*)$ *(where* $\ker(\pi_*) = \{[g] \in H^1(\mathcal{G}, \sigma) \mid \pi_*([g]) = [1]$ *in* $H^1(\mathcal{C}, \sigma)\}$*).*

This lemma is contained in [59, I §5, Proposition 38]. We also sketch its proof. First of all, $\pi(i(n)) = 1$ for all $n \in \mathcal{N}$ implying that $\mathrm{im}(i_*) \subset \ker(\pi_*)$. For the converse let $g \in \mathcal{G}$ be such that $\pi_*([g]) = [1]$; then there is a $c_1 \in \mathcal{C}$ such that $\pi(g) = c_1^{-1}\sigma(c_1)$. Let $g_1 \in \mathcal{G}$ such that $c_1 = \pi(g_1)$. As $\pi(g) = \pi(g_1^{-1}\sigma(g_1))$ we get $g = g_1^{-1}\sigma(g_1)n_1$ for an $n_1 \in \mathcal{N}$. We have $\sigma(g_1)n_1\sigma(g_1)^{-1} = n_2 \in \mathcal{N}$ so that $g = g_1^{-1}n_2\sigma(g_1)$. So n_2 is a cocycle and $i_*([n_2]) = [g]$.
Let $\mathcal{C}^\sigma = \{c \in \mathcal{C} \mid \sigma(c) = c\}$. We now define a right action of $\mathcal{C}^\sigma$ on $H^1(\mathcal{N}, \sigma)$. For $c \in \mathcal{C}^\sigma$ write $c = \pi(g)$ for a $g \in \mathcal{G}$. For $n \in Z^1(\mathcal{N}, \sigma)$ we have that $g^{-1}n\sigma(g) \in Z^1(\mathcal{N}, \sigma)$. We define

$$[n] \cdot c = [g^{-1}n\sigma(g)]. \tag{1.8.1}$$

Short calculations show that $[g^{-1}n\sigma(g)] \in H^1(\mathcal{N}, \sigma)$ is independent of the choice of g and that this indeed defines an action of $\mathcal{C}^\sigma$.

Lemma 1.8.3. *Let $n_1, n_2 \in Z^1(\mathcal{N}, \sigma)$. Then $i_*([n_1]) = i_*([n_2])$ if and only if there exists a $c \in \mathcal{C}^\sigma$ with $[n_1] \cdot c = [n_2]$.*

This lemma is contained in [59, I §5, Proposition 39(ii)]). The proof is again a short verification. If $[n_1]$, $[n_2]$ are equivalent in $H^1(\mathcal{G}, \sigma)$ then there is a $g \in \mathcal{G}$ with $g^{-1}n_1\sigma(g) = n_2$. Applying π we see that this entails $\pi(\sigma(g)) = \pi(g)$, so if $c = \pi(g)$ then $c \in \mathcal{C}^\sigma$ and $[n_1] = [n_2] \cdot c$ in $H^1(\mathcal{N}, \sigma)$. The converse is even more straightforward.
Now fix $[c] \in H^1(\mathcal{C}, \sigma)$; we wish to compute $\pi_*^{-1}([c])$; if we can do this then we can obviously determine $H^1(\mathcal{G}, \sigma)$. We assume that there is a $g_c \in Z^1(\mathcal{G}, \sigma)$ such that $\pi(g_c) = c$. If this is not the case then $\pi_*^{-1}([c]) = \emptyset$. In the sequel we fix such a g_c.
Define the involution $\tau : \mathcal{G} \to \mathcal{G}$ by $\tau(g) = g_c\sigma(g)g_c^{-1}$. Then for $g \in Z^1(\mathcal{G}, \tau)$ we have $gg_c \in Z^1(\mathcal{G}, \sigma)$. Furthermore, if the cocycles $g_1, g_2 \in Z^1(\mathcal{G}, \tau)$ are equivalent, then g_1g_c and g_2g_c are equivalent in $Z^1(\mathcal{G}, \sigma)$. So we get a map $\eta : H^1(\mathcal{G}, \tau) \to H^1(\mathcal{G}, \sigma)$. Its inverse is $[g] \mapsto [gg_c^{-1}]$, hence η is a bijection. Similarly we have a bijection $\xi : H^1(\mathcal{C}, \tau) \to H^1(\mathcal{C}, \sigma)$, $\xi([d]) = [dc]$. We get the following diagram

$$\begin{tikzcd} H^1(\mathcal{N}, \tau) \arrow[r, "i_*"] & H^1(\mathcal{G}, \tau) \arrow[r, "\pi_*"] \arrow[d, "\eta"'] & H^1(\mathcal{C}, \tau) \arrow[d, "\xi"] \\ & H^1(\mathcal{G}, \sigma) \arrow[r, "\pi_*"] & H^1(\mathcal{C}, \sigma) \end{tikzcd}$$

where the square on the right commutes.

Proposition 1.8.4. *Let* $\{[n_1], \ldots, [n_r]\}$ *be a set of* $\mathcal{C}^\tau$*-orbit representatives in* $H^1(\mathcal{N}, \tau)$*, then* $\pi_*^{-1}([c]) = \{[n_1 g_c], \ldots, [n_r g_c]\}$*.*

Proof. Let $M = \{[g] \in H^1(\mathcal{G}, \sigma) \mid \pi_*([g]) = [c]\}$. So $\eta^{-1}(M)$ is the preimage in $H^1(\mathcal{G}, \tau)$ of $\xi^{-1}([c])$. But $\xi^{-1}([c]) = [1]$, so $\eta^{-1}(M) = \ker(\pi_*)$ which by Lemma 1.8.2 is equal to $\mathrm{im}(i_*)$. By Lemma 1.8.3 we see that $\mathrm{im}(i_*)$ corresponds to the $\mathcal{C}^\tau$-orbits on $H^1(\mathcal{N}, \tau)$; the claim follows. □

1.9 Real forms of a complex Lie algebra

When using Galois cohomology (introduced in Section 1.8), we often let σ be the complex conjugation and then deal with complex Lie algebras. In this case, Theorem 1.8.1 gives the real orbits contained in some complex orbits. In this setting, when using Galois cohomology, real Lie algebras arise as a consequence of the application of Theorem 1.8.1. In this perspective, our results and the possible applications concern real Lie algebras. Therefore, in this section we aim to recall basic definitions about real Lie algebras and their connection with the complex ones. Then, we fix some notation in order to make clear the meaning and the outcomes of our work and, practically, allow the reader to better comprehend the results shown in some of our tables (see, for example, the stabilizers in Table 3.3).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

Definition 1.9.1. *A real Lie algebra* $\mathfrak{r}$ *is said to be a real form of* $\mathfrak{g}$ *if* $\mathfrak{g} \cong \mathfrak{r} \otimes_{\mathbb{R}} \mathbb{C}$ *as complex Lie algebras. Equivalently,* $\mathfrak{r}$ *can be viewed as a real Lie subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{r} \oplus \imath\mathfrak{r}$*.*

Moreover, we recall we have two specific types of real forms:

Definition 1.9.2. *A real form* $\mathfrak{r}$ *of a complex semisimple Lie algebra* $\mathfrak{g}$ *is said to be:*

- split *if it contains a Cartan subalgebra* $\mathfrak{h}$*, such that* $\mathrm{ad}h$ *is diagonalizable (over* $\mathbb{R}$*) for any* $h \in \mathfrak{h}$*;*
- compact *if its Killing form is negative definite.*

The existence of such real forms is guaranteed by the following result:

Theorem 1.9.3. *Any complex semisimple Lie algebra contains both a split and a compact real form.*

Clearly, this works in particular for simple Lie algebras. For example, in $\mathfrak{sl}_{n+1}(\mathbb{C})$, we have the split real form $\mathfrak{sl}_{n+1}(\mathbb{R})$ and the compact real form $\mathfrak{su}(n+1)$. The fact that $\mathfrak{sl}_{n+1}(\mathbb{R})$ is a split real form is a direct consequence of the standard decomposition of a complex trace zero matrix into real and imaginary parts, together with the presence of real diagonal matrices of trace zero. Similarly, $\mathfrak{su}(n+1) = \{x \in \mathfrak{sl}_{n+1}(\mathbb{C}) \mid \bar{x}^\top + x = 0\}$ is naturally a real form via the standard skew-Hermitian decomposition $X = U + \imath V$, with $U = (X - X^*)/2$ and $V = -\imath(X + X^*)/2$ belonging to $\mathfrak{su}(n+1)$. Furthermore, its compactness is an immediate consequence of the fact that skew-Hermitian matrices have purely imaginary eigenvalues, making the Killing form $\kappa(A, B) = 2(n+1)\mathrm{Tr}(AB)$ strictly negative definite. With similar techniques, one can show analogous results for the other simple Lie algebras. In $\mathfrak{so}_{2n+1}(\mathbb{C})$ we have the compact form $\mathfrak{so}_{2n+1}(\mathbb{R})$ and the split form $\mathfrak{so}(n+1,n)$; it is similar for $\mathfrak{so}_{2n}(\mathbb{C})$. For $\mathfrak{sp}_{2n}(\mathbb{C})$ the split form is $\mathfrak{sp}(2n,\mathbb{R})$ and the compact one is $\mathfrak{sp}(2n)$. For exceptional Lie algebras we only recall the number of real forms; for G_2, F_4, E_6, E_7, E_8 there are $2, 3, 5, 4, 3$ real forms, respectively. In Table 4.2 the split and compact forms of G_2 appear. They are denoted by $G_{2(2)}$ and G, respectively.

Chapter 2

Component groups of stabilizers of nilpotent orbit representatives in simple Lie algebras

The contents of this chapter are an adaptation of the paper [28].

2.1 Introduction

The theory of the nilpotent orbits of simple Lie algebras has seen tremendous developments over the past decades. For overviews we refer to the book by Collingwood and McGovern ([17]) and the survey paper by Jantzen ([45]). However, for our purposes the basics recalled in Section 1.4 are sufficient. In many contexts an important role is played by the stabilizer of a nilpotent element. Examples are the Springer correspondence ([17, §10.1], [45, §13]) and the theory of W-algebras (see, for example, [50]). In characteristic 0 the identity component of this stabilizer is determined by its Lie algebra, which is the centralizer in the Lie algebra and thus can be determined by linear algebra. So what remains is to determine the component group.
For the Lie algebras of classical type these component groups can be determined by general considerations, see [45, §3] and Section 2.2 below. For the Lie algebras of exceptional type the component groups have first been determined by Alekseevskii, [2] and later by Sommers, [60]. These references provide methods to determine the isomorphism type of the component group, not directly the component group itself. More precisely, with the construction in [60] it is possible to determine an element in each conjugacy class of the component group; but these elements lie in the stabilizer of different nilpotent elements in the same orbit. In order to perform explicit computations with the component group one often needs explicit generators of the component group of the stabilizer of one fixed nilpotent element. In [48] such generators have been determined by impressive hand calculations for the Lie algebras of exceptional type. For the simple Lie algebra of type E_8 we have verified the correctness of the component groups given in [48]. This involved constructing a Lie algebra in GAP4 with the multiplication table given in [49]. This is a rather laborious task, but once completed it is straightforward to check the correctness of the given elements of the component groups by general considerations, independent of the methods developed in the present chapter.

The main results of this chapter are computational methods to obtain the component group of the centralizer of a nilpotent element in a simple Lie algebra over $\mathbb{C}$. For the classical types there is a straightforward method that directly translates the theoretical construction, see Section 2.2. In Section 2.4 we devise a method for the exceptional types using the double centralizer of an $\mathfrak{sl}_2$-triple. Our method gives an independent construction of the component group, that is, it does not depend on the prior knowledge of the size or isomorphism type of the component group. It is based on Theorem 1.4.2 and Proposition 1.5.1. Let $\mathfrak{g}$ be a complex simple Lie algebra and $G = \mathrm{Aut}(\mathfrak{g})^\circ$. Let $\mathfrak{c}$ denote the sum of the centralizer of an $\mathfrak{sl}_2$-triple (h, e, f) and its double centralizer (that is, the centralizer of the centralizer). Let V denote a complement to $\mathfrak{c}$ in $\mathfrak{g}$ that is perpendicular to $\mathfrak{c}$ with respect to the Killing form. Then by a direct case by case computation, of which the details are given in Section 2.5, we establish that V is a direct sum of pairwise non-isomorphic $\mathfrak{c}$-modules. We then exploit the symmetries that exist among the highest weights of the irreducible summands of V along with the symmetries of $\mathfrak{c}$ (that is, its automorphisms) to construct all elements of the component group of the centralizer of the $\mathfrak{sl}_2$-triple.

A technical ingredient, described in Section 2.3, is an algorithm for finding an explicit automorphism conjugating two nilpotent elements that lie in the same orbit. This algorithm is very useful on many occasions, see Example 2.3.2. However, as it involves Gröbner basis computations, it does not always terminate in reasonable time, see Example 2.3.3. We also remark that if we had an efficient solution to this conjugacy problem then it would be easy to determine the component group of the centralizer of a nilpotent element e. Firstly we could determine the nilpotent e' from the list in [48] that is conjugate to e. Then we find an explicit automorphism σ mapping e' to e. Finally the elements of the component group of $Z_G(e)$ are of the form $\sigma\tau\sigma^{-1}$ where τ runs over the elements of the component group of $Z_G(e')$ as given in [48]. Secondly, with the method of Sommers ([60]) it is possible to determine nilpotent elements $e_1, \dots, e_r \in \mathfrak{g}$, all conjugate to e, and an element σ_i in the component group of $Z_G(e_i)$. By conjugating e_i and e we can then find a corresponding element in the component group of $Z_G(e)$. By the results of Sommers, this way we find an element in each conjugacy class of the component group of $Z_G(e)$, and hence these generate the component group. We have tried the latter approach for several nilpotent orbits. In some cases we found the component group this way, but in quite a few other cases this did not work due to the difficulty of the Gröbner basis computations involved.

We have implemented our algorithms in the computational algebra system GAP4, with the help of the SLA package. For the algorithm of Sections 2.3, 2.4 we need Gröbner basis computations. For these we have used the systems Magma and Singular. For the exceptional types we have constructed the component groups for all nilpotent orbits, using the representatives given in [24]. The component groups have been made available in the latest release (version 1.6.2) of the SLA package, [23].

2.2 Simple Lie algebras of classical type

In this section we let $\mathfrak{g}$ be a simple Lie algebra of classical type, that is, of type A_ℓ ($\ell \geq 1$), B_ℓ ($\ell \geq 2$), C_ℓ ($\ell \geq 3$), D_ℓ ($\ell \geq 4$). Let $e \in \mathfrak{g}$ be a nilpotent element and (h, e, f) an $\mathfrak{sl}_2$-triple. For the Lie algebras of type A_ℓ the stabilizer $Z_G(h, e, f)$ is always connected (see [60, §3.5]) so we deal with the Lie algebras of type B, C, D. For these algebras we describe an algorithm to obtain the component group of $Z_G(h, e, f)$ closely following the construction in [45, §3.1].

We will focus on the Lie algebras of type B_ℓ and D_ℓ; in Remark 2.2.2 we indicate what has to be changed for type C_ℓ. We will first work with the groups $O(n,\varphi)$ (defined below) instead of the inner automorphism group G. Later we will pass to G.
Let V be an n-dimensional vector space over $\mathbb{C}$. Let $\varphi : V \times V \to \mathbb{C}$ be a nondegenerate bilinear form. Set

$$\widehat{G}(V,\varphi) = \{g \in \mathrm{GL}(V) \mid \varphi(gv, gw) = \varphi(v,w) \text{ for all } v, w \in V\}.$$

This is an algebraic subgroup of $\mathrm{GL}(V)$ with Lie algebra

$$\mathrm{g}(V,\varphi) = \{x \in \mathrm{End}(V) \mid \varphi(xv, w) + \varphi(v, xw) = 0 \text{ for all } v, w \in V\}.$$

The group $\widehat{G}(V,\varphi)$ acts on the Lie algebra $\mathrm{g}(V,\varphi)$ by $g \cdot x = gxg^{-1}$. This yields a map from $\widehat{G}(V,\varphi)$ to the automorphism group of $\mathrm{g}(V,\varphi)$.
If φ is symmetric (that is $\varphi(v,w) = \varphi(w,v)$ for all $v, w \in V$) then $\widehat{G}(V,\varphi) = \mathcal{O}(V,\varphi)$ is the orthogonal group and we write $\mathfrak{o}(V,\varphi) = \mathrm{g}(V,\varphi)$ for its Lie algebra. If φ is alternating (that is, $\varphi(v,w) = -\varphi(w,v)$ for all $v,w \in V$) then we write $\mathrm{Sp}(V,\varphi) = \widehat{G}(V,\varphi)$ which is the symplectic group. In that case we write $\mathfrak{sp}(V,\varphi) = \mathrm{g}(V,\varphi)$.
In the remainder of this section we let φ be symmetric. If $n \geq 5$ is odd then $\mathfrak{o}(V,\varphi)$ is simple of type B_ℓ where $n = 2\ell + 1$. If $n \geq 8$ is even then it is simple of type D_ℓ where $n = 2\ell$.
The group $\mathcal{O}(V,\varphi)$ has two connected components. The identity component is $\mathrm{SO}(V,\varphi) = \{g \in \mathcal{O}(V,\varphi) \mid \det(g) = 1\}$. The second component is $\{g \in \mathcal{O}(V,\varphi) \mid \det(g) = -1\}$. We now describe a simple method to find an element in this second component. Let $v_1, \dots, v_n$ be a basis of V. Let $w_n \in V$ be such that $\varphi(w_n, w_n) \neq 0$. This element can be found as follows: if there is an index i with $\varphi(v_i, v_i) \neq 0$ then we set $w_n = v_i$. Otherwise there are $i < j$ with $\varphi(v_i, v_j) \neq 0$ and we set $w_n = v_i + v_j$. Let $w_1, \dots, w_{n-1} \in V$ be a basis of the subspace $\{v \in V \mid \varphi(w_n, v) = 0\}$. Then $w_1, \dots, w_n$ is a basis of V. Let $g \in \mathrm{GL}(V)$ be the map that sends $w_i \mapsto w_i$ for $1 \leq i \leq n-1$ and $w_n \mapsto -w_n$. Then $\varphi(gw_i, gw_j) = \varphi(w_i, w_j)$ (for $i, j \leq n-1$ and $i = j = n$ this is clear, for $i \leq n-1$, $j = n$ both are 0). Hence $g \in \mathcal{O}(V,\varphi)$ and $\det(g) = -1$.
Let (h,e,f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{o}(V,\varphi)$ and let $\mathfrak{a}$ be the subalgebra spanned by it. Then $\mathfrak{a}$ acts on V and V splits as a direct sum of irreducible $\mathfrak{a}$-modules, $V = V_1 \oplus \cdots \oplus V_m$. Write $d_i = \dim V_i$. From the representation theory of $\mathfrak{sl}_2$ (see [42, §II.7]) it follows that each V_i has a unique (up to nonzero scalar multiples) element $\hat{v}_i$ such that $f \cdot \hat{v}_i = 0$, $h \cdot \hat{v}_i = (-d_i + 1)\hat{v}_i$, and $\{e^k \cdot \hat{v}_i \mid 0 \leq k \leq d_i - 1\}$ forms a basis of V_i, and $e^{d_i} \cdot \hat{v}_i = 0$. For $s \geq 1$ we let M_s be the space spanned by all $\hat{v}_i$ such that $d_i = s$. Note that the direct sum decomposition of V is not unique in general, but the space M_s is uniquely determined.
Write $\widehat{G} = \mathcal{O}(V,\varphi)$. Let $g \in Z_{\widehat{G}}(h,e,f)$ then g stabilizes each space M_s. Moreover $g \cdot (e^k \cdot \hat{v}_i) = e^k(g \cdot \hat{v}_i)$. It follows that the action of g on each M_s determines g. So we get an injective homomorphism

$$Z_{\widehat{G}}(h,e,f) \to \mathrm{GL}(M_1) \times \mathrm{GL}(M_2) \times \cdots. \tag{2.2.1}$$

On the space M_s define a bilinear form ψ_s by $\psi_s(v,w) = \varphi(v, e^{s-1}w)$. In [45, §3.7] it is shown that ψ_s is symmetric if s is odd and alternating if s is even. So for s odd we get the orthogonal group $\mathcal{O}(M_s,\psi_s)$ and for s even we get the symplectic group $\mathrm{Sp}(M_s,\psi_s)$. Now [45, §3.8, Proposition 2] states that the above injective homomorphism is an isomorphism onto

$$\prod_{s \text{ odd}} \mathcal{O}(M_s,\psi_s) \times \prod_{s \text{ even}} \mathrm{Sp}(M_s,\psi_s). \tag{2.2.2}$$

For t odd let $\hat{g}_t$ be a fixed element in the non-identity component of $\mathcal{O}(M_t,\psi_t)$. Above we have outlined how to find such an element. Let g_t be the preimage of $1 \times \cdots \times 1 \times \hat{g}_t \times 1 \times \cdots \times 1$ (where in the product decomposition (2.2.2) we set the identity for all $s \neq t$). Then

from the considerations above it follows that the component group of $Z_{\widehat{G}}(h,e,f)$ is the elementary abelian 2-group generated by g_t for all odd t.
Setting $\mathrm{SO}(V,\varphi) = \{g \in \mathcal{O}(V,\varphi) \mid \det(g) = 1\}$ we have that $\mathrm{SO}(V,\varphi)$ is connected and the map $\mathcal{O}(V,\varphi) \to \mathrm{Aut}(\mathfrak{so}(V,\varphi))$ restricts to a surjective map $\mathrm{SO}(V,\varphi) \to G$ (where, as usual, G is the identity component of $\mathrm{Aut}(\mathfrak{so}(V,\varphi))$). It follows that the component group of $\{g \in Z_{\widehat{G}}(h,e,f) \mid \det(g) = 1\}$ surjects to the component group of $Z_G(h,e,f)$.
This leads to an immediate algorithm for computing the component group of $Z_G(h,e,f)$ of an $\mathfrak{sl}_2$-triple in $\mathfrak{o}(V,\varphi)$, which we summarize in the following steps:

1. Let $\mathfrak{a}$ be the subalgebra spanned by h,e,f. Compute the direct sum decomposition of the $\mathfrak{a}$-module $V = V_1 \oplus \cdots \oplus V_m$.
2. In each V_i compute the unique (up to scalar multiples) $\hat{v}_i$ such that $f \cdot \hat{v}_i = 0$.
3. For $s \geq 1$ let M_s be the space spanned by the vectors $\hat{v}_i$ such that $\dim V_i = s$.
4. For each odd s compute the reflection $\hat{g}_s \in \mathcal{O}(M_s, \psi_s)$ with determinant -1.
5. Construct $g_s \in \mathcal{O}(V,\varphi)$ in the following way:
   (a) For each i with $\dim V_i = s$ compute a basis of V_i of the form $\hat{v}_i, e\cdot\hat{v}_i, \ldots, e^{s-1}\cdot\hat{v}_i$.
   (b) Set $g_s \cdot (e^k \hat{v}_i) = e^k \cdot (\hat{g}_s \cdot \hat{v}_i)$.
   (c) Let g_s be the identity on the spaces V_j with $\dim V_j \neq s$.
6. Let $\mathcal{H}$ be the group generated by all g_s, for s odd. Compute $\mathcal{H}_1 = \{g \in \mathcal{H} \mid \det(g) = 1\}$.
7. For $g \in \mathcal{H}_1$ compute the automorphism $\sigma_g \in \mathrm{Aut}(\mathfrak{o}(V,\varphi))$ given by $\sigma_g(x) = gxg^{-1}$.
8. Using the algorithm IsEltOf (with input $\mathfrak{z}_{\mathfrak{g}}(h,e,f)$ where $\mathfrak{g} = \mathfrak{o}(V,\varphi)$ and σ_g) remove the automorphisms σ_g that lie in the identity component of $Z_G(h,e,f)$. The remaining elements σ_g generate the component group.

We have implemented this algorithm in the language of the computational algebra system GAP4. This implementation allows us to perform challenging calculations in a shorter time. For example, for the simple Lie algebra of type B_{10} which is isomorphic to $\mathfrak{o}(21,\varphi)$ the program needed 182 seconds to compute the component groups for all 195 nilpotent orbits.

Example 2.2.1. Here we show how the algorithm works for a simple Lie algebra of type B_2. Let $V = \mathbb{C}^5$ and $\varphi_A(v,w) = v^\top Aw$, where $v,w \in \mathbb{C}^5$ and A is the 5×5 matrix such that $A_{i,5-i+1} = 1$ and $A_{i,j} = 0$ elsewhere. Observe that φ_A is symmetric, since A is a symmetric matrix, so that $\widehat{G} = \mathcal{G}(\mathbb{C}^5,\varphi_A) = O(\mathbb{C}^5,\varphi_A)$ and its Lie algebra is $\mathrm{g}(\mathbb{C}^5,\varphi_A) = \mathfrak{so}(\mathbb{C}^5,\varphi_A)$, which is indeed of type B_2. We fix the following $\mathfrak{sl}_2$-triple:

$$e = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix}.$$

At this point we can easily compute the action of the nilpotent element e on the canonical basis $\{v_1,\ldots,v_5\}$ of $\mathbb{C}^5$: $ev_5 = -v_3$, $ev_3 = v_1$, $ev_1 = 0$, $ev_4 = 0$ and $ev_2 = 0$. Under the action of the triple, $\mathbb{C}^5$ decomposes as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3$ and v_2, v_4, v_5 are the corresponding minimal vectors. We then have $M_1 = \langle v_2, v_4 \rangle$ and $M_3 = \langle v_5 \rangle$ and we fix the respective bases $\mathcal{B}_1 = \{v_2, v_4\}$ and $\mathcal{B}_3 = \{v_5\}$. At this point we need to compute the reflections:

since M_3 is 1-dimensional, $\hat{g}_3$ will just map $v_5 \to -v_5$; for $\hat{g}_1$ we start observing that $\varphi_A\big|_{M_1}(v_2, v_2) = \varphi_A\big|_{M_1}(v_4, v_4) = 0$. Then we set $w_1 := v_2 - v_4$ and $w_2 := v_2 + v_4$ and consider the map that sends $w_1 \to w_1$, $w_2 \to -w_2$. In this way, it is straightforward to get that $\hat{g}_1$ has to be the 2×2 matrix with zero diagonal and -1 elsewhere (it is just the matrix associated to the endomorphism with respect to $\mathcal{B}_1$, it can be computed by basic linear algebra techniques). At this point we can use the isomorphism defined by (2.2.1), (2.2.2).

$$Z_{\widehat{G}}(h,e,f) \xrightarrow{\sim} O\left(M_1, \varphi_A\big|_{M_1}\right) \times O\left(M_3, \varphi_A\big|_{M_3}\right).$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{h_1} \longleftarrow \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\hat{g}_1} \times \underbrace{\begin{pmatrix} 1 \end{pmatrix}}_{id_{M_3}}$$

$$\underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}}_{h_3} \longleftarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{id_{M_1}} \times \underbrace{\begin{pmatrix} -1 \end{pmatrix}}_{\hat{g}_3}$$

We get that $\langle h_1, h_3 \rangle = \{id, h_1, h_3, h_1h_3\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the component group of $Z_{\widehat{G}}(h,e,f)$. Following the final steps of our algorithm, it is then straightforward to see that $\{id, h_1h_3\} \simeq \mathbb{Z}/2\mathbb{Z}$ is the component group of $Z_G(h,e,f)$ (just observe that h_1h_3 is the only nontrivial element with determinant 1).

Remark 2.2.2. Suppose that φ is alternating, and hence the group we consider is $\mathrm{Sp}(V,\varphi)$. Then everything works in the same way. The main difference is that in this case ψ_s is symmetric if s is even and alternating if s is odd. So we get the same algorithm, except that the roles of s even/odd are interchanged.

2.3 Conjugacy of $\mathfrak{sl}_2$-triples

In this section we let $\mathfrak{g}$ be an arbitrary complex semisimple Lie algebra. We use the notation introduced in Sections 1.1, 1.3. In particular, we let $\mathfrak{h}$ denote a fixed Cartan subalgebra and denote the corresponding root system by Φ. We also fix a basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$.
Throughout this section we let (h_i, e_i, f_i), $i = 1, 2$, be two $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ where e_1, e_2 lie in the same G-orbit. As remarked in Section 1.4 this implies that there exist $\sigma \in G$ with $\sigma(h_1) = h_2$, $\sigma(e_1) = e_2$, $\sigma(f_1) = f_2$. In this section we describe computational methods to find such an automorphism σ. We remark that for our calculations we will only use the algorithm described in this section for a very specific case (see Remark 2.3.1). However, the algorithm is interesting per se and can have potential different applications (the computation of component groups for exceptional Lie algebras 2.4 can be tackled using this method, even though for that specific situation we could provide a more efficient

method). We note that it suffices to find σ with $\sigma(h_1) = h_2$ and $\sigma(e_1) = e_2$ because then automatically $\sigma(f_1) = f_2$ (Proposition 1.4.3). First we assume that $h_1, h_2 \in \mathfrak{h}$. The Weyl group W of Φ acts on $\mathfrak{h}$, as described in Section 1.4. This action can also be realized differently. Let h_i^*, $x_{\pm\alpha_i}$ for $1 \le i \le \ell$ be a canonical generating set of $\mathfrak{g}$. For $t \in \mathbb{C}^\times$ define $w_{\alpha_i}(t) = \exp(t\mathrm{ad}x_{\alpha_i})\exp(-t^{-1}\mathrm{ad}x_{-\alpha_i})\exp(t\mathrm{ad}x_{\alpha_i})$ and set $\dot{s}_{\alpha_i} = w_{\alpha_i}(1)$. Then from [25, Lemma 5.2.13] it follows that $\dot{s}_{\alpha_i}(h) = s^\vee_{\alpha_i}(h)$ for $h \in \mathfrak{h}$. We also remark that the action of W preserves the space $\mathfrak{h}_\mathbb{Q}$.

If $\bar{h} \in \mathfrak{h}_\mathbb{Q}$ then there is a unique $\hat{h}$ in the W-orbit of $\bar{h}$ such that $\alpha_i(\hat{h}) \ge 0$ for $1 \le i \le \ell$ (this is shown in the same way as the corresponding statement for weights, see [42, Lemma 13.2A]). This element can be found in the following way. We define a sequence $\bar{h}_1, \dots, \bar{h}_r$. Set $\bar{h}_1 = \bar{h}$ and for $k \ge 1$ we do the following. If $\alpha_i(\bar{h}_k) \ge 0$ for all i then we set $r = k$ and we stop. Otherwise we set $\bar{h}_{k+1} = s^\vee_{\alpha_i}(\bar{h}_k)$ where i is such that $\alpha_i(\bar{h}_k) < 0$. We also set $k := k+1$ and continue. A first observation is that this sequence is always finite. Indeed, we have a partial order on $\mathfrak{h}_\mathbb{Q}$ defined by $u \le u'$ if $u' - u = \sum a_i \bar{h}_i$ with $a_i \in \mathbb{Q}$ and $a_i \ge 0$ for all i. It follows that $\bar{h}_{k+1} > \bar{h}_k$ for $k \ge 1$. Since the Weyl group is finite and we compute an increasing series in the partial order, the sequence has to be finite. It is clear that $\hat{h} = \bar{h}_r$ is the element that we require.

Thanks to Proposition 1.4.11, h_1, h_2 are W-conjugate. By the above procedure we can compute $w_1, w_2 \in W$ such that $\hat{h}_i = w_i(h_i)$. By the uniqueness of the elements $\hat{h}_i$ it follows $\hat{h}_1 = \hat{h}_2$. Then $w_2^{-1}w_1(h_1) = h_2$. Because the action of the simple reflection s_{α_i} is induced by $\dot{s}_{\alpha_i}$ and every $w \in W$ is a product of simple reflections, we can thus find an automorphism $\tau \in G$ such that $\tau(h_1) = h_2$.

Set $h_1' = \tau(h_1) = h_2$, $e_1' = \tau(e_1)$, $f_1' = \tau(f_1)$. We want to find an element in G that stabilizes h_2 and maps e_1' to e_2. For that we consider the group $Z_G(h_2) = \{g \in G \mid g(h_2) = h_2\}$. This group is connected by [64, Corollary 3.11]. Its Lie algebra is $\mathfrak{z}_\mathfrak{g}(h_2) = \{x \in \mathfrak{g} \mid [x, h_2] = 0\}$. Let $\Psi = \{\alpha \in \Phi \mid \alpha(h_2) = 0\}$. Then Ψ is a root subsystem of Φ and

$$\mathfrak{z}_\mathfrak{g}(h_2) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha.$$

In particular, $\mathfrak{z}_\mathfrak{g}(h_2)$ is reductive. Now we consider the Bruhat decomposition of $Z_G(h_2)$. This is described as follows. Let W_0 be the Weyl group of Ψ. Let $\Pi = \{\beta_1, \dots, \beta_r\}$ be a fixed basis of simple roots of Ψ. Let $x_{\pm\beta_i}$ be the root vectors in a canonical generating set of (the semisimple part of) $\mathfrak{z}_\mathfrak{g}(h_2)$. For $1 \le i \le r$ define $\dot{s}_{\beta_i}$ as above. For $w \in W_0$ let $w = s_{\beta_{i_1}} \cdots s_{\beta_{i_k}}$ be a fixed reduced expression and define $\dot{w} = \dot{s}_{\beta_{i_1}} \cdots \dot{s}_{\beta_{i_k}}$. Let $\{\beta_1, \dots, \beta_m\}$ be the positive roots of Ψ (corresponding to Π). Let U be the subgroup of $Z_G(h_2)$ consisting of all

$$\exp(s_1 \mathrm{ad}x_{\beta_1}) \cdots \exp(s_m \mathrm{ad}x_{\beta_m}) \text{ for } s_1, \dots, s_m \in \mathbb{C}.$$

For $w \in W_0$ let Ψ_w be the set of all β_i, $1 \le i \le m$, such that $w(\beta_i)$ is a negative root. Write $\Psi_w = \{\beta_{i_1}, \dots, \beta_{i_n}\}$. Then U_w is defined to be the subgroup of $Z_G(h_2)$ consisting of all

$$\exp(u_1 \mathrm{ad}x_{\beta_{i_1}}) \cdots \exp(u_n \mathrm{ad}x_{\beta_{i_n}}) \text{ for } u_1, \dots, u_n \in \mathbb{C}.$$

For $1 \le i \le \ell$ and $t \in \mathbb{C}^\times$ define $h_i(t) = w_{\alpha_i}(t) w_{\alpha_i}(1)^{-1}$. Let $H = \{h_1(t_1) \cdots h_\ell(t_\ell) \mid t_i \in \mathbb{C}^\times\}$. Then H is a maximal torus in $Z_G(h_2)$ with Lie algebra $\mathfrak{h}$. We have that $Z_G(h_2)$ is the disjoint union of the sets

$$C_w = UH\dot{w}U_w$$

where w runs over W_0 (see e.g., [25, Proposition 5.1.10, Theorem 5.8.3]). Now in order to find our element of $Z_G(h_2)$ we run over W_0. For each $w \in W_0$ we write $g \in C_w$ as

$$g = \exp(s_1 \mathrm{ad}x_{\beta_1}) \cdots \exp(s_m \mathrm{ad}x_{\beta_m}) h_1(t_1) \cdots h_\ell(t_\ell) \dot{w} \exp(u_1 \mathrm{ad}x_{\beta_{i_1}}) \cdots \exp(u_n \mathrm{ad}x_{\beta_{i_n}})$$

where we take the elements s_i, u_j, t_k to be indeterminates of a polynomial ring. Then the equation $g(e_1) = e_2$ is tantamount to a set of polynomial equations in these indeterminates. We add auxiliary indeterminates $a_1, \dots, a_\ell$ and the polynomial equations $a_i t_i = 1$ for $1 \le i \le \ell$ that express the invertibility of the elements t_i. With the technique of Gröbner bases we check whether these have a solution. For at least one w a solution must exist because e_1' and e_2 are $Z_G(h_2)$-conjugate. So by composing τ with the element previously found we obtain a $\sigma \in G$ with $\sigma(h_1) = h_2$, $\sigma(e_1) = e_2$. But then necessarily also $\sigma(f_1) = f_2$.
The second case occurs when h_1, h_2 do not lie in $\mathfrak{h}$. Then we first find Cartan subalgebras $\mathfrak{h}_i$ containing h_i. We compute the root systems of $\mathfrak{g}$ with respect to the Cartan subalgebras $\mathfrak{h}_i$, and corresponding canonical sets of generators. Mapping these canonical generators to our fixed set of canonical generators yields automorphisms of $\mathfrak{g}$ mapping $\mathfrak{h}_i \to \mathfrak{h}$. By the algorithm IsEltOf we can check whether these are inner. If they are not we compose with a diagram automorphism that stabilizes $\mathfrak{h}$. This way we obtain elements of G mapping h_i into $\mathfrak{h}$. Subsequently we continue as above.

Remark 2.3.1. Let $\mathfrak{z}_\mathfrak{g}(h,e,f) = \{x \in \mathfrak{g} \mid [x,h] = [x,e] = [x,f] = 0\}$. This is the Lie algebra of $Z_G(h,e,f)$. So if $\mathfrak{z}_\mathfrak{g}(h,e,f) = 0$ then $Z_G(h,e,f)$ is a finite group, equalling the component group. In that case we can find the component group by applying the above algorithm with $(h_i, e_i, f_i) = (h,e,f)$ for $i = 1,2$ and finding all solutions of the resulting polynomial equations.
We can also consider the group $A = \mathrm{Aut}(\mathfrak{g})$ and try to find the component group of $Z_A(h,e,f)$. We have that $G = A^\circ$ so $\mathfrak{z}_\mathfrak{g}(h,e,f)$ is also the Lie algebra of $Z_A(h,e,f)$. Therefore if $\mathfrak{z}_\mathfrak{g}(h,e,f) = 0$ also $Z_A(h,e,f)$ is a finite group. We can find it by an extension of the previous method in the following way. By listing the diagram automorphisms of $\mathfrak{g}$ we can find $\sigma_1, \dots, \sigma_r \in A$ such that the components of A precisely are $G\sigma_i$ for $1 \le i \le r$. Fix an index i with $1 \le i \le r$. Write $h' = \sigma_i(h)$, $e' = \sigma_i(e)$, $f' = \sigma_i(f)$. By computing weighted Dynkin diagrams (see [25, §8.2]) we can check whether e, e' are G-conjugate. If they are not we discard σ_i. Otherwise by the above methods we compute the set of all $\phi \in G$ such that $\phi(h', e', f') = (h,e,f)$ (note that this is a finite set because the stabilizer of such a triple is finite by our assumption). Then $\phi\sigma_i$ where ϕ runs over the previously computed set, is the set of all elements of $G\sigma_i$ stabilizing (h,e,f).

Example 2.3.2. Let $\mathfrak{g}$ be the simple Lie algebra of type F_4 and $e \in \mathfrak{g}$ a representative of the nilpotent orbit with Bala-Carter label $F_4(a_3)$ (see [17, §8.4]). Let (h,e,f) be an $\mathfrak{sl}_2$-triple. Then $\mathfrak{z}_\mathfrak{g}(h,e,f) = 0$. The centralizer $\mathfrak{z}_\mathfrak{g}(h)$ is of type $A_1 + A_2 + T_1$ (where T_1 indicates a 1-dimensional center). So $\mathfrak{z}_\mathfrak{g}(h)$ has four positive roots and $|W_0| = 12$. Hence the method of this section yields 12 systems of polynomial equations in 16 indeterminates. It took Magma 552 seconds to compute the Gröbner bases of all systems with respect to a lexicographical order. From these Gröbner bases the solutions are readily determined. There are 24 solutions and the stabilizer of (h,e,f) is isomorphic to S_4.

Example 2.3.3. Let $\mathfrak{g}$ be the simple Lie algebra of type E_8 and $e \in \mathfrak{g}$ a representative of the nilpotent orbit with Bala-Carter label $E_8(a_7)$. Let (h,e,f) be an $\mathfrak{sl}_2$-triple. Then $\mathfrak{z}_\mathfrak{g}(h,e,f) = 0$. The centralizer $\mathfrak{z}_\mathfrak{g}(h)$ is of type $A_3 + A_4 + T_1$ (where T_1 indicates a 1-dimensional center). So $\mathfrak{z}_\mathfrak{g}(h)$ has 16 positive roots and $|W_0| = 2880$. Hence the method of this section yields 2880 systems of polynomial equations in 48 indeterminates. With two exceptions Magma could not compute the Gröbner bases of these polynomial systems. For this reason we resort to different methods.
The first of these is based on the operations of factorization and reduction. For a set P of multivariate polynomials denote the set of their common zeros in $\mathbb{C}$ by $Z(P)$. By the division algorithm described in Section 1.2, it is straightforward to compute a set P' such that no leading monomial of an element of P' divides a leading monomial of another element of P' and such that P, P' generate the same ideal. We say that P' is a *reduction* of P. We have that $Z(P) = Z(P')$.

Let Q be a set of polynomials; we want to determine $Z(Q)$. Throughout our procedure we let A be a set of sets of polynomials such that $Z(Q)$ is the union of all $Z(Q')$ for $Q' \in A$. Let $P \in A$ and suppose that there is a polynomial $f \in P$ with a nontrivial factorization $f = f_1^{e_1} \cdots f_k^{e_k}$ with f_i irreducible. Then we replace P in A with reductions of the sets $P_i = (P \setminus \{f\}) \cup \{f_i\}$. If an element of A contains a nonzero constant then we discard it. We continue this process until no further factorizations are possible. This way we can decide for many of our 2880 polynomial systems that they have no common zeros. More precisely, after performing this procedure we are left with 507 sets of polynomials to be checked.
Another idea is to look for elements of order 2, since $Z_G(e)$ is isomorphic to S_5 (see [60]) which is generated by elements of order 2. We construct elements of the form

$$A = \exp(s_1 \mathrm{ad} x_{\beta_1}) \cdots \exp(s_m \mathrm{ad} x_{\beta_m}) h_1(t_1) \cdots h_\ell(t_\ell) \dot{w} \exp(u_1 \mathrm{ad} x_{\beta_{i_1}}) \cdots \exp(u_n \mathrm{ad} x_{\beta_{i_n}}).$$

The inverse of such an element is

$$A^{-1} = \exp(-u_n \mathrm{ad} x_{\beta_{i_n}}) \cdots \exp(-u_1 \mathrm{ad} x_{\beta_{i_1}}) \dot{w}^{-1} h_1(t_1^{-1}) \cdots h_\ell(t_\ell^{-1}) \\ \exp(-s_m \mathrm{ad} x_{\beta_m}) \cdots \exp(-s_1 \mathrm{ad} x_{\beta_1}).$$

The condition that A be of order 2 is the same as $A = A^{-1}$ yielding a set of polynomial equations in the same indeterminates for each $w \in W_0$. To these sets we applied the factorization procedure as outlined above. We then added these sets to the original sets for the same $w \in W$. Like this we found one polynomial set whose solutions yielded six elements of order 2. These elements generate a group of order 120. Since in this case the stabilizer is known to be of order 120, we are done.

2.4 Using double centralizers

Let $\mathfrak{g}$ be a complex simple Lie algebra of exceptional type. In this section we outline a computational method for determining the component group of $Z_G(h,e,f)$. For this we use the centralizer $\mathfrak{c}_1 = \mathfrak{z}_{\mathfrak{g}}(h,e,f)$ and the double centralizer $\mathfrak{c}_2 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1)$. The sum $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2$ is reductive in $\mathfrak{g}$ (see Lemma 2.4.1 below). We write $\mathfrak{g} = \mathfrak{c} \oplus V$, where V is a $\mathfrak{c}$-module (see below for its definition). The method is based on the following facts that we will prove later: for the restriction of an automorphism $\sigma \in Z_G(h,e,f)$ to $\mathfrak{c}$ there are a finite number of possibilities (modulo $Z_G(h,e,f)^\circ$); a second observation is that the $\mathfrak{c}$-module V is multiplicity free. This forces an automorphism σ to permute the simple summands of V.
Throughout we let $\mathfrak{g}$ be a simple Lie algebra of exceptional type. Secondly, we assume that e is such that the component group is nontrivial. It is known for which nilpotent orbits that happens and those are the cases of interest (see [60]). Section 2.5 has tables listing data on the algebras $\mathfrak{c}_1$, $\mathfrak{c}_2$ and the module V used to underpin the method described in this section. However, we remark that this method also works for the cases where the component group is trivial, see Remark 2.4.9.
Let κ denote the Killing form of $\mathfrak{g}$ (see Section 1.1). In that section we also define reductive Lie algebras. Here we bring a more specific result that we need in the following.

Lemma 2.4.1. *Let $\mathfrak{u}$ be reductive in $\mathfrak{g}$.*

- $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \oplus \mathfrak{z}(\mathfrak{u})$ *where* $\mathfrak{z}(\mathfrak{u})$ *is the center of* $\mathfrak{u}$*; moreover, for* $x \in \mathfrak{z}(\mathfrak{u})$ *the map* $\mathrm{ad}_{\mathfrak{g}} x$ *is semisimple.*

- *Let $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{u}) = \{x \in \mathfrak{g} \mid [x,y] = 0$ for all $y \in \mathfrak{u}\}$ be the centralizer of $\mathfrak{u}$ in $\mathfrak{g}$. Then the restriction of κ to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{u})$ is non-degenerate and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{u})$ is reductive in $\mathfrak{g}$.*

Proof. For the decomposition of $\mathfrak{u}$ see Proposition 1.1.2. The fact that $\mathrm{ad}x$ is semisimple for $x \in \mathfrak{z}(\mathfrak{u})$ follows from [44, Chapter III, Theorem 10]. The second part is shown in the same way as the first part of the proof of [25, Lemma 8.3.9]. □

Now fix a nilpotent $e \in \mathfrak{g}$ lying in the $\mathfrak{sl}_2$-triple (h,e,f). Let $\mathfrak{c}_1 = \mathfrak{z}_{\mathfrak{g}}(h,e,f) = \{x \in \mathfrak{g} \mid [x,h] = [x,e] = [x,f] = 0\}$. Here we suppose that $\dim \mathfrak{c}_1 > 0$. If $\dim \mathfrak{c}_1 = 0$ then we can determine $Z_G(h,e,f)$ as in Remark 2.3.1. By the previous lemma $\mathfrak{c}_1$ is reductive in $\mathfrak{g}$. Set $\mathfrak{c}_2 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1)$. Again by the previous lemma $\mathfrak{c}_2$ is reductive in $\mathfrak{g}$. For $i = 1,2$ set $\mathfrak{c}_i' = [\mathfrak{c}_i, \mathfrak{c}_i]$ which is the semisimple part of $\mathfrak{c}_i$, so that $\mathfrak{c}_i = \mathfrak{c}_i' \oplus \mathfrak{z}(\mathfrak{c}_i)$.

Lemma 2.4.2. *We have $h,e,f \in \mathfrak{c}_2$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_2) = \mathfrak{c}_1$ and $\mathfrak{c}_1 \cap \mathfrak{c}_2 = \mathfrak{z}(\mathfrak{c}_1) = \mathfrak{z}(\mathfrak{c}_2)$.*

Proof. The first statement is obvious. It is equally obvious that $\mathfrak{c}_1 \subset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_2)$. On the other hand, if $x \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_2)$ then $[x,h] = [x,e] = [x,f] = 0$ as $h,e,f \in \mathfrak{c}_2$, implying $x \in \mathfrak{c}_1$. We conclude that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_2) = \mathfrak{c}_1$. We have

$$\mathfrak{c}_1 \cap \mathfrak{c}_2 = \{x \in \mathfrak{c}_1 \mid [x, \mathfrak{c}_1] = 0\} = \mathfrak{z}(\mathfrak{c}_1).$$

Similarly

$$\mathfrak{c}_1 \cap \mathfrak{c}_2 = \{x \in \mathfrak{c}_2 \mid [x, \mathfrak{c}_2] = 0\} = \mathfrak{z}(\mathfrak{c}_2).$$

□

In the sequel we set $\mathfrak{t} = \mathfrak{z}(\mathfrak{c}_1) = \mathfrak{z}(\mathfrak{c}_2)$. The Lie algebra of the stabilizer $Z_G(h,e,f) = \{\sigma \in G \mid \sigma(h) = h, \sigma(e) = e, \sigma(f) = f\}$ is $\mathrm{ad}\mathfrak{c}_1 = \{\mathrm{ad}x \mid x \in \mathfrak{c}_1\}$. Hence $\mathrm{ad}\mathfrak{t} = \{\mathrm{ad}x \mid x \in \mathfrak{t}\}$ is the Lie algebra of a central torus T of $Z_G(h,e,f)$.

Lemma 2.4.3. *Let $\sigma \in Z_G(h,e,f)$. Then σ stabilizes $\mathfrak{c}_1'$, $\mathfrak{c}_2'$ and $\mathfrak{t}$.*

Proof. Since the Lie algebra of $Z_G(h,e,f)$ is $\mathrm{ad}\mathfrak{c}_1$ we have $\sigma(\mathrm{ad}\mathfrak{c}_1)\sigma^{-1} = \mathrm{ad}\mathfrak{c}_1$. But $\sigma(\mathrm{ad}x)\sigma^{-1} = \mathrm{ad}\sigma(x)$. As the adjoint representation of $\mathfrak{g}$ is faithful it follows for $x \in \mathfrak{c}_1$ that $\sigma(x) \in \mathfrak{c}_1$. For $x \in \mathfrak{c}_1$ and $y \in \mathfrak{c}_2$ we have $[x, \sigma(y)] = \sigma([\sigma^{-1}(x), y]) = 0$ as $\sigma^{-1}(x) \in \mathfrak{c}_1$ by the first part. It follows that $\sigma(y) \in \mathfrak{c}_2$. So σ stabilizes $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Because σ is an automorphism it stabilizes $\mathfrak{c}_i'$ and $\mathfrak{t} = \mathfrak{z}(\mathfrak{c}_i)$ as well. □

Let $z \in \mathfrak{t}$, $x,y \in \mathfrak{c}_1$. Then $\kappa([x,y],z) = \kappa(x,[y,z]) = 0$ (for the first equality see [42, §4.3]). It follows that $\kappa(\mathfrak{c}_1', \mathfrak{t}) = 0$. Since the restriction of κ to $\mathfrak{c}_1$ is non-degenerate (Lemma 2.4.1) also its restrictions to $\mathfrak{c}_1'$, $\mathfrak{t}$ are non-degenerate. Similarly we see that the restriction of κ to $\mathfrak{c}_2'$ is non-degenerate. Set $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2 = \mathfrak{c}_1' \oplus \mathfrak{c}_2' \oplus \mathfrak{t}$; then $\mathfrak{c}$ is reductive in $\mathfrak{g}$ and the restriction of κ to $\mathfrak{c}$ is non-degenerate.
Write $\mathfrak{c}^\perp = \{x \in \mathfrak{g} \mid \kappa(x,y) = 0$ for all $y \in \mathfrak{c}\}$. As κ is non-degenerate on $\mathfrak{c}$ we have $\mathfrak{c} \cap \mathfrak{c}^\perp = 0$ so that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{c}^\perp$.

Lemma 2.4.4. *We have $[\mathfrak{c}, \mathfrak{c}^\perp] \subset \mathfrak{c}^\perp$.*

Proof. Let $x,y \in \mathfrak{c}$ and $z \in \mathfrak{c}^\perp$. Then $\kappa(x,[y,z]) = \kappa([x,y],z) = 0$. Hence $[y,z] \in \mathfrak{c}^\perp$. □

Let $V = \mathfrak{c}^\perp$ which is a $\mathfrak{c}$-module by Lemma 2.4.4. For $x \in \mathfrak{c}$ and $v \in V$ we write $x \cdot v = [x,v]$. The next lemma is proved by explicit computation, going through the list of nilpotent orbits of the Lie algebras of exceptional type and for each case calculating V and its decomposition as a $\mathfrak{c}$-module. We have performed these computations in the computer algebra system GAP4, using the package SLA. For the cases where the stabilizer has a non-trivial component group we have listed the highest weights of the irreducible summands in the tables in Section 2.5.

Lemma 2.4.5. *Let $\mathfrak{g}$ be of exceptional type. Then V is multiplicity free.*

Next we show how to determine a finite set $\mathcal{S}$ of automorphisms of $\mathfrak{g}$, stabilizing $\mathfrak{c}_1'$, $\mathfrak{t}$, $\mathfrak{c}_2'$ and such that each component of $Z_G(h,e,f)$ has an element lying in $\mathcal{S}$.
First consider $\mathfrak{c}_1'$ and write $A_1 = \mathrm{Aut}(\mathfrak{c}_1')$. This is a semisimple Lie algebra, so A_1 is described as in (1.3.2). In particular, we fix diagram automorphisms $\theta_1,\dots,\theta_p$ of $\mathfrak{c}_1'$ such that $\theta_i A_1^\circ$ for $1 \le i \le p$ are the connected components of A_1.

Lemma 2.4.6. *Each component of $Z_G(h,e,f)$ has an element whose restriction to $\mathfrak{c}_1'$ is equal to one of the θ_i.*

Proof. The identity component A_1° has Lie algebra $\mathrm{ad}_{\mathfrak{c}_1'}\mathfrak{c}_1' = \{\mathrm{ad}_{\mathfrak{c}_1'}x \mid x \in \mathfrak{c}_1'\}$. The identity component $Z_G(h,e,f)^\circ$ has Lie algebra $\mathrm{ad}\mathfrak{c}_1 = \{\mathrm{ad}x \mid x \in \mathfrak{c}_1\}$. Now $\mathrm{ad}_{\mathfrak{c}_1'}\mathfrak{c}_1'$ is isomorphic to the subalgebra $\mathrm{ad}\mathfrak{c}_1' = \{\mathrm{ad}x \mid x \in \mathfrak{c}_1'\}$ of $\mathrm{ad}\mathfrak{c}_1$. Let C_1' denote the connected algebraic subgroup of $Z_G(h,e,f)^\circ$ whose Lie algebra is $\mathrm{ad}\mathfrak{c}_1'$. We have a morphism $C_1' \to A_1^\circ$, mapping $\phi \in C_1'$ to its restriction to $\mathfrak{c}_1'$ (note that by Lemma 2.4.3 ϕ stabilizes $\mathfrak{c}_1'$). We have that A_1° is generated by elements $\exp(\mathrm{ad}_{\mathfrak{c}_1'}x)$ where $x \in \mathfrak{c}_1'$ is nilpotent. Such an element is the restriction of $\exp(\mathrm{ad}x) \in C_1'$. Hence the map $C_1' \to A_1^\circ$ is surjective. Let $\sigma \in Z_G(h,e,f)$ then the restriction σ' of σ to $\mathfrak{c}_1'$ lies in A_1. Hence there is a $\phi' \in A_1^\circ$ such that $\sigma'\phi'$ is equal to θ_i for an i with $1 \le i \le p$. Let $\phi \in C_1'$ map to ϕ'. Then the restriction of $\sigma\phi$ to $\mathfrak{c}_1'$ is equal to θ_i. Furthermore, $\sigma\phi$ lies in the same component of $Z_G(h,e,f)$ as σ. □

We want to compute one element of each component of $Z_G(h,e,f)$. Therefore we may assume that the restriction of such an element to $\mathfrak{c}_1'$ is equal to one of the θ_i.
Note that $h,e,f \in \mathfrak{c}_2'$. Furthermore, $\mathfrak{z}_\mathfrak{g}(h,e,f) = \mathfrak{c}_1$, so that $\mathfrak{z}_{\mathfrak{c}_2'}(h,e,f) = 0$. Let $A_2 = \mathrm{Aut}(\mathfrak{c}_2')$. Let $Z_{A_2}(h,e,f)$ denote the stabilizer of h,e,f in A_2. The Lie algebra of $Z_{A_2}(h,e,f)$ is $\{\mathrm{ad}_{\mathfrak{c}_2'}x \mid x \in \mathfrak{z}_\mathfrak{g}(h,e,f) \cap \mathfrak{c}_2'\}$. Hence this Lie algebra is trivial, and it follows that $Z_{A_2}(h,e,f)$ is finite. Therefore we can determine $Z_{A_2}(h,e,f)$ as seen in Remark 2.3.1. Since the restriction of an element of $Z_G(h,e,f)$ to $\mathfrak{c}_2'$ lies in $Z_{A_2}(h,e,f)$ we can determine a finite set of automorphisms $\eta_1,\dots,\eta_q$ of $\mathfrak{c}_2'$ such that the restriction of an automorphism $\sigma \in Z_G(h,e,f)$ to $\mathfrak{c}_2'$ is equal to one of the η_i.
Let $h_1,\dots,h_s$, $x_1,\dots,x_s$, $y_1,\dots,y_s$ be a canonical generating set of the semisimple subalgebra $\mathfrak{c}' = \mathfrak{c}_1' \oplus \mathfrak{c}_2'$. Here the elements h_i span a fixed Cartan subalgebra of $\mathfrak{c}'$, the x_i are root vectors of $\mathfrak{c}'$ corresponding to the simple positive roots, and the y_i are root vectors corresponding to the simple negative roots. As seen in Section 1.7 we can compute a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_m$, where each V_i is irreducible. Moreover, because of Lemma 2.4.5 this decomposition is uniquely determined. For each V_i we fix a highest weight vector v_i.
Write $d = \dim \mathfrak{t}$ (from the tables in Section 2.5 we see that $d \le 2$ if the component group of $Z_G(h,e,f)$ is nontrivial). Fix a basis $h_{s+1},\dots,h_{s+d}$ of $\mathfrak{t}$. From the tables in Section 2.5 we can read off the values ν_{ij} with $h_i \cdot v_j = \nu_{ij}v_j$, $1 \le i \le s+d$, $1 \le j \le m$.
Fix two automorphisms θ_u and η_v. Next we study properties of the automorphism $\sigma \in G$ whose restrictions to $\mathfrak{c}_1'$, $\mathfrak{c}_2'$ are θ_u, η_v respectively. In this case, we say that σ is a (u,v)-*extension.* We remark that for given u,v the set of all (u,v)-extensions could be empty.
First we show that we can determine the restriction to $\mathfrak{t}$ of any (u,v)-extension. Note that any (u,v)-extension automatically lies in $Z_G(h,e,f)$.

Remark 2.4.7. It may happen that $\dim \mathfrak{c}_1' = 0$ (for an example see Example 2.4.10). In that case there is no θ_u. However, everything that follows works unchanged and instead of (u,v)-extensions we use the term (v)-extensions.

Fix u and v and let $\sigma \in G$ be a (u,v)-extension. Set $\bar{h}_i = \sigma(h_i)$, $\bar{x}_i = \sigma(x_i)$ and $\bar{y}_i = \sigma(y_i)$. Except $\bar{h}_i$ for $s+1 \le i \le s+d$ these elements can be computed from the knowledge of u,v because we know the restrictions of σ to $\mathfrak{c}_1'$, $\mathfrak{c}_2'$. For $1 \le j \le m$ let $\bar{v}_j$ be a nonzero element

of V_j with $\bar{x}_i \cdot \bar{v}_j = 0$ for $1 \le i \le s$. Then $\bar{v}_j$ is a highest weight vector of V_j with respect to the new canonical generating set consisting of the elements $\bar{h}_i$, $\bar{x}_i$, $\bar{y}_i$. Hence, up to scalar multiples, it is uniquely determined. Since the decomposition of V is unique, we must have that $\sigma(v_j)$ is a nonzero scalar multiple of some $\bar{v}_l$. We have $h_i \cdot v_j = \nu_{ij} v_j$ for $1 \le i \le s$. Applying σ we see that $\bar{h}_i \cdot \sigma(v_j) = \nu_{ij}\sigma(v_j)$ for $1 \le i \le s$. Let P_{uv} be the set of permutations π of $\{1,\dots,m\}$ such that $\bar{h}_i \cdot \bar{v}_{\pi(j)} = \nu_{ij}\bar{v}_{\pi(j)}$ for $1 \le i \le s$ and $1 \le j \le m$. There can be more than one such permutation, but if there is no such permutation then we can immediately conclude that there are no (u,v)-extensions. On the other hand if there is a (u,v)-extension σ then $\sigma(v_j) = \lambda_j \bar{v}_{\pi(j)}$ for some $\pi \in P_{uv}$ and nonzero $\lambda_j \in \mathbb{C}$.
Now suppose that $P_{uv} \ne \varnothing$ and let $\pi \in P_{uv}$. Let σ be a (u,v)-extension with $\sigma(v_j) = \lambda_j \bar{v}_{\pi(j)}$ for $1 \le j \le m$. We call such an automorphism σ a *(u,v,π)-extension*. We write $\bar{h}_{s+i} = \sigma(h_{s+i}) = \sum_{k=1}^{d} a_{ik} h_{s+k}$. The elements of $\mathfrak{t}$ are central in $\mathfrak{c}$ and therefore they act as scalar multiplication on irreducible $\mathfrak{c}$-modules (this is Schur's lemma, cf. [42, §6.1]). It follows that $h_{s+i} \cdot \bar{v}_j = \nu_{s+i,j}\bar{v}_j$ for $1 \le i \le d$, $1 \le j \le m$. Now $h_{s+i} \cdot v_j = \nu_{s+i,j} v_j$ upon application of σ implies that $\bar{h}_{s+i} \cdot \bar{v}_j = \nu_{s+i,\pi^{-1}(j)}\bar{v}_j$ for $1 \le j \le m$. We see that $\sum_{k=1}^{d} a_{ik}\nu_{s+k,j} = \nu_{s+i,\pi^{-1}(j)}$ for $1 \le i \le d$ and $1 \le j \le m$. The values of the $\nu_{s+i,j}$ can be read off from the tables in Section 2.5. From that we see that the vectors $(\nu_{s+1,j},\dots,\nu_{s+d,j})$ for $1 \le j \le m$ span $\mathbb{C}^d$. Hence either the equations give a unique solution for the coefficients a_{ik}, or no solution. In the latter case we conclude that there are no (u,v,π)-extensions. In the former case we can uniquely compute the restriction of any (u,v,π)-extension to $\mathfrak{t}$.
Let σ be a (u,v,π)-extension. Let $\phi \in T$; then the restriction of ϕ to $\mathfrak{c}$ is the identity. Because V is multiplicity free we have that $\phi(v_j)$ is a scalar multiple of v_j for $1 \le j \le m$. Hence $\sigma\phi$ also is a (u,v,π)-extension. We see that if there are (u,v,π)-extensions and $\dim T \ge 1$ then there are infinitely many such extensions. Our next goal is to determine a finite set of (u,v,π)-extensions such that every (u,v,π)-extension is equal, modulo $Z_G(h,e,f)^\circ$, to an element of this set.
Now we consider the group

$$H = \{\sigma \in G \mid \sigma(x) = x \text{ for all } x \in \mathfrak{c}\}.$$

This is an algebraic subgroup of G with Lie algebra $\mathrm{ad}\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$. But the latter is $\mathrm{ad}\mathfrak{t}$ by Lemma 2.4.2. It follows that $H^\circ = T$. If $\sigma \in H$ then $\sigma(v_i)$ is a highest weight vector with the same weight as v_i. So because V is multiplicity free it follows that $\sigma(v_i) = \chi_i(\sigma)v_i$, where $\chi_i : H \to \mathbb{C}^\times$ is a character of H. The differential $\mathrm{d}\chi_i : \mathfrak{t} \to \mathbb{C}$ is a linear map; it can be read off from the tables in Section 2.5. From those tables it is also easily seen that in all cases there are d of such characters $\chi_{i_1},\dots,\chi_{i_d}$ such that the intersection of the kernels of the differentials $\mathrm{d}\chi_{i_l}$, $1 \le l \le d$ is 0 (if $d = 1$ this just means that there is an index i_1 such that $\mathrm{d}\chi_{i_1}$ is nonzero). This implies that $K = \{\sigma \in H \mid \chi_{i_l}(\sigma) = 1 \text{ for } 1 \le l \le d\}$ is a finite group.

Lemma 2.4.8. *Let σ be a (u,v,π)-extension. Then there is a $\phi \in T$ such that $\sigma\phi(v_{i_l}) = \bar{v}_{\pi(i_l)}$ for $1 \le l \le d$.*

Proof. Define $\xi : T \to (\mathbb{C}^\times)^d$ by $\xi(h) = (\chi_{i_1}(h),\dots,\chi_{i_d}(h))$. Since the intersections of the kernels of the maps $\mathrm{d}\chi_i$ is zero, we have that $\mathrm{d}\xi : \mathfrak{t} \to \mathbb{C}^d$ is bijective. It follows that $\xi(T)$ is an algebraic subgroup of $(\mathbb{C}^\times)^d$ whose Lie algebra is $\mathbb{C}^d$. In particular the dimension of $\xi(T)$ is d, which implies that $\xi(T) = (\mathbb{C}^\times)^d$.
We have that $\sigma(v_j) = \lambda_j \bar{v}_{\pi(j)}$ for $1 \le j \le m$, where the $\lambda_j \in \mathbb{C}$ are nonzero. Let $\phi \in T$ be such that $\xi(\phi) = (\lambda_{i_1}^{-1},\dots,\lambda_{i_d}^{-1})$. Then $\sigma\phi$ has the desired property. ☐

Let $\sigma_1,\sigma_2 \in G$ be two (u,v,π)-extensions and write $\sigma_i(v_j) = \lambda_j^i \bar{v}_{\pi(j)}$ for $1 \le j \le m$. Suppose that $\lambda_{i_l}^r = 1$ for $1 \le l \le d$ and $r = 1,2$. Then $\sigma_2^{-1}\sigma_1$ lies in H and $\chi_{i_l}(\sigma_2^{-1}\sigma_1) = 1$

for $1 \le l \le d$. Hence $\sigma_2^{-1}\sigma_1$ lies in K which is a finite group. We conclude that there is a finite number of (u, v, π)-extensions σ with $\sigma(v_{i_l}) = \bar{v}_{\pi(i_l)}$ for $1 \le l \le d$.
We remark that the sets

$$\{h_1, \dots, h_{s+d}, x_1, \dots, x_s, y_1, \dots, y_s, v_1, \dots, v_m\},$$
$$\{\bar{h}_1, \dots, \bar{h}_{s+d}, \bar{x}_1, \dots, \bar{x}_s, \bar{y}_1, \dots, \bar{y}_s, \bar{v}_1, \dots, \bar{v}_m\}$$

generate $\mathfrak{g}$. So an automorphism $\sigma \in G$ with $\sigma(h_i) = \bar{h}_i$, $\sigma(x_i) = \bar{x}_i$, $\sigma(y_i) = \bar{y}_i$, $\sigma(v_j) = \lambda_j \bar{v}_j$ is uniquely determined. Furthermore, we can compute the matrix of σ with respect to a given basis of $\mathfrak{g}$. This matrix has entries that are polynomials in the λ_j. We compute those polynomials. Then the requirement that σ be an automorphism yields polynomial equations in the λ_j. We compute these equations, where we now view the λ_i as indeterminates. Furthermore we set $\lambda_j = 1$ (that is, we add the equations $\lambda_j - 1 = 0$) for $j \in \{i_1, \dots, i_d\}$. Finally we add equations of the form $\lambda_i \hat{\lambda}_i = 1$, where the $\hat{\lambda}_j$ are auxiliary indeterminates (these equations make sure that the value of λ_j is nonzero). As seen above these equations have a finite number of solutions. We determine these solutions, using the technique based on computing a Gröbner basis. Doing this for all u, v, π we obtain a finite set of automorphisms of G such that every component has one element in the set. By the algorithm IsEltOf we get rid of pairs of automorphisms that are equal modulo $Z_G(h, e, f)^\circ$.

Remark 2.4.9. Let e be a nilpotent element of $\mathfrak{g}$ lying in an $\mathfrak{sl}_2$-triple (h, e, f) such that the component group of $Z_G(h, e, f)$ is trivial. We consider the subalgebras $\mathfrak{c}_1'$, $\mathfrak{t}$, $\mathfrak{c}_2'$. By direct computation we have verified the following statements. If $\mathfrak{g}$ is of type G_2 or F_4 then $\dim \mathfrak{t} = 0$. If $\mathfrak{g}$ is of type E_6 then $\dim \mathfrak{t} \le 1$. Furthermore, in all cases where $\dim \mathfrak{t} = 1$ we can find a character χ_{i_1} such that $d\chi_{i_1}$ is nonzero (notation as above). If $\mathfrak{g}$ is of type E_7 or E_8 then $\dim \mathfrak{t} = 0$. This means that the method outlined in this section also works when the component group is trivial.

Example 2.4.10. Here we look at the simple Lie algebra $\mathfrak{g}$ of type E_6 and the nilpotent orbit with label $D_4(a_1)$. In this case $\mathfrak{c}_1 = \mathfrak{t}$ which is of dimension 2. Hence $\mathfrak{c}_1' = 0$. Furthermore $\mathfrak{c}_2'$ is semisimple of type D_4. For the Dynkin diagram of $\mathfrak{c}_2'$ we use the following enumeration of the vertices:

$$\begin{array}{ccccc} & & 4 & & \\ & & | & & \\ 1 & — & 2 & — & 3 \end{array}$$

In GAP4 we have explicitly determined the $\mathfrak{c}$-module V and its decomposition. It turns out that V splits as a direct sum of six irreducible modules with highest weights

$$\begin{array}{lccccccclccccccc}
 & & & 0 & & & & & & & & 0 & & & & \\
 & & & | & & & & & & & & | & & & & \\
V_1: & 1 & — & 0 & — & 0 & (-2,0) & & V_2: & 1 & — & 0 & — & 0 & (2,0) & \\
 & & & 1 & & & & & & & & 1 & & & & \\
 & & & | & & & & & & & & | & & & & \\
V_3: & 0 & — & 0 & — & 0 & (-1,-3) & & V_4: & 0 & — & 0 & — & 0 & (1,3) & \\
 & & & 0 & & & & & & & & 0 & & & & \\
 & & & | & & & & & & & & | & & & & \\
V_5: & 0 & — & 0 & — & 1 & (-1,3) & & V_6: & 0 & — & 0 & — & 1 & (1,-3) &
\end{array}$$

Here the eigenvalues of the Cartan elements of $\mathfrak{c}_2'$ are shown on the Dynkin diagram of $\mathfrak{c}_2'$. This is followed by a vector of length 2 containing the eigenvalues of the elements of a fixed basis of $\mathfrak{t}$ on the highest weight vector.
With the method indicated in Remark 2.3.1 it is straightforward to compute $Z_{A_2}(h,e,f)$ where $A_2 = \mathrm{Aut}(\mathfrak{c}_2')$. The symmetry group of the Dynkin diagram is exactly the group S_X permuting the vertices in $X = \{1,3,4\}$. For each diagram automorphism σ_τ of $\mathfrak{c}_2'$, with $\tau \in S_X$, there is exactly one inner automorphism ϕ such that $\phi(\sigma_\tau(h), \sigma_\tau(e), \sigma_\tau(f)) = (h,e,f)$. Hence $Z_{A_2}(h,e,f)$ is of order 6 and is isomorphic to S_X.
We consider $\tau = (1,3,4) \in S_X$ and let η_1 be the corresponding element of $Z_{A_2}(h,e,f)$. We want to determine the possible extensions of η_1 to automorphisms of $\mathfrak{g}$. That is, we want to determine the (1)-extensions (cf. Remark 2.4.7). We fix canonical generators h_i, x_i, y_i $1 \le i \le 4$ of $\mathfrak{c}_2'$ and let $\bar{h}_i = \eta_1(h_i)$, $\bar{x}_i = \eta_1(x_i)$, $\bar{y}_i = \eta_1(y_i)$. Let v_j for $1 \le j \le 6$ denote a fixed highest weight vector of V_j and let $\bar{v}_j$ be a nonzero element of V_j such that $\bar{x}_i \cdot \bar{v}_j = 0$ for $1 \le i \le 4$. If $h_i \cdot v_j = \nu_{i,j} v_j$ then $\bar{h}_i \cdot \bar{v}_j = \nu_{\tau^{-1}(i),j} \bar{v}_j$. It follows that if $\sigma \in G$ extends η_1 then $\sigma(v_j) = \lambda_i \bar{v}_{\pi(j)}$ where π is a permutation of $\{1,\dots,6\}$ mapping

$$
\begin{aligned}
1 &\mapsto 5 \text{ or } 6 \\
2 &\mapsto 5 \text{ or } 6 \\
3 &\mapsto 1 \text{ or } 2 \\
4 &\mapsto 1 \text{ or } 2 \\
5 &\mapsto 3 \text{ or } 4 \\
6 &\mapsto 3 \text{ or } 4
\end{aligned}
$$

So we have eight possible permutations.
Let $\sigma \in G$ extend τ and suppose that $\sigma(v_1) = \lambda_1 \bar{v}_5$, $\sigma(v_3) = \lambda_3 \bar{v}_1$ and $\sigma(v_5) = \lambda_5 \bar{v}_4$. Let h_5, h_6 be the basis elements of $\mathfrak{t}$ with respect to which the above weights have been computed. As remarked above, Schur's lemma implies that h_5, h_6 have the same eigenvalues on $\bar{v}_i$ as on v_i. Write $\sigma(h_5) = ah_5 + bh_6$, $\sigma(h_6) = ch_5 + dh_6$. Applying σ to the equality $h_5 \cdot v_1 = -2v_1$ we see that $(ah_5 + bh_6) \cdot \bar{v}_5 = -2\bar{v}_5$, implying $-a + 3b = -2$. Applying σ to $h_5 \cdot v_3 = -v_3$ we obtain $-2a = -1$ and we see that $a = \frac{1}{2}$, $b = -\frac{1}{2}$. But applying σ to $h_5 \cdot v_5 = -v_5$ we get $-a - 3b = -1$ and we have a contradiction. The conclusion is that this permutation does not work.
We apply the above procedure to all eight permutations. It turns out that only one of them yields a solution for a, b, c, d namely

$$\pi: \ 1 \mapsto 6,\ 2 \mapsto 5,\ 3 \mapsto 2,\ 4 \mapsto 1,\ 5 \mapsto 3,\ 6 \mapsto 4.$$

In this case $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $c = -\frac{3}{2}$, $d = -\frac{1}{2}$.
So we try to find $\sigma \in G$ with $\sigma(h_i) = \bar{h}_i$, $\sigma(x_i) = \bar{x}_i$, $\sigma(y_i) = \bar{y}_i$ for $1 \le i \le 4$, $\sigma(h_5) = ah_5 + bh_6$, $\sigma(h_6) = ch_5 + dh_6$ (with a, b, c, d as above) and $\sigma(v_j) = \lambda_j \bar{v}_{\pi(j)}$ with π as above. In other words, we try to find all $(1,\pi)$-extensions.
We have written an ad-hoc program in GAP4 for computing the polynomial equations that the λ_i have to satisfy in order that σ be an automorphism. It took the program somewhat more than 13 seconds to compute these equations. The ensuing Gröbner basis computation terminated almost immediately. Setting $\lambda_1 = \lambda_3 = 1$, the equations yield exactly one solution.
Doing this for all elements of $Z_{A_2}(h,e,f)$ we obtain exactly six elements of $Z_G(h,e,f)$ and it follows that the component group is isomorphic to S_X.

2.5 Tables for the exceptional types

In this section we give some data relative to the nilpotent orbits of the Lie algebras of exceptional type, whose stabilizers have nontrivial component group (Tables 2.1, 2.2, 2.3, 2.4, 2.5). The first column of these tables displays the Bala-Carter label of the orbit (see Section 1.4). The second and fourth columns have respectively the isomorphism type of the subalgebras $\mathfrak{c}_1'$ and $\mathfrak{c}_2'$. Here a notation like $2A_2$ means a direct sum of two subalgebras, both of type A_2. The third column has the dimension of the center $\mathfrak{t}$. The fifth column has the highest weights of the $\mathfrak{c}$-module V. Here the notation works as follows. The canonical generators of $\mathfrak{c}_1'$ are h_i, x_i, y_i with $1 \le i \le a$. The basis elements of $\mathfrak{t}$ are h_i with $a+1 \le i \le a+d$. The canonical generators of $\mathfrak{c}_2'$ are h_i, x_i, y_i with $a+d+1 \le i \le r$. The weights are written as $(w_1; w_2; w_3)$ where w_1 has the eigenvalues of the h_i, $1 \le i \le a$ on the highest weight vector, w_2 (written in boldface) has the eigenvalues of the h_i, $a+1 \le i \le a+d$ on the highest weight vector, w_3 has the eigenvalues of the h_i, $a+d+1 \le i \le r$ on the highest weight vector. If $\dim \mathfrak{c}_1' = 0$ or $\dim \mathfrak{t} = 0$ then we just write $(w_2; w_3)$, respectively $(w_1; w_3)$. The eigenvalues of direct summands of $\mathfrak{c}_1'$, $\mathfrak{c}_2'$ are separated by commas. For the enumeration of the nodes of the Dynkin diagrams of the simple Lie algebra we follow the conventions in [13, Planche I-IX]. All weights are of multiplicity one (i.e., each weight belongs to a unique irreducible summand of V). The last column has the isomorphism type of the component group of the stabilizer (which is contained in the tables of [17, Chapter 8], [60]). The data in the second, third, fourth and fifth columns have been computed with GAP4, with the help of the SLA package.

Table 2.1: Nilpotent orbits in the Lie algebra of type G_2 whose stabilizer has a nontrivial component group

label	$\mathfrak{c}_1'$	$\dim \mathfrak{t}$	$\mathfrak{c}_2'$	A
$G_2(a_1)$	0	0	$\mathfrak{g}$	S_3

Table 2.2: Nilpotent orbits in the Lie algebra of type F_4 whose stabilizer has a nontrivial component group

label	$\mathfrak{c}_1'$	$\dim \mathfrak{t}$	$\mathfrak{c}_2'$	weights of V	A
$\widetilde{A}_1$	A_3	0	A_1	$(100;1)$, $(001;1)$, $(010;2)$	S_2
A_2	A_2	0	A_2	$(20;01)$, $(02;10)$	S_2
B_2	$2A_1$	0	B_2	$(1{,}1;10)$, $(1{,}0;01)$, $(0{,}1;01)$	S_2
$C_3(a_1)$	A_1	0	C_3	$(1;001)$	S_2
$F_4(a_3)$	0	0	$\mathfrak{g}$		S_4
$F_4(a_2)$	0	0	$\mathfrak{g}$		S_2

$F_4(a_1)$	0	0	$\mathfrak{g}$		S_2

Table 2.3: Nilpotent orbits in the Lie algebra of type E_6 whose stabilizer has a nontrivial component group

label	$\mathfrak{c}_1'$	$\dim \mathfrak{t}$	$\mathfrak{c}_2'$	weights of V	A
A_2	$2A_2$	0	A_2	$(10,01;01)$, $(01,10;10)$	S_2
$D_4(a_1)$	0	2	D_4	$(-\mathbf{2},\mathbf{0};1000)$, $(\mathbf{2},\mathbf{0};1000)$	S_3
				$(-\mathbf{1},-\mathbf{3};0010)$, $(\mathbf{1},\mathbf{3};0010)$	
				$(-\mathbf{1},\mathbf{3};0001)$, $(\mathbf{1},-\mathbf{3};0001)$	
$E_6(a_3)$	0	0	$\mathfrak{g}$		S_2

Table 2.4: Nilpotent orbits in the Lie algebra of type E_7 whose stabilizer has a nontrivial component group

label	$\mathfrak{c}_1'$	$\dim \mathfrak{t}$	$\mathfrak{c}_2'$	weights of V	A
A_2	A_5	0	A_2	$(00010;10)$, $(01000;01)$	S_2
A_2+A_1	A_3	1	A_1+A_2	$(000;\mathbf{4};0,01)$, $(000;-\mathbf{4};0,10)$	S_2
				$(010;-\mathbf{2};0,01)$, $(010;\mathbf{2};0,10)$	
				$(001;-\mathbf{3};1,00)$, $(100;\mathbf{3};1,00)$	
				$(001;\mathbf{1};1,01)$, $(100;-\mathbf{1};1,10)$	
$D_4(a_1)$	$3A_1$	0	D_4	$(1,1,0;1000)$, $(1,0,1;0010)$, $(0,1,1;0001)$	S_3
$D_4(a_1)+A_1$	$2A_1$	0	A_1+D_4	$(1,1;0,1000)$, $(1,0;1,0001)$, $(0,1;1,0010)$	S_2
A_3+A_2	A_1	1	A_2+A_3	$(0;-\mathbf{4};01,000)$, $(0;\mathbf{4};10,000)$	S_2
				$(1;-\mathbf{3};00,001)$, $(1;\mathbf{3};00,100)$	
				$(0;\mathbf{2};01,010)$, $(0;-\mathbf{2};10,010)$	
				$(1;-\mathbf{1};01,100)$, $(1;\mathbf{1};10,001)$	
A_4	A_2	1	A_4	$(00;-\mathbf{3};0001)$, $(00;\mathbf{3};1000)$	S_2

				$(01;\mathbf{2};0001),\ (10;-\mathbf{2};1000)$	
				$(01;-\mathbf{1};0010),\ (10;\mathbf{2};0100)$	
A_4+A_1	0	2	A_1+A_4	$(\mathbf{3},-\mathbf{1};1,0000),\ (-\mathbf{3},\mathbf{1};1,0000)$	S_2
				$(\mathbf{0},-\mathbf{2};0,1000),\ (\mathbf{0},\mathbf{2};0,0001)$	
				$(-\mathbf{2},\mathbf{2};0,1000),\ (\mathbf{2},-\mathbf{2};0,0001)$	
				$(\mathbf{1},\mathbf{1};1,1000),\ (-\mathbf{1},-\mathbf{1};1,0001)$	
				$(-\mathbf{2},\mathbf{0};0,0100),\ (\mathbf{2},\mathbf{0};0,0010)$	
				$(\mathbf{1},-\mathbf{1};1,0100),\ (-\mathbf{1},\mathbf{1};1,0010)$	
$D_5(a_1)$	A_1	1	D_5	$(0;\mathbf{2};100000),\ (0;-\mathbf{2};10000)$	S_2
				$(1;-\mathbf{1};000010),\ (0;\mathbf{1};00001)$	
$E_6(a_3)$	A_1	0	F_4	$(2;0001)$	S_2
$E_7(a_5)$	0	0	$\mathfrak{g}$		S_3
$E_6(a_1)$	0	1	E_6	$(-\mathbf{2};000001),\ (\mathbf{2};100000)$	S_2
$E_7(a_3)$	0	0	$\mathfrak{g}$		S_2

Table 2.5: Nilpotent orbits in the Lie algebra of type E_8 whose stabilizer has a nontrivial component group

label	$\mathfrak{c}_1'$	$\dim\mathfrak{t}$	$\mathfrak{c}_2'$	weights of V	A
A_2	E_6	0	A_2	$(100000;10),\ (000001;01)$	S_2
A_2+A_1	A_5	0	A_1+A_2	$(10000;1,10),\ (00001;1,01)$	S_2
				$(01000;0,01),\ (00010;0,10)$	
				$(00100;1,00)$	
$2A_2$	$2G_2$	0	A_1	$(10,10;2),\ (10,00;4),\ (00,10;4)$	S_2
$D_4(a_1)$	D_4	0	D_4	$(1000;1000),\ (0010;0010),\ (0001;0001)$	S_3
$D_4(a_1)+A_1$	$3A_1$	0	A_1+D_4	$(1,1,0;0,1000),\ (1,0,1;0,0010),\ (0,1,1;0,0001)$	S_3
				$(1,0,0;1,0001),(0,1,0;1,0010),\ (0,0,1;1,1000)$	

				$(1,1,1;1,0000)$	
A_3+A_2	B_2	1	A_2+B_2	$(00;-\mathbf{4};01,00)$, $(00;\mathbf{4};10,00)$	S_2
				$(00;\mathbf{2};01,10)$, $(00;-\mathbf{2};10,10)$	
				$(10;\mathbf{2};01,00)$, $(10;-\mathbf{2};10,00)$	
				$(01;\mathbf{3};00,01)$, $(01;-\mathbf{3};00,01)$	
				$(01;-\mathbf{1};01,01)$, $(01;\mathbf{1};10,01)$	
				$(10;\mathbf{0};00,10)$	
A_4	A_4	0	A_4	$(1000;0100)$, $(0001;0010)$	S_2
				$(0010;1000)$, $(0100;0001)$	
$D_4(a_1)+A_2$	A_2	0	A_1	$(30;4)$, $(03;4)$, $(22;2)$, $(11;6)$	S_2
A_4+A_1	A_2	1	A_1+A_4	$(00;-\mathbf{6};0,0001)$, $(00;\mathbf{6};0,1000)$	S_2
				$(01;-\mathbf{5};1,0000)$, $(10;\mathbf{5};1,0000)$	
				$(10;-\mathbf{4};0,1000)$, $(01;\mathbf{4};0,0001)$	
				$(00;-\mathbf{3};1,0100)$, $(00;\mathbf{3};1,0010)$	
				$(10;-\mathbf{1};1,0001)$, $(01;\mathbf{1};1,1000)$	
				$(01;-\mathbf{2};0,0010)$, $(10;\mathbf{2};0,0100)$	
$D_5(a_1)$	A_3	0	D_5	$(100;00010)$, $(001;00001)$	S_2
				$(010;10000)$	
A_4+2A_1	A_1	1	A_1+A_4	$(1;-\mathbf{5};1,0000)$, $(1;\mathbf{5};1,0000)$	S_2
				$(0;-\mathbf{4};0,0100)$, $(0;\mathbf{4};0,0010)$	
				$(2;-\mathbf{2};0,1000)$, $(2;\mathbf{2};0,0001)$	
				$(0;-\mathbf{2};2,1000)$, $(0;\mathbf{2};2,0001)$	
				$(1;-\mathbf{3};1,0001)$, $(1;\mathbf{3};1,1000)$	
				$(1;-\mathbf{1};1,0010)$, $(1;\mathbf{1};1,0100)$	
				$(2;\mathbf{0};2,0000)$	
D_4+A_2	A_2	0	A_2+G_2	$(20;10,00)$, $(02;01,00)$	S_2

				$(10;01{,}10),\ (01;10{,}10)$	
				$(11;00{,}10)$	
$E_6(a_3)$	G_2	0	F_4	$(10;0001)$	S_2
$D_6(a_2)$	$2A_1$	0	D_6	$(10;000010),\ (01;000001)$	S_2
				$(11;000001)$	
$E_6(a_3)+A_1$	A_1	0	A_1+F_4	$(3;1{,}0000),\ (2;0{,}0001),\ (1;1{,}0001)$	S_2
$E_7(a_3)$	A_1	0	E_7	$(1;0000001)$	S_3
$E_8(a_7)$	0	0	$\mathfrak{g}$		S_5
$D_6(a_1)$	$2A_1$	0	D_6	$(1{,}1;100000),\ (1{,}0;000010),\ (0{,}1;000001)$	S_2
$E_7(a_4)$	A_1	0	E_7	$(1;0000001)$	S_2
$E_6(a_1)$	A_2	0	E_6	$(10;000001),\ (01;100000)$	S_2
D_5+A_2	0	1	A_2+D_5	$(-\mathbf{4};01{,}00000),\ (\mathbf{4};10{,}00000)$	S_2
				$(-\mathbf{3};00{,}00001),\ (\mathbf{3};00{,}00010)$	
				$(-\mathbf{2};10{,}10000),\ (\mathbf{2};01{,}10000)$	
				$(-\mathbf{1};01{,}00010),\ (\mathbf{1};10{,}00001)$	
$D_7(a_2)$	0	1	D_7	$(-\mathbf{2};1000000),\ (\mathbf{2};1000000)$	S_2
				$(-\mathbf{1};0000010),\ (\mathbf{1};0000001)$	
$E_6(a_1)+A_1$	0	1	A_1+E_6	$(-\mathbf{3};1{,}000000),\ (\mathbf{3};1{,}000000)$	S_2
				$(-\mathbf{2};0{,}000001),\ (\mathbf{2};0{,}100000)$	
				$(-\mathbf{1};1{,}100000),\ (\mathbf{1};1{,}000001)$	
$E_7(a_3)$	A_1	0	E_7	$(1;0000001)$	S_2
$E_8(b_6)$	0	0	$\mathfrak{g}$		S_3
$D_7(a_1)$	0	1	D_7	$(-\mathbf{2};100000),\ (\mathbf{2};1000000)$	S_2
				$(-\imath;000001),\ (\imath;01000010)$	
$E_8(a_6)$	0	0	$\mathfrak{g}$		S_3
$E_8(b_5)$	0	0	$\mathfrak{g}$		S_3

$E_8(a_5)$	0	0	$\mathfrak{g}$		S_2
$E_8(b_4)$	0	0	$\mathfrak{g}$		S_2
$E_8(a_4)$	0	0	$\mathfrak{g}$		S_2
$E_8(a_3)$	0	0	$\mathfrak{g}$		S_2

Chapter 3

Classification of nilpotent and semisimple 4-vectors of a real 8-dimensional space

The contents of this chapter are an adaptation of joint work with Andrea Santi ([29]). We also thank Simon Salamon for very helpful discussions and for communicating the content of Remark 3.3.2 to us.

3.1 Introduction

Antonyan classified the orbits of the group $\mathrm{SL}(8,\mathbb{C})$ acting on the space $\bigwedge^4 \mathbb{C}^8$ ([3], see [52] for a translation). For this he used Vinberg's theory of θ-groups (see Section 1.6). Here we briefly summarize this, and give more details in the next section. The starting point is a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the simple complex Lie algebra $\mathfrak{g}$ of type E_7. Here $\mathfrak{g}_0 \cong \mathfrak{sl}(8,\mathbb{C})$ and $\mathfrak{g}_1 \cong \bigwedge^4 \mathbb{C}^8$ (as $\mathfrak{sl}(8,\mathbb{C})$-module). Let G be the adjoint group of $\mathfrak{g}$ and G_0 the connected subgroup corresponding to the subalgebra $\mathfrak{g}_0$. There is a surjective morphism $\mathrm{SL}(8,\mathbb{C}) \to G_0$ by which $\mathfrak{g}_1$ becomes a $\mathrm{SL}(8,\mathbb{C})$-module isomorphic to $\bigwedge^4 \mathbb{C}^8$. A first remark is that $\mathfrak{g}_1$ is closed under the Jordan decomposition of $\mathfrak{g}$ (see Theorem 1.6.1). This divides the elements of $\bigwedge^4 \mathbb{C}^8$, and hence the $\mathrm{SL}(8,\mathbb{C})$-orbits, into three groups: nilpotent, semisimple and mixed. Antonyan showed that there are 94 nonzero nilpotent orbits. There are infinitely many semisimple orbits. However, these can be divided into 32 classes, including the class consisting just of 0. The 31 nonzero classes contain an infinite number of orbits and their representatives depend on up to seven parameters. The representatives of each class have the same stabilizer in $\mathrm{SL}(8,\mathbb{C})$. Finally, for each semisimple class C it is possible to find a finite list L of nilpotent elements such that the elements $p + e$ ($p \in C$, $e \in L$) are representatives of the orbits of mixed type. In [3] the list L is given for 7 of the 31 nonzero classes of semisimple elements.
In this chapter we consider the classification of the $\mathrm{SL}(8,\mathbb{R})$-orbits in the space $\bigwedge^4 \mathbb{R}^8$. We use the methods developed in [9, 27]. Our main technical tool is Galois cohomology (see Section 1.8). We use the algorithms of [8] to compute the necessary Galois cohomology sets. In order to use these algorithms we need to determine the components of the stabilizers of the various elements that we are interested in. The determination of these

components is the main part of the work that went into this chapter. Our methods for tackling this question are described in Sections 3.5, 3.6. Based on our results we can formulate the following theorem.

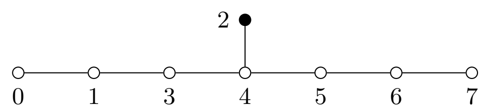
Theorem 3.1.1. *In* $\bigwedge^4 \mathbb{R}^8$ *there are 258 (nonzero) nilpotent* $\mathrm{SL}(8,\mathbb{R})$*-orbits. Including the zero class there are 1452 classes of semisimple* $\mathrm{SL}(8,\mathbb{R})$*-orbits. The nonzero semisimple classes depend on up to seven parameters. The representatives of the orbits in a fixed class have the same stabilizer in* $\mathrm{SL}(8,\mathbb{R})$.

In order to classify the mixed orbits as well one would need to classify the elements that can be the nilpotent part of a mixed element with semisimple part in one of the 1452 nonzero semisimple classes. With our current methods this is a huge, if not impossible, task.
The chapter is organized as follows. The next section describes the basic setup and recalls a number of results from the literature. Section 3.3 has two tables: one for the nilpotent orbits, and one with data relative to the semisimple orbits. In Section 3.4 we give some more info concerning Galois cohomology in the specific case we are dealing with. Then in Sections 3.5, 3.6 we go into the classification of the nilpotent and semisimple orbits. In Section 3.7 we show how we classified the Cartan subspaces in $\bigwedge^4 \mathbb{R}^8$. This can be seen as a very rough classification of the semisimple orbits: each semisimple orbit has a point in (at least) one of the five (up to the action of two automorphisms) given Cartan subspaces. The appendix contains descriptions of the classes of semisimple elements in which the complex semisimple orbits are divided.
The computations of the components of the stabilizers were mostly performed with the help of the computer algebra system GAP4. In many cases it sufficed to compute a Gröbner basis of an ideal to determine the components. On many occasions we used the computer algebra system Magma ([10]) for computing and working with Gröbner bases. The Galois cohomology sets were computed in GAP4 using the implementation of the algorithms of [8] in that system.

3.2 Preliminaries

3.2.1 A specific construction of θ-group

In this chapter we are concerned with one example of a θ-group. It is constructed from a $\mathbb{Z}/2\mathbb{Z}$-grading of the simple Lie algebra $\mathfrak{g}$ of type E_7. We number the nodes of the extended Dynkin diagram of $\mathfrak{g}$ as follows:



Here the node labeled 0 corresponds to the lowest root α_0 of the root system. The remaining nodes correspond to the simple positive roots $\alpha_1, \dots, \alpha_7$. For $0 \le i \le 7$ we fix a nonzero root vector e_i in $\mathfrak{g}_{\alpha_i}$. Then $e_0, \dots, e_7$ generate $\mathfrak{g}$. By [46, Theorem 8.6] sending $e_i \mapsto e_i$ for $i \ne 2$ and $e_2 \mapsto -e_2$ uniquely defines an automorphism θ of $\mathfrak{g}$ of order 2. By [46, Proposition 8.6] the subalgebra $\mathfrak{g}_0$ is simple of type A_7. Hence we have an isomorphism $\psi : \mathfrak{sl}(8,\mathbb{C}) \to \mathfrak{g}_0$. Using this isomorphism we can make $\mathfrak{g}_1$ into an $\mathfrak{sl}(8,\mathbb{C})$-module. Moreover, as $\mathrm{SL}(8,\mathbb{C})$ is simply connected, the isomorphism lifts to a surjective homomorphism $\psi : \mathrm{SL}(8,\mathbb{C}) \to G_0$. We can use this to make $\mathfrak{g}_1$ into a $\mathrm{SL}(8,\mathbb{C})$-module.

Again using [46, Proposition 8.6] we see that the $\mathrm{SL}(8,\mathbb{C})$-modules $\mathfrak{g}_1$ and $\bigwedge^4\mathbb{C}^8$ are isomorphic. In the sequel we identify these two modules and we give their elements as linear combinations of the standard basis of $\bigwedge^4\mathbb{C}^8$. For the latter, if $v_1,\dots,v_8$ denotes the standard basis of $\mathbb{C}^8$, we write

$$e_{ijkl} = v_i \wedge v_j \wedge v_k \wedge v_l.$$

3.2.2 Orbits of a θ-group

Thanks to Theorem 1.6.1, the elements of $\mathfrak{g}_1$ (and hence the G_0-orbits) are divided into three groups: semisimple, nilpotent and mixed. Here we briefly comment on the methods to classify the nilpotent and semisimple orbits.

There are finitely many nilpotent orbits (as expected, see Proposition 1.6.4) and there are several algorithms to classify them ([25, 67]). In our main example, there are ninety-four nonzero nilpotent orbits, see [3], [52].

For the semisimple orbits the key notion is that of a *Cartan subspace*: this is a maximal subspace of $\mathfrak{g}_1$ consisting of commuting semisimple elements. Vinberg has shown that two Cartan subspaces in $\mathfrak{g}_1$ are G_0-conjugate ([66, Theorem 1]). Hence every semisimple orbit has a point in a fixed Cartan subspace $\mathfrak{h}$. Furthermore, consider the following groups:

$$Z_{G_0}(\mathfrak{h}) = \{g \in G_0 \mid g(x) = x \text{ for all } x \in \mathfrak{h}\},$$
$$N_{G_0}(\mathfrak{h}) = \{g \in G_0 \mid g(x) \in \mathfrak{h} \text{ for all } x \in \mathfrak{h}\}$$

and $W_0(\mathfrak{h}) = N_{G_0}(\mathfrak{h})/Z_{G_0}(\mathfrak{h})$. Then $W_0(\mathfrak{h})$ is a finite group of linear transformations of $\mathfrak{h}$, called the *little Weyl group* of the graded Lie algebra $\mathfrak{g}$. Furthermore, two elements of $\mathfrak{h}$ are G_0-conjugate if and only if they are conjugate under $W_0(\mathfrak{h})$ ([66, Theorem 2]). So a set of representatives of the $W_0(\mathfrak{h})$-orbits on $\mathfrak{h}$ is also a set of representatives of the semisimple G_0-orbits. However, this can be refined.

Write $W = W_0(\mathfrak{h})$ and for $p \in \mathfrak{h}$ let $W_p = \{w \in W \mid w(p) = p\}$ be its stabilizer. For a subgroup H of W define

$$\mathfrak{h}_H = \{p \in \mathfrak{h} \mid H \subset W_p\},$$
$$\mathfrak{h}_H^\circ = \{p \in \mathfrak{h} \mid H = W_p\}.$$

Then $\mathfrak{h}_H$ is the intersection of the subspaces of $\mathfrak{h}$ defined by the equations $(w-1)p = 0$ for $w \in H$; so it is a subspace of $\mathfrak{h}$. Furthermore, $\mathfrak{h}_H^\circ$ is the subset of $\mathfrak{h}_H$ defined by the inequalities $(w-1)p \neq 0$ for $w \in W \setminus H$; so it is a Zariski open subset of $\mathfrak{h}_H$. Also note that $p \in \mathfrak{h}$ lies in $\mathfrak{h}_H^\circ$ where $H = W_p$. So the sets $\mathfrak{h}_H^\circ$ for H a subgroup of W cover all of $\mathfrak{h}$. For subgroups $H, H' \subset W$ and $w \in W$ we have (see [27, (3.3)])

$$w\mathfrak{h}_H^\circ = \mathfrak{h}_{H'}^\circ \text{ if and only if } wHw^{-1} = H'. \tag{3.2.1}$$

We say that a subgroup $H \subset W$ is a stabilizer if there is a $p \in \mathfrak{h}$ with $H = W_p$. Let $H_1,\dots,H_r$ be the subgroups of W, up to conjugacy, that are stabilizers. Then it follows that a semisimple G_0-orbit in $\mathfrak{g}_1$ has a point in exactly one of the $\mathfrak{h}_{H_i}^\circ$. Furthermore, it follows that two elements of $\mathfrak{h}_{H_i}^\circ$ are G_0-conjugate if and only if they are conjugate under $\Gamma_{H_i} = N_W(H_i)/H_i$. We conclude that the union of the sets of representatives of the Γ_{H_i}-orbits on $\mathfrak{h}_{H_i}^\circ$ is also a set of representatives of the semisimple G_0-orbits in $\mathfrak{g}_1$. Furthermore, we have the following theorem ([32, Lemma 2.9], [27, Corollary 3.13]).

Theorem 3.2.1. *Let $x, x' \in \mathfrak{h}_{H_i}^\circ$. Then $Z_{G_0}(x) = Z_{G_0}(x')$.*

In our main example a Cartan subspace of $\mathfrak{g}_1$ has dimension 7. A particular Cartan subspace $\mathfrak{h}$, found by Antonyan ([3]) is spanned by

$$\begin{aligned}
p_1 &= e_{1234} + e_{5678},\\
p_2 &= e_{1357} + e_{2468},\\
p_3 &= e_{1256} + e_{3478},\\
p_4 &= e_{1368} + e_{2457},\\
p_5 &= e_{1458} + e_{2367},\\
p_6 &= -e_{1467} - e_{2358},\\
p_7 &= -e_{1278} - e_{3456}.
\end{aligned}$$

(Here we could of course erase the minuses in the last two elements; however we preferred to stick to the exact basis given by Antonyan.)
So in this case $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. The little Weyl group, $W = W_0(\mathfrak{h})$, is equal to the Weyl group of the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (see [32, Lemma 2.6]). It has thirty-two conjugacy classes of subgroups that are stabilizers of elements.

3.2.3 Real forms

Throughout this chapter we write $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(8,\mathbb{C})$ and $\widehat{G}_0 = \mathrm{SL}(8,\mathbb{C})$. The construction of the $\widehat{G}_0$-module $\bigwedge^4 \mathbb{C}^8$ using a $\mathbb{Z}/2\mathbb{Z}$-grading of the simple Lie algebra $\mathfrak{g}$ of type E_7 has been used by Antonyan ([3], see also [52]) to classify the $\widehat{G}_0$-orbits in this module. Also we write $\widehat{G}_0(\mathbb{R}) = \mathrm{SL}(8,\mathbb{R})$. In this chapter we classify the nilpotent and semisimple $\widehat{G}_0(\mathbb{R})$-orbits on $\bigwedge^4 \mathbb{R}^8$. For this reason we describe the real forms of the various objects that we use.
We consider a fixed Chevalley basis of $\mathfrak{g}$ (see [42, Theorem 25.2]) and let the generators $e_0,\dots,e_7$ of Section 3.2.1 be elements of that basis. Let $f_0,\dots,f_7$ be the elements of the Chevalley basis lying in the root spaces of $-\alpha_0,\dots,-\alpha_7$. Denote the 8×8-matrix with a 1 on position (i,j) and zeros elsewhere by e_{ij}. Then mapping $e_{12}, e_{23},\dots,e_{78}$ to, respectively, $e_0,e_1,e_3,e_4,\dots,e_7$ and $e_{21},e_{32},\dots,e_{87}$ to respectively $f_0,f_1,f_3,f_4,\dots,f_7$ extends to a unique isomorphism $\psi : \hat{\mathfrak{g}}_0 \to \mathfrak{g}_0$. This isomorphism lifts to a unique surjective homomorphism $\psi : \widehat{G}_0 \to G_0$. We let $\mathfrak{g}^{\mathbb{R}}$ be the real span of the fixed Chevalley basis, and let $\mathfrak{g}_0^{\mathbb{R}}$, $\mathfrak{g}_1^{\mathbb{R}}$ be the intersections of $\mathfrak{g}^{\mathbb{R}}$ with $\mathfrak{g}_0$, $\mathfrak{g}_1$ respectively. Then ψ restricts to an isomorphism $\psi : \mathfrak{sl}(8,\mathbb{R}) \to \mathfrak{g}_0^{\mathbb{R}}$. We let $G(\mathbb{R})$ be the set of elements of G that map $\mathfrak{g}^{\mathbb{R}}$ to itself (recall that G is the inner automorphism group of $\mathfrak{g}$), and $G_0(\mathbb{R})$ is the intersection of G_0 and $G(\mathbb{R})$. We also consider the restriction of the group homomorphism $\psi : \widehat{G}_0(\mathbb{R}) \to G_0(\mathbb{R})$. We remark that this map is not necssarily surjective as $G_0(\mathbb{R})$ could be non-connected. However, the map ψ makes $\mathfrak{g}_1^{\mathbb{R}}$ into a $\widehat{G}_0(\mathbb{R})$-module isomorphic to $\bigwedge^4 \mathbb{R}^8$.
We have a canonical conjugation of all complex objects involved. In all cases we denote this map by σ. On the real spaces $\mathfrak{g}^{\mathbb{R}}$, $\mathfrak{g}_0^{\mathbb{R}}$, $\mathfrak{g}_1^{\mathbb{R}}$, $\bigwedge^4 \mathbb{R}^8$ it is the identity, and it is extended to the corresponding complex spaces by sending complex coefficients to their complex conjugates. On $\widehat{G}_0$ it is defined by sending the matrix entries of an element to their complex conjugates. On G_0 it is defined by considering the matrix of a $g \in G_0$ with respect to a basis of $\mathfrak{g}^{\mathbb{R}}$ and sending its matrix coefficients to their complex conjugates.

3.2.4 Two module automorphisms

Here we define two automorphisms of the $\mathrm{SL}(8,\mathbb{C})$-module $\bigwedge^4 \mathbb{C}^8$. They are defined over $\mathbb{R}$ so that they restrict to automorphisms of the $\mathrm{SL}(8,\mathbb{R})$-module $\bigwedge^4 \mathbb{R}^8$.

For the first map let

$$g_0 = \operatorname{diag}(-1,1,1,1,1,1,1,1) \text{ and } g_1 = \operatorname{diag}(-\omega,\omega,\omega,\omega,\omega,\omega,\omega,\omega)$$

where ω is a primitive 16-th root of unity with $\omega^4 = \imath$. We define $\nu : \mathrm{SL}(8,\mathbb{C}) \to \mathrm{SL}(8,\mathbb{C})$ by $\nu(g) = g_0 g g_0^{-1}$. Then ν is an automorphism of $\mathrm{SL}(8,\mathbb{C})$ that restricts to an automorphism of $\mathrm{SL}(8,\mathbb{R})$. Obviously g_0 can be seen as a linear map $\mathbb{C}^8 \to \mathbb{C}^8$ and we define the map $\nu : \bigwedge^4 \mathbb{C}^8 \to \bigwedge^4 \mathbb{C}^8$ by $\nu(u_1 \wedge u_2 \wedge u_3 \wedge u_4) = (g_0u_1) \wedge (g_0u_2) \wedge (g_0u_3) \wedge (g_0u_4)$. Then ν restrictes to a linear map of $\bigwedge^4 \mathbb{R}^8$. For $g \in \mathrm{SL}(8,\mathbb{C})$ and $v \in \bigwedge^4 \mathbb{C}^8$ we have $\nu(g \cdot v) = \nu(g) \cdot \nu(v)$. So ν maps the $\mathrm{SL}(8,\mathbb{C})$-orbit of v to the $\mathrm{SL}(8,\mathbb{C})$-orbit of $\nu(v)$. If $u \in \bigwedge^4 \mathbb{R}^8$ then the same holds with $\mathrm{SL}(8,\mathbb{C})$ replaced by $\mathrm{SL}(8,\mathbb{R})$.
Define ν_1 in the same way as ν but with g_0 replaced by g_1. Then ν_1 has the same properties, but it obviously does not restrict to maps of $\mathrm{SL}(8,\mathbb{R})$ and $\bigwedge^4 \mathbb{R}^8$. Note that $g_1 \in \mathrm{SL}(8,\mathbb{C})$ so ν_1 maps the $\mathrm{SL}(8,\mathbb{C})$-orbit of $v \in \bigwedge^4 \mathbb{C}^8$ to itself. We have that $\nu_1\nu$ is the identity on $\mathrm{SL}(8,\mathbb{C})$ and maps $v \mapsto \imath v$ for $v \in \bigwedge^4 \mathbb{C}^8$. It follows that the $\mathrm{SL}(8,\mathbb{C})$-orbit of $\nu(v)$ is the $\mathrm{SL}(8,\mathbb{C})$-orbit of $\imath v$. So also ν maps the $\mathrm{SL}(8,\mathbb{R})$-orbits contained in $\mathrm{SL}(8,\mathbb{C}) \cdot v \cap \bigwedge^4 \mathbb{R}^8$ to the $\mathrm{SL}(8,\mathbb{R})$-orbits contained in $\mathrm{SL}(8,\mathbb{C}) \cdot \imath v \cap \bigwedge^4 \mathbb{R}^8$.
For the second map we consider the centralizer G^θ of θ in G. Some calculations show that $G^\theta = G_0 \cup \varphi G_0$. We have that φ is of order 2 and restricts to an outer automorphism of $\mathfrak{g}_0$. It lifts to an automorphism of $\mathrm{SL}(8,\mathbb{C})$ and for $g \in \mathrm{SL}(8,\mathbb{C})$ we have

$$\varphi(g) = Qg^{-\top}Q^{-1}$$

with

$$Q = \operatorname{antidiag}(1,-1,1,-1,1,-1,1,-1)$$

(the anti-diagonal goes from the top right to the bottom left). Furthermore, φ stabilizes $\mathfrak{g}_1$ and hence can be seen as a map $\bigwedge^4 \mathbb{C}^8 \to \bigwedge^4 \mathbb{C}^8$. Since it is of order 2 we have that $\bigwedge^4 \mathbb{C}^8$ is the direct sum of the eigenspaces with eigenvalue 1 and -1 respectively.
A direct computation shows that a basis of the eigenspace with eigenvalue 1 is as follows

$$\begin{gathered}
e_{1234}, e_{1235}, e_{1246}, e_{1256}, e_{1347}, e_{1357}, e_{1467}, e_{1567},\\
e_{2348}, e_{2358}, e_{2468}, e_{2568}, e_{3478}, e_{3578}, e_{4678}, e_{5678},\\
e_{1245}+e_{1236}, e_{1345}+e_{1237}, e_{1346}+e_{1247}, e_{1356}+e_{1257},\\
e_{1456}+e_{1267}, e_{1457}+e_{1367}, e_{2345}+e_{1238}, e_{2346}+e_{1248},\\
e_{2347}+e_{1348}, e_{2356}+e_{1258}, e_{2357}+e_{1358}, e_{2367}+e_{1458},\\
e_{2456}+e_{1268}, e_{2457}+e_{1368}, e_{2458}+e_{2368}, e_{2467}+e_{1468},\\
e_{2567}+e_{1568}, e_{3456}+e_{1278}, e_{3457}+e_{1378}, e_{3458}+e_{2378},\\
e_{3467}+e_{1478}, e_{3468}+e_{2478}, e_{3567}+e_{1578}, e_{3568}+e_{2578},\\
e_{4567}+e_{1678}, e_{4568}+e_{2678}, e_{4578}+e_{3678}.
\end{gathered}$$

For the elements listed here of the form $e_{ijkl} + e_{pqrs}$ we have $\varphi(e_{ijkl}) = e_{pqrs}$ and vice versa. So with the above list we can determine the image of φ on every element of $\bigwedge^4 \mathbb{C}^8$. It is clear that φ restricts to automorphisms of $\mathrm{SL}(8,\mathbb{R})$ and $\bigwedge^4 \mathbb{R}^8$. As φ is an automorphism of $\mathfrak{g}$ we also have $\varphi(g \cdot v) = \varphi(g) \cdot \varphi(v)$ for g in $\mathrm{SL}(8,\mathbb{C})$ or $\mathrm{SL}(8,\mathbb{R})$ and v in $\bigwedge^4 \mathbb{C}^8$ or in $\bigwedge^4 \mathbb{R}^8$. So it maps $\mathrm{SL}(8,\mathbb{C})$-orbits to $\mathrm{SL}(8,\mathbb{C})$-orbits and $\mathrm{SL}(8,\mathbb{R})$-orbits to $\mathrm{SL}(8,\mathbb{R})$-orbits.
A quick check shows that for the basis vectors $p_1, \dots, p_7$ of the Cartan subspace $\mathfrak{h}$ we have $\varphi(p_i) = p_i$ for all i. Hence φ maps a semisimple $\mathrm{SL}(8,\mathbb{C})$-orbit to itself. (But it could of course permute the real orbits contained in the same complex semisimple orbits.)
Some computations show that the group generated by ν and φ has the following eight elements

$$1,\ \varphi,\ \nu,\ \varphi\nu,\ \nu\varphi,\ \varphi\nu\varphi,\ \nu\varphi\nu,\ \nu\varphi\nu\varphi$$

with the relation $\nu\varphi\nu\varphi = \varphi\nu\varphi\nu$ (which makes it isomorphic to the Weyl group of the root system of type B_2).

3.3 Tables

Table 3.1 contains representatives of the real nilpotent orbits. We briefly explain its contents. The first column has the label of the complex orbit as in Table 10 of [52]. We have followed the numbering of that table, except that our 83, 84, 85 is 85, 83, 84 respectively in [52]. The reason for this change is the following. Consider the map φ from Section 3.2.4. Let $\beta_1, \dots, \beta_7$ be the simple roots of the root system of $\mathfrak{sl}(8, \mathbb{C})$ with respect to the standard Cartan subalgebra consisting of diagonal matrices. Let $e \in \mathfrak{g}_1$ be nilpotent lying in a homogeneous $\mathfrak{sl}_2$-triple (h, e, f). We view h as element of $\mathfrak{sl}(8, \mathbb{C}) \cong \mathfrak{g}_0$. After replacing e by a G_0-conjugate we may assume that h is diagonal (that is, it lies in the Cartan subalgebra that we use) such that $\gamma_i(h) \geq 0$. The 7-tuple $(\gamma_1(h), \dots, \gamma_7(h))$ is called the *characteristic* of the orbit of e. The characteristic uniquely determines the nilpotent orbit. Now $\varphi(e)$ lies in the homogeneous $\mathfrak{sl}_2$-triple $(\varphi(h), \varphi(e), \varphi(f))$. From the explicit description of φ it easily follows that $(\gamma_1(\varphi(h)), \dots, \gamma_7(\varphi(h))) = (\gamma_7(h), \gamma_6(h), \dots, \gamma_1(h))$. In other words, the characteristic of the orbit of $\varphi(e)$ is the reverse of the characteristic of the orbit of e. If e and $\varphi(e)$ lie in different orbits (in other words, the characteristic is not invariant under reversion) then φ also maps the real $\widehat{G}_0(\mathbb{R})$-orbits contained in the $\widehat{G}_0$-orbit of e to the real orbits contained in the $\widehat{G}_0$-orbit of $\varphi(e)$. In this case we just list one of the two orbits in Table 3.1 and we note that the number of complex nilpotent orbits is 94, so the last orbit in the table is also followed by its image under φ. In Table 10 of [52] the reverse of a characteristic is always listed immediately after it. So, for example, orbit number 11 is the image under φ of orbit number 10 (and in our table we do not have orbit number 11). The only exception is orbit number 82 whose image (with the reverse characteristic) is orbit number 85. For this reason, we have changed the numbering, so as to have the reverse of orbit 82 immediately below it as orbit 83.
We also consider the action of ν on the nilpotent orbits. In Section 3.5 we show that ν maps a complex nilpotent orbit to itself. However, it can nontrivially permute the real nilpotent orbits contained in a given complex nilpotent orbit. If this happens then of each pair of ν-conjugates we just list one element. We indicate this situation by an asterisk, that is, if a representative e of a nilpotent orbit in Table 3.1 is followed by a $*$ then $\nu(e)$ is a representative of a second real nilpotent $\mathrm{SL}(8, \mathbb{R})$-orbit. In Section 3.5 we also describe how to compute the image of ν on a real nilpotent orbit.
In the second column of the table we give in row k representatives of the real orbits contained in the complex orbit number k. This always starts with the same representative as in [52]. The third column then lists the isomorphism type of the real stabilizer $\mathfrak{z}_{\mathfrak{g}_0}(h, e, f) = \{x \in \mathfrak{g}_0 \mid [x, h] = [x, e] = [x, f] = 0\}$. This stabilizer is reductive, so it is the sum of a semisimple subalgebra and a toral center. For the center we use the following notation: $\mathfrak{t}(k)$ denotes a k-dimensional Lie algebra whose elements are commuting semisimple elements with real eigenvalues, and $\mathfrak{u}(k)$ denotes the same, with the difference that the eigenvalues are all purely imaginary. The last column has the isomorphism type of the component group $Z_{G_0}(h, e, f)/Z_{G_0}(h, e, f)^{\circ}$.

Table 3.1: Nilpotent orbits representatives

N	Real representative	$\mathfrak{z}_{\widehat{G}_0(\mathbb{R})}(h,e,f)$	K
1	e_{1234}	$2\mathfrak{sl}(4,\mathbb{R})$	$\{1\}$
2	$e_{1234}+e_{1256}$	$2\mathfrak{sl}(2,\mathbb{R})+\mathfrak{so}(2,3)+\mathfrak{t}(1)$	$\{1\}$
3	$e_{1234}+e_{1256}+e_{1278}$ *	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{sp}(3,\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
5	$e_{1235}+e_{1246}+e_{1347}$	$\mathfrak{sl}(3,\mathbb{R})+\mathfrak{t}(2)$	$\{1\}$
6	$e_{1235}+e_{1246}+e_{1347}+e_{2348}$ *	$\mathfrak{sl}(4,\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
7	$e_{1345}+e_{1236}+e_{1247}+e_{1258}$ *	$\mathfrak{sl}(3,\mathbb{R})+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
9	$e_{1234}+e_{1567}$ $-\frac{1}{2}e_{1234}-\frac{1}{2}e_{1236}+\frac{1}{2}e_{1247}-\frac{1}{2}e_{1267}$ $-\frac{1}{2}e_{1345}-\frac{1}{2}e_{1356}+\frac{1}{2}e_{1457}-\frac{1}{2}e_{1567}$	$2\mathfrak{sl}(3,\mathbb{R})+\mathfrak{t}(1)$ $\mathfrak{sl}(3,\mathbb{C})+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
10	$e_{1245}+e_{1367}+e_{1238}$ $-e_{1238}-\frac{1}{2}e_{1245}-\frac{1}{2}e_{1247}+\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1267}+\frac{1}{2}e_{1345}-\frac{1}{2}e_{1347}+\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1367}$ *	$2\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(2)$ $\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
12	$e_{1345}+e_{2346}+e_{1256}+e_{1237}+e_{1248}$ * $-e_{1237}-e_{1248}-e_{1256}-e_{1345}-e_{2346}$ *	$2\mathfrak{sl}(2,\mathbb{R})$ $2\mathfrak{sl}(2,\mathbb{R})$	$(\mathbb{Z}/2\mathbb{Z})^2$
13	$e_{1246}+e_{1357}+e_{1238}+e_{1458}$ $-e_{1238}-\frac{1}{2}e_{1246}+\frac{1}{2}e_{1247}+\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1257}+\frac{1}{2}e_{1346}+\frac{1}{2}e_{1347}+\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-e_{1458}$ *	$2\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$ $2\mathfrak{su}(2,\mathbb{R})+\mathfrak{t}(1)$	$\{1\}$
15	$e_{2345}+e_{1246}+e_{1357}+e_{1238}$ $-e_{1238}-\frac{1}{2}e_{1246}-\frac{1}{2}e_{1247}-\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1257}+\frac{1}{2}e_{1346}-\frac{1}{2}e_{1347}-\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-e_{2345}$ * $-e_{1238}-\frac{1}{2}e_{1246}+\frac{1}{2}e_{1247}+\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1257}+\frac{1}{2}e_{1346}+\frac{1}{2}e_{1347}+\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-e_{2345}$ *	$\mathfrak{t}(3)$ $\mathfrak{u}(2)+\mathfrak{t}(1)$ $\mathfrak{u}(2)+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$

16	$e_{1246}+e_{1357}+e_{1238}+e_{1458}+e_{1678}$ * $-e_{1238}-\frac{1}{2}e_{1246}+\frac{1}{2}e_{1247}+\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1257}+\frac{1}{2}e_{1346}+\frac{1}{2}e_{1347}+\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-e_{1458}-e_{1678}$ *	$G_{2(2)}$ G	$\mathbb{Z}/4\mathbb{Z}$
18	$e_{2345}+e_{1246}+e_{1357}+e_{1238}+e_{1458}$ $-e_{1238}-\frac{1}{2}e_{1246}+\frac{1}{2}e_{1247}+\frac{1}{2}e_{1256}$ $+\frac{1}{2}e_{1257}+\frac{1}{2}e_{1346}+\frac{1}{2}e_{1347}+\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-e_{1458}-e_{2345}$ * $-e_{1238}-2e_{1246}+2e_{1357}-2e_{1458}-\frac{1}{2}e_{2345}$ $\frac{1}{4}e_{1238}+\frac{1}{8}e_{1246}-4e_{1247}-\frac{1}{8}e_{1256}$ $-4e_{1257}-e_{1346}-\frac{1}{8}e_{1347}-e_{1356}$ $+\frac{1}{8}e_{1357}+4e_{1458}+4e_{2345}$ *	$2\mathfrak{sl}(2,\mathbb{R})$ $2\mathfrak{su}(2)$ $2\mathfrak{sl}(2,\mathbb{R})$ $2\mathfrak{su}(2)$	$\mathbb{Z}/4\mathbb{Z}$
20	$e_{1256}+e_{1347}+e_{2348}$	$3\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	$\{1\}$
21	$e_{2345}+e_{1347}+e_{1567}+e_{1268}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(2)$	$\{1\}$
22	$e_{2456}+e_{2347}+e_{1357}+e_{1348}+e_{1268}$	$\mathfrak{t}(2)$	$\{1\}$
23	$e_{3456}+e_{1347}+e_{1257}+e_{2348}+e_{1268}$ *	$2\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
25	$e_{2456}+e_{2347}+e_{1457}+e_{1367}+e_{1358}+e_{1268}$	$\mathfrak{t}(1)$	$\{1\}$
26	$e_{2456}+e_{2357}+e_{1457}+e_{1367}$ $+e_{2348}+e_{1358}+e_{1268}$ *		$\mathbb{Z}/8\mathbb{Z}$
27	$e_{1345}+e_{2346}+e_{1257}+e_{1268}$	$2\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	$\{1\}$
28	$e_{2345}+e_{1347}+e_{1267}+e_{1258}+e_{1368}$ $\frac{1}{2}e_{1247}-\frac{1}{2}e_{1248}-\frac{1}{2}e_{1257}-\frac{1}{2}e_{1258}$ $-e_{1267}-\frac{1}{2}e_{1347}-\frac{1}{2}e_{1348}-\frac{1}{2}e_{1357}$ $+\frac{1}{2}e_{1358}-e_{1368}-e_{2345}$ $-\frac{1}{2}e_{1247}-\frac{1}{2}e_{1248}+\frac{1}{2}e_{1257}-\frac{1}{2}e_{1258}$ $-e_{1267}-\frac{1}{2}e_{1347}+\frac{1}{2}e_{1348}-\frac{1}{2}e_{1357}$ $-\frac{1}{2}e_{1358}-e_{1368}-e_{2345}$	$\mathfrak{t}(2)$ $\mathfrak{u}(1)+\mathfrak{t}(1)$ $\mathfrak{u}(1)+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$

30	$e_{3456}+e_{1247}+e_{1357}+e_{2358}$ $+e_{1268}+e_{1368}+e_{2368}$ * $-e_{1247}-e_{1268}-e_{1357}-e_{1368}$ $-e_{2358}+\frac{1}{2}e_{2367}-e_{2368}-e_{3456}$ *		$\mathbb{Z}/4\mathbb{Z}\times S_3$
32	$e_{2356}+e_{2347}+e_{1457}+e_{1367}+e_{1248}+e_{1358}$	$\mathfrak{t}(1)$	$\{1\}$
33	$e_{1356}+e_{2456}+e_{1347}+e_{1257}+e_{2348}+e_{1268}$ *	$\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/4\mathbb{Z}$
34	$e_{1256}+e_{3456}+e_{1347}+e_{2348}$ *	$3\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
36	$e_{1256}+e_{2347}+e_{1357}+e_{1348}$	$\mathfrak{t}(3)$	$\{1\}$
37	$e_{2345}+e_{1357}+e_{1467}+e_{1348}+e_{1268}$	$\mathfrak{t}(2)$	$\{1\}$
39	$e_{2456}+e_{2347}+e_{1357}+e_{1467}+e_{1348}+e_{1268}$ *	$\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
40	$e_{2356}+e_{1456}+e_{2347}+e_{1357}+e_{1348}+e_{1278}$ $-e_{1278}-e_{1348}+e_{1357}-e_{1456}-e_{2347}-e_{2356}$	$\mathfrak{t}(1)$ $\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
42	$e_{1456}+e_{2347}+e_{1267}+e_{1358}+e_{2358}+e_{1268}$ $-e_{1267}-e_{1268}-\frac{1}{2}e_{1347}-\frac{1}{2}e_{1348}+\frac{1}{2}e_{1357}$ $-\frac{1}{2}e_{1358}-e_{1456}-e_{2347}-e_{2358}$	$\mathfrak{t}(1)$ $\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
43	$e_{2345}+e_{1356}+e_{1347}+e_{1267}+e_{1248}$	$\mathfrak{t}(2)$	$\{1\}$
44	$e_{1246}+e_{1357}+e_{2348}+e_{2358}$ $-e_{1246}-\frac{1}{2}e_{1256}+\frac{1}{2}e_{1258}-e_{1357}-e_{2348}$ $-\frac{1}{2}e_{2356}-\frac{1}{2}e_{2358}$	$\mathfrak{t}(3)$ $\mathfrak{t}(3)$	S_3
45	$e_{1258}+e_{1267}+e_{1347}+e_{1368}+e_{1456}+e_{2345}$ * $\frac{1}{2}e_{1247}-\frac{1}{2}e_{1248}-\frac{1}{2}e_{1257}-\frac{1}{2}e_{1258}$ $-e_{1267}-\frac{1}{2}e_{1347}-\frac{1}{2}e_{1348}-\frac{1}{2}e_{1357}$ $+\frac{1}{2}e_{1358}-e_{1368}-e_{1456}-e_{2345}$ * $-\frac{1}{2}e_{1247}-\frac{1}{2}e_{1248}+\frac{1}{2}e_{1257}-\frac{1}{2}e_{1258}$ $-e_{1267}-\frac{1}{2}e_{1347}+\frac{1}{2}e_{1348}-\frac{1}{2}e_{1357}$ $-\frac{1}{2}e_{1358}-e_{1368}-e_{1456}-e_{2345}$ *	$\mathfrak{t}(1)$ $\mathfrak{u}(1)$ $\mathfrak{u}(1)$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

47	$e_{2345}+e_{1347}+e_{1267}+e_{1258}+e_{1368}+e_{2378}$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/4\mathbb{Z}$
	$-\frac{1}{4}e_{1247}-\frac{1}{8}e_{1248}+e_{1257}-\frac{1}{2}e_{1258}$ $-e_{1267}-\frac{1}{2}e_{1347}+\frac{1}{4}e_{1348}-2e_{1357}$ $-e_{1358}-e_{1368}-e_{2345}-e_{2378}$	$\mathfrak{su}(2)$	
	$2e_{1258}-2e_{1267}-2e_{1347}-2e_{1368}-\frac{1}{2}e_{2345}$ $-e_{2378}$	$\mathfrak{sl}(2,\mathbb{R})$	
	$\frac{1}{8}e_{1247}+2e_{1248}-\frac{1}{8}e_{1257}+2e_{1258}$ $+\frac{1}{4}e_{1267}+2e_{1347}-\frac{1}{8}e_{1348}+2e_{1357}$ $+\frac{1}{8}e_{1358}+\frac{1}{4}e_{1368}+4e_{2345}+\frac{1}{4}e_{2378}$	$\mathfrak{su}(2)$	
49	$e_{2356}+e_{1456}+e_{2347}+e_{1357}$ $+e_{1267}+e_{1348}+e_{1258}$ *		$(\mathbb{Z}/4\mathbb{Z})^2$
	$-2e_{1258}-2e_{1267}+2e_{1348}+4e_{1357}$ $-2e_{1456}-\frac{1}{2}e_{2347}+\frac{1}{2}e_{2356}$ *		
50	$e_{1256}+e_{1357}+e_{1467}+e_{2348}$	$\mathfrak{sl}(3,\mathbb{R})+\mathfrak{t}(1)$	$\{1\}$
51	$e_{2456}+e_{1357}+e_{1467}+e_{2348}+e_{1268}$	$\mathfrak{t}(2)$	$\{1\}$
52	$e_{3456}+e_{2357}+e_{1457}+e_{1367}+e_{2348}+e_{1268}$	$\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
	$-e_{1268}-e_{1367}-e_{1457}-e_{2348}-e_{2357}-e_{3456}$	$\mathfrak{t}(1)$	
54	$e_{1258}+e_{1267}+e_{1348}+e_{1356}+e_{2348}+e_{2457}$ *	$\mathfrak{t}(1)$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
	$-e_{1258}-e_{1267}-e_{1348}-\frac{1}{2}e_{1356}$ $+\frac{1}{2}e_{1357}+\frac{1}{2}e_{1456}+\frac{1}{2}e_{1457}-e_{2348}$ $+\frac{1}{2}e_{2356}+\frac{1}{2}e_{2357}+\frac{1}{2}e_{2456}-\frac{1}{2}e_{2457}$ *	$\mathfrak{u}(1)$	
	$-e_{1258}-e_{1267}-e_{1348}-\frac{1}{2}e_{1356}$ $-\frac{1}{2}e_{1357}-\frac{1}{2}e_{1456}+\frac{1}{2}e_{1457}-e_{2348}$ $+\frac{1}{2}e_{2356}-\frac{1}{2}e_{2357}-\frac{1}{2}e_{2456}-\frac{1}{2}e_{2457}$ *	$\mathfrak{u}(1)$	

56	$e_{1356}+e_{2457}+e_{1267}+e_{3467}+e_{1348}+e_{2348}$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/4\mathbb{Z}$
	$-e_{1267}-e_{1348}-\frac{1}{2}e_{1356}+\frac{1}{2}e_{1357}$ $+\frac{1}{2}e_{1456}+\frac{1}{2}e_{1457}-e_{2348}+\frac{1}{2}e_{2356}$ $+\frac{1}{2}e_{2357}+\frac{1}{2}e_{2456}-\frac{1}{2}e_{2457}-e_{3467}$	$\mathfrak{su}(2)$	
	$2e_{1267}+2e_{1348}+2e_{1356}+\frac{1}{2}e_{2348}$ $+\frac{1}{2}e_{2457}-e_{3467}$	$\mathfrak{sl}(2,\mathbb{R})$	
	$4e_{1267}+4e_{1348}+\frac{1}{8}e_{1356}-4e_{1357}$ $-e_{1456}-\frac{1}{8}e_{1457}+4e_{2348}-\frac{1}{8}e_{2356}$ $-4e_{2357}-e_{2456}+\frac{1}{8}e_{2457}+\frac{1}{4}e_{3467}$	$\mathfrak{su}(2)$	
58	$e_{2356}+e_{1456}+e_{1257}+e_{1367}+e_{2348}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
	$-e_{1257}-e_{1367}-e_{1456}-e_{2348}-e_{2356}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	
60	$e_{2356}+e_{1456}+e_{1357}+e_{1267}+e_{2348}+e_{1258}$ *	$\mathfrak{t}(1)$	$(\mathbb{Z}/2\mathbb{Z})^2$
	$-e_{1258}-e_{1267}-e_{1357}-e_{1456}-e_{2348}-e_{2356}$ *	$\mathfrak{t}(1)$	
61	$e_{1356}+e_{2457}+e_{1267}+e_{1348}+e_{2348}$	$\mathfrak{t}(2)$	$\mathbb{Z}/2\mathbb{Z}$
	$-e_{1267}-e_{1348}-\frac{1}{2}e_{1356}-\frac{1}{2}e_{1357}$ $-\frac{1}{2}e_{1456}+\frac{1}{2}e_{1457}-e_{2348}+\frac{1}{2}e_{2356}$ $-\frac{1}{2}e_{2357}-\frac{1}{2}e_{2456}-\frac{1}{2}e_{2457}$	$\mathfrak{u}(1)+\mathfrak{t}(1)$	
	$-e_{1267}-e_{1348}-\frac{1}{2}e_{1356}+\frac{1}{2}e_{1357}$ $+\frac{1}{2}e_{1456}+\frac{1}{2}e_{1457}-e_{2348}+\frac{1}{2}e_{2356}$ $+\frac{1}{2}e_{2357}+\frac{1}{2}e_{2456}-\frac{1}{2}e_{2457}$	$\mathfrak{u}(1)+\mathfrak{t}(1)$	
63	$e_{2356}+e_{1457}+e_{1367}+e_{2348}+e_{1358}+e_{1268}$	$\mathfrak{t}(1)$	$\mathbb{Z}/4\mathbb{Z}$
	$2e_{1268}-e_{1358}+2e_{1367}+2e_{1457}$ $+\frac{1}{2}e_{2348}+\frac{1}{2}e_{2356}$	$\mathfrak{t}(1)$	

65	$e_{3456}+e_{1357}+e_{2467}+e_{1348}$ $+e_{2348}+e_{1258}+e_{1268}$ $-e_{1258}-e_{1268}-e_{1348}-\frac{1}{2}e_{1357}$ $+\frac{1}{2}e_{1367}+\frac{1}{2}e_{1457}+\frac{1}{2}e_{1467}-e_{2348}$ $+\frac{1}{2}e_{2357}+\frac{1}{2}e_{2367}+\frac{1}{2}e_{2457}-\frac{1}{2}e_{2467}$ $-e_{3456}$ $4e_{1258}+4e_{1268}+4e_{1348}+\frac{1}{8}e_{1357}$ $-2e_{1367}-2e_{1457}-\frac{1}{8}e_{1467}+4e_{2348}$ $-\frac{1}{8}e_{2357}-2e_{2367}-2e_{2457}+\frac{1}{8}e_{2467}$ $+\frac{1}{4}e_{3456}$ $-4e_{1258}-4e_{1268}-4e_{1348}-4e_{1357}$ $-\frac{1}{4}e_{2348}-\frac{1}{4}e_{2467}-\frac{1}{4}e_{3456}$		$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
67	$e_{2456}+e_{2357}+e_{1457}+e_{1367}$ $+e_{2348}+e_{1358}+e_{1278}$ * $-4e_{1278}-4e_{1358}-4e_{1367}-4e_{1457}$ $-\frac{1}{4}e_{2348}-\frac{1}{4}e_{2357}-\frac{1}{4}e_{2456}$ *		$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
69	$e_{2456}+e_{2357}+e_{1467}+e_{2348}+e_{1358}+e_{1268}$	$\mathfrak{t}(1)$	$\{1\}$
70	$e_{2456}+e_{1457}+e_{1367}+e_{2348}+e_{1258}+e_{1358}$ $-e_{1258}-e_{1358}-e_{1367}-e_{1457}-e_{2348}-e_{2456}$	$\mathfrak{t}(1)$ $\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
72	$e_{2346}+e_{1456}+e_{1357}+e_{1267}+e_{1248}+e_{1258}$ * $-e_{1248}-e_{1258}-e_{1267}-e_{1357}-e_{1456}-e_{2346}$ *	$\mathfrak{t}(1)$ $\mathfrak{t}(1)$	$(\mathbb{Z}/2\mathbb{Z})^2$
73	$e_{2346}+e_{1456}+e_{1257}+e_{1348}+e_{1358}$ * $-\frac{1}{2}e_{1247}+\frac{1}{2}e_{1248}-e_{1257}-\frac{1}{2}e_{1347}$ $-\frac{1}{2}e_{1348}-e_{1358}-e_{1456}-e_{2346}$ *	$\mathfrak{t}(2)$ $\mathfrak{u}(1)+\mathfrak{t}(1)$	$(\mathbb{Z}/2\mathbb{Z})^2$
75	$e_{2456}+e_{2357}+e_{1457}+e_{1367}$ $+e_{2348}+e_{1258}+e_{1358}$ * $-e_{1258}-e_{1358}-e_{1367}-e_{1457}$ $-e_{2348}+e_{2357}-e_{2456}$ *		$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

77	$e_{2456}+e_{1357}+e_{1267}+e_{2348}$ $+e_{1458}+e_{1368}+e_{1468}$ $-2e_{1258}-e_{1267}-e_{1357}-e_{1368}$ $-e_{1458}-e_{1468}-e_{2348}-e_{2456}$ $-4e_{1267}-4e_{1357}-4e_{1368}-4e_{1458}$ $-4e_{1468}-\frac{1}{4}e_{2348}-\frac{1}{4}e_{2456}$ $8e_{1258}+4e_{1267}+4e_{1357}+4e_{1368}$ $+4e_{1458}+4e_{1468}+\frac{1}{4}e_{2348}+\frac{1}{4}e_{2456}$		$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
79	$e_{2456}+e_{2347}+e_{1567}+e_{1348}+e_{1358}+e_{1268}$	$\mathfrak{t}(1)$	$\{1\}$
80	$e_{2456}+e_{2347}+e_{1367}+e_{1348}+e_{1258}+e_{1358}$ *	$\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
82	$e_{2456}+e_{2347}+e_{1457}+e_{1367}$ $+e_{1348}+e_{1258}+e_{1358}$ * $-4e_{1258}-4e_{1348}-4e_{1358}-4e_{1367}$ $-4e_{1457}-\frac{1}{4}e_{2347}-\frac{1}{4}e_{2456}$ *		$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
84	$e_{2456}+e_{2357}+e_{1467}+e_{2348}$ $+e_{1358}+e_{1368}+e_{1278}$ $-4e_{1278}-4e_{1358}-4e_{1368}-4e_{1467}$ $-\frac{1}{4}e_{2348}-\frac{1}{4}e_{2357}-\frac{1}{4}e_{2456}$		$\mathbb{Z}/4\mathbb{Z}$
86	$e_{2356}+e_{2457}+e_{1367}+e_{2348}$ $+e_{1458}+e_{1268}+e_{1278}$ $-4e_{1268}-4e_{1278}-4e_{1367}-4e_{1458}$ $-\frac{1}{4}e_{2348}-\frac{1}{4}e_{2356}-\frac{1}{4}e_{2457}$		$\mathbb{Z}/4\mathbb{Z}$
88	$e_{3456}+e_{2457}+e_{2367}+e_{1467}$ $+e_{2348}+e_{1358}+e_{1268}$ $-4e_{1268}-4e_{1358}-4e_{1467}-\frac{1}{4}e_{2348}$ $-\frac{1}{4}e_{2367}-\frac{1}{4}e_{2457}-\frac{1}{4}e_{3456}$		$\mathbb{Z}/4\mathbb{Z}$
90	$e_{1356}+e_{2456}+e_{2347}+e_{1257}+e_{1248}$ *	$\mathfrak{t}(2)$	$\mathbb{Z}/2\mathbb{Z}$
92	$e_{1256}+e_{2347}+e_{1357}+e_{1467}+e_{1348}$ *	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	$\mathbb{Z}/2\mathbb{Z}$
93	$e_{2345}+e_{1356}+e_{1457}+e_{1267}+e_{1348}+e_{1258}$ * $-e_{1258}-e_{1267}-e_{1348}+e_{1356}-e_{1457}-e_{2345}$ *	$\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

Table 3.2 shows some data concerning the complex and real semisimple orbits. As noted above, the Weyl group W has thirty-two conjugacy classes of subgroups that are stabilizers of elements. We denote these by $H_1, \dots, H_{32}$. Here $H_1 = W$ and $H_{32} = 1$. The first column of Table 3.2 has the index i. The second column has a basis of $\mathfrak{h}_{H_i}$. In some cases these bases are different from the ones found by Antonyan (in other words, we work with different subgroups of W). The reason is the following. Write $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(8, \mathbb{C})$ and for $p \in \mathfrak{h}$ set $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(p) = \{x \in \hat{\mathfrak{g}}_0 \mid x \cdot p = 0\}$. Then with the bases in Table 3.2 we have that $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(p)$ has a Cartan subalgebra consisting of diagonal matrices, so that we do not have to diagonalize when working with it. This simplifies a lot of computations later on. The fifth column of Table 3.2 has a description of the isomorphism type of the component group of the stabilizer $Z_{\widehat{G}_0}(p)$ for $p \in \mathfrak{h}^\circ_{H_i}$. The numbers in the last column have the following meaning. The space $\mathfrak{h}_{H_i}$ has a finite number of subsets $U_1, \dots, U_r$ (where each U_j is the real span of a basis of $\mathfrak{h}_{H_i}$) such that the set of $p \in \mathfrak{h}^\circ_{H_i}$ with the property that p is $\widehat{G}_0$-conjugate to a $q \in \mathfrak{g}_1$ which is real (i.e., lies in $\mathfrak{g}_1^{\mathbb{R}}$), is exactly the union of $U_1^\circ, \dots, U_r^\circ$, where U_i° is an open set in U_i (see Theorem 3.6.2, and the explanation below it). Let $p \in U_j^\circ$ be conjugate to $q \in \mathfrak{g}_1^{\mathbb{R}}$. Then the $\widehat{G}_0$-orbit of q contains k_j real $\widehat{G}_0(\mathbb{R})$-orbits (which we have determined using Galois cohomology). The last column of Table 3.2 has the r-tuple $(k_1, \dots, k_r)$. This means that the single class of semisimple $\widehat{G}_0$-orbits $\mathfrak{h}^\circ_{H_i}$ contains $k_1 + \dots + k_r$ classes of real $\widehat{G}_0(\mathbb{R})$-orbits.

Table 3.2: Classes of semisimple elements

i	basis of $\mathfrak{h}_{H_i}$	$\mathfrak{z}_{\mathfrak{g}_0}(a)$	$\lvert H_i\rvert$	K	
1	-	A_7	2 903 040	$\{1\}$	-
2	$p_1 + p_2 + p_4$	C_4	51 840	$\mathbb{Z}/2\mathbb{Z}$	$(2,2)$
3	p_1	$2A_3$	23 040	$\mathbb{Z}/2\mathbb{Z}$	$(2,2)$
4	$p_1 + p_2$	$2B_2 + \mathfrak{t}(1)$	3 840	$\mathbb{Z}/2\mathbb{Z}$	$(3,3)$
5	$-2p_1 - p_2 + p_3 + p_5$	B_3	5 040	$\mathbb{Z}/2\mathbb{Z}$	$(4,4)$
6	$\frac{1}{2}p_1 - \frac{1}{2}p_2 + \frac{1}{2}p_3 + p_5 + \frac{1}{2}p_6 + p_7$	$A_3 + \mathfrak{t}(1)$	1 440	$\mathbb{Z}/2\mathbb{Z}$	$(4,4)$
7	$3p_1 + 2p_2 - p_3 + p_7$	$B_2 + A_1$	720	$\mathbb{Z}/2\mathbb{Z}$	$(6,6)$
8	$-2p_1 - p_2 + p_3$	$3A_1 + \mathfrak{t}(1)$	288	$\mathbb{Z}/2\mathbb{Z}$	$(7,7)$
9	$p_1 + p_2, p_4$	$2B_2$	1 920	$(\mathbb{Z}/2\mathbb{Z})^2$	$(4,1,1,4)$
10	p_1, p_2	$4A_1 + \mathfrak{t}(1)$	384	$(\mathbb{Z}/2\mathbb{Z})^2$	$(6,3,1,6)$
11	$p_1 + p_2 + p_4, p_3 + p_5 - p_6 + p_7$	A_3	720	$(\mathbb{Z}/2\mathbb{Z})^2$	$(6,4,4,6)$
12	$p_4, -p_1 + p_3 + p_5$	A_3	720	$(\mathbb{Z}/2\mathbb{Z})^2$	$(3,1,1,3)$

13	$\frac{1}{2}p_1 - \frac{1}{2}p_2 + \frac{1}{2}p_4 + p_5$, $-p_1 - p_3 + p_7$	$B_2 + \mathfrak{t}(1)$	240	$(\mathbb{Z}/2\mathbb{Z})^2$	$(8,8)$
14	$-2p_1 + p_2 + 2p_3 + 2p_4 + p_6 + p_7$, $p_1 - p_3 - p_4 + p_5$	$3A_1$	144	$(\mathbb{Z}/2\mathbb{Z})^2$	$(10,3,3,10)$
15	$p_6, -2p_1 + p_2 - p_3 + p_4 + p_7$	$2A_1 + \mathfrak{t}(2)$	96	$(\mathbb{Z}/2\mathbb{Z})^2$	$(5,5,5,5)$
16	$-2p_1 - p_2 + p_3, p_1 + p_2 + p_4$	$2A_1 + \mathfrak{t}(1)$	72	$(\mathbb{Z}/2\mathbb{Z})^2$	$(4,12,12,4)$
17	$p_1 + p_2, -p_1 + p_3$	$A_1 + \mathfrak{t}(3)$	48	$(\mathbb{Z}/2\mathbb{Z})^2$	$(1,13,1,13)$
18	p_1, p_2, p_4	$4A_1$	192	$(\mathbb{Z}/2\mathbb{Z})^3$	$(8,2,2,2,2,8)$
19	$p_1 + p_2, p_4, p_1 - p_3 + p_7$	B_2	120	$(\mathbb{Z}/2\mathbb{Z})^3$	$(10,2,2,10)$
20	$p_1 - p_2 + p_3, p_1 - p_2 + p_4, p_2 + p_7$	$2A_1 + \mathfrak{t}(1)$	48	$(\mathbb{Z}/2\mathbb{Z})^3$	$(4,14,3,6,6,3,14,4)$
21	$p_4, -p_2 + p_5 + p_6, p_2 - p_3 + p_7$	$2A_1 + \mathfrak{t}(1)$	48	$(\mathbb{Z}/2\mathbb{Z})^3$	$(13,2,3,3,2,13)$
22	$-2p_1 - p_2 + p_3, p_1 + p_2 + p_4, p_5$	$2A_1$	36	$(\mathbb{Z}/2\mathbb{Z})^3$	$(6,2,16,4,4,2,16,6)$
23	$p_1 + p_2, -p_1 + p_3, p_4$	$A_1 + \mathfrak{t}(2)$	24	$(\mathbb{Z}/2\mathbb{Z})^3$	$(18,5,2,2,2,2,5,18)$
24	p_2, p_3, p_6	$\mathfrak{t}(4)$	16	$(\mathbb{Z}/2\mathbb{Z})^3$	$(20,7,2,7,2,20)$
25	$p_1 + p_2, p_4, -p_1 - p_3 + p_5$, $p_1 + p_3 + p_7$	$2A_1$	24	$(\mathbb{Z}/2\mathbb{Z})^4$	$(4,18,4,1,4,5,1,5,4,$ $4,4,18)$
26	$p_1 + p_2, -p_1 + p_3, p_4, p_7$	$A_1 + \mathfrak{t}(1)$	12	$(\mathbb{Z}/2\mathbb{Z})^3$	$(8,2,2,2,2,8)$
27	p_1, p_4, p_6, p_7	$\mathfrak{t}(3)$	8	$(\mathbb{Z}/2\mathbb{Z})^4$	$(27,7,3,3,1,7,3,27)$
28	$-p_1 + p_4, p_5, p_1 + p_6, p_1 + p_7$	$\mathfrak{t}(3)$	8	$(\mathbb{Z}/2\mathbb{Z})^4$	$(4,28,4,8,4,4,10,4,4,$ $10,8,28)$
29	$p_1 + p_2, -p_1 + p_3, p_4, p_5, p_7$	A_1	6	$(\mathbb{Z}/2\mathbb{Z})^5$	$(2,2,8,32,2,2,8,32)$
30	p_1, p_2, p_3, p_4, p_7	$\mathfrak{t}(2)$	4	$(\mathbb{Z}/2\mathbb{Z})^5$	$(38,10,2,6,2,6,3,6,11,$ $11,3,2,6,2,6,10,6,38)$
31	$p_1, p_3, p_4, p_5, p_6, p_7$	$\mathfrak{t}(1)$	2	$(\mathbb{Z}/2\mathbb{Z})^6$	$(52,14,3,4,1,4,4,4,3,$ $14,52)$
32	$p_1, p_2, p_3, p_4, p_5, p_6, p_7$	0	1	U_8	$(72,72,6,6,6,6,20,20,$ $2,2)$

Remark 3.3.1. The group U_8 on line 32 of this table is of order 2^8. It has a center of order 4, generated by $\mathrm{diag}(\imath, \dots, \imath)$. The quotient of U_8 by its center is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$. This is in accordance with [55, Table 1], where the author determines the stabilizer of a generic element. However, the group that he considers is not $\mathrm{SL}(8,\mathbb{C})$ but rather its image in $\mathrm{GL}(\bigwedge^4 \mathbb{C}^8)$.

From Table 3.2 we see that there are very many classes of semisimple orbits. For this reason we cannot list them all. Table 3.3 contains representatives of the semisimple orbits that have a stabilizer of large dimension (≥ 10) (with the exception of the semisimple orbits with number 18 in Table 3.2, as here the representatives turned out to be rather bulky). Also we have given representatives of the orbits up to the action of the group generated by φ and ν (Section 3.2.4). In Table 3.3 the first column has the index of the corresponding complex semisimple orbit (as in Table 3.2). The second column has the representatives of the real semisimple orbits. As outlined before, these are representatives of classes of orbits. They can depend on parameters, and the restrictions on the parameters are also listed. Furthermore, if u is a representative in the table then the corresponding class is given by au where $a \in \mathbb{R} \setminus \{0\}$. So we see that a class in the table can depend on up to three real parameters. All elements of a given class share the same stabilizer in $\widehat{G}_0(\mathbb{R})$ and $\mathfrak{sl}(8,\mathbb{R})$. In the third column we give the isomorphism type of the stabilizer in $\mathfrak{sl}(8,\mathbb{R})$. The last column has the orbit of the element under the group generated by φ and ν. If u is the given representative, then "I" means that the orbit is $\eta(u)$ for $\eta \in \{1, \nu, \varphi\nu, \nu\varphi\nu\}$. For "II" and "III" this last set is $\{1,\nu\}, \{1\}$ respectively. By putting all these orbits together we get representatives of the semisimple $\mathrm{SL}(8,\mathbb{R})$-orbits. We refer to Remark 3.6.4 for the methods we used to compute the images of the maps ν and φ.

Table 3.3: representatives of real semisimple orbits with large stabilizer

i	representative	stabilizer	orbit
2	$e_{1234} + e_{1357} + e_{1368} + e_{2457} + e_{2468} + e_{5678}$	$\mathfrak{sp}(4,\mathbb{R})$	I
3	$(5+3\sqrt{2})e_{1234} - 3e_{1237} + 3e_{1248} + 5e_{1278} + 3e_{1346} - 5e_{1367} + 5e_{1468} + (3+5\sqrt{2})e_{1678} - (3+5\sqrt{2})e_{2345} - 5e_{2357} + 5e_{2458} - 3e_{2578} + 5e_{3456} + 3e_{3567} - 3e_{4568} + (5+3\sqrt{2})e_{5678}$	$\mathfrak{sl}(4,\mathbb{C})$	II
	$e_{1234} + e_{5678}$	$2\mathfrak{sl}(4,\mathbb{R})$	II
4	$-2e_{1234} - e_{1357} + e_{1368} + e_{1458} + e_{1467} + e_{2358} + e_{2367} + e_{2457} - e_{2468} - 2e_{5678}$	$\mathfrak{so}(5,\mathbb{C}) + \mathfrak{u}(1)$	I
	$e_{1234} + e_{1357} + e_{2468} + e_{5678}$	$2\mathfrak{so}(2,3) + \mathfrak{t}(1)$	II
5	$-2e_{1234} + e_{1256} - e_{1357} + e_{1458} + e_{2367} - e_{2468} + e_{3478} - 2e_{5678}$	$\mathfrak{so}(4,3)$	I
	$e_{1234} - e_{1256} + e_{1278} + e_{1357} + e_{1368} - e_{1458} + e_{1467} + e_{2358} - e_{2367} + e_{2457} + e_{2468} + e_{3456} - e_{3478} + e_{5678}$	$\mathfrak{so}(7)$	I

6	$e_{1234}+e_{1256}-2e_{1278}-e_{1357}+2e_{1458}-e_{1467}-e_{2358}+2e_{2367}-e_{2468}-2e_{3456}+e_{3478}+e_{5678}$	$\mathfrak{sl}(4,\mathbb{R})+\mathfrak{t}(1)$	II
	$-2e_{1234}-e_{1256}+e_{1278}+2e_{1357}-e_{1458}+e_{1467}+e_{2358}-e_{2367}+2e_{2468}+e_{3456}-e_{3478}-2e_{5678}$	$\mathfrak{sl}(4,\mathbb{R})+\mathfrak{u}(1)$	II
	$80e_{1234}-48e_{1237}+64e_{1256}-128e_{1278}+48e_{1345}+80e_{1357}+128e_{1458}-64e_{1467}-4e_{2358}+8e_{2367}-5e_{2468}+3e_{2678}-8e_{3456}+4e_{3478}-3e_{4568}+5e_{5678}$	$\mathfrak{sl}(2,\mathbb{H})+\mathfrak{u}(1)$	I
7	$3e_{1234}-e_{1256}-e_{1278}+2e_{1357}+2e_{2468}-e_{3456}-e_{3478}+3e_{5678}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{so}(2,3)$	I
	$-3e_{1234}+e_{1256}+e_{1278}-e_{1357}+e_{1368}-e_{1458}-e_{1467}-e_{2358}-e_{2367}+e_{2457}-e_{2468}+e_{3456}+e_{3478}-3e_{5678}$	$\mathfrak{su}(2)+\mathfrak{so}(4,1)$	I
	$-3e_{1234}+e_{1256}+e_{1278}-e_{1357}+e_{1368}+e_{1458}+e_{1467}+e_{2358}+e_{2367}+e_{2457}-e_{2468}+e_{3456}+e_{3478}-3e_{5678}$	$\mathfrak{su}(2)+\mathfrak{so}(5)$	I
8	$-2e_{1234}+e_{1256}-e_{1357}-e_{2468}+e_{3478}-2e_{5678}$	$3\mathfrak{sl}(2,\mathbb{R})+\mathfrak{t}(1)$	II
	$e_{1234}+e_{1256}+2e_{1357}+3e_{1368}+3e_{1458}+3e_{2367}+3e_{2457}+2e_{2468}+e_{3478}+e_{5678}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	I
	$-128e_{1234}+64e_{1256}-17e_{1357}+15e_{1358}+15e_{1367}+17e_{1368}+15e_{1457}+17e_{1458}+17e_{1467}-15e_{1468}-15e_{2357}+17e_{2358}+17e_{2367}+15e_{2368}+17e_{2457}+15e_{2458}+15e_{2467}-17e_{2468}+4e_{3478}-8e_{5678}$	$\mathfrak{su}(2)+\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	I
	$e_{1234}-e_{1256}+e_{1278}+e_{1357}+e_{1368}+e_{1467}+e_{2358}+e_{2457}+e_{2468}+e_{3456}-e_{3478}+e_{5678}$	$3\mathfrak{su}(2)+\mathfrak{t}(1)$	I
9	$e_{1234}+e_{1357}+e_{2468}+e_{5678}+b(e_{2457}+e_{1368});\ b\neq 0,1,-1$	$2\mathfrak{so}(2,3)$	I
	$-e_{1357}+e_{1467}+4e_{2358}-4e_{2468}+b(e_{1278}-4e_{3456});\ b\neq 0,1,-1$	$2\mathfrak{so}(2,3)$	II
	$-2e_{1234}-(1-b)e_{1357}+(1-b)e_{1368}+(1+b)e_{1458}+(1+b)e_{1467}+(1+b)e_{2358}+(1+b)e_{2367}+(1-b)e_{2457}-(1-b)e_{2468}-2e_{5678};\ b\neq 0,1,-1$	$\mathfrak{so}(5,\mathbb{C})$	I
10	$e_{1234}+e_{5678}+b(e_{1357}+e_{2468});\ b\neq 0,-1,1$	$4\mathfrak{sl}(4,\mathbb{R})+\mathfrak{t}(1)$	II
	$-5e_{1234}-3e_{1236}-3e_{1247}+5e_{1267}+3e_{1348}-5e_{1368}-5e_{1478}-3e_{1678}+3e_{2345}-5e_{2356}-5e_{2457}-3e_{2567}+5e_{3458}+3e_{3568}+3e_{4578}-5e_{5678}+b(8e_{1357}+8e_{2468});\ b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	I

	$-3e_{1356}+5e_{1357}+5e_{1368}+3e_{1378}+5e_{1456}+3e_{1457}+3e_{1468}-5e_{1478}$ $-5e_{2356}+3e_{2357}+3e_{2368}+5e_{2378}+3e_{2456}+5e_{2457}+5e_{2468}-3e_{2478}$ $+b(32e_{1234}+2e_{5678});\ b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	I
	$-(1+b)e_{1234}+(1-b)e_{1278}-(1+b)e_{1357}-(1-b)e_{1458}$ $-(1-b)e_{2367}-(1+b)e_{2468}+(1-b)e_{3456}-(1+b)e_{5678};\ b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{C})+\mathfrak{t}(1)$	II
	$5e_{1234}+3e_{1236}+3e_{1247}-5e_{1267}-3e_{1348}+5e_{1368}+5e_{1478}+3e_{1678}$ $-3e_{2345}+5e_{2356}+5e_{2457}+3e_{2567}-5e_{3458}-3e_{3568}-3e_{4578}+5e_{5678}$ $+b(-8e_{1357}+8e_{2468});\ b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	III
	$-5e_{1357}-3e_{1358}-3e_{1367}+5e_{1368}-3e_{1457}+5e_{1458}+5e_{1467}+3e_{1468}$ $-3e_{2357}-5e_{2358}-5e_{2367}+3e_{2368}-5e_{2457}+3e_{2458}+3e_{2467}+5e_{2468}$ $+b(8e_{1234}+8e_{5678});\ b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{C})+\mathfrak{u}(1)$	III
	$e_{1234}+e_{5678}+b(e_{1357}-e_{2468});\ b\neq 0,1,-1$	$4\mathfrak{sl}(4,\mathbb{R})+\mathfrak{t}(1)$	III
	$2e_{1234}+4e_{1278}+e_{3456}+2e_{5678}+b(-2e_{1346}+4e_{1678}-e_{2345}+2e_{2578});$ $b\neq 0,1,-1$	$2\mathfrak{sl}(2,\mathbb{R})$ + $\mathfrak{sl}(2,\mathbb{C})+\mathfrak{t}(1)$	III
11	$e_{1256}-e_{1278}+e_{1458}+e_{1467}+e_{2358}+e_{2367}-e_{3456}+e_{3478}+b(e_{1234}$ $+e_{1357}+e_{1368}+e_{2457}+e_{2468}+e_{5678});\ b\neq 0,1,-1$	$\mathfrak{sl}(4,\mathbb{R})$	I
	$-e_{1256}+e_{1357}+e_{1458}+e_{2367}+e_{2468}-e_{3478}+b(e_{1234}-e_{1278}$ $-e_{1368}+e_{1467}+e_{2358}-e_{2457}-e_{3456}+e_{5678});\ b\neq 0,1,-1$	$\mathfrak{su}(4)$	I
	$(1+b)e_{1357}+(1+b)e_{1368}+e_{1458}+e_{1467}+e_{2358}+e_{2367}$ $-(1-b)e_{2457}+(1+b)e_{2468}+b(e_{1234}+e_{5678});\ b\neq 0,1,-1$	$\mathfrak{su}(2,2)$	I
	$e_{1257}-e_{1268}+e_{1458}-e_{1467}+e_{2358}-e_{2367}-e_{3457}+e_{3468}+b(e_{1234}$ $+e_{1356}-e_{1378}+e_{2456}-e_{2478}+e_{5678});\ b\neq 0,1,-1$	$\mathfrak{sl}(4,\mathbb{R})$	I
	$-e_{1236}-e_{1238}+4e_{1245}-4e_{1247}+e_{1257}-4e_{1268}+4e_{1345}-4e_{1347}$ $-e_{1357}-4e_{1368}+4e_{1456}+4e_{1458}-4e_{1467}+4e_{1478}+4e_{1567}+4e_{1578}$ $+e_{2346}+e_{2348}-e_{2356}+e_{2358}-e_{2367}-e_{2378}+e_{2457}+4e_{2468}-e_{2568}$ $+e_{2678}-e_{3457}+4e_{3468}+e_{3568}-e_{3678}+4e_{4567}+4e_{4578}+b(-4e_{1235}$ $-4e_{1237}+4e_{1246}-4e_{1248}+2e_{1258}+2e_{1267}-4e_{1346}+4e_{1348}-2e_{1356}$ $+2e_{1378}-e_{1568}-e_{1678}-4e_{2345}-4e_{2347}-2e_{2456}+2e_{2478}+e_{2567}$ $-e_{2578}+2e_{3458}+2e_{3467}+e_{3567}-e_{3578}+e_{4568}+e_{4678});\ b\neq 0,1,-1$	$\mathfrak{su}(1,3)$	I
12	$(1+b)e_{1234}-(1+b)e_{1256}(1-3b)e_{1368}-(1+b)e_{1458}-(1+b)e_{2367}$ $(1-3b)e_{2457}-(1+b)e_{3478}(1+b)e_{5678};\ b\neq 0,1,-1,\frac{1}{3}$	$\mathfrak{sl}(4,\mathbb{R})$	II

	$-e_{1234}+e_{1256}+e_{1458}+e_{2367}+e_{3478}-e_{5678}+b(e_{1368}+e_{2457})$; $b\neq 0,1,-1,3$	$\mathfrak{sl}(4,\mathbb{R})$	II
	$-16e_{1234}+16e_{1256}+16e_{1458}+e_{2367}+e_{3478}-e_{5678}-b(16e_{1368}+e_{2457})$; $b\neq 0,1,-1,3$	$\mathfrak{sl}(4,\mathbb{R})$	II
	$-256e_{1258}-(768-512\sqrt{2})e_{1267}+(768-512\sqrt{2})e_{1456}-(768-512\sqrt{2})e_{1478}-e_{2356}+e_{2378}-e_{3458}-(3+-2\sqrt{2})e_{3467}+b(16e_{1256}+48e_{1278}-16e_{1458}-(48-32\sqrt{2})e_{1467}+(\frac{3}{16}+\frac{1}{8}\sqrt{2})e_{2358}+\frac{1}{16}e_{2367}-\frac{3}{16}e_{3456}-\frac{1}{16}e_{3478})$; $b\neq 0,\pm(16-16\sqrt{2}),48-48\sqrt{2}$	$\mathfrak{sl}(4,\mathbb{R})$	III
	$(-\frac{17}{64}+\frac{3}{16}\sqrt{2})e_{1256}-(\frac{17}{64}-\frac{3}{16}\sqrt{2})e_{1357}-(\frac{17}{64}-\frac{3}{16}\sqrt{2})e_{1467}-e_{2358}-e_{2468}-e_{3478}+b((-\frac{3}{8}+\frac{1}{4}\sqrt{2})e_{1234}+(24+16\sqrt{2})e_{5678})$; $b\neq 0,\pm\alpha,3\alpha,\ \alpha=-\frac{3}{8}+\frac{1}{4}\sqrt{2}$	$\mathfrak{sl}(4,\mathbb{R})$	III
13	$(1+b)e_{1234}+(1+b)e_{1256}-(1-b)e_{1278}+2e_{1458}+2e_{2367}-(1-b)e_{3456}+(1+b)e_{3478}+(1+b)e_{5678}$; $b\neq 0,1,-1,-2$	$\mathfrak{so}(2,3)+\mathfrak{t}(1)$	I
	$e_{1256}-e_{1368}-2e_{1458}-(1-b)e_{1467}-(1-b)e_{2358}-2e_{2367}-e_{2457}+e_{3478}+b(e_{1278}+e_{1357}+e_{2468}+e_{3456})$; $b\neq 0,1,-1,2$	$\mathfrak{so}(2,3)+\mathfrak{u}(1)$	I
	$(1+b)e_{1234}-e_{1357}+e_{1368}+2e_{1458}+2e_{2367}+e_{2457}-e_{2468}+(1+b)e_{5678}+b(e_{1256}+e_{1278}+e_{3456}+e_{3478})$; $b\neq 0,1,-1,-2$	$\mathfrak{so}(4,1)+\mathfrak{u}(1)$	I
	$(1+b)e_{1234}+e_{1256}+e_{1278}-e_{1357}+e_{1368}+(1+b)e_{1458}+(1-b)e_{1467}+(1-b)e_{2358}+(1+b)e_{2367}+e_{2457}-e_{2468}+e_{3456}+e_{3478}+(1+b)e_{5678}$; $b\neq 0,1,-1,-2$	$\mathfrak{so}(5)+\mathfrak{u}(1)$	I
19	$(1+c)e_{1234}+e_{1357}+e_{2468}+(1+c)e_{5678}+b(e_{1368}+e_{2457})-c(e_{1256}+e_{1278}+e_{3456}+e_{3478})$; $b\neq 0,\pm 1,\ c\neq 0,-1,\ \varepsilon b+2\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(2,3)$	I
	$(2+c)e_{1234}-(1+b)e_{1256}-(1+b)e_{1278}+(1-b+c)e_{1357}-(1-b-c)e_{1368}-(1-b-c)e_{2457}+(1-b+c)e_{2468}-(1+b)e_{3456}-(1+b)e_{3478}+(2+c)e_{5678}$; $b\neq 0,\pm 1,\ c\neq 0,-2,\ \varepsilon b+\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(2,3)$	I
	$(1+c)e_{1234}+e_{1357}+e_{2468}+(1+c)e_{5678}+b(e_{1368}+e_{2457})-c(e_{1256}+e_{1278}+e_{3456}+e_{3478})$; $b\neq 0,\pm 1,\ c\neq 0,-1,\ \varepsilon b+2\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(2,3)$	I

	$(1-b+c)e_{1234}-(1+b+c)e_{1256}+2e_{1357}-(1-b)e_{1368}$ $+(1+b)e_{1458}+(1+b)e_{2367}-(1-b)e_{2457}+2e_{2468}-(1+b+c)e_{3478}$ $+(1-b+c)e_{5678}-c(e_{1278}+e_{3456})$; $b\neq 0,\pm 1$, $c\neq 0,-2$, $\varepsilon b+\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(4,1)$	I
	$-(2+c)e_{1234}+(b-1)e_{1357}-(b-1)e_{1368}+(b+1)e_{1458}$ $+(b+1)e_{1467}+(b+1)e_{2358}+(b+1)e_{2367}-(b-1)e_{2457}+(b-1)e_{2468}$ $-(2+c)e_{5678}+c(e_{1256}+e_{1278}+e_{3456}+e_{3478})$; $b\neq 0,\pm 1$, $c\neq 0,-2$, $\varepsilon b+\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(5)$	I
	$(4+c)e_{1257}-4e_{1367}+e_{2458}-(1+c)e_{3468}+b(-4e_{1478}+e_{2356})$ $+c(2e_{1234}+4e_{1268}-e_{3457}+2e_{5678})$; $b\neq 0,\pm 1$, $c\neq 0,-1$, $\varepsilon b+2\delta c\neq 1$ where $\varepsilon,\delta\in\{1,-1\}$	$\mathfrak{so}(2,3)$	I

Remark 3.3.2. The fourvectors in Table 3.3 that have a compact stabilizer play an important role in differential geometry. There are four of them, corresponding to the lines 5, 7, 11 and 19. Their stabilizers are Lie algebras of Riemannian holonomy groups that occur in Berger's classification, [4]. They define Spin(7), quaternion-Kaehler, Calabi-Yau, and hyperkaehler geometries respectively, all described in [57]. We look at each of them in somewhat more detail.
Set

$$\begin{aligned}\Phi_5 = & e_{1234}-e_{1256}+e_{1278}+e_{1357}+e_{1368}-e_{1458}+e_{1467}+e_{2358}-e_{2367}\\ & +e_{2457}+e_{2468}+e_{3456}-e_{3478}+e_{5678}\end{aligned}$$

which equals $\frac{1}{2}(-\omega_1^2+\omega_2^2-\omega_3^2)$ where

$$\omega_1=e_{12}-e_{34}+e_{56}-e_{78},\ \omega_2=e_{13}-e_{42}+e_{57}-e_{86},\ \omega_3=e_{14}-e_{23}+e_{58}-e_{67}.$$

We decompose $\mathbb{R}^8=\mathbb{R}^4+\mathbb{R}^4$ such that ω_i is the sum of two anti-self-dual (ASD) 2-forms on each summand. The stabilizer of Φ_5 is the subgroup $\mathrm{Spin}(7)\subset\mathrm{SO}(8)$ resulting from its spin representation. We note that Φ_5 is equivalent to the fundamental 4-form Φ_K considered in [14, Definition 2.9]. Indeed, the element $g=\mathrm{diag}(1,1,1,1,1,1,-1,-1)$ maps Φ_5 to Φ_K (see [14, (2,23)]).
Next we look at

$$\begin{aligned}\Phi_7 = & -3e_{1234}+e_{1256}+e_{1278}-e_{1357}+e_{1368}+e_{1458}+e_{1467}+e_{2358}+e_{2367}\\ & +e_{2457}-e_{2468}+e_{3456}+e_{3478}-3e_{5678}\end{aligned}$$

which equals $-\frac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2)$ where

$$\omega_1=e_{12}+e_{34}+e_{65}+e_{87},\ \omega_2=-e_{13}-e_{42}+e_{68}-e_{75},\ \omega_3=-e_{14}-e_{23}+e_{67}+e_{58}.$$

Here each ω_i is the sum of two ASD 2-forms as above (after applying the double transposition $(5,6)(7,8)$ which is induced by an element of $\mathrm{SL}(8,\mathbb{R})$). The stabilizer of Φ_7 is the subgroup $\mathrm{Sp}(2)\mathrm{Sp}(1)\cong(\mathrm{Sp}(2)\times\mathrm{Sp}(1))/Z_2\subset\mathrm{SO}(8)$ where $Z_2=\{(1,1),(-1,-1)\}$. The subgroup $\mathrm{Sp}(1)=\mathrm{SU}(2)$ acts as $\mathrm{SO}(3)$ rotating the ω_i. Linear deformations of Φ_7 are investigated in [18].

Now we consider

$$\begin{aligned}\Phi_{11} = &- e_{1256} + e_{1357} + e_{1458} + e_{2367} + e_{2468} - e_{3478}\\ &+ b(e_{1234} - e_{1278} - e_{1368} + e_{1467} + e_{2358} - e_{2457} - e_{3456} + e_{5678}),\ b \neq 0, 1, -1.\end{aligned}$$

We have that Φ_{11} equals $\frac{1}{2}\omega^2 - b\mathrm{Re}(\Omega)$ where $\omega = e_{15} - e_{37} + e_{26} - e_{48}$ is a symplectic 2-form with stabilizer $\mathrm{Sp}(4,\mathbb{R})$ and $\Omega = (e_1 + ie_5)(e_3 - ie_7)(e_2 + ie_6)(e_4 - ie_8)$ is a complex volume form with stabilizer $\mathrm{SL}(4,\mathbb{C})$. The intersection of these two groups is $\mathrm{SU}(4)$, a subgroup of $\mathrm{Spin}(7)$. If $b = \pm 1$ then the form is conjugate to $-\Phi_5$. Indeed $\mathrm{diag}(-1,-1,-1,1,-1,1,1,1)$ (if $b = 1$) and $\mathrm{diag}(1,-1,-1,1,1,1,1,1)$ (if $b = -1$) map Φ_{11} to $-\Phi_5$.
Finally, let

$$\begin{aligned}\Phi_{19} = &- (2+c)e_{1234} + (b-1)e_{1357} - (b-1)e_{1368} + (b+1)e_{1458} + (b+1)e_{1467}\\ &+ (b+1)e_{2358} + (b+1)e_{2367} - (b-1)e_{2457} + (b-1)e_{2468} - (2+c)e_{5678}\\ &+ c(e_{1256} + e_{1278} + e_{3456} + e_{3478}),\\ &b \neq 0, \pm 1,\ c \neq 0, -2,\ \varepsilon b + \delta c \neq 1 \ \text{ where } \varepsilon, \delta \in \{1, -1\}.\end{aligned}$$

With $\omega_1, \omega_2, \omega_3$ as for Φ_7 we have $\Phi_{19} = \frac{1}{2}(-c\omega_1^2 + (b-1)\omega_2^2 - (b+1)\omega_3^2)$.

3.4 Galois cohomology

In this section we make some comments on the use of Galois cohomology in the situation that we consider. For this we revert to the notation of Section 3.2. Let $x \in \mathfrak{g}_1^{\mathbb{R}}$ and let $\mathcal{O} = \widehat{G}_0 \cdot x$ be its $\widehat{G}_0$-orbit. For the involution σ we use complex conjugation, that is we use the maps described in Section 3.2.3. We note that $H^1 \widehat{G}_0$ is trivial (see [5, Corollary III.8.26]). So by Theorem 1.8.1 there is a 1-to-1 correspondence between the elements of $H^1(Z_{\widehat{G}_0}(x))$ and the $\widehat{G}_0(\mathbb{R})$-orbits in $\mathcal{O}$. In the following sections we will see how this is applied to nilpotent and semisimple orbits. In both cases there are some differences to the straightfoirward approach just outlined. For the nilpotent orbits we prefer to work with $\mathfrak{sl}_2$-triples and the actions of the various groups on them. For the semisimple orbits we have the added difficulty that they come in infinite parametrized classes. Finally, we compute Galois cohomology sets with the algorithm of [8]. For this we need one element of each component of $Z_{\widehat{G}_0}(x)$ and one of the main problems we will be concerned with in the next sections is to determine such elements. We also remark that [8] contains algorithms to determine $g \in \widehat{G}_0$ such that $g^{-1}\overline{g} = c$, where c is a given cocycle in $\widehat{G}_0$.

3.5 Real nilpotent orbits

In this section we describe the classification of real nilpotent orbits. It is based on the fact that also real nilpotent orbits are classified by $\mathfrak{sl}_2$-triples. More precisely, we have the following theorem, for which we use the notation introduced in Section 3.2.3. Note that the map ψ of Section 3.2.3 yields a $\widehat{G}_0$-action on $\mathfrak{g}_0$ and $\mathfrak{g}_1$, and a $\widehat{G}_0(\mathbb{R})$ action on $\mathfrak{g}_0^{\mathbb{R}}$ and $\mathfrak{g}_1^{\mathbb{R}}$.

Theorem 3.5.1. *Let $e \in \mathfrak{g}_1^{\mathbb{R}}$ be nilpotent, then there are $h \in \mathfrak{g}_0^{\mathbb{R}}$ and $f \in \mathfrak{g}_1^{\mathbb{R}}$ such that (h, e, f) is an $\mathfrak{sl}_2$-triple. Moreover, $e, e' \in \mathfrak{g}_1^{\mathbb{R}}$ lying in $\mathfrak{sl}_2$-triples (h, e, f), (h', e', f') are $\widehat{G}_0(\mathbb{R})$-conjugate if and only if there is a $g \in \widehat{G}_0(\mathbb{R})$ such that $g \cdot h' = h$, $g \cdot e' = e$, $g \cdot f' = f$.*

Proof. The proof is analogous to the proof of Theorem 1.6.3. The existence of the $\mathfrak{sl}_2$-triple is shown as in [25, Lemma 8.3.5]; in that proof the field is assumed to be algebraically closed, but that assumption plays no role in the proof.
Let (h'', e, f'') be an $\mathfrak{sl}_2$-triple with $h'' \in \mathfrak{g}_0^{\mathbb{R}}$, $e, f'' \in \mathfrak{g}_1^{\mathbb{R}}$. Then the arguments in the proofs of [25, Proposition 8.1.3, Theorem 8.3.6] show that there is a nilpotent $z \in \mathfrak{g}_0^{\mathbb{R}}$ such that with $\tau = \exp(\mathrm{ad} z)$ we have $\tau(h'') = h$, $\tau(e) = e$, $\tau(f'') = f$.
Suppose that there is a $g_0 \in \widehat{G}_0(\mathbb{R})$ with $g_0(e') = e$. Then with $h'' = g_0 \cdot h'$, $f'' = g_0 \cdot f'$ we have that (h'', e, f'') is an $\mathfrak{sl}_2$-triple. Let $\tau = \exp(\mathrm{ad} z)$ be as above. Let $\hat{z}$ be a pre-image of z under φ in $\mathfrak{sl}(8, \mathbb{R})$. Then also $\hat{z}$ is nilpotent and hence $\hat{\tau} = \exp(\hat{z}) \in \widehat{G}_0(\mathbb{R})$. Furthermore, $\psi(\hat{\tau}) = \exp(\mathrm{ad} z)$. So with $g = \hat{\tau} g_0$ we get the second statement of the theorem. □

Now let $\mathcal{T}$ be the set of all $\mathfrak{sl}_2$-triples (h, e, f) with $h \in \mathfrak{g}_0$, $e, f \in \mathfrak{g}_1$. Let $\mathcal{T}^{\mathbb{R}}$ be the set of all $\mathfrak{sl}_2$-triples (h, e, f) with $h \in \mathfrak{g}_0^{\mathbb{R}}$, $e, f \in \mathfrak{g}_1^{\mathbb{R}}$. Consider the conjugation σ of Section 3.2.3. Define $\mathcal{T}^{\sigma} = \{(h, e, f) \in \mathcal{T} \mid \sigma(h) = h, \sigma(e) = e, \sigma(f) = f\}$. Then $\mathcal{T}^{\sigma} = \mathcal{T}^{\mathbb{R}}$. Furthermore $\widehat{G}_0$ acts on $\mathcal{T}$ by $g \cdot (h, e, f) = (g \cdot h, g \cdot e, g \cdot f)$. The action of $\widehat{G}_0(\mathbb{R})$ on $\mathcal{T}^{\mathbb{R}}$ is defined by the same formula. Hence, Theorems 1.8.1, 3.5.1 and the fact that the first Galois cohomology set of $\widehat{G}_0$ is trivial immediately imply the following theorem.

Theorem 3.5.2. *For $(h, e, f) \in \mathcal{T}^{\mathbb{R}}$ set*

$$Z_{\widehat{G}_0}(h, e, f) = \{g \in \widehat{G}_0 \mid g \cdot h = h, g \cdot e = e, g \cdot f = f\}.$$

Then the $\widehat{G}_0(\mathbb{R})$-orbits contained in $\widehat{G}_0 \cdot e$ are in bijection with the Galois cohomology set $H^1(Z_{\widehat{G}_0}(h, e, f), \sigma)$. In particular, a class $[c]$ in $H^1(Z_{\widehat{G}_0}(h, e, f), \sigma)$ corresponds to the $\widehat{G}_0(\mathbb{R})$-orbit of $g \cdot e$, where $g \in \widehat{G}_0$ is such that $g^{-1}\sigma(g) = c$.

In order to apply this theorem we need to compute the Galois cohomology sets of the stabilizers $Z_{\widehat{G}_0}(h, e, f)$. The algorithm of [8] for computing the first Galois cohomology set of a linear algebraic group requires as input a basis of the Lie algebra of the group, as well as one element of each component of the group. Writing $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(8, \mathbb{C})$ we have that the Lie algebra of $Z_{\widehat{G}_0}(h, e, f)$ is

$$\mathfrak{z}_{\hat{\mathfrak{g}}_0}(h, e, f) = \{x \in \hat{\mathfrak{g}}_0 \mid x \cdot h = x \cdot e = x \cdot f = 0\},$$

which we can compute by linear algebra techniques.
So the first problem is to compute one element of each component of the stabilizers $Z_{\widehat{G}_0}(h, e, f)$ where e runs through a set of representatives of the nilpotent orbits. We use the representatives listed in [52, Table 10]. Observe that in the cited list all representatives are real (which is necessary to use Theorem 3.5.2). Note that the equations $g \cdot h = h$, $g \cdot e = e$, $g \cdot f = f$ directly translate to polynomial equations on the entries of $g \in \widehat{G}_0$. So we consider these polynomials in 64 indeterminates, add the polynomial corresponding to the condition $\det(g) = 1$, and compute a Gröbner basis (for which we used the computer algebra system Magma). By studying the Gröbner basis it is often possible to identify elements of $\widehat{G}_0$ such that $Z_{\widehat{G}_0}(h, e, f)$ is the union of the components containing these elements. With the algorithm IsEltOf, it is then possible to see which elements lie in the same component. For the majority of the nilpotent orbits we were able to determine elements of each component of $Z_{\widehat{G}_0}(h, e, f)$ from this Gröbner basis.
However, due to the high complexity of the Gröbner basis algorithm, for some of the orbits this computation failed. For those cases we turned to a slightly different strategy that we describe as follows.
Fix an $\mathfrak{sl}_2$-triple $t = (h, e, f)$ with $h \in \mathfrak{g}_0$, $e, f \in \mathfrak{g}_1$. Consider the stabilizer $Z_{\widehat{G}_0}(t)$ and its Lie algebra $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(t)$. The latter is reductive (cf. [25, Lemma 8.3.9]), so it can be written as the direct sum of its semisimple part $\mathfrak{s}$ and its center $\mathfrak{c}$. We assume that $\mathfrak{s}$ is nonzero,

and we fix a Cartan subalgebra of $\mathfrak{s}$, a set of simple roots in the root system and a set of canonical generators $e_1, \dots, e_s$, $f_1, \dots, f_s$, $h_1, \dots, h_s$. Relative to these choices we consider the group $\Gamma(\mathfrak{s})$ of diagram automorphisms. Also we remark that a $g \in Z_{\widehat{G}_0}(t)$ acts on $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(t)$ by the adjoint action (cf. [6, §3.13]). Furthermore, since $\mathfrak{s}$ is the derived subalgebra of $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(t)$ it is stabilized by the adjoint action of the elements of $Z_{\widehat{G}_0}(t)$.

Proposition 3.5.3. *Each component of $Z_{\widehat{G}_0}(t)$ has an element whose action on $\mathfrak{s}$ induces an element of $\Gamma(\mathfrak{s})$.*

Proof. Let $S \subseteq Z^\circ_{\widehat{G}_0}(t)$ be connected with Lie algebra $\mathfrak{s}$. The action of S on $\mathfrak{s}$ induces a morphism of algebraic groups $S \to \mathrm{Aut}(\mathfrak{s})^\circ$. Since the latter is the adjoint group of $\mathfrak{s}$ this morphism is surjective ([62, Chapter V, Corollary 1]). Let $g \in Z_{\widehat{G}_0}(t)$ and consider the action of g on $\mathfrak{z}_{\mathfrak{g}_0}(t)$. Clearly, this induces an automorphism of $\mathfrak{s}$, say σ_g. As $\mathrm{Aut}(\mathfrak{s}) = \Gamma(\mathfrak{s}) \ltimes \mathrm{Aut}(\mathfrak{s})^\circ$ we can write $\sigma_g = \sigma_\pi \varphi$ with $\varphi \in \mathrm{Aut}(\mathfrak{s})^\circ$ and $\sigma_\pi \in \Gamma(\mathfrak{s})$. Let $h \in S$ be a preimage of φ. Then gh^{-1} lies in the same component as g and acts on $\mathfrak{s}$ as σ_π. □

In other words, the above proposition shows that representatives of the component group are contained in the set of solutions of the systems of equations given by $g \cdot h_i = h_{\pi(i)}$, $g \cdot e_i = e_{\pi(i)}$ and $g \cdot f_i = f_{\pi(i)}$ (one for each possible diagram automorphism π). These equations yield additional polynomial conditions on the 64 indeterminates a_{ij}. Write $\hat{h}_i = \psi^{-1}(h_i)$. Then the equations $g \cdot h_i = h_{\pi(i)}$ are equivalent to $g\hat{h}_i = \hat{h}_{\pi(i)} g$. We just take these equations and add those deriving from $g \cdot h = h$, $g \cdot e = e$ $g \cdot f = f$. This leads to larger systems of equations which are often easier to compute. Note that the polynomials corresponding to $g \cdot h_i = h_{\pi(i)}$ are linear, thus essentially reducing the number of indeterminates.
Using these procedures we managed to determine the components of all $Z_{\widehat{G}_0}(h, e, f)$ for all representatives e of the nilpotent orbits. Using Theorem 3.5.2 and the algorithms of [8] we were able to find the nilpotent orbits of $\widehat{G}_0(\mathbb{R})$.

Example 3.5.4. Let $e = e_{1246} + e_{1357} + e_{1258} + e_{1458} + e_{1678}$ and let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple containing e (this is number 16 in Table 3.1). Computing a Gröbner basis of the ideal defining $Z_{\widehat{G}_0}(h, e, f)$ proved computationally too difficult. The centralizer $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(h, e, f)$ is simple of type G_2. Since this Lie algebra has no diagram automorphisms every component of $Z_{\widehat{G}_0}(h, e, f)$ has an element that acts as the identity on $\mathfrak{z}_{\hat{\mathfrak{g}}_0}(h, e, f)$. Adding the polynomials that are equivalent to this condition we did get an easily computable Gröbner basis. It consists of a_{ij} for $i \neq j$ and

$$a_{11} - a_{88},\ a_{22} - a_{55}a_{77}a_{88}^3,\ a_{33} - a_{44}a_{66}a_{88}^3,\ a_{44}a_{55} - a_{88}^2,\ a_{66}a_{77} - a_{88}^2,\ a_{88}^4 - 1.$$

We see that the set of matrices satisfying these conditions forms a 3-dimensional variety of diagonal matrices consisting of four components. It is straightforward to write down an element of each component. Using IsEltOf it is then a small calculation to show that these elements lie in different components of $Z_{\widehat{G}_0}(h, e, f)$. We see that the component group is cyclic of order 4.

We end this section by showing how to compute the image of the map ν defined in Section 3.2.4 on a complex and real nilpotent orbit.

Lemma 3.5.5. *We have that ν maps a complex nilpotent $\widehat{G}_0$-orbit to itself.*

Proof. Let $e \in \bigwedge^4 \mathbb{C}^8$ be a representative of such an orbit, lying in the homogeneous $\mathfrak{sl}_2$-triple (h, e, f). We may choose e such that h lies in the Cartan subalgebra of $\mathfrak{sl}(8, \mathbb{C})$ consisting of all traceless diagonal matrices. Then $\nu(h) = h$. So $(h, \nu(e), \nu(f))$ is a homogeneous $\mathfrak{sl}_2$-triple containing $\nu(e)$. By Theorem 1.6.3 it follows that e and $\nu(e)$ are $\widehat{G}_0$-conjugate. □

Now we show how we can compute the image of ν on a real nilpotent orbit. Let $e \in \bigwedge^4 \mathbb{R}^8$ lie in the homogeneous $\mathfrak{sl}_2$-triple (h,e,f). First of all we note that the element h spans the Lie algebra of a 1-dimensional torus H in $\widehat{G}_0$. Morever, as shown in [8, Section 6] we can compute an explicit isomorphism $\chi : \mathbb{C}^\times \to H$. Then for $t \in \mathbb{C}^\times$ we have $\chi(t)\cdot e = t^m \cdot e$ for a certain $m \in \mathbb{Z}$. We can find a $t_0 \in \mathbb{C}$ such that $t_0^m = \imath$, and thus we have the element $\chi(t_0) \in \widehat{G}_0$ with $\chi(t_0)\cdot e = \imath e$.
Write $Z = Z_{\widehat{G}_0}(h,e,f)$ and $H^1 Z = \{[c_1],\dots,[c_r]\}$. Let $g_i \in \widehat{G}_0$ be such that $g_i^{-1}\sigma(g_i) = c_i$. Then the real $\widehat{G}_0(\mathbb{R})$-orbits contained in the orbit $\widehat{G}_0 \cdot e$ have representatives $g_1\cdot e,\dots,g_r\cdot e$. By Lemma 3.5.5 it follows that ν permutes these orbits. Using the notation of Section 3.2.4 we compute

$$\nu(g_i\cdot e) = g_0 g_i g_0^{-1} g_0 \cdot e = g_0 g_i g_0^{-1} g_1^{-1} g_1 g_0 \cdot e = g_0 g_i g_0^{-1} g_1^{-1} \cdot \imath e = g_0 g_i g_0^{-1} g_1^{-1} \chi(t_0)\cdot e.$$

Set $\hat{g}_i = g_0 g_i g_0^{-1} g_1^{-1}\chi(t_0)$; then $\hat{g}_i \in \widehat{G}_0$ and $\nu(g_i\cdot e) = \hat{g}_i \cdot e$. Set $\hat{c}_i = \hat{g}_i^{-1}\sigma(\hat{g}_i)$ then $\hat{c}_i$ is a cocyle in Z. By algorithms of [8] we can find j such that $\hat{c}_i$ and c_j are equivalent cocycles. Then ν maps the $\widehat{G}_0(\mathbb{R})$-orbit of $g_i\cdot e$ to the $\widehat{G}_0(\mathbb{R})$-orbit of $g_j\cdot e$.

3.6 Real semisimple orbits

In this section we describe our methods for classifying the real semisimple orbits. We use the complex classification by Antonyan ([52, Table 1]), in which there are 32 distinct classes of complex semisimple orbits. However, in many cases we work with a different space $\mathfrak{h}_{H_i}$, see Table 3.2.

3.6.1 Determining the components of a stabilizer

The first step is to compute the component group of the stabilizer $Z_{\widehat{G}_0}(p)$, where $p \in \mathfrak{h}_{H_i}^\circ$. By Theorem 3.2.1 all elements of $\mathfrak{h}_{H_i}^\circ$ have the same stabilizer, so it does not matter which p in that set we choose. Computing representatives of the elements of the component group works exactly as in Section 2 for most cases, that is, we use direct Gröbner basis computations. However, there are some exceptions that require a different approach. In these cases the standard procedure does not work because the centralizer is a torus, so we cannot work with diagram automorphisms to simplify the computations. The classes for which this happens are the ones numbered 24, 27, 28, 30, 31, 32. Here we show how to deal with the first five, postponing the last case to Section 3.7.
Let $p \in \mathfrak{h}_{H_i}^\circ$ and set $\mathfrak{c} = \mathfrak{z}_{\hat{\mathfrak{g}}_0}(p)$, so that $\mathrm{Lie}(Z_{\widehat{G}_0}(p)) = \mathfrak{c}$, and $\mathfrak{c}' = \mathfrak{z}_{\hat{\mathfrak{g}}_0}(\mathfrak{c})$. Then we have the following.

Proposition 3.6.1. *With the notation above, for each $i \in \{1,\dots,32\}$ and $p \in \mathfrak{h}_{H_i}^\circ$, we have that $g \in Z_{\widehat{G}_0}(p)$ normalizes both $\mathfrak{c}$ and $\mathfrak{c}'$.*

Proof. At first, let $g \in Z_{\widehat{G}_0}(p)$ and observe that, if $c \in \mathfrak{c}$, then $gcg^{-1} \in \mathfrak{c}$, as it is just the result of the action of $Z_{\widehat{G}_0}(p)$ on its Lie algebra. Moreover, if $t \in \mathfrak{c}'$ and $c \in \mathfrak{c}$, we have $[gtg^{-1}, c] = g[t, g^{-1}cg]g^{-1} = 0$, which means that g normalizes $\mathfrak{c}'$. □

Let p lie in $\mathfrak{h}_{H_i}^\circ$ with $i = 24$ or $i = 28$. Then a direct computation shows that $\mathfrak{c}'$ is a Cartan subalgebra of $\hat{\mathfrak{g}}_0$. In fact, it is the standard Cartan subalgebra consisting of diagonal matrices. Since $\mathfrak{c}'$ is a Cartan subalgebra of $\hat{\mathfrak{g}}_0$, $N_{\widehat{G}_0}(\mathfrak{c}')/Z_{\widehat{G}_0}(\mathfrak{c}') \cong W_0$, the Weyl group of $\hat{\mathfrak{g}}_0$. Let $\beta_1,\dots,\beta_7$ be the simple roots of the root system of $\hat{\mathfrak{g}}_0$ with respect to $\mathfrak{c}'$. Fix a corresponding canonical generating set $x_1,\dots,x_7$, $y_1,\dots,y_7$, $h_1,\dots,h_7$. Set $s_i =$

$\exp(x_i)\exp(-y_i)\exp(x_i)$; then $s_i \in N_{\widehat{G}_0}(\mathfrak{c}')$ induces the simple reflection corresponding to β_i. For $w \in W_0$ we fix a reduced expression as product of simple reflections and let $\dot{w}$ be the analogous product of the s_i. Proposition 3.6.1 assures that each element of $Z_{\widehat{G}_0}(p)$ lies in $N_{\widehat{G}_0}(\mathfrak{c}')$ and hence can be written as $\dot{w}z$ with $z \in Z_{\widehat{G}_0}(\mathfrak{c}')$. Since $\mathfrak{c}'$ is the standard Cartan subalgebra of $\hat{\mathfrak{g}}_0$ consisting of traceless diagonal matrices we have that $Z_{\widehat{G}_0}(\mathfrak{c}')$ is the maximal torus of $\widehat{G}_0$ consisting of diagonal matrices. Now let $w \in W_0$ and $z \in Z_{\widehat{G}_0}(\mathfrak{c}')$ be such that $\dot{w}z \cdot p = p$. By Proposition 3.6.1 $\dot{w}z$ normalizes $\mathfrak{c}$. Since $\mathfrak{c}$ is abelian we have $\mathfrak{c} \subset \mathfrak{c}'$ so that also z normalizes (in fact, centralizes) $\mathfrak{c}$. It follows that $\dot{w}$ normalizes $\mathfrak{c}$. We remark that $|W_0| = 8!$. We have written a little code checking which $\dot{w}$ normalizes $\mathfrak{c}$. It turns out that there are 48 such elements. Now we fix one of these 48 elements $\dot{w}$ and try to determine all $z \in Z_{\widehat{G}_0}(\mathfrak{c}')$ such that $\dot{w}z \cdot p = p$. We claim that if $z_1, z_2 \in Z_{\widehat{G}_0}(\mathfrak{c}')$ satisfy this, then $(\dot{w}z_1)^{-1}(\dot{w}z_2)$ lies in the identity component of $Z_{\widehat{G}_0}(p)$. Indeed, $(\dot{w}z_1)^{-1}(\dot{w}z_2) = z_1^{-1}z_2 \in Z_{\widehat{G}_0}(p)$, hence, in particular, $z_1^{-1}z_2 \in Z_{\widehat{G}_0}(p) \cap Z_{\widehat{G}_0}(\mathfrak{c}') = \{h \in Z_{\widehat{G}_0}(\mathfrak{c}') \mid hp = p\}$. It is straightforward to compute the defining equations of the latter variety. An immediate computation in Magma shows that the associated ideal is prime, so that the variety $Z_{\widehat{G}_0}(p) \cap Z_{\widehat{G}_0}(\mathfrak{c}')$ is connected ([40, Corollary 1.4]) and hence contained in the identity component of $Z_{\widehat{G}_0}(p)$. Therefore, for our purposes it is sufficient to check, for each w among the 48 candidates, whether there exists an element $z \in Z_{\widehat{G}_0}(\mathfrak{c}')$ such that $\dot{w}z \cdot p = p$. We write z as a diagonal matrix with indeterminates on the diagonal and consider the polynomial equations equivalent to $(\dot{w}z) \cdot p = p$. We compute a Gröbner basis and, if that is nontrivial, a solution exists. With this strategy, we get a set of 8 elements, say $\dot{w}_1, \dots, \dot{w}_8$ such that the equations have solutions. However, it turns out that $\dot{w}_i \cdot p = p$ for $i = 1, \dots, 8$. The set of the classes represented by the $\dot{w}_i's$ is in fact a group of order 8. We can then conclude just observing that the $\dot{w}_i$ are distinct modulo $Z_{\widehat{G}_0}(p)^\circ$ because they are distinct modulo $Z_{\widehat{G}_0}(\mathfrak{c}')$ and $Z_{\widehat{G}_0}(p)^\circ \subseteq Z_{\widehat{G}_0}(\mathfrak{c}')$. So we get a component group of order 8.

Now let p lie in $\mathfrak{h}^\circ_{H_i}$ with $i = 27, 30$ or 31. Again we have that $\mathfrak{c}$ is a torus (of dimension 3,2,1 respectively), but this time $\mathfrak{c}' = 2A_3 + \mathfrak{c}$ (for $i = 31$), or $\mathfrak{c}' = 4A_1 + \mathfrak{t}''$, where $\mathfrak{c} \subseteq \mathfrak{t}''$ and $\mathfrak{t}''$ is a torus of dimension 3 (for $i = 27, 30$). From now on we fix $i = 31$; the procedure is similiar for p in the other two subalgebras. First observe that there is a Cartan subalgebra of $\hat{\mathfrak{g}}_0$, say $\mathfrak{h}'$, such that $\mathfrak{c} \subseteq \mathfrak{h}' \subseteq \mathfrak{c}'$ (clearly, $\mathfrak{h}'$ is also a Cartan subalgebra of $\mathfrak{c}'$). Again this implies that $N_{\widehat{G}_0}(\mathfrak{h}')/Z_{\widehat{G}_0}(\mathfrak{h}') \cong W_0$. In the sequel we write $H' = Z_{\widehat{G}_0}(\mathfrak{h}')$; then H' is the connected torus whose Lie algebra is $\mathfrak{h}'$. This also implies that $H' \subset Z_{\widehat{G}_0}(\mathfrak{c})$. For each $w \in W_0$ we fix, in the same way as above, a $\dot{w} \in N_{\widehat{G}_0}(\mathfrak{h}')$ inducing w. We also observe that, as $\mathfrak{c}$ is a torus we have that the group $Z_{\widehat{G}_0}(\mathfrak{c})$ is connected ([64, Corollary 3.11]).

Let $g \in Z_{\widehat{G}_0}(p)$; by Proposition 3.6.1 g normalizes $\mathfrak{c}$ and $\mathfrak{c}'$, so that $g\mathfrak{h}'g^{-1} \subseteq \mathfrak{c}'$. Then, since $g\mathfrak{h}'g^{-1}$ is another Cartan subalgebra of $\mathfrak{c}'$, it has to be $Z_{\widehat{G}_0}(\mathfrak{c})$-conjugate to $\mathfrak{h}'$; in other words, there is an $h \in Z_{\widehat{G}_0}(\mathfrak{c})$ with $h(g\mathfrak{h}'g^{-1})h^{-1} = \mathfrak{h}'$. Hence $hg \in N_{\widehat{G}_0}(\mathfrak{h}')$ so that $hg = t\dot{w}$ for some $w \in W_0$ and $t \in H'$. As $h \in Z_{\widehat{G}_0}(\mathfrak{c})$ we have $(hg)\mathfrak{c}(hg)^{-1} = \mathfrak{c}$, hence $t\dot{w}\mathfrak{c}\dot{w}^{-1}t^{-1} = \mathfrak{c}$, which is equivalent to $\dot{w}\mathfrak{c}\dot{w}^{-1} = \mathfrak{c}$. As in the previous paragraph, we can then compute the subset W^* of W_0 consisting of all $w \in W_0$ such that $\dot{w}$ normalizes $\mathfrak{c}$. It turns out that $|W^*| = 1152$. Summarizing: for any $g \in Z_{\widehat{G}_0}(p)$ we have $g = k\dot{w}$, where $k \in Z_{\widehat{G}_0}(\mathfrak{c})$ and $w \in W^*$. With the algorithm IsEltOf we can check for each pair $w_1, w_2 \in W^*$ whether $\dot{w}_1$, $\dot{w}_2$ are equal modulo $Z_{\widehat{G}_0}(\mathfrak{c})$, that is, whether $\dot{w}_1\dot{w}_2^{-1} \in Z_{\widehat{G}_0}(\mathfrak{c})$. It turns out that W^* splits into two disjunct classes, $W^* = W_0^* \cup W_1^*$ where $|W_0^*| = |W_1^*| = 512$ and, if $w_1, w_2 \in W_0^*$, or $w_1, w_2 \in W_1^*$, then $\dot{w}_1$ and $\dot{w}_2$ are equal modulo $Z_{\widehat{G}_0}(\mathfrak{c})$. For $w \in W^*$ set $X_w = \{k\dot{w} \mid k \in Z_{\widehat{G}_0}(\mathfrak{c}), (k\dot{w}) \cdot p = p\}$. It is straightforward to see that if $\dot{w}_1$, $\dot{w}_2$ are equal modulo $Z_{\widehat{G}_0}(\mathfrak{c})$ then $X_{w_1} = X_{w_2}$. The set W_0^* contains the identity and let w_1 be a fixed element of W_1^*. Then it follows that $Z_{\widehat{G}_0}(p) = X_1 \cup X_{w_1}$. Therefore, we deduce that it is sufficient to solve the two systems of equations $kp = p$ and $k\dot{w}_1 p = p$ for $k \in Z_{\widehat{G}_0}(\mathfrak{c})$. By inspecting the Lie algebra $\mathfrak{c}'$ of $Z_{\widehat{G}_0}(\mathfrak{c})$ we see that the latter group

$Z_{\widehat{G}_0}(\mathfrak{c}) = (\mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(4,\mathbb{C})) \cdot T_1$ where each $\mathrm{SL}(4,\mathbb{C})$ occupies a 4×4-block in the matrix of an element of $Z_{\widehat{G}_0}(\mathfrak{c})$ and T_1 is a 1-dimensional torus with Lie algebra $\mathfrak{c}$. So we can write an element k of $Z_{\widehat{G}_0}(\mathfrak{c})$ as a matrix depending on 33 parameters $a_1, \ldots, a_{32}, s_1$. Then, we compute kp and $k\dot{w}_1 p$, two elements in $\bigwedge^4 \mathbb{C}^8$ of the form $\sum_{1 \le i<j<k<l \le 8} \gamma^{ijkl} e_{ijkl}$, where $\gamma^{ijkl} \in \mathbb{C}[a_1, \ldots, a_{32}, s_1]$; now, if $p = \sum_{1 \le i<j<k<l \le 8} q^{ijkl} e_{ijkl}$, we deduce the equations $\gamma^{ijkl} = q^{ijkl}$ (at this point, we also add the equations expressing that the determinants of the two 4×4-blocks is 1). In particular, we will have two large systems of equations that we can reduce computing their Gröbner bases. Now, observe that by the construction above the number of solutions is not finite. Indeed, $T_1 = Z_{\widehat{G}_0}(p)^\circ \subset Z_{\widehat{G}_0}(\mathfrak{c})$ (as $\mathfrak{c} \subset \mathfrak{c}'$) so if $g \in T_1$ then g is a solution of the first set of equations and $g\dot{w}_1^{-1}$ is a solution of the second set of equations. However, as $T_1 = Z_{\widehat{G}_0}(p)^\circ$ we can fix the parameter s_1 arbitrarily, after which the solution sets are finite. It turns out that the first system gives 64 solutions, the second one 128. After that, we can just conclude computing the group generated by those solutions and then computing it modulo the identity component $Z_{\widehat{G}_0}(\mathfrak{c})^\circ$, using again the algorithm IsEltOf. In the end we find a component group of 64 elements.

3.6.2 Determining the semisimple orbits

Now we turn to the problem of determining the semisimple $\widehat{G}_0(\mathbb{R})$-orbits. We fix a subgroup H of W and write $\mathcal{A} = \mathfrak{h}_H^\circ$. We wish to determine the $\widehat{G}_0(\mathbb{R})$-orbits $\widehat{G}_0(\mathbb{R}) \cdot q$ such that the complex orbit $\widehat{G}_0 \cdot q$ has a point q' in $\mathcal{A}$. The problem is that q' may be non-real, but we need the real point q in order to use Galois cohomology. We use the theorem below to deal with this. In order to formulate it we first need some notation. Set

$$N_{\widehat{G}_0}(\mathcal{A}) = \{g \in \widehat{G}_0 \mid g \cdot p \in \mathcal{A} \text{ for all } p \in \mathcal{A}\},$$
$$Z_{\widehat{G}_0}(\mathcal{A}) = \{g \in \widehat{G}_0 \mid g \cdot p = p \text{ for all } p \in \mathcal{A}\}.$$

Let $N_W(H)$ denote the normalizer of H in W. Let $g \in N_{\widehat{G}_0}(\mathcal{A})$. Then there is a $w_g \in N_W(H)$ such that $g \cdot p = w_g \cdot p$ for all $p \in \mathcal{A}$ ([27, Lemma 4.13]). We set $\Gamma_H = N_W(H)/H$ and define a map $\varphi : N_{\widehat{G}_0}(\mathcal{A}) \to \Gamma_H$ by $\varphi(g) = w_g H$. By [27, Lemma 4.13] this is well defined and a surjective group homomorphism with kernel $Z_{\widehat{G}_0}(\mathcal{A})$. From now on we use the bar notation to indicate the complex conjugation. Now we can state the theorem; for the proof we refer to [27, Proposition 4.14].

Theorem 3.6.2. *Let $H^1\Gamma_H = \{[\gamma_1], \ldots, [\gamma_r]\}$. Then:*

1. *Let $\mathcal{O}$ be a $\widehat{G}_0$-orbit with a representative in $\mathcal{A}$. If $\mathcal{O}$ has real points then there is a $q \in \mathcal{A} \cap \mathcal{O}$ and an i with $1 \le i \le r$ such that $\overline{q} = \gamma_i^{-1} q$.*
2. *Let $q \in \mathcal{A}$ be such that $\overline{q} = \gamma_i^{-1} q$ for some $1 \le i \le r$. Then the orbit of q has real points if and only if there exists a cocycle $c_i \in Z^1(N_{\widehat{G}_0}(\mathcal{A}))$ such that $\varphi(c_i) = \gamma_i$. If the latter holds, then gq is a real point of the orbit of q, where $g \in \widehat{G}_0$ is such that $g^{-1}\overline{g} = c_i$.*

We note that the element $g \in \widehat{G}_0$ in the second statement exists because $H^1\widehat{G}_0$ is trivial. To apply this theorem, we have to accomplish the following tasks:

1. Compute $H^1\Gamma_H = \{[\gamma_1], \ldots, [\gamma_r]\}$.
2. For $1 \le i \le r$ describe the sets $\mathcal{A}_{\gamma_i} = \{q \in \mathcal{A} \mid \overline{q} = \gamma_i^{-1} q\}$.
3. For each i find a cocycle $c_i \in Z^1 N_{\widehat{G}_0}(\mathcal{A})$ with $\varphi(c_i) = \gamma_i$, or decide that no such cocycle exists.

4. Compute a $g_{\gamma_i} \in \widehat{G}_0$ such that $g_{\gamma_i}^{-1}\overline{g}_{\gamma_i} = c_i$ and describe the set $g_{\gamma_i} \cdot \mathcal{A}_{\gamma_i}$.

Write, as in the theorem, $H^1\Gamma_H = \{[\gamma_1], \dots, [\gamma_r]\}$. Then every orbit with real points and a representative in $\mathcal{A}$ has a representative in $\mathcal{A}_{\gamma_i}$ for some i. Furthermore, if $q \in \mathcal{A}_{\gamma_i}$ then $g_{\gamma_i} \cdot q$ is a real point of the orbit of q.

We first comment on the steps above. Then we say how we find the real orbits. The first step is easy: Γ_H is a finite group of size bounded by the size of W (which is 2903040). So we can compute $H^1\Gamma_H$ by brute force (first listing all cocycles in Γ_H, then dividing them into equivalence classes). For the second step we note that $\mathcal{A}$ is an open set in the subspace $\mathfrak{h}_H$. Let $q_1, \dots, q_k$ be a basis of $\mathfrak{h}_H$; it is given in Table 3.2. From that table we see that the basis consists of real elements. We now view $\mathfrak{h}_H$ as a vector space over $\mathbb{R}$ of dimension $2k$. We note that the matrix of γ_i with respect to the given basis of $\mathfrak{h}_H$ has real coefficients. So we can write $\gamma_i^{-1} \cdot q_j = \sum_{s=1}^k \delta_{sj} q_s$ with $\delta_{sj} \in \mathbb{R}$. Then the set of $q \in \mathfrak{h}_H$ with $\bar{q} = \gamma_i^{-1} q$ consists of the linear combinations $\sum_{j=1}^k (a_j + \imath b_j) q_j$ where $a_j, b_j \in \mathbb{R}$, the vector with coordinates $a_1, \dots, a_k$ is an eigenvector with eigenvalue 1 of the matrix (δ_{sj}) and the vector with coordinates $b_1, \dots, b_k$ is an eigenvector of the same matrix with eigenvalue -1. Since (δ_{sj}) is real and γ_i is a cocycle it follows that the matrix is of order 2. Hence the solution space has $\mathbb{R}$-dimension k. It also follows that $\gamma_i = \gamma_i^{-1}$ and that the solution space is closed under complex conjugation. So we can find a basis $q_1', \dots, q_k'$ of the real space $\{q \in \mathfrak{h}_H \mid \bar{q} = \gamma_i^{-1} q\}$ and $\mathcal{A}_{\gamma_i}$ is an open set in it.

For the third step we note that even if the cocycle c_i exists it may not be easy to find it because the set $\varphi^{-1}(\gamma_i)$ is not easy to describe. We used a few ad-hoc methods that are not guaranteed to succeed, but for us they always did. We also remark that if such methods do not succeed in finding c_i then it is possible to use the more systematic methods of [7]. For our first method we let $\delta : N_{\widehat{G}_0}(\mathfrak{h}) \to W$ be the usual projection. Let $w_i \in N_W(H)$ be such that $w_i H = \gamma_i$. Then $\delta^{-1}(w_i) \subset \varphi^{-1}(\gamma_i)$. Indeed, let $g \in N_{\widehat{G}_0}(\mathfrak{h})$ be such that $\delta(g) = w_i$; as $w_i \in N_W(H)$ by (3.2.1) we see that g stabilizes $\mathfrak{h}_H^\circ$ and acts on it in the same way as w_i, so that $g \in \varphi^{-1}(\gamma_i)$. Now suppose that $[\gamma_i]$ has m elements $\gamma_i^1, \dots, \gamma_i^m$ (observe that we computed these in the first step). Let $w_i^j \in W$ be such that $\gamma_i^j = w_i^j H$. Then for each i, j and for each $w' \in w_i^j H$ we computed an element of $\delta^{-1}(w')$. In the majority of cases at least one of these elements turned out to be a cocycle. For the remaining cases we succeeded with brute force: taking a general matrix $g \in \widehat{G}_0$ whose entries are indeterminates and considering the polynomial equations that are equivalent to $g \cdot q_i = \gamma_i \cdot q_i$ for $1 \le i \le r$. We computed a Gröbner basis of the ideal generated by these polynomials, and looked for a cocycle in the solution set.

The fourth step can be tackled with algorithms given in [8]. However, in our case the following simple method always worked. Let $c_j' = P^{-1} c_j P$ be a diagonal matrix, where P is the invertible matrix whose columns are the eigenvectors of c_j. In all cases P turns to be a real matrix, so that we can proceed in the following way. It is a straightforward calculation to find a diagonal matrix $s \in \widehat{G}_0$ such that $s^{-1}\overline{s} = c_j'$ and such that the determinant of sP^{-1} equals 1, and then $(sP^{-1})^{-1}\overline{sP^{-1}} = Ps^{-1}\overline{s}P^{-1} = Pc_j'P^{-1} = c_j$. So we can set $g_{\gamma_i} = sP^{-1}$. Furthermore $g_{\gamma_i} \cdot \mathcal{A}_{\gamma_i}$ is an open set in the subspace spanned by $g_{\gamma_i} \cdot q_1', \dots, g_{\gamma_i} \cdot q_k'$.

Let $p \in g_{\gamma_i} \cdot \mathcal{A}_{\gamma_i}$; then p is real. By Theorem 1.8.1 the $\widehat{G}_0(\mathbb{R})$-orbits contained in the $\widehat{G}_0$-orbit of p are in bijection with $H^1 Z_{\widehat{G}_0}(p)$. Furthermore, all elements $q \in \mathcal{A}_{\gamma_i}$ have the same stabilizer in $\widehat{G}_0$ (Theorem 3.2.1). Denote this stabilizer by Z. Above we have commented on how we determined it. Then $Z_{\widehat{G}_0}(p) = g_{\gamma_i} Z g_{\gamma_i}^{-1}$.

Example 3.6.3. Let $H = H_{10}$ then from Table 3.2 we see that $\mathfrak{h}_H$ is spanned by p_1, p_2. Furthermore, $\mathcal{A} = \mathfrak{h}_H^\circ$ consists of $xp_1 + yp_2$ with $xy(x^2 - y^2) \neq 0$. The group Γ_H is of order

8 and the actions of two generators on $\mathfrak{h}_H$ is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These two matrices are real, so the first Galois cohomology of Γ_H consists of the conjugacy classes of elements of order dividing 2. Representatives of these classes are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Denoting these elements by $\gamma_1, \dots, \gamma_4$ we have that the real spaces $\{q \in \mathfrak{h}_H \mid \bar{q} = \gamma_i^{-1} q\}$ are spanned by respectively $\{p_1, p_2\}$, $\{\imath p_1, \imath p_2\}$, $\{\imath p_1, p_2\}$, $\{p_1 - p_2, \imath(p_1 + p_2)\}$.
Let us consider the third case. We have that $\mathcal{A}_{\gamma_3}$ consists of $a\imath p_1 + bp_2$ with $a, b \in \mathbb{R}$ and $ab(a^2 + b^2) \neq 0$. Since $a, b \in \mathbb{R}$ the latter is equivalent to $ab \neq 0$. Set

$$c_3 = \operatorname{diag}(1, -1, 1, 1, 1, 1, 1, -1),\ g_3 = \operatorname{diag}(1, \imath, 1, 1, 1, 1, 1, -\imath).$$

Then $c_3, g_3 \in \widehat{G}_0$, c_3 is a cocycle inducing the action of γ_3 on $\mathfrak{h}_H$ and $g_3^{-1}\bar{g}_3 = c_3$. In fact,

$$g_3 \cdot \imath p_1 = -e_{1234} + e_{5678},\ g_3 \cdot p_2 = e_{1357} + e_{2468}.$$

Write $q_1' = e_{1234} - e_{5678}$, $q_2' = e_{1357} + e_{2468}$. Then $g_3 \cdot \mathcal{A}_{\gamma_3}$ consists of $aq_1' + bq_2'$ with $a, b \in \mathbb{R}$ and $ab \neq 0$. Let $Z = Z_{\widehat{G}_0}(q)$ with $q \in \mathcal{A}_{\gamma_3}$. Then Z is of type $4A_1 + \mathfrak{t}(1)$ and has four components. We have that $Z_{\widehat{G}_0}(q')$ with $q' \in g_3 \cdot \mathcal{A}_{\gamma_3}$ is $g_3 Z g_3^{-1}$. We computed its Galois cohomology with the algorithms of [8]. The first Galois cohomology set has three elements. One is the class of the identity leading to the orbits with representatives $aq_1' + bq_2'$ with $ab \neq 0$. Set

$$h_2 = \begin{pmatrix}
\frac{5}{4} + \frac{3}{4}\imath & 0 & 0 & 0 & \frac{3}{4} + \frac{5}{4}\imath & 0 & 0 & 0 \\
0 & \frac{1}{2} - \frac{1}{2}\imath & 0 & 0 & 0 & -\frac{1}{2} - \frac{1}{2}\imath & 0 & 0 \\
0 & 0 & -\frac{1}{2}\imath & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} - \frac{1}{2}\imath & 0 & 0 & 0 & -\frac{1}{2} - \frac{1}{2}\imath \\
\frac{3}{4} + \frac{5}{4}\imath & 0 & 0 & 0 & \frac{5}{4} + \frac{3}{4}\imath & 0 & 0 & 0 \\
0 & -\frac{1}{2} - \frac{1}{2}\imath & 0 & 0 & 0 & \frac{1}{2} - \frac{1}{2}\imath & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}\imath & 0 \\
0 & 0 & 0 & -\frac{1}{2} - \frac{1}{2}\imath & 0 & 0 & 0 & \frac{1}{2} - \frac{1}{2}\imath
\end{pmatrix}.$$

The second element of $H^1 Z_{\widehat{G}_0}(q')$ is $h_2^{-1}\bar{h}_2$. Set $q_1'' = -8h_2 \cdot q_1'$ and $q_2'' = -h_2 \cdot q_2'$ (here the coefficients -8 and -1 have been added to make the formulas below come out a bit nicer). Then

$$\begin{aligned}
q_1'' &= 5e_{1234} + 3e_{1238} - 3e_{1247} - 5e_{1278} + 3e_{1346} + 5e_{1368} - 5e_{1467} - 3e_{1678} - 3e_{2345} \\
&\quad + 5e_{2358} - 5e_{2457} + 3e_{2578} + 5e_{3456} - 3e_{3568} + 3e_{4567} - 5e_{5678} \\
q_2'' &= e_{1357} + e_{2468}.
\end{aligned}$$

The third element of the first Galois cohomology set is $h_3^{-1}\bar{h}_3$. We do not report h_3 here, but if we set $q_1''' = h_3 \cdot q_1'$ and $q_2''' = 16h_3 \cdot q_2'$ then

$$\begin{aligned}
q_1''' &= 4e_{1234} - \tfrac{1}{4}e_{5678} \\
q_2''' &= 17e_{1357} - 15e_{1358} - 15e_{1367} - 17e_{1368} - 15e_{1457} - 17e_{1458} - 17e_{1467} + 15e_{1468} \\
&\quad + 15e_{2357} - 17e_{2358} - 17e_{2367} - 15e_{2368} - 17e_{2457} - 15e_{2458} - 15e_{2467} + 17e_{2468}.
\end{aligned}$$

We conclude that the $\widehat{G}_0$-orbit of $a\imath p_1 + bp_2$ with $a, b \in \mathbb{R}$, $ab \neq 0$ has three real $\widehat{G}_0(\mathbb{R})$-orbits with representatives $-aq_1' + bq_2'$, $\frac{1}{8}aq_1'' - bq_2''$, $-aq_1''' + \frac{1}{16}bq_3'''$. In particular, the element γ_3 corresponds to three classes of real orbits and the elements of each class share the same stabilizer in $\widehat{G}_0(\mathbb{R})$. The elements γ_1, γ_2, γ_4 yield 6,6,1 classes respectively. We conclude that the single complex class $\mathfrak{h}_H^\circ$ gives rise to 16 classes of real orbits.

Remark 3.6.4. Now we show how to determine the images of the semisimple orbits of the maps ν and φ from Section 3.2.4. We use the notation from above and consider the semisimple orbits that lie in the complex orbits of the sets $g_{\gamma_j} \cdot \mathcal{A}_{\gamma_j}$ for $1 \le j \le r$.
As seen in Section 3.2.4 ν maps the real orbits contained in the $\widehat{G}_0$-orbit of a $v \in \bigwedge^4 \mathbb{C}^8$ to the real orbits contained in the $\widehat{G}_0$-orbit of $\imath v$. It is often the case that $\mathcal{A}_{\gamma_l} = \imath \mathcal{A}_{\gamma_j}$ for certain l, j. In that case we have that the map ν from Section 3.2.4 maps the $\mathrm{SL}(8,\mathbb{C})$-orbit of an element in $\mathcal{A}_{\gamma_j}$ to the $\mathrm{SL}(8,\mathbb{C})$-orbit of an element of $\mathcal{A}_{\gamma_l}$.
As φ is the identity on the Cartan subspace $\mathfrak{h}$, it follows that φ maps each semisimple complex orbit to itself. Consider an element of a set $\mathcal{A}_{\gamma_j}$. We can compute the action of φ on the real orbits contained in the complex orbit of a $u \in \mathcal{A}_{\gamma_j}$ as follows. Let $h = g_{\gamma_j}$ then $h \cdot u$ is real. Let $H^1 Z_{\widehat{G}_0}(h \cdot u) = \{[c_1], \dots, [c_r]\}$ with $c_i \in \widehat{G}_0$ such that $c_i \sigma(c_i) = 1$. Let $g_i \in \widehat{G}_0$ be such that $g_i^{-1}\sigma(g_i) = c_i$; then $g_1 h \cdot u, \dots, g_r h \cdot u$ are representatives of the real $\widehat{G}_0(\mathbb{R})$-orbits in the $\widehat{G}_0$-orbit of u. Applying φ we get $\varphi(g_k h \cdot u) = \varphi(g_k h) h^{-1} \cdot hu$. Write $a_k = \varphi(g_k h) h^{-1}$ and $b_k = a_k^{-1}\sigma(a_k)$. Then the orbit of $\varphi(g_k h \cdot u)$ corresponds to the cocycle b_k. We can compute l such that b_k is equivalent to c_l in $Z^1 Z_{\widehat{G}_0}(h \cdot u)$. Then φ maps the orbit with representative $g_k h \cdot u$ to the orbit with representative $g_l h \cdot u$.

Remark 3.6.5. The numbers displayed in the sixth column of Table 3.2 often display a certain symmetry. This is explained by the previous remark. Consider the sets $\mathcal{A}_{\gamma_1}, \dots, \mathcal{A}_{\gamma_r}$. Then the list in the sixth column is $(k_1, \dots, k_r)$. This means that the $\widehat{G}_0$-orbit of an element in $\mathcal{A}_{\gamma_j}$ contains k_j real $\mathrm{SL}(8,\mathbb{R})$-orbits for $1 \le j \le r$. If $\mathcal{A}_{\gamma_l} = \imath \mathcal{A}_{\gamma_j}$ for certain l, j then ν maps the $\widehat{G}_0$-orbit of an element in $\mathcal{A}_{\gamma_j}$ to the $\widehat{G}_0$-orbit of an element of $\mathcal{A}_{\gamma_l}$. So since ν also maps real orbits to real orbits we have $k_l = k_j$. In many cases we have that multiplication by i permutes the sets $\mathcal{A}_{\gamma_j}$. So if that happens we see that the list $(k_1, \dots, k_r)$ has a nontrivial symmetry of order 2.

3.7 The Cartan subspaces in $\mathfrak{g}_1^{\mathbb{R}}$

A Cartan subspace in $\mathfrak{g}_1^{\mathbb{R}}$ is a maximal abelian subspace consisting of semisimple elements. In this section we classify the Cartan subspaces in $\mathfrak{g}_1^{\mathbb{R}}$. Let $\mathfrak{c}$ be a subspace of $\mathfrak{g}_1^{\mathbb{R}}$ and let $\mathfrak{c}^{\mathbb{C}}$ be the subspace of $\mathfrak{g}_1$ spanned by $\mathfrak{c}$. From [27] we recall that $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{g}_1$ if and only if $\mathfrak{c}^{\mathbb{C}}$ is a Cartan subspace of $\mathfrak{g}_1$.
For the remainder of this section we set $Z = Z_{\widehat{G}_0}(\mathfrak{h})$ and $N = N_{\widehat{G}_0}(\mathfrak{h})$. By [27, Theorem 4.7] and the fact that $H^1 \widehat{G}_0 = 1$ we have the following theorem.

Theorem 3.7.1. *There is a bijection between the elements of $H^1 N$ and the Cartan subspaces in $\mathfrak{g}_1^{\mathbb{R}}$. More precisely, let $[c] \in H^1 N$ and let $g \in \widehat{G}_0$ be such that $g^{-1}\bar{g} = c$; then $[c]$ corresponds to the Cartan subspace $g \cdot \mathfrak{h}$.*

It follows that we need to determine $H^1 N$. The group N is finite but very large (below we show that it is of order $2^8 \cdot 2903040$), so we cannot compute $H^1 N$ by brute force. Therefore we use the strategy outlined in Section 1.8 using the exact sequence

$$1 \longrightarrow Z \longrightarrow N \xrightarrow{j} W \longrightarrow 1$$

where $W = N/Z$ which is equal to the Weyl group of the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. So we know W, and now we outline how we computed Z. First we observe that $G_0 \subset G^\theta = \{g \in G \mid g\theta = \theta g\}$ and both groups have the same Lie algebra (which is $\mathfrak{g}_0$). Set $H = Z_G(\mathfrak{h})$ then H is a connected torus and since its Lie algebra is $\mathfrak{h}$, which is contained in $\mathfrak{g}_1$, the intersection of H with G^θ is a finite group. We first compute

this intersection. Let $x_1, \dots, x_7$ be the positive simple root vectors in a fixed canonical generating set of $\mathfrak{g}$ relative to the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Define the elements $w_i(t) = \exp(t\mathrm{ad}x_i)\exp(-t^{-1}\mathrm{ad}x_i)\exp(t\mathrm{ad}x_i)$, $h_i(t) = w_i(t)w_i(1)^{-1}$ where $t \in \mathbb{C}^\times$. Then $H = \{h_1(t_1)\cdot\dots\cdot h_7(t_7) \mid t_i \in \mathbb{C}^\times\}$ (cf. [62, Chapter 3, Lemma 28]). So we can write an arbitrary element of H as a 133×133 matrix depending on seven parameters $t_1, \dots, t_7$. Then the condition that an element of H commutes with θ translates to polynomial equations in $t_1, \dots, t_7$. We treat $t_1, \dots, t_7$ as indeterminates and solve these equations by computing a Gröbner basis and find the set $H \cap G^\theta$ consisting of 128 elements. Now with the algorithm IsEltOf, we find the subset of elements that lie in G_0; it has 64 elements. Now we consider the map $\psi : \widehat{G}_0 \to G_0$ (see Section 3.2) and observe that $\psi(Z) = H \cap G_0$. Let $g \in H \cap G_0$ then we compute a preimage $\psi^{-1}(g)$ in the following way. By the algorithm of [16] (see also [25, §5.7]) we can write $g = \exp(\mathrm{ad}z_1)\cdot\ldots\cdot\exp(\mathrm{ad}z_s)$ where $z_1, \dots, z_j \in \mathfrak{g}_0$ are nilpotent. Let $\hat{z}_i \in \hat{\mathfrak{g}}_0$ be such that $\psi(\hat{z}_i) = z_i$. Then $\exp(\hat{z}_1)\cdot\ldots\cdot\exp(\hat{z}_s)$ is a preimage of g. So we get 64 preimages. To this set we add the kernel of ψ which is of order 4 and generated by the 8×8 diagonal matrix with $\imath$ on the diagonal. As a result we find the group Z of order 2^8.
Thanks to the results described in Section 3.4, we were able to develop an algorithm for computing $H^1 N$, which we summarize in the following steps:

1. Compute $H^1 W$. Since the complex conjugation acts trivially on W, the cohomology classes coincide with the conjugacy classes of elements of order 2 in W. These are easily calculated by GAP. In particular, it turns out that $H^1 W$ has exactly ten elements, and we denote fixed representatives of the classes in $H^1 W$ by π_i for $1 \le i \le 10$.

2. Determine cocycles $n_i \in N$ such that $j(n_i) = \pi_i$; note that $j^{-1}(\pi_i)$ has 2^8 elements, and so for each $j^{-1}(\pi_i)$ we easily find such a cocycle; in particular, we can pick cocycles represented by diagonal matrices.

3. For each i define $\tau_i : N \mapsto N$ with $\tau_i(n) = n_i \bar{n} n_i^{-1}$. This clearly induces a map on W which we denote again by τ_i, by abuse of notation, such that $\tau_i(w) = \pi_i w \pi_i^{-1}$.

4. By direct brute force computation we compute $H^1(Z, \tau_i)$ and $W^{\tau_i} = \{w \in W \mid \tau_i(w) = w\}$.

5. Observe that W^{τ_i} acts on $H^1(Z, \tau_i)$ as described in (1.8.1) and, in particular, it turns out (again by direct computation) that the action is transitive. This implies, thanks to Proposition 1.8.4, that $j_*^{-1}([\pi_i]) = \{[n_i]\}$.

6. We conclude that $H^1 N = \{[n_1], \dots, [n_{10}]\}$.

Proceeding as in Section 3.6, for each i we compute $g_i \in \widehat{G}_0$ with $g_i^{-1}\bar{g}_i = n_i$. Then $g_i \cdot \mathfrak{h}$ is the Cartan subspace corresponding to $[n_i]$. It follows that $\mathfrak{g}_1^{\mathbb{R}}$ has ten Cartan subspaces up to conjugacy.
We consider computing the action of the map ν on the set of Cartan subspaces. By the above algorithm, the elements n_i and g_i can be chosen so that they are diagonal matrices. Hence $\nu(g_i \cdot \mathfrak{h}) = g_0 g_i g_0^{-1} \bar{g}_1^{-1} g_1 g_0 \cdot \mathfrak{h} = g_i g_1^{-1} \cdot \imath\mathfrak{h} = g_i g_1^{-1} \cdot \mathfrak{h}$ (as $\imath\mathfrak{h} = \mathfrak{h}$). Write $h_i = g_i g_1^{-1}$, then, as all involved matrices commute, $h_i^{-1}\bar{h}_i = g_i^{-1}\bar{g}_i g_1 \bar{g}_1^{-1} = n_i g_1 \bar{g}_1^{-1}$. But $g_1\bar{g}_1^{-1} = \mathrm{diag}(\omega^2, \dots, \omega^2)$ (notation as in Section 3.2.4). Hence $g_1\bar{g}_1^{-1} \cdot u = -u$ for all $u \in \bigwedge^4 \mathbb{C}^8$. Now the longest element w_0 of the Weyl group acts as -1 on $\mathfrak{h}$. It follows that the restriction of $h_1^{-1}\bar{h}_i$ to $\mathfrak{h}$ is equal to $\pi_i w_0$. So the cocycle corresponding to $\nu(g_i \cdot \mathfrak{h})$ lies in the conjugacy class of $\pi_i w_0$. A computation shows that the conjugacy classes of π_i and $\pi_i w_0$ are always different. Hence up to the action of ν only 5 Cartan subspaces remain. Their bases are as follows:

1. $e_{1234} + e_{5678}, e_{1357} + e_{2468}, e_{1256} + e_{3478}, e_{1368} + e_{2457}, e_{1458} + e_{2367}, e_{1467} + e_{2358}$, $e_{1278} + e_{3456}$;
2. $e_{1257} - 4e_{3468}, e_{1358} - 4e_{2467}, e_{1268} - 4e_{3457}, e_{1456} - 4e_{2378}, e_{1478} - 4e_{2356}, e_{1367} - 4e_{2458}$, $e_{1234} + 16e_{5678}$;
3. $e_{1234} + e_{5678}, e_{1257} - e_{3468}, e_{1356} + e_{2478}, e_{1268} - e_{3457}, e_{1458} - e_{2367}, e_{1467} - e_{2358}$, $e_{1378} + e_{2456}$;
4. $e_{1256} + 128e_{3478}, e_{1368} + 128e_{2457}, e_{1357} + 128e_{2468}, e_{1467} + 128e_{2358}, e_{1234} - 128e_{5678}$, $e_{1278} + 128e_{3456}, e_{1458} + 128e_{2367}$;
5. $e_{1256} + 128e_{3478}, e_{1678} - 192e_{2345}, e_{1458} + 128e_{2367}, e_{1346} + 192e_{2578}, e_{1238} + 192e_{4567}$, $e_{1247} + 192e_{3568}, e_{1357} + 128e_{2468}$.

Chapter 4

Classification of the orbits of $\mathrm{Spin}(7,7)$ acting on its semispinor module

4.1 Introduction

The orbits of the group $\mathrm{Spin}(14,\mathbb{C})$ acting on its semispinor module have been classified by Popov ([55]) and a bit later by Kac and Vinberg ([37]). The latter used Vinberg's theory of θ-groups to classify the orbits.
The setting of this chapter is as follows. Let $\mathfrak{g}$ be the simple complex Lie algebra of type E_8, endowed with a $\mathbb{Z}$-grading where the only nonzero graded subspaces are the $\mathfrak{g}_i$'s with $i = -2,-1,0,1,2$. Let G be the adjoint group of $\mathfrak{g}$ and G_0 the connected subgroup corresponding to the subalgebra $\mathfrak{g}_0$. In particular, G_0 is reductive with a nontrivial center. Namely, $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus \mathfrak{t}$, where $\mathfrak{t}$ is a 1-dimensional center and $\tilde{\mathfrak{g}}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$. Let $\widetilde{G}_0$ be the subgroup of G_0 corresponding to $\tilde{\mathfrak{g}}_0$. It turns out that $\tilde{\mathfrak{g}}_0 \cong \mathfrak{o}(14,\mathbb{C})$, $\widetilde{G}_0 \cong \mathrm{Spin}_{14}$. Moreover, $\mathfrak{g}_1$ is isomorphic, as an $\mathfrak{o}(14,\mathbb{C})$-module, to the semispinor module S^+.
Vinberg classified the orbits into nine families, denoting them by Roman numerals. The families I to VIII each correspond to a single nilpotent orbit. The family IX corresponds to a 1-parameter space of closed orbits (see Table 4.1). Here we use the same setup, along with Galois cohomology, to classify the orbits of the real form $\mathrm{Spin}(7,7)$ on its semispinor module. For this purpose, we use Theorem 1.8.1. To determine the explicit correspondence, we use the algorithm of [8]. For this purpose, we need to compute the component groups of the stabilizers of orbit representatives. The approach is, in fact, similar to what we show in Chapter 3. However, in this chapter we study the action of a proper subgroup of a θ-group, leading to some peculiarities. Also, when using Galois cohomology, we have to deal with the fact that this time $H^1\widetilde{G}_0$ is not trivial.
Let e be an orbit representative among the ones listed in Table 4.1. Let $Z_{\widetilde{G}_0}(e)$ be, as usual, the stabilizer of e in $\widetilde{G}_0$. Thanks to Proposition 1.6.5, we equivalently work with $Z_{\widetilde{G}_0}(h,e,f)$, (h,e,f) being an $\mathfrak{sl}_2$-triple. To determine the component group of $Z_{\widetilde{G}_0}(h,e,f)$ (Section 4.5), we first determine the component group of $Z_{G_0}(h,e,f)$ (Section 4.4). The orbits of type IX are treated separately in Section 4.6. Finally, in Section 4.7 we describe how to determine the real orbits. We show our results in the tables of Section 4.3. We find that there are 12 real orbits among types I to VIII. For type IX, our conclusion is that there are three distinct 1-parameter families of real orbits.
To perform our computations we used the computer algebra systems GAP4 ([34]) and

Magma ([10]). In particular, in GAP4 we mostly used the packages SLA and CoReLG to deal with Lie theory.

4.2 Preliminary constructions

Throughout this chapter we use the implementations of many algorithms in GAP4 and Magma. In particular, two algorithms for algebraic groups will play an important role. One is the well-established IsEltOf. The second takes as input a basis of a Lie subalgebra $\mathfrak{t}$ of $\mathfrak{gl}(n,\mathbb{C})$, consisting of pairwise commuting semisimple matrices. We assume that $\mathfrak{t}$ is algebraic, i.e., that it is the Lie algebra of an algebraic group $T \subset \mathrm{GL}(n,\mathbb{C})$. Then the algorithm explicitly computes an isomorphism $\chi : (\mathbb{C}^\times)^d \to T$. The algorithm is described in [8, §6.5]; here we call it TorusPar.

4.2.1 The Clifford algebra and the groups $\mathrm{Spin}(14,\mathbb{C})$ and $\mathrm{Spin}(7,7)$

Here we describe the construction of the groups in the title, using a Clifford algebra. This is based on [12, Chapter VIII, §13.4] and [38, Chapter 6]. We specialize all constructions for the case that we consider.
Define the 14×14-matrix

$$A = \begin{pmatrix} 0 & \dots & 1 \\ & \iddots & \\ 1 & \dots & 0 \end{pmatrix}.$$

Let FA be the free associative algebra over $\mathbb{C}$ with one with 14 generators denoted $v_1, \dots, v_{14}$. We let V be the subspace spanned by $v_1, \dots, v_{14}$ and let Ψ be the bilinear form on V with matrix A, that is $\Psi(v_i, v_j) = A_{ij}$. Then the Clifford algebra C is the quotient of FA by the ideal I generated by the elements $v_i v_j + v_j v_i - \Psi(v_i, v_j) \cdot 1$ for $1 \le i, j \le 14$. By v_i we also denote the image of this element in C. By V we also denote the space in C spanned by the v_i. It is known that the products $v_{i_1} \cdots v_{i_k}$ for $k \ge 0$ and $i_1 < i_2 < \dots < i_k$ form a basis of C (see, e.g., [38, Proposition 6.1.5]; it can also be shown by observing that the given relations form a Gröbner basis of I in FA).
There is an antiautomorphism τ of FA with $\tau(v_i) = v_i$ for all i. It leaves the generators of I invariant, and hence induces an antiautomorphism, also denoted τ, of C. Let C^+ be the even subalgebra of C; namely, each element in C^+ can be written as a linear combination of products of an even number of elements of C. We have $\tau(v_{i_1} \cdots v_{i_k}) = v_{i_k} v_{i_{k-1}} \cdots v_1$. Now we set

$$\mathrm{Spin}(14,\mathbb{C}) = \{u \in C^+ \mid u\tau(u) = 1 \text{ and } uV\tau(u) = V\}.$$

This is a subgroup of the group of invertible elements of C. As a subset of C^+ it is given by a set of polynomial equations in the coefficients of a $u \in C^+$. Hence it is a linear algebraic group. By [38, Corollay 6.3.9] it is simple and simply connected of type D_7.
By $C_\mathbb{R}$ we denote the real span of the standard basis of C (consisting of the elements $v_{i_1} \cdots v_{i_k}, i_1 < \dots < i_k$). This is a real form of C. Let $V_\mathbb{R}$ be the real span of $v_1, \dots, v_{14}$. Then the bilinear form Ψ, restricted to $V_\mathbb{R}$, has signature $(7,7)$. Hence we have

$$\mathrm{Spin}(7,7) = \mathrm{Spin}(14,\mathbb{C}) \cap C_\mathbb{R}$$

(see [38, Section 6.4.3]). This is a real form of $\mathrm{Spin}(14,\mathbb{C})$. As $\mathrm{Spin}(14,\mathbb{C})$ is simply connected we have that $\mathrm{Spin}(7,7)$ is connected (see [54, Proposition 7.6]).
Let $\mathfrak{o}(14,\mathbb{C}) = \{x \in \mathfrak{gl}(14,\mathbb{C}) \mid x^\intercal A + Ax = 0\}$. Then $\mathfrak{o}(14,\mathbb{C})$ is a simple Lie algebra of type D_7. Define the map $f : \mathfrak{o}(14,\mathbb{C}) \to C$ by $f(x) = \frac{1}{2}\sum_{i=1}^{14}(xv_i)v_{15-i}$. Then f is linear

and $f([x,y]) = [f(x), f(y)]$ where the latter is the commutator in C (see [12, Lemme 1, §VIII.13 no 2]). In fact, f is an isomorphism of $\mathfrak{o}(14,\mathbb{C})$ onto a Lie subalgebra of C. The image of f is the Lie algebra of $\mathrm{Spin}(14,\mathbb{C})$ ([38, Theorem 6.3.6]).
A Cartan subalgebra of $\mathfrak{o}(14,\mathbb{C})$ is spanned by $a_j = e_{j,j} - e_{15-j,15-j}$ for $1 \le j \le 7$ (where $e_{k,l}$ denotes the 14×14-matrix with a 1 on position (k,l) and zeros elsewhere). We have $f(a_j) = v_j v_{15-j} - \frac{1}{2}$. A canonical generating set is given by $\hat{e}_i = e_{i,i+1} - e_{14-i,15-i}$ for $1 \le i \le 6$, $\hat{e}_7 = e_{6,8} - e_{7,9}$, $\hat{f}_i = \hat{e}_i^\top$, $\hat{h}_i = [\hat{e}_i, \hat{f}_i]$ for $1 \le i \le 7$. For $1 \le i \le 6$ we have $f(\hat{e}_i) = v_i v_{14-i}$, $f(\hat{f}_i) = v_{i+1} v_{15-i}$, $f(\hat{h}_i) = v_i v_{15-i} - v_{i+1} v_{14-i}$ and $f(\hat{e}_7) = v_6 v_7$, $f(\hat{f}_7) = v_8 v_9$, $f(\hat{h}_7) = v_6 v_9 + v_7 v_8 - 1$.

4.2.2 The spinor modules

Here we define the (semi) spinor module of C, which is then naturally also a module of $\mathrm{Spin}(14,\mathbb{C})$.
Write $F = \{v_1 \dots, v_7\}$ and $F' = \{u_1, \dots, u_7\}$ where $u_i = v_{7+i}$ for $1 \le i \le 7$. Let U denote the span of F' and E the exterior algebra of U, i.e.,

$$E = \bigwedge^0 U \oplus \bigwedge^1 U \oplus \cdots \oplus \bigwedge^7 U.$$

For $u \in F'$ and $v \in F$ we define the endomorphisms $\lambda(u), \lambda(v)$ of E by

$$\lambda(u)(u_{i_1} \wedge \cdots \wedge u_{i_k}) = u \wedge u_{i_1} \wedge \cdots \wedge u_{i_k},$$
$$\lambda(v)(u_{i_1} \wedge \cdots \wedge u_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} \Psi(u_{i_j}, v) u_{i_1} \wedge \cdots \wedge u_{i_{j-1}} \wedge u_{i_{j+1}} \wedge \cdots \wedge u_{i_k}.$$

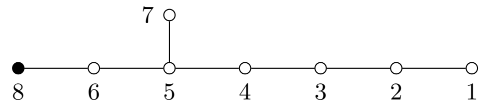
Then, the map $\lambda : F \cup F' \to \mathrm{End}(E)$ extends to a homomorphism $\lambda : C \to \mathrm{End}(E)$ making E into a C-module ([12, §VIII.13 no 2]). Hence, it is also a $\mathrm{Spin}(14,\mathbb{C})$-module and a $\mathfrak{o}(14,\mathbb{C})$-module. It is known that E splits into the direct sum of two irreducible modules S^+ and S^- which are the sum of the $\bigwedge^k U$ with k even, respectively odd. Such modules are called *semispinor modules*. We have

$$S^+ = \bigwedge^0 U \oplus \bigwedge^2 U \oplus \bigwedge^4 U \oplus \bigwedge^6 U,$$
$$S^- = \bigwedge^1 U \oplus \bigwedge^3 U \oplus \bigwedge^5 U \oplus \bigwedge^7 U.$$

Consider the basis of S^+ consisting of $u_{i_1} \wedge \dots \wedge u_{i_k}$ with k even. Let $S^+_{\mathbb{R}}$ be the real span of this basis. Then $S^+_{\mathbb{R}}$ is an irreducible $\mathrm{Spin}(7,7)$-module.
In the sequel we will denote the basis element $u_{i_1} \wedge \dots \wedge u_{i_k}$ of S^+ (or $S^+_{\mathbb{R}}$) by $e_{i_1 i_2 \cdots i_k}$. We remark that they are weight vectors for the Cartan subalgebra determined in Section 4.2.1. We denote the basis element of $\bigwedge^0 U$ by q_0. Then q_0 is a highest weight vector of S^+ and we have $\hat{h}_i \cdot q_0 = 0$ for $1 \le i \le 6$ and $\hat{h}_7 \cdot q_0 = q_0$ (where $\hat{h}_i$ are the elements defined in Section 4.2.1).

4.2.3 The construction via θ-groups

For the problem studied in this chapter we use the theoretical setting introduced in Section 1.6. We let $\mathfrak{g}$ be the simple complex Lie algebra of type E_8 and consider a $\mathbb{Z}$-grading of $\mathfrak{g}$ defined as follows. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\{\alpha_1, \dots, \alpha_8\}$ be a set of simple roots, ordered according to the Dynkin diagram below. We define the degree of a root $\alpha \in \Phi$ by writing it as $\alpha = \sum_{i=1}^{8} k_i \alpha_i$ and stipulating $\deg(\alpha) = k_8$. In the diagram below the black node corresponds to the simple root with degree 1 and the other nodes to the simple roots with degree 0.



Then, denoting as usual by $\mathfrak{g}_\alpha$ the root space corresponding to the root $\alpha \in \Phi$, we let $\hat{\mathfrak{g}}_i := \langle \mathfrak{g}_\alpha \mid \deg(\alpha) = i \rangle$ and set

$$\mathfrak{g}_i := \begin{cases} \hat{\mathfrak{g}}_i & i \neq 0, \\ \hat{\mathfrak{g}}_i \oplus \mathfrak{h} & i = 0. \end{cases}$$

Since the maximal root of $\mathfrak{g}$ has degree 2, it is clear that $\mathfrak{g}_i = 0$ for $i < -2$ and $i > 2$. So the grading reads

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \tag{4.2.1}$$

It is a basic fact that for any $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ (see, for example, [42, Proposition 8.4(d)]). Since $\deg(\alpha+\beta) = \deg(\alpha) + \deg(\beta)$ it follows that (4.2.1) is a $\mathbb{Z}$-grading.

As a consequence of Theorem 1.6.1, $\mathfrak{g}_0$ is reductive. We set $\tilde{\mathfrak{g}}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ and let $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}_0)$ be the center. The latter is 1-dimensional. Furthermore, $\tilde{\mathfrak{g}}_0$ is simple of type D_7. Indeed, its root system is the subsystem of Φ spanned by the simple roots corresponding to the white nodes of the above Dynkin diagram. Let e_i, f_i, h_i, for $1 \leq i \leq 8$, denote the elements of a canonical set of generators of $\mathfrak{g}$ (where again we use the numbering as in the above Dynkin diagram). Then e_i, f_i, h_i for $i \neq 8$ form a canonical set of generators of $\tilde{\mathfrak{g}}_0$. In particular $\tilde{\mathfrak{g}}_0$ is isomorphic to $\mathfrak{o}(14, \mathbb{C})$. We fix an isomorphism $\mathfrak{o}(14,\mathbb{C}) \to \tilde{\mathfrak{g}}_0$ (mapping the Cartan subalgebra of $\mathfrak{o}(14,\mathbb{C})$ determined in Section 4.2.1 to the Cartan subalgebra of $\tilde{\mathfrak{g}}_0$ spanned by $h_1, \ldots, h_7$). As $\mathfrak{g}_1$ is a $\tilde{\mathfrak{g}}_0$-module this isomorphism makes it also into a $\mathfrak{o}(14,\mathbb{C})$-module.

The following proposition is well-known (it is used, for example, in [37]). However, we could not find an explicit proof in the literature, so we include one here.

From now on, we will denote by Φ^n, for $-2 \leq n \leq 2$, the subset of Φ that contains the roots of the form $\alpha = n\alpha_8 + \sum_{j=1}^{7} k_j \alpha_j$, where k_j's are integers (all nonnegative if $n = 1, 2$ or all nonpositive if $n = -1, -2$); equivalently, $\alpha \in \Phi^n$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{g}_n$.

Proposition 4.2.1. *The subspace $\mathfrak{g}_1$ is an irreducible $\mathfrak{o}(14,\mathbb{C})$-module isomorphic to S^+.*

Proof. We have $e_8 \in \mathfrak{g}_1$ and it is a minimal vector for the given canonical generating set of $\tilde{\mathfrak{g}}_0$, that is, $[f_i, e_8] = 0$ for $1 \leq i \leq 7$. This implies that e_8 generates an irreducible $\tilde{\mathfrak{g}}_0$-submodule U of $\mathfrak{g}_1$. Let $\alpha \in \Phi^1$ and assume $\alpha \neq \alpha_8$. By [42, Corollary 10.2], $\alpha = \alpha_{j_1} + \cdots + \alpha_{j_l}$ such that α_{j_i} is a simple root and $\alpha_{j_1} + \cdots + \alpha_{j_m}$ is a positive root for any $m \leq l$. Among the $\alpha'_{j_i}s$, the simple root α_8 appears exactly once, say $\alpha_{j_k} = \alpha_8$. Then $\beta = \alpha_{j_1} + \cdots + \alpha_{j_{k-1}} \in \Phi^0$ and $[\mathfrak{g}_{\alpha_{j_l}}, [\ldots [\mathfrak{g}_{\alpha_{j_{k+1}}}, [\mathfrak{g}_\beta, \mathfrak{g}_{\alpha_8}]] \ldots]] = \mathfrak{g}_\alpha$, with $\mathfrak{g}_\beta, \mathfrak{g}_{\alpha_{j_{k+1}}}, \ldots, \mathfrak{g}_{\alpha_{j_l}} \subset \tilde{\mathfrak{g}}_0$. We see that $\mathfrak{g}_\alpha \subset U$. Hence, $\mathfrak{g}_1 = U$ and $\mathfrak{g}_1$ is an irreducible $\tilde{\mathfrak{g}}_0$-module.

Using the relations (1.3.1), we deduce that e_8 has weight $\mu = (0,0,0,0,0,-1,0)$ (the entries follow the ordering of the diagram above from 1 to 7). The highest weight of $\mathfrak{g}_1$ is $-w_0 \cdot \mu$ where w_0 is the longest element in the Weyl group of $\tilde{\mathfrak{g}}_0$. By [11, Planche IV, (XI)] this means that the highest weight of $\mathfrak{g}_1$ is $(0,0,0,0,0,0,1)$. Since this is the same weight of the maximal vector of the $\mathfrak{o}(14,\mathbb{C})$-module S^+ (see Section 4.2.2), the proof is concluded. □

As above we let G be the (inner) automorphism group of $\mathfrak{g}$ and we let G_0 be the connected subgroup of G with Lie algebra $\mathrm{ad}\mathfrak{g}_0$. We also let $\widetilde{G}_0$ be the connected subgroup of G with Lie algebra $\mathrm{ad}\tilde{\mathfrak{g}}_0$. The weights of the $\mathfrak{o}(14,\mathbb{C})$-module S^+ span the weight lattice of $\mathfrak{o}(14,\mathbb{C})$ (this can be verified by a direct check, it is also well known and can immediately

be deduced from [39, Chapter 3, Theorem 2.18]). So by Proposition 4.2.1 the same holds for the weights of the module $\mathfrak{g}_1$. Hence $\widetilde{G}_0$ is simply connected.
As seen in Section 4.2.1 the Lie algebra of $\mathrm{Spin}(14,\mathbb{C})$ is isomorphic to $\mathfrak{o}(14,\mathbb{C})$. So also $\mathrm{Lie}(\mathrm{Spin}(14,\mathbb{C}))$ and $\tilde{\mathfrak{g}}_0$ are isomorphic. We fix an isomorphism $\psi : \mathrm{Lie}(\mathrm{Spin}(14,\mathbb{C})) \to \tilde{\mathfrak{g}}_0$ mapping $f(\hat{e}_i)$, $f(\hat{f}_i)$, $f(\hat{h}_i)$ (as defined in Section 4.2.1) to e_i, f_i, h_i respectively. Since both $\mathrm{Spin}(14,\mathbb{C})$ and $\widetilde{G}_0$ are simply connected, this isomorphism lifts to an isomorphism of algebraic groups $\Psi : \mathrm{Spin}(14,\mathbb{C}) \to \widetilde{G}_0$ (see [62, Chapter 5, Corollary 1]). Hence $\mathfrak{g}_1$ is also a $\mathrm{Spin}(14,\mathbb{C})$-module, isomorphic to S^+.
Let $\mathfrak{g}_\mathbb{R}$ be the real subalgebra of $\mathfrak{g}$ generated by the canonical generating set consisting of e_i, f_i, h_i, $1 \le i \le 8$. Then $\mathfrak{g}_\mathbb{R}$ is the split real form of $\mathfrak{g}$. We set $\mathfrak{g}_{i,\mathbb{R}} = \mathfrak{g}_i \cap \mathfrak{g}_\mathbb{R}$. Furthermore, by $G(\mathbb{R})$ we denote the subset of G consisting of all elements that send $\mathfrak{g}_\mathbb{R}$ to itself, and $G_0(\mathbb{R}) = G(\mathbb{R}) \cap G_0$, $\widetilde{G}_0(\mathbb{R}) = G(\mathbb{R}) \cap \widetilde{G}_0$.
We identify the root systems of $\mathrm{Spin}(14,\mathbb{C})$ and $\widetilde{G}_0$ with Φ^0. The algebraic groups $\mathrm{Spin}(14,\mathbb{C})$ and $\widetilde{G}_0$ are generated by unipotent subgroups whose elements we denote by $x_\alpha(t)$ and $x'_\alpha(t)$ respectively (for $t \in \mathbb{C}$, $\alpha \in \Phi^0$). The isomorphism Ψ, by construction, maps $x_\alpha(t)$ to $x'_\alpha(t)$ (see [62, Chapter 5, Corollary 1]). However, since these groups are simply connected, the real forms $\mathrm{Spin}(7,7)$ and $\widetilde{G}_0(\mathbb{R})$ are generated by $x_\alpha(t)$ and, respectively, $x'_\alpha(t)$ with $t \in \mathbb{R}$ (see [62, Chapter 5, Corollary 3]). It follows that the restriction of Ψ to $\mathrm{Spin}(7,7)$ is an isomorphism $\Psi : \mathrm{Spin}(7,7) \to \widetilde{G}_0(\mathbb{R})$.

Remark 4.2.2. The theory developed in [62] concerns algebraic groups of linear transformations. The group $\mathrm{Spin}(14,\mathbb{C})$ is defined as a subgroup of the group of units of the associative algebra C^+. However, it is straightforward to transform it into a group of linear transformations by considering the left regular representation of C^+. Then all theorems in [62] are immediately applicable.

4.2.4 The complex orbits

In this section we describe the orbits of the groups G_0 and $\widetilde{G}_0$ in $\mathfrak{g}_1$. Because of the isomorphism Ψ the $\widetilde{G}_0$-orbits in $\mathfrak{g}_1$ are the same as the $\mathrm{Spin}(14,\mathbb{C})$-orbits in S^+.
Since we are working with a $\mathbb{Z}$-grading of $\mathfrak{g}$ all elements of $\mathfrak{g}_1$ are nilpotent. The classification of the G_0-orbits in $\mathfrak{g}_1$ has been worked out in [37]. There are nine such orbits denoted I to IX.
Now we consider the $\widetilde{G}_0$-orbits. Their classification has also been determined in [37]. It can be deduced from the classification of the G_0-orbits in the following way. As in Section 4.2.3 we write $\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus \mathfrak{t}$. Let $T \subset G_0$ be the connected subgroup with Lie algebra $\mathrm{ad}\mathfrak{t}$. Then T is a central torus and $G_0 = \widetilde{G}_0 \cdot T$. We have that T acts on $\mathfrak{g}_1$ as scalar multiplication. Let e be a representative of a G_0-orbit. It follows that the $\widetilde{G}_0$-orbits contained in $G_0 \cdot e$ have representatives $\nu \cdot e$ for $\nu \in \mathbb{C}^\times$. For the orbits I to VIII there is an $\mathfrak{sl}_2$-triple (h,e,f) with $h = h_1 + h_2$ where $h_1 \in \tilde{\mathfrak{g}}_0$ and $h_2 \in \mathfrak{t}$. Let $T_1 \subset \widetilde{G}_0$ be the connected subgroup whose Lie algebra is spanned by $\mathrm{ad}h_1$. Then the elements of T_1 multiply e by a nonzero scalar. It follows that in this case all $\nu \cdot e$ are $\widetilde{G}_0$-conjugate. Therefore each G_0-orbit from I to VIII corresponds to exactly one $\widetilde{G}_0$-orbit. Now let e be a representative of orbit IX, then we have an $\mathfrak{sl}_2$-triple (h,e,f) with $h \in \mathfrak{t}$. So the above argument does not work in this case. In fact, setting $N_{\widetilde{G}_0}(e) = \{g \in \widetilde{G}_0 \mid g \cdot e = \nu \cdot e \text{ for some } \nu \in \mathbb{C}^\times\}$ and $Z_{\widetilde{G}_0}(e) = \{g \in \widetilde{G}_0 \mid g \cdot e = e\}$ we have $N_{\widetilde{G}_0}(e) = Z_{\widetilde{G}_0}(e) \rtimes \mu_8$, where μ_8 is a cyclic group multiplying e by an eighth root of unity. It follows that the $\widetilde{G}_0$-orbits lying in $G_0 \cdot e$ have representatives $\alpha \cdot e$ and $\alpha \cdot e$, $\alpha' \cdot e$ are $\widetilde{G}_0$-conjugate if and only if $\alpha = \zeta\alpha'$ with $\zeta^8 = 1$.
Representatives of the $\widetilde{G}_0$-orbits in $\mathfrak{g}_1$ are contained in Table 4.1.

4.2.5 Classification of the orbits of $\mathrm{Spin}(7,7)$

In this section we outline our strategy to classify the orbits of $\mathrm{Spin}(7,7)$ acting on $S^+_{\mathbb{R}}$. As seen in Section 4.2.3 these are the same as the $\widetilde{G}_0(\mathbb{R})$-orbits in $\mathfrak{g}_{1,\mathbb{R}}$. The idea is to take a complex orbit $\widetilde{G}_0 \cdot e$ and to classify the $\widetilde{G}_0(\mathbb{R})$-orbits contained in it. For this purpose, we will use the techniques illustrated in Section 1.8. However, we will also need an additional result (we refer to Sansuc, [58, Lemme 1.13], see also [8, Proposition 10.1]).

Lemma 4.2.3 (Sansuc)**.** *Let G be a linear algebraic group defined over $\mathbb{R}$ and let U be a normal unipotent subgroup, also defined over $\mathbb{R}$. Let $\pi : G \to \bar{G} = G/U$ be the quotient homomorphism. Then the induced map $\pi_* : H^1G \to H^1\bar{G}$ is bijective.*

We first look at the structure of the stabilizer of an $e \in \mathfrak{g}_1$. As illustrated in Section 1.6, e lies in a homogeneous $\mathfrak{sl}_2$-triple (h, e, f).
An immediate application of Lemma 4.2.3 is that, in combination with Proposition 1.6.5 and Theorem 1.8.1, it implies that in order to classify the $\widetilde{G}_0(\mathbb{R})$-orbits inside the $\widetilde{G}_0$-orbit of a given real $e \in \mathfrak{g}_1$ we can compute the Galois cohomology of $Z_{\widetilde{G}_0}(h, e, f)$, where (h, e, f) is a homogeneous $\mathfrak{sl}_2$-triple containing e.
A first problem that has to be solved is to find a real element in each complex orbit. For the orbits I to VIII Table 4.1 already has a real representative. The orbits IX are a parametrized family of orbits with representatives $\alpha(1 + e_{123456} + e_{1237} + e_{4567})$ for $\alpha \in \mathbb{C}$. It can happen that an orbit with this representative has real points also if α is not real. In Section 4.7 we determine the α such that the corresponding orbit has real points.
We use the algorithm of [8] to compute $H^1 Z_{\widetilde{G}_0}(h, e, f)$. The input to this algorithm consists of a basis of the Lie algebra of $Z_{\widetilde{G}_0}(h, e, f)$ along with one element of each component of $Z_{\widetilde{G}_0}(h, e, f)$. The Lie algebra is

$$\{x \in \tilde{\mathfrak{g}}_0 \mid [x, h] = [x, e] = [x, f] = 0\}$$

and can easily be computed using linear algebra methods. So the problem remains to compute one element of each component of $Z_{\widetilde{G}_0}(h, e, f)$. We will outline our methods for that in Sections 4.5, 4.6.

4.3 Tables

In this section we give the tables of orbit representatives of the group $\mathrm{Spin}(14, \mathbb{C})$ acting on S^+ (Table 4.1) and of $\mathrm{Spin}(7,7)$ acting on $S^+_{\mathbb{R}}$ (Table 4.2). The representatives of the orbits of $\mathrm{Spin}(14, \mathbb{C})$ are taken from [37]. The representatives of the orbits of $\mathrm{Spin}(7,7)$ have been calculated with the methods described in this chapter.
Table 4.1 has five columns. The first has the number of the orbit, where we follow [37]. The second column has a representative of each orbit, also taken from [37]. The third and fourth column have the isomorphism type of the stabilizer of an $\mathfrak{sl}_2$-triple h, e, f (where e is the representative in the second column) in respectively $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$. Here $\mathfrak{t}(k)$ denotes a k-dimensional Lie algebra whose elements are commuting semisimple elements with real eigenvalues. The last column has the component group of $Z_{\widetilde{G}_0}(h, e, f)$.

Table 4.1: Representatives of the orbits of $\mathrm{Spin}(14,\mathbb{C})$

N	Complex representative	$\mathfrak{z}_{\mathfrak{g}_0}(h,e,f)$	$\mathfrak{z}_{\tilde{\mathfrak{g}}_0}(h,e,f)$	K
I	1	$A_6 + \mathfrak{t}(1)$	A_6	1
II	$1 + e_{1234}$	$B_3 + A_2 + \mathfrak{t}(1)$	$B_3 + A_2$	1
III	$1 + e_{1234} + e_{3456}$	$C_3 + \mathfrak{t}(2)$	$C_3 + \mathfrak{t}(1)$	1
IV	$1 + e_{123456}$	$A_5 + \mathfrak{t}(1)$	A_5	$\mathbb{Z}/2\mathbb{Z}$
V	$1 + e_{1234} + e_{3456} + e_{1367}$	$A_3 + \mathfrak{t}(1)$	A_3	1
VI	$1 + e_{123456} + e_{1237}$	$A_2 + A_2 + \mathfrak{t}(1)$	$A_2 + A_2$	$\mathbb{Z}/2\mathbb{Z}$
VII	$1 + e_{123456} + e_{1237} + e_{1567}$	$B_2 + A_1 + \mathfrak{t}(1)$	$B_2 + A_1$	1
VIII	$1 + e_{123456} + e_{1237} + e_{1567} + e_{2467}$	$G_2 + \mathfrak{t}(1)$	G_2	1
IX	$\alpha(1 + e_{123456} + e_{1237} + e_{4567})$	$G_2 + G_2$	$G_2 + G_2$	1

The first column of Table 4.2 has again the number of the complex orbit. However, each $\mathrm{Spin}(14,\mathbb{C})$-orbit contains a finite number of real $\mathrm{Spin}(7,7)$-orbits. The second column has representatives of these orbits. The last column has the isomorphism type of the stablizer $\mathfrak{z}_{\tilde{\mathfrak{g}}_{0,\mathbb{R}}}(h,e,f)$. We see that the different orbits are distinguished by their stabilizers.

Table 4.2: Real orbits representatives

N	Real representative	$\mathfrak{z}_{\tilde{\mathfrak{g}}_{0,\mathbb{R}}}(h,e,f)$
I	1	$2\mathfrak{sl}(7,\mathbb{R})$
II	$1 + e_{1234}$	$\mathfrak{sl}(3,\mathbb{R}) + \mathfrak{so}(4,3)$
III	$1 + e_{1234} + e_{3456}$	$\mathfrak{sp}(3,\mathbb{R})$
IV	$1 + e_{123456}$	$\mathfrak{sl}(6,\mathbb{R})$
	$559e_{45} - 497e_{36} + 463e_{3456} + 463e_{12}$ $-497e_{1245} - 559e_{1236} + 529e_{123456}$	$\mathfrak{su}(3,3)$
V	$1 + e_{1234} + e_{3456} + e_{1367}$	$\mathfrak{sl}(4,\mathbb{R})$

VI	$1+e_{123456}+e_{1237}$	$\mathfrak{sl}(3,\mathbb{R})+\mathfrak{sl}(3,\mathbb{R})$
	$1+e_{36}+e_{24}+e_{2346}-e_{15}-e_{1356}$ $+e_{1245}-2e_{1237}+e_{123456}$	$\mathfrak{sl}(3,\mathbb{C})$
VII	$1+e_{123456}+e_{1237}+e_{1567}$	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{so}(2,3)$
	$\frac{1}{2}+\frac{1}{2}e_{35}+\frac{1}{2}e_{26}-\frac{1}{2}e_{2356}-e_{1567}-\frac{1}{2}e_{14}$ $-\frac{1}{2}e_{1345}-\frac{1}{2}e_{1246}-e_{1237}+\frac{1}{2}e_{123456}$	$\mathfrak{su}(2)+\mathfrak{so}(4,1)$
VIII	$1+e_{1237}+e_{1567}+e_{2467}+e_{123456}$	$G_{2(2)}$
	$-\frac{1}{2}+\frac{1}{2}e_{36}-\frac{1}{2}e_{25}-e_{2467}+\frac{1}{2}e_{2356}-e_{1567}$ $+\frac{1}{2}e_{14}-\frac{1}{2}e_{1346}+\frac{1}{2}e_{1245}-e_{1237}-\frac{1}{2}e_{123456}$	G
IX	$\mu(1+e_{123456}+e_{1237}+e_{4567})$, $\mu\in\mathbb{R},\mu>0$	$G_{2(2)}+G_{2(2)}$
	$\mu(6e_{47}+\frac{1}{64}e_{46}-\frac{1}{64}e_{45}-6e_{4567}-6e_{37}-\frac{1}{64}e_{36}$ $-\frac{1}{64}e_{35}-6e_{3567}+2e_{27}-\frac{3}{64}e_{26}-\frac{3}{64}e_{25}+2e_{2567}$ $+2e_{2347}-\frac{3}{64}e_{2346}+\frac{3}{64}e_{2345}-2e_{234567}-2e_{17}-\frac{3}{64}e_{16}$ $+\frac{3}{64}e_{15}+2e_{1567}+2e_{1347}+\frac{3}{64}e_{1346}+\frac{3}{64}e_{1345}+2e_{134567}$ $-6e_{1247}+\frac{1}{64}e_{1246}+\frac{1}{64}e_{1245}-6e_{124567}-6e_{1237}+\frac{1}{64}e_{1236}$ $-\frac{1}{64}e_{1235}+6e_{123567})$, $\mu\in\mathbb{R},\mu>0$	$G+G$
	$\mu(6-4\sqrt{2}+\frac{1}{2}e_{4567}+\frac{1}{2}e_{1567}-\frac{1}{2}e_{2467}-\frac{1}{2}e_{3457}$ $-e_{123456}-\frac{1}{2}e_{1267}-\frac{1}{2}e_{1357}+\frac{1}{2}e_{2347}+\frac{1}{2}e_{1237})$, $\mu\in\mathbb{R},\mu>0$	$G_2(\mathbb{C})$

Remark 4.3.1. It is known that the invariant ring $\mathbb{C}[S^+]^{\mathrm{Spin}(14,\mathbb{C})}=\{f\in\mathbb{C}[S^+]\mid f(gv)=f(v)$ for all $g\in\mathrm{Spin}(14,\mathbb{C}),v\in S^+\}$ is generated by a single homogeneous invariant J_8 of degree 8 (see, for example, [1, 37]). Denote the representatives of the orbits of type IX in Table 4.2 by $\mu_i e_i$ for $i=1,2,3$, where $\mu_i\in\mathbb{R}$, $\mu_i>0$. We normalize J_8 so that $J_8(e_1)=1$. Then $J_8(\mu_1 e_1)=\mu_1^8$. Furthermore, e_1 and e_2 are $\mathrm{Spin}(14,\mathbb{C})$-conjugate. Hence $J_8(\mu_2 e_2)=\mu_2^8$. Furthermore, e_3 is $\mathrm{Spin}(14,\mathbb{C})$-conjugate to $(1+(1-\sqrt{2})\imath)e_1$ (see Section 4.7), so that $J_8(\mu_3 e_3)=\mu_3^8(1+(1-\sqrt{2})\imath)^8=\mu_3^8(-1088+768\sqrt{2})$. The real number in brackets is between -1 and -2. It follows that $S^+_{\mathbb{R}}$ is a union of three sets $S_{>0},S_0,S_{<0}$ consisting respectively of the elements on which the value of J_8 is >0, 0 and <0. The first set $S_{>0}$ is the union of the orbits of $\mu_1 e_1$, $\mu_2 e_2$, the set S_0 is the union of the orbits numbered I to VIII (and 0), and $S_{<0}$ is the union of the orbits of $\mu_3 u_3$.

4.4 The components of $Z_{G_0}(h,e,f)$ - orbits I to VIII

Let e be a nilpotent element among the representatives of the orbits listed in Table 4.1. Fix an $\mathfrak{sl}_2$-triple (h,e,f) with $h\in\mathfrak{g}_0$ and $e,f\in\mathfrak{g}_1$. As a first step towards computing the components of $Z_{\widetilde{G}_0}(h,e,f)$ we compute the components of $Z_{G_0}(h,e,f)$. In this section we show how that can be done for the specific setting of this chapter, using an approach

that is broadly similar to the one used in Section 2.4.
Let $\mathfrak{c} = \mathfrak{z}_{\mathfrak{g}_0}(h,e,f)$ then $\mathfrak{c}$ is the Lie algebra of $Z_{G_0}(h,e,f)$. We have that $\mathfrak{c}$ is reductive in $\mathfrak{g}_0$. Write $\mathfrak{c} = \mathfrak{s}_\mathfrak{c} \oplus \mathfrak{t}_\mathfrak{c}$ where $\mathfrak{s}_\mathfrak{c}$ is its semisimple part and $\mathfrak{t}_\mathfrak{c}$ is the center. In Table 4.1 we list the type of $\mathfrak{s}_\mathfrak{c}$ as well as the dimension of $\mathfrak{t}_\mathfrak{c}$.

Proposition 4.4.1. *Use the notation above. Let $g \in Z_{G_0}(h,e,f)$; then g stabilizes both $\mathfrak{c}$ and $\mathfrak{s}_\mathfrak{c}$.*

Proof. If $x \in \mathfrak{c}$, then $g(x) \in \mathfrak{c}$, as it is just the result of the action of $Z_{G_0}(h,e,f)$ on its Lie algebra. Moreover, if $x \in \mathfrak{s}_\mathfrak{c}$, i.e. $x = [a,b]$ for some $a,b \in \mathfrak{c}$, then $g(x) = g([a,b]) = [g(a), g(b)] \in \mathfrak{s}_\mathfrak{c}$. □

Now we fix a Cartan subalgebra $\mathfrak{h}_\mathfrak{s}$ of $\mathfrak{s}_\mathfrak{c}$. We also fix a basis of simple roots of the root system of $\mathfrak{s}_\mathfrak{c}$ with respect to $\mathfrak{h}_\mathfrak{s}$ and a canonical generating set $\{x_i, h_i, y_i \mid i = 1,\dots,\ell\}$ of $\mathfrak{s}_\mathfrak{c}$.

Proposition 4.4.2. *Every component of $Z_{G_0}(h,e,f)$ contains a diagram automorphism, that is, an element g such that $g(x_i) = x_{\pi(i)}, g(y_i) = y_{\pi(i)}, g(h_i) = h_{\pi(i)}$ for some permutation π of the integers $1,\dots,\ell$.*

Proof. Let $z \in Z_{G_0}(h,e,f)$ and by Proposition 4.4.1 we deduce that $z\big|_{\mathfrak{s}_\mathfrak{c}}$ is an automorphism of $\mathfrak{s}_\mathfrak{c}$. Since $\mathfrak{s}_\mathfrak{c}$ is semisimple we have that $\mathrm{Aut}(\mathfrak{s}_\mathfrak{c}) = \mathrm{Aut}(\mathfrak{s}_\mathfrak{c})^\circ \rtimes \Gamma$, where $\mathrm{Aut}(\mathfrak{s}_\mathfrak{c})^\circ$ is the inner automorphism group of $\mathfrak{s}_\mathfrak{c}$ and Γ is the finite group of diagram automorphisms of $\mathfrak{s}_\mathfrak{c}$. In particular, we get that $z = \hat{z}\sigma_\pi$ for some $\hat{z} \in \mathrm{Aut}(\mathfrak{s}_\mathfrak{c})^\circ$ and $\sigma_\pi \in \Gamma$. Let $S \subset Z_{G_0}(h,e,f)^\circ$ be the connected algebraic subgroup with Lie algebra $\mathfrak{s}_\mathfrak{c}$. By Proposition 4.4.1 we have a homomorphism of algebraic groups $S \to \mathrm{Aut}(\mathfrak{s}_\mathfrak{c})^\circ$ mapping a $h \in S$ to its restriction to $\mathfrak{s}_\mathfrak{c}$. As $\mathrm{Aut}(\mathfrak{s})^\circ$ is the adjoint group of $\mathfrak{s}$ it follows that this homomorphism is surjective (cf. [62, Chapter 5, Corollary 1]). So there is a $g' \in Z_{G_0}(h,e,f)^\circ$ whose restriction to $\mathfrak{s}_\mathfrak{c}$ is $\hat{z}$. So $g = (g')^{-1}z$ lies in the same component as z and clearly has the stated properties. □

As $\mathfrak{s}_\mathfrak{c} \subset \mathfrak{g}_0$ we see that $\mathfrak{g}_1, \mathfrak{g}_{-1}$ are $\mathfrak{s}_\mathfrak{c}$-modules. Let $\mathfrak{g}_1 = V_1^1 \oplus \cdots \oplus V_1^k$ and $\mathfrak{g}_{-1} = V_{-1}^1 \oplus \cdots \oplus V_{-1}^k$ be decompositions into irreducible $\mathfrak{s}_\mathfrak{c}$-modules and $v_1^1, \dots, v_1^k$, $v_{-1}^1, \dots, v_{-1}^k$ the corresponding highest weight vectors. Let $\mu_1^1, \dots, \mu_1^k$ and $\mu_{-1}^1, \dots, \mu_{-1}^k$ be the corresponding weights (so $[h, v_{\pm1}^i] = \mu_{\pm1}^i(h) v_{\pm1}^i, h \in \mathfrak{h}_\mathfrak{s}$). We say that a weight μ_1^j has multiplicity m if there are exactly m weights $\mu_1^{j_1}, \dots, \mu_1^{j_m}$ with $\mu_1^j = \mu_1^{j_r}$ for $1 \le r \le m$. Denoting the Killing form of $\mathfrak{g}$ by κ we have that the restriction $\kappa : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathbb{C}$ is non-degenerate. This implies that $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$ (the dual module) as $\mathfrak{g}_0$-modules. So also $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$ as $\mathfrak{s}_\mathfrak{c}$-modules. Hence we can order the V_{-1}^j so that $V_{-1}^j \cong (V_1^j)^*$.
Let $g \in Z_{G_0}(h,e,f)$ be as in Proposition 4.4.2, that is, its restriction to $\mathfrak{s}_\mathfrak{c}$ is a diagram automorphism of $\mathfrak{s}_\mathfrak{c}$ induced by the permutation π. As g permutes the x_i we have that $g(v_{\pm1}^j)$ is also a highest weight vector in $\mathfrak{g}_{\pm1}$. Then, the corresponding weight maps h_i to $\mu_{\pm1}^j(h_{\pi^{-1}(i)})$. We denote such a weight by $\mu_{\pm1}^{j,\pi}$. It follows that $\mu_{\pm1}^{j,\pi}$ appears among the weights $\mu_{\pm1}^n$ for $1 \le n \le k$ and that it has the same multiplicity as $\mu_{\pm1}^j$. We say that the permutation π is admissible if this holds for all j with $1 \le j \le k$. We conclude that if $g \in Z_{G_0}(h,e,f)$ restricts to the diagram automorphism of $\mathfrak{s}_\mathfrak{c}$ corresponding to π then π is admissible.
Now let π be an admissible permutation of $1,\dots,\ell$. We want to decide if there is a $g \in Z_{G_0}(h,e,f)$ that restricts to $\mathfrak{s}_\mathfrak{c}$ as the diagram automorphism corresponding to π, and in the affirmative case we want to determine such a g up to equivalence by $Z_{G_0}(h,e,f)^\circ$ (where g_1, g_2 are eqivalent if $g_1^{-1}g_2 \in Z_{G_0}(h,e,f)^\circ$). For this purpose we first have a straightforward lemma.

Lemma 4.4.3. *Let X_{-1}, X_0, X_1 be subsets of $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$ respectively. Suppose that $\mathfrak{g}$ is generated by $X_{-1} \cup X_0 \cup X_1$. Let $g \in G$ have the property that $g(x) \in \mathfrak{g}_m$ for all $x \in X_m$ and $m \in \{-1,0,1\}$. Then $g \in G_0$.*

Proof. There is a basis of $\mathfrak{g}$ whose elements consist of products of elements of $X_{-1} \cup X_0 \cup X_1$. Let x denote an element of this basis. Then $x \in \mathfrak{g}_m$ for a certain $m \in \{-2, \dots, 2\}$. Because of the hypothesis on g also $g(x) \in \mathfrak{g}_m$. It follows that g stabilizes all subspaces $\mathfrak{g}_m$. Define $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\theta(y) = 2^m y$ for $y \in \mathfrak{g}_m$ and extend θ to $\mathfrak{g}$ by linearity. Then θ is an automorphism of $\mathfrak{g}$, i.e., $\theta \in G$. Let G^θ be the subgroup consisting of all elements commuting with θ. As g stabilizes the subspaces $\mathfrak{g}_m$ we have $g \in G^\theta$.
An $h \in G$ lies in G^θ if and only if $h\theta = \theta h$. So G^θ is given, as a subgroup of G, by linear equations. Hence the Lie algebra of G^θ is given, as subalgebra of $\mathrm{ad}\mathfrak{g}$, by the same linear equations. For $y \in \mathfrak{g}$ we have $\theta(\mathrm{ad}y)\theta^{-1} = \mathrm{ad}\theta(y)$. It follows that the Lie algebra of G^θ is $\mathrm{ad}\mathfrak{g}^\theta$ where $\mathfrak{g}^\theta = \{y \in \mathfrak{g} \mid \theta(y) = y\}$. But the latter is $\mathfrak{g}_0$. Since G is simply connected we have that G^θ is connected ([63, Theorem 8.1]). So because G_0 and G^θ have the same Lie algebra we conclude that $G_0 = G^\theta$ and therefore $g \in G_0$. □

Observe that e, f span 1-dimensional submodules of $\mathfrak{g}_1$, $\mathfrak{g}_{-1}$ respectively. So we may assume that e, f appear among the v_1^j and v_{-1}^j respectively.
Let $j_1, \dots, j_m$ be such that the $\mu_{\pm1}^{j_1}, \dots, \mu_{\pm1}^{j_m}$ are exactly the highest weights of multiplicity 1. By explicit computation we have established that, with the exception of when e lies in the orbit of type III, we have that

$$X = \{h_i, x_i, y_i \mid 1 \le i \le l\} \cup \{v_{\pm1}^{j_r} \mid 1 \le r \le m\} \cup \{e, f\}$$

generates $\mathfrak{g}$. So first we suppose that we are not dealing with orbit III; we say what we do in that case in Remark 4.4.4. Moreover, it turns out that $e, f \notin \{v_{\pm1}^{j_r} \mid 1 \le r \le m\}$ for the orbits of type III, IV, VI, VIII while $e, f \in \{v_{\pm1}^{j_r} \mid 1 \le r \le m\}$ for the orbits of type I, II, V, VII, IX. In the latter case, of course the set $\{e, f\}$ is redundant; however, this does not affect the following general procedure.
Let π be an admissible permutation of $1, \dots, \ell$. For $j \in \{j_1, \dots, j_m\}$ we let $v_{\pm1}^{j,\pi}$ be the element of X whose weight is $\mu_{\pm1}^{j,\pi}$. Let $g \in Z_{G_0}(h, e, f)$ be such that its restriction to $\mathfrak{s}_\mathfrak{c}$ is the diagram automorphism induced by π. Then $g(v_{\pm1}^{j_r}) = a_{\pm1}^{j_r} v_{\pm1}^{j_r,\pi}$ with $a_{\pm1}^{j_r} \in \mathbb{C}^\times$. Of course $g(e) = e$ and $g(f) = f$. We see that the image of every element of X is a scalar multiple of a fixed other element of X. In some cases the scalar is 1, in other cases we just know that it is nonzero and we denote it by $a_{\pm1}^{j_r}$.
Now we treat the $a_{\pm1}^{j_r}$ as indeterminates. We compute a basis $\{b_i \mid 1 \le i \le 248\}$ of $\mathfrak{g}$ containing X and such that $b_i = [x_{i_1}, [\dots [x_{i_{t-1}}, x_{i_t}] \dots]]$, for some $x_{i_1}, \dots, x_{i_t} \in X$. We denote this basis by $\mathcal{B}$. There is a straightforward algorithm to compute $\mathcal{B}$, of which we omit the details. Since we know $g(x)$ for $x \in X$ there is an equally obvious algorithm to compute $g(b_i)$ for all i. We write $g(b_i) = \sum_{j=1}^{248} \gamma_{ij} b_j$, where the γ_{ij} are polynomials in the indeterminates $a_{\pm1}^{j_r}$. By linear algebra we can compute the structure constants $\delta_{ij}^k \in \mathbb{C}$ such that $[b_i, b_j] = \sum_{k=1}^{248} \delta_{ij}^k b_k$. Then the condition $[g(b_i), g(b_j)] = g([b_i, b_j])$ amounts to

$$\sum_{s=1}^{248} \sum_{t=1}^{248} \delta_{st}^r \gamma_{is} \gamma_{jt} = \sum_{s=1}^{248} \delta_{ij}^s \gamma_{sr} \text{ for } 1 \le r \le 248.$$

So we get a number of polynomials in the polynomial ring $\mathbb{C}[a_{\pm1}^{j_r} \mid 1 \le r \le m]$. Every element of the zero locus of such polynomials yields an automorphism of $\mathfrak{g}$. By Proposition 4.4.2 the set of all these automorphisms contains at least one element of each component of $Z_{G_0}(h, e, f)$. However, the zero locus of the polynomials is infinite. Indeed, let $T_\mathfrak{c} \subset G_0$ be the connected algebraic subgroup whose Lie algebra is $\mathfrak{t}_\mathfrak{c}$. Then if g is a solution of the equations, so are all elements of $gT_\mathfrak{c}$. Secondly, if g_1, g_2 are two solutions then some power of $g_1^{-1} g_2$ lies in $T_\mathfrak{c}$. It follows that the solution set is 1-dimensional and can be written $g_1 T_\mathfrak{c} \cup \dots \cup g_s T_\mathfrak{c}$. Note that all elements of $g_i T_\mathfrak{c}$ lie in the same component as g_i. Furthermore, all elements of $T_\mathfrak{c}$ map the $v_{\pm1}^{j_r}$ to nonzero scalar multiples of themselves.

So there are characters $\chi_{\pm 1}^{j_r} : T_{\mathfrak{c}} \to \mathbb{C}^\times$ with $t \cdot v_{\pm 1}^{j_r} = \chi_{\pm 1}^{j_r}(t) v_{\pm 1}^{j_r}$. We select an index j_r such that $\chi_1^{j_r}$ is nontrivial. Then every set $g_i T_{\mathfrak{c}}$ contains a g with $g(v_{\pm 1}^{j_r}) = v_{\pm 1}^{j_r,\pi}$, i.e., such that $a_{\pm 1}^{j_r} = 1$. By adding the polynomial $a_{\pm 1}^{j_r} - 1$ to the set of polynomials we obtain a 0-dimensional system of which we can determine the solution set by Gröbner basis techniques (see [20]).

We remark that if g_1, g_2 are two solutions then g_1 and g_2 lie in the same component if and only if $g_1^{-1} g_2$ lie in the identity component. This can be decided with the algorithm IsEltOf.

The results of our computations is that for the orbits considered here $Z_{G_0}(h,e,f)$ is connected, except for the orbits IV and VI, where the component group has order 2. Moreover, we have computed explicit elements of the non-identity components which we denote by g^{IV} and g^{VI} respectively.

Remark 4.4.4. For the orbit of type III the above procedure does not work because $\mathfrak{g}$ is not generated by a canonical generating set along with the highest weight vectors of $\mathfrak{g}_1$, $\mathfrak{g}_{-1}$ of multiplicity 1.

In this case $\mathfrak{s}_{\mathfrak{c}}$ is of type C_3 and $\mathfrak{t}_{\mathfrak{c}}$ is of dimension 2. Let $\{x_i, h_i, y_i \mid i = 1, \dots, 3\}$ be a fixed canonical generating set of $\mathfrak{s}$. We have the decompositions $\mathfrak{g}_1 = V_1^1 \oplus \cdots \oplus V_1^{10}$ and $\mathfrak{g}_{-1} = V_{-1}^1 \oplus \cdots \oplus V_{-1}^{10}$ of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ respectively into irreducible $\mathfrak{s}_{\mathfrak{c}}$-modules. As before we let $v_{\pm 1}^j$ be a fixed highest weight vector of $V_{\pm 1}^j$. It turns out that in each decomposition there are 4 non-isomorphic $\mathfrak{s}$-modules of multiplicity $1, 2, 3$ and 4, respectively. After reordering the decomposition we have that $V_{\pm 1}^1$ is of multiplicty 1, $V_{\pm 1}^2 \cong V_{\pm 1}^3$ of multiplicity 2, $V_{\pm 1}^4 \cong V_{\pm 1}^5 \cong V_{\pm 1}^6$ of multiplicity 3, $V_{\pm 1}^7 \cong V_{\pm 1}^8 \cong V_{\pm 1}^9 \cong V_{\pm 1}^{10}$ of multiplicity 4. Here we list the corresponding highest weights:

$$V^1: \quad 0 \text{ --- } 0 \Leftarrow 1 \qquad\qquad V^2: \quad 0 \text{ --- } 1 \Leftarrow 0$$

$$V^4: \quad 1 \text{ --- } 0 \Leftarrow 0 \qquad\qquad V^7: \quad 0 \text{ --- } 0 \Leftarrow 0$$

Taking the highest weight vectors of the first three modules we obtain a generating set of $\mathfrak{g}$. In other words, $D := \{x_i, h_i, y_i, v_{\pm 1}^1, v_{\pm 1}^2, v_{\pm 1}^3, e, f \mid i = 1, \dots, 3\}$ is a generating set for $\mathfrak{g}$.

Now observe that $\mathfrak{s}_{\mathfrak{c}}$ has no diagram automorphisms. So every component of $Z_{G_0}(h,e,f)$ has an element restricting to the identity on $\mathfrak{s}_{\mathfrak{c}}$. Moreover $g \in Z_{G_0}(h,e,f)$ has to map $v_{\pm 1}^1$ to a multiple of itself, and both $v_{\pm 1}^2$ and $v_{\pm 1}^3$ to linear combinations of these elements. Summarizing we have the following equations $g(e) = e, g(e) = f$, $g(x_i) = x_i, g(h_i) = h_i, g(y_i) = y_i$ for $i = 1, \dots, 3$, $g(v_{\pm 1}^1) = \lambda_1^{\pm 1} v_{\pm 1}^1$, $g(v_{\pm 1}^2) = \lambda_{2,2}^{\pm 1} v_{\pm 1}^2 + \lambda_{2,3}^{\pm 1} v_{\pm 1}^3$, $g(v_{\pm 1}^3) = \lambda_{3,2}^{\pm 1} v_{\pm 1}^2 + \lambda_{3,3}^{\pm 1} v_{\pm 1}^3$.

In the same way as above we obtain a set of polynomial equations in the indeterminates $\lambda_1^{\pm 1}$, $\lambda_{ij}^{\pm 1}$. We add two extra inteterminates, say $\delta_{\pm 1}$ and add the equations $\delta_{\pm 1}(\lambda_{2,2}^{\pm 1}\lambda_{3,3}^{\pm 1} - \lambda_{2,3}^{\pm 1}\lambda_{3,2}^{\pm 1}) = 1$, expressing that these determinants must be nonzero.

By Gröbner basis computations we established that these equations only yield the trivial solution. Hence the component group is trivial in this case.

4.5 The components of $Z_{\widetilde{G}_0}(h,e,f)$ - orbits I to VIII

Let h, e, f be as in the previous section with e as in Table 4.1, lying in one of the orbits in the title of the section. In this section we deal with the problem to compute one element of each component of $Z_{\widetilde{G}_0}(h,e,f)$, knowing one element of each component of $Z_{G_0}(h,e,f)$.

Observe that, since $Z_{\widetilde{G}_0}(h,e,f) \subseteq Z_{G_0}(h,e,f)$, each component of $Z_{\widetilde{G}_0}(h,e,f)$ is contained in a component of $Z_{G_0}(h,e,f)$. So for each component of $Z_{G_0}(h,e,f)$ we have to determine its intersection with $\widetilde{G}_0$.

4.5.1 Computing $Z_{G_0}(h,e,f)^\circ \cap \widetilde{G}_0$

Here we consider the identity component of $Z_{G_0}(h,e,f)$ and compute $Z_{G_0}(h,e,f)^\circ \cap \tilde{G}_0$. As in the previous section we write $\mathfrak{z}_{\mathfrak{g}_0}(h,e,f) = \mathfrak{s}_\mathfrak{c} \oplus \mathfrak{t}_\mathfrak{c}$, where $\mathfrak{s}_\mathfrak{c}$ is semisimple and $\mathfrak{t}_\mathfrak{c}$ is the center. Let S,T be the connected subgroups of G whose Lie algebras are $\mathrm{ad}\mathfrak{s}_\mathfrak{c}$ and $\mathrm{ad}\mathfrak{t}_\mathfrak{c}$ respectively. We have $\mathfrak{s}_\mathfrak{c} \subset \tilde{\mathfrak{g}}_0$, so that $S \subset \widetilde{G}_0$. Furthermore, $Z_{G_0}(h,e,f)^\circ = ST$ hence for any $x \in Z_{G_0}(h,e,f)^\circ$ we have $x = st$ for some $s \in S, t \in T$. Since $S \subseteq \widetilde{G}_0$ it follows that requiring $x \in \widetilde{G}_0$ is equivalent to requiring $t \in \widetilde{G}_0$. So the problem is to determine the intersection $T \cap \widetilde{G}_0$.
First suppose that the orbit of e is not of type III. Then $\mathfrak{t}_\mathfrak{c}$ is 1-dimensional. With our chosen representative it happens that $\mathfrak{t}_\mathfrak{c}$ (and hence T as well) consists of diagonal matrices. Let T_0 be the standard maximal torus of $\widetilde{G}_0$ consisting of the diagonal matrices in $\widetilde{G}_0$. Then of course $T \cap \widetilde{G}_0 = T \cap T_0$. The Lie algebra of T_0 is spanned by $\mathrm{ad}h_i$ for $1 \le i \le 7$ (with h_i as in Section 4.2.3). With the algorithm TorusPar we compute explicit isomorphisms $\chi : \mathbb{C}^\times \to T$ and $\chi_0 : (\mathbb{C}^\times)^7 \to T_0$. Then the equation $\chi(z) = \chi(y_1,\dots,y_7)$ amounts to a finite number of polynomial equations in $z, y_1, \dots, y_7$. We view these as indeterminates, and compute the zero locus of the polynomials by Gröbner basis techniques. Note that the zero locus is 0-dimensional as $\mathfrak{t}_\mathfrak{c} \cap \tilde{\mathfrak{g}}_0 = 0$. Hence we find a finite number of solutions. With the algorithm IsEltOf it is checked that all these elements lie in $Z_{\widetilde{G}_0}(h,e,f)^\circ$.
Second, suppose that e lies in the orbit of type III. In this case $\dim \mathfrak{t}_\mathfrak{c} = 2$. We have that $\mathfrak{t}_1 = \mathfrak{t}_\mathfrak{c} \cap \tilde{\mathfrak{g}}_0$ is 1-dimensional and algebraic. Let $T_1 \subset \widetilde{G}_0$ be the corresponding connected subgroup. We determine a subalgebra $\mathfrak{t}_2 \subset \mathfrak{t}_\mathfrak{c}$ such that $\mathfrak{t}_\mathfrak{c} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ and $\mathfrak{t}_2$ is algebraic as well. Let $T_2 \subset G_0$ denote the corresponding connected subgroup. Then $T = T_1T_2$. Moreover, it is clear that $T \cap \widetilde{G}_0 = T_2 \cap \widetilde{G}_0$. We now continue as in the previous case, finding the same conclusion, namely that $Z_{G_0}(h,e,f)^\circ \cap \widetilde{G}_0 = Z_{\widetilde{G}_0}(h,e,f)^\circ$.

4.5.2 The non-identity component of $Z_{G_0}(h,e,f)$

Here we let e be an element of one of the orbits IV, VI and set $g = g^{\mathrm{IV}}, g^{\mathrm{VI}}$ accordingly. The problem is to determine $gZ_{G_0}(h,e,f)^\circ \cap \tilde{G}_0$. With $\mathfrak{c}, \mathfrak{s}_\mathfrak{c}, \mathfrak{t}_\mathfrak{c}, S, T$ as above we have $Z_{G_0}(h,e,f)^\circ = TS$. For $t \in T$ and $s \in S$ we have $gts \in \widetilde{G}_0$ if and only if $gt \in \widetilde{G}_0$. Hence the problem is to determine $gT \cap \widetilde{G}_0$.
First by a linear algebra computation we determine a basis of $\mathfrak{k} = \{x \in \mathfrak{g}_0 \mid g(x) = x\}$. We have that $\mathrm{ad}\mathfrak{k}$ is the Lie algebra of $Z_{G_0}(g) = \{a \in G_0 \mid ag = ga\}$. Secondly, the elements g that we found have finite order and hence are semisimple. Therefore $\mathfrak{k}$ is reductive (see [41, Corollary 26.2 A]). In both cases that we consider we have $\mathfrak{t}_\mathfrak{c} \subset \mathfrak{k}$. We let $\mathfrak{u} = \mathfrak{z}_\mathfrak{k}(\mathfrak{t}_\mathfrak{c})$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{u}$. Since $[\hat{t}, h] = 0$ for all $\hat{t} \in \mathfrak{t}_\mathfrak{c}, h \in \mathfrak{h} \subseteq \mathfrak{u}$, then $\hat{t} \in \mathfrak{h}$, i.e. $\mathfrak{t}_\mathfrak{c} \subseteq \mathfrak{h}$. This implies that $\mathfrak{z}_\mathfrak{k}(\mathfrak{h}) \subseteq \mathfrak{z}_\mathfrak{k}(\mathfrak{t}_\mathfrak{c}) = \mathfrak{u}$, hence $\mathfrak{z}_\mathfrak{k}(\mathfrak{h}) = \mathfrak{z}_\mathfrak{u}(\mathfrak{h}) = \mathfrak{h}$, i.e. $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{k}$. Let $H \subset G$ be the connected subgroup of G with Lie algebra $\mathrm{ad}\mathfrak{h}$. Then H is a maximal torus of $Z_{G_0}(g)$. Also note that $T \subset H$. Furthermore $g \in Z_{G_0}(g)^\circ$ ([65, Proposition 28.3.1(iii)]). As g is semisimple and commutes with all elements of $Z_{G_0}(g)$ it follows that $g \in H$. So the problem is to determine all $t \in T$ such that $gt \in \widetilde{G}_0 \cap H$.
In both cases that we consider it turns out that $\mathfrak{h}$ is of dimension 8, and hence it is a Cartan subalgebra of $\mathfrak{g}_0$. Secondly, $\tilde{\mathfrak{g}}_0 \cap \mathfrak{h}$ is of dimension 7 and hence it is a Cartan subalgebra of $\tilde{\mathfrak{g}}_0$. Let $H_1 \subset \widetilde{G}_0$ be the corresponding connected subgroup. We claim that $\widetilde{G}_0 \cap H = H_1$. To see this, let $H_2 \subset G_0$ be the connected subgroup corresponding to the center of $\mathfrak{g}_0$. Then $H = H_1H_2$. Let $a \in \widetilde{G}_0 \cap H$ and write $a = a_1a_2$ with $a_1 \in H_1$ and

$a_2 \in H_2$. As $a_1 \in \widetilde{G}_0$ we see that $a_2 \in \widetilde{G}_0$ as well. So $a_2 \in Z_{\widetilde{G}_0}(H_1)$. But H_1 is a maximal torus of $\widetilde{G}_0$ hence $Z_{\widetilde{G}_0}(H_1) = H_1$ ([61, Corollary 7.6.4(ii)]). It follows that $a_2 \in H_1$ and our claim is proved.
As before we use the algorithm TorusPar to compute isomorphisms $\chi : \mathbb{C}^\times \to T$ and $\chi_0 : (\mathbb{C}^\times)^7 \to H_1$. Then the set of $t \in T$ with $gt \in H_1$ corresponds to the zero locus of the polynomial equations that are equivalent to $g\chi(z) = \chi_0(y_1, \dots, y_7)$. Again we can determine this zero locus with Gröbner basis techniques. We find two and three distinct elements $t \in T$ such that $gt \in H_1$, respectively for $g = g^{\mathrm{IV}}$ and $g = g^{\mathrm{VI}}$. However, it turns out that these are equal modulo $Z_{\widetilde{G}_0}(h,e,f)^\circ$. Hence $gZ_{G_0}(h,e,f)^\circ \cap \widetilde{G}_0$ consists of one component in both cases.

4.6 The components of $Z_{\widetilde{G}_0}(h,e,f)$ - orbit IX

Let e be a representative of one of the orbits IX. We first want to determine the components of the group $N_{\widetilde{G}_0}(e) = \{g \in \widetilde{G}_0 \mid g(e) = \nu e \text{ for a } \nu \in \mathbb{C}\}$. An explicit computation shows that its Lie algebra $\mathfrak{n}_{\tilde{\mathfrak{g}}_0}(e) = \{x \in \tilde{\mathfrak{g}}_0 \mid [x,e] = \nu e \text{ for a } \nu \in \mathbb{C}\}$ is semisimple of type $G_2 + G_2$ (a fact known in the literature, see [35, Example 21.1]). We fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{n}_{\tilde{\mathfrak{g}}_0}(e)$. A computation shows that its centralizer $\mathfrak{z}_{\tilde{\mathfrak{g}}_0}(\mathfrak{h}_0)$ in $\tilde{\mathfrak{g}}_0$ is equal to a Cartan subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{g}}_0$.

Proposition 4.6.1. *Every component of $N_{\widetilde{G}_0}(e)$ has an element lying in $N_{\widetilde{G}_0}(\mathfrak{h}_0)$.*

Proof. We let $\ell = \dim(\mathfrak{h}_0)$ and fix a canonical generating set $\{x_i, h_i, y_i, 1 \le i \le \ell\}$, with $h_i \in \mathfrak{h}_0$. Since $\mathfrak{n}_{\tilde{\mathfrak{g}}_0}(e)$ is semisimple, we can proceed as for $\mathfrak{s}_\mathfrak{c}$ in the proof of Proposition 4.4.2. Then, we obtain that in each connected component of $N_{\widetilde{G}_0}(e)$ there is an element acting on the h_i's as a permutation induced by a diagram automorphism of $G_2 + G_2$. In particular, every component of $N_{\widetilde{G}_0}(e)$ contains an element that stabilizes $\mathfrak{h}_0$. □

Thanks to Proposition 4.6.1 and the basic fact that $N_{\widetilde{G}_0}(\mathfrak{h}_0) \subset N_{\widetilde{G}_0}(\mathfrak{h})$, we can proceed as follows. Let $H \subset \widetilde{G}_0$ be the connected subgroup with Lie algebra $\mathfrak{h}$. Then $Z_{\widetilde{G}_0}(H) = H$ and $N_{\widetilde{G}_0}(\mathfrak{h})/H \cong W$, where W is the Weyl group of the root system Ψ of $\tilde{\mathfrak{g}}_0$ with respect to $\mathfrak{h}$. This isomorphism is concretely realized as follows. Let $\Pi = \{\beta_1, \dots, \beta_7\}$ be a fixed basis of simple roots of Ψ. Let h_i, x_i, y_i, $1 \le i \le 7$ be the elements of a canonical generating set of $\tilde{\mathfrak{g}}_0$, where $x_i \in (\tilde{\mathfrak{g}}_0)_{\beta_i}$, $y_i \in (\tilde{\mathfrak{g}}_0)_{-\beta_i}$. Then set

$$\dot{w}_i = \exp(\mathrm{ad} x_i)\exp(-\mathrm{ad} y_i)\exp(\mathrm{ad} x_i).$$

Denote the simple reflections in W by s_i for $1 \le i \le 7$. Let $w \in W$ have reduced expression $w = s_{i_1} \cdots s_{i_r}$. We set $\dot{w} = \dot{w}_{i_1} \cdots \dot{w}_{i_r}$. Then the isomorphism $N_{\widetilde{G}_0}(\mathfrak{h})/H \to W$ maps the coset $\dot{w}H$ to w.
Since the elements of H act trivially on $\mathfrak{h}$ we see that a coset $\dot{w}H$ has an element stabilizing $\mathfrak{h}_0$ if and only if $\dot{w}$ stabilizes $\mathfrak{h}_0$. Now we run over all elements of W (we have $|W| = 322560$). For each $w \in W$ we construct the element $\dot{w}$ and check whether $\dot{w}(\mathfrak{h}_0) = \mathfrak{h}_0$. It turns out that 288 elements of W satisfy this requirement. With the algorithm IsEltOf it is established that 36 of these elements lie in $N_{\widetilde{G}_0}(e)^\circ$. So the 288 elements are divided into 8 cosets modulo $N_{\widetilde{G}_0}(e)^\circ$.
Fix a representative $\dot{w}$ of one of these cosets. We need to determine the $h \in H$ such that $\dot{w}h(e) = \nu e$ for a $\nu \in \mathbb{C}$. For this we determine a subalgebra $\mathfrak{h}_1 \subset \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ and such that $\mathfrak{h}_1$ is algebraic. Let H_0, H_1 be the connected subgroups of $\widetilde{G}_0$ with Lie algebras $\mathrm{ad}\mathfrak{h}_0$ and $\mathrm{ad}\mathfrak{h}_1$ respectively. Then $H = H_1 H_0$. As $H_0 \subset N_{\widetilde{G}_0}(e)$ we need to determine the

$h \in H_1$ such that $\dot{w}h(e) = \nu e$. With the algorithm TorusPar we can compute an explicit isomorphism $\chi : (\mathbb{C}^\times)^3 \to H_1$. Then the equation $\dot{w}\chi(y_1, y_2, y_3) - z \cdot e = 0$ is equivalent to a number of polynomial equations in the indeterminates y_1, y_2, y_3, z. By Gröbner basis techniques we determine the zero locus of this set of polynomials, and thus find all $h \in H_1$ such that $\dot{w}h(e) = \nu e$ for a $\nu \in \mathbb{C}$. It turns out that all eight elements $\dot{w}$ yield such elements in H_1. However, many are equivalent modulo $N_{\widetilde{G}_0}(e)^\circ$ (which is established using the algorithm IsEltOf). In the end eight elements remain and the component group is cyclic of order 8. A generator of this group of order 8 interchanges the two factors of type G_2 of $\mathfrak{n}_{\tilde{\mathfrak{g}}_0}(e)$ and multiplies e by a primitive eighth root of unity ζ. It follows that $N_{\widetilde{G}_0}(e) = N_{\widetilde{G}_0}(e)^\circ \ltimes (\mathbb{Z}/2\mathbb{Z})^8$.
It turns out that $\mathfrak{n}_{\tilde{\mathfrak{g}}_0}(e) = \mathfrak{z}_{\tilde{\mathfrak{g}}_0}(h, e, f)$. Hence $Z_{\widetilde{G}_0}(h, e, f)^\circ = N_{\widetilde{G}_0}(e)^\circ$. So the elements of a non-identity component of $N_{\widetilde{G}_0}(e)$ all map e to $\zeta^i e$ for $1 \le i \le 7$. We conclude that $Z_{\widetilde{G}_0}(h, e, f)$ is connected (a result that is also well known, see, e.g, [36]).

4.7 The classification of the orbits of $\mathrm{Spin}(7, 7)$

In order to classify the real $\mathrm{Spin}(7, 7)$-orbits we first need to determine which orbits have real representatives. For orbits I to VIII this is obvious as the given representatives are real. The orbits of type IX have representatives αe with $\alpha \in \mathbb{C}^\times$ and $e = 1 + e_{123456} + e_{1237} + e_{4567}$ (Table 4.1). If $\alpha \in \mathbb{R}$ then obviously αe is real. However, it also happens that the orbit of αe has a real representative if $\alpha \notin \mathbb{R}$.
A first observation is that if the orbit of αe has a real representative then $\alpha e, \bar{\alpha} e$ lie in the same orbit. Indeed, if $g(\alpha e)$ is real then $\bar{g}^{-1} g$ maps αe to $\bar{\alpha} e$. It is known that αe and βe lie in the same orbit if and only if $\alpha^8 = \beta^8$ (see [37] - it also follows from the computation of the normalizer in Section 4.6). So if the orbit of αe has a real element then $\alpha^8 = \bar{\alpha}^8$ or $\bar{\alpha} = \zeta^k \alpha$ where $\zeta = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\imath$ and $0 \le k \le 7$. We show that only $k = 0, 1$ are of interest here. Let $\alpha, \beta \in \mathbb{C}$ be such that $\bar{\alpha} = \zeta^k \alpha$ and $\bar{\beta} = \zeta^l \beta$ with $k > l$. Write $z = \frac{\alpha}{\beta}$; then $\bar{z} = \zeta^m z$ with $m = k - l > 0$. Suppose that m is even. Then we can have $m = 2, 4, 6$. This yields $z = \mu(1 - \imath)$ if $m = 2$, $z = \mu\imath$ if $m = 4$, $z = \mu(1 + \imath)$ if $m = 6$ with $\mu \in \mathbb{R}$ in all cases. Then $\alpha^8 = (\sqrt{2}\mu)^8 \beta^8$ if $m = 2, 6$ and $\alpha^8 = \mu^8 \beta^8$ if $m = 4$. In the first two cases we see that βe lies in the orbit of $\sqrt{2}\mu\alpha e$ and in the last case βe lies in the orbit of $\mu\alpha e$.
Secondly, for $k = 0$ we have $\alpha \in \mathbb{R}$ and for $k = 1$ we have $\alpha = \mu(1 + (1 - \sqrt{2})\imath)$ with $\mu \in \mathbb{R}$. As already noted, in the first case we have a real representative αe.
In the second case we need to find a real representative. For this purpose we note that there is a $g_0 \in \widetilde{G}_0$ such that $g_0 \cdot e = \zeta e$. In Section 4.6 it is shown how to determine g_0 explicitly. In the sequel we work with our explicitly determined element g_0.
Now let $e' = (1 + (1 - \sqrt{2})\imath)e$. Then $g_0 \cdot e' = \zeta e' = \bar{e}'$. It also turns out that g_0^{-1} is a cocycle, i.e., $g_0^{-1} \bar{g}_0^{-1} = 1$. The Galois cohomology set $H^1 \widetilde{G}_0$ has three elements. It turns out that g_0^{-1} is an element of the trivial class and with the algorithms of [8] we can find a $h \in \widetilde{G}_0$ with $h^{-1}\bar{h} = g_0^{-1}$. Then

$$\overline{h \cdot e'} = \bar{h} \cdot \bar{e}' = h g_0^{-1} \bar{e}' = h e'$$

and we see that he' is a real element of the orbit of e'. It is given below as the third representative of orbits numbered IX.
Let e be a real representative of a $\mathrm{Spin}(14, \mathbb{C})$-orbit, or equivalently of a $\widetilde{G}_0$-orbit. By linear algebra computations we can determine a basis of the Lie algebra $\mathfrak{z}_{\tilde{\mathfrak{g}}_0}(h, e, f)$. In the previous sections we have shown how to determine one element of each component of $Z_{\widetilde{G}_0}(h, e, f)$. These two ingredients serve as input to the algorithms of [8] by which we

can compute the Galois cohomology $H^1 Z_{\widetilde{G}_0}(h,e,f)$. By Lemma 4.2.3 this is the same as $H^1 Z_{\widetilde{G}_0}(e)$. The Galois cohomology set $H^1\widetilde{G}_0$ has three elements. The algorithms of [8] also allow us to compute the kernel of the map $H^1 Z_{\widetilde{G}_0}(e) \to H^1\widetilde{G}_0$. The elements of this kernel yield the real $\widetilde{G}_0(\mathbb{R})$-orbits that are contained in the $\widetilde{G}_0$-orbit of e (or, equivalently, the real $\mathrm{Spin}(7,7)$-orbits contained in the $\mathrm{Spin}(14,\mathbb{C})$-orbit of e). The obtained representatives are written in Table 4.2.

Following this procedure we established that the $\mathrm{Spin}(14,\mathbb{C})$-orbits I to VIII contain 1,1,1,2,1,2,2,2 $\mathrm{Spin}(7,7)$-orbits respectively. Let $e = 1 + e_{123456} + e_{1237} + e_{4567}$ be as above. For the orbits of type IX we find that the $\mathrm{Spin}(14,\mathbb{C})$-orbits with representatives αe with $\alpha \in \mathbb{R}$ contain two $\mathrm{Spin}(7,7)$-orbits, whereas the $\mathrm{Spin}(14,\mathbb{C})$-orbits with representative $\alpha(1 + (1-\sqrt{2})\imath)e$ with $\alpha \in \mathbb{R}$ contain one $\mathrm{Spin}(7,7)$-orbit.

Write the orbits numbered IX as $\mu_i e_i$ (so $e_1 = 1 + e_{123456} + e_{1237} + e_{4567}$). Then $\mu_2 e_2$ is $\mathrm{Spin}(14,\mathbb{C})$-conjugate to $\mu_2 e_1$. Furthermore, $\mu_3 e_3$ is $\mathrm{Spin}(14,\mathbb{C})$-conjugate to $\mu_3(1 + 1 - \sqrt{2})\imath)e_1$. Fix i with $1 \le i \le 3$. We see that if $\mu_i e_i$ is $\mathrm{Spin}(7,7)$-conjugate to $\nu_i e_i$ for some $\mu_i, \nu_i \in \mathbb{R}$ then $\mu_i e_1$, $\nu_i e_1$ are $\mathrm{Spin}(14,\mathbb{C})$-conjugate. This implies that $\left(\frac{\mu_i}{\nu_i}\right)^8 = 1$. Since $\mathbb{R}$ has only two eighth roots of unity this entails $\nu_i = \pm\mu_i$. Again let $g_0 \in \tilde{G}_0$ generate the component group of $Z_{\widetilde{G}_0}(h,e,f)$. A computation shows that g_0^4 is real and maps w to $-w$ for all $w \in \mathfrak{g}_1$. It follows that we may assume $\mu_i > 0$ and if $\mu_i, \nu_i > 0$ then $\mu_i e_i$ and $\nu_i e_i$ are not $\mathrm{Spin}(7,7)$-conjugate. This concludes our classification and exhaustively justifies the outcomes shown in Table 4.2.

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Appendix

In this appendix we describe the classes $\mathfrak{h}_H^\circ$ of semisimple orbits defined in Chapter 3. Excluding the class consisting only of 0, there are 31 such classes, numbered $2, \ldots, 32$. In each case we use the basis $u_1, \ldots, u_k$ given in Table 3.2. In each case $\mathfrak{h}_H^\circ$ consists of $\lambda_1 u_1 + \cdots + \lambda_k u_k$ where the coefficients λ_i are such that certain polynomials in the λ_i are nonzero. We give these polynomials, along with generators and the size of the group Γ_H. The classes 2-9 are all 1-dimensional. Let q be a basis element. Then in each case $\mathfrak{h}_H^\circ$ consists of μq with $\mu \neq 0$ and $\Gamma_H = \{1, -1\}$.

Class 10 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Order of Γ_H: 8. Polynomial conditions:

$$\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2) \neq 0.$$

Class 11 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Order of Γ_H: 4. Polynomial conditions:

$$\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2) \neq 0.$$

Class 12 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}.$$

Order of Γ_H: 12. Polynomial conditions:

$$\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - 9\lambda_2)^2 \neq 0.$$

Class 13 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Order of Γ_H: 2. Polynomial conditions:

$$\lambda_1 \lambda_2 (\lambda_1^2 - 4\lambda_2)^2 \neq 0.$$

Class 14 Generators of Γ_H:

$$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}.$$

Order of Γ_H: 4. Polynomial conditions:

$$\lambda_1 \lambda_2 (3\lambda_1 - 2\lambda_2)(\lambda_1 - \lambda_2)(2\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_2) \neq 0.$$

Class 15 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Order of Γ_H: 4. Polynomial conditions:

$$\lambda_1 \lambda_2 (\lambda_1^2 - 4\lambda_2^2)(\lambda_1^2 - 16\lambda_2^2) \neq 0.$$

Class 16 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

Order of Γ_H: 4. Polynomial conditions:

$$\lambda_1\lambda_2(2\lambda_1-\lambda_2)(3\lambda_1-2\lambda_2)(\lambda_1-2\lambda_2)(\lambda_1-\lambda_2)\neq 0.$$

Class 17 Generators of Γ_H:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Order of Γ_H: 12. Polynomial conditions:

$$\lambda_1\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_1-2\lambda_2)(2\lambda_1-\lambda_2)\neq 0.$$

Class 18 Generators of Γ_H:

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Order of Γ_H: 48. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\neq 0.$$

Class 19 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

Order of Γ_H: 12. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_2^2)(\lambda_1+\lambda_3)\neq 0, \lambda_1+\epsilon\lambda_2+2\delta\lambda_3\neq 0 \text{ where } \epsilon,\delta\in\{1,-1\}.$$

Class 20 Generators of Γ_H:

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Order of Γ_H: 8. Polynomial conditions:

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3(\lambda_2-\lambda_3)(\lambda_1^2-\lambda_3^2)(\lambda_1+\lambda_2)\neq 0,\\ &(\lambda_1+\lambda_2-\lambda_3)(3\lambda_1+2\lambda_2-\lambda_3)\neq 0,\ \lambda_3\neq\pm(\lambda_1+2\lambda_2).\end{aligned}$$

Class 21 Generators of Γ_H:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 3 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Order of Γ_H: 48. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3(\lambda_2^2-\lambda_3^2)\neq 0,\ \lambda_1-\delta(3\lambda_2-\lambda_3)\neq 0,\ \lambda_1-\delta(\lambda_2-3\lambda_3)\neq 0,\ \lambda_1+\delta\lambda_2+\epsilon\lambda_3\neq 0,$$

for all $\delta,\epsilon\in\{1,-1\}$.

Class 22 Generators of Γ_H:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\2&-1&0\\0&0&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&0&1\\-1&2&1\\3&0&-1\end{pmatrix}.$$

Order of Γ_H: 24. Polynomial conditions:

$\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_3^2)(9\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_2)(2\lambda_1-\lambda_2) \neq 0$, $\lambda_1-2\lambda_2+\delta\lambda_3 \neq 0$, $3\lambda_1-2\lambda_2+\delta\lambda_3 \neq 0$,

for all $\delta \in \{1,-1\}$.

Class 23 Generators of Γ_H:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&-1&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&1&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Order of Γ_H: 8. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_2)(2\lambda_1-\lambda_2) \neq 0,\ \lambda_1-2\lambda_2+\delta\lambda_3 \neq 0,\ \lambda_1+\delta\lambda_2+\epsilon\lambda_3 \neq 0,$$

for all $\epsilon, \delta \in \{1,-1\}$.

Class 24 Generators of Γ_H:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

Order of Γ_H: 48. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2) \neq 0,\ \lambda_1+\delta\lambda_2+\epsilon\lambda_3 \neq 0,$$

for all $\epsilon, \delta \in \{1,-1\}$.

Class 25 Generators of Γ_H:

$$\begin{pmatrix}0&0&0&-1\\0&1&0&0\\-1&0&1&-1\\-1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\1&0&-1&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&-1&-1&2\\-1&1&-1&2\\0&0&2&0\\1&1&1&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&-1&1&0\\-2&0&2&0\\1&1&1&0\\1&1&-1&2\end{pmatrix}.$$

Order of Γ_H: 96. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_1-\lambda_3)(\lambda_1+\lambda_4)(\lambda_3-\lambda_4)(\lambda_1-\lambda_3+\lambda_4) \neq 0,$$
$$\lambda_1+\delta\lambda_2+\epsilon\lambda_3 \neq 0,\ \lambda_1+\delta\lambda_2-3\lambda_3+2\lambda_4 \neq 0,\ \lambda_1+\delta\lambda_2+\epsilon(\lambda_3-2\lambda_4) \neq 0,$$

for all $\epsilon, \delta \in \{1,-1\}$.

Class 26 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix}1&-1&-1&-1\\-1&1&-1&-1\\-1&-1&1&-1\\-1&-1&-1&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}-1&1&1&-1\\1&-1&1&-1\\1&1&-1&-1\\-1&-1&-1&-1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Order of Γ_H: 48. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_1-\lambda_2)(\lambda_2^2-\lambda_4^2)(\lambda_1^2-\lambda_3^2) \neq 0,$$
$$\lambda_1+\delta\lambda_2+\epsilon\lambda_3+\nu\lambda_4 \neq 0,\ \lambda_1-2\lambda_2+\delta\lambda_3 \neq 0,\ 2\lambda_1-\lambda_2+\delta\lambda_4 \neq 0,$$

for all $\epsilon, \delta, \nu \in \{1,-1\}$.

Class 27 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix}-1&-1&-1&-1\\-1&1&-1&1\\1&-1&-1&1\\1&1&-1&-1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}-1&1&1&1\\1&-1&1&1\\1&1&1&-1\\-1&-1&1&-1\end{pmatrix}.$$

Order of Γ_H: 1152. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_3^2-\lambda_4^2)\neq 0,$$
$$\lambda_1+\delta\lambda_2+\epsilon\lambda_3+\nu\lambda_4\neq 0,$$

for all $\epsilon,\delta,\nu\in\{1,-1\}$.

Class 28 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix}-2&-1&1&1\\0&0&-2&-2\\-2&1&1&1\\0&-1&1&-1\end{pmatrix},\ \begin{pmatrix}0&0&0&-1\\0&-1&0&0\\1&0&-1&-1\\-1&0&0&0\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}1&1&-1&-2\\2&0&-2&0\\1&-1&-1&-2\\-1&1&-1&0\end{pmatrix}.$$

Order of Γ_H: 96. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_1-\lambda_3)(\lambda_1^2-\lambda_4^2)(\lambda_3+\lambda_4)(\lambda_1-2\lambda_3-\lambda_4)(\lambda_1-\lambda_3-\lambda_4)\neq 0,$$
$$\lambda_2+\delta\lambda_3+\epsilon\lambda_4\neq 0,\ \lambda_1+\delta\lambda_2+\epsilon\lambda_3\neq 0,\ \lambda_1+\delta\lambda_2-\lambda_3-2\lambda_4\neq 0,\ 2\lambda_1+\delta\lambda_2-\lambda_3-\lambda_4\neq 0,$$

for all $\epsilon,\delta\in\{1,-1\}$.

Class 29 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix}1&0&1&1&0\\2&0&0&0&0\\0&0&0&0&2\\1&-2&1&-1&0\\-1&2&1&-1&0\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}1&0&1&-1&0\\0&1&0&-1&1\\2&-1&0&1&1\\-1&-1&1&0&1\\1&-2&-1&-1&0\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}-1&0&1&1&0\\-1&1&1&0&-1\\-1&-1&-1&0&-1\\-2&1&0&-1&1\\-1&2&-1&1&0\end{pmatrix}.$$

Order of Γ_H: 1440. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5(\lambda_1-\lambda_2)\neq 0,$$
$$\lambda_2+\delta\lambda_4+\epsilon\lambda_5\neq 0,\ \lambda_1+\delta\lambda_3+\epsilon\lambda_4\neq 0,\ \lambda_1-2\lambda_2+\delta\lambda_3+\epsilon\lambda_4\neq 0,$$
$$\lambda_1+\delta\lambda_2+\epsilon\lambda_3+\nu\lambda_5\neq 0,\ 2\lambda_1-\lambda_2+\delta\lambda_4+\epsilon\lambda_5\neq 0,$$

for all $\epsilon,\delta,\nu\in\{1,-1\}$.

Class 30 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix}-2&0&0&0&0\\0&-1&1&-1&-1\\0&1&1&-1&1\\0&1&1&1&-1\\0&-1&1&1&1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}-2&0&0&0&0\\0&1&-1&1&1\\0&1&-1&-1&-1\\0&-1&-1&-1&1\\0&1&1&-1&1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}2&0&0&0&0\\0&1&1&1&-1\\0&1&1&-1&1\\0&1&-1&1&1\\0&-1&1&1&1\end{pmatrix}.$$

Order of Γ_H: 768. Polynomial conditions:

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5(\lambda_2^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)\neq 0,$$
$$\lambda_2+\delta\lambda_3+\epsilon\lambda_4+\nu\lambda_5\neq 0,\ \lambda_1+\delta\lambda_4+\epsilon\lambda_5\neq 0,\ \lambda_1+\delta\lambda_3+\epsilon\lambda_4\neq 0,$$
$$\lambda_1+\delta\lambda_2+\epsilon\lambda_5\neq 0,\ \lambda_1+\delta\lambda_2+\epsilon\lambda_3\neq 0,$$

for all $\epsilon,\delta,\nu\in\{1,-1\}$.

Class 31 Generators of Γ_H:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Order of Γ_H: 23040. Polynomial conditions:

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6 \neq 0, \\ &\lambda_i + \delta\lambda_j + \epsilon\lambda_k \neq 0,\ (i,j,k) \in \{(1,2,5),(1,4,6),(2,3,6),(3,4,5)\}, \\ &\lambda_i + \delta\lambda_j + \epsilon\lambda_k + \nu\lambda_l \neq 0,\ (i,j,k,l) \in \{(1,2,3,4),(1,3,5,6),(2,4,5,6)\}, \end{aligned}$$

for all $\epsilon, \delta, \nu \in \{1,-1\}$.

Class 32 Generators of Γ_H:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Order of Γ_H: 2903040. Polynomial conditions:

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7 \neq 0, \\ &\lambda_i + \delta\lambda_j + \epsilon\lambda_k + \nu\lambda_l \neq 0,\ (i,j,k,l) \in \{(1,2,3,6),(1,2,5,7), \\ &(1,3,4,5),(1,4,6,7),(2,3,4,7),(2,4,5,6),(3,5,6,7)\}, \end{aligned}$$

for all $\epsilon, \delta, \nu \in \{1,-1\}$.