INFINITE DIMENSIONAL OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE FUNCTION AND THE TRACE FORMULA FOR THE HEAT SEMIGROUP.

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ABSTRACT. Infinite dimensional oscillatory integrals with a polynomially growing phase function with a small parameter ϵ are studied by means of an analytic continuation technique, as well as their asymptotic expansion in the limit $\epsilon \downarrow 0$. The results are applied to the study of the semiclassical behavior of the trace of the heat semigroup with a polynomial potential.

Key words: Infinite dimensional oscillatory integrals, asymptotics, stationary phase method, Laplace method, degenerate phase, Heat kernels, polynomial potential, semiclassical limit.
AMS classification : 28C20, 34E05, 35K05, 35C15, 35C20.

1. Introduction

Oscillatory integrals on finite dimensional Hilbert spaces, an expression of this form

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\epsilon}\Phi(x)} g(x)dx, \tag{1}$$

(where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the phase function and $\epsilon \in \mathbb{R}^+$ a real positive parameter) are a classical topic of investigation , having several applications, e.g. in electromagnetism, optics and acoustics. They are part of the general theory of Fourier integral operators [24, 31]. Particularly interesting is the study of the asymptotic behavior of these integrals in the limit $\epsilon \downarrow 0$. The generalization of the definition of oscillatory integrals to the case where the integration is performed on an infinite dimensional space, in particular a space of continuous functions, presents a particular interest in connection with applications to quantum theory such as the mathematical realization of Feynman path integrals [1, 8]. In the case where the integration is performed on such spaces and on general real separable Hilbert spaces, the theory was for a long time restricted to oscillatory integrals with phase functions Φ which can be written as sums of a quadratic form and a bounded function belonging to the class of Fourier transforms of complex measures.

In [9, 10] these results have been generalized to phase functions with quartic polynomial growth. In this paper we consider a generalization of the oscillatory integral (1) and its infinite dimensional analogue, in the case where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int e^{-\frac{s}{\epsilon}\Phi(x)} g(x)dx. \tag{2}$$

Strictly speaking $I(s)$ has an oscillatory behavior only for s being a pure imaginary number. By generalizing the results of [10], we prove (in section 2) a representation formula which allows to compute an infinite dimensional oscillatory integral of the form (2), with a phase function Φ having an arbitrary even polynomial growth, in terms of a Gaussian integral. In the non degenerate case (i.e. when the Hessian of the phase function is non degenerate), we compute (in section 3) the asymptotic expansion of the integral as $\epsilon \downarrow 0$ in powers of ϵ. In the degenerate case the situation is more involved. In section 4 we handle in detail a particular example and apply this result to the study of the asymptotic behavior of the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, in the case where H is the essentially self-adjoint operator on $C_0^\infty \equiv C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ given on the functions $\phi \in C_0^\infty$ by

$$H\phi(x) = \Big(-\frac{\hbar^2}{2}\Delta_x + V(x)\Big)\phi(x), \tag{3}$$

where $\hbar > 0$ and V is a polynomially growing potential of the form $V(x) = |x|^{2N}$, $x \in \mathbb{R}^d$. This corresponds to exhibiting the detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, "near the classical limit". Indeed H can be interpreted as a Schrödinger Hamiltonian, (in which case $\hbar$ is the reduced Planck's constant) and consequently $e^{-\frac{t}{\hbar}H}$ as a Schrödinger semigroup (heat semigroup). In recent years a particular interest has been devoted to the study of the trace of the heat semigroup and of the corresponding Schrödinger group $e^{-\frac{it}{\hbar}H}$, $t \in \mathbb{R}$, (related to the heat semigroup by analytic continuation in the "time variable" t) and their asymptotics in the "semiclassical limit $\hbar \downarrow 0$" (see, e.g., [40], [1, 3, 4] and also [15, 13, 14, 17] for related problems). In particular one is interested in the proof of a trace formula of Gutzwiller's type, relating the asymptotics of the trace of the Schrödinger group and the spectrum of the quantum mechanical energy operator H with the classical periodic orbits of the system. Gutzwiller's heuristic trace formula, which is a basis of the theory of quantum chaotic systems, is the quantum mechanical analogue of Selberg's trace formula, relating the spectrum of the Laplace-Beltrami operator on manifolds with constant negative

curvature with the periodic geodesics (see, e.g., [21] and [2, 3, 4].
In the case where the potential V is the sum of an harmonic oscillator part and a bounded perturbation V_0 that is the Fourier transform of a complex (bounded variation) measure on $\mathbb{R}^d$, some rigorous results on the asymptotics of the trace of the Schrödinger group and the heat semigroup have been obtained in [3, 4] by means of an infinite dimensional version of the stationary phase method for infinite dimensional oscillatory integrals (see [8] for a review of this topic).
The paper is organized as follows. In section 2 we give the definition and the main results on infinite dimensional oscillatory integrals of the form (2) with a polynomial phase function Φ, in section 3 we study the asymptotic expansion of the integral in the case where the origin is a non degenerate critical point of Φ, while in section 4 we study a degenerate case and apply these results to the asymptotics of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, as $\hbar \downarrow 0$.

2. Infinite dimensional oscillatory integrals

The present section is devoted to the study of the *oscillatory integrals with complex parameter s*. In the following we shall denote by $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ a real separable infinite dimensional Hilbert space, s will be a complex number such that $\mathrm{Re}(s) \geq 0$, $g : \mathcal{H} \to \mathbb{C}$ a Borel function.
Let us consider the generalization of the oscillatory integral (1) to the case (2) where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C}\ :\ \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int_{\mathbb{R}^n} e^{-\frac{s}{\epsilon}\Phi(x)} g(x)dx. \tag{4}$$

In the case where s is a pure imaginary number, by exploiting the oscillatory behavior of the integrand, the oscillatory integral (4) can still be defined as an improper Riemann integral even if the (continuous) function g is not summable. In the case where the phase function Φ is a quadratic form, the integral (4) is called *Fresnel integral*. We propose here for the general case (4) a modification of the Hörmander's definition [24], also considered in [18, 6] in connection to the generalization to the infinite dimensional case. This modification is as follows:

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function, $s \in \mathbb{C}^+$ a complex parameter. Let $\mathcal{S}$ be a subset of the space of the Schwartz test functions $S(\mathbb{R}^n)$. If for each $\phi \in \mathcal{S}$ such that $\phi(0) = 1$ the integrals*

$$I_\delta(f, \phi) := \int_{\mathbb{R}^n} (2\pi s^{-1})^{-n/2} e^{-\frac{s}{2}|x|^2} f(x)\phi(\delta x)dx$$

exist for all $\delta > 0$ *and* $\lim_{\delta\to 0} I_\delta(f,\phi)$ *exists and is independent of* ϕ, *then this limit is called the Fresnel integral of* f *with parameter* s *(with respect to the space* $\mathcal{S}$ *of regularizing functions) and denoted by*

$$\mathcal{F}^s(f) \equiv \widetilde{\int_{\mathbb{R}^n}^s} e^{-\frac{s}{2}|x|^2} f(x)dx. \tag{5}$$

By an adaptation of the definition of infinite dimensional oscillatory integrals given in [18] it is possible to define the *oscillatory integral with parameter* s on the Hilbert space $\mathcal{H}$, namely

$$I(s) = \widetilde{\int_{\mathcal{H}}^s} e^{-\frac{s}{2}\|x\|^2} g(x)dx \tag{6}$$

as the limit of a sequence of (suitably normalized) finite dimensional approximations [4].

Definition 2. *A Borel measurable function* $f : \mathcal{H} \to \mathbb{C}$ *is called* $\mathcal{F}^s$ *integrable if for each sequence* $\{P_n\}_{n\in\mathbb{N}}$ *of projectors onto n-dimensional subspaces of* $\mathcal{H}$, *such that* $P_n \le P_{n+1}$ *and* $P_n \to I$ *strongly as* $n \to \infty$ *(*I *being the identity operator in* $\mathcal{H}$*), the finite dimensional approximations of the Fresnel integral of* f, *with parameter* s,

$$\mathcal{F}^s_{P_n}(f) \equiv \widetilde{\int_{P_n\mathcal{H}}^s} e^{-\frac{s}{2}\|P_n x\|^2} f(P_n x) d(P_n x) \tag{7}$$

exists (in the sense of definition 1) and the limit $\lim_{n\to\infty} \mathcal{F}^s_{P_n}(g)$ *exists and is independent of the sequence* $\{P_n\}$.
In this case the limit is called the infinite dimensional Fresnel integral of f *with parameter* s *and is denoted by*

$$\widetilde{\int_{\mathcal{H}}^s} e^{-\frac{s}{2}\|x\|^2} f(x)dx.$$

f *is then said to be integrable (in the sense of Fresnel integrals with parameter* s*).*

The description of the largest class of functions which are integrable in this sense is an open problem, even in the finite dimensional case. Clearly it depends on the class $\mathcal{S}$ of the regularizations. The common choice is $\mathcal{S} \equiv S(\mathbb{R}^n)$, [18, 6]. In this case [18, 8, 6] the space of integrable functions includes (in finite as well as in infinite dimensions) the Fresnel class $\mathcal{F}(\mathcal{H})$, that is the set of functions $f : \mathcal{H} \to \mathbb{C}$ that are Fourier transforms of complex bounded variation measures on $\mathcal{H}$:

$$f(x) = \int_{\mathcal{H}} e^{i\langle y,x\rangle} d\mu_f(y) \equiv \hat{\mu}_f(x), \qquad x \in \mathcal{H}$$

$$\sup \sum_i |\mu_f(E_i)| < \infty,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$.
In fact for any $f \in \mathcal{F}(\mathcal{H})$ it is possible to prove a Parseval type equality that allows to compute the infinite dimensional oscillatory integral of f (with purely imaginary parameter s) in terms of an absolute convergent integral with respect to the associated measure μ_f [18, 6]. Indeed given a self-adjoint trace-class operator $B : \mathcal{H} \to \mathcal{H}$, such that $(I - B)$ is invertible, a function $f \in \mathcal{F}(\mathcal{H})$, $f = \hat{\mu}_f$ and a positive parameter $\hbar \in \mathbb{R}^+$, it is possible to prove that the function $e^{-\frac{i}{2\hbar}\langle x, Bx\rangle} f(x)$ is Fresnel integrable and the corresponding Fresnel integral with parameter $s = -i/\hbar$ is given by

$$\widetilde{\int_{\mathcal{H}}^{-i/\hbar}} e^{\frac{i}{2\hbar}\|x\|^2} e^{-\frac{i}{2\hbar}\langle x, Bx\rangle} e^{i\langle x, y\rangle} f(x) dx$$
$$= (\det(I - B))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha + y, (I-B)^{-1}(\alpha + y)\rangle} \mu_f(d\alpha) \quad (8)$$

where $\det(I - B) = |\det(I - B)| e^{-\pi i \, \mathrm{Ind}\,(I-B)}$ is the Fredholm determinant of the operator $(I - B)$, $|\det(I - B)|$ its absolute value and $\mathrm{Ind}((I - B))$ is the number of negative eigenvalues of the operator $(I - B)$, counted with their multiplicity.

Let us also recall, for later use, a known result on infinite dimensional oscillatory integrals.

Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. Let also $\|\cdot\|$ be an equivalent norm on $\mathcal{H}$ with scalar product denoted by $\langle \cdot, \cdot \rangle$. Let us denote the new Hilbert space by $\tilde{\mathcal{H}}$. Let us assume moreover that

$$\langle x_1, x_2\rangle = (x_1, x_2) + (x_1, T x_2), \qquad x_1, x_2 \in \tilde{\mathcal{H}}$$
$$\|x\|^2 = |x|^2 + (x, Tx), \qquad x \in \tilde{\mathcal{H}}$$

where T is a self-adjoint trace class operator on $\mathcal{H}$. The following holds (see [4, 5]):

Theorem 1. *Let $f : \mathcal{H} \to \mathbb{C}$ be a Borel function. f is integrable on $\mathcal{H}$ (in the sense of definition 2) if and only if f is integrable on $\tilde{\mathcal{H}}$ and in this case*

$$\widetilde{\int_{\tilde{\mathcal{H}}}} e^{-\frac{s}{2}|x|^2} f(x) dx = \det(I + T)^{1/2} \widetilde{\int_{\mathcal{H}}} e^{-\frac{s}{2}|x|^2} f(x) dx \quad (9)$$

Recently the class of "Fresnel integrable function" in the sense of definition 2 has been further enlarged. In particular in [9] the Parseval type equality (8) has been generalized to the case where $\mathcal{H}$ is finite dimensional but the phase function is an even degree (not necessarily second order) polynomial, while in [10] a corresponding result has been proved for infinite dimensional Hilbert spaces and phase functions which are the sum of a quadratic and a quartic term.
Let us also remark that definition 2 can be seen as an extension of a line of development relating infinite dimensional integrals of probabilistic and oscillatory type, going back to Cameron, see, e.g., [16], [32] and corresponding references under "analytic approach" in [1, 8].

In the following we shall extend these results to infinite dimensional Hilbert spaces and suitable polynomial phase functions of higher degrees. The main idea is a generalization of Parseval-type equality, obtained by modifying the definition 1 by restricting the class of regularizing functions to a class $\mathcal{S}$ of analytic functions.
Let $\alpha \in \mathbb{R}$, in the following I_α will denote the open interval $(0, \alpha)$ if $\alpha > 0$ and $(\alpha, 0)$ if $\alpha < 0$; D_α will denote the sector of the complex $z-$ plane

$$D_\alpha := \{z = |z|e^{i\varphi} \in \mathbb{C} \ : \ |z| > 0, \, \varphi \in I_\alpha\},$$

and $\mathcal{S}_\alpha(\mathbb{R}^n)$ will denote the space of functions $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the following assumptions:

(1) for any $x \in \mathbb{R}^n$ the function

$$z \mapsto \phi(zx), \qquad z \in \mathbb{R}, \quad x \in \mathbb{R}^n$$

can be extended to an analytic function in D_α and continuous in the closure $\bar{D}_\alpha$ of D_α.

(2) for any $z \in \bar{D}_\alpha$ the map

$$x \mapsto \phi^z(x) := \phi(zx), \qquad z \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

is bounded

Clearly $\mathcal{S}_\beta(\mathbb{R}^n) \subset \mathcal{S}_\alpha(\mathbb{R}^n)$ if $\alpha < \beta$. As an example the function $x \in \mathbb{R}^n \mapsto e^{-\|x\|^2}$ is an element of $\mathcal{S}_{\pi/4}(\mathbb{R}^n)$.

Given a real separable Hilbert space of $\mathcal{H}$, with inner product $\langle \ , \ \rangle$ and norm $\| \ \|$, let us consider the *abstract Wiener space* $(\mathcal{H}, \mathcal{B})$ built on $\mathcal{H}$, where $(\mathcal{B}, | \ |)$ is the Banach space completion of $\mathcal{H}$ with respect to the measurable norm $| \ |$ and let μ be the standard Gaussian measure on $\mathcal{B}$ associate with $\mathcal{H}$ (see [22, 29] and the Appendix of the present paper). $\mathcal{H}$ is sometimes called the reproducing kernel Hilbert space of

$\mathcal{B}$. Let us denote by c the norm of the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$.

Theorem 2. *Let $s, r \in \mathbb{C}$, $s = |s|e^{i\alpha}$ and $r = |r|e^{i\beta}$, with $\alpha, \beta \in [-\pi/2, \pi/2]$. Let us assume that for any φ belonging to the closure $\bar{I}_{-\alpha/2}$ of $I_{-\alpha/2}$, the angle $\beta + 2N\varphi$ is included in the interval $[-\pi/2, \pi/2]$. Let $B : \mathcal{H} \to \mathcal{H}$ be a trace class symmetric operator such that $(I - B)$ is strictly positive. Let $V_{2N} : \mathcal{H} \to \mathbb{R}$ be a positive, continuous in the $|\ |$-norm and homogeneous function of order $2N$, i.e. $V_{2N}(\lambda x) = \lambda^{2N} V_{2N}(x)$, for any $\lambda \in \mathbb{R}$, $x \in \mathcal{H}$. Let $g : \mathcal{H} \to \mathbb{C}$ satisfy the following assumptions:*

- *for any $x \in \mathcal{H}$ the map*
$$z \mapsto g(zx), \qquad z \in \mathbb{R}, \quad x \in \mathcal{H}$$
can be extended to an analytic function on $D_{-\alpha/2}$ and continuous in $\bar{D}_{-\alpha/2}$.
- *$\exists K_1 > 0$, $\exists K_2 \in (0, 1/c^2)$, $\forall x \in \mathcal{H}$*
$$|g(zx)| \leq K_1 |e^{\frac{sz^2}{2}(K_2|x|^2 - \langle x, Bx \rangle)}|, \qquad \forall z \in \bar{D}_{-\alpha/2} \tag{10}$$
- *the function $x \mapsto g^\alpha(x) \equiv g(e^{-i\alpha/2}x)$, $x \in \mathcal{H}$, is continuous in the $|\cdot|$-norm.*

Then the infinite dimensional oscillatory integral with parameter s and regularizing class $\mathcal{S}_{-\alpha/2}$ of the function $f : \mathcal{H} \to \mathbb{C}$
$$f(x) = e^{\frac{s}{2}\langle x, Bx \rangle - rV_{2N}(x)} g(x), \qquad x \in \mathcal{H}, \tag{11}$$
is well defined and it is given by
$$\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{s}{2}\langle x, (I-B)x \rangle - rV_{2N}(x)} g(x) dx = \int_{\mathcal{B}} e^{\frac{1}{2}\langle \omega, B\omega \rangle - rs^{-N}\tilde{V}_{2N}(\omega)} \tilde{g}^\alpha(|s|^{-1/2}\omega) d\mu(\omega), \tag{12}$$
$\tilde{V}_{2N}$ resp. $\tilde{g}^\alpha$ being the stochastic extensions of V_{2N} resp. g^α to $\mathcal{B}$.

Proof: The right hand side of (12) is well defined, indeed under the assumption of $|\ |$-norm continuity, the functions V_{2N} and g^α can be extended by continuity to random variables $\bar{V}_{2N}$ and $\bar{g}^\alpha$ on $\mathcal{B}$, which coincide with the stochastic extensions of $\tilde{V}_{2N}$ and $\tilde{g}^\alpha$ of V_{2N} and g^α μ-a.e. Moreover for any $\lambda \in \mathbb{C}^+$ and for any increasing sequence of $n-$dimensional projectors P_n in $\mathcal{H}$, the family of bounded random variables $e^{-\lambda V_{2N} \circ \tilde{P}_n(\cdot)} \equiv e^{-\lambda V_{2N}^n(\cdot)}$ ($\tilde{P}_n$ being the stochastic extension of P_n to $\mathcal{B}$)converges $\mu-$a.e. to $e^{-\lambda \bar{V}_{2N}(\cdot)}$.

As B is symmetric trace class, the quadratic form on $\mathcal{H} \times \mathcal{H}$:
$$x \in \mathcal{H} \mapsto \langle x, Bx \rangle$$

can be extended to a random variable on $\mathcal{B}$, denoted again by $\langle \cdot, B \cdot \rangle$. Moreover the random variable $e^{\frac{1}{2}\langle \cdot, B \cdot \rangle}$ is in $L^1(\mu)$ (see appendix). The bound (10) for $z = s^{-1/2}$ extends by continuity on $\tilde{g}^\alpha : \mathcal{B} \to \mathbb{C}$ and by Fernique's theorem the integral on the right hand side of (12) is convergent.
Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of finite dimensional projection operators on $\mathcal{H}$ converging strongly to the identity as $n \to \infty$. Let $\phi \in \mathcal{S}_{-\alpha/2}(\mathbb{R}^n)$ be a regularizing function. For any $\delta > 0$ let us consider the regularized finite dimensional approximations

$$(2\pi s^{-1})^{-n/2} \int_{P_n\mathcal{H}} e^{-\frac{s}{2}\langle P_n x,(I-B)P_n x\rangle - rV_{2N}(P_n x)} g(P_n x)\phi(\delta P_n x) d(P_n x). \tag{13}$$

For any $z \in \mathbb{R}^+$ the integral (13) is equal to

$$\left(\frac{z^2 s}{2\pi}\right)^{n/2} \int_{P_n\mathcal{H}} e^{-\frac{sz^2}{2}\langle P_n x,(I-B)P_n x\rangle - rz^{2N}V_{2N}(P_n x)} g(zP_n x)\phi(z\delta P_n x) d(P_n x). \tag{14}$$

By the assumptions on the functions ϕ, g, as well as on the parameters s and r, by Fubini and Morera theorems the integral (14) is an analytic function of the variable z in the sector $D_{-\alpha/2}$ and continuous on $\bar{D}_{-\alpha/2}$ and coincides with the value of the integral (13) on $\mathbb{R}^+$. By a straightforward application of the reflection principle [30] it is a constant function on the whole closed sector $\bar{D}_{-\alpha/2}$. In particular for $z = s^{-1/2} := |s|^{-1/2}e^{-i\alpha/2}$, we conclude that

$$(2\pi s^{-1})^{-n/2} \int_{P_n\mathcal{H}} e^{-\frac{s}{2}\langle P_n x,(I-B)P_n x\rangle - rV_{2N}(P_n x)} g(P_n x)\phi(\delta P_n x) d(P_n x)$$
$$= (2\pi)^{-n/2} \int_{P_n\mathcal{H}} e^{-\frac{1}{2}\langle P_n x,(I-B)P_n x\rangle - rs^{-N}V_{2N}(P_n x)} g(s^{-1/2}P_n x)\phi(s^{-1/2}\delta P_n x) d(P_n x)$$

By letting $\delta \downarrow 0$ and by the dominated convergence theorem the latter is equal to

$$\int_{P_n\mathcal{H}} e^{\frac{1}{2}\langle P_n x, BP_n x\rangle - rs^{-N}V_{2N}(P_n x)} g(s^{-1/2}P_n x) \frac{e^{-\frac{1}{2}\|P_n x\|^2}}{(2\pi)^{n/2}} d(P_n x)$$
$$= \int_{\mathcal{B}} e^{\frac{1}{2}\langle P_n x, BP_n x\rangle - rs^{-N}\tilde{V}_{2N}(P_n x)} \tilde{g}^\alpha(|s|^{-1/2}P_n x) d\mu(x)$$

By letting $n \to \infty$ and by the dominated convergence theorem the latter converges to the right hand side of (12) □

Remark 1. *Theorem 2 generalizes the results obtained in* [10] *concerning the oscillatory integrals of the form*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle x,x\rangle} e^{i\lambda V_4(x)} g(x) dx. \tag{15}$$

Indeed the Parseval type equality (12) *allows one to compute explicitly infinite dimensional oscillatory integrals with polynomial phase of higher degree, provided that the parameter s has a non vanishing real part. For instance one can compute infinite dimensional oscillatory integrals of the form*

$$\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{|s|e^{i\alpha}}{2}\langle x,x\rangle} e^{i\lambda V_{2N}(x)} g(x) dx$$

with $\lambda \in \mathbb{R}^+$ *and* $\alpha \in [-\pi/N, 0]$.

Remark 2. *In the case* $s \in \mathbb{R}^+$*, theorem 2 relates a Gaussian integral on the Banach space* B *with an integral on its reproducing kernel Hilbert space* $\mathcal{H}$.

If the operator $(I - B) : \mathcal{H} \to \mathcal{H}$ is not strictly positive, formula (12) does not hold. In the following we shall generalize the results of theorem 2 to the case where $(I - B)$ has non positive eigenvalues, by restricting the class of polynomial phase functions V_{2N}.
Given a trace class symmetric operator $B : \mathcal{H} \to \mathcal{H}$, then the number of non positive eigenvalues of $(I - B)$ (counted with their multiplicity) is finite. We shall denote by $\mathcal{H}_0$ the kernel of $I - B$, by $\mathcal{H}_-$ the subspace of $\mathcal{H}$ where $I - B$ is negative definite, and by $\mathcal{H}_+$ the subspace of $\mathcal{H}$ where $I - B$ is positive definite. We have $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$. Let us introduce the notation $\mathcal{H}_1 \equiv \mathcal{H}_- \oplus \mathcal{H}_0$, $\mathcal{H}_2 \equiv \mathcal{H}_+$ and $x \in \mathcal{H} = x_1 + x_2$, with $x_i \in \mathcal{H}_i$, $i = 1, 2$. Clearly $\dim(\mathcal{H}_1) < +\infty$ and this fact will be used in the following. Let us denote by $(\mathcal{H}_2, \mathcal{B}_2)$ the abstract Wiener space associated with $\mathcal{H}_2$ and by μ_2 the Gaussian measure on $\mathcal{B}_2$ associated with $\mathcal{H}_2$.

Theorem 3. *Let* $s, r \in \mathbb{C}$*,* $s = |s|e^{i\alpha}$ *and* $r = |r|e^{i\beta}$*, with* $\alpha, \beta \in [-\pi/2, \pi/2]$*. Let us assume that for any* $\varphi \in \bar{I}_{-\alpha/2}$ *, the angle* $\beta + 2N\varphi$ *is included in the interval* $(-\pi/2, \pi/2)$.
Let $B : \mathcal{H} \to \mathcal{H}$ *and* $V_{2N} : \mathcal{H} \to \mathbb{R}$ *satisfy the assumptions of theorem 2. Let us assume moreover that there exists a constant* K_3 *such that for any* $x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$ *one has* $V_{2N}(x_1 + x_2) - V_{2N}(x_1) \geq K_3$*. Let* $g : \mathcal{H} \to \mathbb{C}$ *satisfy the following assumptions:*

- *for any $x \in \mathcal{H}$ the map*

$$z \mapsto g(zx), \qquad z \in \mathbb{R}, \quad x \in \mathcal{H}$$

can be extended to a function which is analytic in $D_{-\alpha/2}$ and continuous in $\bar{D}_{-\alpha/2}$.

- $\exists K_4, K_5, \delta > 0$, $\exists K_6 \in (0, 1/c^2)$, $\forall x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$, $\forall z \in \bar{D}_{-\alpha/2}$*:*

$$|g(z(x_1 + x_2))| \leq K_4 |e^{K_5|zx_1|^{2N-\delta} + \frac{sz^2}{2}(K_6|x_2|^2_{\mathcal{B}_2} - \langle x_2, Bx_2\rangle)}|, \qquad (16)$$

- *the function $x \mapsto g^\alpha(x) \equiv g(e^{-i\alpha/2}x)$, $x \in \mathcal{H}$, is continuous in the $|\;|$-norm.*

Then the infinite dimensional oscillatory integral with parameter s and regularizing class $\mathcal{S}_{-\alpha/2}$ of the function (11) *is well defined and it is given by*

$$\widetilde{\int_{\mathcal{H}}^{s}} e^{-\frac{s}{2}\langle x,(I-B)x\rangle - rV_{2N}(x)} g(x)dx = (2\pi)^{-dim(\mathcal{H}_1)/2} \int_{\mathcal{H}_1 \times \mathcal{B}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle}$$
$$e^{\frac{1}{2}\langle \omega_2, B\omega_2\rangle - rs^{-N}\tilde{V}_{2N}(x_1+\omega_2)} \tilde{g}^\alpha(|s|^{-1/2}(x_1 + \omega_2)) d\mu(\omega_2) \times dx_1 \quad (17)$$

Proof: The proof is completely analogous to the proof of theorem 2. Let us consider a sequence $\{P_n\}_{n\in\mathbb{N}}$ of finite dimensional projection operators on $\mathcal{H}_2$ converging strongly to the identity as $n \to \infty$. Because of the conditions on the parameters $s, r \in \mathbb{C}$, the regularized finite dimensional approximations of the integral

$$(2\pi s^{-1})^{-(n+dim(\mathcal{H}_1))/2} \int_{\mathcal{H}_1 \times P_n\mathcal{H}_2} e^{-\frac{s}{2}\langle x_1,(I-B)x_1\rangle} e^{-\frac{s}{2}\langle P_n x_2,(I-B)P_n x_2\rangle - rV_{2N}(P_n x_2 + x_1)}$$
$$g(P_n x_2 + x_1)\phi(\delta(P_n x_2 + x_1)) dx_1 \times d(P_n x_2)$$

are equal to

$$(2\pi)^{-dim(\mathcal{H}_1)/2} \int_{\mathcal{H}_1 \times P_n\mathcal{H}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} e^{\frac{1}{2}\langle P_n x_2, BP_n x_2\rangle - rs^{-N}V_{2N}(P_n x_2 + x_1)}$$
$$g(s^{-1/2}(P_n x_2 + x_1)) \frac{e^{-\frac{1}{2}||P_n x_2||^2}}{(2\pi)^{n/2}} dx_1 \times d(P_n x_2)$$
$$= (2\pi)^{-dim(\mathcal{H}_1)/2} \int_{\mathcal{H}_1 \times \mathcal{B}_2} e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle} e^{\frac{1}{2}\langle P_n x_2, BP_n x_2\rangle - rs^{-N}\tilde{V}_{2N}(P_n x_2 + x_1)}$$
$$\tilde{g}^\alpha(|s|^{-1/2}(P_n x_2 + x_1)) dx_1 \times d\mu(\omega_2). \quad (18)$$

As by hypothesis

$$|e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle}e^{\frac{1}{2}\langle P_nx_2,BP_nx_2\rangle-rs^{-N}\tilde{V}_{2N}(P_nx_2+x_1)}\tilde{g}^\alpha(|s|^{-1/2}(P_nx_2+x_1))|$$
$$\leq K_4e^{K_5|s^{-1/2}x_1|^{2N-\delta}}e^{-\frac{1}{2}\langle x_1,(I-B)x_1\rangle-|r||s|^{-N}\cos(\beta-N\alpha)V_{2N}(x_1)}$$
$$e^{-|r||s|^{-N}\cos(\beta-N\alpha)K_3}e^{\frac{K_6}{2}|P_nx_2|^2_{\mathcal{B}_2}}$$

the dominated convergence theorem can be applied and by letting $n\to\infty$ the integral (18) converges to the right hand side of (17). □

Remark 3. *In theorem 3 the convergence of the integral on the subspace $\mathcal{H}_1$ is due to the fast decreasing behavior of the function $e^{-rs^{-N}V_{2N}}$ instead of $e^{-\frac{1}{2}\langle\cdot,(I-B)\cdot\rangle}$, as the latter has an exponential growth on $\mathcal{H}_1$. For this reason the assumptions of theorem 3 include the condition that for any $\varphi\in\bar{I}_{-\alpha/2}$, the angle $\beta+2N\varphi$ is included in the open interval $(-\pi/2,\pi/2)$, instead of the closed one (as in theorem 2). On the other hand this restriction allows us to admit a stronger growth of the function g on the subspace $\mathcal{H}_1$ and to replace condition* (10) *of theorem 2 by condition* (16).

3. The asymptotic expansion

In the following we shall put $s:=\frac{s'}{\epsilon}$, $s'=|s'|e^{i\alpha}$, $r:=\frac{r'}{\epsilon}$, with $\epsilon\in\mathbb{R}^+$ and s',r' satisfying the assumptions of theorem 2 and we shall study the asymptotic behavior of the integral

$$I(\epsilon):=\widetilde{\int_{\mathcal{H}}^{s}}e^{-\frac{s'}{2\epsilon}\langle x,(I-B)x\rangle-\frac{r'}{\epsilon}V_{2N}(x)}g(x)dx \tag{19}$$

in the limit $\epsilon\downarrow 0$. Let us assume that the operator $B:\mathcal{H}\to\mathcal{H}$ be of trace class, symmetric and such that $I-B>0$ and the functions V_{2N},g satisfy the assumptions of the theorem 2. Let us denote by $g_{s'}:\mathcal{B}\to\mathbb{C}$ the function given by $g_{s'}(\omega):=\tilde{g}^\alpha(|s'|^{-1/2}\omega)$ ($\tilde{g}^\alpha$ being the stochastic extension of $x\mapsto g(e^{-i\alpha/2}x)$, $x\in\mathcal{H}$). Assume that g_s satisfies the following hypothesis:

(1) $\forall\omega\in\mathcal{B}$, the function $\lambda\mapsto g_{s'}(\lambda\omega)$ is $2m-$times continuously differentiable in $\lambda\in\mathbb{R}$.
(2) $\forall k=1,\dots,2m$, $\exists$ a polynomial Q_k in the variables $|\bar{\lambda}|$ and $\omega|$ such that $\forall\omega\in\mathcal{B}$, $\forall\bar{\lambda}\in\mathbb{R}$

$$\left|\frac{d^k}{d\lambda^k}g_{s'}(\lambda\omega)_{|\lambda=\bar{\lambda}}\right|\leq Q_k(|\bar{\lambda}|,|\omega|)$$

For notational simplicity in the following we shall adopt the short writing

$$g^{(k)}(\bar{\lambda},\omega) := \frac{d^k}{d\lambda^k} g_{s'}(\lambda\omega)_{|\lambda=\bar{\lambda}}.$$

The following holds:

Theorem 4. *Under the assumptions above the integral $I(\epsilon)$ admits the following asymptotic expansion*

$$I(\epsilon) = \sum_{n=0}^{m-1} \epsilon^n C_n + O(\epsilon^m) \tag{20}$$

and the leading term is $C_0 = \det(I-B)^{-1/2}\tilde{g}(0)$.

Proof: By equation (12) the integral $I(\epsilon)$ is equal to

$$\int_{\mathcal{B}} e^{\frac{1}{2}\langle\omega,B\omega\rangle - r's'^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega)} g_{s'}(\sqrt{\epsilon}\omega)d\mu(\omega) \tag{21}$$

For any $\omega \in \mathcal{B}$, let us consider the function $f : \mathbb{R}^+ \to \mathbb{C}$, given by

$$f(\epsilon) := e^{-r's'^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega)} g_{s'}(\sqrt{\epsilon}\omega), \qquad \epsilon \in \mathbb{R}^+$$

By expanding $f(\epsilon)$ in power series of $\sqrt{\epsilon}$ we get

$$f(\epsilon) = \sum_{n}^{2m-1} c_n \sqrt{\epsilon}^n + R_{2m}(\sqrt{\epsilon})$$

where

$$c_n = \sum_{k,l\,:\,k+(2N-2)l=n} \frac{1}{l!k!} g^{(k)}(0,\omega)(-r's'^{-N})^l \tilde{V}_{2N}(\omega)^l$$

and $R_{2m} = \frac{\epsilon^m}{(2m-1)!}\int_0^1 f^{(2m)}(t\sqrt{\epsilon})(1-t)^{2m-1}dt$, with

$$f^{(2m)}(\lambda) = \sum_{k=0}^{2m} \frac{2m!}{k!(2m-k)!} g^{(k)}(\lambda,\omega) P_{2m-k}(\lambda,\omega) e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)},$$

and $P_k(\bar{\lambda},\omega)$ are polynomials (in λ and $V(\omega)$) defined by $\frac{d^k}{d\lambda^k}_{|\lambda=\bar{\lambda}} e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)} = P_k(\bar{\lambda},\omega)e^{-r's'^{-N}\lambda^{2N-2}\tilde{V}_{2N}(\omega)}$. By substituting into (21) we get

$$I(\epsilon) = \sum_{n=0}^{m-1} C_n \epsilon^n + \mathcal{R}_m(\epsilon)$$

$$C_n = \sum_{k,l\,:\,k+(2N-2)l=2n} \frac{(-r')^l s'^{-Nl}}{l!k!} \int_{\mathcal{B}} e^{\frac{1}{2}\langle\omega,B\omega\rangle} g^{(k)}(0,\omega)\tilde{V}_{2N}(\omega)^l d\mu(\omega) \tag{22}$$

and

$$\mathcal{R}_m(\epsilon) = \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle} f^{(2m)}(t\sqrt{\epsilon})(1-t)^{2m-1}dt\, d\mu(\omega).$$

By the assumptions on the function g, the integrals in the formula (22) are well defined, as well as the remainder that satisfies the following estimate

$$\begin{aligned}|\mathcal{R}_m(\epsilon)| &\leq \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}|f^{(2m)}(t\sqrt{\epsilon})|(1-t)^{2m-1}dt\, d\mu(\omega)\\ &= \frac{\epsilon^m}{(2m-1)!}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\sum_{k=0}^{2m}\frac{2m!}{k!(2m-k)!}|g^{(k)}(t\sqrt{\epsilon},\omega)|\\ &\quad |P_{2m-k}(t\sqrt{\epsilon},\omega)|e^{-r's'^{-N}t\epsilon^{N-1}\tilde{V}_{2N}(\omega)}(1-t)^{2m-1}dt\, d\mu(\omega)\\ &\leq \epsilon^m\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\mathcal{P}_m(t\sqrt{\epsilon},|\omega|)(1-t)^{2m-1}dt\, d\mu(\omega) \qquad (23)\end{aligned}$$

where $\mathcal{P}_m(\lambda,|\omega|)$ denotes a polynomial in the variables λ, $|\omega|$ and

$$\lim_{\epsilon\downarrow 0}\int_{\mathcal{B}}\int_0^1 e^{\frac{1}{2}\langle\omega,B\omega\rangle}\mathcal{P}_m(t\sqrt{\epsilon},|\omega|)(1-t)^{2m-1}dt\, d\mu(\omega) < \infty.$$

The leading term is given by

$$C_0 = \tilde{g}(0)\int_{\mathcal{B}} e^{\frac{1}{2}\langle\omega,B\omega\rangle}d\mu(\omega) = \tilde{g}(0)\det(I-B)^{-1/2},$$

with $\det(I-B)$ being the Fredholm determinant of the operator $I-B$ (see Appendix). □

Remark 4. *Theorem 4 allows one to handle the asymptotic behavior of infinite dimensional integrals with a complex phase function Φ of the form*

$$\Phi(x) := -\frac{s'}{2}\langle x,(I-B)x\rangle - r'V_{2N}(x), \qquad x\in\mathcal{B}.$$

It generalizes both the Laplace method (for the study of the asymptotics of integrals with real phase functions) and the stationary phase method (for the study of the asymptotics of integrals with purely imaginary phase functions). According to theorem 4, the only critical point contributing to the asymptotic behavior is the origin $x=0$. Indeed one can easily verify that the only real stationary point of the phase functional is $x=0$ and formula (20) is the asymptotic expansion around this critical point.

If the operator $(I-B):\mathcal{H}\to\mathcal{H}$ is not strictly positive, the results of theorem 4 are no longer valid. For instance, in the case where $(I-B)$ has a non trivial kernel, the phase function $\Phi:\mathcal{H}\to\mathbb{C}$,

$$\Phi(x):=-\frac{s'}{2}\langle x,(I-B)x\rangle-r'V_{2N}(x)$$

has a degenerate critical point in $x=0$, i.e. $\Phi'(0)=0$ and $Ker\Phi''(0)\neq\{0\}$. In the case where the negative eigenspace of the operator $I-B$ is not empty, the phase function Φ could have critical points $x_c\in\mathcal{H}$ different from 0 and the asymptotic behavior of the integral should be determined by these critical points. Let us consider for instance a factorisable integral of the following form:

$$I(\epsilon):=\widetilde{\int_{\mathcal{H}_1\times\mathcal{H}_2}^{s}}e^{-\frac{s'}{2\epsilon}\langle x_1,(I-B)x_1\rangle-\frac{s'}{2\epsilon}\langle x_2,(I-B)x_2\rangle-\frac{r'}{\epsilon}V_{2N}(x_1)-\frac{r'}{\epsilon}V_{2N}(x_2)}dx_1dx_2 \tag{24}$$

where $\dim\mathcal{H}_1=1$. By theorem 3 $I(\epsilon)=I_1(\epsilon)I_2(\epsilon)$, with $I_2(\epsilon)=\int_{\mathcal{B}_2}e^{\frac{1}{2}\langle\omega_2,B\omega_2\rangle}e^{-r'(s')^{-N}\epsilon^{N-1}\tilde{V}_{2N}(\omega_2)}d\mu_2(\omega_2)$ satisfies the assumptions of theorem 4, and I_1 is of the form $I_1(\epsilon)=\int_{\mathbb{R}}e^{\frac{a^2}{2\epsilon}y^2-\frac{\lambda}{\epsilon}y^{2N}}dy$, with $a\geq 0$ and $\lambda\in\mathbb{C}^+$. In particular if $a=0,\lambda=1$, then $I_1(\epsilon)=\epsilon^{1/2N}\frac{\Gamma(1/2N)}{N}$, while if $a=1,\lambda=1/2N$, then $I_1(\epsilon)\sim e^{\frac{N-1}{2N\epsilon}}$ (where $\sim$ means that the quotient of both sides tends to 1 as $\epsilon\downarrow 0$). In the non factorisable case the situation is more involved. Indeed in principle one should apply an infinite dimensional version of the saddle point method and analyze the behavior of the integral around *non real* stationary points. Actually a detailed treatment of the saddle point method in the case the dimension of the space on which the integral is performed is greater than 1 is still lacking (see however [28]). In the following we give an example of the study of the asymptotics of the integral in a degenerate (non factorisable case) and apply this result to the study of the trace of the heat semigroup with a polynomial potential.

4. A DEGENERATE CASE

Let $(\mathcal{H}_{p,t},\langle\ ,\ \rangle,\|\|)$ be the Hilbert space

$$\mathcal{H}_{p,t}:=\{\gamma\in H^1([0,t];\mathbb{R}^d):\ \gamma(0)=\gamma(t)\}$$

with inner product

$$\langle\gamma_1,\gamma_2\rangle=\int_0^t\dot{\gamma}_1(\tau)\dot{\gamma}_2(\tau)d\tau+\int_0^t\gamma_1(\tau)\gamma_2(\tau)d\tau.$$

The present section is devoted to the study of the asymptotic behavior as $\epsilon\downarrow 0$ of an infinite dimensional Fresnel integral (with parameter s/ϵ)

of the form

$$I(\epsilon) := \widetilde{\int_{\mathcal{H}_{p,t}}^{s/\epsilon}} e^{-\frac{s}{2\epsilon}\int_0^t \dot\gamma(\tau)^2 d\tau - \frac{r}{\epsilon}\int_0^t |\gamma(\tau)|^{2N} d\tau} d\gamma \tag{25}$$

with $N \in \mathbb{N}$, $N \geq 2$, and $s, r \in \mathbb{C}^+$ satisfy the assumptions of theorem 3.

Heuristically the integral (25) can be written as "$\widetilde{\int}_{\mathcal{H}_{p,t}}^{1/\epsilon} e^{\frac{1}{\epsilon}\Phi(\gamma)} d\gamma$", where the phase function $\Phi : \mathcal{H}_{p,t} \to \mathbb{R}$ is given by

$$\Phi(\gamma) = -\frac{s}{2}\int_0^t \dot\gamma(\tau)^2 d\tau - r\int_0^t |\gamma(\tau)|^{2N} d\tau \tag{26}$$

and the asymptotic behavior of $I(\epsilon)$ should be determined by the stationary points of the phase functional Φ, i.e. the points such that

$$\Phi'(\gamma)(\phi) = 0, \qquad \forall \phi \in \mathcal{H}_{p,t},$$

Φ' being the Fréchet derivative. One can easy verify that the null path $\gamma = 0$ is a stationary point of Φ and it is degenerate, namely Ker $(\Phi''(0))$ is not trivial. Indeed

$$\langle \Phi''(0)(\phi), \psi\rangle = -s\int_0^t \dot\phi(\tau)\dot\psi(\tau) d\tau := -s\langle \phi, (I+L)\psi\rangle \tag{27}$$

where L is the unique self-adjoint operator on $\mathcal{H}_{p,t}$ defined by the quadratic form

$$\langle \phi, L\psi\rangle = -\int_0^t \phi(\tau)\psi(\tau) d\tau.$$

We easily see that L for any $\psi \in \mathcal{H}_{p,t}$ is given by:

$$L\psi(\tau) = \int_0^\tau \sinh(\tau-u)\psi(u)du - \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(\tau-u)\psi(u)du + \\ + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t \sinh(t+\tau-u)\psi(u)du, \tag{28}$$

The kernel of $I + L$ is given by the solution of the equation

$$\psi(\tau) + \frac{1}{(1-e^t)(1-e^{-t})}\int_0^t (\sinh(t+\tau-u) - \sinh(\tau-u))\psi(u)du + \\ + \int_0^\tau \sinh(\tau-u)\psi(u)du = 0 \tag{29}$$

with the periodic condition $\psi(0) = \psi(t)$. By differentiating (29) twice, it is easy to see that if ψ satisfies (29) then

$$\ddot\psi(\tau) = 0, \qquad \forall \tau \in [0, t],$$

so that the only solutions of (29) satisfying the periodic condition $\psi(0) = \psi(t)$ are the constant paths. From (27) the kernel of $\Phi''(0)$ is the $d-$ dimensional subspace:

$$Ker[\Phi''(0)] = \{\gamma \in \mathcal{H}_{p,t} : \ \gamma(\tau) = x \ \forall \tau \in [0,t], \ x \in \mathbb{R}^d\}.$$

Let us decompose the Hilbert space $\mathcal{H}_{p,t}$ into the direct sum $\mathcal{H}_{p,t} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = Ker[\Phi''(0)]$ and $\mathcal{H}_2 = Ker[\Phi''(0)]^\perp$, $\gamma(\tau) = \gamma_1(\tau)+\gamma_2(\tau)$, $\gamma_1(\tau) = t^{-1}\int_0^t \gamma(u)du$, $\gamma_2(\tau) = \gamma(\tau)-\gamma_1(\tau)$. In particular

$$\mathcal{H}_2 = \{\gamma \in \mathcal{H}_{p,t} \ : \ \int_0^t \gamma(\tau)d\tau = 0\}.$$

As one can easily verify that for any $\gamma_2 \in \mathcal{H}_2, \gamma_1 \in \mathcal{H}_1$ one has

$$V_{2N}(\gamma_1+\gamma_2) - V_{2N}(\gamma_1) = \int_0^t |\gamma_1(\tau)+\gamma_2(\tau)|^{2N}d\tau - \int_0^t |\gamma_1(\tau)|^{2N}d\tau \geq 0,$$

the assumptions of theorem 3 are satisfied and

$$I(\epsilon) = (2\pi)^{-d/2}\int_{\mathcal{B}_2\times\mathcal{H}_1} e^{-\frac{1}{2}\langle\omega_2,L\omega_2\rangle-\lambda\epsilon^{N-1}\int_0^t|\gamma_1(\tau)+\omega_2(\tau)|^{2N}d\tau}d\mu_2(\omega_2)\times d\gamma_1$$
$$(2\pi)^{-d/2}\int_{\mathcal{B}_2\times\mathbb{R}^d} e^{-\frac{1}{2}\langle\omega_2,L\omega_2\rangle-\lambda\epsilon^{N-1}\int_0^t|\frac{y}{t}+\omega_2(\tau)|^{2N}d\tau}d\mu_2(\omega_2)\times dy, \quad (30)$$

where $\lambda = rs^{-N}$ and $(\mathcal{H}_2, \mathcal{B}_2)$ is the abstract Wiener space built on $\mathcal{H}_2$. By putting $x := \sqrt{\epsilon}y/t$ and expanding the term $|\sqrt{\epsilon}\omega_2(\tau) + x|^{2N}$ we have

$$I(\epsilon) = \Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{t\lambda}{\epsilon}|x|^{2N}}f(x,\epsilon)dx,$$

where

$$f(x,\epsilon) = \int_{\mathcal{B}_2} e^{-(\frac{1}{2}\langle\omega_2,L\omega_2\rangle+\frac{\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\omega_2(\tau)+x|^{2N}d\tau-\frac{\lambda t}{\epsilon}|x|^{2N})}d\mu(\omega_2)$$
$$= \widetilde{\int_{\mathcal{H}_2}} e^{-(\frac{1}{2}\int_0^t\dot\gamma_2(\tau)^2d\tau+\frac{\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\gamma_2(\tau)+x|^{2N}d\tau-\frac{\lambda t}{\epsilon}|x|^{2N})}d\gamma_2.$$

The asymptotic behavior of $f(x,\epsilon)$ as $\epsilon \downarrow 0$ can be simply determined by expanding the integrand in powers of ϵ. Indeed

$$f(x,\epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-\frac{\lambda}{\epsilon}P_{2N}(x,\sqrt{\epsilon}\gamma_2)}d\gamma_2,$$

where $L_x : \mathcal{H}_2 \to \mathcal{H}_2$ is the unique bounded self adjoint operator determined by the quadratic form

$$\langle \phi, (I + L_x)\psi \rangle = \int_0^t \dot{\phi}(\tau)\dot{\psi}(\tau)d\tau + 2N\lambda|x|^{2N-2} \int_0^t \phi(\tau)\psi(\tau)d\tau$$
$$+ 4N(N-1)\lambda|x|^{2N-4} \int_0^t x\phi(\tau)\, x\psi(\tau)d\tau, \qquad \phi, \psi \in \mathcal{H}_2, \quad (31)$$

and one can easily see that L_x is given by given by

$$L_x\psi(\tau) = B \int_0^\tau \sinh(u-\tau)\psi(u)du + \frac{B}{(1-e^t)(1-e^{-t})} \int_0^t \sinh(\tau-u)\psi(u)du +$$
$$- \frac{B}{(1-e^t)(1-e^{-t})} \int_0^t \sinh(t+\tau-u)\psi(u)du, \quad (32)$$

where B is the $d \times d$ matrix defined by $B := A^2(x) - 1_{d\times d}$ and

$A^2(x)_{i,j} = 2N\lambda|x|^{2N-2}\delta_i^j + 4N(N-1)\lambda|x|^{2N-4}x_i x_j, \qquad i, j = 1, \dots, d.$

Moreover

$$P_{2N}(x, \sqrt{\epsilon}\gamma_2) = \int_0^t |\sqrt{\epsilon}\gamma_2(\tau)+x|^{2N} d\tau - t|x|^{2N} - 2N|x|^{2N-2} \int_0^t \sqrt{\epsilon}x\gamma_2(\tau)d\tau$$
$$-\epsilon N|x|^{2N-2} \int_0^t |\gamma(\tau)|^2 d\tau - 2N(N-1)\epsilon|x|^{2N-4} \int_0^t (x\gamma(\tau))^2 d\tau := \epsilon^{3/2} g(x, \epsilon, \gamma_2) \quad (33)$$

(where we have used the fact that $\int_0^t \gamma_2(\tau)d\tau = 0$ as $\gamma_2 \in \mathcal{H}_2$), and for any x, γ_2 we have

$$\lim_{\epsilon\downarrow 0} g(x, \epsilon, \gamma_2) = \frac{N!}{(N-3)!3!} 8|x|^{2N-6} \int_0^t (x\gamma_2(s))^3 ds +$$
$$+ 2N(N-1)|x|^{2N-4} \int_0^t x\gamma_2(s)|\gamma_2(s)|^2 ds. \quad (34)$$

By expanding $e^{-\lambda\epsilon^{1/2}g(x,\epsilon,\gamma_2)}$ around $\epsilon = 0$:

$$f(x, \epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle \gamma_2, (I+L_x)\gamma_2 \rangle} e^{-\lambda\epsilon^{1/2} g(x,\epsilon,\gamma_2)} d\gamma_2 = f_1(x, \epsilon) - \lambda\epsilon^{1/2} f_2(x, \epsilon), \quad (35)$$

where

$$f_1(x, \epsilon) = \widetilde{\int_{\mathcal{H}_2}} e^{-\frac{1}{2}(\langle \gamma_2, (I+L_x)\gamma_2 \rangle} d\gamma_2 = \det(I + L_x)^{-1/2}$$

and

$$f_2(x,\epsilon)=\widetilde{\int_{\mathcal{H}_2}} g(x,\epsilon,\gamma_2)e^{-\frac{1}{2}(\langle\gamma_2,(I+L_x)\gamma_2\rangle}e^{-u\lambda\epsilon^{1/2}g(x,\epsilon,\gamma_2)}d\gamma_2, \tag{36}$$

with $u\in(0,1)$.
For the calculation of the spectrum $\sigma(L_x)$ of L_x, it is convenient to replace the standard basis of $\mathbb{R}^d$ with an orthonormal basis which diagonalizes the symmetric matrix $A^2(x)$. By denoting its eigenvalues by a_i^2, $i=1,\dots,d$, it is easy to verify that the spectrum of L_x is given by $\sigma(L_x)=\{\lambda_{i,n}, i=1,\dots,d,\ n=1,2,\dots\}$, where

$$\lambda_{i,n}=\frac{a_i^2-1}{1+\frac{4\pi^2n^2}{t^2}},\qquad i=1,\dots,d,\quad n=1,2,\dots$$

are eigenvalues of multiplicity 2. By applying Lidskij's theorem [41] and the Hadamard factorization theorem (see [42], theorem 8.24) one gets

$$\det(I+L_x)=\begin{cases}\det\left(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\right), & \text{for } x\neq 0\\ (2\cosh t-2)^{-d}, & \text{for } x=0\end{cases}$$

The next result follows easily by the integral representation (36) of the function f_2.

Lemma 1. *$f_2(x,\epsilon)$ is a C^∞ function of both $x\in\mathbb{R}^d$ and $\epsilon:=\sqrt{\epsilon}\in\mathbb{R}^+$. Moreover for any $x\in\mathbb{R}^d$, $f_2(x,0)=0$ and $\lim_{\epsilon\downarrow 0}\frac{f_2(x,\epsilon)-f_2(x,0)}{\epsilon^{1/2}}=C(x)$, where C is a positive function of $x\in\mathbb{R}^d$.*

Proof: First of all we have

$$f_2(x,\epsilon)=\widetilde{\int_{\mathcal{H}_2}} e^{\frac{u\lambda t|x|^{2N}}{\epsilon}}g(x,\epsilon,\gamma_2)e^{-\frac{1}{2}\int_0^t\dot\gamma_2^2(s)ds}e^{-\frac{u\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\gamma_2(s)+x|^{2N}ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t|\gamma(s)|^2ds+4N(N-1)|x|^{2N-4}\int_0^t(x\gamma(s))^2ds\right)}d\gamma_2. \tag{37}$$

By expressing the infinite dimensional integral on the Hilbert space $\mathcal{H}_2$ as an integral on the abstract Wiener space $(i,\mathcal{H}_2,\mathcal{B}_2)$ associated with $\mathcal{H}_2$ one gets:

$$f_2(x,\epsilon)=e^{\frac{u\lambda t|x|^{2N}}{\epsilon}}\int_{\mathcal{B}_2}\tilde g(x,\epsilon,\omega_2)e^{\frac{1}{2}\langle\omega_2,L_0\omega_2\rangle}e^{-\frac{u\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\omega_2(s)+x|^{2N}ds}$$
$$e^{-\frac{1-u}{2}\left(2N|x|^{2N-2}\int_0^t|\omega_2(s)|^2ds+4N(N-1)|x|^{2N-4}\int_0^t(x\omega_2(s))^2ds\right)}d\mu(\omega_2), \tag{38}$$

where the functions

$$\omega_2\mapsto\tilde g(x,\epsilon,\omega_2)$$

$$\omega_2 \mapsto \langle \omega_2, L_0\omega_2\rangle$$

$$\omega_2 \mapsto \int_0^t |\sqrt{\epsilon}\omega_2(s) + x|^{2N} ds$$

$$\omega_2 \mapsto 2N|x|^{2N-2}\int_0^t |\omega_2(s)|^2 ds + 4N(N-1)|x|^{2N-4}\int_0^t (x\omega_2(s))^2 ds$$

represent the stochastic extensions to $\mathcal{B}_2$ of the corresponding functions on $\mathcal{H}_2$. The stochastic extensions are well defined because of the regularity of the functions involved. Analogously

$$f_2(x,\epsilon) = \int_{\mathcal{B}_2} \tilde{g}(x,\epsilon,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} e^{-u\lambda\epsilon^{1/2}\tilde{g}(x,\epsilon,\omega_2)} d\mu(\omega_2). \tag{39}$$

Representation (38) shows the absolute convergence of the integrals involved, while representation (39) shows the regularity of f_2 as a function of $\sqrt{\epsilon}$.

By a direct computation we obtain

$$f_2(x,0) = \int_{\mathcal{B}_2} \tilde{g}(x,0,\omega_2) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2),$$

where

$$\tilde{g}(x,0,\omega_2) = \begin{cases} \frac{N!}{(N-3)!3!} 8|x|^{2N-6}\int_0^t (x\omega_2(s))^3 ds + \\ +2N(N-1)|x|^{2N-4}\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, \qquad 2N \geq 6 \\ \\ 4\int_0^t x\omega_2(s)|\omega_2(s)|^2 ds, \qquad 2N = 4 \end{cases} \tag{40}$$

and

$$\lim_{\epsilon\downarrow 0} \frac{f_2(x,\epsilon) - f_2(x,0)}{\epsilon^{1/2}} = \int_{\mathcal{B}_2} g_4(\omega_2,x) e^{-\frac{1}{2}(\langle \omega_2, L_x\omega_2\rangle} d\mu(\omega_2) < \infty, \tag{41}$$

where

$$g_4(\omega_2,x) = \begin{cases} \int_0^t |\omega_2(s)|^4 ds, \qquad 2N = 4 \\ \\ 3|x|^2\int_0^t |\omega_2(s)|^4 ds + 12\int_0^t (x\omega_2(s))^2|\omega_2(s)|^2 ds, \qquad 2N = 6 \\ \\ \binom{N}{2}|x|^{2N-4}\int_0^t |\omega_2(s)|^4 ds + 4\binom{N}{2}\binom{N-2}{1}|x|^{2N-6}\int_0^t (x\omega_2(s))^2|\omega_2(s)|^2 ds + \\ +16\binom{N}{4}|x|^{2N-8}\int_0^t (x\omega_2(s))^4 ds, \qquad 2N \geq 8 \end{cases} \tag{42}$$

□

By equation (35), the integral $I(\epsilon)$ can be represented as the sum $I(\epsilon) =$

$I_1(\epsilon)+I_2(\epsilon)$, where

$$I_1(\epsilon)=\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{t\lambda}{\epsilon}|x|^{2N}}f_1(x,\epsilon)dx,$$

$$I_2(\epsilon)=-\lambda\epsilon^{1/2}\Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{t\lambda}{\epsilon}|x|^{2N}}f_2(x,\epsilon)dx$$

Lemma 2. *$I_2(\epsilon)=O(\epsilon^{\frac{4-d}{2}-\frac{4-d}{2N}})$, as $\epsilon\downarrow 0$.*

Proof: By scaling

$$\begin{aligned}I_2(\epsilon)&=-\lambda\epsilon^{1/2}t^d(2\pi)^{-d/2}\epsilon^{d/2N-d/2}\int_{\mathbb{R}^d}e^{-t\lambda|x|^{2N}}f_2(\epsilon^{1/2N}x,\epsilon)dx\\&=-\lambda t^d(2\pi)^{-d/2}\epsilon^{d/2N-d/2+1/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}\tilde g(\epsilon^{1/2N}x,\epsilon,\omega_2)\\&e^{-\frac{1-u}{2}\left(2N|\epsilon^{1/2N}x|^{2N-2}\int_0^t|\omega_2(s)|^2ds+4N(N-1)|\epsilon^{1/2N}x|^{2N-4}\int_0^t(\epsilon^{1/2N}x\omega_2(s))^2ds\right)}\\&\qquad e^{-\frac{u\lambda}{\epsilon}\int_0^t|\sqrt{\epsilon}\omega_2(s)+\epsilon^{1/2N}x|^{2N}ds}e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx\end{aligned}\tag{43}$$

By the dominated convergence theorem, the definition (33) of the function g, lemma 1 and equation 41 we get:

$$\lim_{\epsilon\downarrow 0}\frac{I_2(\epsilon)}{\epsilon^{\frac{3-d}{2}-\frac{3-d}{2N}}}=-\lambda t^d(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}\tilde g(x,0,\omega_2)e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx=0,\tag{44}$$

where $\tilde g(x,0,\omega_2)$ is given by (40), and

$$\lim_{\epsilon\downarrow 0}\frac{I_2(\epsilon)}{\epsilon^{\frac{4-d}{2}-\frac{4-d}{2N}}}=-\lambda t^d(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-t\lambda(1-u)|x|^{2N}}\int_{\mathcal{B}_2}g_4(\omega_2,x)e^{\frac12\langle\omega_2,L_0\omega_2\rangle}d\mu(\omega_2)dx<\infty,\tag{45}$$

where $g_4(\omega_2,x)$ is given by (42) □

Lemma 3. *$I_1(\epsilon)=\epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)}+O(\epsilon^{(2-d)\frac{N-1}{2N}})$ as $\epsilon\downarrow 0$.*

Proof:

$$\begin{aligned} I_1(\epsilon) &= \Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\epsilon}|x|^{2N}} \det(I+L_x)^{-1/2}dx \\ &= \Big(\frac{2\pi\epsilon}{t^2}\Big)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\epsilon}|x|^{2N}} \det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)(\cosh t-1)}\Big)^{-1/2}dx \\ = t^d\Big(\frac{\cosh t-1}{2\pi\epsilon}\Big)^{d/2} &\int_{\mathbb{R}^d} e^{-\frac{\lambda t}{\epsilon}|x|^{2N}} \det\Big(\frac{\cosh(A(x)t)-1}{A^2(x)}\Big)^{-1/2}dx \end{aligned} \quad (46)$$

By scaling

$$\begin{aligned} I_1(\epsilon) &= C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \det\Big(\frac{\cosh(A(\epsilon^{1/2N}x)t)-1}{A^2(\epsilon^{1/2N}x)}\Big)^{-1/2}dx \\ &= C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \det\Big(\frac{\cosh(\epsilon^{(N-1)/2N}A(x)t)-1}{\epsilon^{(N-1)/N}A^2(x)}\Big)^{-1/2}dx \end{aligned}$$

with $C_t = t^d\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}$. Let $a_i^2(x)$, $i=1,...,d$ be the eigenvalues of the matrix $A^2(x)$. Then

$$\begin{aligned} I_1(\epsilon) &= C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \frac{\epsilon^{\frac{d(N-1)}{2N}}\prod_i a_i(x)}{\prod_i\sqrt{\cosh(\epsilon^{(N-1)/2N}a_i(x)t)-1}}dx \\ &= C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}} \frac{2^{d/2}t^{-d}}{\prod_i\sqrt{1+\frac{\cosh(\theta_i)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2}}dx \\ = C_t\epsilon^{\frac{d}{2N}-\frac{d}{2}}2^{d/2}t^{-d} &\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}}\prod_i\Big(1-\frac{\frac{\cosh(\theta_i)}{24}\epsilon^{(N-1)/N}a_i^2(x)t^2}{(1+\frac{\xi_i\cosh(\theta_i)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2)^{3/2}}\Big)dx \end{aligned}$$

with $\theta_i \in (0, \epsilon^{(N-1)/2N}a_i(x)t)$ and $\xi_i \in (0,1)$. We have

$$I_1(\epsilon) = I_{1,1}(\epsilon) + I_{1,2}(\epsilon),$$

where the first term is equal to

$$\begin{aligned} I_{1,1}(\epsilon) &= \epsilon^{-d\frac{N-1}{2N}}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}\int_{\mathbb{R}^d} e^{-\lambda t|x|^{2N}}dx \\ &= \epsilon^{-d\frac{N-1}{2N}}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\int_{\mathbb{R}^d} e^{-|x|^{2N}}dx \\ &\qquad = \epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)}, \end{aligned}$$

and the second term is equal to

$$I_{1,2}(\epsilon)=\Big(\frac{\cosh t-1}{2\pi\epsilon}\Big)^{d/2}\epsilon^{d/2N}2^{d/2}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}$$
$$\Big(\prod_i\Big(1-\frac{\frac{\cosh(\theta_i\epsilon^{(N-1)/2N}a_i(x)t)}{24}\epsilon^{(N-1)/N}a_i^2(x)t^2}{(1+\frac{\xi_i\cosh(\theta_i\epsilon^{(N-1)/2N}a_i(x)t)}{12}\epsilon^{(N-1)/N}a_i^2(x)t^2)^{3/2}}\Big)-1\Big)dx$$

and it satisfies the following relation

$$\lim_{\epsilon\downarrow 0}\frac{I_{1,2}(\epsilon)}{\epsilon^{-d\frac{N-1}{2N}+\frac{N-1}{N}}}=-\frac{t^2}{24}\Big(\frac{\cosh t-1}{2\pi}\Big)^{d/2}2^{d/2}\int_{\mathbb{R}^d}e^{-\lambda t|x|^{2N}}\sum_i a_i^2(x)dx<\infty.$$

□

By combining lemma 2 and 3 we get:

Theorem 5. *As $\epsilon\downarrow 0$ the infinite dimensional oscillatory integral $I(\epsilon)$ (25) has the following asymptotic behavior:*

$$I(\epsilon)=\epsilon^{-d\frac{N-1}{2N}}(\cosh t-1)^{d/2}2^{d/2}t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{N\Gamma(d/2)}+O(\epsilon^{(2-d)\frac{N-1}{2N}})\tag{47}$$

The latter result can be applied to the study of the asymptotic behavior of the trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ of the heat semigroup, where $H:D(H)\subset L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is the quantum mechanical Hamiltonian given on the dense set of vectors $\psi\in S(\mathbb{R}^d)$ by

$$H\psi(x)=-\frac{\hbar^2}{2}\Delta_x\psi(x)+V(x)\psi(x),\tag{48}$$

with $V(x)=\lambda|x|^{2N}$, $N\in\mathbb{N}$, $N\geq 2$, $\lambda>0$, $x\in\mathbb{R}^d$, $N\in\mathbb{N}$.
It is well known that H is an essentially self adjoint operator on $C_0^\infty(\mathbb{R}^d)$ (see [37], theorem X.28). H is a positive operator and is the generator of an analytic semigroup, denoted by $e^{-\frac{t}{\hbar}H}$, $t\geq 0$ (the "heat semigroup" with potential V). Its trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ is well defined as $V(x)$ is smooth and increases at least quadratically at infinity, hence the spectrum of H consists of (real positive) eigenvalues $\lambda_{\bar n}$, $\bar n\in\mathbb{N}^d$. By a standard WKB argument one can deduce that there exist a positive constant α (depending on N) with

$$\liminf_{|\bar n|\to\infty}\frac{\lambda_{\bar n}}{|\bar n|^\alpha}>0.$$

Theorem 6. *The trace of the heat semigroup* $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t>0$, *for* H *as in equation* (48), *is given by the infinite dimensional Fresnel integral (with parameter* $s=1/\hbar$, *in the sense of definition 2)*

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = (2\cosh t - 2)^{-d/2}\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2\hbar}\int_0^t \dot\gamma(s)^2 ds - \frac{\lambda}{\hbar}\int_0^t \gamma(s)^{2N} ds} d\gamma \tag{49}$$

For $\hbar \downarrow 0$ *the following asymptotics holds:*

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \hbar^{-d\frac{N-1}{2N}} t^{-d/2N}\lambda^{-d/2N}\frac{\Gamma(d/2N)}{2^{d/2}N\Gamma(d/2)} + O(\hbar^{(2-d)\frac{N-1}{2N}}) \tag{50}$$

Proof: The proof of (49) is divided into 3 steps.
1_{st} *Step:* By Feynman-Kac formula (see e.g.[40, 41]) $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ is given, for $t>0$ by:

$$\begin{aligned}\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] &= \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x)ds} d\mu_(\alpha)\\ &= \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\int_{C_{[0,t]}} e^{-\lambda\hbar^{N-1}\int_0^t |\alpha(s)+x|^{2N} ds} d\mu_(\alpha)\end{aligned} \tag{51}$$

where $C_{[0,t]}$ is the space of continuous paths $\alpha:[0,t]\to\mathbb{R}^d$ such that $\alpha(0)=\alpha(t)$ and μ is the Brownian bridge probability measure on it (see, e.g. [40] for this concept).
Let us introduce the Hilbert spaces $Y_{0,t}$ and $Y_{p,t}$ of paths ,

$$\{\gamma\in H^1(0,t;\mathbb{R}^d):\ \gamma(0)=\gamma(t)=0\}$$

with norms

$$\|\gamma\|^2_{Y_{0,t}} \equiv |\gamma| = \int_0^t \dot\gamma(s)^2 ds.$$

$$\|\gamma\|^2_{Y_{p,t}} \equiv \|\gamma\| = \int_0^t \dot\gamma(s)^2 ds + \int_0^t \gamma(s)^2 ds.$$

It is well known that $(i, Y_{0,t}, C_{[0,t]})$ is an abstract Wiener space.
First of all (see remark 2) the integral in (51) on $C_{[0,t]}$ with respect to the Brownian bridge measure can also be written in terms on an infinite dimensional integral (with parameter $s=1$) on the Hilbert space $Y_{0,t}$ (in the sense of definition 2):

$$\int_{C_{[0,t]}} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\alpha(s)+\sqrt{\hbar}x)ds} d\mu_(\alpha) = \widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma,$$

so that

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d}\frac{dx}{(2\pi t)^{d/2}}\widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma \tag{52}$$

2_{nd} *Step:* By the transformation formula relating infinite dimensional integral on Hilbert spaces with varying norms (theorem 1), we get a relation between the integral on $Y_{0,t}$ and the integral on $Y_{p,t}$. Indeed

$$\|\gamma\|^2 = |\gamma|^2 + (\gamma, T\gamma)$$

where T is the unique self-adjoint trace class operator on $Y_{0,t}$ defined by the quadratic form

$$(\gamma_1, T\gamma_2) = \int_0^t \gamma_1(s)\gamma_2(s)ds,$$

Indeed (see [4] for details) $\eta = T\gamma$, $\gamma \in Y_{0,t}$ if and only if

$$\begin{cases} \ddot{\eta}(s) + \gamma(s) = 0, & s \in [0,t] \\ \dot{\eta}(0) = 0 \\ \dot{\eta}(t) = 0 \end{cases} \tag{53}$$

and $\det(I + T) = \left(\frac{\sinh t}{t}\right)^d$. By inserting this into equation (9) we obtain:

$$\widetilde{\int_{Y_{0,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$

$$= \left(\frac{t}{\sinh t}\right)^{d/2} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma$$

and by equation (52)

$$\mathrm{Tr}[e^{-\frac{t}{\hbar}H}] = \int_{\mathbb{R}^d} \frac{dx}{(2\pi \sinh t)^{d/2}} \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}|\gamma|^2} e^{-\frac{1}{\hbar}\int_0^t V(\sqrt{\hbar}\gamma(s)+\sqrt{\hbar}x)ds} d\gamma \tag{54}$$

3_{rd} *Step:* The final step is a transformation of variable formula for integrals on the Hilbert space $\mathcal{H}_{p,t}$. $Y_{p,t}$ can be regarded as a subspace of $\mathcal{H}_{p,t}$ and any vector $\gamma \in \mathcal{H}_{p,t}$ can be written as a sum of a vector $\eta \in Y_{p,t}$ and a constant in the following way:

$$\gamma(s) = \eta(s) + x, \qquad s \in [0,t],\ \gamma \in \mathcal{H}_{p,t},\ \eta \in Y_{p,t},\ x = \gamma(0).$$

We have to compute a constant C_t such that for integrable functions f

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma) d\gamma = C_t \int_{\mathbb{R}^d} dx \widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+x\|^2} f(\eta + x) d\eta.$$

By Fubini theorem

$$\widetilde{\int_{\mathcal{H}_{p,t}}} e^{-\frac{1}{2}\|\gamma\|^2} f(\gamma) d\gamma = \widetilde{\int_{Y_{p,t}^\perp}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta+\xi\|^2} f(\eta+\xi) d\eta\Big) d\xi, \tag{55}$$

where $Y_{p,t}^{\perp}$ is the space orthogonal to $Y_{p,t}$ in $\mathcal{H}_{p,t}$. One can easily verify that $Y_{p,t}^{\perp}$ is $d-$dimensional and it is generated by the vectors $\{v_i\}_{i=1,\dots,d}$, with $v_i(s) = \hat{e}_i \Big(\frac{e^s(1-e^{-t})+e^{-s}(e^t-1)}{2\sqrt{2}\sqrt{\sinh t(\cosh t-1)}} \Big)$, $s \in [0,t]$, $\hat{e}_i$ being the i_{th} vector of the canonical basis in $\mathbb{R}^d$. The right hand side of (55) is equal to

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_i) d\eta \Big) dy,$$

where $\xi(s) = \sum_i y_i v_i(s)$, $i = 1, \dots, d$. By writing the finite dimensional approximation of $\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_1) d\eta$, by the formula for the change of variables in finite dimensional integrals and by noting that

$$\langle u_j, v_i \rangle_{\mathcal{H}_{p,t}} = \delta_i^j \frac{\sqrt{2\cosh t - 2}}{\sqrt{\sinh t}},$$

where $u_j \in \mathcal{H}_{p,t}$ is the vector given by $u_j(s) = \hat{e}_j$, $s \in [0,t]$, we get

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i y_i v_i\|^2} f(\eta + \sum_i y_i v_i) d\eta \Big) dy$$

$$= \Big(\frac{\sqrt{2\cosh t - 2}}{\sqrt{\sinh t}} \Big)^d \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \Big(\widetilde{\int_{Y_{p,t}}} e^{-\frac{1}{2}\|\eta + \sum_i x_i u_i\|^2} f(\eta + \sum_i x_i u_i) d\eta \Big) dx$$

so that the constant C_t is equal to $\Big(\frac{\sqrt{2\cosh t - 2}}{\sqrt{2\pi \sinh t}} \Big)^d$.
By combining these results we get equation (49).
The asymptotic behavior of the trace $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$ as $\hbar \downarrow 0$ follows by equation (49) and theorem 5. □

Remark 5. *In* [4, 7] *the representation* (49) *is proved for the case where V is a quadratic function plus a bounded perturbation (which is the Fourier transform of a complex measure) by means of a different technique (a Fubini theorem for infinite dimensional oscillatory integrals with respect to non-degenerate quadratic forms), that cannot be applied in our present case. Indeed the quadratic part of the phase function appearing in the integral on the right hand side of* (49) *can be written as*

$$\int_0^t \dot{\gamma}^2(s) ds = -\langle \gamma, L\gamma \rangle,$$

with $L : \mathcal{H}_{p,t} \to \mathcal{H}_{p,t}$ is the operator (28). *As we have seen, L is not invertible and* $\det L = 0$. *This fact forbids the application of the Fubini* *theorem as stated in* [4, 7] *and a direct application of the methods of* [4, 7].

Remark 6. *A representation equivalent to* (51) *is discussed in* [40] *for other continuous potential* V *with* $e^{-V} \in L^1$. *However the limit* $\hbar \downarrow 0$ *discussed in* [40] *is not the semiclassical limit we discuss here. To the best of our knowledge our limit for our type of polynomially growing potentials has not been rigorously discussed before. In addition our result on this problem, besides coming as a direct application of a study concerning oscillatory integrals, also provides a method to derive an explicit fractional power expansion formula in* $\hbar$ *in terms of classical orbits (we shall however do not give here details of this, contenting ourselves to have indicated the method how to obtain them).*

Appendix A. Abstract Wiener spaces

Let $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ be a real real separable Hilbert space. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{1}{2}\|x\|^2}$. Let $|\ |$ be a "measurable" norm on $\mathcal{H}$, that is $|\ |$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has

$$\nu(\{x \in \mathcal{H} |\ |P(x)| > \epsilon\}) < \epsilon,$$

where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $|\ |$ is weaker than $\|\ \|$. Denoted by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $|\ |$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* (see, e.g., [22, 29]). Given $y \in \mathcal{B}^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify $\mathcal{B}^*$ as a subset of $\mathcal{H}$. Moreover $\mathcal{B}^*$ is dense in $\mathcal{H}$ and we have the dense continuous inclusions $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$. Each element $y \in \mathcal{B}^*$ can be regarded as a random variable $n(y)$ on $(\mathcal{B}, \mu)$. A direct computation shows that $n(y)$ is normally distributed, with covariance $\|y\|^2$. More generally, given $y_1, y_2 \in \mathcal{B}^*$, one has

$$\int_{\mathcal{B}} n(y_1)n(y_2)d\mu = \langle y_1, y_2\rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(\mathcal{B}, \mu)$, because $\mathcal{B}^*$ is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x\rangle e_i$$

for some orthonormal $e_1,\dots,e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on $\mathcal{B}$ is well defined by

$$\tilde{P}(\,\cdot\,)=\sum_{i=1}^{n} n(e_i)(\,\cdot\,)e_i.$$

Given a function $f:\mathcal{H}\to\mathcal{B}_1$, where $(\mathcal{B}_1,\|\ \|_{\mathcal{B}_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to $\mathcal{B}$ exists if the functions $f\circ\tilde{P}:\mathcal{B}\to\mathcal{B}_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g:\mathcal{B}\to\mathcal{B}_1$ is continuous and $f:=g|_{\mathcal{H}}$, then one can prove [22] that the stochastic extension of f is well defined and it is equal to g $\mu-$a.e. Moreover for any $h\in\mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{n} h_i n(e_i), \qquad h_i=\langle e_i,h\rangle$$

converges in $L^2(\mathcal{B},\mu)$, and by subsequences μ a.e., to the random variable $n(h)$.
Given a self-adjoint trace class operator $B:\mathcal{H}\to\mathcal{H}$, the quadratic form on $\mathcal{H}\times\mathcal{H}$:

$$x\in\mathcal{H}\mapsto\langle x,Bx\rangle$$

can be extended to a random variable on $\mathcal{B}$, denoted again by $\langle\,\cdot\,,B\,\cdot\,\rangle$. Indeed for each increasing sequence of finite dimensional projectors P_n converging strongly to the identity, $P_n(x)=\sum_{i=1}^n e_i\langle e_i,x\rangle$ ($\{e_i\}$ being a CONS in $\mathcal{H}$), the sequence of random variables

$$\omega\in\mathcal{B}\mapsto\sum_{i,j=1}^{n}\langle e_i,Be_j\rangle n(e_i)(\omega)n(e_j)(\omega)$$

is a Cauchy sequence in $L^1(\mathcal{B},\mu)$. By passing if necessary to a subsequence, it converges to $\langle\,\cdot\,,B\,\cdot\,\rangle$ $\mu-$a.e.
Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that $(I-B)$ is strictly positive). Then one can prove that the random variable $g(\,\cdot\,):=e^{\frac{1}{2}\langle\,\cdot\,,B\,\cdot\,\rangle}$ is μ-summable. Indeed by considering a CONS $\{e_i\}$ made of eigenvectors of the operator B, b_i being the corresponding eigenvalues, the sequence of random variables

$$g_n:\mathcal{B}\to\mathbb{C}, \qquad \omega\mapsto g_n(\omega)=e^{\frac{1}{2}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2},$$

converges to $g(\omega)$ μ-a.e..
On the other hand one has

$$\int_{\mathcal{B}} g_n(\omega)d\mu(\omega)=\prod_{i=1}^{n}\int\frac{e^{-\frac{1}{2}(1-b_i)x_i^2}}{\sqrt{2\pi}}dx_i=(\prod_{i=1}^{n}(1-b_i))^{-1/2}$$

so that $\int g_n d\mu$ converges, as $n \to \infty$, to $(\det(I-B))^{-1/2}$, where $\det(I-B)$ denotes the Fredholm determinant of $(I-B)$, which is well defined as B is trace class. Moreover $0 \leq g_n \leq g_{n+1}$ for each n . It follows that, as $n \to \infty$, $\int g_n d\mu \to \int g d\mu = (\det(I-B))^{-1/2}$. By an analogous reasoning one can prove that for any $y \in \mathcal{H}$, the sequence of random variables f_n:

$$\omega \mapsto f_n(\omega) = e^{\sum_{i=1}^n y_i n(e_i)(\omega)} e^{\frac{1}{2}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2}$$

where $y_i = \langle y, e_i \rangle$, converges $\mu-$a.e. as n goes to ∞ to the random variable $f(\,\cdot\,) = e^{n(y)(\,\cdot\,)} e^{\frac{1}{2}\langle\,\cdot\,, B\,\cdot\,\rangle}$ and that

$$\int f_n d\mu \to \int f d\mu = (\det(I-B))^{-1/2} e^{\frac{1}{2}\langle y, (I-B)^{-1} y\rangle}. \tag{56}$$

(see [29, 26]).

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