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Addendum to: *Bose-Einstein Condensate and Spontaneous Breaking of Conformal Symmetry on Killing Horizons*, J. Math. Phys. 46, 062303 (2005).

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Abstract. In the paper cited in the title local scalar QFT (in Weyl algebraic approach) has been constructed on degenerate semi-Riemannian manifolds $\mathbb{S}^1 \times \Sigma$ corresponding to the extension of Killing horizons by adding points at infinity to the null geodesic forming the horizon. It has been proved that the theory admits a natural representation of $PSL(2,\mathbb{R})$ in terms of $*$-automorphisms and this representation is unitarily implementable if referring to a certain invariant state λ. Among other results it has been proved that the theory admits a class of inequivalent algebraic (coherent) states $\{\lambda_\zeta\}$, with $\zeta \in L^2(\Sigma)$, which break part of $PSL(2,\mathbb{R})$ symmetry. These states, if restricted to suitable portions of $\mathbb{M}$ are invariant and extremal KMS states with respect a surviving one-parameter group symmetry. In this *addendum* we clarify the nature of that $PSL(2,\mathbb{R})$ symmetry braking. We show that, in fact, *spontaneous* symmetry breaking occurs in the natural sense of algebraic quantum field theory: if $\zeta \neq 0$, there is no unitary representation of whole group $PSL(2,\mathbb{R})$ which implements the $*$-automorphism representation of $PSL(2,\mathbb{R})$ itself in the GNS representation of λ_ζ.

1 Summary of some achieved results.

In the paper cited in the title [1] local scalar QFT (in Weyl algebraic approach) is constructed on degenerate semi-Riemannian manifolds $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ corresponding to the extension of future Killing horizons by adding points at infinity to the null geodesic forming the horizon. Above the transverse manifold Σ has a Riemannian metric inducing the volume form ω_Σ, whereas $\mathbb{S}^1$ is equipped with the null metric. To go on, fix a standard frame (see section II A of [1]) $\theta \in (-\pi, \pi]$ on $\mathbb{S}^1$ of $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ – so that $\mathbb{S}^1$ is realized as $(-\pi, \pi]$ with the endpoints identified – and consider the C^*-algebra of Weyl $\mathcal{W}(\mathbb{M})$ generated by non-vanishing elements $V(\omega)$ in Eq. (12) in section II E in [1], the smooth forms ω of the space $\mathcal{D}(\mathbb{M})$ being the space of forms ϵ_ψ defined in Eq.(1) in section II A in [1]. We shall exploit the group of *-automorphisms α defined in Eq. (19) in [1] representing the Möbius group $PSL(2,\mathbb{R})$ viewed as a subgroup of diffeomorphisms of $\mathbb{S}^1$ and thus of $\mathbb{M}$ (see Section III A of [1]):

$$\alpha_g\left(V(\omega)\right) = V(\omega^{(g^{-1})})\,, \tag{1}$$

$\omega^{(g)} := g^*\omega$ being the natural pullback action of $g \in PSL(2,\mathbb{R})$ on forms. In the following $\{\alpha_t^{(\mathcal{X})}\}_{t\in\mathbb{R}}$ indicates the one-parameter subgroup associated with the vector field $\mathcal{X}$ on $\mathbb{M}$ corresponding to an element of the Lie algebra of $PSL(2,\mathbb{R})$. In particular the vector field on $\mathbb{M}$, $\mathcal{D}$, generating a one-parameter subgroup of $PSL(2,\mathbb{R})$, is that defined in in Eq. (16) in [1]. If $\zeta \in L^2(\Sigma,\omega_\Sigma)$ and λ_ζ is the pure coherent state on $\mathcal{W}(\mathbb{M})$ defined by means of (33) of Section IV B in [1], we indicate its GNS triple by $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$. Those states are constructed as follows with respect to $\lambda := \lambda_0$. The map $V(\omega) \mapsto V(\omega)\, e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})}$, $\omega \in \mathcal{D}(\mathbb{M})$, uniquely extends to a *-automorphism γ_ζ on $\mathcal{W}(\mathbb{M})$ such that $\gamma_\zeta \circ \alpha_t^{(\mathcal{D})} = \alpha_t^{(\mathcal{D})} \circ \gamma_\zeta$ for all $t \in \mathbb{R}$. The function $\Gamma := \ln|\tan(\theta/2)|$ and the $\mathcal{D}$-positive-frequency part of ω, ω_+, are respectively defined and discussed in section IV B (Eq. (32)) and in Lemma 3.1 of section III B of [1].

$$\lambda_\zeta(w) \quad := \quad \lambda(\gamma_\zeta w)\,, \quad \text{for all } w \in \mathcal{W}(\mathbb{M})\,. \tag{2}$$

Among other results it has been proved that (theorems 4.1, 4.2, 5.1) the pure states λ_ζ are inequivalent states and, they are restricted to the algebra localized at the "half circle times Σ", they give rise to different extremal KMS states at rationalized Hawking temperature $T = 1/2\pi$ with respect to $\{\alpha_t^{(\mathcal{D})}\}_{t\in\mathbb{R}}$. If one is dealing with a bifurcate Killing Horizon of a black hole the "half circle times Σ" is noting but $\mathbb{F}_+$, the future right branch of the Killing horizon (see Section I and figure in [1]). In this case $-\zeta^{-1}\mathcal{D}$, with $\zeta^{-1} = \kappa$ being the surface gravity of the examined black hole, can be recognized as the restriction to the horizon to the Killing vector field defining Schwarzschild time, kT becomes proper Hawking's temperature and the order parmeter associated with the beakdown of symmetry can be related with properties of the black hole.

2 Spontaneous breaking of $PSL(2,\mathbb{R})$ symmetry.

In the physical literature there are several, also strongly inequivalent, definitions of *spontaneous* breaking of symmetry related to different approaches to quantum theories. We adopt the following elementary definition which is quite natural in algebraic QFT.

Definition 1. *Referring to a C^*-algebra $\mathcal{A}$, one says that:*
(a) *A Lie group $\mathcal{G}$ is a* **group of symmetries** *for $\mathcal{A}$ if there is a representation β of $\mathcal{G}$ made of $*$-automorphisms of $\mathcal{A}$. If some notion of time evolution is provided, it is required that it corresponds to a one parameter subgroup of $\mathcal{G}$. (This subgroup may or may not belong to the center of the group, in the latter case, the self-adjoint generators of unitary representations of $\mathcal{G}$ implementing β define constant of motion which depend explicitly on time)*
(b) *Assuming that (a) is valid,* **spontaneous breaking of $\mathcal{G}$ symmetry** *occurs with respect to an algebraic state μ on $\mathcal{A}$ and β, if there are elements $g \in \mathcal{G}$ such that β_g are not implementable*

unitarily in the GNS representation of μ.

Considering the inequivalent GNS triples $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$, theorems 3.2[1] and 3.3 in [1] show that, if $\zeta = 0$ the group of automorphisms α representing $PSL(2,\mathbb{R})$ can be unitarily implemented in the space $\mathfrak{H} = \mathfrak{H}_0$ and the cyclic vector $\Psi := \Psi_0$ of the GNS representation is invariant under that (strongly continuous) unitary representation of $PSL(2,\mathbb{R})$. Conversely, if $\zeta \neq 0$, $PSL(2,\mathbb{R})$ symmetry turns out to be broken. Indeed, theorem 4.1 states that each state λ_ζ with $\zeta \neq 0$ is invariant under $\{\alpha_t^{(\mathcal{D})}\}_{t\in\mathbb{R}}$, but it is not under any other one-parameter subgroup of α (barring those associated with $c\mathcal{D}$ for $c \in \mathbb{R}$ constant). In the general case this is not enough to assure occurrence of *spontaneous* breaking of $PSL(2,\mathbb{R})$ symmetry as defined in Def.1. (However, it is possible to show that – see section III.3.2 of [2] – if $\mathcal{G}$ is Poincaré group or an internal symmetry group for a special relativistic system and the reference state μ is a primary vacuum state over the net of algebras of observables, then non-$\mathcal{G}$ invariance of the vacuum state implies – and in fact is equivalent to – spontaneous breaking of $\mathcal{G}$ symmetry. Our considered case is far from that extent and thus there is no *a priori* guarantee for the occurrence of spontaneous breaking of $PSL(2,\mathbb{R})$ symmetry for states λ_ζ with $\zeta \neq 0$ and the issue deserve further investigation. The following theorem give an answer to the issue.

Theorem 1. *If $L^2(\Sigma, \omega_\Sigma) \ni \zeta \neq 0$, spontaneous breaking of $PSL(2,\mathbb{R})$-symmetry occurs with respect to λ_ζ and the representation α. (In particular, there is no unitary implementation of the nontrivial elements of the subgroup of $PSL(2,\mathbb{R})$ generated from the vector field $\frac{\partial}{\partial\theta}$.)*

Proof. Referring to the GNS triple of λ_ζ define $\hat{V}_\zeta(\omega) := \Pi_\zeta(V(\omega))$. The existence of a unitary implementation, $L_g : \mathfrak{H}_\zeta \to \mathfrak{H}_\zeta$, of α in the GNS triple $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$ implies, in particular, that

$$L_g \hat{V}_\zeta(\omega) L_g^\dagger = \alpha_g\left(\hat{V}_\zeta(\omega)\right) , \quad \text{for all } \omega \in \mathcal{D}(\mathbb{M}) \text{ and every } g \in PSL(2,\mathbb{R}) . \tag{3}$$

By construction, $(\mathfrak{H}_\zeta, \Pi, \Psi_\zeta)$ (notice that we wrote Π instead of Π_ζ) is a GNS triple of λ (notice that we wrote λ instead of λ_ζ) if

$$\Pi : V(\omega) \mapsto \hat{V}(\omega) := \hat{V}_\zeta(\omega) e^{-i\int_{\mathbb{M}} \Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})} \tag{4}$$

In this realization α can be unitarily implemented (theorem 3.2 in [1]): There is a (strongly continuous) unitary representation U of $PSL(2,\mathbb{R})$ such that

$$U_g \hat{V}(\omega) U_g^\dagger = \alpha_g\left(\hat{V}(\omega)\right) , \quad \text{for all } \omega \in \mathcal{D}(\mathbb{M}) \text{ and every } g \in PSL(2,\mathbb{R}) . \tag{5}$$

Suppose that α can be implemented also in $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$, where now $\Pi_\zeta : V(\omega) \mapsto \hat{V}_\zeta(\omega)$, and let L be the corresponding unitary representation of $PSL(2,\mathbb{R})$ satisfying (3). That equation

[1] In the proof theorem 3.2 in all occurrences of the symbol $PSL(2,\mathbb{R})$ before the statement "...is in fact a representation of $PSL(2,\mathbb{R})$." it has to be replaced by $\widetilde{PSL}(2,\mathbb{R})$, it denoting the universal covering of $PSL(2,\mathbb{R})$.

together with (4) entail that the unitary operator $S_g := U_g^\dagger L_g$ satisfies

$$\begin{aligned} S_g V(\omega) S_g^\dagger &= e^{ic_{g,\omega}} V(\omega)\,, \\ c_{g,\omega} &:= \int_{\mathbb{M}} \left(\zeta \left(\omega^{(g^{-1})}\right)_+ + \overline{\zeta\left(\omega^{(g^{-1})}\right)_+} - \zeta\omega_+ - \overline{\zeta\omega_+} \right) \Gamma\,. \end{aligned} \tag{6}$$

Now, dealing with exactly as in the proof of (ii) of (b) of theorem 4.1 (where the role of our S_g was played by the operator U and the role of $c_{g,\omega}$ was played by the simpler phase $\int_{\mathbb{M}}(\zeta\omega_+ + \overline{\zeta\omega_+})\Gamma$) one finds that (6) entails that $\langle \Psi_\zeta, S_g \Psi_\zeta \rangle \neq 0$ and

$$||S_g\Psi_\zeta||^2 = |\langle \Psi_\zeta, S_g \Psi_\zeta \rangle|^2 \; e^{\sum_{n,j}^\infty |\lambda_{n,j}|^2}\,, \tag{7}$$

$$\lambda_{n,j} := -2i \int_{\mathbb{M}} \Gamma(\theta)\overline{\zeta(s)} u_j(s)\, \frac{\partial}{\partial\theta} \frac{e^{i\theta_g(\theta)} - e^{i\theta}}{\sqrt{4\pi n}}\, d\theta d\omega_\Sigma(s)\,.$$

Above the real compactly-supported functions u_j defines a Hilbert base in $L^2(\Sigma, \omega_\Sigma)$, and $\theta_g(\theta)$ is the θ component of the point $g(\theta, s) \in \mathbb{M}$ obtained by the action of $g \in PSL(2,\mathbb{R})$ on (θ, s). Now take $g \in \{\alpha_t^{(\mathcal{K})}\}_{t\in\mathbb{R}}$, the one-parameter subgroup of $PSL(2,\mathbb{R})$ generated by the vector field $\mathcal{K} := \frac{\partial}{\partial\theta}$, and realize the factor $\mathbb{S}^1$ of $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ as $[-\pi, \pi)$ with the identification of its endpoints. In this case, obviously, $\theta_g(\theta) = \theta + t$. A direct computation shows that, for some j_0 with $\int_\Sigma u_j(s)\overline{\zeta(s)}\omega_\Sigma(s) = 0$ (which does exist otherwise $\zeta = 0$ almost everywhere)

$$|\lambda_{2n+1,j_0}|^2 = C \left[\frac{1}{2n+1} - \frac{\cos((2n+1)t)}{2n+1} \right]$$

for some constant $C > 0$ independent form n. The series of elements $-\cos((2n+1)t)/(2n+1)$ converges for $t \neq 0, \pm\pi$ (it diverges to $+\infty$ for $t = \pm\pi$), whereas that of elements $1/(2n+1)$ diverges to $+\infty$. Thus the exponent in (7) and L_g cannot exist, if $g = \alpha_t^{(\mathcal{K})}$ with $t \neq 0$. $\Box$

As a final comment we notice that the automorphism γ_ζ is a symmetry of the system because it commutes with time evolution α_t. (It is worth noticing that, if restricting to real functions ζ, $\zeta \mapsto \gamma_\zeta$ defines a group of automorphisms.) Since $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)} \neq \lambda_{\zeta'} \restriction_{\mathcal{W}(\mathbb{F}_+)}$ for $\zeta \neq \zeta'$, and all these states are extremal $\alpha^{(\mathcal{D})}$-KMS states at the same temperature, following Haag (V.I.5 in [2]), we can say that *spontaneous symmetry breaking* with respect to γ_ζ occours in the context of extremal KMS states theory.

References

[1] V. Moretti and N. Pinamonti: *"Bose-Einstein condensate and Spontaneous Breaking of Conformal Symmetry on Killing Horizons"*, J. Math. Phys. 46, 062303 (2005). [hep-th/0407256]

[2] R. Haag, *"Local quantum physics: Fields, particles, algebras"*, Second Revised and Enlarged Edition. Springer Berlin, Germany (1992).