

# Lagrange Interpolation in Satellite Positioning for Inter-Vehicle Distance Estimation: A Case Study

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## Abstract

Cooperative Inter-vehicle Distance (IVD) estimation algorithms such as Absolute Position Differencing (APD), Single-Differencing (SD), and Double-Differencing (DD) are promising and cost-effective solutions thanks to Global Navigation Satellite System (GNSS) observables. These algorithms directly utilize pseudorange measurements, i.e., the estimated distance between the antennas of satellites orbiting the Earth and the GNSS receiver installed on the vehicle. However, an accurate IVD estimate using these techniques is dependent on exact satellite coordinates, as any GNSS pseudorange requires precise satellite positions. To compute satellite positions, the satellite's distributed navigation message or interpolation methods can be employed. This paper examines the performance of the Lagrange interpolation approach for estimating satellite locations epoch-by-epoch in a real-world experiment in the IVD estimation problem. The experimental results demonstrate that the Lagrange interpolation method performs effectively with sub-centimeter accuracy in the IVD estimation problem. Furthermore, the results indicate that even in a short study duration of 15 minutes, using outdated fixed-satellite positions influences IVD estimation accuracy and causes increased uncertainty.

## Keywords

Inter-Vehicle Distance (IVD), Absolute Position Differencing (APD), Single-Differencing (SD), Double-Differencing (DD), Satellite Position, Lagrange Interpolation

## 1. Introduction

Autonomous Vehicles (AVs) play a pivotal role in future mobility. It promises several advantages, including simplified driving, reduced traffic congestion and accidents, increased safety, and improved energy efficiency of the transportation system [1]. As a key component of AVs, robust and precise localization has been widely investigated in recent years [1,2]. The architecture supporting autonomous driving generally comprises five functional systems: localization, perception, planning, control, and system management [3]. These systems need precise information

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*Proceedings of the Work-in-Progress Papers at the 13th International Conference on Indoor Positioning and Indoor Navigation (IPIN-WiP 2023), September 25–28, 2023, Nuremberg, Germany*

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CEUR Workshop Proceedings (CEUR-WS.org)

on the vehicle's position and Inter-Vehicle Distance (IVD) measurements [1].

The global navigation satellite system (GNSS), which estimates the vehicle position from the pseudorange (an estimate of the distance between a satellite and a GNSS receiver installed on the vehicle) measurements from several satellites, is the most popular method for vehicle localization [2]. However, due to existing errors such as satellite clock error, multipath error, and ionospheric delay of pseudorange, the GNSS positioning performance is not satisfactory [2]. There is a growing body of literature that recognizes cooperative localization methods as an alternative solution for improving positioning accuracy by sharing localization information between two or more sources, i.e., vehicles and infrastructure, via emerging vehicular communication technologies [4–6]. Cooperative IVD estimation algorithms can be classified as ranging-based or non-ranging-based [2]. For IVD estimation in ranging-based methods, signal strength variations such as radio signal strength [7], Time of Arrival [8], round trip time [9] or Time Difference of Arrival [10] can be used. However, these approaches are often costly since they require additional infrastructure and hardware to be implemented. In addition, the fast vehicle speed may also introduce noise or errors in estimated distances [2]. The non-ranging cooperative localization algorithm that directly utilizes each vehicle's pseudorange measurements can be used as a cost-effective alternative for vehicle localization and IVD estimation thanks to GNSS observables, i.e., pseudorange [2],[6],[11,12].

Any pseudorange in GNSS requires the computation of the satellite location, and the methodologies for doing so are well-known in the literature [13]. We may use Kepler's law to determine satellite locations utilizing distributed navigation messages from the satellite, such as RINEX (Receiver Independent Exchange Format) or RTCM (Radio Technical Commission for Maritime Services) [13,14]. To this end, the receiver observation data (e.g., RTCM) should be converted to RINEX format for post-processing and RTKLIB, an open-source program package for GNSS positioning, can be used [15]. Interestingly, using archival data from the International GNSS Service (IGS) [16] is an easy way to use satellite locations in post-processing. However, since IGS data is often provided in 15-minute intervals and we do not have access to satellite positions epoch-by-epoch, several interpolation algorithms have been proposed including Lagrange interpolation [17], Chebyshev polynomial fitting [18], Newton's divided difference interpolation polynomial [19], and Cubic spline interpolation [19]. According to a comparison study [20] of different interpolation techniques for estimating satellite locations, Lagrange interpolation performed well with higher precision. Additionally, it can be used to compute satellite coordinates with only two known satellite positions. The literature [21–23] earlier proposed the application of the Lagrange interpolation algorithm in GNSS orbit interpolation and gave performance evaluations. However, this is based on mathematical and theoretical studies rather than a real-world assessment. The motivation for this investigation stems from the need to evaluate the performance of the Lagrange interpolation approach in a real-world application. In this study, we examine the performance of Lagrange interpolation in cooperative IVD estimation techniques using real-world measurements. The remainder of this paper is structured as follows. A description of the mathematical formulation of the GNSS pseudorange measurements and cooperative IVD estimation algorithms is provided in Section 2. Section 3 is concerned with the methodology used for this study. A discussion of the estimated IVD based on fixed satellite coordinates and epoch-by-epoch satellite locations obtained via Lagrange interpolation is presented in Section 4. Finally, Section 5, summarizes the work.

## 2. Problem Formulation

### 2.1. GNSS pseudorange measurement model

The GNSS observables (raw code pseudorange) denoted by  $\rho$ , are defined as the estimated distance between the GNSS receiver installed on vehicle  $V \in \{V_1, V_2, V_3, \dots, V_n\}$  and a satellite  $S \in \{S_1, S_2, S_3, \dots, S_A\}$  at any time-step  $k$ , which are modeled as follows [2],[6]:

$$\rho_V^S(k) = R_V^S(k) + t_V^S(k) + \varepsilon_c^S(k) + \varepsilon_u^S(k) \quad (1)$$

where  $R_V^S(k) = \|\vec{P}_S(k) - \vec{P}_V(k)\|$  is the true geometric range between vehicle  $V$  and satellite  $S$ , the symbol  $\|\cdot\|$  represents the  $l_2$  norm operation,  $\vec{P}_S(k) = [x_S(k), y_S(k), z_S(k)]^T$  is the position vector of satellite  $S$ ,  $\vec{P}_V(k) = [x_V(k), y_V(k), z_V(k)]^T$  is the true position vector of vehicle  $V$  on the Earth-centered, Earth-fixed (ECEF) coordinate system,  $t_V^S(k)$  is the clock misalignment error between the GNSS receiver installed on the vehicle  $V$  and satellite  $S$ ,  $\varepsilon_c^S(k)$  indicates the correlated (common) uncertainty induced by the ephemeris and the atmosphere, and finally,  $\varepsilon_u^S(k)$  denotes the uncorrelated uncertainty, which includes the multi-path error, the thermal noise, and other residual errors [6].

### 2.2. Cooperative IVD Estimation Algorithms

#### 2.2.1. Absolute Position Differencing (APD)

The GNSS receiver installed on each vehicle is able to compute an estimate of its absolute position vector in ECEF coordinates after acquiring and tracking the GNSS signal of at least four satellites. The absolute position differencing (APD) method calculates the estimated distance between two vehicles at any time-step  $k$  denoted by  $\hat{d}_{ij}(k) = \|\vec{P}_{V_j}(k) - \vec{P}_{V_i}(k)\|$ , i.e.

$$\hat{d}_{ij}(k) = \sqrt{(z_{V_j} - z_{V_i})^2 + (y_{V_j} - y_{V_i})^2 + (x_{V_j} - x_{V_i})^2} \quad (2)$$

where  $\vec{P}_{V_i}(k) = [x_{V_i}(k), y_{V_i}(k), z_{V_i}(k)]^T$  and  $\vec{P}_{V_j}(k) = [x_{V_j}(k), y_{V_j}(k), z_{V_j}(k)]^T$  are the estimated position vectors of vehicle  $i$  and vehicle  $j$  obtained at time-step  $k$  from the GNSS in ECEF coordinates, respectively.

#### 2.2.2. Single-Differencing (SD)

Fig.1 depicts the single differencing used for the IVD. The SD method estimates the IVD by subtracting the pseudorange measurements of two vehicles from the same satellite. This approach can eliminate both the clock imperfect synchronization between the vehicles as well as the atmospheric delay error. Given that the satellite  $S$  is sufficiently far from vehicles, the pseudorange measurements from each vehicle toward the satellite  $S$  are considered to be parallel (see Fig.1) [2],[6]. More precisely, given (1) for two vehicles  $V_i$  and  $V_j$ , when computing the difference we have:

$$\Delta \rho_{V_i V_j}^S(k) = \rho_{V_i}^S(k) - \rho_{V_j}^S(k) = \Delta R_{V_i V_j}^S(k) + \Delta t_{V_i V_j}(k) + \Delta \varepsilon_{u_0}(k) \quad (3)$$

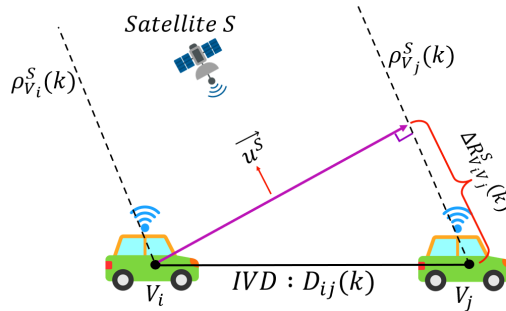
where  $\Delta R_{V_i V_j}^S(k) = R_{V_i}^S(k) - R_{V_j}^S(k)$  defines the difference between the true distance of vehicle  $V_i$  and vehicle  $V_j$  from the satellite  $S$ ,  $\Delta t_{V_i V_j}(k) = t_{V_i}^S(k) - t_{V_j}^S(k)$  denotes the time delay error, and  $\Delta \varepsilon_{u_0}(k) = \varepsilon_{u_{V_i}}^S(k) - \varepsilon_{u_{V_j}}^S(k)$  represents all the remaining uncertainties, usually dubbed *unusual error* [2],[6]. Due to the difference among the measured pseudoranges, the unusual error appears to be increasing [2]. Since the true distances between the vehicles and the satellites ( $R_{V_i}^S(k)$  and  $R_{V_j}^S(k)$ ), are much larger than the distance between the vehicles, we can estimate the  $\Delta R_{V_i V_j}^S(k)$  as follows [2],[6]:

$$\Delta R_{V_i V_j}^S(k) = [\vec{u}^S]^T \vec{D}_{ij}(k) \quad (4)$$

where  $\vec{u}^S = \frac{\vec{P}_S(k) - \vec{P}_{V_i}(k)}{\|\vec{P}_S(k) - \vec{P}_{V_i}(k)\|}$  is the Line-Of-Sight (LOS) unit vector from vehicle  $V_i$  to satellite  $S$ ,  $\vec{D}_{ij}(k)$  indicates the vehicle distance vector,  $\vec{P}_S(k)$  represents the position vector of the satellite  $S$  and  $\vec{P}_{V_i}(k)$  defines the position vector of the reference vehicle  $V_i$  at time-step  $k$  (see Fig.1 for reference). By considering  $N$  common visible satellites for the two vehicles and using (3), we can build the following measurement matrix:

$$\begin{bmatrix} \Delta \rho_{V_i V_j}^1(k) \\ \Delta \rho_{V_i V_j}^2(k) \\ \vdots \\ \Delta \rho_{V_i V_j}^N(k) \end{bmatrix} \approx \begin{bmatrix} [u^1]^T & 1 \\ [u^2]^T & 1 \\ \vdots & \vdots \\ [u^N]^T & 1 \end{bmatrix} \begin{bmatrix} \vec{D}_{ij}(k) \\ \Delta t_{V_i V_j}(k) \end{bmatrix} \quad (5)$$

yielding the SD estimates [2],[6]. Next, with an initial estimation of the position of the reference vehicle  $V_i$ , Eq.5 can be solved iteratively, resulting in an estimate of  $D_{ij}(k)$ , which can then be used to determine the distance between both vehicles for each time instant  $k$  [6]. Notice that the vehicle distance vector  $\vec{D}_{ij}(k)$ , obtained via matrix inversion (Least Square Method) of Eq.5 consists of three distance vector components, i.e.,  $(x, y, z)$  and one time delay component. For IVD, we used the  $l_2$  norm of the first three components.



**Figure 1:** Single-Differencing (SD) IVD estimation algorithm

### 2.2.3. Double-Differencing (DD)

In the SD-based algorithm of (5), user clock offsets and common biases among those measurements are still present. To eliminate these uncertainties and also any other common biases, we can utilize a new GNSS measurement and then compute the difference between the SD estimates obtained from two distinct satellites, say  $S_A$  and  $S_B$ . This is referred to as the double-differencing (DD) algorithm and is demonstrated in Fig.2. The DD-based approach assumes that both vehicles can track satellites  $S_A$  and  $S_B$  at the same time. Hence, we first apply an SD-based algorithm to each vehicle toward the satellites  $S_A$  and  $S_B$ , denoted by  $\Delta\rho_{V_iV_j}^{S_A}(k)$  and  $\Delta\rho_{V_iV_j}^{S_B}(k)$ , respectively, which are obtained from (3). Then, each double difference of such quantities defined by  $\nabla\Delta\rho_{V_iV_j}^{S_A S_B}(k)$  is obtained as [11]:

$$\nabla\Delta\rho_{V_iV_j}^{S_A S_B}(k) = \Delta\rho_{V_iV_j}^{S_A}(k) - \Delta\rho_{V_iV_j}^{S_B}(k) = \Delta R_{V_iV_j}^{S_A S_B}(k) + \Delta\varepsilon_{V_iV_j}^{S_A S_B}(k) \quad (6)$$

where  $\Delta R_{V_iV_j}^{S_A S_B}(k) = \Delta R_{V_iV_j}^{S_A}(k) - \Delta R_{V_iV_j}^{S_B}(k)$  and  $\Delta\varepsilon_{V_iV_j}^{S_A S_B}(k) = \Delta\varepsilon_{V_iV_j}^{S_A}(k) - \Delta\varepsilon_{V_iV_j}^{S_B}(k)$ . We can then estimate  $\Delta R_{V_iV_j}^{S_A S_B}(k)$  using the same trigonometric idea of SD, that is illustrated in Fig.2 [2],[6],[11].

$$\Delta R_{V_iV_j}^{S_A S_B}(k) = [\vec{u}^{S_A} - \vec{u}^{S_B}] \vec{D}_{ij}(k) \quad (7)$$

where  $\vec{u}^{S_A}$  and  $\vec{u}^{S_B}$  are computed as in (4). Using (6) is then possible to calculate the distance and the relative positions of two vehicles. Indeed, using the satellite  $A$  as a reference, the solution to the DD-based algorithm according to Fig.2 is given by the matrix form [2],[11]:

$$\begin{bmatrix} \nabla\Delta\rho_{V_iV_j}^{S_1 S_A}(k) \\ \nabla\Delta\rho_{V_iV_j}^{S_2 S_A}(k) \\ \vdots \\ \nabla\Delta\rho_{V_iV_j}^{S_B S_A}(k) \end{bmatrix} \approx \begin{bmatrix} [u^1 - u^A]^T \\ [u^2 - u^A]^T \\ \vdots \\ [u^N - u^A]^T \end{bmatrix} \vec{D}_{ij}(k) \quad (8)$$

Notice that the IVD vector  $\vec{D}_{ij}(k)$  is projected in the direction of the difference satellite unitary vectors  $\vec{u}^{S_{AB}} = \vec{u}^{S_A} - \vec{u}^{S_B}$  for each DD measurement indicated by  $\nabla\Delta\rho_{V_iV_j}^{S_{AB}}(k)$ . Assuming four satellites, say  $S_A$ ,  $S_B$ ,  $S_C$ , and  $S_D$ , and considering  $S_A$  as the reference satellite, the following system of linear equations derived from (8) can be obtained [11]:

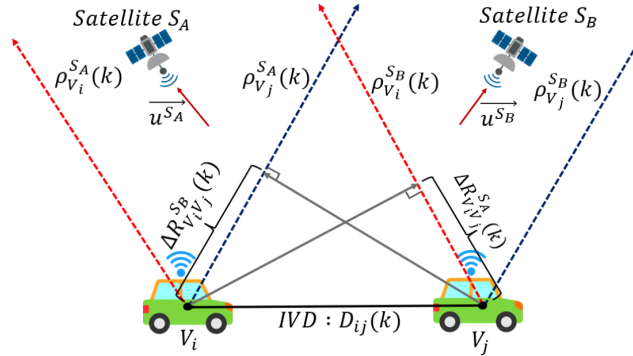
$$\begin{bmatrix} \nabla\Delta\rho_{V_iV_j}^{S_{AB}} \\ \nabla\Delta\rho_{V_iV_j}^{S_{AC}} \\ \nabla\Delta\rho_{V_iV_j}^{S_{AD}} \end{bmatrix} = \begin{bmatrix} u_x^{S_{AB}} & u_y^{S_{AB}} & u_z^{S_{AB}} \\ u_x^{S_{AC}} & u_y^{S_{AC}} & u_z^{S_{AC}} \\ u_x^{S_{AD}} & u_y^{S_{AD}} & u_z^{S_{AD}} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \mathbf{G}_{uu} \vec{D}_{ij}(k) \quad (9)$$

where:

$$\vec{u}^{S_{qr}} = \frac{\vec{S}_q(k) - \vec{P}_{V_i}(k)}{\|\vec{S}_q(k) - \vec{P}_{V_i}(k)\|} - \frac{\vec{S}_r(k) - \vec{P}_{V_i}(k)}{\|\vec{S}_r(k) - \vec{P}_{V_i}(k)\|} = \begin{bmatrix} u_x^{S_q} \\ u_y^{S_q} \\ u_z^{S_q} \end{bmatrix} - \begin{bmatrix} u_x^{S_r} \\ u_y^{S_r} \\ u_z^{S_r} \end{bmatrix}$$

where  $\vec{S}_q$  and  $\vec{S}_r$ ,  $q, r \in \{A, B, C, D\}$  are the satellite position vectors and  $\vec{P}_{V_i}$  is position vector of the vehicle  $V_i$ , all evaluated at the time-step  $k$ . Notice that 4 is the minimum number

of satellites needed to have a solution of the DD-based algorithm, i.e.,  $G_{uu}$  (known as the *geometry matrix*) should be non-singular. Usually, if more than 4 satellites are available, a more precise and effective Least Squares solution is adopted.



**Figure 2:** Double-Differencing (DD) IVD estimation algorithm

### 3. Methods and Methodology

#### 3.1. Real-World Experiment Set-up

To evaluate the performance of Lagrange interpolation in a cooperative IVD estimation problem, we used a real-world experiment scenario in which two static outdoor autonomous vehicles ( $V_1$  and  $V_2$ ) with LOS views toward GNSS satellites are located in the ECEF coordinates, and collected pseudorange measurements from different satellites. Fig.3 shows the 15-minute study interval. As shown in this figure, we picked two available known satellite locations from the IGS data on April 26, 2022, at 12:45 and 13:00 UTC. Then, we employed Lagrange interpolation to compute epoch-by-epoch satellite locations for the whole study interval. To this end, we consider  $L_s \in \{l_0, l_1, l_2, \dots, l_n\}$  to be the values of the satellite locations, i.e.,  $L_s = [x_s, y_s, z_s]$ , in time-step at  $k \in \{k_0, k_1, k_2, \dots, k_n\}$ . The first and final known satellite positions are then used as inputs for the Lagrange approach to calculate the approximate value of  $l$ , denoted by  $L(k)$  at any time of  $k$  as follows [17]:

$$L(k) = a_0 l_0 + a_1 l_1 + a_2 l_2 + \dots + a_n l_n = \sum_{i=0}^n a_i l_i \quad (10)$$

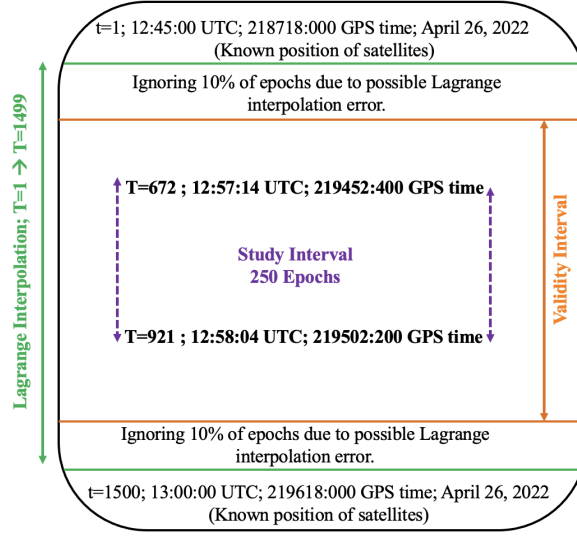
where:

$$a_i = \frac{(k - k_0)(k - k_1) \dots (k - k_{i-1})(k - k_{i+1}) \dots (k - k_n)}{(k_i - k_0)(k_i - k_1) \dots (k_i - k_{i-1})(k_i - k_{i+1}) \dots (k_i - k_n)} \quad (11)$$

Now, by substituting  $k$  in Eq.10 with  $\{k_0, k_1, k_2, \dots, k_n\}$ , we obtain:

$$L(k_0) = l_0, L(k_1) = l_1, \dots, L(k_n) = l_n \quad (12)$$

According to [17], when dealing with Lagrange interpolation (polynomial fitting) for computing satellite coordinates, we typically have an error (Runge's phenomenon) in the beginning



**Figure 3:** An interval study of a real environment experiment.

and ending points of the interpolation. Hence, as proposed in [17], we considered a validity interval for our satellite positioning by ignoring the starting and ending points, i.e., 10% on the data set.

### 3.2. Statistical Measurement Criteria

To quantify the performance of the IVD estimation methods in this study, we compute three statistical criteria, including the root mean squared error (RMSE), average distance error ( $\overline{\Delta d}$ ), and average relative error ( $RE$ ), which are defined as follows:

$$RMSE = \sqrt{\frac{1}{N} \sum_{k=1}^N [D_{ij}(k) - \hat{D}_{ij}(k)]^2} \quad (13)$$

$$\overline{\Delta d} = \frac{\sum_{k=1}^N |D_{ij}(k) - \hat{D}_{ij}(k)|}{N} \quad (14)$$

$$RE = \frac{|D_{ij}(k) - \hat{D}_{ij}(k)|}{D_{ij}(k)} \quad (15)$$

where  $\hat{D}_{ij}(k)$  indicates the average estimated IVD at time-step  $k$  by SD- and DD-based algorithms, and  $N$  represents the number of total epochs during the interval period, which is 250. We assume here that the two autonomous vehicles equipped with the GNSS receivers additionally have a Real-time kinematic (RTK) system that calculates the distance between itself and the broadcasting satellite. Thus, utilizing the RTK data, the average estimated IVD by the APD approach given by Eq.2 is assumed to be the actual ground truth (real distance) between the two vehicles  $V_i$  and  $V_j$  at time-step  $k$  and indicated by  $D_{ij}(k)$  which is 3.354 (m) during the study interval.

**Table 1**  
Best categories of satellites based on MVA and statistical criteria

System Category	Satellite	Best Category	Average Estimated IVD SD/DD (m)	RMSE SD/DD (m)	$\overline{\Delta d}$ SD/DD (m)	RE SD/DD (m)
One System of Satellites	GPS	G5-G16-G18-G31	2.816	1.504	1.343	0.159
	GLONASS	R9-R15-R18-R19	28.326	25.489	24.972	7.443
	<b>Galileo</b>	<b>E33-E31-E24-E26</b>	<b>3.294</b>	<b>0.276</b>	<b>0.024</b>	<b>0.007</b>
	BeiDou	C35-C45-C13-C24	3.947	1.811	1.484	0.174
Two Systems of Satellites	GPS	G18-G5-R9-R15	11.775	8.563	21.052	2.511
	GLONASS					
	GPS	G18-G5-E24-E12	4.292	1.429	2.346	0.280
	<b>Galileo</b>					
	<b>GPS</b>	<b>C35-G5-C24-C45</b>	<b>2.927</b>	<b>1.176</b>	<b>1.066</b>	<b>0.128</b>
	<b>BeiDou</b>					
	Galileo	C35-E1-E24-C29	31.632	32.413	70.696	8.425
	BeiDou	R18-R15-E24-E26	4.478	2.395	2.810	0.333
Three Systems of Satellites	GLONASS	R18-R15-E24-E26	4.478	2.395	2.810	0.333
	BeiDou	C35-R15-C13-C24	3.710	1.659	0.891	0.104
	<b>GLONASS</b>					
	<b>GPS</b>	<b>C35-E1-G5-G27</b>	<b>2.873</b>	<b>0.677</b>	<b>1.201</b>	<b>0.142</b>
	<b>Galileo</b>					
	<b>BeiDou</b>					
Four Systems of Satellites	GPS	G18-G23-E24-G5	5.505	2.220	5.379	0.640
	GLONASS					
	Galileo	C35-E1-E24-C29	31.632	32.413	70.696	8.425
	BeiDou					
Four Systems of Satellites	<b>GLONASS</b>	<b>C35-E24-G5-R15</b>	<b>3.472</b>	<b>1.035</b>	<b>0.294</b>	<b>0.034</b>
	<b>Galileo</b>					
	<b>BeiDou</b>					

### 3.3. Optimal Geometric Satellite Selection Algorithm

Employing multi-GNSS systems can enhance positioning accuracy, and using more satellites may give higher precision in the IVD estimation [12]. However, for the sake of simplicity in analyzing Lagrange interpolation in satellite positioning, we are examining a group of four satellites from the available satellites. To this end, in this study, we use the Maximum Volume Algorithm (MVA) to pick the optimum geometric group of four satellites comprising GPS, GLONASS, Galileo, and BeiDou to use in the SD- and DD-based IVD estimation techniques. The MVA is a four-step heuristic technique based on tetrahedron geometry that consists of the stages listed below [24]:

- **Step.1:** Select the visible satellite  $S_1$  with the largest elevation angle relative to the position of the receiver (in our case,  $V_i$ ).
- **Step.2:** Choose the visible satellite  $S_2$  having the elevation angle to  $S_1$ , i.e.,  $\theta_{S_1 S_2}$ , close to  $109.47^\circ$ . It should be noticed that this elevation angle is obtained from a simple geometric consideration of:  $\cos \theta = -\frac{1}{3}$  as detailed in [24].
- **Step.3:** Pick the visible satellite  $S_3$  that maximizes the volume of the tetrahedron

$$V_A = \frac{1 - a_3}{6} \left[ \sqrt{2(1 - a_2)(1 + a_3)(1 - a_2 a_3 - b_2 b_3)} + |b_2 c_3| \right] \quad (16)$$

where

$$a_2 = \cos \theta_{S_1 S_2}, b_2 = \sin \theta_{S_1 S_2}, a_3 = \cos \theta_{S_1 S_3},$$

$$b_3 = \frac{\cos \theta_{S_2 S_3} - a_2 a_3}{b_2}, c_3 = \pm \sqrt{1 - a_3^2 - b_3^2}.$$

Notice that the tetrahedron is formed by  $S_1, S_2, S_3$ .

- **Step.4:** Select the satellite  $S_4$  from the remaining visible satellites so that it maximizes the volume of the tetrahedron

$$V_B = \frac{1}{6} \det(S) \quad (17)$$

where  $S$  is the matrix that contains the line-of-sight vectors corresponding to  $S_1, S_2, S_3$ , and  $S_4$ .

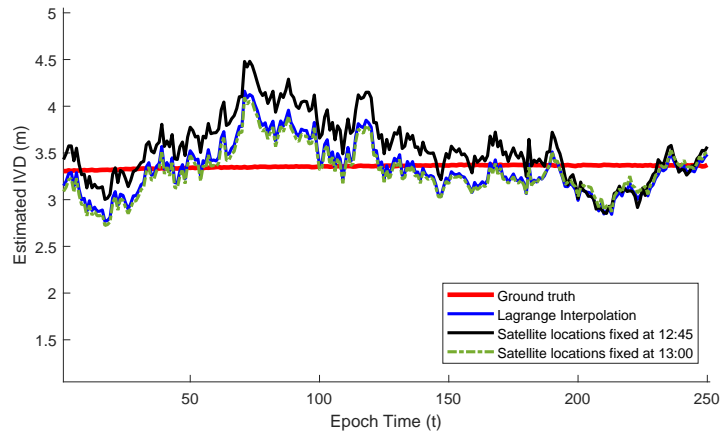
As shown in Fig.4, there are a total of 26 common visible satellites for vehicles  $V_1$  and  $V_2$  for the study interval depicted in Fig.3. Table.1 shows the results of utilizing the MVA to determine the optimal geometric arrangement of four satellites. In Table.1, we highlighted the final best categories of satellites based on the lowest RMSE, lowest average estimated distance, and lowest relative error on DD/SD algorithms for one-, two-, three-, and four-satellite systems, respectively. Notice that as stated in [12], when four satellites are used, there is no significant difference between SD and DD methods, which is also true in our investigation.

## 4. Results and Discussion

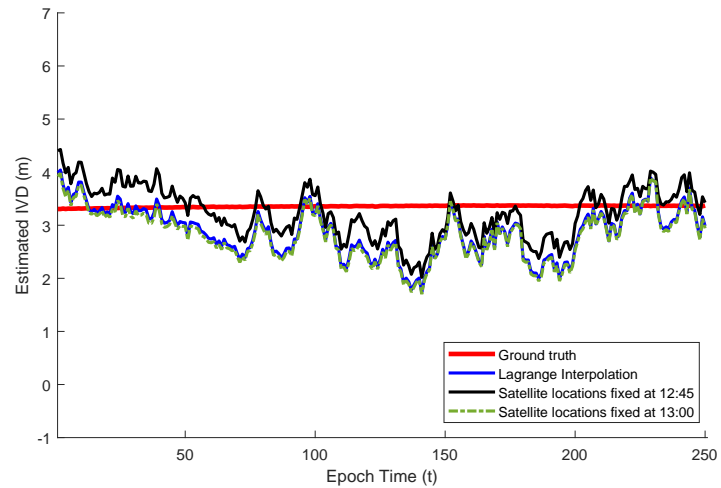
To assess the Lagrange interpolation, we investigated the DD/SD IVD estimation algorithm in three situations, including employing satellite locations derived by the Lagrange method and fixed for 15 minutes at 12:45 and 13:00 UTC. Fig.5 (a) and Fig.5 (b) show the IVD estimate using the DD/SD approach in those three scenarios for one- and three-system satellites, respectively. It is apparent from these figures that DD/SD techniques employing Lagrange interpolation and fixed satellite locations at 13:00 UTC are comparable, thus not justifying the increased complexity of the method.

Furthermore, it can be seen from Fig.5 (a) and Fig.5 (b) that computing the IVD at 12:45 UTC has an influence on the IVD estimate accuracy and produces additional uncertainty. More precisely, consider Table.2 and the definition of an absolute accuracy error, i.e.,  $AE = |D_{ij}(k) - \hat{D}_{ij}(k)|$ . When utilizing Lagrange interpolation and fixed satellite coordinates at 12:45 UTC, the absolute





**Figure.5 (a):** An illustration of the influence of satellite locations on estimated IVD in the DD/SD technique for E33-E31-E24-E26 (One system of satellites)



**Figure.5 (b):** An illustration of the influence of satellite locations on estimated IVD in the DD/SD technique for C35-E1-G5-G27 (Three systems of satellites)

## 5. Conclusion

This study examined the performance of the Lagrange interpolation method for calculating satellite positions to use in cooperative IVD estimation algorithms in a real-world application using a single 15-minute dataset. The experimental results indicated a well-functioning Lagrange interpolation. Moreover, this study demonstrated that utilizing outdated satellite locations for IVD estimates increases IVD uncertainty from 11.8 cm up to 20.1 cm in four systems of satellites and from 6 cm up to 19.1 cm in one system of satellites, which is crucial for fully autonomous vehicles.

Several future works are planned. First, a comparative study will be investigated to analyze

the positions of satellites estimated using Lagrange interpolation in conjunction with broadcast ephemeris, as well as higher-order interpolations of IGS precise orbits. Second, there will be a focus on simulating and assessing the performance of Lagrange interpolation in moving vehicles and for longer study periods exceeding 15 minutes. Finally, as carrier-based techniques are increasingly utilized by mass-market receivers to enhance accuracy, an examination of their applicability to the IVD estimation problem will also be undertaken.

## Acknowledgments

This research was supported by Centro Ricerche Fiat (CRF) and imec-WAVES research group at Ghent University.

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