UNISOLVENT AND MINIMAL PHYSICAL DEGREES OF FREEDOM FOR THE SECOND FAMILY OF POLYNOMIAL DIFFERENTIAL FORMS

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Abstract. The principal aim of this work is to provide a family of unisolvent and minimal physical degrees of freedom, called weights, for Nédélec second family of finite elements. Such elements are thought of as differential forms $\mathcal{P}_r\Lambda^k(T)$ whose coefficients are polynomials of degree r. In this paper we confine ourselves in the two dimensional case $\mathbb{R}^2$, as in this framework the Five Lemma offers a neat and elegant treatment avoiding computations on the middle space. The majority of definitions and constructions are meaningful for $n > 2$ as well and, when possible, they are thus given in such a generality, although more complicated techniques shall be invoked to replace the graceful role of the Five Lemma. In particular, we use techniques of homological algebra to obtain degrees of freedom for the whole diagram

$$\mathcal{P}_r\Lambda^0(T) \to \mathcal{P}_{r-1}\Lambda^1(T) \to \mathcal{P}_{r-2}\Lambda^2(T),$$

being T a 2-simplex of $\mathbb{R}^2$. This work pairs its companions recently appeared for Nédélec first family of finite elements.

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1. Introduction

Degrees of freedom are one of the main ingredients of a finite element triple as defined by Ciarlet [20]. For standard polynomial Lagrange elements over simplices, the classical degrees of freedom are evaluations on the principal lattice $L_r(T)$ of top dimensional simplices T of the triangulation. These degrees of freedom have a clear physical meaning: if u_h is the numerical solution, then degrees of freedom are just the values of the exact solution at some points of the mesh. On the other side, for the polynomial differential forms families $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ described in [7], the standard degrees of freedom are the so called *moments*, that is, integrals against $(d-k)$-forms on d-subsimplices, for $d = k, \ldots, n$, where $n = \dim T$ is the dimension of the domain of the problem. These "test forms" are polynomial forms as well, and as a consequence of this, moments for the space $\mathcal{P}_r\Lambda^k$ involve evaluations or integrals of polynomials of degree up to $2r$ approximately. Thus these degrees of freedom have some disadvantages, which we aim here to improve:

(1) they lack an immediate physical interpretation;
(2) the associated Vandermonde matrix is not well conditioned;

(3) they are difficult to implement; in fact, some FEM libraries (*e.g.* MFEM [5]) use nodal degrees of freedom also for vector valued elements.

To overcome these issues, another choice of degrees of freedom has been proposed in [10]. It consists in considering integrals over k-cells topologically contained in the top dimensional simplices. These degrees of freedom are called *weights* or *physical*, since they have a clear physical interpretation: circulations or fluxes for vector fields (1- and $n-1$-forms) and averages for densities (n-forms). Moreover, weights are a straightforward generalization of the evaluation-type degrees of freedom for scalar functions (for $k = 0$, a k-cell is just a point and the integral is just the evaluation).

A charming theoretical aspect of weights is the following. Physical degrees of freedom show that general finite element systems enjoy the same properties of Whitney forms [36]. In particular, discrete differential forms are cochains of an appropriate finer complex and the discrete exterior derivative is the coboundary operator. As a practical consequence, finite element systems can be used to construct high order Yee-like schemes in the fashion of [28]. We elaborate more on this point in Section 2.3.

Physical degrees of freedom for the first (or *trimmed*) family $\mathcal{P}_r^- \Lambda^k$, whose features in the framework that we adopt here have been pointed out in several works, such as [7,23,27], were studied extensively in [17,34] and more recently in [2,4,13]. On the other side, for the second (or *complete*) family $\mathcal{P}_r \Lambda^k$ the first physical degrees of freedom for the two dimensional case were proposed in [37] only recently, where however unisolvence was not proved, but only checked numerically. In this work, we stick to the two dimensional case and we provide different physical degrees of freedom for the second family and we rigorously prove the unisolvence using cohomological tools. Moreover, we provide numerical evidence of improvements towards the well-conditioning of the associated Vandermonde matrix and we perform some interpolation tests.

We assume that the reader is familiar with standard notions in differential geometry and algebraic topology that are now common in most works on Finite Element Exterior Calculus (FEEC) and Finite Element Systems (FES), such as differential forms, differential complexes, cellular complexes, chains, cochains, cohomology, de Rham maps, and so on. We address to Section 2 of [16] for a concise introduction on these topics. We however recall known and useful facts when setting the notation.

The outline of this work is as follows. In Section 2 we introduce basic definitions and tools. We recall known results and state lemmas that we will use in the subsequent. In Section 3 we state the main results concerning the construction of unisolvent and minimal sets. In particular, confining ourselves in the case of $\mathbb{R}^2$, we identify a unisolvent and minimal sequence for Nédélec second family. We close this section giving an explicit translation of these results to the more classical language of vector calculus. In Section 4 we present some numerical results concerning the generalised Vandermonde matrices associated with the introduced families and the associated interpolators, comparing an example of convergence of a smooth, oscillating form with that of its differential. We summarise conclusions and propose future developments in Section 5.

2. Physical systems of degrees of freedom

In this section we recall the definition of a physical system of degrees of freedom and some results from [37] and show some properties of the associated bases.

Let X be a compatible finite element system in the sense of Christiansen [15,17,18] over the cellular complex $\mathcal{T}$. In particular, for each T in $\mathcal{T}$ (of any dimension[1]), for each $k = 0, 1, 2, \ldots, \dim T$, the following sequence is exact:

$$0 \to \mathbb{R} \hookrightarrow X^0(T) \xrightarrow{\mathrm{d}} X^1(T) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{d} X^{\dim T}(T) \to 0.$$

Here the first arrow is the inclusion and d denotes the exterior derivative. Moreover, for T in $\mathcal{T}$ we denote with $\mathring{X}^k(T)$ the subspace of $X^k(T)$ made of all forms with zero trace on the boundary. Finally, let $X^k(\mathcal{T})$ be

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[1]In the finite element systems framework, one considers spaces of differential forms and degrees of freedom on cells of each dimensions, not only on top dimensional ones.

the global finite element space constructed by "gluing" the local spaces $X^k(T)$, that is, the restriction ω_T of $\omega \in X^k(\mathcal{T})$ to any cell T belongs to $X^k(T)$ and, if S is a subcell of T, the trace of ω_T on S coincides with ω_S.

Definition 2.1. A *system of physical degrees of freedom* (physical sysdofs) $\mathcal{F}$ over X is a choice, for each cell T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots, \dim T$, of a finite set $\mathring{\mathcal{F}}^k(T) \doteq \{s_1, \ldots, s_{\mathring{N}^k(T)}\}$ of non-overlapping k-cells. These cells induce functionals

$$\omega \mapsto w(\omega, s_i) \doteq \int_{s_i} \omega. \tag{2.1}$$

We call $w(\omega, s_i)$ the *weight* of ω on s_i.

The unisolvence of a physical system of degrees of freedom is defined in the obvious way.

Definition 2.2. A physical sysdofs is said to be *unisolvent* if, for each T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots, \dim T$, the only form ω in $\mathring{X}^k(T)$ which satisfies

$$w(\omega, s) = 0, \quad \forall s \in \mathring{\mathcal{F}}^k(T)$$

is the zero form.

Clearly, a unisolvent physical sysdofs must satisfy the trivial necessary condition: for each T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots$, $\mathring{N}^k(T) \geq \dim \mathring{X}^k(T)$ where $\mathring{N}^k(T)$ denotes the cardinality of the set $\mathring{\mathcal{F}}^k(T)$. This motivates the following definition.

Definition 2.3. A physical sysdofs is *minimal* if, for each T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots, \dim T$, the following equality holds:

$$\mathring{N}^k(T) = \dim \mathring{X}^k(T).$$

From the properties of compatible finite element system we obtain the following equivalent definition of unisolvence and minimality, which is closer to classical one found in standard books on finite elements [20]. For each S, T in $\mathcal{T}$ we write $S \leq T$ is S is a subcell of T. Moreover, for T in $\mathcal{T}$ and $k = 0, 1, 2, \ldots, \dim T$, write

$$\mathcal{F}^k(T) \doteq \bigcup_{S \leq T} \mathring{\mathcal{F}}^k(S). \tag{2.2}$$

Lemma 2.4. *If a physical system of degrees of freedom $\mathcal{F}$ is unisolvent, then, for each top dimensional cell T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots, \dim T$, the only form ω in $X^k(T)$ satisfying*

$$w(\omega, s) = 0, \qquad \forall s \in \mathcal{F}^k(T) \tag{2.3}$$

is the zero form. Moreover $\mathcal{F}$ is minimal and unisolvent if and only if the above condition holds and $N^k(T) = \dim X^k(T)$, where $N^k(T)$ denotes the cardinality of $\mathcal{F}^k(T)$.

Proof. Assume that $\mathcal{F}$ is unisolvent. Let $\omega \in X^k(T)$ satisfying condition (2.3). Then let S any k-subcell of T and let $\iota_{S,T} : S \to T$. Clearly $\iota_{S,T}^* \omega$ belongs to $X^k(S)$, but since it is a k-form on a k-cell, its traces on the boundary of S vanish by definition, therefore $\iota_{S,T}^* \omega$ actually belongs to $\mathring{X}^k(S)$. Then, by unisolvence, $\iota_{S,T}^* \omega = 0$. Let now S be a $(k+1)$-subcell of T and S' be a k-cell belonging to the boundary of S. Then $\iota_{S',S}^* \iota_{S,T}^* \omega = \iota_{S',T}^* \omega = 0$ by the previous argument. Therefore $\iota_{S,T}^* \omega$ belongs to $\mathring{X}^k(S)$. Again, by unisolvence, $\iota_{S,T}^* \omega = 0$. Proceeding in this way, we obtain that $\omega \in \mathring{X}^k(T)$. Finally, unisolvence gives $\omega = 0$. For the stronger statement, see Proposition 2.5 in [17]. $\square$

From the computational point of view, one may check an equivalent condition for any given top dimensional cell T and any $k = 0, 1, 2, \ldots, \dim T$. We thus define the *generalised Vandermonde matrix* as the matrix V whose (i, j)-th element is

$$V_{i,j} \doteq \int_{s_i} \omega_j,$$

being $\omega_1, \ldots, \omega_{N^k(T)}$ some basis for $X^k(T)$. We thus have the following.

Lemma 2.5. *A collection of k-cells $\{s_1, \ldots, s_{N^k(T)}\}$ is unisolvent and minimal if and only if V is a square full rank matrix. Such a rank does not depend on the basis $\{\omega_1, \ldots, \omega_{N^k(T)}\}$ chosen for $X^k(T)$.*

2.1. A motivation: the scalar case

To fix ideas, let T be a 2-simplex, *i.e.* a non degenerate triangle. Notice that, for $k = 0$ and $X^0(T) = \mathbb{P}_r(T)$, the problem of deducing unisolvence and minimality is linked to the problem of deducing if a collection of nodes $\mathcal{N}$ in $\mathbb{R}^2$ is *poised*, which means that the only polynomial vanishing on $\mathcal{N}$ is the zero polynomial. Explicitly, for a polynomial $\varphi \in \mathbb{P}_r(\mathbb{R}^2)$ this reads as

$$\varphi(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \mathcal{N} \implies \varphi(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^2.$$

This problem is still unsolved in its greatest generality, however several partial results and conjectures have been offered. A possible approach to a complete understanding of the placement of points in $\mathbb{R}^2$ consists in studying the number of lines that pass through a fixed number of points of $\mathcal{N}$. This does not give *all possible unisolvent sets*, but the conjectural result claims these collections are *all unisolvent*, see [22]. This approach is convenient in this framework, since when considering particular collection of points, such as *principal lattices* or *regular lattices* [19] and some of their subsets, one may reduce the problem.

These considerations clearly also extend to greater k, in this context to $k = 1$ (that is, to edges) and $k = 2$ (that is, to faces). Some numerical results relate these two problems. In particular, for $k = 1$ we address the reader to [4] and for $k = 2$ to [3].

2.2. Interpolators and (co)-homological tools

For each T in $\mathcal{T}$, for each $k = 0, 1, 2, \ldots, \dim T$, a physical sysdofs induces an interpolator $\Pi^k(T) : \Lambda^k(T) \to X^k(T)$ by the equations:

$$w(\omega, s) = w\big(\Pi^k(T)\omega, s\big), \qquad \forall s \in \mathcal{F}^k(T). \tag{2.4}$$

The interpolator is well defined if the physical sysdofs is unisolvent. In fact, assume that $\Pi^k\omega$ and $\tilde{\Pi}^k\omega$ are two interpolators which satisfy (2.4). Then

$$w\Big(\Pi^k\omega - \tilde{\Pi}^k\omega\Big) = 0,$$

and unisolvence gives $\omega = 0$.

We are interested in interpolators that commute with the exterior derivative, that is, such that the following diagram is commutative

$$
\begin{array}{ccc}
\Lambda^k(T) & \xrightarrow{\ \mathrm{d}\ } & \Lambda^{k+1}(T) \\
\downarrow{\scriptstyle \Pi^k(T)} & & \downarrow{\scriptstyle \Pi^{k+1}(T)} \\
X^k(T) & \xrightarrow{\ \mathrm{d}\ } & X^{k+1}(T)
\end{array} \ .
$$

In [37], Zampa *et al.* showed that an interpolator induced by a physical sysdofs commutes with the exterior derivative if and only if the union

$$\mathcal{F}^\bullet(T) \doteq \bigcup_{k=0}^{\dim T} \mathcal{F}^k(T) = \bigcup_{k=0}^{\dim T} \bigcup_{S \leq T} \mathring{\mathcal{F}}^k(S) \tag{2.5}$$

is a cellular complex, that is, if and only if the boundary of a cell in $\mathcal{F}^{k+1}(T)$ is a union of cells in $\mathcal{F}^k(T)$.

If this is the case, we can consider k-chains $C_k(\mathcal{F}^\bullet(T))$ and k-cochains $C^k(\mathcal{F}^\bullet(T))$ over $\mathbb{R}$. Denote with δ the coboundary operator mapping k-cochains to $(k+1)$-cochains [26]. It is natural then to consider the de Rham map [36]

$$
\begin{aligned}
\mathfrak{R}^k : \quad & X^k(T) \to C^k(\mathcal{F}^\bullet(T)) \\
& \omega \mapsto \left(c \mapsto \int_c \omega \right).
\end{aligned}
\tag{2.6}
$$

Stokes theorem [29] implies that the de Rham map commutes with the exterior derivative, that is, it is a chain map. We can then arrange everything in a commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{R} & \lhook\joinrel\longrightarrow & X^0(T) & \overset{d}{\longrightarrow} & X^1(T) & \overset{d}{\longrightarrow} & \dots & \overset{d}{\longrightarrow} & X^{\dim T}(T) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow{\scriptstyle\mathfrak{R}^0} & & \downarrow{\scriptstyle\mathfrak{R}^1} & & & & \downarrow{\scriptstyle\mathfrak{R}^{\dim T}} & & \\
0 & \longrightarrow & \mathbb{R} & \overset{\psi}{\longrightarrow} & C^0(\mathcal{F}^\bullet(T)) & \overset{\delta}{\longrightarrow} & C^1(\mathcal{F}^\bullet(T)) & \overset{\delta}{\longrightarrow} & \dots & \overset{\delta}{\longrightarrow} & C^{\dim T}(\mathcal{F}^\bullet(T)) & \longrightarrow & 0
\end{array}
\tag{2.7}
$$

where ψ is the unique map that makes it commutative, sending 1 to the 0-cochain $c \mapsto 1$. Notice that the top sequence is exact since X is a compatible finite element system. We can thus give an equivalent characterization of unisolvence and minimality in terms of the de Rham map.

Lemma 2.6. *A physical sysdofs $\mathcal{F}$ is unisolvent (unisolvent and minimal) if and only if, for each T in $\mathcal{T}$ and for each k the de Rham map* (2.6) *is injective (an isomorphism of vector spaces).*

In [37], the authors showed that a unisolvent and minimal physical system of degrees of freedom that induces commuting interpolators must satisfy the following condition: the union of all cells in $\mathcal{F}^\bullet(T)$ "paves" (*i.e.* covers with disjoint interiors) T. If this is the case, the bottom sequence in (2.7) is exact. Moreover, the union of all small cells

$$
\mathcal{F}^\bullet(\mathcal{T}) \doteq \bigcup_{k=0}^{\dim \mathcal{T}} \mathcal{F}^k(\mathcal{T}) \doteq \bigcup_{k=0}^{\dim \mathcal{T}} \bigcup_{T \in \mathcal{T}} \mathcal{F}^k(T)
\tag{2.8}
$$

is a cellular complex and, in particular, is a *refinement* of $\mathcal{T}$. We can then consider global cochains $C^k(\mathcal{F}^\bullet(\mathcal{T}))$ over this cellular complex.

2.3. Generalised high order Whitney forms

It is well known that, if $\mathcal{T}$ is a simplicial complex, the complex of Whitney forms $\mathcal{P}_1^- \Lambda^\bullet(\mathcal{T})$ is isomorphic to the cochain complex $C^\bullet(\mathcal{T})$; see, for instance, [7]. In particular, Whitney forms satisfy the following properties:

(1) For each k-simplex T in $\mathcal{T}$, let ω^T be the associated Whitney form, then, for each k-simplex S in $\mathcal{T}$ it holds

$$
\int_S \omega^T = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}
$$

(2) For each k-simplex T and each $(k+1)$-simplex S, it holds

$$
\int_S \mathrm{d}\omega^T = \begin{cases} 1 & \text{if } T \text{ is a subsimplex of } S \text{ and is outward oriented with respect to } S, \\ -1 & \text{if } T \text{ is a subsimplex of } S \text{ and is inward oriented with respect to } T, \\ 0 & \text{if } T \text{ is not a subsimplex of } S. \end{cases}
$$

In other words, the exterior derivative coincides with the coboundary operator $\delta : C^k(\mathcal{T}) \to C^{k+1}(\mathcal{T})$.

This properties can be used to construct Yee-like schemes on simplicial meshes [11]. The following result generalizes such properties to an arbitrary finite element system equipped with an unisolvent and minimal physical system of degrees of freedom.

Lemma 2.7. *Let X be a finite element system and let $\mathcal{F}$ be an unisolvent and minimal physical system of degrees of freedom. Let $k = 0, 1 \ldots, \dim T$ and let $X^k(\mathcal{T})$ be the global discrete space. We denote with $\mathcal{B}X^k(\mathcal{T}) = \{\omega_1, \omega_2, \ldots, \omega_{\dim X^k(\mathcal{T})}\}$ the* generalised Lagrange basis *of $X^k(\mathcal{T})$ defined by*

$$\int_{s_i} \omega_j = \delta_{ij}. \tag{2.9}$$

Then the following statements hold:

(1) *The de Rham map is a bijection between $\mathcal{B}X^k(\mathcal{T})$ and the canonical basis of $\mathcal{C}^k(\mathcal{F}^\bullet(\mathcal{T}))$.*
(2) *The matrix $\mathfrak{D}$ of the exterior derivative d in the bases $\mathcal{B}X^k(\mathcal{T})$ and $\mathcal{B}X^{k+1}(\mathcal{T})$ is given by*

$$\mathfrak{D}_{ij} = \begin{cases} 1 & \text{if } s_j \text{ is a subcell of } s_i \text{ and is outward oriented with respect to } s_i, \\ -1 & \text{if } s_j \text{ is a subcell of } s_i \text{ and is inward oriented with respect to } s_i, \\ 0 & \text{if } s_j \text{ is not a subcell of } s_i. \end{cases} \tag{2.10}$$

Proof. The first claim is a simple consequence of the definition of the de Rham map:

$$[\mathcal{R}\omega_j](s_i) = \int_{s_i} \omega_j = \delta_{ij}.$$

The second claim follows from (2.9) and Stokes Theorem:

$$\mathfrak{D}_{ij} = \int_{s_i} \mathrm{d}\omega_j = \int_{\partial s_i} \omega_j = \sum_{s_k \in \partial s_i} o(s_i, s_k) \int_{s_k} \omega_j = \sum_{s_k \in \partial s_i} o(s_i, s_k) \delta_{kj}.$$

Here $o(s_i, s_k)$ is the relative orientation of s_i and s_k. $\qquad\square$

The first practical consequence of the above lemma is that the exterior derivative of a discrete k-form ω in the generalised Lagrange basis can be computed *explicitly* in a purely *combinatorial* way, while for a general basis one must solve the variational problem: Find $\sigma \in X^{k+1}(\mathcal{T})$ such that

$$(\sigma, \tau)_{L^2\Lambda^{k+1}(\Omega)} = (d\omega, \tau)_{L^2\Lambda^{k+1}(\Omega)} \quad \text{for each } \tau \in X^{k+1}(\mathcal{T}).$$

Here Ω is the domain discretised by $\mathcal{T}$ and $L^2\Lambda^k(\Omega)$ is the L^2 product between differential k-forms. In other words, our approach avoids the inversion of the mass matrix. Similarly, the "generalised stiffness" matrix $\mathfrak{S}$ with entries $\mathfrak{S}_{ij} = (d\omega_i, d\omega_j)_{L^2\Lambda^{k+1}(\Omega)}$ can be directly computed as $\mathfrak{S} = \mathfrak{D}^T \mathfrak{M} \mathfrak{D}$, where $\mathfrak{M}$ is the mass matrix for $(k + 1)$ forms.

Remark 2.8. We remark that these considerations remain true even with a different basis (for instance, the Bernstein one), as one can always change basis to the generalised Lagrange basis, compute the exterior derivative and then go back to the desired basis. The first step requires the inversion of the global generalised Vandermonde matrix, which is much cheaper than the inversion of the mass matrix since it can be computed efficiently with a local elimination procedure as explained in [9, 30].

Remark 2.9. To complete the analogy with Yee-like schemes we need a discrete Hodge star operator. It is well known that the mass matrix $\mathfrak{M}$ can play this role [35], but, in order to maintain the explicit character of the method, we would prefer a diagonal operator or, at least, an operator with sparse inverse and, clearly, such that the accuracy of the method is not degraded. This will be the object of future research.

3. Physical degrees of freedom for the second family

In this section we will construct a physical sysdofs for the finite element system $X^k(T) = \mathcal{P}_{r-k}\Lambda^k(T)$ with $r \geq 2$ in the two-dimensional case. Since the majority of the following results are general, we claim and prove them in the case of an n-simplex T; the specific case of interest here is immediately obtained for $n = 2$. We will exploit features of $\mathbb{R}^2$ only for the definition of $\mathcal{F}$ and hence in Theorem 3.5. We invite the reader to match the following construction with that for Nédélec first family [31] given in [2,13]. Recall that spaces $\mathcal{P}_{r-k}\Lambda^k(T)$ are defined as subspaces of differential k-forms $\Lambda^k(T)$ whose coefficient are polynomials of degree $\leq r - k$. These spaces are sometimes called *complete*, since they are precisely tensor products $\mathbb{P}_{r-k}(T) \otimes \mathrm{Alt}^k(T)$, being $\mathbb{P}_{r-k}(T)$ the space of polynomials of degree at most $r - k$ in n variables defined on T and $\mathrm{Alt}^k(T)$ that of linear alternating k-forms on (the tangent bundle of) T. When $k = 0$, $\mathcal{P}_r\Lambda^0(T)$ thus coincides with $\mathbb{P}_r(T)$. This makes easy the computation of

$$\dim \mathcal{P}_r\Lambda^k(T) = \dim \mathbb{P}_r(T) \cdot \dim \mathrm{Alt}^k(T) = \binom{r + \dim T}{\dim T}\binom{\dim T}{k}.$$

When $n = 2$, proxies of this sequence are known as Nédélec second family [32] and the central space is that of [12]. Note that we use the subscript $r - k$ instead of the classical r found in the literature, since the exterior derivative lowers the polynomial degree at each stage of the complex

$$\mathcal{P}_r\Lambda^0(T) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^1(T) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \mathcal{P}_{r-\dim T}\Lambda^{\dim T}(T).$$

We recall now the definition of *small simplex* from [17]. For $n = \dim T$ and $r \geq 0$ let $\mathcal{I}(n+1, r)$ be the set of multi-indices $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$ with nonnegative components and such that $|\boldsymbol{\alpha}| \doteq \alpha_0 + \ldots + \alpha_n = r$. If T is a simplex of dimension n and vertices $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$, we equip it with barycentric coordinates $\{\lambda_0, \ldots, \lambda_n\}$, *i.e.* the only (up to permutations) non negative degree 1 polynomials defined on T such that

$$\boldsymbol{x} = \sum_{i=0}^n \lambda_i \boldsymbol{x}_i, \qquad \sum_{i=0}^n \lambda_i = 1, \qquad \forall \boldsymbol{x} \in T.$$

For each $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r-1)$ we define the *small n-simplex* $s^{\boldsymbol{\alpha}}$ as the image of T under the homothety

$$z_{\boldsymbol{\alpha}} : \quad \boldsymbol{x} \mapsto z_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \frac{1}{r}\sum_{i=0}^n [\lambda_i(\boldsymbol{x}) + \alpha_i]\, \boldsymbol{x}_i\,. \tag{3.1}$$

Note that (3.1) is just the identity for $r = 1$. Small k-simplices are just k-subsimplices of small n-simplices and we denote them with $\Sigma_r^k(T)$. In particular $\Sigma_r^0(T)$ is the principal lattice $L_r(T)$, that is, the set of points with barycentric coordinates

$$\Sigma_r^0(T) \doteq \frac{1}{r}(\alpha_0, \ldots, \alpha_n), \qquad \boldsymbol{\alpha} \in \mathcal{I}(n+1, r).$$

If the reader is familiar with weights for Nédélec first family they might have noted that a slightly different definition of small simplices is usually provided. In particular, the term $\lambda_i(\boldsymbol{x})$ in (3.1) is usually omitted [17], so that overlappings are avoided. We shall see the reason of such a different choice in the subsequent of this section.

For $\boldsymbol{\xi} \in T$, define the affine tranformation

$$\tau_{\boldsymbol{\xi}} : \quad \boldsymbol{x} \mapsto \lambda_0(\boldsymbol{\xi})\boldsymbol{x} + \sum_{i=1}^n \lambda_i(\boldsymbol{\xi})\boldsymbol{x}_i. \tag{3.2}$$

Note that the map (3.2) is invertible if and only if $\lambda_0(\boldsymbol{\xi}) \neq 0$. We let $\tau_{\boldsymbol{\xi}}^*$ denote the pullback with respect to $\tau_{\boldsymbol{\xi}}$. We have the following.

Lemma 3.1. *Let $\omega \in \mathcal{P}_{r-\dim T}\Lambda^{\dim T}(T)$ be such that*

$$\int_T \tau_{\boldsymbol{\xi}^*}\omega = 0, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^{\dim T}.$$

Then $\omega = 0$.

Proof. This is a direct consequence of Lemma 3.12 from [17]. $\qquad\square$

It is well known that, when T is a non degenerate simplex in $\mathbb{R}^{\dim T}$, the principal lattice $L_r(T)$ is a poised set for $\mathbb{P}_r(\mathbb{R}^{\dim T})$; see [19,33]. Since $\dim \mathbb{P}_r(\mathbb{R}^{\dim T}) = \dim \mathcal{P}_r\Lambda^{\dim T}(T)$, one has $|L_r(T)| > \dim \mathcal{P}_{r-\dim T}\Lambda^{\dim T}(T)$. Thus, there exist subsets of $L_r(T)$ that are poised for $\mathbb{P}_{r-\dim T}(\mathbb{R}^{\dim T})$. With a slight abuse of notation, if Γ is a subset of $L_r(T)$ which is poised for $\mathbb{P}_{r-\dim T}(\mathbb{R}^{\dim T})$, we call such a set a *poised subset* of $L_r(T)$.

Theorem 3.2. *Let $\Gamma = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N^{\dim T}(T)}\}$ be a poised subset of $L_r(T)$ with $N^{\dim T}(T) = \dim \mathbb{P}_{r-\dim T}(\mathbb{R}^{\dim T})$ such that $\lambda_0(\boldsymbol{\xi}_i) > 0$ for $i = 1, \ldots, N^{\dim T}(T)$. Let $\omega \in \mathcal{P}_{r-\dim T}\Lambda^{\dim T}(T)$ be such that*

$$\int_{\tau_{\boldsymbol{\xi}}(T)} \omega = 0, \qquad \forall \boldsymbol{\xi} \in \Gamma.$$

Then $\omega = 0$.

Proof. The map

$$\boldsymbol{\xi} \mapsto \int_{\tau_{\boldsymbol{\xi}}(T)} \omega$$

is a polynomial of degree $r - \dim T$ in $\dim T$ variables which belongs to $\mathbb{P}_{r-\dim T}(\mathbb{R}^{\dim T})$. Such a polynomial vanishes by hypothesis on $|\Gamma| = \dim \mathbb{P}_{r-\dim T}(T)$ points of a poised set for $\mathbb{P}_{r-\dim T}(\mathbb{R}^{\dim T})$, therefore it is zero for each $\boldsymbol{\xi} \in \mathbb{R}^{\dim T}$. It follows that

$$\int_{\tau_{\boldsymbol{\xi}}(T)} \omega = \int_T \tau_{\boldsymbol{\xi}}^*\omega = 0, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^{\dim T}$$

and hence Lemma 3.1 implies that $\omega = 0$. $\qquad\square$

As an example of set Γ, we may pick any set satisfying the GC condition [14] (see also [6,21] for higher dimensional counterparts). Some explicit examples can be found in [8,24] and we offer more in a recursive fashion in the following.

We define $\mathcal{F}$ as follows. Let T be a 2-simplex. For $k = 0$, $\mathcal{F}^0(T)$ is just the principal lattice $L_r(T)$. For $k = 2$ we consider the GC set $\Gamma_r = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N^2(T)}\}$, which is a subset of $L_r(T)$ of cardinality $N^2(T) = \dim \mathcal{P}_{r-2}\Lambda^2(T) = \frac{r(r-1)}{2}$. For $i = 1, \ldots, N^2(T)$, define the subset

$$\Gamma_r(i) \doteq \{\boldsymbol{\xi} \in \Gamma \mid \lambda_0(\boldsymbol{\xi}) < \lambda_0(\boldsymbol{\xi}_i)\}$$

We define $\mathcal{F}^2(T)$ as the set $\{s_1, \ldots, s_{N^2(T)}\}$ where

$$s_i = \overline{\tau_{\boldsymbol{\xi}_i}(T) \setminus \left(\bigcup_{\boldsymbol{\xi} \in \Gamma_r(i)} \tau_{\boldsymbol{\xi}}(T)\right)}. \tag{3.3}$$

The closure is needed to preserve the structure of cell. Finally, we define $\mathcal{F}^1(T)$ as the subset of $\Sigma_r^1(T)$ made of those small 1-simplices that are on the boundary of cells in $\mathcal{F}^2(T)$. Note that this choice ensures that $\mathcal{F}^\bullet(T)$ is a cellular complex.
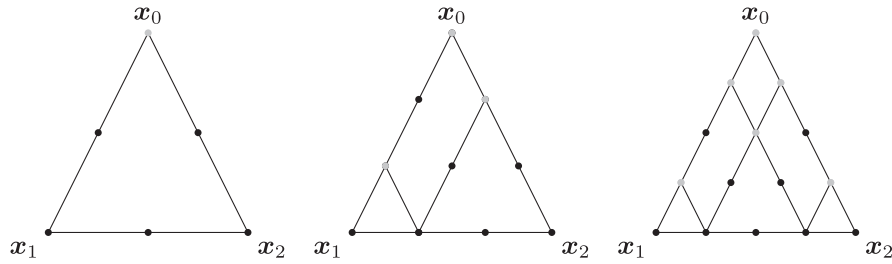
FIGURE 1. Cells of $\mathcal{F}$ for $r = 2$, $r = 3$ and $r = 4$, *left* to *right*. Gray dots represent the set Γ_r, that is, vertices of the triangles considered as (small) 2-simplices.

We now propose a possible choice of Γ_r for each polynomial degree r. We identify each point x of T with the triple $(\lambda_0(x), \lambda_1(x), \lambda_2(x))$ (*e.g.* the barycenter is $(1/3, 1/3, 1/3)$). Let

$$\Gamma_r = \begin{cases} \{(1,0,0)\} & \text{if } r = 2, \\ \tau_{\boldsymbol{\zeta}_r}(\Gamma_{r-1}) \cup \Delta_r & \text{if } r > 2, \end{cases} \tag{3.4}$$

where

$$\boldsymbol{\zeta}_r = \begin{cases} \left(\frac{r-1}{r}, 0, \frac{1}{r}\right) & \text{if } r \text{ is odd}, \\ \left(\frac{r-1}{r}, \frac{1}{r}, 0\right) & \text{if } r \text{ is even}, \end{cases}$$

$$\Delta_r = \begin{cases} \left\{\frac{1}{r}(i, 1-i, 0) \text{ for } i = 1, \dots, r, \ i \neq \frac{r+1}{2}\right\}, & \text{if } r \text{ is odd}, \\ \left\{\frac{1}{r}(i, 0, 1-i) \text{ for } i = 1, \dots, r, \ i \neq \frac{r}{2}\right\}, & \text{if } r \text{ is even}. \end{cases}$$

For example Γ_3 is

$$\Gamma_3 = \left\{\left(\frac{2}{3}, 0, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, 0\right), (1, 0, 0)\right\},$$

since $\tau_{(2/3,0,1/3)}$ maps $(1,0,0)$ to $(2/3,0,1/3)$ and $\Delta_3 = \{(1/3, 2/3, 0), (1, 0, 0)\}$. Similarly, Γ_4 is given by

$$\Gamma_4 = \left\{\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}, 0\right), \left(\frac{3}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, 0, \frac{3}{4}\right), \left(\frac{3}{4}, 0, \frac{1}{4}\right), (1, 0, 0)\right\}.$$

See Figure 1 for a depiction of the set Γ_r and the resulting cells $\mathcal{F}$ for $r = 2$, 3 and 4.

Remark 3.3. The recursiveness in the definition of Γ_r gives a hierarchy on the weights associated with these cells. In fact, as degree r is increased by one, the associated family $\mathcal{F}$ is obtained by adding a stripe on one side of the triangle, as shown in Figure 2.

Before proving unisolvence, we check that Γ_r has the right cardinality. This is immediate from Remark 3.3.

Lemma 3.4. *The set Γ_r has cardinality $|\Gamma_r|$ equal to the dimension of $\mathcal{P}_{r-2}\Lambda^2(T)$, that is*

$$|\Gamma_r| = \frac{r(r-1)}{2}.$$

Proof. We use induction on r. The result clearly holds for $r = 2$, see Figure 1. For $r > 2$ the sets $\tau_{\boldsymbol{\zeta}_r}(\Gamma_{r-1})$ and Δ_r are disjoint, therefore the cardinality of Γ_r is given by

$$|\Gamma_r| = |\Gamma_{r-1}| + |\Delta_r| = \frac{(r-1)(r-2)}{2} + r - 1 = \frac{r(r-1)}{2}.$$
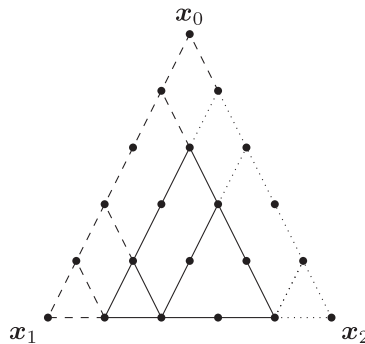
This concludes the proof. $\qquad\square$

FIGURE 2. Cells of $\mathcal{F}$ for $r = 5$. Step from $r = 3$ to $r = 4$ are obtained by adding the dotted part, step from $r = 4$ to $r = 5$ is obtained by adding the dashed part.

To prove unisolvence of weights here defined we shall work as follows. Consider the sequence

$$\mathcal{P}_r \Lambda^0(T) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1} \Lambda^1(T) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-2} \Lambda^2(T). \tag{3.5}$$

The first and the last space are isomorphic under the action of the (smooth) Hodge star operator $\star$ [1]. This rather easy fact induces an interesting consequence, which consists in the fact that techniques adopted to prove unisolvence of the spaces at the extremity of (3.5) are very close. On the contrary, unisolvence for the central space is obtained without direct computations but just relying on the structure of the sequence (3.5) itself.

We are ready to prove the unisolvence of $\mathcal{F}$.

Theorem 3.5. *Let $\mathcal{F}$ be the physical system of degrees of freedom defined above, that is, $\mathcal{F}^0(T) = L_r(T)$, $\mathcal{F}^2(T)$ is defined by formula (3.3) where Γ_r is given by (3.4) and $\mathcal{F}^1(T)$ is uniquely determined by the requirement that $\mathcal{F}^\bullet(T)$ is a cellular complex. If the assumptions of Theorem 3.2 hold, then $\mathcal{F}$ is a unisolvent and minimal physical sysdofs.*

Proof. The minimality holds by construction for $k = 0$ and $k = 2$. For $k = 0$, unisolvence is just the standard Lagrange unisolvence on poised sets. For $k = 2$, let $\omega \in \mathcal{P}_{r-2} \Lambda^2(T)$ and assume that $w(\omega, s) = 0$ for each $s \in \mathcal{F}^2(T)$. Then, by linearity of the integral, it follows that

$$\int_{\tau_{\boldsymbol{\xi}}(T)} \omega = \int_T \tau_{\boldsymbol{\xi}}^* \omega = 0, \quad \forall \boldsymbol{\xi} \in \Gamma.$$

Then Theorem 3.2 implies $\omega = 0$. Finally, for $k = 1$, consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \overset{\iota}{\hookrightarrow} & \mathcal{P}_r \Lambda^0(T) & \overset{\mathrm{d}}{\longrightarrow} & \mathcal{P}_{r-1} \Lambda^1(T) & \overset{\mathrm{d}}{\longrightarrow} & \mathcal{P}_{r-2} \Lambda^2(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow{\scriptstyle\mathfrak{R}^0} & & \downarrow{\scriptstyle\mathfrak{R}^1} & & \downarrow{\scriptstyle\mathfrak{R}^2} & & \\
0 & \longrightarrow & \mathbb{R} & \overset{\psi}{\longrightarrow} & C^0(\mathcal{F}^\bullet(T)) & \overset{\delta}{\longrightarrow} & C^1(\mathcal{F}^\bullet(T)) & \overset{\delta}{\longrightarrow} & C^2(\mathcal{F}^\bullet(T)) & \longrightarrow & 0.
\end{array}
$$

We already know that the rows are exact and we have just showed that the maps $\mathfrak{R}^0$ and $\mathfrak{R}^2$ are isomorphisms. Then, by the Five Lemma (see [26], Section 2.1) it follows that also $\mathfrak{R}^1$ is an isomorphism. In particular minimality holds also for $k = 1$. $\qquad\square$

The idea of proving the unisolvence of the intermediate space $k = 1$ using the Five Lemma appeared for the first time in [37], but it was not exploited since a proof of the unisolvence for the case $k = 2$ was lacking. The

problem with the physical sysdofs defined in [37] is that the 2-cells cannot be written as differences of small 2-simplices as in (3.3) and therefore Theorem 3.2 does not apply.

We remark an interesting aspect. For $k = 1$, the set involved is the set of small simplices defined in [4], which is a subset of that of *small simplices* introduced by Bossavit in [34]. Interestingly, for $k = 2$ one does not find its 2-dimensional counterpart, but the set $\Sigma_r^2(T)$ defined in [2]. To the authors' knowledge, this is the first construction in which those sets appears paired in such a natural fashion.

3.1. Translation to the language of vector calculus

In order to translate this this work to the classical language of vector calculus, we shall express the sequence

$$\mathcal{P}_r\Lambda^0(T) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^1(T) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-2}\Lambda^2(T)$$

in terms of the corresponding *proxy fields* and the associated differential operators. This yields two [7] sequences,

$$\mathbb{P}_r(T) \xrightarrow{\nabla} [\mathbb{P}_{r-1}(T)]^2 \xrightarrow{\mathrm{rot}} \mathbb{P}_{r-2}(T) \tag{3.6}$$

and

$$\mathbb{P}_r(T) \xrightarrow{\mathbf{rot}} [\mathbb{P}_{r-1}(T)]^2 \xrightarrow{\mathrm{div}} \mathbb{P}_{r-2}(T), \tag{3.7}$$

with $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $\mathrm{rot} = \frac{\partial}{\partial y} - \frac{\partial}{\partial x}$, $\mathbf{rot} = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)$ and $\mathrm{div} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. These two sequences are related by the Hodge star operator, the duality $\star : \Lambda^1(T) \to \Lambda^{n-1}(T)$, which in two dimensions is represented by the rotation matrix

$$\star = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note, in particular, that $\mathbf{rot} = \star \circ \nabla$ and $\mathrm{rot} = \mathrm{div} \circ \star$. This also gives the correspondence, in the context of $n - 1 = 1$ and $k = 1$, between Nédélec elements of second kind for $H(\mathrm{rot})$ and Brezzi-Douglas-Marini elements for $H(\mathrm{div})$. It is well known that degrees of freedom for the former family involve line integrals of *tangential* components, whereas the latter line integrals of *normal* components. As a consequence, if we identify the 1-form $\omega = p(x,y)\mathrm{d}x + q(x,y)\mathrm{d}y$ with the vector field $\mathbf{w} = (p(x,y), q(x,y))$, weights for the central space of (3.6) read as

$$\int_e \mathbf{w} \cdot t_e, \quad e \in \mathcal{F}^1(T),$$

whereas if we identify ω with $\mathbf{w} = (q(x,y), -p(x,y))$, weights for the central space of (3.7) read as

$$\int_e \mathbf{w} \cdot n_e, \quad e \in \mathcal{F}^1(T).$$

Here t_e and n_e denote the tangent and the normal vector to the small edge e respectively. In both cases unisolvence is provided by Theorem 3.5 and we may read the above definitions simply as different realisations of the same differential complex. This is peculiar to the case $n - 1 = 1$.

4. Numerical tests

We offer a computational proof of unisolvence exploiting Lemma 2.3. We compute the conditioning number of the Vandermonde matrices of the sequence

$$\mathcal{P}_r\Lambda^0(T) \to \mathcal{P}_{r-1}\Lambda^1(T) \to \mathcal{P}_{r-2}\Lambda^2(T),$$

for $r - 2 = 1, \ldots, 4$. These quantities are reported in Table 1 and confirm, up to the considered degree, the theoretical statement proved in Theorem 3.5. The basis chosen for such computations is the monomial one, and barycentric coordinates offer a compact way to visualise it. In particular, when $k = 0$, it is defined as $\boldsymbol{\lambda^\alpha}$

TABLE 1. Conditioning number of the Vandermonde matrix for $k = 0, 1, 2$ computed with respect to the monomial basis.

r	$k = 0$	$k = 1$	$k = 2$
1	3.7320×10^0	4.4985×10^0	3.1682×10^1
2	3.0969×10^1	2.3281×10^1	5.2130×10^2
3	3.1245×10^2	8.6268×10^1	9.3809×10^3
4	3.4290×10^3	5.6267×10^2	1.3525×10^6
5	3.9513×10^4	2.9791×10^3	–
6	4.7004×10^5	–	–

TABLE 2. Conditioning number of the Vandermonde matrix for $k = 0, 1, 2$ computed with respect to the Bernstein basis.

r	$k = 0$	$k = 1$	$k = 2$
1	1.0000×10^0	4.4985×10^0	9.2035×10^0
2	3.3319×10^0	1.3274×10^1	8.0907×10^1
3	8.1677×10^0	2.6596×10^1	8.4783×10^2
4	2.5917×10^1	6.9064×10^1	1.0571×10^5
5	6.6004×10^1	1.8071×10^2	–
6	1.8279×10^2	–	–

with $|\boldsymbol{\alpha}| = r$. When $k = 1$ it is defined as $\{\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \mathrm{d}\lambda_1, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \mathrm{d}\lambda_2\}$ with $|\boldsymbol{\alpha}| = r$. Finally, for $k = 2$, such a basis is $\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2$, again with $|\boldsymbol{\alpha}| = r$. Results for $r = 1, \ldots, 6$ are reported in Table 1. To improve condition numbers Bernstein bases or orthogonal polynomials shall be taken into account. In fact, we report in Table 2 the same computations of Table 1 performed with respect to the Bernstein basis. These results show a significant improvement of the conditioning of the generalised Vandermonde matrix. This is in accordance with what has been already observed for Nédélec first family $\mathcal{P}_r^- \Lambda^k(T)$, see [13]. However, as stated in Lemma 2.5, unisolvence is independent from the choice of the basis for $\mathcal{P}_r \Lambda^k(T)$, and thus even more convenient basis may be found. We leave the problem of further optimising such a quantity to future investigations.

Remark 4.1. We stress that Tables 1 and 2 shall be read *diagonally*. In particular, when the degree for $k = 0$ is r, the corresponding data for $k = 1$ and $k = 2$ are, respectively, those associated with $r - 1$ and $r - 2$.

4.1. Some interpolation tests

In Section 2.2 we have defined how an interpolator can be constructed using weights and we have briefly discussed its features. In particular, we showed that under the hypothesis of Theorem 3.5 such an operator is well defined and commutes with the exterior derivative. We now give an explicit meaning of this fact, using weights to interpolate a 0-form ω and its differential $\mathrm{d}\omega \in \Lambda^1(T)$. For ease of the reader we deal with the standard 2-simplex. This is not restrictive, since one may always reduce to this case by passing to barycentric coordinates. We thus consider a 0-form

$$\omega = e^x \sin(\pi y),$$

whence

$$\mathrm{d}\omega = e^x \sin(\pi y)\mathrm{d}x + \pi e^x \cos(\pi y)\mathrm{d}y.$$

TABLE 3. Trend of $\omega - \Pi\omega$ with respect to the 0-norm for the 1-form ω above defined and its potential. The 0-norm for of the function $k = 0$ is approximately 1.7319 whereas $\|\omega\|_0 \sim 2.5334$ for the case $k = 1$.

	$k = 0$	$k = 1$
r	$\|\omega - \Pi\omega\|_0$	$\|\omega - \Pi\omega\|_0$
1	–	2.5334
2	0.3377×10^0	1.1224
3	0.6967×10^{-1}	0.4292
4	0.1792×10^{-1}	0.0782
5	0.1600×10^{-2}	0.0171
6	0.4314×10^{-3}	–
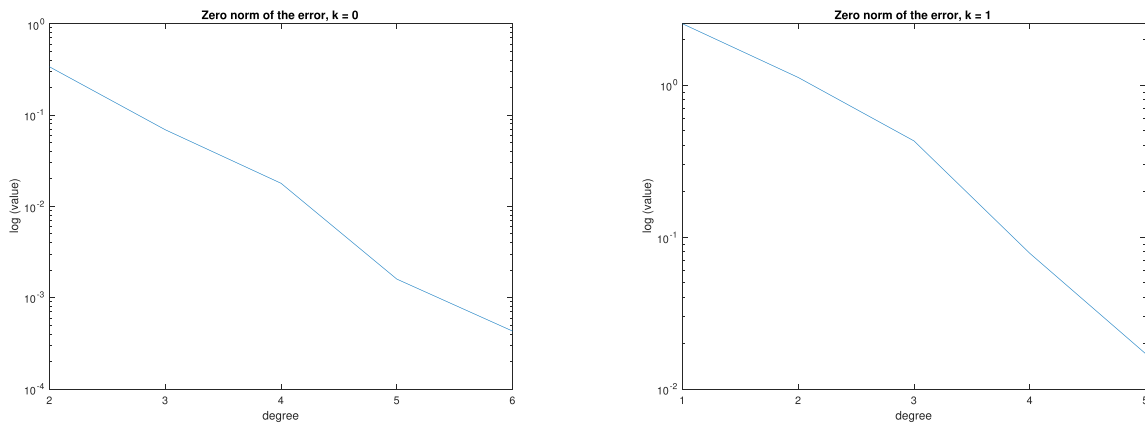
FIGURE 3. Plot of the convergence, a comparison for the nodal case $k = 0$ and the simplicial case $k = 1$ in semi-logarithmic scale. *Left*, the case for $k = 0$ and *right*, that for $k = 1$. Notice the degree shift, explained by the sequence.

We interpolate by means of the interpolator (2.4) and study the convergence as r increases. The most informative norm for such a situation is the 0-norm [25], which is defined as

$$\|\omega\|_0 \doteq \sup_{c \in \mathcal{C}_k(T)} \frac{1}{|c|_0} \left| \int_c \omega \right|, \tag{4.1}$$

being $|c|_0$ the k-th volume of the k-simplex T and $\mathcal{C}_k(T) \doteq \mathcal{C}_k(\mathcal{F}^\bullet(T))$ the set of all possible k-chains supported in T. Results are reported in Table 3, where a comparison with the corresponding points for $k = 0$ is included, and shown in Figure 3.

5. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we have proposed new physical degrees of freedom for the second family $\mathcal{P}_{r-k}\Lambda^k$ in the two dimensional case. We have proved rigorously their unisolvence and we have showed their effectiveness with an interpolation test.

The three dimensional case is trickier. In principle one could use the same technique to construct unisolvent and minimal physical degrees of freedom for the case $k = 3$, but unisolvence and minimality of the intermediate spaces in the sequence, that is $k = 1$ and $k = 2$, will not follow trivially since the Five Lemma cannot be applied

in this situation. Note that, sacrificing the request of minimality, one may exploit inclusions $\mathcal{P}_{r-k}\Lambda^k(T) \subseteq \mathcal{P}_{r+1}^-\Lambda^k(T)$ and consider, for each k and n, the family of simplices $\Sigma_r^k(T)$ defined in Section 2.4 of [13] and the corresponding cells studied in Section 4.2.2 of [13]. In such a framework, unisolvence can be proved without using the Five Lemma. However, this choice would be far from being optimal, since the number of redundant k-simplices increases with $n - k$. This will be the object of future research.

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