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Do Inconsistency Indices Measure Inconsistency of Preferences?

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ABSTRACT

In this manuscript, we question the capacity of well-known inconsistency indices to measure, in a cardinal sense, the inconsistency of preferences. We argue that, at present, axiomatic properties of inconsistency indices only guarantee the good behaviour of an index when it comes to rank matrices according to their inconsistency, and the measurement of the intensity of preferences is not guaranteed. We propose to adapt the concept of measurability, established by Multiple Attribute Value Theory, to define measurable inconsistency indices. We conclude by presenting some methods to obtain measurable inconsistency indices.

1 | Introduction

Multiple Criteria Decision Analysis (MCDA) methods have shown their strength in representing subjective preferences and combining them with subjective and objective information on criteria and alternatives with the goal of ranking, sorting, or clustering the latter ones (Belton and Stewart 2002; Greco et al. 2016).

Most of these methods follow a divide and conquer logic according to which an otherwise complex problem is divided into smaller and more tractable subproblems. Some of these methods, such as the Analytic Hierarchy Process (AHP) (Saaty 1977), the Best Worst Method (Rezaei 2015) and some belonging to Multi-Attribute Value Theory (MAVT) (Keeney and Raiffa 1993) face the problem of finding a suitable weight vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ whose components are nonnegative and sum up to 1. Using the above-mentioned divide and conquer logic, these methods ask the Decision Makers (DMs), or some experts, to elicitate judgements over ratios between weights, mostly of criteria, but possibly also of alternatives (see, for example, Corrente et al. 2024). This significantly increases the tractability of the problem as each question regards only two entities.

Some specific mathematical objects have been proposed in the literature to collect pairwise statements. For example, given an index set $N = \{1, \ldots, n\}$ of attributes, pairwise judgements expressing ratios between their weights $\omega_1, \ldots, \omega_n$ can be collected into a positive *pairwise comparison matrix* $\mathbf{A} = (a_{ij})_{n \times n}$ satisfying reciprocity, $a_{ij} = 1/a_{ji} \ \ \forall i, j \in N$, that is,

$$\mathbf{A} = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \cdots & \omega_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \end{array} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}.$$

Hereafter, we denote the set of all pairwise comparison matrices by $\mathcal{A}$. To keep the exposition simple we will assume that there are no missing entries: that is, pairwise comparison matrices are *complete*. However, as we shall recall later, our discussion can be extended to incomplete matrices too. Furthermore, we do not add any a priori constraint on the values that can be assigned to the entries $a_{ij} > 0$: in practical situations, as for example in the AHP, some restricted scales are used, for example, $[1/9, 9]$. Nevertheless, this loss of generality, similarly to the assumption of completeness of preferences, does not affect the validity of the discussion.

As mentioned, the entries of the matrix $\mathbf{A}$ are assumed to approximate ratios between positive weights, such that

$$a_{ij} \approx \frac{\omega_i}{\omega_j} \quad \forall i < j. \tag{1}$$

A matrix $\mathbf{A}$ is *consistent* if there exists a (priority) vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ such that

$$a_{ij} = \frac{\omega_i}{\omega_j} \quad \forall i < j. \tag{2}$$

Equivalently, (2) can be restated as follows:

$$a_{ij} a_{jk} = a_{ik} \quad \forall i < j < k. \tag{3}$$

More often than not, in real-world situations, an expert cannot be completely consistent and a vector $\boldsymbol{\omega}$ satisfying (3) does not exist. In these cases, it is pragmatic to still try to find a vector $\boldsymbol{\omega}$ such that the ratios between its components are, at least, good approximations of the entries of $\mathbf{A}$, that is, such that (1) holds. The most applied method, Saaty's eigenvector method (Saaty 1977), suggests to use the Perron-Frobenius eigenvector of $\mathbf{A}$ to represent the priorities $\boldsymbol{\omega}$. The other most widely adopted method, the geometric mean method, was initially proposed by Rabinowitz (1976) and then studied by Crawford and Williams (1985) and alternatively formulated as a logarithmic least squares minimisation problem (Csató 2019b). It suggests to estimate weights as row-wise geometric means of entries in $\mathbf{A}$. Namely, for all i

$$\omega_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}}. \tag{4}$$

Evaluating inconsistency—the deviation from the consistency conditions—is considered an important step in most of the methods employing pairwise comparisons. For example, evaluations of inconsistency of preferences have been used to weight experts in group decision making according to their rationality (Koksalmis and Kabak 2019) and to guide experts in the process of revising their judgements (Mazurek 2023), just to cite two tangential applications.

An *inconsistency index* is a function $I: \mathcal{A} \to \mathbb{R}$ and the preorder $\precsim^I$ induced by I ranks matrices according to their inconsistency level, so that

$$\mathbf{A} \precsim^I \mathbf{B} \;\Leftrightarrow I(\mathbf{A}) \leq I(\mathbf{B}) \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{A}.$$

In words, $\mathbf{A} \precsim^I \mathbf{B}$ means that $\mathbf{B}$ is at least as inconsistent as $\mathbf{A}$. Simply put, the greater the value of I, the greater the inconsistency of $\mathbf{A}$. Furthermore, if we decompose $\precsim^I$ into its symmetric and asymmetric parts, then we obtain

$$\mathbf{A} \begin{Bmatrix} \succ^I \\ \sim^I \\ \prec^I \end{Bmatrix} \mathbf{B} \;\Leftrightarrow I(\mathbf{A}) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} I(\mathbf{B}) \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{A}. \tag{5}$$

Brunelli (2018) offered a survey of inconsistency indices, and Mazurek (2023) reported the definitions of 17 distinct inconsistency indices, but it is safe to say that even more indices have been proposed in the literature. Consequently, in the last years, some proposals have sprouted to make order among inconsistency indices through the introduction of a few reasonable properties. Recently, Brunelli and Fedrizzi (2024) proposed to characterise the idea of a reasonable inconsistency index by means of a single property based on the Pareto dominance principle, and Pant et al. (2025) presented an overview of different sets of reasonable properties. Even considering alternative sets of properties (Brunelli 2017; Koczkodaj and Urban 2018), one sees that the common goal is to define indices that can reasonably classify matrices from the most to the least inconsistent, according to some agreeable properties. Another purpose of formal studies of properties of inconsistency indices is to use them to characterise axiomatically some indices. This was done, for example, by Csató (2018, 2019a).

Currently, in the literature, inconsistency indices are considered equivalent if they imply the same order relation $\precsim^I$. This was formalised by Csató (2018) thanks to his definition of inconsistency-ranking, that is, the order relation $\precsim^I$. Coherently, all studies on formal properties of inconsistency indices have focused on ordinal properties. The consequence is that, at present, indices are only required to *order* matrices according to their inconsistency, and the interpretation of their value does not go beyond the basic deduction: 'the greater the value returned by the function I, the greater the inconsistency of the pairwise comparison matrix'. To illustrate this idea, one can consider three pairwise comparison matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and assume $I(\mathbf{A}) = 1$, $I(\mathbf{B}) = 2$, $I(\mathbf{C}) = 6$. These values only allow us to deduce that $\mathbf{A} \prec^I \mathbf{B} \prec^I \mathbf{C}$, but we cannot deduce anything on the extent of the intensity of inconsistency, as we are limited to simple statements based on order relations. That is, without further investigation, even if

$$\underbrace{I(\mathbf{C}) - I(\mathbf{B})}_{=4} > \underbrace{I(\mathbf{B}) - I(\mathbf{A})}_{=1},$$

we cannot claim that the increase of inconsistency due to moving from the preferences in $\mathbf{B}$ to those in $\mathbf{C}$ is greater than the increase obtained by passing from the preferences in $\mathbf{A}$ to those in $\mathbf{B}$.

The implications of this fact are non-negligible. For example, one should not blindly consider the well-known Saaty's consistency ratio CR (Saaty 1977), defined as the ratio between the inconsistency of a matrix $\mathbf{A}$ and the average inconsistency of reciprocal random matrices of the same order as $\mathbf{A}$, in a cardinal sense. In principle, without further scrutiny, $\mathrm{CR}(\mathbf{A}) = 0.1$ does *not* imply that the level of inconsistency of $\mathbf{A}$ is the 10% of the inconsistency of an average random matrix. Theory alone allows us only to say that the inconsistency of $\mathbf{A}$ is lower than that of an average random matrix. That is, unless measurability is assured, the two concepts may be disjoint and inconsistency indices should be used only in an *ordinal* sense.

In this manuscript we shall draw on the idea of measurable value function coming from MAVT and transpose it to the case of inconsistency indices. In this way we will be able to formulate the concept of measurability for inconsistency indices, and state that if an index is measurable, then its value can be interpreted in a cardinal sense too.

The rest of the paper is organised as follows. Section 2 recalls some of the most important inconsistency indices and proposes to classify them into two categories. Section 3 recalls the relevant results from MAVT with a particular focus on the idea of measurability. Section 4 borrows the concept of measurability and applies it to inconsistency indices, by interpreting them as special cases of value functions. Furthermore, it discusses a framework to estimate measurable inconsistency indices. Finally, Section 5 wraps up the manuscript.

2 | Some Inconsistency Indices

The most prominent and widely-used inconsistency index is the CI index proposed by Saaty (1977) in his seminal work on the Analytic Hierarchy Process (AHP) and can be formulated as follows

$$\mathrm{CI}(\mathbf{A}) = \frac{1}{\binom{n}{2}} \sum_{i<j} \left(a_{ij}\frac{\omega_j}{\omega_i} + \frac{1}{a_{ij}\frac{\omega_j}{\omega_i}} - 2 \right) \tag{6}$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is the Perron-Frobenius eigenvector of $\mathbf{A}$. This index is more often written in the equivalent formulation

$$\mathrm{CI}(\mathbf{A}) = \frac{\lambda_{\max} - n}{n-1}$$

where $\lambda_{\max}$ is the Perron-Frobenius eigenvalue of $\mathbf{A}$.

The Geometric Consistency Index, initially proposed by Crawford and Williams (1985) and then studied by Aguarón and Moreno-Jiménez (2003), was defined as

$$\mathrm{GCI}(\mathbf{A}) = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2 a_{ij}\frac{\omega_j}{\omega_i}, \tag{7}$$

where $\boldsymbol{\omega}$ is estimated with the geometric mean method, and has often been proposed as an alternative to CI thanks to its analytic formulation.

Takeda (1993) proposed using the following index

$$T(\mathbf{A}) = \frac{1}{\binom{n}{2}} \sum_{i<j} \left(a_{ij}\frac{\omega_j}{\omega_i} + \frac{1}{a_{ij}\frac{\omega_j}{\omega_i}} \right) \tag{8}$$

where also in this case, $\boldsymbol{\omega}$ is estimated with the geometric mean method. Let us observe that, as noted by Brunelli (2018), the index T was later independently reintroduced by Wu and Xu (2012). Note that T differs from CI in that (i) they employ two different methods to find the weight vector and (ii) one formulation is a translation of the other.

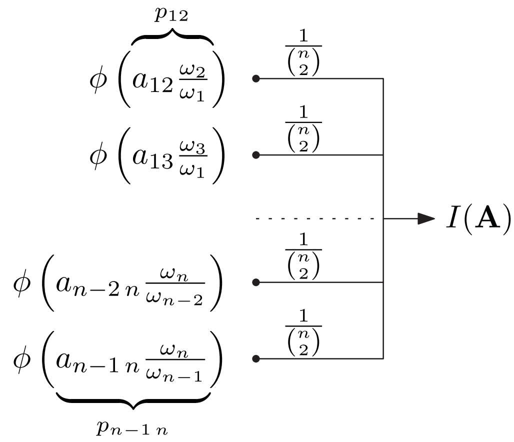

FIGURE 1 | General framework for the inconsistency indices CI, GCI and T.

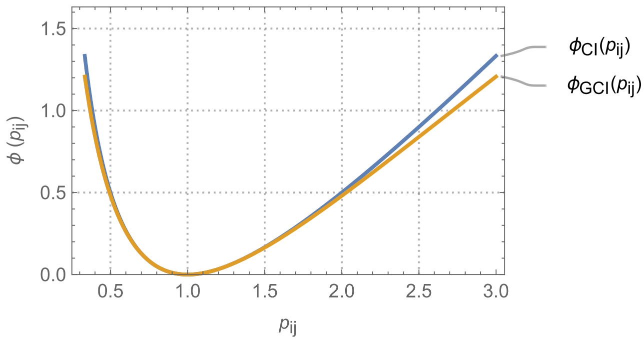

FIGURE 2 | Plots of local evaluations of inconsistency according to CI, GCI and T.

Up to multiplication by a positive scalar, all the indices presented until now have a structure similar to the one presented in Figure 1, where $p_{ij} := a_{ij}\frac{\omega_j}{\omega_i}$ and $\phi: \mathbb{R}^+ \to \mathbb{R}$ is a local quantification of the inconsistency associated with p_{ij}.

If we consider a specific inconsistency index I, then we add it as the suffix of ϕ and we can call $\phi_I(p_{ij})$ its local evaluation of inconsistency stemming from p_{ij}. We can plot the behaviours of different ϕ_I as univariate functions, as shown in Figure 2.

Some other indices are based on the divergence from the consistency condition (3). Duszak and Koczkodaj (1994) introduced

$$K(\mathbf{A}) = \max_{i<j<k} \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\} \tag{9}$$

to account for the maximum local inconsistency. Another index was proposed by Peláez and Lamata (2003)

$$\mathrm{PL}(A) = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right) \tag{10}$$

which was then proven to be proportional to another index formulated by Shiraishi et al. (1998). Kułakowski and

Szybowski (2014) proposed a general triad-based approach to inconsistency evaluation and introduced some indices. One of them was later studied by Grzybowski (2016):

$$\mathrm{ATI}(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\}. \tag{11}$$

Note that, if $n = 3$, then $\mathrm{ATI}(\mathbf{A}) = K(\mathbf{A})$.

Aguarón et al. (2020) showed that GCI can also be rewritten as

$$\mathrm{GCI}(\mathbf{A}) = \frac{1}{3\binom{n}{3}} \sum_{i<j<k} \ln^2 \frac{a_{ij}a_{jk}}{a_{ik}} \tag{12}$$

making explicit that it can also be interpreted as a quantification of the violation of the Condition (3). A general formulation was proposed by Kazibudzki (2019), within which he considered the following two indices:

$$\mathrm{ALTI}_1(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \left| \ln \frac{a_{ij}a_{jk}}{a_{ik}} \right| \tag{13}$$

$$\mathrm{ALTI}_2(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \ln^2 \frac{a_{ij}a_{jk}}{a_{ik}}. \tag{14}$$

It can be shown that ALTI_1 is equivalent to the index proposed by Cavallo and D'Apuzzo (2009) in the case of additive preference relations. Similarly, ALTI_2 is functionally related to GCI, as it appears when comparing its formulation with (12).

Inconsistency indices (10–14) are examples of triad-based inconsistency indices and fit into the framework represented in Figure 3, where $t_{ijk} := \frac{a_{ij}a_{jk}}{a_{ik}}$ and $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a function characterising the local evaluation of the inconsistency of a triad.

If, similarly to what was done before, we call $\psi_I(t_{ijk})$ the evaluation made by index I of the local inconsistency stemming from t_{ijk}, we can again plot them as univariate functions, as shown in Figure 4. From this figure one can see that the behaviour of distinct indices, also in terms of second derivatives, can differ.

Incidentally, from a qualitative inspection of Figures 2 and 4, we can see the greater variability of local inconsistency evaluations in the case of inconsistency indices based on the consistency Condition (3).

Recently, Bortot et al. (2023) proposed an overarching approach thanks to the two parametric indices.

$$I_\alpha(\mathbf{A}) = \frac{1}{\binom{n}{2}} \sum_{i<j} \begin{cases} \frac{1}{\alpha^2}\left(\left(a_{ij}\frac{\omega_j}{\omega_i} \right)^{\alpha} - \left(a_{ij}\frac{\omega_j}{\omega_i} \right)^{-\alpha} - 2 \right) & \alpha \neq 0 \\ \ln^2 a_{ij}\frac{\omega_j}{\omega_i} & \alpha = 0 \end{cases} \tag{15}$$

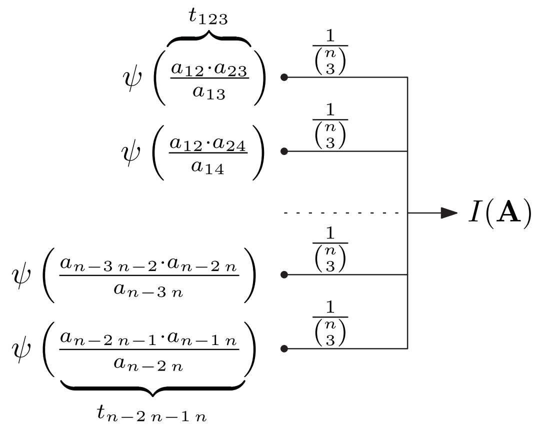

FIGURE 3 | General framework underlying the inconsistency indices PL, ATI, ALTI_1 and ALTI_2.

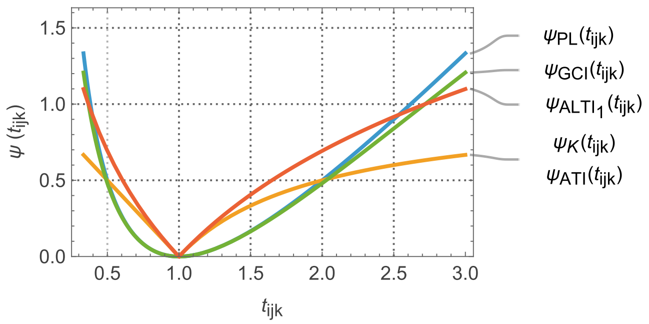

FIGURE 4 | Plots of local evaluations of inconsistency according to the indices PL, ATI, K, GCI and ALTI_1.

$$J_\alpha(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \begin{cases} \frac{1}{\alpha^2}\left(\left(\frac{a_{ij}a_{jk}}{a_{ik}} \right)^{\alpha} - \left(\frac{a_{ij}a_{jk}}{a_{ik}} \right)^{-\alpha} - 2 \right) & \alpha \neq 0 \\ \ln^2 \frac{a_{ik}}{a_{ij}a_{jk}} & \alpha = 0 \end{cases} \tag{16}$$

based on Conditions (2) or (3), respectively. Hereafter, we shall call ϕ_α and ψ_α the local evaluations of inconsistency that are considered as terms in the sums of $I_\alpha(\mathbf{A})$ and $J_\alpha(\mathbf{A})$, respectively.

The parametric inconsistency indices I_α and J_α are so general that Bortot et al. (2023) showed that they encompass some well-known inconsistency indices but, at present, nobody has yet proposed a procedure to estimate the parameter α characterising them.

Interestingly, many inconsistency indices share the same structural form, to the extent that the same inconsistency index, adhering to one of the two structural forms, has often been introduced independently.

There are many other inconsistency indices, but Brunelli and Fedrizzi (2024) showed that all those explicitly mentioned above respect some basic properties and, therefore, are particularly relevant. Moreover, following the previous considerations, without further scrutiny we cannot, a priori, interpret their values in a cardinal sense.

3 | Measurable Multi-Attribute Value Functions

In this section, we shall recall some notions from multi-attribute value theory. All these concepts can also be found in Dyer and Sarin (1979) and Smith and Dyer (2021). We start considering a multi-attribute decision problem with m attributes, taking values in the sets $X_1, \ldots, X_m$, respectively, and we call $X := X_1 \times \cdots \times X_m$ the space of all possible alternatives. MAVT is then concerned with the definition of a representative *value function* $v: X \to \mathbb{R}$ such that

$$\mathbf{x} \precsim \mathbf{y} \Leftrightarrow v(\mathbf{x}) \leq v(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in X. \tag{17}$$

It can be shown that, under the assumption of preference independence (see Appendix A) (Keeney and Raiffa 1993; Wakker 1989), there exist functions $v_i: X_i \to \mathbb{R}$ and scalars $w_i \geq 0$ with $w_1 + \cdots + w_m = 1$ such that the *additive value function*

$$v(\mathbf{x}) = v(x_1, \ldots, x_m) = \sum_{i=1}^{m} w_i v_i(x_i) \tag{18}$$

can represent the preferences of a DM. This means that (17) holds, and the additive value function (18) can be effectively used to rank alternatives in X. The structure of an additive value function is sketched in Figure 5.

In most of the cases, ordering the alternatives is not enough and it is also desirable to attach an interpretation to the values $v(\mathbf{x})$, $\forall \mathbf{x} \in X$, in terms of their intensity. That is, we want v to be a *measurable value function*. This requirement was formalised as follows:

$$(\mathbf{x} \to \mathbf{x}') \precsim_E (\mathbf{x}'' \to \mathbf{x}''') \Leftrightarrow v(\mathbf{x}') - v(\mathbf{x}) \leq v(\mathbf{x}''') - v(\mathbf{x}'') \tag{19}$$

for all $\mathbf{x}, \mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in X$, where $\precsim_E$ is an order relation on exchanges between alternatives. More precisely, $(\mathbf{x} \to \mathbf{x}') \precsim_E (\mathbf{x}'' \to \mathbf{x}''')$ states that switching from alternative $\mathbf{x}$ to alternative $\mathbf{x}'$ yields a not greater increase in the perceived value than switching from $\mathbf{x}''$ to $\mathbf{x}'''$.

As shown by Dyer and Sarin (1979), the existence of a measurable additive value function—a function (18) respecting (19)—is guaranteed if three conditions are satisfied (there are a few more topological conditions, but they are beyond the objective of this discussion): mutual preference independence, difference consistency, difference independence. Details on these three conditions can be found in Appendix A.

In practical situations, difference consistency is always satisfied and, as suggested by Smith and Dyer (2021), it is sufficient to

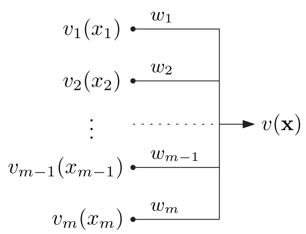

FIGURE 5 | A representation of the additive value function (18).

check the other two properties by answering the following two questions:

Mutual preference independence: Do tradeoffs between two attributes depend on the levels of the other attributes?

Difference independence: Does the improvement associated with changes in one attribute depend on the levels of the other attributes?

The second condition is usually verified by asking a DM questions on the strength of preference over single attributes. If we consider the generic i th attribute, these statements are in the form

$$(x_i \to x_i') \left\{ \begin{matrix} \succsim_E \\ \sim_E \\ \precsim_E \end{matrix} \right\} (x_i'' \to x_i''') \tag{20}$$

where x_i, x_i', x_i'', x_i''' are attribute levels in X_i. If the DM feels confident about her statements on intensities of preference and does not perceive that they should be dependent on the levels of the other attributes, then difference independence can be considered satisfied.

In a nutshell, the whole point is that if mutual preference independence pushes us to aggregate additively some attribute value functions, the obtained value function can be measurable only if the attribute value functions themselves are measurable. Consequently, if the attribute value functions are not measurable, then we should not expect that the value functions themselves are measurable. Next, we shall see how this simple consideration can affect the measurability of inconsistency indices.

4 | On Measurable Inconsistency Indices

According to MAVT, as well as to the vast majority of MCDA methods, the global value of an alternative depends on the extent to which it fulfils the attribute levels. Likewise, in the case of inconsistency, as corroborated by the inconsistency indices recalled in Section 2, it is common to consider the magnitude of the phenomenon as a function of local violations of the consistency Conditions (2) and (3). Inconsistency indices are often nothing more than sums of local violations of Conditions (2) and (3), and their similarity with the additive value framework is corroborated by the common structure of Figures 1, 3, and 5.

Similarly to what was proposed for value functions, we define a *measurable inconsistency index* as an inconsistency index satisfying the condition

$$(\mathbf{A} \to \mathbf{B}) \precsim_E^I (\mathbf{C} \to \mathbf{D}) \quad \Leftrightarrow \quad I(\mathbf{B}) - I(\mathbf{A}) \leq I(\mathbf{D}) - I(\mathbf{C}) \tag{21}$$

for all $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{A}$. In this context, the left hand side of (21) means that the subjective feeling of increase in inconsistency associated by changing the preferences in $\mathbf{A}$ with those in $\mathbf{B}$ is perceived as smaller than, or equal to, the one felt by changing the preferences in $\mathbf{C}$ with those in $\mathbf{D}$. By requiring Condition (21), we enable the value of an index to be interpreted in a cardinal sense.

$$
\begin{aligned}
\min_{\delta \geq 0, \alpha} \quad & \delta \\
\text{s. t.} \quad & -\delta \leq \phi_{I_\alpha}\left(p_{ij}^{(2)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(1)}\right) + \phi_{I_\alpha}\left(p_{ij}^{(3)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(4)}\right) \leq \delta \quad \text{if} \quad \left(p_{ij}^{(1)} \to p_{ij}^{(2)}\right) \sim_E^p \left(p_{ij}^{(3)} \to p_{ij}^{(4)}\right) \\
& \phi_{I_\alpha}\left(p_{ij}^{(4)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(3)}\right) + \phi_{I_\alpha}\left(p_{ij}^{(1)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(2)}\right) \leq \delta \quad \text{if} \quad \left(p_{ij}^{(1)} \to p_{ij}^{(2)}\right) \gtrsim_E^p \left(p_{ij}^{(3)} \to p_{ij}^{(4)}\right) \text{ and } p_{ij}^{(3)} \precsim^p p_{ij}^{(4)} \\
& \phi_{I_\alpha}\left(p_{ij}^{(4)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(3)}\right) + \phi_{I_\alpha}\left(p_{ij}^{(1)}\right) - \phi_{I_\alpha}\left(p_{ij}^{(2)}\right) \leq \delta - \gamma \quad \text{if} \quad \left(p_{ij}^{(1)} \to p_{ij}^{(2)}\right) \succ_E^p \left(p_{ij}^{(3)} \to p_{ij}^{(4)}\right) \text{ and } p_{ij}^{(3)} \precsim^p p_{ij}^{(4)}
\end{aligned}
\tag{22}
$$

The first, and most intuitive, way to find a measurable inconsistency index is to simply inspect and/or draw/sketch the function ϕ if the index is based on p_{ij} and the function ψ if the index is based on t_{ijk}, making sure that these local evaluations respect the cardinal interpretation of the DM of different inconsistency levels. If, on top of having found a measurable function for the local inconsistencies, we also know that the condition of preference independence holds, then we know that using an additive approach to aggregate these local evaluations with appropriate scaling constants leads to a measurable inconsistency index.

Using a parametric inconsistency index and finding a suitable value of the parameter determining its shape could be another viable solution. Hence, the problem becomes that of finding such a parameter.

Let us first consider the case of the violation of (C1). We could have a list of statements on quadruples of pairs $\left(a_{ij}, \omega_i / \omega_j\right)$ in terms of intensities of preferences. Bearing in mind that $p_{ij} := a_{ij} \frac{\omega_j}{\omega_i}$, such statements are in the form

$$
\left(p_{ij}^{(1)} \to p_{ij}^{(2)}\right) \begin{Bmatrix} \succ_E^p \\ \sim_E^p \\ \prec_E^p \end{Bmatrix} \left(p_{ij}^{(3)} \to p_{ij}^{(4)}\right).
$$

An analyst could consider a parametric inconsistency index in the family of parameters $\boldsymbol{\alpha}$ and, then, formulate an optimisation problem to find the parameters $\boldsymbol{\alpha}$ that best fits the statements on the intensities of inconsistency in p_{ij}.

Then, we formulate the optimisation problem (22)—where γ is a small positive real number and $p_{ij}^{(3)} \precsim^p p_{ij}^{(4)}$ indicates that $p_{ij}^{(3)}$ is not more inconsistent than $p_{ij}^{(4)}$—to find the α that best fits the statements of an analyst on inconsistency intensities on p_{ij}.

Optimisation problem (22) is very general and its simplest declination is perhaps the one using only the first family of the constraints assuming that $p_{ij}^{(2)} = p_{ij}^{(3)}$ is determined by the expert/analyst. We borrow, once again, from MAVT: this can be seen as an adaptation of the bisection method that is used to estimate attribute value functions (French 1986, 86) (Eisenführ et al. 2010, 119) (von Winterfeldt and Edwards 1986, 235) (Keeney and Raiffa 1993, 120). Suppose that we call $p_x = p_{ij}^{(2)} = p_{ij}^{(3)}$, then, a question that could be asked to a DM is to find p_x such that

$$
\left(p_{ij}^{(1)} \to p_x\right) \sim_E^p \left(p_x \to p_{ij}^{(4)}\right). \tag{23}
$$

The question can be interpreted as in Figure 6, where the DM manipulates the pairwise comparison $a_{ij}^{(x)}$ so that the value p_x establishes an indifference relation—in terms of inconsistency—between the two exchanges $\left(p_{ij}^{(1)} \to p_x\right)$ and $\left(p_x \to p_{ij}^{(4)}\right)$. It is important to keep in mind that $a_{ij}^{(x)}$ does not refer to a given decision problem with specific i and j, but only exemplifies a hypothetical pairwise comparison of two entities. If we consider the parametric inconsistency index I_α, recalled in (15) which is an average of parametrized local inconsistencies $\phi_{I_\alpha}(p_{ij})$, then, the optimisation problem (22) collapses into the following.

$$
\begin{aligned}
\min_{\alpha, \delta \geq 0} \quad & \delta \\
\text{s. t.} \quad & -\delta \leq -\phi_{I_\alpha}\left(p_{ij}^{(1)}\right) + \phi_{I_\alpha}(p_x) + \phi_{I_\alpha}(p_x) - \phi_{I_\alpha}\left(p_{ij}^{(4)}\right) \leq \delta
\end{aligned}
\tag{24}
$$

Example 1. To give a concrete example, we can consider the same values from Figure 6, in which $a_{ij}^{(1)} = \omega_i / \omega_j = 1, a_{ij}^{(x)} = 9/4$, $a_{ij}^{(4)} = 4$. In this case, (24) becomes

$$
\begin{aligned}
\min_{\alpha, \delta \geq 0} \quad & \delta \\
\text{s. t.} \quad & -\delta \leq -\phi_{I_\alpha}(1) + \phi_{I_\alpha}\left(\frac{9}{4}\right) + \phi_{I_\alpha}\left(\frac{9}{4}\right) - \phi_{I_\alpha}(4) \leq \delta
\end{aligned}
$$

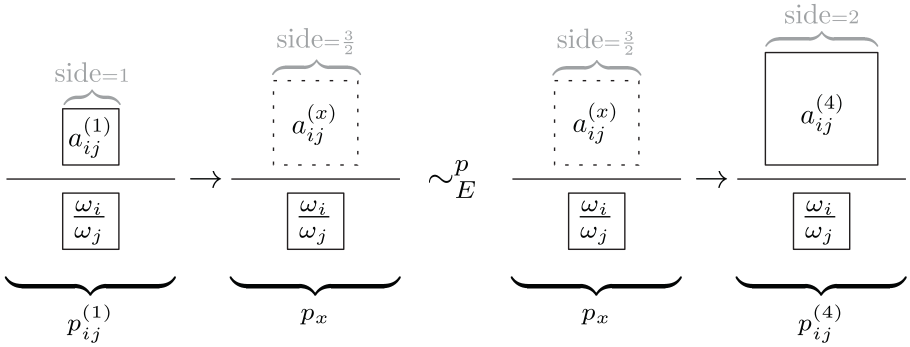

FIGURE 6 | Graphical interpretation of the question on the indifference between inconsistency exchanges ($\sim_E^p$). The goal of the question is the elicitation of the (area) value for $a_{ij}^{(x)}$. The area values in this figure comply with the numbers used in Example 1.

whose optimal value is $\delta^* = 0.6066$ with $\alpha^* = 0$. This means that the parametric function cannot accommodate the statement of the DM. This is shown in Figure 7 where the most suitable function ϕ_0 is plotted, but still shows that even in this (best) case, $\phi_{I_0}(9/4) - \phi_{I_0}(1) \neq \phi_{I_0}(4) - \phi_{I_0}(9/4)$. However, note that the fact that we could not reach $\delta^* = 0$ with $p_x = 9/4$ did not mean that the indifference statement given by the DM was inherently wrong, but only that the parametric inconsistency index is not sufficiently general to accommodate it.

Example 2. On the contrary, with the same data as in Example 1 but $a_{ij}^{(x)} = 49/16$, the solution of

$$\begin{aligned} \min_{\delta,\alpha} \quad & \delta \\ \text{s.t.} \quad & -\delta \leq -\phi_{I_\alpha}(1) + \phi_{I_\alpha}\left(\frac{49}{16}\right) + \phi_{I_\alpha}\left(\frac{49}{16}\right) - \phi_{I_\alpha}(4) \leq \delta \end{aligned}$$

is $\delta^* = 0$ with $\alpha^* = 2.3236$, meaning that the local inconsistency estimator $\phi_{I_{2.3236}}$ fully reflects the indifference statement. Figure 8 shows, as claimed, that, if $p_x = 49/16$, then the parametric function ϕ_{I_α} is sufficiently general to accommodate the indifference statement with $\alpha = 2.3236$. This can be seen from the plot of $\phi_{I_{2.3236}}(p_{ij})$ where

$$\phi_{I_{2.3236}}\left(\frac{49}{16}\right) - \phi_{I_{2.3236}}(1) = \phi_{I_{2.3236}}(4) - \phi_{I_{2.3236}}\left(\frac{49}{16}\right).$$

Note that the optimisation problem (22) is not convex and therefore it may have multiple local optima. However, this should not be considered a major problem. For each value of the variable α it is possible to determine the associated value of δ minimising the objective function and therefore the optimisation problem can also be interpreted as a univariate problem and easily plotted. Moreover, if we fix a bound to the value of α, we can apply Weierstrass theorem and have the guarantee of the existence of a global optimum.

The same approach can be carried out for triads t_{ijk}, and given their similarity, we omit the discussion for this latter case based on (3).

5 | Discussion

The literature has shown some attempts to define the concept of inconsistency index by restricting the search among functions $I: \mathcal{A} \to \mathbb{R}$ that respect some basic properties, mostly defined using the order relation $\succsim^I$ induced on the set $\mathcal{A}$. Such properties are objective and not context dependent, as they should not depend on the analyst who uses the method. We reckoned that these properties are useful to help define inconsistency indices that are sound from an ordinal point of view, but they do not have a grip when it comes to define them from the cardinal point of view.

To overcome this possible shortcoming, we borrowed the idea of measurable value function from MAVT and we applied it in this context. We did so by analysing the structure of some of the most common inconsistency indices and drawing a parallel between them and additive measurable value functions. Thus, if some regularity conditions are satisfied, measurability of the inconsistency index comes from the measurability of the local estimates of inconsistency. Statements on the intensity like 'one of the comparisons is much more inconsistent than the others' (Wedley 1993) or the concept of being 'close' to the condition of consistency are meaningful only if the inconsistency index is measurable. The properties used to define the basic requirements for $\precsim^I$ could only exclude some manifestly unfit indices or, in the case of their use to characterise a specific index, uniquely determine the relation $\precsim^I$. Either way, with the introduction of the concept of measurability we help analysts refine the search for the most suitable index.

We conclude with the following remarks:

- All the inconsistency indices that are not expressed as an average function of local inconsistencies implicitly claim that the condition of preference independence does *not* apply. Hence, if someone believes that the condition of preference independence should hold, then she should not use indices that cannot be expressed in an additive form. Index K, presented in (9), considers the maximum of local inconsistencies and is an example of an index that conflicts with the property of preference independence. We remark that preference independence is a property that has not an intrinsic desirability, and it simply describes the attitude of DMs. Hence, its violation could be used as a further argument in favour or against such inconsistency indices, depending on whether one believes that preference independence should hold or not. It is also reasonable that testing the attitude of a decision analyst

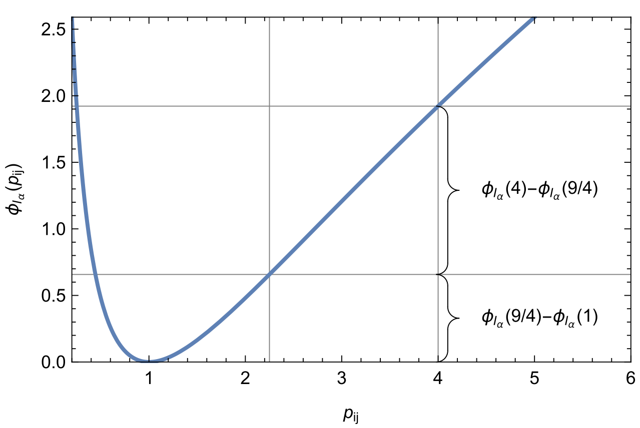

FIGURE 7 | Plot of $\phi_{I_0}(p_{ij})$.

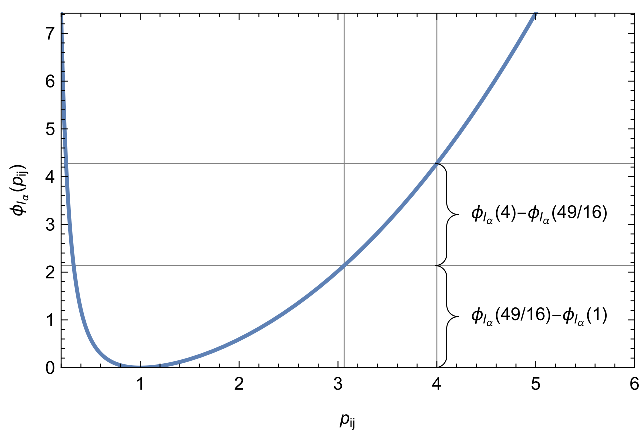

FIGURE 8 | Plot of $\phi_{I_{2.3236}}(p_{ij})$.

towards preference independence could help her choose between arithmetic mean based inconsistency indices and other types of inconsistency indices, for example, K.

- For pair- and triad-based inconsistency indices in the forms identified in Figures 1 and 3 the 'weights' assigned to the local evaluations of inconsistency were equal, for example, $1/\binom{n}{2}$ or $1/\binom{n}{3}$. This can be seen comparing Figures 1 and 3 with Figure 5. Such choice was due to the fact that the number of arguments of the weighted sums are $\binom{n}{2}$ and $\binom{n}{3}$, respectively, and thus such scaling factors lead to an (unweighted) arithmetic mean-based formulation of the indices. Nevertheless, for measurability purposes, the values of distinct scaling constants could be different. So, why did we impose that they must be equal? There are two reasons. The first one, loosely based on the principle of insufficient reasons, is that it is difficult to envision situations where—only on the ground of the entities involved in the considered comparisons—one pair or one triad of indices should be more valuable than others when it comes to assess the global inconsistency of a matrix. On the other hand, it may be reasonable to assign weights on the ground of the value of the local inconsistency. An example is index K, which assigns weight 1 to the largest local inconsistency and 0 to all the others. This, however, is done with respect to the value of the local inconsistencies and it is not based on the identity of the compared entities (alternatives or criteria). A second argument, related to the previous one, is that all the axiomatic approaches proposed so far in the literature—for example, Brunelli and Fedrizzi (2024), Koczkodaj and Urban (2018)—require a basic property which says that, if alternatives are rearranged in the rows and columns of the matrix $\mathbf{A}$ and, after this relabelling, a new matrix $\mathbf{A}_\sigma$ is obtained, then $I(\mathbf{A}) = I(\mathbf{A}_\sigma)$. Now, if preference independence holds we should aggregate local inconsistencies using an additive function and a non-uniform assignment of 'weights' on pairs and triads would lead to a violation of the above-mentioned widely accepted requirement. Hence, when we discussed the use of an additive aggregation of local inconsistencies we imposed that all weights are equal.

- In spite of some parts of the scientific community claiming the necessity of normalising inconsistency indices (see Koczkodaj et al. (2017) for the range $[0, 1]$), if we do not consider restrictions on the values of the entries of the matrix $\mathbf{A}$, then all the inconsistency indices presented in this paper, except K and ATI, have $\mathbb{R}_0^+$ as their image. Given an inconsistency index I of this second type, it would be natural to normalise it into a new I_N using a monotone increasing bijection $I^N = I/(1+I)$. Note that I_N induces the same order relation of I and, therefore, both of them can be used interchangeably to order preferences according to their inconsistency. Nevertheless, it is known that only positive affine transformations preserve the measurability of value functions. That is, if $v(x)$ is a measurable value function, then $\hat{v}(x)$ is measurable if and only if there exists $a > 0$ and $b \in \mathbb{R}$ such that $\hat{v}(x) = a \cdot v(x) + b$. Hence, we conclude that the bijection $I^N = I/(1+I)$ does not preserve measurability. Furthermore, as there is not any positive affine transformation that can map the set of positive real numbers into a bounded interval, we conclude that a measurable inconsistency index I with unbounded image, for example, $\mathbb{R}_0^+$, *cannot* be normalised without losing the property of measurability.

- We employed a promising parametric family of inconsistency indices, but it would have been unreasonable to expect that it could have covered all possible attitudes towards inconsistency: a given parametric family of inconsistency indices can cover some attitudes towards inconsistency and leave some others uncovered. Possible future research may focus on the study of parametric inconsistency indices in relation to their capacity of measuring inconsistency.

- The idea of measurability used in MAVT suggested the decomposition of global inconsistency indices into local contributions. This could support the use of more transparent and easily interpretable inconsistency indices. Indeed, the property of measurability is easier to be tested on local indicators—and then inherited by the global measure—than be tested directly on the global measure of inconsistency. Consider, for example, the harmonic consistency index (Stein and Mizzi 2007) whose interpretation is much more challenging.

Although, for simplicity, our manuscript considered only complete pairwise comparison matrices, we must recall that our discussion can be straightforwardly extended to incomplete pairwise comparison matrices too. Nowadays, there are extensions of the eigenvector (Bozóki et al. 2010; Harker 1987) and the geometric mean method (Bozóki et al. 2010), as well as of many inconsistency indices (Kułakowski and Talaga 2020), that make the complete case appear as a special case of the incomplete one. For example, for the eigenvector method, Bozóki et al. (2010) took inspiration from Shiraishi et al. (1998) and Shiraishi and Obata (2002), and studied and implemented the minimisation of $\lambda_{\max}$ keeping the missing entries as variables, whereas Harker (1987) devised a method that allows to find the largest eigenvector of matrices with missing entries too.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A

Independence Properties

Given an index set of some attributes, $M = \{1, \dots, m\}$, we consider $I \subseteq M$ and we call X_I the set of attribute levels of attributes with indices in I, and $\overline{X}_I$ its complement with respect to M. Furthermore, we call $(\mathbf{x}_I, \overline{\mathbf{y}}_I)$ the alternative combining the evaluations of $\mathbf{x}$ for the attributes in I and the evaluations of $\mathbf{y}$ for the attributes in $\overline{I}$. The set of attribute levels X_I is preferentially independent of $\overline{X}_I$ if

$$(\mathbf{y}_I, \overline{\mathbf{x}}_I) \precsim (\mathbf{x}_I, \overline{\mathbf{x}}_I) \text{ for all } \mathbf{x}_I, \mathbf{y}_I \in X_I \text{ and } \overline{\mathbf{x}}_I \in \overline{X}_I$$

implies

$$(\mathbf{y}_I, \overline{\mathbf{y}}_I) \precsim (\mathbf{x}_I, \overline{\mathbf{y}}_I) \text{ for all } \overline{\mathbf{y}}_I \in \overline{X}_I.$$

In other words, given two alternatives with some equal attribute levels, their ranking is uniquely determined by the unequal attribute levels. If in a decision problem each attribute set X_I is preferentially independent of its complement, then the set of attributes is called *mutually preferentially independent*.

A set of preferences is *difference consistent* if, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$,

$$\mathbf{z} \precsim \mathbf{y} \precsim \mathbf{x} \Rightarrow (\mathbf{z} \to \mathbf{y}) \precsim_E (\mathbf{z} \to \mathbf{x}) \quad \text{(A1)}$$

Next, consider the two alternatives

$$\mathbf{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \quad \text{(A2)}$$

$$\mathbf{y} = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \quad \text{(A3)}$$

differing only in the ith attribute. Consider also a second pair of alternatives

$$\mathbf{x}' = (x'_1, \dots, x'_{i-1}, x_i, x'_{i+1}, \dots, x'_n) \quad \text{(A4)}$$

$$\mathbf{y}' = (x'_1, \dots, x'_{i-1}, y_i, x'_{i+1}, \dots, x'_n) \quad \text{(A5)}$$

with the same characteristic. The attribute $i \in M$ is *difference independent* if and only if $(\mathbf{x} \to \mathbf{y}) \sim_E (\mathbf{x}' \to \mathbf{y}')$, for all possible choices of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x}'$, $\mathbf{y}'$ as long as the alternatives are structured as in (A2) and (A4).

As written by Eisenführ et al. (2010, 132) in their textbook:

Hence, our earlier statement that 'preferential independence' is the necessary condition for the additive model was very sloppy. More exactly, we have to require mutual preferential independence and, if we ask for a measurable value function (as we usually do), we also need difference independence.

As recalled by Vilkkumaa et al. (2014), these three conditions 'can be usually achieved through careful problem structuring'.