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Morphisms between Grassmannians, II

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Abstract. Denote by $\mathbb{G}(k, n)$ the Grassmannian of linear subspaces of dimension k in $\mathbb{P}^n$. We show that if $\varphi : \mathbb{G}(l, n) \to \mathbb{G}(k, n)$ is a nonconstant morphism and $l \neq 0, n-1$, then $l = k$ or $l = n-k-1$ and φ is an isomorphism.

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1. Introduction. In [10], Tango proved that there are no nonconstant morphisms from $\mathbb{P}^m$ to the Grassmannian $\mathbb{G}(k, n)$ if $m > n$, and later, in [11], considered the case $m = n$, proving the same result for kn even, $(k, n) \neq (2, 5)$, $k \notin \{0, n-1\}$. The result in [10] has been generalized in [8] to the case of morphisms $\varphi : \mathbb{G}(l, m) \to \mathbb{G}(k, n)$, with $m > n$, and later to a more general setting (see Theorem 2.3 and references therein).

The aim of the present paper is to generalize the results in [11], considering morphisms $\mathbb{G}(l, n) \to \mathbb{G}(k, n)$, and proving the following:

Theorem 1.1 *If $\varphi : \mathbb{G}(l, n) \to \mathbb{G}(k, n)$ is a nonconstant morphism and $l \neq 0, n-1$, then $l = k$ or $l = n-k-1$ and φ is an isomorphism.*

In a nutshell, the idea in [8,10] to prove the constancy of a morphism $\varphi : M \to \mathbb{G}(k, n)$ was to consider the relation among Chern classes coming from the universal sequence on $\mathbb{G}(k, n)$, pull it back via φ, and show that this leads to a contradiction, via a study of the Chow ring $\mathrm{A}^\bullet(M)$.

In [7], this idea was refined and reinterpreted geometrically, considering the Schubert varieties $X_H, X_p \subset \mathbb{G}(k, n)$, parametrizing linear spaces contained in a hyperplane H or passing through a point p; then $[X_H] \in \mathrm{A}^{k+1}(\mathbb{G}(k, n))$, $[X_p] \in \mathrm{A}^{n-k}(\mathbb{G}(k, n))$, and clearly $[X_H] \cdot [X_p] = 0$. If the pullback via φ of one of the two cycles is zero, one can construct a morphism from M to a smaller Grassmannian which factors via φ, allowing inductive arguments. Else, one obtains two effective nonzero cycles whose product is zero in $\mathrm{A}^{n+1}(M)$. The proof can then be finished by showing that there are no such pairs in $\mathrm{A}^{\bullet}(M)$. This is the idea that led to the notion of effective good divisibility of a variety (see Sect. 2.1).

The last step of the above argument could be further refined: in fact, it is enough to show that for every effective nonzero $x \in \mathrm{A}^{k+1}(M), y \in \mathrm{A}^{n-k}(M)$, it holds $x \cdot y \neq 0$. We fulfill this task, in our setup, by characterizing the pairs of effective nonzero cycles of total codimension $n+1$ in $\mathrm{A}^{\bullet}(\mathbb{G}(l,n))$ whose product is zero, called maximal disjoint pairs (Corollary 3.3). In particular, for $\mathbb{G}(l,n)$, such a pair consists of cycles of codimensions $l+1$ and $n-l$, forcing $l=k$ or $n-k-1$ in order for $\varphi : \mathbb{G}(l,n) \to \mathbb{G}(k,n)$ to be nonconstant. In these cases, we conclude that φ is an isomorphism using a Remmert-Van de Ven type theorem for rational homogeneous varieties due to Hwang and Mok ([6, Main Theorem]).

2. Preliminaries.

2.1. Effective good divisibility. The notion of effective good divisibility of a smooth complex projective variety M (see [7, Section 2.1]) is related to the total codimension of effective zero divisors in the Chow ring $\mathrm{A}^{\bullet}(M)$.

Definition 2.1 The effective good divisibility of M, denoted by e.d.(M), is the maximum integer s such that, given effective cycles $x_i \in \mathrm{A}^i(M)$, $x_j \in \mathrm{A}^j(M)$ with $i+j \leq s$ and $x_i x_j = 0$, then either $x_i = 0$ or $x_j = 0$.

Example 2.2 The effective good divisibility is known for the wide class of rational homogeneous manifolds: it was computed for Grassmannians in [8], for varieties of classical type independently in [5,7], for varieties of exceptional type in [5]. In this paper, we are mostly interested in Grassmannians $\mathbb{G}(k,n)$, parametrizing linear subspaces of dimension k in $\mathbb{P}^n$, for which e.d.$(\mathbb{G}(k,n)) = n$.

Knowledge of the effective good divisibility can be used to prove the non existence of nonconstant morphisms, as exemplified by the following result.

Theorem 2.3 (cf. [7, Theorem 1.3], [5, Theorem 1.4]). *Let M be a smooth complex projective variety, and let M' be a rational homogeneous manifold of classical type such that* e.d.$(M) >$ e.d.(M'). *Then there are no nonconstant morphisms from M to M'.*

In order to prove the nonexistence of nonconstant morphisms between two Grassmannians $\mathbb{G}(l,n)$ and $\mathbb{G}(k,n)$, which have the same effective good divisibility, we need to characterize the pairs of effective zero divisors (x_i, x_j) of minimal total codimension $i+j$; they have been introduced in [7, Definition 2.3]. In the definition given below, we consider also the type of the pair, which keeps track of the codimensions of x_i and x_j.

Definition 2.4 A (non-ordered) pair $\{x_i, x_j\}$ with $x_i \in \mathrm{A}^i(M)$, $x_j \in \mathrm{A}^j(M)$ nonzero effective cycles such that $x_i x_j = 0$ and $i+j =$ e.d.$(M)+1$ will be called a maximal disjoint pair (md-pair for short) of type $\{i,j\}$ in M.

We will prove Theorem 1.1 using the fact that two nonisomorphic Grassmannians $\mathbb{G}(l, n)$ and $\mathbb{G}(k, n)$ do not possess md-pairs of the same type, so we review in the next subsection basic facts about the generators of the Chow ring $\mathrm{A}^\bullet(\mathbb{G}(k, n))$.

2.2. Schubert varieties. We will recall some basic facts about Schubert calculus in $\mathbb{G}(k, n)$. We refer to [2,3] for the proofs and for a complete account on the subject.

Let us identify $\mathbb{G}(k, n)$ with the Grassmannian $G(k+1, V)$ of vector subspaces of dimension $k+1$ in a vector space V of dimension $n+1$ and consider a complete flag $\mathcal{V}$ of vectors subspaces of V:

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \subsetneq V.$$

Given a sequence of integers $I = \{0 < i_1 < \cdots < i_{k+1} \leq n+1\}$, called a *Schubert symbol*, we define the Schubert variety X_I as

$$X_I = \{W \in G(k+1, V) \mid \dim(V_{i_j} \cap W) \geq j \text{ for all } j\}. \tag{1}$$

To the Schubert variety X_I one can associate a *Young diagram* λ_I in the following way: consider a rectangle with $k+1$ rows and $n-k$ columns and the path from the lower-left corner to the upper-right corner consisting of $n+1$ steps, where the i-th step is vertical if $i \in I$ and horizontal if $i \notin I$. The Young diagram is the part of the rectangle that is top-left of this path.

Identifying the Young diagram λ_I with the *partition* $(\lambda_1, \ldots, \lambda_{k+1})$, where λ_i is the number of boxes in row i, we have

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k+1} \geq 0 \quad \text{and} \quad \lambda_j = i_{k+2-j} - (k+2-j).$$

By abuse, we denote by λ_I also the corresponding partition $(\lambda_1, \ldots, \lambda_{k+1})$, and we set $|I| := |\lambda_I| = \sum \lambda_i$; this number is the dimension of the subvariety X_I.

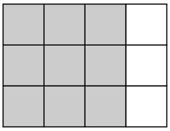
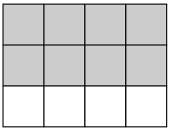
The *Bruhat order* on the set of Schubert symbols is defined as follows: $I \leq L$ if and only if $i_j \leq l_j$ for every $j = 1, \ldots, n+1$. Notice that $I \leq L$ if and only if the Young diagram of I is a subdiagram of the Young diagram of L.

The class $[X_I] \in \mathrm{A}^\bullet(\mathbb{G}(k, n))$ does not depend on the choice of the flag $\mathcal{V}$, and will be called a Schubert cycle. The Schubert cycles form a basis of $\mathrm{A}^\bullet(\mathbb{G}(k, n))$.

Example 2.5 Let $H = \mathbb{P}(V_n)$; the Schubert variety $X_H := X_{I_H}$, corresponding to the Schubert symbol $I_H = \{n-k < n-k+1 < \cdots < n\}$ parametrizes $(k+1)$-dimensional subspaces of V_n, or equivalently k-dimensional linear subspaces of $\mathbb{P}(V)$ contained in $H = \mathbb{P}(V_n)$. The associated Young diagram λ_{I_H} is the diagram whose partition is $(n-k-1, \ldots, n-k-1)$.

Let $p = \mathbb{P}(V_1)$; the Schubert variety $X_p := X_{I_p}$, corresponding to the Schubert symbol $I_p = \{1 < n-k+2 < \cdots < n < n+1\}$, parametrizes $(k+1)$-dimensional subspaces of V containing V_1, or equivalently k-dimensional linear subspaces of $\mathbb{P}(V)$ containing p. The associated Young diagram λ_{I_p} is the diagram whose partition is $(n-k, \ldots, n-k, 0)$. Let us show the diagrams

λ_{I_H} and λ_{I_p} for $n = 6, k = 2$.

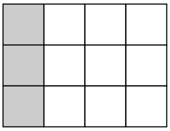
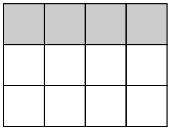
$\lambda_{I_H} =$  $\lambda_{I_p} =$ 

A dual description of Schubert varieties, which highlights the codimension of the variety rather than the dimension, is possible. Given a Schubert symbol $I = \{0 < i_1 < \cdots < i_{k+1} \leq n+1\}$, the *dual Schubert symbol* is defined by setting $I^\vee = \{n+2-i_{k+1} < \cdots < n+2-i_1\}$. The associated Young diagram corresponds to the partition $(\lambda_1^\vee, \ldots, \lambda_{k+1}^\vee)$, where

$$\lambda_j^\vee = n + 2 - i_j - (k + 2 - j) = n - k - \lambda_{k+2-j}. \tag{2}$$

In particular, the diagram $\lambda_{I^\vee}$ is obtained from the diagram λ_I by taking the complement and rotating it by 180°. Since $|\lambda_I| + |\lambda_{I^\vee}| = (k+1)(n-k) = \dim \mathbb{G}(k,n)$, we see that $|I^\vee| = |\lambda_{I^\vee}| = \operatorname{codim} X_I$.

Example 2.6 Let I_p and I_H be the Schubert symbols introduced in Example 2.5. Then the Young diagram of $\lambda_{I_H^\vee}$ corresponds to the partition $(1, \ldots, 1)$, while the one of $\lambda_{I_p^\vee}$ corresponds to the partition $(n-k, 0, \ldots, 0)$. Again, we show the diagrams for the case $n = 6, k = 2$.

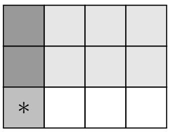
$\lambda_{I_H^\vee} =$  $\lambda_{I_p^\vee} =$ 

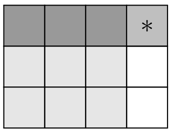
Given a Schubert index I, we define the *opposite Schubert variety* X^I to be $w_0 X_{I^\vee}$, where w_0 is the longest element of the Weyl group of $\mathrm{GL}(n+1, \mathbb{C})$, which acts on the canonical basis as $w_0(\mathbf{e}_i) = \mathbf{e}_{n+1-i}$. Clearly we have an equality of cycles $[X^I] = [X_{I^\vee}] \in \mathrm{A}^{|I|}(\mathbb{G}(k,n))$.

We are going to use the following well-known fact [1, Section 1.3 and Proposition 1.3.2] or [9, Lemma 3.1 (3)]:

Proposition 2.7 *The intersection of a Schubert variety X_J and an opposite Schubert variety X^I is nonempty if and only if $I \leq J$. This is the case if and only if the intersection product $[X^I] \cdot [X_J]$ is nonzero.*

Example 2.8 Let I_H and I_p be as in Example 2.5. Clearly, recalling the geometric descriptions of X_{I_H} and X_{I_p}, we have that $[X_{I_H}] \cdot [X_{I_p}] = 0$, but we can obtain this also using Proposition 2.7 since $I_H^\vee \not\leq I_p$ (and dually $I_p^\vee \not\leq I_H$).

$I_H^\vee \not\leq I_p :$ 

$I_p^\vee \not\leq I_H :$ 

3. Md-pairs in Grassmannians. In this section, we will show that the only md-pair for $\mathbb{G}(k,n)$, $1 \le k \le n-2$, is essentially the one described in Example 2.5; let us start by considering md-pairs whose elements are Schubert cycles.

Theorem 3.1 *In the Grassmannian $\mathbb{G}(k,n)$, with $1 \le k \le n-2$, let X_I, X_J be Schubert varieties with $\operatorname{codim}(X_I) + \operatorname{codim}(X_J) \le n+1$. Then $[X_I] \cdot [X_J] = 0$ if and only if $\{I, J\} = \{I_H, I_p\}$.*

Proof Since $[X_I] = [X^{I^\vee}]$, by Proposition 2.7, we know that $[X_I] \cdot [X_J] \ne 0$ if and only if $I^\vee \le J$. This in turn happens if and only if the Young diagram of $I^\vee$ is contained in the Young diagram of J, i.e., the associated partitions $\lambda_{I^\vee}$ and μ_J satisfy $\lambda_i^\vee \le \mu_i$ for $1 \le i \le k+1$. Note that the assumption on the codimensions can be rewritten as

$$|\lambda_{I^\vee}| + (\dim \mathbb{G}(k,n) - |\mu_J|) \le n+1$$

which can be restated as

$$|\lambda_{I^\vee}| \le |\mu_J| - k(n-k) + (k+1). \tag{3}$$

Now, from Proposition 3.2 below, we get $(\lambda_{I^\vee}, \mu_J) = ((n-k, 0, \ldots, 0), (n-k-1, \ldots, n-k-1))$ or $((1, \ldots, 1), (n-k, \ldots, n-k, 0))$ and from Example 2.5, we obtain the statement. □

Proposition 3.2 *Given integers n, k such that $1 \le k \le n-2$, let λ and μ be two partitions such that*

$$n-k \ge \lambda_1 \ge \cdots \ge \lambda_{k+1} \ge 0, \qquad n-k \ge \mu_1 \ge \cdots \ge \mu_{k+1} \ge 0, \tag{4}$$

and

$$|\lambda| \le |\mu| - k(n-k) + (k+1). \tag{5}$$

Then $\lambda \not\le \mu$ if and only if $\lambda = (n-k, 0, \ldots, 0)$, $\mu = (n-k-1, \ldots, n-k-1)$ or $\lambda = (1, \ldots, 1)$, $\mu = (n-k, \ldots, n-k, 0)$.

Proof In order to have $\lambda \not\le \mu$, we must have $\lambda_{h+1} > \mu_{h+1}$ for some $h \in \{0, \ldots, k\}$. We write $\lambda_{h+1} = \mu_{h+1} + m$ for some positive integer m; let us distinguish two cases: $h > 0$ or $h = 0$.

$\boxed{h = 0}$

Using the inequalities (4), we see that

$$|\mu| - \lambda_1 \le k\mu_1 - m = k\lambda_1 - (k+1)m \le k(n-k) - (k+1)m, \tag{6}$$

hence, using (5), we obtain

$$0 \le \sum_{j \ge 2} \lambda_j = |\lambda| - \lambda_1 \le (k+1)(1-m), \tag{7}$$

forcing $m = 1$ and $\lambda_j = 0$ for $j \ge 2$. Moreover, all inequalities in (6) and (7) are equalities. In particular,

$$\sum_{j \ge 2} \mu_j - 1 = |\mu| - \lambda_1 = k(n-k-1) - 1$$

so that $\mu_j = n-k-1$ for every $j \geq 2$. Now, since $\mu_2 \leq \mu_1 < \lambda_1 \leq n-k$, we get $\mu_1 = n-k-1$ and $\lambda_1 = n-k$ (Note that here we are using the assumption $k \geq 1$). We have thus proved that $\lambda = (n-k, 0, \ldots, 0)$, $\mu = (n-k-1, \ldots, n-k-1)$.

$\boxed{h > 0}$

We use inequalities (4) to obtain

$$\sum_{j>h+1} \mu_j \leq (k-h)\mu_{h+1} \leq (k-h)(n-k-m) \qquad \text{and} \qquad \sum_{j\leq h} \mu_j \leq h(n-k).$$

Combining the two inequalities, we get

$$|\mu| - \lambda_{h+1} \leq -(k-h+1)m + k(n-k). \tag{8}$$

Now, from (5), we obtain

$$|\lambda| - \lambda_{h+1} \leq -(k-h+1)m + k + 1 = hm + (k+1)(1-m). \tag{9}$$

On the other hand, using again (4), we get

$$|\lambda| - \lambda_{h+1} \geq \sum_{j\leq h} \lambda_j \geq h\lambda_{h+1} \geq hm. \tag{10}$$

Combining (9) and (10), we obtain that $m = 1$; moreover equality holds everywhere in (8), (9), and (10). In particular,

$$\lambda_{h+1} = 1, \quad |\mu| = k(n-k) - (k-h), \qquad \text{and} \qquad |\lambda| = h + 1. \tag{11}$$

Since $\lambda_{h+1} = 1$, we have $\mu_{h+1} = 0$, hence $\mu_j = 0$ for $j \geq h+1$. Then

$$|\mu| = \sum_{j\leq h} \mu_j \leq h(n-k),$$

which, combined with (11), recalling that $k \leq n-2$, gives $h = k$ and $\mu_j = n-k$ for every $j \leq k$, so $\mu = (n-k, \ldots, n-k, 0)$. By (11), we have $|\lambda| = k+1$. Recalling that $\lambda_j \leq \lambda_{k+1} = 1$ for every j, we conclude that $\lambda = (1, \ldots, 1)$. □

We can now describe the md-pairs of $\mathbb{G}(k,n)$.

Corollary 3.3 *Let k, n be integers such that $1 \leq k \leq n-2$, and let $[\Gamma] \in \mathrm{A}^i(\mathbb{G}(k,n)), [\Delta] \in \mathrm{A}^j(\mathbb{G}(k,n))$ be effective nonzero cycles such that $[\Gamma]\cdot[\Delta] = 0$ and $i + j \leq n+1$. Then $\{[\Gamma], [\Delta]\} = \{a[X_H], b[X_p]\}$ with X_H and X_p as in Example 2.5. In particular, all the md-pairs in $\mathbb{G}(k,n)$ have type $\{k+1, n-k\}$.*

Proof By [4, Corollary of Theorem 1], the cones of effective classes of a fixed codimension in $\mathbb{G}(k,n)$ are polyhedral cones generated by the Schubert classes of the same codimension. Therefore we can write $[\Gamma]$ and $[\Delta]$ as linear combinations with nonnegative coefficients:

$$[\Gamma] = \sum_{|K^\vee|=i} \gamma_K [X_K], \qquad [\Delta] = \sum_{|L^\vee|=j} \delta_L [X_L].$$

Moreover, every product $[X_K]\cdot[X_L]$ is a combination of Schubert cycles with nonnegative coefficients due to the Littlewood–Richardson rule. Then, if $\gamma_K\delta_L \neq 0$, we must have $[X_K]\cdot[X_L] = 0$, and the statement now follows from Theorem 3.1. □

4. Morphisms to Grassmannians. In this section, we will prove Theorem 1.1. We state and prove first two auxiliary results that could be useful to study morphisms to Grassmannians from other kinds of varieties.

Lemma 4.1 *Let k, n be integers such that $1 \leq k \leq n-2$, and let $Y \subset \mathbb{G}(k,n)$ be a positive dimensional closed irreducible subvariety. Then:*

- *If $[Y]\cdot[X_H] = 0$, there is a nonconstant morphism $\psi : Y \to \mathbb{G}(k-1, n-1)$.*
- *If $[Y] \cdot [X_p] = 0$, there is a nonconstant morphism $\psi : Y \to \mathbb{G}(k, n-1)$.*

Proof In the first case, for a general hyperplane H', Y does not meet the subvariety $X_{H'} \subset \mathbb{G}(k,n)$ parametrizing linear spaces contained in H'. We can take ψ to be the restriction to Y of the morphism $\pi_H : \mathbb{G}(k,n)\backslash X_{H'} \to \mathbb{G}(k-1, n-1)$, which sends Λ to $\Lambda \cap H'$. Since the fibers of this morphism are affine (see [7, Example 5.5]), we get that ψ is not constant.

The argument in the second case is similar: for a general point $q \in \mathbb{P}^n$, Y does not meet the subvariety $X_q \subset \mathbb{G}(k,n)$ parametrizing linear spaces passing through q. We can take ψ to be the restriction to Y of the morphism $\pi_q : \mathbb{G}(k,n) \setminus X_q \to \mathbb{G}(k, n-1)$, which sends Λ to the linear projection of Λ from q to a hyperplane. Again the fibers of this morphism are affine and ψ is not constant. □

Proposition 4.2 *Let k, n be integers such that $1 \leq k \leq n-2$ and let M be a smooth variety with* e.d.$(M) = n$ *in which there is no md-pair of type $\{k+1, n-k\}$. Then every morphism $\varphi : M \to \mathbb{G}(k,n)$ is constant.*

Proof Let $[X_H], [X_p] \in \mathrm{A}^\bullet(\mathbb{G}(k,n))$ be as in Example 2.5. For general $g, g' \in \mathrm{GL}(n+1, \mathbb{C})$, gX_H and $g'X_p$ are disjoint and generically transverse to φ, that is, $\varphi^{-1}(gX_H), \varphi^{-1}(g'X_p)$ are generically reduced and of the same codimensions as $X_H, X_p \subset X$ (see [2, Theorem 1.7]).

By [2, Theorem 1.23], we have that $\varphi^*[X_H] = [\varphi^{-1}(gX_H)] \in \mathrm{A}^{k+1}(M)$ and $\varphi^*[X_p] = [\varphi^{-1}(g'X_p)] \in \mathrm{A}^{n-k}(M)$. In particular,

$$\operatorname{codim} \varphi^*[X_H] + \operatorname{codim} \varphi^*[X_p] = n + 1 = \text{e.d.}(M) + 1$$

and $\varphi^*[X_H] \cdot \varphi^*[X_p] = 0$. By assumption, $(\varphi^*[X_H], \varphi^*[X_p])$ cannot be an md-pair for M, so one of the two cycles is zero.

If φ were not constant, by Lemma 4.1, we would get a nonconstant morphism $\psi \circ \varphi$ from M to $\mathbb{G}(k-1, n-1)$ or $\mathbb{G}(k, n-1)$, contradicting [8, Proposition 2.4]. □

Proof of Theorem 1.1 Since the Picard number of $\mathbb{G}(l,n)$ is one, a nonconstant morphism must be finite. In fact, if this were not the case, the pullback of an ample line bundle on the image will be ample on $\mathbb{G}(l,n)$, but trivial on the contracted curves, contradicting Kleiman's criterion. If $k = 0, n-1$, we then have a morphism $\varphi : \mathbb{G}(l,n) \to \mathbb{P}^n$ which is constant since $\dim \mathbb{G}(l,n) > n$. Else, by Proposition 4.2, if φ is not constant, then $l = k$ or $n-k-1$. In this case, since the dimensions of the domain and the codomain are equal, φ must be surjective. We conclude that φ is an isomorphism by [6, Main Theorem]. □

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