XXIX Ciclo della Scuola di Dottorato

Tesi di Dottorato

Some aspects in Cosmology: Quantum fluctuations in non flat FLRW space-time and Gravitational mimetic models

Università degli Studi di TRENTO
Facoltà di Scienze Matematiche, Fisiche e Naturali
Dipartimento di Fisica

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Relatore:
Prof. Sergio Zerbini

Candidata:
Yevgeniya Rabochaya

The Universe looks more and more like a great thought
rather than a great machine.
James H.Jeans

О сколько нам открытий чудных
Готовит просвещенья дух,
И опыт, сын ошибок трудных,
И гений - парадоксов друг,
И случай, Бог изобретатель.
А.С.Пушкин

Introduction

This work is mainly divided in two parts and deals with some aspects of quantum field theory in curved space-time and some open problems related with the cosmological history of our Universe.

The Hawking discovery of the black hole radiation [1] is considered one of the most important predictions of quantum field theory in curved space-time and its effects have been studied in several papers and are quite robust [2]–[11]. In fact, the issue according to which the black holes are not so black but may emit radiation due to quantum effects near to their horizon permits to exploring some important thermodynamic proprieties related to the black hole solutions. Given the existence of a trapping horizon one may compute the surface gravity, the Misner Sharp Mass of the black hole and the entropy, that in General Relativity satisfies the Area Law and is equal to one quarter of the area of the horizon itself. An investigation of the First Law of thermodynamics, or, alternatively, a direct computation of Hawking radiation make possible the identification of the surface gravity with the temperature of the black hole itself.

It may be interesting to know if the formalism is gauge coordinate invariant, or, in other words, if, given a metric with a related horizon, the surface gravity can still be interpreted as a temperature. This is the case of the de Sitter metric, where the inverse of the Hubble parameter corresponds to the radius of a cosmological horizon which traps outside the radiation. In order to investigate the thermodynamic proprieties of de Sitter metric, one must deal with quantum field theory in curved space-time.

In our work, we approach the problem by making use of the Unruh-De Witt detector in curved space-time, which is a generalization of the well-known Unruh detector in Minkowski space [5]. An idealized detector is a quantum system with different internal energy states and coupled with a scalar field in its vacuum state along a given trajectory. Thus, a transition of the detector from its ground state to an excited one may be interpreted as the detection of a particle with the corresponding energy and associated temperature. The Unruh effect in Minkowski space-time predicts that the temperature revealed by the detector is proportional to the acceleration of the detector itself. However, in the presence of gravity, we expect an additional contribution from the gravitational field. We carefully investigate the Wightman function in curved Friedmann-Lemaitre-Robertson-Walker metric, which is the starting point to derive the response function of the detector. After that, we apply the result to the De Sitter space. From the response function and the transition probability of the detector, we read off the temperature. Here, we remember that the homogeneity of the Friedmann-Lemaitre-Robertson-Walker metric leads to three possible topologies for the spatial geometry, namely the Euclidean flat space, the spherical space, and the hyperbolic space. The computation of the flat case is quite easy and show that the detected temperature is proportional to the acceleration of the detector plus an additional term corresponding to the surface gravity of the de Sitter horizon, confirming in this way the expected result. In the non trivial curved case the calculation is quite involved, and the result can be obtained by posing the acceleration of the detector equal to zero, such that only the temperature of the de Sitter space can be revealed. Once again, we confirm that the temperature is identified with the surface gravity of the de Sitter horizon, either in spherical and hyperbolic spatial cases.

As a more general approach, we can compute the temperature of a gravitational manifold by starting from the vacuum fluctuations of a massless scalar field in its ground state. We use this method explore the proprieties of Friedmann-Lemaitre-Robertson-Walker space-time and to confirm the results obtained for the de Sitter space. Interesting, in this case the contribution of the acceleration of the observer can be taken into account in the non-flat solution as well. Moreover, by using our general results we are able to show that the intrinsic temperature of the Milne Universe is equal to zero. This is not surprising, since the Milne Universe corresponds to the Minkowski space-time in expanding coordinates [12].

Today we got the evidence that our Universe is dominated by two dark components [13]: dark matter [14]–[18] and dark energy [19]–[23]. The dark matter is a form of non-baryonic matter which cannot be detected since does not couple with electromagnetic fields and its presence was first revealed ninety years ago from the observation of the velocity rotation spectrum of the galaxies. Dark energy is strictly connected by the more recent discovery of the accelerated expansion of our Universe which requires the presence of a repulsive form of energy.

Dark energy and dark matter are still under investigation in the standard cosmological model, since some unsolved issues concerning their interpretation are present. In fact, it is known that dark energy effect is well parametrized by the inclusion of a very small and positive cosmological constant in the framework of General Relativity. As a consequence, we are able to describe the accelerated expansion of the Universe today, but the coincidence and cosmological constant problems arise. In other words, the physical origin of the cosmological constant and the reason for which the amount of dark energy in the present Universe is on the same order of magnitude of (dark) matter amount remain unknown. Moreover, in the elementary particle physics exist few candidates for dark matter, but they have not been found experimentally and several alternatives are allowed.

Today it is also well accepted the idea according to which the Universe underwent another period of strong accelerated expansion after the Big Bang, namely the inflation. The existence of the primordial acceleration was first proposed by Guth [24] and Sato [25] in 1981 in order to solve the problem of initial conditions of our Friedmann Universe and to explain the thermalization of the observable Universe [26]–[29], and in the past years several approaches to the inflationary paradigm have been proposed (see Refs. [12, 30, 31] for some reviews).

In this context, the theories of modified gravity where the Hilbert-Einstein action of General Relativity is modified with the introduction of more general functions of the curvature invariants appear quite promising in the attempt of finding alternative descriptions of the dark energy, the inflation and the dark matter phenomenologies [32]-[49].

A particularly interesting theory of modified gravity which has emerged in the past few years is represented by mimetic gravity, first introduced by Chamseddine and Mukhanov in Ref. [50]. Mimetic gravity is a Weyl-symmetric extension of General Relativity, related to the latter by a singular disformal transformation, wherein the appearance of a dust-like perfect fluid can mimic cold dark matter. The physical metric is parametrized by an auxiliary metric and a scalar field, such that the conformal degree of freedom results to be isolated. The scalar field, dubbed "mimetic field", leads to a dark-matter contribution in the field equations and at the cosmological level is identified with the time. Within the framework of mimetic gravity it is also possible to provide a unified geometrical explanation for several cosmological scenarios (like the late-time and the early-time acceleration), making it a very attractive theory [51] (see also Ref. [52] or Ref. [53] for a general review). Thus, in mimetic gravity we can describe the dark components of the Universe as a purely geometrical effect, without the need of introducing additional matter fields departing from standard matter and radiation.

Another modification of General Relativity is given by the scalar-tensor theories of gravity, where a scalar field is coupled with the gravitational invariants (Ricci scalar, Einstein's tensor...) inside the action. For example, the first model of scalar-tensor theory was proposed by Brans & Dicke [54] in 1961, trying to incorporate the Mach's principle into the theory of gravity. The field equations of such theories are higher order differential equations and result to be much more complicated respect to the those of General Relativity. However, in 1974, Horndeski [55] proposed a class of scalar-tensor models where the field equations remain at the second order like in the theory of Einstein. Due to this peculiarity together with the possibility to reproduce a huge variety of interesting cosmological solutions, in the second part of this work we consider a class of mimetic Horndeski theory, where we identify the Horndeski field with a mimetic field thanks to the introduction of a constraint in the Lagrangian. Thus, we proceed to the investigation of the model in the Friedmann-Lemaitre-Robertson-Walker space-time. In this respect, by making use of a suitable potential for the mimetic field, several exact solutions for the inflation, the cosmological bounce and the dark energy together with the dark matter description are found.

Horndeski mimetic models are affected by a non-trivial problem, namely inside their framework the cosmological perturbations around the Friedmann Universe cannot propagate [56]. The study of cosmological perturbations is crucial especially in the context of the early-time expansion of the Universe, since the perturbations left at the end of inflation are at the origin of the anisotropies and permit to describing the structure of the galaxies. The problem arises from the fact that the mimetic constraint on the scalar field takes out the degree of freedom associated to the field itself, and it is not possible to have oscillating wave-like solutions for perturbations. We solve the problem by breaking the Horndeski structure of the model. In this way, the field equations result to be at higher order like in a general theory of modified gravity, but thanks to an accurate mechanism we can still deal with second order differential equations in the case of Friedmann-Lemaitre-Robertson-Walker space-time. As a consequence, all the Friedmann solutions found in the Horndeski framework can be recovered, but now it is also possible to propagate the perturbations and the theory is viable at the cosmological level.

To conclude our work, we study the cosmological bounce as an alternative description of the Big Bang theory: instead from an initial singularity, the expanding Universe emerges from a cosmological bounce following a contracting phase (see Ref. [57] for a review). We show that the modified Friedmann equations may lead to the cosmological bounce in the presence of loop quantum gravity effects or with a non-flat spatial topology. Here, we should mention that in the early phases of expansion of our Universe the spatial curvature could be different to zero.

The work is organized as follows.

In **Chapter 1** we will review and discuss the physics of the black holes. Given a trapping horizon, some fundamental thermodynamic quantities can be computed, namely the energy, the surface gravity and the entropy. Moreover, several attempts to define the black hole temperature can be carried out. In the specific, it is possible to derive a First law of thermodynamics for static and dynamic black holes by starting from the Einstein's field equations, or infer the temperature from the particle emission rate of Hawking radiation. As a result, one finds that the black hole temperature is proportional to the surface gravity.

Chapter 2 contains a study of the quantum field theory in non-flat Friedmann-Lemaitre- Robertson-Walker space-time with the aim of understanding some thermodynamical aspects of De Sitter Universe. Making use of quantum field theory, it is possible to compute the Unruh effect in Minkowski space and generalize it to curved space-time with the Unruh-De Witt detector. We calculate the Wightman function for non-flat Friedmann-Lemaitre- Robertson-Walker metric obtaining a general expression of the probability transition of the detector. We pose our attention in non-flat de Sitter space and we show that its temperature is proportional to the surface gravity of the de Sitter horizon. The same result can be reached by computing the vacuum expectation value of massless conformally coupled scalar field in non-flat Friedmann-Lemaitre-Robertson-Walker space-time. We discuss the case of the Milne Universe, recovering the pure Unruh effect of Minkowski space-time.

In **Chapter 3** we briefly review the open problems of modern cosmology and we introduce the mimetic gravity. We show how this kind of theory brings to the appearance of the dark matter at the cosmological level and discuss the problem of perturbations.

Chapter 4 is devoted to the investigation of a class of mimetic Horndeski models of gravity. Several exact Friedmann-Lemaitre- Robertson-Walker solutions are found by using different methods, like the so-called adiabatic invariant method, either in the presence and in the absence of ordinary matter, while the inclusion of the dark matter naturally emerges from the theory. Our models provide solutions for inflation, dark energy, finite future-time singularities and cosmological bounce. Finally, we show how modify the theory in order to describe the cosmological perturbations without destroying the feature of the model in the Friedmann Universe.

In **Chapter 5** we study the cosmological bounce as an alternative description of the Big bang theory. Bounce solutions of loop quantum gravity are compared with the ones obtained in non-flat Friedmann-Lemaitre- Robertson-Walker Universe.

Conclusions and final remarks are given in **Chapter 6**.

The present work is based on the following papers published in the referred journal and conference proceedings (Refs.[58]-[61]):

- Y. Rabochaya and S. Zerbini, "Quantum detectors in generic non flat FLRW space-times," Int. J. Theor. Phys. **55** (2016) no.5, 2682 [arXiv:1505.00998 [gr-qc]];
- Y. Rabochaya and S. Zerbini, "A note on a mimetic scalar-tensor cosmological model," Eur. Phys. J. C **76** (2016) no.2, 85 [arXiv:1509.03720 [gr-qc]];
- Y. Rabochaya, "Quantum fluctuations in FRLW space-time," to appear in the proceedings of 14th Marcel Grossmann Meeting on General Relativity (MGXIV), Rome,Italy,12-18 Jul 2015, arXiv:1511.07040 [gr-qc];
- I. Brevik, A. V. Timoshkin, Y. Rabochaya and S. Zerbini, "Turbulence Accelerating Cosmology from an Inhomogeneous Dark Fluid," Astrophys. Space Sci. **347** (2013) 203 [arXiv:1307.6006 [gr-qc]].

Units: We use units of $k_{\rm B} = c = \hbar = 1$. Moreover, when is explicitly present, the gravitational constant is given by $G_N^{-1/2} = M_{\rm Pl}$, where $M_{\rm Pl} \equiv 1.2 \times 10^{19}$GeV is the Planck mass.

Contents

Chapter 1

Black hole thermodynamics

Spherically symmetric metrics are using to describe the spherical galactic objects like the surfaces of planets, stars and so on. If the metric possesses a horizon located in the external space, we are dealing with a black hole. A black hole can be studied by starting from some thermodynamic quantities related to its horizon, namely the energy, the surface gravity and the entropy. Thus, several attempts to define a black hole temperature can be carried out. In the specific, one may derive a First law of thermodynamics for static and dynamic black holes or infer the temperature from the particle emission rate of Hawking radiation. As a result, we find that the black hole temperature is proportional to its surface gravity.

1.1 Metric horizons and black holes

Let us start by reviewing some basis about the geometry of spherically symmetric space-time. Any spherically symmetric four dimensional metric $g_{\mu\nu}$ can be locally expressed in the following form:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ij}(x^i)dx^i dx^j + \mathcal{R}^2(x^i)d\Omega_2^2\,, \qquad i,j \in \{0,1\}\,, \tag{1.1}$$

where $d\Omega_2^2$ is the metric of a generic two-dimensional maximally symmetric space (for example, the usual two sphere S^2, but also flat or hyperbolic topologies are allowed). In the above expression, the two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j \tag{1.2}$$

is referred to the space-time orthogonal to the maximally symmetric one. The related coordinates are $\{x^i\}$, while $\mathcal{R}(x^i)$ is the areal radius, and can be treated as a scalar field in the two dimensional normal space. A relevant scalar quantity in the reduced normal space is given by

$$\chi(x^i) = \gamma^{ij}(x^i)\partial_i\mathcal{R}(x^i)\partial_j\mathcal{R}(x^i)\,. \tag{1.3}$$

The two sphere with areal radius $\mathcal{R}(x^i)$ is called untrapped when $0 < \chi(x^i)$ and trapped when $\chi(x^i) < 0$ [62]. Thus, a trapping horizon, if exists, is the closure of a hypersurface foliated by marginal surfaces, on wich (marginal sphere),

$$\chi(x^i)\Big|_H = 0\,, \tag{1.4}$$

provided that $\partial_i\chi(x^i)|_H \neq 0$ (we use the suffix H for all quantities evaluated on the horizon). If the condition (1.4) is satisfied and $0 < \partial_i\chi(x^i)|_H$ (i.e. the trapped sphere remains inside), we will refer to the metric as a black hole solution[1]. The quasi-local Misner-Sharp gravitational energy is defined by [63]

$$E_{MS}(x^i) := \frac{1}{2G_N}\mathcal{R}(x^i)\left[1-\chi(x^i)\right]\,. \tag{1.5}$$

Here, G_N denotes the Newton's constant. E_{MS} is an invariant quantity on the normal space. In particular, the energy associated to the black hole horizon reads

$$E \equiv E_{MS}|_H = \frac{1}{2G_N}\mathcal{R}(x^i)|_H\,. \tag{1.6}$$

In the static case one can introduce the Killing vector fields $\xi_\mu(x^\nu)$ by using the covariant derivatives ∇_μ,

$$\nabla_\mu\xi^\nu(x^\nu) + \nabla^\nu\xi_\mu(x^\nu) = 0\,. \tag{1.7}$$

In this (stationary) case, thanks to the symmetry respect to the time, one has the time-like Killing vector field $\xi^\mu(x^\nu) = (1,0,0,0)$, from which it is possible to define a Killing surface gravity κ_K as

$$\kappa_K\xi^\mu(x^\nu) = \xi^\nu\nabla_\nu\xi^\mu(x^\nu)\,. \tag{1.8}$$

In the spherically symmetric, dynamical case, the real geometric object which generalizes the time-like Killing vector field is the Kodama vector field $\mathcal{K}(x^i)$ [64]. Given the metric in the form of (1.1), it is defined by

$$\mathcal{K}^i(x^i) := \frac{1}{\sqrt{-\gamma}}\,\varepsilon^{ij}\partial_j\mathcal{R}(x^i)\,,\; i=0,1 \qquad \mathcal{K}^i := 0\,,\; i\neq 0,1\,, \tag{1.9}$$

ε^{ij} being the completely antisymmetric Levi-Civita tensor on the normal space and γ the determinant associated with the metric γ_{ij}. In such a case, Hayward [65] proposed a surface gravity associated with dynamical horizon as

$$\kappa_H := \frac{1}{2}\Box_\gamma\mathcal{R}(x^i)\Big|_H\,, \tag{1.10}$$

where $\Box_\gamma$ is the Laplacian corresponding to the γ metric.

[1] In the Schwarzschild solution with $\gamma_{00}(r) = (1-2MG_N/r)$ and $\gamma_{11}(r) = (1-2MG_N/r)^{-1}$, one gets $\mathcal{R} = r$ and $\chi(r) = (1-2MG_N/r)$. In such a case, the marginal surface is located at $r_H = 2MG_N$, the trapped sphere is given by $r < 2MG_N$, while the untrapped one is given by $2MG_N < r$.

An interesting geometric dynamical identity holds true. This can be derived from the Einstein's equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \,, \tag{1.11}$$

where $G_{\mu\nu}$ is the Einstein's tensor inferred from the Ricci tensor $R_{\mu\nu}$ and its contraction $R = g^{\mu\nu} R_{\mu\nu}$, namely the Ricci scalar, while $T_{\mu\nu}$ is the stress-energy tensor of the matter contents of the space-time.

Let us introduce the normal space invariant

$$\mathrm{T}^{(2)}(x^i) = \gamma^{ij} T_{ij}(x^i) \,, \tag{1.12}$$

namely the reduced trace of the matter stress energy tensor. Then, making use of the components of the Einstein's equation, it is possible to show that, on the dynamical horizon [65],

$$\frac{\kappa_H}{G_N} = \frac{1}{2\mathcal{R}_H G_N} + 2\pi \mathcal{R}_H \frac{\mathrm{T}^{(2)}_H}{G_N} \,, \tag{1.13}$$

where $\mathcal{R}_H = \mathcal{R}(x^i)|_H$ and $\mathrm{T}^{(2)}_H = \mathrm{T}^{(2)}(x^i)|_H$. Introducing the area of the horizon $\mathcal{A}_H$ and the (formal) three-volume V_H enclosed by the horizon, with their respective 'thermodynamical' differentials $d\mathcal{A}_H = 8\pi \mathcal{R}_H d\mathcal{R}_H$ and $dV_H = 4\pi \mathcal{R}_H^2 d\mathcal{R}_H$ (in this moment, for the sake of simplicity, we are assuming a horizon with the topology of a sphere such that $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$), we get

$$\frac{\kappa_H}{8\pi G_N} d\mathcal{A}_H = d\left(\frac{\mathcal{R}_H}{2G_N}\right) + \frac{\mathrm{T}^{(2)}_H}{2G_N} dV_H \,. \tag{1.14}$$

This equation can be recast in the form of a geometrical identity, once the Misner-Sharp energy at the horizon (1.6) has been introduced. It reads

$$dE = \frac{\kappa_H}{2\pi} d\left(\frac{\mathcal{A}_H}{4G_N}\right) - \frac{\mathrm{T}^{(2)}_H}{2G_N} dV_H \,. \tag{1.15}$$

Thus, in analogy with the thermodynamic, since $\mathrm{T}^{(2)}_H/(2G_N)$ is a pressure term and, according with the area law $\mathcal{A}_H/(4G_N)$ is the entropy of the black hole, we may identify $\kappa_H/(2\pi)$ as the temperature associated to the horizon. This identification is better justified by making use of standard derivations of Hawking radiation [1, 66] (see also Refs. [67, 68]), as we will see in the next section.

1.2 The Hawking radiation

It is known that black holes are not so black. They may emit radiation due to quantum effects near to the horizon. To compute the emitted radiation, we present a short review of the tunneling method. This method is covariant and can be extended to the dynamical case [69, 70, 71], and to the study of decay of massive particles and particle creation by naked singularities [72].

In 2000 Parikh and Wilczek [68] introduced the so-called tunneling approach for investigating Hawking radiation. Here, we will consider a variant of their method, called Hamilton-Jacobi tunneling method [67, 73, 74, 75]. The tunneling method is based on the computation of the classical action I along a trajectory starting slightly behind the trapping horizon but ending in the bulk, and the associated amplitude in the form (in WKB approximation with the light speed $c = 1$ and $\hbar$ the Planck constant),

$$\text{Amplitude} \propto e^{i\frac{I}{\hbar}} \,. \tag{1.16}$$

Note that for classic evaporating black holes such a trajectory is forbidden. Thus, the related semi-classical emission rate Γ reads

$$\Gamma \propto |\text{Amplitude}|^2 \propto e^{-2\frac{\Im I}{\hbar}} \,. \tag{1.17}$$

The imaginary part of the classical action is due to deformation of the integration path according to the Feynman prescription, in order to avoid the divergence present on the horizon. As a result, one asymptotically gets a Boltzmann factor β, and a (Killing) energy ω_K appears, namely

$$\Gamma \propto e^{-\frac{\beta}{\hbar}\omega_K} \,. \tag{1.18}$$

Thus, the Hawking temperature T_K may be identified as

$$T_K = \frac{1}{\beta} \,. \tag{1.19}$$

It is crucial that the argument of the exponent be a coordinate scalar (invariant quantity), since otherwise no physical meaning can be addressed to the emission rate Γ.

To evaluate the action I, let us start with a generic static, spherically symmetric solution in four dimensions, written in Eddington-Finkelstein gauge, which, as it is well known, is regular gauge on the horizon,

$$ds^2 = -B(r)e^{2\alpha(r)}dv^2 + 2e^{\alpha(r)}dv\,dr + r^2 d\Omega_2^2 \,, \tag{1.20}$$

where $B(r)$ and $\alpha(r)$ are functions of the radial coordinate $r = \mathcal{R}$ only. We note that the use of singular gauges, as the Schwarzschild gauge, leads, in general, to ambiguities (it is also useless in the dynamical case). We also mention that in Parikh and Wilczek work one can find the treatment of back-reaction on the metric (see also Refs. [76, 77]), based on energy conservation, but here we will neglect it by assuming a short number of emitted particles.

From (1.20) we infer the following Killing surface gravity (1.8) and Hayward surface gravity (1.10),

$$\kappa_K = \mathrm{e}^{\alpha(r_H)}\frac{dB(r)}{dr}|_{r=r_H} \,, \quad \kappa_H = \frac{dB(r)}{dr}|_{r=r_H} \,. \tag{1.21}$$

Since we are dealing with static, spherically symmetric solution space-times, one may restrict to radial trajectories, and only the two-dimensional normal metric is relevant, such that the Hamilton-Jacobi equation for a (massless) particle reads

$$\gamma^{ij}\partial_i I \partial_j I = 2e^{\alpha(r)}\partial_v I \partial_r I + e^{2\alpha(r)}B(r)(\partial_r I)^2 = 0\,. \tag{1.22}$$

Thus, it turns out that

$$\partial_r I = \frac{2\omega_K}{e^{\alpha(r)}B(r)}\,. \tag{1.23}$$

Here, $\omega_K = -\partial_v I$ is the Killing energy of the emitted particle. In the near horizon approximation,

$$B(r) \simeq \left(\frac{dB(r)}{dr}\right)_H (r - r_H)\,. \tag{1.24}$$

Now, making use of Feynman prescription for the simple pole in $(r - r_H)$, one has

$$I = \int dr \partial_r I = \int dr \frac{2\omega_K}{e^{\alpha(r)}(dB(r)/dr)_H (r - r_H - i\varepsilon)}\,, \tag{1.25}$$

where the range of integration over r contains the location of the horizon r_H and $\epsilon \to 0^+$ has been introduced for removing the singularity at $r = r_H$. The imaginary part of the action results to be

$$\Im I = \frac{2\pi\omega_K}{e^{\alpha(r_H)}(dB(r)/dr)_H}\,, \tag{1.26}$$

and, from (1.18)–(1.19), the Hawking-Killing temperature finally reads

$$T_K = \frac{\kappa_K}{2\pi} = \frac{e^{\alpha(r)_H}(dB(r)/dr)_H}{4\pi}\,. \tag{1.27}$$

However, if one had introduced the Kodama energy $\omega_H = e^{-\alpha(r)_H}\omega_K$, one would have obtained the Hayward temperature

$$T_H = \frac{\kappa_H}{2\pi} = \frac{(dB(r)/dr)_H}{4\pi}\,. \tag{1.28}$$

Nevertheless, the emission rate is independent whatever the choice of Killing or Hayward prescription. Moreover, in the vacuum static case of General Relativity, $\alpha(r) = 0$ and the two definition coincide[2]. A detailed discussion about this issue can be found also in Ref. [78].

[2] In the static vacuum case of General Relativity one has the Schwarzschild solution with $\alpha(r) = 0$ and $B(r) = (1 - 2MG_N/r)$. The integration constant M corresponds to the Misner-Sharp mass of the black hole solution whose radius of horizon is located at $r_H = 2MG_N$.

In conclusion, let us return to Equation (1.15). In the static case, we may write

$$dE = T_K dS\,, \quad T_K = \frac{\kappa_K}{2\pi}\,, \tag{1.29}$$

where we introduced the black hole entropy (namely the well known Area Law) as

$$S = \frac{\mathcal{A}_H}{4G_N}\,. \tag{1.30}$$

More in general, for the dynamical case, if one uses the Hayward temperature and the Hayward surface gravity, one obtains

$$dE = T_H dS + p dV\,, \quad T_H = \frac{\kappa_H}{2\pi}\,, \tag{1.31}$$

where the matter pressure term is given by

$$p = \frac{\mathrm{T}^{(2)}_H}{2G_N}\,. \tag{1.32}$$

In the next section, we will try to generalize the formalism to the horizon associated to the dynamic patch of de Sitter (dS) space-time.

1.3 Dynamical horizons of de Sitter space-times

One of the important example that demonstrates the gauge independence of the formalism is given by de Sitter space-time. If a cosmological horizon exists (this is the case $\partial_i\chi(x^i)|_H < 0$ in (1.4), such that the trapped region remains outside), one may try to derive some thermodynamical proprieties. Let us consider three different patches, or coordinate systems, of dS space-time as vacuum solutions of Einstein's equation with Cosmological Constant Λ,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \quad \Lambda = 3H_0^2\,, \tag{1.33}$$

with H_0 a (positive) constant. The static patch reads,

$$ds^2 = -dt^2(1 - H_0^2 r^2) + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega_2^2\,. \tag{1.34}$$

Here, the areal radius is $\mathcal{R}(r) = r$ and the horizon is located at $r_H = 1/H_0$ with the related (Hayward) surface gravity $\kappa_H = H_0$.

The second patch is given by the expanding coordinates of the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + \mathrm{e}^{2H_0 t}\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)\,, \tag{1.35}$$

where the scale factor reads $a(t) = \exp[H_0 t]$ and H_0 coincides with the constant Hubble parameter, $H_0 = \dot{a}(t)/a(t)$, the dot denoting the derivative with respect to the cosmological time. In this case, $\mathcal{R}(t) = e^{H_0 t} r$, the dynamical horizon is $\mathcal{R}_H = 1/H_0$, and for the surface gravity one has $\kappa_H = H_0$.

Then, the global patch in non-flat (spherical) FLRW metric is given by

$$ds^2 = -dt^2 + \cosh^2(H_0 t)\left(\frac{dr^2}{(1 - H_0^2 r^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1.36}$$

with $a(t) = \cosh(H_0 t)$, H_0 being a mass scale for the radius. In this case the Hubble parameter reads $H = H_0 \tanh(H_0 t)$, the areal radius is $\mathcal{R}(t) = r\cosh(H_0 t)$ and (again) $\mathcal{R}_H = 1/H_0$ and $\kappa_H = H_0$.

In Fig. 1.1 we depict the behavior of the scale factor in the flat and in the non-flat spherical de Sitter Universe. In the first case we have an eternal expansion. In the second case we see that an expanding phase follows a contracting one and the scale factor reaches a minimum value at $t = 1/H_0$ (bounce solution).

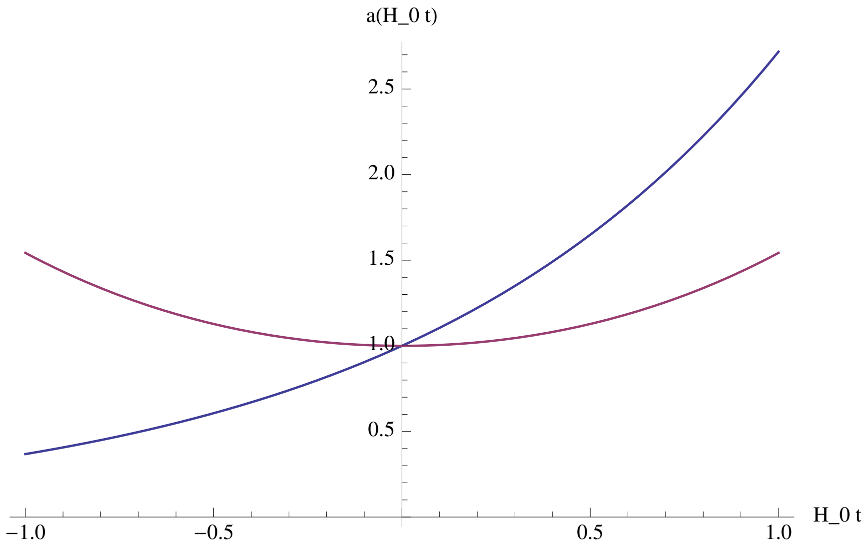

Figure 1.1: Scale factor of De Sitter Universe in unit of H_0 for flat (blue line) and spherical (red line) topologies.

In the preceding section, we have seen that the semiclassical WKB method leads to an asymptotic particle production rate, involving the Boltzmann factor and related "temperature". In the dynamical case, but for slow changes in the geometry, the question is: would possibly the quantity

$$T_H = \frac{\kappa_H}{2\pi} = \frac{H_0}{2\pi}, \tag{1.37}$$

be interpreted as a dynamical Hawking temperature? Probably, a complete answer is still missing (see Ref. [79, 80] for recent discussions), but we may provide a quantum theoretical approach to confirm such a result.

Chapter 2

Quantum fluctuations in non-flat de Sitter space-time

The De Sitter space-time possesses a cosmological horizon which traps the radiation outside and to which it is possible to assign a surface gravity. Can be such a surface gravity be interpreted as a temperature of the horizon? In this chapter, we will try to substantiate this idea, showing the gauge coordinate invariance of the black hole formalism. The Unruh-de Witt detector is a quantum system with different internal energy states. A transition of the detector from its ground state to an exited one can be interpreted as the detection of a particle with a given energy and temperature. The temperature will be proportional to the acceleration of the detector through the considered space-time, but will also get a contribute from the gravitational field, namely -in our case- from the temperature associated to the de Sitter horizon. To calculate this temperature, we will make use of the Wightman function in FLRW space-time. We will review the result for flat de Sitter metric and we will extend it to the non-flat case. We will show that the temperature of the de Sitter horizon is proportional to its surface gravity. In the last part of the chapter, we will get the same result by calculating the temperature of a massless scalar field at the thermal equilibrium in FLRW space-time. The specific case of the de Sitter metric will be point out. Interesting, we can use this approach to demonstrate how in the Milne Universe with negative spatial curvature the intrinsic temperature vanishes: this is not surprising, since the Milne Universe corresponds to the Minkowski space-time in expanding coordinates.

2.1 Quantum field theory in non-flat FLRW space-time

First of all, we review the quantization of a conformally coupled massless scalar field in a generic FRWL space-time [81]–[95]. In order to do it, it is convenient to introduce the conformal time η by means

$$d\eta = \frac{dt}{a(t)}\,, \tag{2.1}$$

$a \equiv a(t)$ being the scale factor of expanding Universe. Thus, the FRWL metric reads

$$ds^2 = a^2(\eta)(-d\eta^2 + d\Sigma_3^2)\,, \tag{2.2}$$

where the metric of the spatial section may be written as

$$d\Sigma_3^2 = \frac{dr^2}{1 - kh_0^2 r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{2.3}$$

Here, $k = 0, \pm 1$ determines the spatial topology (spherical for $k = 1$, flat for $k = 0$, and hyperbolic for $k = -1$) and h_0 is a mass scale, related to the scalar Ricci curvature as

$$R = 6\left(\frac{1}{a^3}\frac{d^2a}{d\eta^2} + k\frac{h_0^2}{a^2}\right)\,. \tag{2.4}$$

An equivalent form for the spatial section is given by

$$d\Sigma_3^2 = d\chi^2 + h_k^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{2.5}$$

where

$$\chi(k=1) = \frac{1}{h_0}\sin^{-1}[h_0 r]\,, \quad \chi(k=0) = r\,, \quad \chi(k=-1) = \frac{1}{h_0}\sinh^{-1}[h_0 r]\,, \tag{2.6}$$

and

$$h_1(\chi) = \frac{\sin h_0\chi}{h_0}\,, \quad h_0(\chi) = \chi\,, \quad h_{-1}(\chi) = \frac{\sinh h_0\chi}{h_0}\,, \tag{2.7}$$

depending on the choice of the topology. In what follows, we will work in the spherical case and then later we generalize the result to $k = 0, -1$. For $k = 1$ the metric reads:

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\chi^2 + \frac{1}{h_0^2}\sin^2[h_0\chi]\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\,, \quad k = 1\,. \tag{2.8}$$

In the case of a free massless scalar field $\phi(x)$ which is conformally coupled to gravity the related Wightman function,

$$W(x, x') =< \phi(x)\phi(x') >\,, \tag{2.9}$$

can be computed in an exact way (x, x' denote general coordinates of the space-time and for simplicity we omit to write the indexes). Wightman functions are well-defined distributions, namely they can be represented as a distributional limit of regular analytic functions, as we will see in the next sections. For the sake of completeness, we derive their general forms for FLRW space-time. In this respect, a complete computation is presented in Appendix A, while here we give a derivation based on conformal invariance. The quantum field $\phi(x)$ has the usual expansion

$$\phi(x) = \sum_\alpha f_\alpha(x)a_\alpha + f_\alpha^*(x)a_\alpha^+\,, \tag{2.10}$$

where a_α and a_α^+ are the destruction/creation operators associated to the state 'α' and the modes functions $f_\alpha(x)$ satisfy the conformally invariant equation,

$$\left(\Box - \frac{R}{6}\right) f_\alpha(x) = 0 \,. \tag{2.11}$$

Here, $\Box = \nabla_\mu \nabla^\mu$ represents the d'Alambertian operator in curved space-times, ∇_μ being the covariant derivative. Defining the vacuum state by $a_\alpha|0>=0$, the Wightman function turns to be

$$W(x,x') = \sum_\alpha f_\alpha(x) f_\alpha^*(x') \,, \tag{2.12}$$

such that (once x' is fixed),

$$\left(\Box - \frac{R}{6}\right) W(x,x') = 0 \,. \tag{2.13}$$

If we make a conformal transformation of the metric,

$$ds^2 = \Omega(x)^2 ds_0^2 \,, \tag{2.14}$$

where $\Omega(x)$ is a generic function of the space-time coordinates which relates the metric ds_0^2 with the metric ds^2, we obtain for the field

$$\phi(x) = \frac{1}{\Omega(x)} \phi_0(x) \,, \tag{2.15}$$

and the Wightman functions transform as

$$W(x,x') = \frac{1}{\Omega(x)\Omega(x')} W_0(x.x') \,. \tag{2.16}$$

Now we observe that the metric (2.8) is conformally related to a static Einstein-space $\mathcal{R} \times S_3$ with metric

$$ds_E^2 = -d\eta^2 + d\chi^2 + \frac{1}{h_0^2} \sin^2 h_0 \chi \left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{2.17}$$

by means

$$ds^2 = \Omega^2(\eta) ds_E^2 \,, \quad \Omega(\eta) = a(\eta) \,. \tag{2.18}$$

On the other hand, the metric (2.17) is conformally related to Minkowski space-time, since

$$ds_E^2 = \Omega^2(x)\left(-dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \,, \quad \Omega(x) = 2\cos\left[h_0 \frac{\eta+\chi}{2}\right] \cos\left[h_0 \frac{\eta-\chi}{2}\right] \,, \tag{2.19}$$

with the Minkowski coordinates given by

$$t \pm r = \frac{1}{h_0} \tan\left[h_0 \frac{\eta \pm \chi}{2}\right] \,. \tag{2.20}$$

Due to the homogeneity and isotropy of FLRW space-times, we may take $W(x,x') = W(x',x) = W(\eta-\eta', \chi-\chi')$ along a radial trajectory. Thus, since the Minkowski Wightman function is known [96],

$$W_M(x,x') = -\frac{1}{4\pi^2}\frac{1}{(t-t'-i\epsilon)^2-(r-r')^2}\,, \tag{2.21}$$

the Einstein-space Wightman function for radial separation is computed as

$$W_E(x,x') = \frac{h_0^2}{8\pi^2}\frac{1}{\cos[h_0(\eta-\eta')]-\cos[h_0(\chi-\chi')]}\,, \tag{2.22}$$

where we used relation (2.16). In the same way, one finally has for the spherical FLRW metric,

$$W(x,x') = \frac{h_0^2}{8\pi^2 a(\eta)a(\eta')}\frac{1}{\cos[h_0(\eta-\eta')]-\cos[h_0(\chi-\chi')]}\,, \quad k=1\,. \tag{2.23}$$

Furthermore, the Wightman function related to FRW hyperbolic spatial section with $k = -1$ can be obtained by the substitution $h_0 \to ih_0$, and reads

$$W(x,x') = -\frac{h_0^2}{8\pi^2 a(\eta)a(\eta')}\frac{1}{\cosh[h_0(\eta-\eta')]-\cosh[h_0(\chi-\chi')]}\,, \quad k=-1\,. \tag{2.24}$$

Finally, the Wightman function related to FRW flat spatial section $k = 0$ corresponds to the limit $h_0 \to 0$,

$$W(x,x') = \frac{1}{4\pi^2 a(\eta)a(\eta')}\frac{1}{(-(\eta-\eta')^2+(\chi-\chi')^2)}\,, \quad k=0\,. \tag{2.25}$$

These results are in agreement with the ones obtained in Appendix A.

2.2 The Unruh-DeWitt detector in FLRW space-time

The Unruh-DeWitt detector approach is a well known and used technique for exploring quantum field theoretical aspects in curved space-time. Basically, the Unruh-DeWitt detector is a "quantum thermometer" whose response function (that is, loosely speaking, the number of clicks per unit proper time it detects as it is carried around the Universe) can be used to exploring the temperature of curved space-time. For a recent review see Ref. [97]. Here, we present the basic formulas following Ref. [89, 98].

In 1976 Unruh [5] found that a uniformly accelerated observer moving through empty Minkowski space sees a thermal radiation with a temperature directly proportional to its proper acceleration. DeWitt extended the approach to curved space-times [2, 99], where the temperature gets a contribution from the expansion of the manifold. An idealized quantum detector is a quantum mechanical system with different internal energy states and coupled with a scalar field in its vacuum state along a given trajectory. Thus, a transition of the detector from its ground state to an exited one may be interpreted as the detection of a particle with the corresponding energy and associated temperature. The reaction of the detector is formulated via a response function. If at the proper time $\tau = \tau_0$ the detector is in the ground state, the probability to find the detector in the exited state $0 < E$ at the proper time $\tau = \tau_0 + \Delta\tau\,, 0 < \Delta\tau$, along a trajectory and through a massless scalar field $\phi(x)$ is given by

$$F(E,\tau) = \frac{1}{2\pi^2}\mathrm{Re}\left[\int_{\tau_0}^{\tau} d\tau' \int_0^{\Delta\tau} \mathrm{e}^{-iEs}(4\pi^2)W(\tau',\tau'-s)ds\right]\,, \tag{2.26}$$

where $W(\tau,\tau')$ is the Wightman function associated to the scalar field $\phi(\tau)$ at two different proper times for a given trajectory. In analogy with the thermodynamics, one may use the variation of the response function with respect to the proper time to infer a temperature, namely

$$F(E,\tau)_\tau \equiv \frac{d}{d\tau}F(E,\tau) \propto E\,\mathrm{e}^{-E/T_H}\,. \tag{2.27}$$

In other words, the detector feels the vacuum as a mixed quasi-thermal state.

The transition probability per unit proper time is given by

$$F(E,\tau)_\tau = \frac{1}{2\pi^2}\mathrm{Re}\left[\int_0^{\Delta\tau} \mathrm{e}^{-iEs}(4\pi^2)W(\tau,\tau-s)ds\right]\,. \tag{2.28}$$

If one deals with flat FLRW space-time with Wightman function (2.25), one has [89],

$$F(E,\tau)_\tau = \frac{1}{2\pi^2}\mathrm{Re}\left[\int_0^{\Delta\tau} \frac{ds}{a(\tau)a(\tau-s)}\frac{e^{-iEs}}{[\Delta\chi(s)^2-(\Delta\eta(s)-i\epsilon)^2]}\right]\,, \tag{2.29}$$

while for the spherical case with (2.23),

$$F(E,\tau)_\tau = \frac{1}{4\pi^2}\mathrm{Re}\left[\int_0^{\Delta\tau} \frac{ds}{a(\tau)a(\tau-s)}\frac{h_0^2 e^{-iEs}}{[-\cos[h_0\Delta\chi(s)]+\cos[h_0\Delta\eta(s)-i\epsilon]]}\right]\,. \tag{2.30}$$

In the above expressions, $\Delta\chi(s) = \chi(\tau) - \chi(\tau - s)$, and $\Delta\eta(s) = \eta(\tau) - \eta(\tau - s)$, while the $i\epsilon$-prescription with $\epsilon \to 0^+$ is necessary in order to deal with the second order pole at $s = 0$.

By making use of the condition for radial trajectory,

$$a^2(\tau)\left(\left(\frac{d\eta}{d\tau}\right)^2 - \left(\frac{d\chi}{d\tau}\right)^2\right) = 1\,, \tag{2.31}$$

and its first derivative,

$$\frac{da}{d\tau}\frac{1}{a} + a^2\left(\frac{d^2\eta}{d\tau^2}\frac{d\eta}{d\tau} - \frac{d^2\chi}{d\tau^2}\frac{d\chi}{d\tau}\right) = 0\,, \tag{2.32}$$

it is easy to show that, after the Taylor expansions

$$\Delta\chi(s) = \frac{d\chi}{d\tau}s - \frac{1}{2}\frac{d^2\chi}{d\tau^2}s^2 , \quad \Delta\eta(s) = \frac{d\eta}{d\tau}s - \frac{1}{2}\frac{d^2\eta}{d\tau^2}s^2 ,$$
$$a(\tau - s) = a(\tau)\left(1 - \frac{1}{a(\tau)}\frac{da}{d\tau}s + \frac{1}{2a(\tau)}\frac{d^2a}{d\tau^2}s^2\right) , \tag{2.33}$$

the leading singularity in the coincidence limit, namely for small s, of the Wightman function is in the form (see also § 2.5),

$$W(\tau, \tau - s) \simeq -\frac{1}{4\pi^2 s^2} + \frac{B}{48\pi^2} + O(s^2) , \tag{2.34}$$

where B is a regular part depending on the topology. As a result, one may try to avoid the awkward limit $\epsilon \to 0^+$ by omitting the ϵ-terms but subtracting the leading pole at $s = 0$ (see [98] for details). Thus, by introducing the quantity

$$\Sigma^2(\tau, s) = a(\tau)a(\tau - s)\left[\Delta\chi^2(s) - \Delta\eta^2(s)\right] , \quad k = 0 , \tag{2.35}$$

$$\Sigma^2(\tau, s) \equiv a(\tau)a(\tau - s)\frac{2}{h_0^2}\left[\cos[h_0\Delta\eta(s)] - \cos[h_0\Delta\chi(s)]\right] , \quad k = 1 , \tag{2.36}$$

one can present the detector transition probability per unit time in the form

$$F(E, \tau)_\tau = \frac{1}{2\pi^2}\int_0^\infty ds\, \cos(Es)\left(\frac{1}{\Sigma^2(\tau, s)} + \frac{1}{s^2}\right) + J_\tau(E, \tau) , \tag{2.37}$$

where the "tail" or finite time fluctuating term is given by

$$J_\tau(E, \tau) := -\frac{1}{2\pi^2}\int_{\Delta\tau}^\infty ds\, \frac{\cos(E\, s)}{\Sigma^2(\tau, s)} . \tag{2.38}$$

In the important stationary cases (examples are the static black hole or the FLRW de Sitter space), one has $\Sigma(\tau, s)^2 = \Sigma^2(s) = \Sigma^2(-s)$, and Eq. (2.37) simply becomes

$$F(E, \tau)_\tau = \frac{1}{4\pi^2}\int_{-\infty}^\infty ds\, \cos[Es]\left(\frac{1}{\Sigma^2(s)} + \frac{1}{s^2}\right) + J_\tau(E, \tau) = F(E)_\tau + J_\tau(E, \tau) . \tag{2.39}$$

The first term is independent on τ, and all the time dependence is contained only in the fluctuating tail. Thus, one may infer the temperature as in (2.27) from the first term only.

2.3 Hawking effect in the flat de Sitter Universe

Let us focus our attention on the De Sitter space-time with various topologies described by the metrics in (1.35)–(1.36), or, if we use the form (2.2) with (2.5) and (2.7),

$$ds^2 = e^{2H_0 t(\eta)}\left(-d\eta^2 + d\chi^2 + \frac{1}{h_0^2}\chi^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) , \quad k = 0 , \tag{2.40}$$

$$ds^2 = \cosh^2[H_0 t(\eta)] \left(-d\eta^2 + d\xi^2 + \frac{1}{h_0^2} \sin^2[h_0 \chi] \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) , \quad k = 1 , \tag{2.41}$$

where appears the cosmological time t as an expression of the conformal time η.

In a flat de Sitter Universe, the calculation of $\Sigma^2(\tau, s)$ in (2.35) is quite simple and can be found in Ref. [89]. The Kodama observer at a given time is determined by the areal radius

$$\mathcal{R}(t) \equiv R_0 = a(t) h_k(\chi) , \tag{2.42}$$

where R_0 is a constant (it means that the Kodama observer is accelerating unless $\chi(t) = 0$) and $h_k(\chi)$ is given by (2.7). In this case, $h_0(\chi) = \chi$. For flat de Sitter space-time (2.40), one has

$$\chi(t) = \mathrm{e}^{-H_0 t} \mathcal{R}_0 . \tag{2.43}$$

For a radial trajectory we get

$$d\tau^2 = a^2(\eta)(d\eta^2 - d\chi^2) , \tag{2.44}$$

such that

$$d\tau^2 = dt^2 \left(1 - a^2 \left(\frac{d\chi}{dt} \right)^2 \right) , \tag{2.45}$$

or, by using (2.43),

$$\tau = t\sqrt{V_0} , \tag{2.46}$$

where V_0 is the Tolman factor

$$V_0 = 1 - \mathcal{R}_0^2 H_0^2 . \tag{2.47}$$

Thus, the conformal time results to be

$$\eta(\tau) = -\frac{1}{H_0} \mathrm{e}^{-\frac{H_0 \tau}{\sqrt{V_0}}} . \tag{2.48}$$

By using these expressions in (2.35) and by taking into account that $h_0 = H_0$, we get

$$\Sigma^2(s)^2 = -\frac{4V_0}{H_0^2} \sinh^2 \left[\frac{H_0 s}{2\sqrt{V_0}} \right] . \tag{2.49}$$

Now $F(E)_\tau$ in (2.39) reads

$$F(E)_\tau = \frac{H_0}{8\pi^2\sqrt{V_0}} \int_{-\infty}^{\infty} dx \, \cos[2\sqrt{V_0} E x / H_0] \left(-\frac{1}{\sinh^2[x]} + \frac{1}{x^2} \right) , \quad x = \frac{H_0 s}{2\sqrt{V_0}} . \tag{2.50}$$

This integral can be solved by making use of the residue theorem and by separating it in two parts as

$$\begin{aligned} F(E)_\tau &= \frac{H_0}{16\pi^2\sqrt{V_0}} \int_{-\infty}^{\infty} dx \, \exp[2i\sqrt{V_0} E x / H_0] \left(-\frac{1}{\sinh^2[x]} + \frac{1}{x^2} \right) \\ &+ \frac{H_0}{16\pi^2\sqrt{V_0}} \int_{-\infty}^{\infty} dx \, \exp[-2i\sqrt{V_0} E x / H_0] \left(-\frac{1}{\sinh^2[x]} + \frac{1}{x^2} \right) . \end{aligned} \tag{2.51}$$

Since we are interested in the case of $E > 0$, we can close the contour of integration of the first integral in the lower half complex plane, summing over the residues of the double poles (with the exception of the $z = x = 0$ pole which has been removed) $z = ik\pi\,, k = -1, -2, -3, ...$ In a hasty manner, the second integral is solved by closing the contour of integration in the upper half complex plane and by summing over the residues of the double poles in $z = ik\pi\,, k = 1, 2, 3, ...$ As a result, one finds

$$F(E)_\tau = \frac{E}{2\pi}\sum_{n=1}^{+\infty} \mathrm{e}^{-\frac{2\pi\sqrt{V_0}E\,n}{H_0}} = \frac{E}{2\pi}\frac{1}{\mathrm{e}^{\frac{2\pi\sqrt{V_0}E}{H_0}} - 1}\,. \tag{2.52}$$

The physical interpretation of the result is that the system detects a quantum system in thermal equilibrium at a temperature,

$$T_K = \frac{H_0}{2\pi\sqrt{V_0}}\,, \tag{2.53}$$

which encodes the Hawking effect and the de Sitter temperature. We note that the four velocity of a Kodama observer with radial trajectory is given by

$$u^\mu = \frac{1}{\sqrt{V_0}}\left(\frac{1}{a(t)}, -H_0\chi(t), 0, 0\right)\,, \tag{2.54}$$

while the four acceleration results to be

$$a^\mu \equiv \frac{1}{\sqrt{V_0}}\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = \left(\frac{H_0^3\mathcal{R}_0^2}{V_0 a(t)}, -\frac{H_0^2\mathcal{R}_0}{V_0 a(t)}, 0, 0\right)\,, \tag{2.55}$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols associated to the metric. Finally, the norm of the four acceleration of a Kodama observer in a flat De Sitter space-time has the expression,

$$A^2 = \frac{H_0^4\mathcal{R}_0^2}{V_0}\,, \tag{2.56}$$

showing that

$$T_K = \frac{\sqrt{H_0^2 + A^2}}{2\pi}\,. \tag{2.57}$$

Thus, when $\mathcal{R}_0 = 0$, namely, when the detector is co-moving with the metric, the four-acceleration goes to zero. In this case, the Dopler shift due to the proper motion of the detector, and therefore the pure Unruh effect, disappear ($V_0 = 1$) and the classical Gibbons-Hawking result (1.37) is recovered according with the prediction. This is the de Sitter version [100] of a formula discovered by Deser and Levine for detectors in anti-de Sitter space. [101]. Alternatively , one may understand the result as the transition from cosmological energy E, conjugated to cosmic de Sitter time t, to the energy $V_0\,E$ as measured locally by Kodama's observers, with proper time τ of Eq. (2.46).

2.4 Hawking effect in the non-flat de Sitter Universe

Let us calculate the detector transition probability for the (non-flat) de Sitter space-time (2.41), generalizing the previous result.

First of all, we need an expression for the proper time as a function of the time. Recall that, in a generic (non-flat) FRWL space-time, the Kodama observer at a given time is determined by (2.42), namely, by using the relation (2.7) for $k = 1$,

$$\mathcal{R}(t) = \mathcal{R}_0 = \frac{a(t)}{h_0} \sin h_0 \chi \,, \tag{2.58}$$

where $\mathcal{R}_0$ is again a constant. For a radial trajectory we can use (2.44)–(2.45) with (2.58), and the proper time in the non-flat FRWL is related to the cosmological time as

$$d\tau^2 = dt^2 \left(1 - \frac{a^2(t) R_0^2 H^2(t)}{a^2(t) - R_0^2 h_0^2} \right) \,, \tag{2.59}$$

where $H(t) = \dot{a}(t)/a(t)$ is the usual Hubble parameter. As a result, the conformal time along a Kodama trajectory is given by

$$\eta(\tau) = \int d\tau \, \frac{\sqrt{a^2(\tau) - \mathcal{R}_0^2 h_0^2}}{a(\tau)\sqrt{a^2(\tau) - \mathcal{R}_0^2 h_0^2 - a^2(\tau) R_0^2 H^2(\tau)}} \,, \tag{2.60}$$

$a(\tau)\,, H(\tau)$ being the scale factor and the Hubble parameter as functions of the proper time.

For a non-flat de Sitter space-time one has $H_0 \equiv h_0$ again, but now $H_0 \neq H(\tau)$ and it is not easy to get an explicit expression for $\eta(\tau)$. For this reason we may consider the $R_0 = 0$ case (comoving Kodama observer) such that $d\tau = dt$. Since $a(t) = \cosh H_0 t = \cosh H_0 \tau$, we obtain

$$\eta(\tau) = \frac{2}{H_0} \arctan \mathrm{e}^{H_0 \tau} \,. \tag{2.61}$$

We have to compute (2.36), namely ($\Delta\chi = 0$),

$$\Sigma^2(\tau, s) = \frac{2}{H_0^2} \cosh[H_0 \tau] \cosh[H_0(\tau - s)] \left(\cos(H_0 \Delta\eta) - 1 \right) \,, \tag{2.62}$$

or, equivalently,

$$\Sigma^2(\tau, s) = -\frac{4}{H_0^2} \cosh[H_0 \tau] \cosh H_0[\tau - s] \sin^2 \left[\frac{H_0 \Delta\eta(\tau, s)}{2} \right] \,. \tag{2.63}$$

Now, by using the trigonometric formula $(\arctan x - \arctan y) = \arctan[(x - y)/(1 + xy)]$, from (2.61) one obtains the following identity

$$H_0 \Delta\eta(\tau, s) = -2 \arctan \left(\frac{\sinh\left[-\frac{H_0 s}{2}\right]}{\cosh H_0[\tau - \frac{s}{2}]} \right) \,, \tag{2.64}$$

and, by taking into account that $\arctan x = \arcsin[x/\sqrt{1+x^2}]$,

$$H_0 \Delta\eta(\tau, s) = -2 \arcsin \left(\frac{\sinh\left[-\frac{H_0 s}{2}\right]}{\sqrt{\cosh^2 H_0 \left[\tau - \frac{s}{2}\right] + \sinh^2[-\frac{H_0 s}{2}]}} \right) . \tag{2.65}$$

Therefore, since

$$\sin^2 \frac{H_0 \Delta\eta(\tau, s)}{2} = \frac{\sinh^2\left[-\frac{H_0 s}{2}\right]}{\cosh^2 H_0 \left[\tau - \frac{s}{2}\right] + \sinh^2\left[-\frac{H_0 s}{2}\right]} , \tag{2.66}$$

and

$$\begin{aligned} \cosh^2 H_0 \left[\tau - \frac{s}{2}\right] + \sinh^2\left[-\frac{H_0 s}{2}\right] &= \\ \cosh^2[H_0\tau]\cosh[H_0 s] - \sinh[H_0\tau]\cosh[H_0\tau]\sinh[H_0 s] &= \\ \cosh[H_0\tau]\cosh[H_0(\tau - s)] , \end{aligned} \tag{2.67}$$

the invariant distance turns to be

$$\Sigma^2(s) = -\frac{4}{H_0^2} \sinh^2\left[\frac{H_0\, s}{2}\right] . \tag{2.68}$$

The detector transition probability can be now computed by starting from (2.39), namely

$$F(E)_\tau = \frac{H_0}{8\pi^2} \int_{-\infty}^{\infty} dx\, \cos[2Ex/H_0] \left(-\frac{1}{\sinh^2[x]} + \frac{1}{x^2} \right) , \quad x = \frac{H_0 s}{2} . \tag{2.69}$$

This integral is analogue to the one in (2.50) when $V_0 = 1$ and the solution is

$$F(E)_\tau = \frac{1}{2\pi} \frac{E}{e^{\frac{2\pi E}{H_0}} - 1} , \tag{2.70}$$

which shows that the Unruh-DeWitt thermometer in the FLRW de Sitter space detects a quantum system in thermal equilibrium at a temperature (1.37), confirming the Gibbons-Hawking result of the flat case. This is an important check of the approach, since it demonstrates the coordinate independence of the result for the important case of de Sitter space.

The response function for unite proper time, in the stationary cases we have considered, gives information about the equilibrium temperature via the Planckian distribution. We also may argue it as follows. In the stationary case, in the limit $\tau \to \infty$, one has [102]

$$F(E)_\tau = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} ds\, e^{-iEs} \left(\frac{1}{\Sigma^2(s)} + \frac{1}{s^2} \right) = \frac{1}{2\pi} \frac{E}{\exp\left[\frac{E}{T_K}\right] - 1} . \tag{2.71}$$

Note also that in this case we get

$$\frac{F(E)_\tau}{F(-E)_\tau} = e^{-\frac{E}{T_K}} . \tag{2.72}$$

Vice versa, if the above relations hold, then $F(E)_\tau$ is the Plank distribution. Thus, we may define the local equilibrium temperature by means of

$$T_K = \frac{E}{\ln F(-E)_\tau - \ln F(E)_\tau} , \tag{2.73}$$

or

$$T_K = \frac{E}{\ln\left(1 + \frac{E}{2\pi(F(E)_\tau)}\right)} , \tag{2.74}$$

which shows in which sense an Unruh-DeWitt detector is a quantum thermometer.

2.4.1 d-dimensional generalization

In the de Sitter case, it is possible to generalize the computation of the response function per unit time to the massive non-conformally coupled d-dimensional scalar field case [103]. One may directly obtain

$$F_{d,\nu}(E)_\tau = \frac{H_0^{d-3} e^{\frac{\pi E}{H_0}}}{8\pi^{(d+1)/2}\Gamma\left[\frac{d-1}{2}\right]} |\Gamma\left[\frac{d-1}{4} + \frac{\nu}{2} + i\frac{E}{2H_0}\right] \Gamma\left[\frac{d-1}{4} - \frac{\nu}{2} + i\frac{E}{2H_0}\right] |^2 , \tag{2.75}$$

where $\Gamma[z]$ is the Euler function and

$$\nu = \sqrt{\frac{(d-1)^2}{4} - \frac{m^2}{H_0^2} - \xi d(d-1)} , \tag{2.76}$$

with m^2 the mass of the scalar field and ξ the coupling constant with the Ricci curvature, being the conformal coupling

$$\xi_c = \frac{d-2}{4(d-1)} . \tag{2.77}$$

As a consequence, one has

$$F_{d,\nu}(-E)_\tau = e^{\frac{E}{T}} F_{d,\nu}(E)_\tau . \tag{2.78}$$

For example, the massless conformally coupled case in d-dimension corresponds to $\nu = 1/2$. In this case, for $d = 4$, one recovers

$$F_{4,\frac{1}{2}}(E)_\tau = \frac{1}{2\pi} \frac{E}{e^{\frac{2\pi E}{H_0}} - 1} . \tag{2.79}$$

Furthermore, making use of the identity [104],

$$\Gamma(z)|^2 |\Gamma(1-z)|^2 = \frac{2\pi^2 |z|^2}{\cosh(2\pi \mathrm{Im} z) - \cos(2\pi \mathrm{Re} z)} , \tag{2.80}$$

one obtains for $d = 3$ and $\nu = 1/2$,

$$F_{3,\frac{1}{2}}(E)_\tau = \frac{1}{16}\left(1+\frac{E^2}{H_0^2}\right)\frac{1}{e^{\frac{2\pi E}{H_0}}+1}\,, \tag{2.81}$$

in which the well known phenomenon of the inversion of the statistic for odd dimensional spaces is manifest.

The other interesting case is the minimally coupled massless scalar field, for which $\nu = (d-1)/2$ and $m = 0$ again. In particular, for $d = 4$, one gets

$$F_{4,\frac{3}{2}}(E)_\tau = \frac{H_0^2}{8\pi^3}\left(1+\frac{E^2}{H_0^2}\right)\frac{1}{E\left(e^{\frac{2\pi E}{H_0}}-1\right)}\,, \tag{2.82}$$

and the well known infrared problem associated with this case appears in the bad behavior for small E.

2.5 Quantum Thermometers

Another proposal to detect local temperature associated with stationary space-time admitting an event horizon has been put forward by Buchholz and collaborators [105] (see also Ref. [106]). The idea may be substantiated by the following argument. We start with a free massless quantum scalar field $\phi(x)$ in thermal equilibrium at temperature T in flat Minkowski space-time. It is well known that finite temperature field theory effects of this kind may be investigated by considering the scalar field defined in the Euclidean manifold $S_1 \times R^3$, where one has introduced the imaginary time $\tilde{\tau} = -it$, compactified in the circle S_1, with period (see for example Ref. [107]),

$$\beta = \frac{1}{T_K}\,. \tag{2.83}$$

Let us see how we can get this result. We consider the local quantity $< \phi(x)^2 >$. Formally, this is a divergent quantity, due to product of valued operator distribution in the same point x, and a regularization and renormalization are necessary. A simple and powerful way to deal with a regularized quantity is to make use of zeta-function regularization (see for example Refs. [107, 108, 109, 110, 111] and references therein). Within zeta-function regularization, one has

$$< \phi(x)^2 >= \zeta(1|L_\beta)(x)\,, \tag{2.84}$$

where $\zeta(z|L_\beta)(x)$ is the analytic continuation of the local zeta-function associated with the operator L_β which is

$$L_\beta = -\partial^2_{\tilde{\tau}} - \nabla^2\,, \quad \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2\,, \tag{2.85}$$

defined on $S_1 \times R^3$. The local zeta-function possesses $\mathrm{Re}\, z$ sufficiently large by means

$$\zeta(z|L_\beta)(x) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}K_t(x,x)dt\,, \tag{2.86}$$

where $\Gamma(z)$ is the Euler function and the heat-kernel on the diagonal is given by

$$K_t(x,x) =< x|e^{-tL_\beta}|x >= \frac{1}{\beta(4\pi t)^{3/2}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2}{\beta^2}n^2} \,. \tag{2.87}$$

In (2.84), the local $\zeta(z|L_\beta)(x)$ with analytical continuation is assumed to be regular at $z = 1$ (in our case, it will be truth). On the other side, if the analytic continuation has a simple pole in $z = 1$, the prescription has to be modified (see Refs. [106, 112]).

A standard computation, which makes use of the Jacobi-Poisson formula, leads to

$$K_t(x,x) = \frac{1}{(4\pi t)^2} \sum_n e^{-\frac{\beta^2 n^2}{4t}} \,. \tag{2.88}$$

Let us plug this expression in (2.86). The term $n = 0$ leads to a formally divergent integral $\int_0^\infty dt(t^{z-3})$, but this is zero in the sense of Gelfand analytic continuation, and it can be neglected. Thus, the analytic continuation of the local zeta-function results to be

$$\zeta(z|L_\beta)(x) = \frac{\Gamma(2-z)}{8\pi^2\Gamma(z)} \left(\frac{\beta^2}{4}\right)^{z-2} \zeta_R(4-2z) \,, \tag{2.89}$$

where $\zeta_R(z)$ is the (usual) Riemann zeta-function. It is easy to see that the analytic continuation of the local zeta-function is regular at $z = 1$, and from (2.84), recalling that $\zeta_R(2) = \pi^2/6$, one has

$$< \phi(x)^2 >= \frac{1}{12\beta^2} = \frac{T_K^2}{12} \,. \tag{2.90}$$

Thus, the regularized vacuum expectation value of the observable ϕ^2 gives the temperature of the quantum field in thermal equilibrium, namely one is dealing with a quantum thermometer. Motivated by this argument, we will consider again a conformal coupled scalar field in a general de Sitter space-time.

2.6 Quantum fluctuations in FLRW space-time

We have seen that the off-diagonal Wigthmann function for generic (flat/spherical) FLRW space-time can be written as

$$W(x,x') =< \phi(x)\phi(x') >= W(\tau, \tau - s) = \frac{1}{4\pi^2} \frac{1}{\Sigma^2(\tau, \tau - s)} \,, \tag{2.91}$$

where $(\chi' - \chi) = \chi(\tau) - \chi(\tau - s)$, and $(\eta' - \eta) = \eta(\tau) - \eta(\tau - s)$, such that $\Sigma^2(\tau, \tau - s)$ is given by (2.35)–(2.36). In the limit $s \to 0$ formally one obtains

$$< \phi(x)^2 >= W(\tau, \tau) \,, \tag{2.92}$$

but $W(\tau, \tau)$ is ill defined, and one has to regularized and then renormalize this object. In our case, we may make use of the simple point splitting procedure [4], namely we will consider $W(\tau,\tau-s)$ and evaluate the limit $s\to 0$. To implement this limit procedure, one needs several identities. For radial time-like separation, the starting point is (2.31) whose first derivative with respect to the proper time is given by (2.32) and the second derivative reads

$$a^2\left(\frac{d\eta}{d\tau}\left(\frac{d^3\eta}{d\tau^3}\right)-\frac{d\chi}{d\tau}\left(\frac{d^3\chi}{d\tau^3}\right)\right)=-a^2\left(\frac{d^2\eta}{d\tau^2}\right)^2+a^2\left(\frac{d^2\chi}{d\tau^2}\right)^2-\frac{1}{a}\frac{d^2a}{d\tau^2}+3\left(\frac{1}{a}\right)^2\left(\frac{da}{d\tau}\right)^2 . \tag{2.93}$$

In what follows, we will rederive the correlation function for flat FLRW space-time following Ref. [89]. In parallel, we will present the generalization of the result for spherical FLRW space-time. At first, by making use of the the Taylor expansions,

$$\cos[h_0\Delta\eta]\simeq 1-\frac{h_0^2}{2}\Delta\eta^2+\frac{h_0^4}{24}\Delta\eta^4+\mathcal{O}(\Delta\eta^6)\,,\quad \cos[h_0\Delta\chi]\simeq 1-\frac{h_0^2}{2}\Delta\chi^2+\frac{h_0^4}{24}\Delta\chi^4+\mathcal{O}(\Delta\chi^6)\,, \tag{2.94}$$

one immediately observes that $\Sigma^2(\tau,\tau-s)$ with $k=1$ in (2.36) is related to $\Sigma^2(\tau,\tau-s)$ with $k=0$ in (2.35) as

$$\Sigma^2(\tau,\tau-s)|_{k=1}\simeq\Sigma^2(\tau,\tau-s)|_{k=0}+\frac{h_0^2}{12}a(\tau)a(\tau-s)(\Delta\eta^4-\Delta\chi^4)\,. \tag{2.95}$$

Thus, when $|s|\ll 1$, one has,

$$\begin{aligned}\Sigma^2(\tau,\tau-s) \;\simeq\; & a^2\left(1-\frac{da}{d\tau}\frac{s}{a}+\frac{d^2a}{d\tau^2}\frac{s^2}{2a}\right)\left(\left(\frac{d\chi}{d\tau}\right)^2 s^2-\frac{d\chi}{d\tau}\frac{d^2\chi}{d\tau^2}s^3+\left(\frac{d^2\chi}{d\tau^2}\right)^2\frac{s^4}{4}+\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3}\frac{s^4}{3}\right.\\ & \left.-\left(\frac{d\eta}{d\tau}\right)^2 s^2+\frac{d\eta}{d\tau}\frac{d^2\eta}{d\tau^2}s^3-\left(\frac{d^2\eta}{d\tau^2}\right)^2\frac{s^4}{4}-\frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\frac{s^4}{3}\right)+\mathcal{O}(s^5)\,,\quad k=0\,,\end{aligned} \tag{2.96}$$

$$\begin{aligned}\Sigma^2(\tau,\tau-s) \;\simeq\; & a^2\left(1-\frac{da}{d\tau}\frac{s}{a}+\frac{d^2a}{d\tau^2}\frac{s^2}{2a}\right)\left(\left(\frac{d\chi}{d\tau}\right)^2 s^2-\frac{d\chi}{d\tau}\frac{d^2\chi}{d\tau^2}s^3+\left(\frac{d^2\chi}{d\tau^2}\right)^2\frac{s^4}{4}+\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3}\frac{s^4}{3}\right.\\ & \left.-\left(\frac{d\eta}{d\tau}\right)^2 s^2+\frac{d\eta}{d\tau}\frac{d^2\eta}{d\tau^2}s^3-\left(\frac{d^2\eta}{d\tau^2}\right)^2\frac{s^4}{4}-\frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\frac{s^4}{3}\right)\\ & +\frac{h_0^2}{12}a^2\left(\left(\frac{d\eta}{d\tau}\right)^4-\left(\frac{d\chi}{d\tau}\right)^4\right)s^4+\mathcal{O}(s^5)\,,\quad k=1\,,\end{aligned} \tag{2.97}$$

where $a\equiv a(\tau)$ and it is understood that the power-law terms of s higher than four can be avoided. Now, by using (2.31, 2.32, 2.93), one finds

$$\Sigma^2(\tau,\tau-s)=-s^2\left(1+\frac{B}{12}s^2+O(s^3)\right)\,, \tag{2.98}$$

with

$$B = -a^2\left(\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3} - \frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\right) - 3\left(-\frac{d^2a}{d\tau^2}\frac{1}{a} + \frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2\right), \quad k=0\,, \tag{2.99}$$

$$\begin{aligned} B &= -a^2\left(\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3} - \frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\right) - 3\left(-\frac{d^2a}{d\tau^2}\frac{1}{a} + \frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2\right) \\ & -h_0^2a^2\left(\left(\frac{d\eta}{d\tau}\right)^4 - \left(\frac{d\chi}{d\tau}\right)^4\right)s^2\,, \quad k=1\,. \end{aligned} \tag{2.100}$$

One can also introduce the Hubble parameter H and its derivative with respect to the cosmological time t, $\dot{H}$, in the following identities,

$$\frac{1}{a}\frac{da}{d\tau} = H\frac{dt}{d\tau}\,, \quad \frac{1}{a}\frac{d^2a}{d\tau^2} = \dot{H}\left(\frac{dt}{d\tau}\right)^2 + H\frac{d^2t}{d\tau^2} + H^2\left(\frac{dt}{d\tau}\right)^2, \tag{2.101}$$

such that

$$B = -a^2\left(\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3} - \frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\right) + 3\left(\dot{H}\left(\frac{dt}{d\tau}\right)^2 + H\frac{d^2t}{d\tau^2}\right), \quad k=0\,, \tag{2.102}$$

$$\begin{aligned} B &= -a^2\left(\frac{d\chi}{d\tau}\frac{d^3\chi}{d\tau^3} - \frac{d\eta}{d\tau}\frac{d^3\eta}{d\tau^3}\right) + 3\left(\dot{H}\left(\frac{dt}{d\tau}\right)^2 + H\frac{d^2t}{d\tau^2}\right) \\ & -h_0^2a^2\left(\left(\frac{d\eta}{d\tau}\right)^4 - \left(\frac{d\chi}{d\tau}\right)^4\right)s^2\,, \quad k=1\,. \end{aligned} \tag{2.103}$$

Finally, since the following relations are valid in FLRW space-time,

$$\frac{d\eta}{d\tau} = \frac{1}{a}\frac{dt}{d\tau}\,, \quad \frac{d\chi}{d\tau} = \frac{1}{a}\sqrt{\left(\frac{dt}{d\tau}\right)^2 - 1}\,, \tag{2.104}$$

we derive

$$B = H^2 + A^2 + 2\dot{H}\left(\frac{dt}{d\tau}\right)^2, \quad k=0\,, \tag{2.105}$$

$$B = H^2 + A^2 + 2\dot{H}\left(\frac{dt}{d\tau}\right)^2 + \frac{h_0^2}{a^2}\left(1 - 2\left(\frac{dt}{d\tau}\right)^2\right), \quad k=1\,. \tag{2.106}$$

Here, A^2 is the square of the four-acceleration along the time-like trajectory, which reads, in FLRW coordinates,

$$A^2 = \frac{1}{(dt/d\tau)^2 - 1}\left(\frac{d^2t}{d\tau^2} + H\left(\left(\frac{dt}{d\tau}\right)^2 - 1\right)\right)^2. \tag{2.107}$$

In (2.106), the last term is the new one with respect to the flat case. Thus, as anticipated in (2.34), the point splitting gives

$$W(\tau,\tau-s)=-\frac{1}{4\pi^2 s^2}+\frac{B}{48\pi^2}+O(s^2)\,. \tag{2.108}$$

With regard to the renormalization, we simply subtract the first divergent term for $s\to 0$. The physical meaning of this subtraction has been discussed in detail in Ref. [89], and it amounts to subtract the contribution related to an inertial trajectory in Minkowski space-time. Thus, the renormalized quantum fluctuation finally reads,

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\dot{H}\left(\frac{dt}{d\tau}\right)^2\right)\,,\quad k=0\,, \tag{2.109}$$

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\dot{H}\left(\frac{dt}{d\tau}\right)^2+\frac{h_0^2}{a^2}\left(1-2\left(\frac{dt}{d\tau}\right)^2\right)\right)\,,\quad k=1\,. \tag{2.110}$$

Moreover, the quantum fluctuations for a FLRW space-time with topology $k=-1$ can be derived from (2.110) with $h_0\to ih_0$, namely

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\dot{H}\left(\frac{dt}{d\tau}\right)^2-\frac{h_0^2}{a^2}\left(1-2\left(\frac{dt}{d\tau}\right)^2\right)\right)\,,\quad k=-1\,. \tag{2.111}$$

In terms of the derivatives with respect to the proper time only, this formulas read

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\left(\frac{dH}{d\tau}\right)\left(\frac{dt}{d\tau}\right)\right)\,,\quad k=0\,, \tag{2.112}$$

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\left(\frac{dH}{d\tau}\right)\left(\frac{dt}{d\tau}\right)+\frac{h_0^2}{a^2}\left(1-2\left(\frac{dt}{d\tau}\right)^2\right)\right)\,,\quad k=1\,, \tag{2.113}$$

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H^2+A^2+2\left(\frac{dH}{d\tau}\right)\left(\frac{dt}{d\tau}\right)-\frac{h_0^2}{a^2}\left(1-2\left(\frac{dt}{d\tau}\right)^2\right)\right)\,,\quad k=-1\,. \tag{2.114}$$

Thus, here we presented the generalization of the result obtained in the flat FLRW space-time in Ref. [89] and within Unruh-de Witt detector in Ref. [88].

2.6.1 Quantum fluctuations in De Sitter Universe

As a first important check, let us consider again the de Sitter space-time in the flat case (2.40) and in the global patch (2.41). In this stationary cases, the fluctuations act as a quantum thermometer according with (2.90).

In the flat case with $H(t)=H_0$, we immediately have

$$<\phi^2>_R=\frac{1}{48\pi^2}\left(H_0^2+A^2\right)\,, \tag{2.115}$$

such that the temperature is given by (2.57) and we recover the Unruh effect in de Sitter space-time [100]. Note that, given the Kodama observer at $\mathcal{R} = \mathcal{R}_0$ and relation (2.46), we obtain again the expression in (2.56) and (2.47) for the acceleration.

In the spherical case one has $h_0 = H_0$ and the following relations hold true

$$a(t) = \cosh H_0 t \,, \quad H(t) = H_0 \frac{\sinh H_0 t}{\cosh H_0 t} \,, \quad \frac{dH}{d\tau} = \frac{H_0^2}{(\cosh H_0 t)^2} \frac{dt}{d\tau} \,. \tag{2.116}$$

As a consequence, we find (2.115) again confirming the result for non-flat case. Now, the acceleration can be computed, since, for a Kodama observer at $\mathcal{R} = \mathcal{R}_0$, one obtains from Eq. (2.59),

$$\left(\frac{dt}{d\tau}\right)^2 - 1 = \frac{R_0^2 H^2(t)}{1 - R_0^2 H_0^2} \,, \tag{2.117}$$

and

$$\frac{d^2 t}{d\tau^2} = \frac{R_0^2}{1 - R_0^2 H_0^2} H(t) \dot{H}(t) \,, \tag{2.118}$$

where we remember that $\dot{H} = (dH/d\tau)(d\tau/dt)$. Taking equation (2.107) into account, one gets (2.56) again. Thus, (2.115) can be written as

$$< \phi^2 >_R = \frac{1}{48\pi^2} \frac{H_0^2}{1 - R_0^2 H_0^2} \,, \tag{2.119}$$

and for a comoving Kodama observer $R_0 = 0$, and from (2.90) one finds the Gibbons-Hawking temperature associated with de Sitter space-time (1.37).

2.6.2 Quantum fluctuations in non-flat FRLW form of Minkowski space-time

As a second check of our formalism, we may consider the Milne Universe, which is a vacuum FLRW solution with $k = -1$ of Einstein's equation. On the other hand, it is well known that the Minkowski space-time

$$ds^2 = -d\tau^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right) , \tag{2.120}$$

may be written in a FRLW form with hyperbolic section $k = -1$, which is exactly the Milne Universe with

$$ds^2 = -dt^2 + t^2 \left(d\chi^2 + \sinh^2 \chi \left(d\theta^2 + \sin^2 d\phi^2\right)\right) \,, \tag{2.121}$$

where

$$\tau = t \cosh \chi \,, \quad r = t \sinh \chi \,. \tag{2.122}$$

Here, the Hubble parameter is given by $H = 1/t$, but, making use of the Hayward formalism, it is easy to verify that there is no dynamical horizon and the surface gravity is vanishing. In fact, if we compute the quantum fluctuations for a Kodama observer at $\mathcal{R} = \mathcal{R}_0 = r$ with $h_0 = 1$ and[1]

$$\left(\frac{dt}{d\tau}\right)^2 = \left(\frac{t^2 + \mathcal{R}_0^2}{t^2}\right), \tag{2.123}$$

by using (2110) one obtains

$$< \phi^2 >_R = \frac{A^2}{48\pi^2}, \tag{2.124}$$

namely only the radial acceleration A^2 is present and the temperature is defined as

$$T_K = \frac{A}{2\pi}, \tag{2.125}$$

recovering the well known Unruh effect in Minkowski space-time. A computation of A^2 leads to

$$A^2 = \frac{\mathcal{R}_0^2 \left(-1 + \frac{1}{t}\sqrt{t^2 + \mathcal{R}_0^2}\right)^2}{t^2(\mathcal{R}_0^2 + t^2)}. \tag{2.126}$$

Correctly, when $\mathcal{R} = 0$, one has $T = 0$.

[1]Note that the proper time of the Kodama observer corresponds to the time in the reference system of Minkowski.

Chapter 3

Mimetic gravity models

In this chapter, we briefly review some unsolved issues of modern cosmology. The complete picture of our Friedmann Universe requires the existence of an early-time accelerated epoch after the Big Bang (the inflation) to explain the thermalization of the observable Universe. Moreover, cosmological data say that the Universe undergoes today an accelerated expansion (the dark energy issue). General Relativity cannot explain the inflationary expansion with standard matter and radiation. To solve the problem, one may modify the theory of Einstein and the modified theories of gravity reached today a high degree of corroboration. The data also confirm the existence of an amount of "dark" matter larger than the detected one. Recently, Mukhanov and Chamseddine introduced a parametrization of the metric by using an auxiliary metric and a scalar field in order to isolate its conformal degree of freedom. In this way, the Einstein's field equations acquire a dark matter-like contribute given by the field. This theory was dubbed "mimetic gravity" and can be easily extended to depict a huge variety of cosmological scenarios (inflation, cosmological bounce, dark energy...) including the dark matter phenomenology.

3.1 Modified theories of gravity

General Relativity (GR), first formulated by Albert Einstein in 1915, is an extremely successful and predictive theory, and together with Quantum Field Theory forms one of the basis of modern physics. We already introduced the well-known Einstein's field equations in Chapter one. Such equations are derived by making use of the variational principle from the Hilbert-Einstein action,

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R}{2} + \mathcal{L}_m\right) , \tag{3.1}$$

where $\mathcal{M}$ represents the space-time manifold, g is the determinant of the metric tensor $g_{\mu\nu}$, and R, as usually, is the Ricci scalar, while $\mathcal{L}_m$ is the Lagrangian of the matter contents of the space-time. In these chapters we will pose the Newton's constant G_N (and therefore the Planck Mass M_{Pl}) at

$$8\pi G_N \equiv \frac{8\pi}{M_{Pl}^2} = 1\,, \tag{3.2}$$

such that the Einstein's equations in (1.11) simply read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}\,. \tag{3.3}$$

We should note that these equations are at the second order respect to the metric. Despite to the fact that as early as 1919 (four years after GR had been formulated), proposals started to be put forward as to how to extend GR, the theory of Einstein gained successful results in describing the Universe and the Solar System. Thus, for several years GR has been considered as the correct theory able to explain the history of our Universe.

According with the Cosmological Principle, the Universe is homogeneous and isotropic and, since we know that its topology is flat, is described by a flat Friedmann-Lemaitre-Robertson Walker (FLRW) metric in the form,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{3.4}$$

where $a = a(t)$ is the scale factor depending on the cosmological time. In this case, the only two independent components of Einstein's field equations read

$$3H^2 = \rho\,, \tag{3.5}$$

$$-(2\dot{H} + 3H^2) = p\,, \tag{3.6}$$

where $H = \dot{a}/a$ is the Hubble parameter, the dot denoting the time derivative, and $\rho \equiv \rho(t)$ and $p \equiv p(t)$ are the matter energy density and pressure, respectively.

The continuity equation of the matter[1], $\nabla^\mu T_{\mu\nu} = 0$, leads to

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{3.7}$$

This equation is a direct consequence of (3.5)–(3.6) together. We also may introduce an Equation of State (EoS) parameter

$$\omega = \frac{p}{\rho}\,. \tag{3.8}$$

For a perfect fluid $\omega =$ const. In the specific, for matter $\omega = 0$ and for radiation $\omega = 1/3$. In such a case, the solution of (3.7) is given by

$$\rho(t) = \rho_0 a(t)^{-3(1+\omega)}\,, \tag{3.9}$$

with ρ_0 positive constant. Now, one may use the Friedmann equations to arrive directly to the explicit solution for $a(t)$, a textbook result,

$$a(t) = a_0(t_0 - t)^{\frac{2}{3(1+\omega)}}\,, \quad \omega \neq -1\,, \tag{3.10}$$

[1]This is a consequence of the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$.

$$a(t) = a_0 \mathrm{e}^{H_0(t)}\,, \quad \omega = -1\,, \tag{3.11}$$

with $a_0\,, t_0\,, H_0$ constants and $0 < a$. In an expanding Universe it must be $0 < \dot{a}$ (and therefore $0 < H$). If $-1 < \omega$ we can set $t_0 = 0$ and take $\tilde{a}_0 = a_0(-1)^{2/(3(1+\omega))}$ positive and real. We find

$$H(t) = \frac{2}{3(1+\omega)t}\,, \tag{3.12}$$

and

$$\frac{\ddot{a}}{a} \equiv H^2 + \dot{H} = -\frac{2(3\omega+1)}{9(1+\omega)^2 t^2}\,, \tag{3.13}$$

such that we have acceleration only if $\omega < -1/3$.

The case when $\omega < -1$ must be treated in a different way. In order to have an expansion, we must choose $0 < t_0$ and $0 < a_0$ in (3.10) with $t < t_0$. Thus,

$$H(t) = -\frac{2}{3(1+\omega)(t_0 - t)}\,, \tag{3.14}$$

and

$$\frac{\ddot{a}}{a} = -\frac{2(3\omega+1)}{9(1+\omega)^2(t_0-t)^2}\,, \tag{3.15}$$

such that now we always have an acceleration. Here, when t is close to t_0 (in fact t_0 can be identified with the total age of the Universe), the Hubble parameter diverges and we have the Big Rip [113].

The special case when $\omega = -1$ in (3.11), whose corresponding fluid was dubbed "dark energy", corresponds to the flat de Sitter solution considered in the preceding chapters. To have an expansion it must be $0 < H_0$ and one gets

$$H(t) = H_0\,, \tag{3.16}$$

and

$$\frac{\ddot{a}}{a} = H_0^2\,, \tag{3.17}$$

such that we obtain an acceleration. A fluid with Equation of State $p = -\rho$ can be easily obtained by introducing in the Einstein's equation (3.3) a cosmological constant Λ as in Eq. (1.33). Therefore, the cosmological constant appears as the energy density of a perfect fluid with EoS parameter $\omega = -1$. Such simple extension of the theory of Einstein brings to the ΛCDM model which unifies in the description the cold dark matter (see later) and the dark energy for supporting the current accelerated expansion of the Universe.

In general, we can derive the following relation from the Friedmann equations (3.5)–(3.6),

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)\,. \tag{3.18}$$

As a consequence, fluids which violate the so called "Strong Energy condition" (SEC),

$$p < -\frac{\rho}{3}\,, \tag{3.19}$$

produce an inflationary expansion (or, in the opposite description, decelerate a contraction). The perfect fluid cases for expanding Universe analyzed above are recovered.

If one supposes that only matter ($\omega = 0$) and radiation ($\omega = 1/3$) perfect fluids are present (producing matter and radiation eras), the emerging picture is the one of a decelerated expanding Universe. However, in the last two decades, we got several evidences related to the existence of an early-time acceleration after the Big Bang, namely the "inflation" [24, 25], and the existence of a current accelerated cosmological expansion [13], namely the "dark energy" (DE) issue. Furthermore, the concordance cosmology model suggests the presence of non-baryonic dark matter (DM) in our Universe. Dark matter shares properties very similar to ordinary matter and satisfies the strong energy condition (it is repulsive from gravitational point of view), but cannot be detected since does not couple with electromagnetic fields. We also mention that the presence of dark matter in the Universe is revealed at the galactic scale (rotation curves of galaxies).

It thus appears that a correct picture of the Universe requires at least three extra (possibly dark) cosmological components: one more dark matter component, some form of dark energy[2], and one or more inflation fields.

For this reasons, today it is well accepted the idea that Einstein's gravity may be not the ultimate theory of gravity, but an extremely good approximation at the present day of detection. In particular, the cosmic acceleration may be due to several factors: modifications of the theory of gravitation, the presence of the Cosmological Constant, the existence of a scalar field (the inflation), or effects of some exotic fluid departing from the standard matter and radiation. In this respect, after the latest accurate measurements of the anisotropy in our observable Universe and the indications that early-time inflation is related to the ultraviolet completeness of Einstein's gravity, the possibilities offered by modified theories of gravity have reached higher degree of corroboration.

By modified theories we mean a generalization of Einstein's gravity, where some combinations of curvature invariants (the Ricci scalar, namely $F(R)$-gravity, contractions of the Riemann and Ricci tensors, Gauss-Bonnet, Weyl tensor, d'Alambertian of Ricci or any other non-local terms...), replace or are added into the classical Hilbert-Einstein action of GR. In this way, the acceleration becomes an effect of the curvature, without the introduction of any dark component.

Constructing metric theories of gravity whose equations differ from those of GR requires at least one of the following to be satisfied (Lovelok's theorem):

- Presence of other fields apart the metric tensor;
- Work in a number of dimensions different from 4;
- Accept metric derivatives of degree higher than 2 in the field equations;
- Give up locality or Lorentz invariance.

[2]Today the Universe is composed for the 71.4% of dark energy, for the 24% of dark matter and only for a 4.6% of ordinary matter [114].

In these chapters we will not consider extra dimension theories and we will preserve the locality and the Lorentz invariance. In the specific, in the attempt to analyze alternative theories for accelerating Universe which also include the dark matter phenomenology, we will deal with mimetic dark matter and Horndeski theories, where, as we will see, the introduction of a scalar (mimetic) field coupled with the curvature invariants still allow to working with second order field equations like in GR. In what follows, we will review the basis of mimetic gravity.

3.2 Mimetic gravity

In 2013 Mukhanov and Chamseddine [50] introduced a parametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi \equiv \phi(x^\mu)$, dubbed mimetic field, in the following form

$$g_{\mu\nu} = -\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,. \tag{3.20}$$

In such a way, the conformal degree of freedom of gravity is explicitly isolated, since the auxiliary metric is invariant under conformal transformations of the type $\tilde{g}_{\mu\nu} \to \Omega(t,\mathbf{x})^2\tilde{g}_{\mu\nu}$, $\Omega(t,\mathbf{x})$ being a function of the space-time coordinates. Thus, as a consistency condition, the mimetic field must satisfy the (mimetic) constraint:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1\,. \tag{3.21}$$

By using the metric parametrization (3.20), the Hilbert-Einstein action (3.1) reads

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g\left(\tilde{g}_{\mu\nu},\phi\right)}\left[\frac{R(\tilde{g}_{\mu\nu},\phi)}{2} + \mathcal{L}_m\right]\,. \tag{3.22}$$

The variation with respect to the physical metric must be done with care, since one has to take into account the dependencies in (3.20). Thus, the variations respect to the auxiliary metric and the mimetic field bring to

$$G_{\mu\nu} = T_{\mu\nu} - (G-T)\partial_\mu\phi\partial_\nu\phi\,, \tag{3.23}$$

where $G(=-R)$ and T are the trace of the Einstein's tensor and the stress energy tensor of matter, respectively, and the auxiliary metric does not appear explicitly in the field equations.

If we confront this expression with the Einstein's equations (3.3), we immediately see that the mimetic field contributes to the right hand side of Einstein's equation through the additional stress-energy tensor component:

$$\tilde{T}_{\mu\nu} = -(G-T)\partial_\mu\phi\partial_\nu\phi\,. \tag{3.24}$$

Both energy-momentum tensor, $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$, are covariantly conserved, i.e. $\nabla^\mu T_{\mu\nu} = \nabla^\mu\tilde{T}_{\mu\nu} = 0$, whereas the continuity equation for $\tilde{T}_{\mu\nu}$ with the mimetic constraint (3.21) leads to:

$$\nabla^\kappa\left((G-T)\partial_\kappa\phi\right) \equiv \frac{1}{\sqrt{-g}}\partial_\kappa\left(\sqrt{-g}(G-T)g^{\kappa\sigma}\partial_\sigma\phi\right) = 0\,. \tag{3.25}$$

Finally, the trace of Eq. (3.23) is derived as

$$(G-T)(1+g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)=0\,, \tag{3.26}$$

and it is automatically satisfied due to (3.21). Thus, the system above admits not trivial solutions even for $G \neq T$ and the conformal degree of freedom becomes dynamical even in the absence of matter ($T = 0$ but $G \neq 0$).

We should note that, since the stress-energy tensor of a perfect fluid with energy density ρ and pressure p is given by

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}\,,\quad u_\mu u^\mu=-1\,, \tag{3.27}$$

the mimetic stress-energy tensor in Eq.(3.24) assumes in fact the same form of the one of a perfect fluid with energy density and pressure,

$$\rho=-(G-T)\,,\quad p=0\,, \tag{3.28}$$

while the gradient of the mimetic field, $\partial_\mu\phi$, turns out to be the four-velocity vector u_μ. In other words, the mimetic field plays the role of pressurless dark matter.

All this is very interesting from cosmological point of view. On FLRW space-time (3.4), the constraint in (3.21) brings to the identification (up to a constant) of the mimetic field with the cosmological time,

$$\phi=t\,, \tag{3.29}$$

and the field can be viewed like a "clock" on an FLRW background. In this case, the energy density of mimetic dark matter $\rho = (G-T)$ decays with the scale factor of the FLRW Universe as $(G-T) \propto 1/a^3$, and we recover (3.9) with $\omega = 0$.

In conclusion, the conformal degree of freedom of gravity can mimic the behavior of dark matter, hence the name "mimetic dark matter". In the present work we will deal with FLRW space-time only. For other applications of mimetic dark matter at the galactic scale see Ref. [115, 116].

An alternative but equivalent formulation of mimetic gravity is given by the Lagrange multiplier approach [51] starting from the action

$$I=\int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{R}{2}+\lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1)+\mathcal{L}_m\right]\,. \tag{3.30}$$

Here, λ is a Lagrange multiplier and the variation of the action respect to it leads to (3.21), while the variation respect to the metric $g_{\mu\nu}$ yields to

$$G_{\mu\nu}-T_{\mu\nu}+\lambda\partial_\mu\phi\partial_\nu\phi=0\,, \tag{3.31}$$

with the trace

$$\lambda=(G-T)\,. \tag{3.32}$$

Thus, one recovers Eq.(3.23) again.

3.2.1 Disformal transformations and mimetic gravity

To conclude this section, it may be interesting to understand the reasons for which the parametrization of the physical metric in (3.20) brings to new solutions respect to GR [117]. General Relativity is invariant under diffeomorphisms, namely we are free to parametrize a metric $g_{\mu\nu}$ in terms of a fiducial metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ. A generic "disformal transformation" is defined by

$$g_{\mu\nu} = \mathcal{A}(\phi, X)\tilde{g}_{\mu\nu} + \mathcal{B}(\phi, X)\partial_\mu\phi\partial_\nu\phi\,, \qquad (3.33)$$

where $X \equiv -\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. The functions $\mathcal{A}(\phi, X) \neq 0$ and $\mathcal{B}(\phi, X)$ are called conformal factor and disformal factor, respectively. In general these functions are arbitrary and, if the transformation is invertible, they bring to the same equations to the those obtained from the metric $g_{\mu\nu}$. However, if the disformal transformation is non-invertible, or singular, we acquire extra degrees of freedom which result in equations of motion. This is the case when

$$\mathcal{B}(X, \phi) = -\frac{\mathcal{A}(X, \phi)}{X} + \mathcal{E}(\phi)\,, \qquad (3.34)$$

where $\mathcal{E}(\phi) \neq 0$ is an arbitrary function. The parametrization in (3.20) for mimetic gravity can be identified with a singular disformal transformation, with $\mathcal{A} = 2X$ and $\mathcal{B} = 0$ in Eq. (3.33), and correspondingly $\mathcal{E}(\phi) = 2$ in Eq. (3.34). As a result, the system possesses additional degree of freedom which in our case mimics a dust component.

3.3 Generalizations of mimetic gravity

By using suitable field potentials into the action, mimetic gravity can be used to unify the dark energy description with new cosmological scenarios absent in GR. In particular, it could be interesting if at an early-time the mimetic field can support an inflationary expansion and only later mimics dark matter.

As we have seen, in FLRW space-time the mimetic field plays the role of "clock" and corresponds to the cosmological time as in (3.29). Thus, at a cosmological level, a field-dependent potential into the action corresponds to a time-dependent potential which brings to a time-varying Hubble parameter (and therefore scale factor). Thus, by using such a mechanism, one can in principle reconstruct any desired expansion history of the Universe [51].

The extended action of mimetic gravity in the Lagrange multiplier formulation of Eq. (3.30) is given by

$$I = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1) - V(\phi) + 2\mathcal{L}_m\right]\,, \qquad (3.35)$$

where $V \equiv V(\phi)$ is a potential for the mimetic field ϕ and $\lambda \to \lambda/2$ respect to (3.30).

The variation with respect to the Lagrange multiplier λ still leads to (3.21), while the variation with respect to the auxiliary metric reads

$$G_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) = T_{\mu\nu}\,, \tag{3.36}$$

which, by taking the trace, can be used to find the Lagrange multiplier, namely

$$\lambda = \frac{1}{2}\left(G - T - 4V\right)\,. \tag{3.37}$$

Moreover, by plugging Eq. (3.37) into Eq. (3.36),we find

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) + T_{\mu\nu}\,, \tag{3.38}$$

and we see that, when a potential is added to the action, the mimetic field contributes to the effective energy density and pressure as

$$\rho = G - T - 3V\,,\quad p = -V\,. \tag{3.39}$$

When the potential disappears, we recover the dark matter behavior (3.28).

Finally, the variation respect to the field reads

$$\nabla^\nu\left[\left(G - T - 4V\right)\partial_\nu\phi\right] = -\frac{\partial V}{\partial\phi}\,, \tag{3.40}$$

where we have taken into account (3.37). In what follows, we will usually neglect the contribution of ordinary matter/radiation ($T_{\mu\nu} = 0$), such that ρ and p in (3.39) correspond to the total effective energy density and pressure of the model.

As we anticipated, on FLRW metric, the mimetic field is identified with the cosmological time as in (3.29) and $V(\phi) = V(t)$, greatly simplifying the equations of motion. In this case, (3.39)–(3.40) reduce to [51],

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3\left(\rho - V\right)\right] = -\frac{dV}{dt}\,, \tag{3.41}$$

which can be integrated to give

$$\rho = \frac{3}{a^3}\int da\; a^2 V\,, \tag{3.42}$$

whereas the pressure remains $p = -V$. From (3.36) one gets the Friedmann-like equations (3.5)–(3.6) where the energy density and pressure read like in (3.39). Thus, the second Friedmann equation can be written as

$$2\dot{H} + 3H^2 = V(t)\,, \tag{3.43}$$

or

$$\ddot{y} - \frac{3}{4}V(t)y = 0\,,\quad y \equiv a^{3/2}\,. \tag{3.44}$$

Let us see some simple examples of potential leading to some interesting solutions for the Hubble parameter.

3.3.1 Quadratic inverse field potential

Let us start with the following potential, first proposed by Chamseddine, Mukhanov and Vikman in Ref. [51],

$$V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2}\,, \tag{3.45}$$

where α is a constant. From (3.44), when $-1/3 \leq \alpha$, one has[3],

$$a(t) = t^{\frac{1}{3}(1+\sqrt{1+3\alpha})}\left(1+\beta t^{-\sqrt{1+3\alpha}}\right)^{\frac{2}{3}}\,, \tag{3.46}$$

where β is an integration constant. Now by using the Eq. (3.42), one may calculate the effective EoS parameter as in (3.8), namely

$$w = -3\alpha\left(1+\sqrt{1+3\alpha}\frac{1-\beta t^{-\sqrt{1+3\alpha}}}{1+\beta t^{-\sqrt{1+3\alpha}}}\right)^{-2}\,. \tag{3.47}$$

We note that, for $|\alpha| \ll 1$, the potential disappears and we recover the dark matter phenomenology, while for $1 \ll \alpha$, ω approaches to minus one, i.e. a cosmological constant, at a late times. Thus, a dynamic behavior for α may help to unify the dark matter and the current accelerated expansion scenarios.

In Fig. 3.1 we depict the potential $V(\phi)$ in (3.45) for different values of α. We see that the potential asymptotically tends to vanish, while for small values of ϕ grows up. The shape is determined by α.

It is also interesting to analyze the case when mimetic field is a subdominant energy component of the Universe, which is instead dominated by another form of matter with EoS $\omega = \tilde{\omega}$ and whose scale factor evolves as

$$a \propto t^{\frac{2}{3(1+\tilde{\omega})}}\,. \tag{3.48}$$

Thus, from Eq. (3.42) we infer the following behavior for the energy density associated to the mimetic field,

$$\rho = -\frac{\alpha}{\tilde{\omega} t^2}\,. \tag{3.49}$$

Given that the pressure of mimetic matter reads $p = -V$, we get the mimetic field EoS parameter

$$\omega = \tilde{\omega}\,, \tag{3.50}$$

which shows that mimetic field, when subdominant, can imitate the EoS of the dominant energy component.

[3]If one chooses $\alpha < -1/3$, the scale factor will describe an oscillating flat Universe with amplitude of oscillations which grows with time, but singularities will appear rendering non-physical the solution.

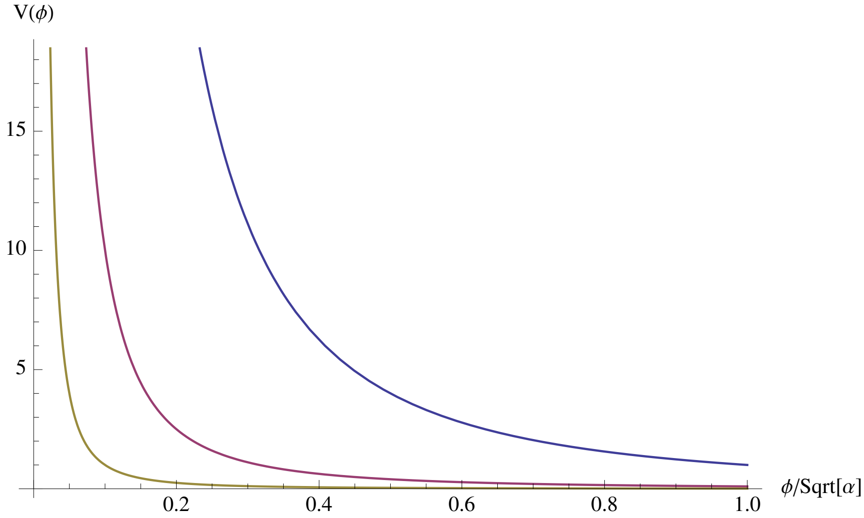

Figure 3.1: $V(\phi)$ *versus* $\phi/\sqrt{\alpha}$ for different values of α, namely $\alpha = 1$ (blue line), $\alpha = 10^{-1}$ (red line), $\alpha = 10^{-2}$ (yellow line) (model (3.45)).

3.3.2 Power law field potential

If one considers the following power-law potential

$$V(\phi) = \alpha\phi^n = \alpha t^n\,, \tag{3.51}$$

$\alpha\,, n$ being generic parameters, the solution of Eq.(3.44) can be written in terms of the Bessel functions $Z_\lambda(t)$ [51],

$$y = t^{\frac{1}{2}} Z_{\frac{1}{n+2}}\left(\frac{\sqrt{-3\alpha}}{n+2} t^{\frac{n+2}{2}}\right)\,. \tag{3.52}$$

When $n = -2$ we recover the case of the preceding subsection. When $n < -2$ the limiting behavior of the scale factor is that of a dust-dominated Universe, with EoS parameter $w = 0$. When $-2 < n$ and $\alpha < 0$, the pressure is positive and we recover the solution for a singular oscillating Universe. When $-2 < n$ and $0 < \alpha$ instead, the pressure is negative and we can find an acceleration. In the specific, for $n = 0$ we get a cosmological constant, since the potential is a constant, whereas $n = 2$ gives an inflationary expansion solution with scale factor,

$$a \propto t^{-\frac{1}{3}} e^{\sqrt{\frac{\alpha}{12}} t^2}\,, \tag{3.53}$$

as in the chaotic inflation with quadratic potential.

In Fig. 3.2 we depict the potential $V(\phi)$ in (3.51) for $n = 2$. We see that when the magnitude of the field is large the potential grows up, while the potential vanishes when the field tends to zero. This behavior is the typical one of a potential for supporting the early-time acceleration of the Universe.

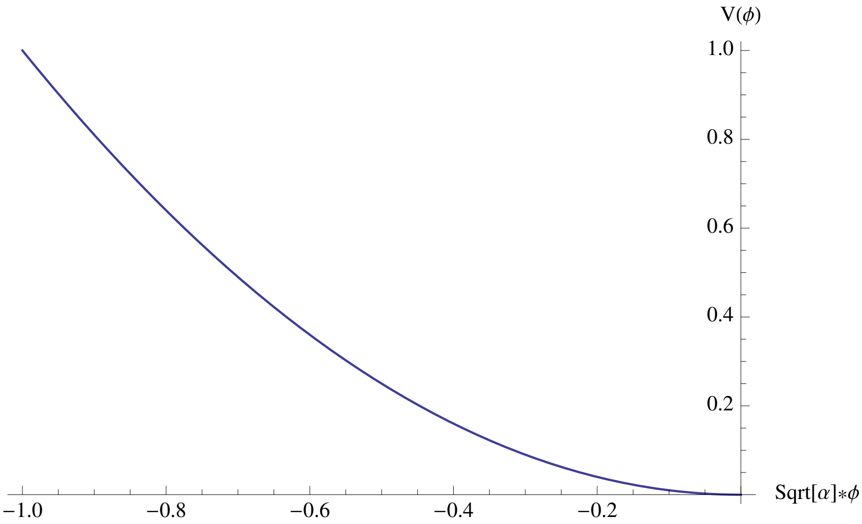

Figure 3.2: $V(\phi)$ *versus* $\sqrt{\alpha}\phi$ for negative values of the field (model (3.51) with $n = 2$).

3.3.3 Inflationary potentials

One may use a reconstruction technique to reproduce with our extended model of mimetic gravity any desidered solution. In particular, one can check for some solution for the early-time inflation, in the attempt to unify the early-time Universe with the matter dominated Universe.

For example, an exponential scale factor ($a \propto e^{-t^2}$, i.e. strongly accelerated solution with graceful exit to matter era) for large negative times, and $a \propto t^{2/3}$ (i.e. matter era) at late time corresponds to the potential [51],

$$V(\phi) = \frac{\alpha\phi^2}{e^{\phi}+1}\,, \tag{3.54}$$

α being a constant, obtained by inverting Eq. (3.44) with $y = a^{3/2}$ and $\phi = t$.

In Fig. 3.3 we depict the potential $V(\phi)$ in (3.54) with $\alpha = 1$. When the magnitude of the field is large, also the potential is large, while tends to vanish with the field. The potential can be used to describe the early-time inflation.

Another interesting possibility is given by the following scale factor,

$$a(t) = \left[\beta J_0\left(\frac{\sqrt{-3\alpha}}{\kappa}e^{-\frac{kt}{2}}\right) + \gamma Y_0\left(\frac{\sqrt{-3\alpha}}{\kappa}e^{-\frac{kt}{2}}\right)\right]^{\frac{2}{3}}\,, \tag{3.55}$$

which is expressed in terms of Bessel functions J_0 and Y_0 of order zero, of the first and second kind respectively [118]. In the expression above, β and γ are integration constants, k a positive number. At early times ($t \to -\infty$) the behavior of the scale factor corresponds to an inflationary solution when $0 < \alpha$, while at late time ($t \to +\infty$) we recover a matter-dominated Universe. The corresponding potential for such a solution is given by

$$V(\phi) = \alpha e^{-\kappa\phi}\,. \tag{3.56}$$

A similar behavior for the scale factor can be obtained with the potential,

$$V(\phi) = \frac{\alpha\phi^{2n}}{e^{\kappa\phi}+1} = \frac{\alpha t^{2n}}{e^{\kappa t}+1}\,, \tag{3.57}$$

$\alpha\,, n\,, k$ being positive parameters. For other examples see Ref. [119].

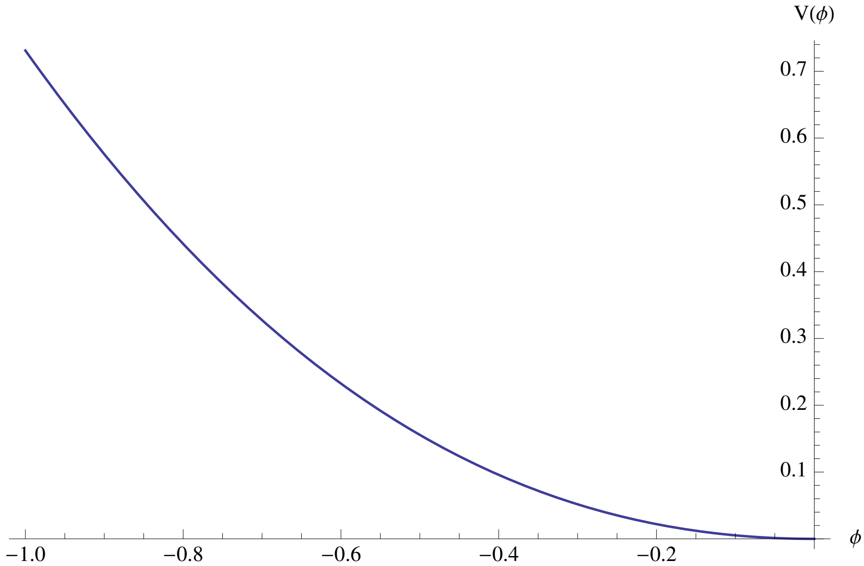

Figure 3.3: $V(\phi)$ *versus* ϕ, for negative values of the field (model (3.54) with $\alpha = 1$).

3.3.4 Bounce potentials

The bounce scenario, where a cosmological contraction ($H < 0$) is followed by an expansion ($0 < H$) at a finite time, was presented as an alternative description respect to the Big Bang theory: instead from an initial singularity, the Universe emerges from a cosmological bounce followed by an accelerating expansion (see Ref. [57] for a review). Once again, we are interested in the reconstruction of mimetic potential for the bouncing Universe with the possibility to recover a matter epoch at the late time. For example, we can consider the following scale factor,

$$a(t) = \left(t^2+1\right)^{\frac{1}{3}}\,, \tag{3.58}$$

with the Hubble parameter

$$H = \frac{2t}{3(t^2+1)}\,. \tag{3.59}$$

At the early time ($t \to -\infty$) and the late time ($t \to +\infty$), one finds a matter contracting and expanding Universe, respectively, while the bounce is located at $t = 0$, when the

Hubble parameter vanishes. The corresponding mimetic potential is given by

$$V(\phi) = \frac{4}{3}\frac{1}{(1+\phi^2)^2}\,. \tag{3.60}$$

A detailed analysis on the possibilities offered from this kind of mimetic model can be found in Ref. [51] again.

3.4 Cosmological perturbations in mimetic gravity

The study of cosmological perturbations is crucial for every theory with the aim of reproducing the inflationary scenario, since it is well known that perturbations left at the end of the early-time expansion of the Universe are at the origin of the anisotropies of our observable Universe. As we will see, the extended models of mimetic gravity suffer of the fact that the sound speed vanishes, and the perturbations cannot propagate, requiring a further modification of the theory. In the following, we will consider the action in (3.35) with $\mathcal{L}_m = 0$.

The perturbations around the FLRW metric (3.4) read, in the Newton's gauge,

$$ds^2 = -(1+2\Phi(t,\mathbf{x}))dt^2 + a^2(t)(1-2\Psi(t,\mathbf{x}))\delta_{ij}dx^i dx^j\,,\quad i,j=1,2,3\,, \tag{3.61}$$

$\Phi \equiv \Phi(t,\mathbf{x})$ and $\Psi \equiv \Psi(t,\mathbf{x})$ being functions of the space-time coordinates such that $|\Phi(t,x)|$, $|\Psi(t,x)| \ll 1$, and $g^{00}(t,x) \simeq -1+2\Phi(t,x)$, $g^{11}(t,x) \simeq a(t)^{-2}(1+2\Psi(t,x))$.

From the (i,j)-components of field equations (3.36), where $i,j=1,2,3$ and $i \neq j$, we immediately get

$$\Phi(t,\mathbf{x}) = \Psi(t,\mathbf{x})\,. \tag{3.62}$$

The perturbation around the field is given by

$$\phi = t + \delta\phi(t,\mathbf{x})\,, \tag{3.63}$$

where $|\delta\phi| \equiv |\delta\phi(t,x)| \ll 1$, which together with Eq.(3.21) leads to

$$\Phi = \delta\dot{\phi}\,. \tag{3.64}$$

Now, if one perturbs the components $(0,i)$, $i=1,2,3$ of the field equations (3.36), by using the perturbed $(0,0)$-component it is possible to derive the closed equation [51],

$$\delta\ddot{\phi} + H\delta\dot{\phi} + \dot{H}\delta\phi = 0\,. \tag{3.65}$$

Given the Laplacian operator, $\triangle \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$, the perturbation around the field (and therefore to the metric) can propagate if in its equation of motion appears a term in the form $c_s^2\Delta\delta\phi$, where c_s^2 is the speed of sound (in the simplest models of chaotic inflation one has $c_s^2 = 1$, while in general $c_s^2 < 1$ and it can depends on the value of the field itself).

Thus, it turns out that in Eq. (3.65) the speed of sound is identically zero. This behavior is confirmed by more general models of mimetic gravity where the equations of motion remain at the second order, namely in the mimetic Horndeski gravity models that we will analyze in the next chapter [56]. Such a feature comes from the fact that the constraint on the scalar field takes out the degree of freedom associated to the field, and it is not possible to have oscillating wave-like solutions.

In order to deal with a theory whose quantum perturbations can be defined in a sensible way, the action (3.35) has to be modified introducing higher derivative terms which lead to higher order field equations. An example has been presented in Ref. [51],

$$I = d^4x\sqrt{-g}\;\left[R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1) - V(\phi) + \frac{1}{2}\gamma(\Box\phi)^2\right]\,, \tag{3.66}$$

which brings to

$$G^\mu_\nu = \left[V + \gamma\left(\partial_\alpha\phi\partial^\alpha\chi\right)\right]\delta^\mu_\nu + 2\lambda\partial^\mu\phi\partial_\nu\phi - \gamma(\partial^\mu\chi\partial_\nu\phi + \partial^\mu\phi\partial_\nu\chi)\,,\quad \chi \equiv \Box\phi\,, \tag{3.67}$$

where γ is a constant.

By using the mimetic constraint $\phi = t$, on FLRW background we obtain

$$\chi = 3H\,, \tag{3.68}$$

$$2\dot{H} + 3H^2 = \frac{2}{2-3\gamma}V\,, \tag{3.69}$$

namely the Freidmann-like equations remain at the second order with the potential renormalized respect to the case $\gamma = 0$, such that one can easily get all the solutions of the original model in (3.35).

However, at the level of perturbations, now we obtain

$$\ddot{\delta\phi} + H\dot{\delta\phi} - \frac{c_s^2}{a(t)^2}\Delta\delta\phi + \dot{H}\delta\phi = 0\,, \tag{3.70}$$

with

$$c_s^2 = \frac{\gamma}{2-3\gamma}\,. \tag{3.71}$$

Therefore, in this case the perturbations can be propagated and the model can be used to produce a realistic description of our Universe when $\gamma \neq 0$.

Chapter 4

Horndeski mimetic gravity models

In this chapter we introduce the Horndeski scalar-tensor theories of gravity, where a scalar field is coupled with the gravitational invariants (the Einstein's tensor, the Ricci scalar...) and the field equations remain at the second order like in General Relativity. If one identifies the scalar field with a mimetic field mimicking the dark matter, the theory can naturally lead to the dark matter phenomenology. We analyze a subclass of mimetic Horndeski gravity models and we find several solutions for the early-time Universe (in particular, the cosmological bounce) and the late-time expansion (the dark energy). Since in mimetic Horndeski gravity the cosmological perturbations at the origin of the galactic inhomogeneities cannot propagate, we suggest a way to modify the theory to correctly reproduce the cosmological perturbations without destroying the solutions obtained for the Friedmann Universe. We remember that the existence of the cosmological perturbations is one of the most important prediction of the early-time inflationary paradigm.

4.1 Horndeski gravity

In 1974 Horndeski [55] showed that the action of the most general 4D local scalar-tensor theory with equations of motion no higher than second order is given by

$$I = \int_{\mathcal{M}} dx^4 \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}_H + L_m \right] , \quad \mathcal{L}_H = \sum_{i=2}^{5} \mathcal{L}_i \,, \tag{4.1}$$

where we work in Planck units (3.2) and

$$\mathcal{L}_2 = P(\phi, X) \,,$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi \,,$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] \,,$$

$$\begin{aligned} \mathcal{L}_5 \;\; = \;\; & G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}[(\Box\phi)^3 - \\ & 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)] \,. \end{aligned} \tag{4.2}$$

In Eq. (4.1), R is the Hilbert-Einstein action of GR, $\mathcal{L}_m$ is again the Lagrangian of the matter contents of the space-time, while L_H includes the higher curvature corrections to GR expressed by (4.2), where we see that a scalar field is coupled with gravity. Here, $P(\phi, X)$ and $G_i(\phi, X)$ with $i = 3, 4, 5$ are functions of the scalar field ϕ and its kinetic energy,

$$X = -\frac{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{2} . \tag{4.3}$$

In particular, $P(\phi, X)$ is the Lagrangian (i.e. the pressure) of the (k-essence) scalar field whose stress-energy tensor reads [120, 121, 122],

$$T^{\mu}_{(\phi)\nu} = (\rho(\phi, X) + p(\phi, X))u^\mu u_\nu + p(\phi, X)\delta^\mu_\nu , \quad u_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}} , \tag{4.4}$$

where u_μ is a four-velocity vector, δ^μ_ν is the four dimensional unit matrix ($diag(\delta^\mu_\nu) = (1, 1, 1, 1)$) and $P(\phi, X) \equiv p(\phi, X)$ corresponds to the effective pressure of k-essence and $\rho(\phi, X)$ to its energy density, due to the fact that the variation respect to the metric leads to

$$\rho(\phi, X) = 2X\frac{\partial p(\phi, X)}{\partial X} - p(\phi, X) . \tag{4.5}$$

For example, for canonical scalar field one has

$$P(\phi, X) \equiv p(\phi, X) = X - V(\phi) , \quad \rho(\phi, X) = X + V(\phi) , \tag{4.6}$$

where $V(\phi)$ is a potential for the field, while generic k-essence Lagrangians contain higher order kinetic term.

In this section, we will consider the following subclass of Horndeski model with fertile applications in cosmology,

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R}{2} + \mathcal{L}_m + P(\phi, X)\right) + I_H , \tag{4.7}$$

where $\mathcal{L}_m$ denotes the matter Lagrangian again, $P(\phi, X)$ is the Lagrangian of a generic k-essence fluid, and

$$I_H = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\alpha\left(G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\right) + \gamma\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \beta\phi\Box\phi\right] , \tag{4.8}$$

with $\alpha\,, \beta\,, \gamma$ constant coefficients, whose dimension, if one reintroduces the Planck Mass, is $[\alpha] = [\gamma] = [M_{Pl}^{-4}]$. The action above corresponds to (4.1)–(4.2) with

$$\mathcal{L}_2 = P(\phi, X) , \quad G_3 = \beta\phi , \quad G_4 = \alpha X , \quad G_5 = \gamma\phi , \tag{4.9}$$

due to the fact that after integration by part we can also write,

$$\begin{aligned}\int_{\mathcal{M}} d^4x\sqrt{-g}G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi &= \int_{\mathcal{M}} d^4x\sqrt{-g}\left(-\frac{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi R}{2} + (\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right) , \\ &\int_{\mathcal{M}} d^4x\sqrt{-g}\left(XR + (\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right) . \end{aligned} \tag{4.10}$$

This equality can be easily demonstrated in the Minkowski space-time, where

$$\int_{\mathcal{M}} d^4x(\Box\phi)^2 = \int_{\mathcal{M}} d^4x\,(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)\ . \tag{4.11}$$

Thus, Equation (4.10) is the generalization to curved space-time [124].

We also should note

$$\int_{\mathcal{M}} d^4x\sqrt{-g}G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi = -\int_{\mathcal{M}} d^4x\sqrt{-g}\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \int_{\mathcal{M}} d^4x\sqrt{-g}G_{\mu\nu}\nabla^\mu[\phi\nabla^\nu\phi]\,, \tag{4.12}$$

but the second term vanishes after an integration by parts due to the fact that $\nabla^\mu G_{\mu\nu} = 0$,

$$\int_{\mathcal{M}} d^4x\sqrt{-g}G_{\mu\nu}\nabla^\mu[\phi\nabla^\nu\phi] = -\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla^\mu G_{\mu\nu}[\phi\nabla^\nu\phi] = 0\,. \tag{4.13}$$

It follows that, in the special case where $\alpha = \gamma$, the corresponding Horndeski contributes must disappear from the Equations of motion. Other examples of Horndeski Lagrangians are been recently considered in Ref. [125].

4.2 Friedmann Universe and cosmological perturbations in Horndeski gravity

In this section, we will briefly review the Friedmann equations and the cosmological perturbations in the Horndeski model (4.7)–(4.8). For the general description see Refs. [123, 124]. On FLRW space-time (3.4) we get

$$X = \frac{\dot\phi^2}{2}\,, \tag{4.14}$$

and the Equations of motion (EOMs) read

$$3H^2(1 - 3\alpha\dot\phi^2 + 3\gamma\dot\phi^2) = \rho + \rho(\phi, X) - \beta\dot\phi^2\,, \tag{4.15}$$

$$\begin{aligned} -(2\dot H + 3H^2) \;=\; & p + p(\phi, X) - \beta\dot\phi + \alpha\dot\phi^2(3H^2 + 2\dot H) - 6\alpha H^2\dot\phi^2 - 4\alpha H\dot\phi\ddot\phi - 4\alpha\dot H\dot\phi^2 \\ & +\gamma(2\dot H\dot\phi^2 + 4H\dot\phi\ddot\phi + 3H^2\dot\phi^2)\,, \end{aligned} \tag{4.16}$$

where, as usually, we adopt the convention according to which the dot denotes the derivative with respect to the time and $\rho\,, p$ are the energy density and pressure of ordinary matter which satisfies the energy conservation law as in (3.7). By combining the Friedmann-like equations we obtain the continuity equation of k-essence coupled with gravity,

$$\begin{aligned} &\dot\rho(\phi, X) + 3H(\rho(\phi, X) + p(\phi, X)) = \\ &-\ddot\phi\dot\phi(-2\beta + 6\alpha H^2 - 6\gamma H^2) - 3H\dot\phi^2(-2\beta + 6\alpha H^2 - 6\gamma H^2) - 12H\dot H\dot\phi^2(\alpha - \gamma)\,, \end{aligned} \tag{4.17}$$

where

$$\rho(\phi, X) + p(\phi, X) = 2Xp_X(\phi, X)\,. \tag{4.18}$$

Despite to the simplified form of the Lagrangian, the considered model of Horndeski gravity offers the possibility to reproduce a huge variety of cosmological solutions. In particular, the higher curvature corrections to Einstein's gravity (maybe related to quantum effects) may support the early-time inflation. In this respect, we should also note that generic models of k-essence are a valid alternative with respect to the classical canonical scalar field (4.6) of the old inflationary scenario, since the presence of higher order kinetic term permits to suppressing the sound speed leading to a negligible tensor-to-scalar ratio according with cosmological observations.

The evolution of inflationary Universe is described by the e-folds number left to the end of inflation,

$$N = \log\left[\frac{a(t_0)}{a(t)}\right]\,, \tag{4.19}$$

where $a(t_0)$ is the scale factor at the time t_0 when the eraly-time acceleration finishes. Thus, by taking into account that $dN = -Hdt$, we derive from (4.15) and (4.17),

$$3H^2 + 9\tilde{\alpha}H^4\phi'^2 = \rho(\phi, X) + \tilde{\beta}H^2\frac{\phi'^2}{2}\,, \tag{4.20}$$

$$-\rho'(\phi, X)+3H^2\phi'^2(p_X(\phi, X)) = H^2\phi'\phi''(\tilde{\beta}-6\tilde{\alpha}H^2)+HH'\phi'^2(\tilde{\beta}-18\tilde{\alpha}H^2)-3H^2\phi'^2(\tilde{\beta}-6\tilde{\alpha}H^2)\,. \tag{4.21}$$

In the expressions above we posed $\rho = p = 0$, since we omit the contribution of ordinary matter in the early-time Universe. The prime denotes the derivative with respect to N and

$$X = \frac{H^2\phi'^2}{2}\,. \tag{4.22}$$

Moreover, for the sake of simplicity we posed

$$\tilde{\alpha} = \gamma - \alpha\,, \quad \tilde{\beta} = -2\beta\,. \tag{4.23}$$

During the primordial accelerated expansion the Universe is in a de Sitter phase and the Hubble parameter is large and almost constant,

$$H \simeq H_{\mathrm{dS}}\,. \tag{4.24}$$

It follows that in a realistic inflationary scenario the field must be also almost a constant, such that

$$X \ll 1\,. \tag{4.25}$$

If we assume

$$H^2\phi'^2 \ll |1/\tilde{\alpha}|\,, \tag{4.26}$$

equation (4.20) reads

$$3H^2 \simeq \rho(\phi, X)\,, \tag{4.27}$$

and the energy density of the field gives the value of the Hubble parameter like in the old inflation scenario. We mention that during inflation the Hubble parameter cannot exceed the Planck Mass ($H < 1$), but the field can acquire superplanckian values. One of the most important test bed for the inflationary models is represented by the reproduction of the exit from the early-time accelerated period. During inflation the field slowly moves and its magnitude typically decreases with the energy density (and therefore the Hubble parameter). In the slow-roll regime with $|H'/H| \ll 1$ and $|\phi''| \ll |\phi'|$, Equation (4.21) leads to

$$\rho'(\phi, X) - 3H^2\phi'^2 p_X(\phi, X) \simeq 3H^2\phi'^2(\tilde{\beta} - 6\tilde{\alpha}H^2) . \tag{4.28}$$

Usually one assumes that at the beginning of inflation the field is negative and its magnitude very large, while at the end, when $N \to 0$, they tend to vanish, i.e. $\phi' < 0$ and $0 < \rho'(\phi, X)$, while $0 < dX/dN$ to exit from inflation. Since

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} , \tag{4.29}$$

the slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} \equiv \frac{H'}{H} , \tag{4.30}$$

is positive and small at the beginning of inflation, while is on the order of the unit when acceleration ends. At that time, the total e-folds

$$\mathcal{N} \equiv N(a(t_{\mathrm{i}})) , \tag{4.31}$$

t_{i} being the initial time of inflation, must be large enough to explain the thermalization of observable Universe,

$$55 < \mathcal{N} < 65 . \tag{4.32}$$

Starting from the formalism and the equations above, one may reproduce some interesting inflationary scenarios (see for Ref. [126]). Here, we would like to point out that, if on one side the inflation must bring to the homogeneity of our Friedmann Universe, on the other side must be able to describe the anisotropies at the origin of the formation of the galaxies. Thus, it is extremely important for every theory of inflation describe the cosmological perturbations and confront the results with cosmological observations.

4.2.1 Cosmological perturbations

Perturbations around flat FLRW metric (3.61) can be parametrized as [123, 124, 127, 128, 129, 130, 131, 132],

$$ds^2 = -[(1 + \alpha(t, \mathbf{x}))^2 - a(t)^{-2}\mathrm{e}^{-2\zeta(t,\mathbf{x})}(\partial\psi(t, \mathbf{x}))^2]dt^2 + 2\partial_i\psi(t, \mathbf{x})dtdx^i + a(t)^2\mathrm{e}^{2\zeta(t,\mathbf{x})}d\mathbf{x}^2 , \tag{4.33}$$

with $\alpha(t,\mathbf{x})\,,\psi(t,\mathbf{x})$ and $\zeta\equiv\zeta(t,\mathbf{x})$ functions of space-time coordinates. Thus, a direct computation inside the action (4.7)–(4.8) with $\mathcal{L}_m=0$ leads to the second order action for perturbations [123]:

$$I=\int_{\mathcal{M}}dx^4a^3\left[A\dot{\zeta}^2-\frac{B}{a^2}(\nabla\zeta)^2\right]\,, \tag{4.34}$$

where

$$\begin{aligned}A\equiv&\frac{\dot{\phi}^2(1+\tilde{\alpha}\dot{\phi}^2)}{2(H+3H\tilde{\alpha}\dot{\phi}^2)^2}\times\\(-6H^2\tilde{\alpha}+p_X(\phi,X)+\tilde{\beta}+p_{XX}(\phi,X)\dot{\phi}^2&+\tilde{\alpha}(18H^2\tilde{\alpha}+p_X(\phi,X)+\tilde{\beta})\dot{\phi}^2+\tilde{\alpha}p_{XX}(\phi,X)\dot{\phi}^4)\,,\\B\equiv&\frac{1}{(H+3H\tilde{\alpha}\dot{\phi}^2)^2}\times\\&\left(-(1+3\tilde{\alpha}\dot{\phi}^2)(-4H^2\tilde{\alpha}^2\dot{\phi}^4+\dot{H}(1+\tilde{\alpha}\dot{\phi}^2)^2)+2H\tilde{\alpha}\dot{\phi}(1+\tilde{\alpha}\dot{\phi}^2)(-1+3\tilde{\alpha}\dot{\phi}^2)\ddot{\phi}\right)\,.\end{aligned} \tag{4.35}$$

It means,

$$I=\int_{\mathcal{M}}dx^4a^3A\left[\dot{\zeta}^2-\frac{c_s^2}{a^2}(\nabla\zeta)^2\right]\,, \tag{4.36}$$

where the square of the sound speed is given by the following expression

$$\begin{aligned}c_s^2\;=\;&\frac{\left(-2(1+3\tilde{\alpha}\dot{\phi}^2)(-4H^2\tilde{\alpha}^2\dot{\phi}^4+\dot{H}(1+\tilde{\alpha}\dot{\phi}^2)^2)+4H\tilde{\alpha}\dot{\phi}(1+\tilde{\alpha}\dot{\phi}^2)(-1+3\tilde{\alpha}\dot{\phi}^2)\ddot{\phi}\right)}{\dot{\phi}^2(1+\tilde{\alpha}\dot{\phi}^2)}\\\times&\frac{1}{\left(-6H^2\tilde{\alpha}+p_X(\phi,X)+\tilde{\beta}+p_{XX}(\phi,X)\dot{\phi}^2+\tilde{\alpha}(18H^2\tilde{\alpha}+p_X(\phi,X)+\tilde{\beta})\dot{\phi}^2+\tilde{\alpha}p_{XX}(\phi,X)\dot{\phi}^4\right)}\,.\end{aligned} \tag{4.37}$$

Without entering in the details, we mention that one must have $0<A,B$ to avoid ghost and instabilities. Therefore, in the limit $|\dot{\phi}^2|\ll1/|\tilde{\alpha}|$, the square of the sound speed reads, in terms of the e-folds (4.19),

$$c_s^2\simeq\frac{2H'}{(H\tilde{\beta}+Hp_X(\phi,X)-6H^3\tilde{\alpha}+p_{XX}(\phi,X)H^3\phi'^2)\phi'^2}\,. \tag{4.38}$$

For example, in the case of the canonical scalar field (4.6) where $p_{XX}(\phi,X)=0$, by using the conservation law in (4.28), one immediately has $c_s^2=1$, while for k-essence with higher order kinetic term and $0<p_{XX}$, the sound speed $c_s^2\simeq2H'/(2H'+p_{XX}H^3\phi'^4)$ results to be smaller than one and can be easily suppressed.

By introducing

$$v\equiv v(t,\mathbf{x})=z(t)\zeta(t,\mathbf{x})\,,\quad z\equiv z(t)=\sqrt{a^3A}\,, \tag{4.39}$$

in (4.36), after integration by part we get

$$I = \int dx^4 \left[\dot{v}^2 - \frac{c_s^2}{a^2}(\nabla v)^2 + \ddot{z}\frac{v^2}{z} \right] , \tag{4.40}$$

and by making use of the variational principle one arrives to the equation of motion for the cosmological perturbations,

$$\ddot{v} - \frac{c_s^2}{a^2} \triangle v - \frac{\ddot{z}}{z} v = 0 . \tag{4.41}$$

By decomposing $v(t, \mathbf{x})$ in Fourier modes $v_k \equiv v_k(t)$ whose explicit dependence on $\mathbf{k}$ is given by $\exp[i\mathbf{kx}]$, one obtains the wave equation

$$\ddot{v}_k + \left(k^2 \frac{c_s^2}{a^2} - \frac{\ddot{z}}{z} \right) v_k = 0 . \tag{4.42}$$

The short-wave solution for $1 \ll k^2/a^2$ reads

$$v_k \simeq c_k \mathrm{e}^{\pm ik \int \frac{c_s}{a} dt} , \tag{4.43}$$

where c_k is a constant. Then, the long-wave solution for $k^2/a^2 \ll 1$ is given by

$$v_k \simeq c_1 z + c_2 z \int \frac{dt}{z^2} , \tag{4.44}$$

where $c_1\,, c_2$ are constants. By matching the two solutions for quasi de-Sitter space-time it is possible to derive [124],

$$v_k(t) \simeq c_0 \sqrt{\frac{a}{2}} \frac{aH}{(c_s k)^{3/2}} \mathrm{e}^{\pm ik \int \frac{c_s}{a} dt} \left(1 + ic_s k \int \frac{dt}{a} \right) , \tag{4.45}$$

and in the given limits one can recovers (4.43)–(4.44) with $c_1 = 0$ again, since $z \simeq \eta^2 \sqrt{a/2}$ and $Ha = -1/\eta$, with $\eta = \int dt/a$ the conformal time. Moreover, the constant c_0 is fixed by the Bunch-Davies vacuum state in the asymptotic past,

$$v_k(t) = \sqrt{a} \exp[\pm ik \int c_s dt/a]/(2\sqrt{c_s \kappa}) , \tag{4.46}$$

such that

$$c_0 = i/\sqrt{2} . \tag{4.47}$$

In conclusion,

$$v_k(t) = i \frac{Ha^{3/2}}{2(c_s k)^{3/2}} \mathrm{e}^{\pm ik \int \frac{c_s}{a} dt} \left(1 + ic_s k \int \frac{dt}{a} \right) , \tag{4.48}$$

and from (4.39) one obtains the expression for the cosmological perturbations during inflation,

$$\zeta_k = i \frac{H}{2\sqrt{A}(c_s k)^{3/2}} \mathrm{e}^{\pm ik \int \frac{c_s}{a} dt} \left(1 + ic_s k \int \frac{dt}{a} \right) . \tag{4.49}$$

The variance of the power spectrum of perturbations on the sound horizon crossing $c_s\kappa \simeq Ha$ reads

$$\mathcal{P}_{\mathcal{R}} \equiv \frac{|\zeta_k|^2 k^3}{2\pi^2}|_{c_s k \simeq Ha} = \frac{H^2}{8\pi^2 c_s^3 A}|_{c_s k \simeq Ha} \,. \tag{4.50}$$

By making use of this expression, we can write the spectral index n_s defined by

$$1 - n_s = -\frac{d\ln \mathcal{P}_{\mathcal{R}}}{d\ln k}|_{k=aH/c_s} \,, \tag{4.51}$$

namely

$$1 - n_s = 2\epsilon + \eta_{sF} + s \,, \tag{4.52}$$

with [124],

$$\epsilon = -\frac{\dot{H}}{H^2} \,, \quad \eta_{sF} = \frac{\dot{\epsilon}_s F + \epsilon_s \dot{F}}{H(\epsilon_s F)} \,, \quad s = \frac{\dot{c}_s}{H c_s} \,, \quad \epsilon_s = \frac{A c_s^2}{F} \,, \tag{4.53}$$

and

$$F = 1 + \alpha\dot{\phi}^2 \,. \tag{4.54}$$

In a similar way, we may derive the tensor-to-scalar ratio r from the tensor perturbations in flat FLRW space-time. Here, we furnish the result,

$$r = 16 c_s \epsilon_s \,. \tag{4.55}$$

We should note that in the case of k-essence models where $c_s < 1$, this quantity tends to vanish.

For our Horndeski Lagrangian (4.7) with $\mathcal{L}_m = 0$ one has, in terms of the e-folds number,

$$\begin{aligned}(n_s - 1) \simeq\; & \Big(\phi' \Big(3HH'' \Big(\tilde{\beta} + H^2 \left(p_{XX}\phi'(t)^2 - 6\tilde{\alpha}\right) + p_X\Big) - \\ & H' \Big(H' \Big(9H^2 \left(p_{XX}\phi'^2 - 6\tilde{\alpha}\right) + 7(\tilde{\beta} + p_X)\Big) \\ & + H\left(H^2 p'_{XX}\phi'^2 + p'_X\right)\Big)\Big) - 2HH'\phi'' \Big(\tilde{\beta} + H^2 \left(2p_{XX}\phi'^2 - 6\tilde{\alpha}\right) + p_X\Big)\Big) \\ & \times \frac{1}{2HH'\phi' \Big(\tilde{\beta} + H^2 \left(p_{XX}\phi'^2 - 6\tilde{\alpha}\right) + p_X\Big)} \,,\end{aligned} \tag{4.56}$$

and

$$r \simeq 16\sqrt{2}\frac{H'}{H}\sqrt{\frac{H'}{H\phi'^2 \Big(\tilde{\beta} + H^2 \left(p_{XX}\phi'^2 - 6\tilde{\alpha}\right) + p_X\Big)}} \,. \tag{4.57}$$

The last Planck satellite data [13] lead to

$$n_s = 0.968 \pm 0.006\,(68\%\,\mathrm{CL})\,, \quad r < 0.11\,(95\%\,\mathrm{CL})\,. \tag{4.58}$$

For instance, if we pose $N \equiv \mathcal{N} \simeq 60$, $\mathcal{N}$ being the total e-folds number in (4.31), the scenario is viable when $(n_s - 1) \simeq -2/N$.

When $p_X = 1$ (canonical scalar field) and $\tilde{\alpha} = \tilde{\beta} = 0$ (chaotic inflation), one can easily verify that

$$(n_s - 1) \simeq -\frac{7H'}{2H} + \frac{3H''}{2H'} - \frac{\phi''}{\phi'}\,, \quad r \simeq 16\sqrt{2}\phi'^2 \left(\frac{H'}{H\phi'^2}\right)^{3/2}\,, \tag{4.59}$$

or, in terms of the cosmological time, by using (4.28),

$$(n_s - 1) \simeq \frac{4\dot{H}}{H^2} - \frac{\ddot{H}}{\dot{H}H^2} = -6\epsilon + 2\eta\,, \quad r \simeq -16\frac{\dot{H}}{H^2} = 16\epsilon\,, \tag{4.60}$$

with

$$\eta = -\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{2\dot{H}H^2}\,, \tag{4.61}$$

which correspond to the usual relations for chaotic canonical scalar field inflation.

4.3 Mimetic Horndeski gravity

One can further considers more general scalar-tensor Horndeski theories, which can be "mimetized" according to the Lagrangian multiplier procedure described in the preceding section. The action in (4.7)–(4.8) is rewritten as

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R}{2} + \lambda\left(X - \frac{1}{2}\right) - V(\phi)\right) + I_H + I_m\,, \tag{4.62}$$

where I_m is the usual matter/radiation action and the higher order contribution I_H is given by (4.8).

Let us study now the dynamics of above model in a flat FLRW space-time. The variation with respect to λ, if one assumes that ϕ depends only on t, gives the mimetic FLRW contraint (3.29) which identifies the field with the cosmological time. Thus, when the constants β, α, and γ are vanishing, the above model reduces to the original extended mimetic gravity theory (3.30) proposed by Chamseddine, Mukhanov and Vikman in Ref. [51].

The first Einstein equation is the generalized Friedmann equation and reads (when $\phi \equiv t$),

$$3H^2(1 - 3\alpha + 3\gamma) - V + \beta - \rho = \lambda\,. \tag{4.63}$$

Here, ρ is the usual matter energy density, and by introducing a constant EoS parameter ω as in (3.8) the pressure reads

$$p = \omega\rho\,, \tag{4.64}$$

and the conservation law (3.7) is given by

$$\dot{\rho} + 3H(1 + \omega)\rho = 0\,, \quad \rho(t) = \rho_0 a(t)^{-3(1+\omega)} \tag{4.65}$$

where we reported the implicit solution (3.9). Then, the second Einstein equation reads

$$c_1(2\dot{H} + 3H^2) = V + \beta - \omega\rho\,, \tag{4.66}$$

with

$$c_1 = (1 - \alpha + \gamma)\,. \tag{4.67}$$

We would like to note that the equation of motion associated with ϕ is a consequence of the other equations of motion, and it is trivially satisfied.

To look for cosmological solutions we will start from Equation (4.66), since the Lagrangian multiplier is absent. This is a non linear first order Riccati differential equation and in order to deal with it we have several possibilities.

4.3.1 Absence of matter

Cosmological scenarios where matter is avoidable are interesting at high energy, at the time of the Big Bang or other alternative descriptions like the cosmological bounce that we will consider in this subsection.

In absence of matter, the Riccati equation may be recasted in an homogeneous linear second order differential equation. If we introduce the variable $y \equiv y(t)$ as

$$y = a^{3/2}\,, \tag{4.68}$$

such that

$$H = \frac{2\dot{y}}{3y}\,, \tag{4.69}$$

we obtain

$$\ddot{y} - \frac{3}{4c_1}(V(t) + \beta)y = 0\,. \tag{4.70}$$

We may consider now a quadratic potential in the form

$$V = -\beta + 2V_0 + \frac{3V_0^2}{c_1}(\phi - \phi_0)^2\,, \tag{4.71}$$

where $V_0\,, \phi_0$ are positive constants. After the identification of the mimetic field with the cosmological time and the replace $\phi_0 = t_0$ we have

$$V(t) = -\beta + 2V_0 + \frac{3V_0^2}{c_1}(t - t_0)^2\,. \tag{4.72}$$

In this case, the solution of (4.70) is given by

$$y(t) = y_0 \mathrm{e}^{\frac{3V_0}{4c_1}(t-t_0)^2}\,, \tag{4.73}$$

and the scale factor in (4.68) and the Hubble parameter (4.69) read

$$a(t) = a_0 \mathrm{e}^{\frac{V_0}{2c_1}(t-t_0)^2}\,, \quad H(t) = \frac{V_0}{c_1}(t - t_0)\,. \tag{4.74}$$

In the expressions above, $y_0\,, a_0$ are constants related to each others. This is an example of cosmological bounce (see also §3.3.4): at the time $t = t_0$ the Hubble parameter vanishes with $0 < \dot{H}(t_0)$. When $t < t_0$, the Hubble parameter is negative and the Universe underwent a contracting phase ($\dot{a}(t) < 0$). On the other side, when $t_0 < t$ the Hubble parameter is positive and we have an expansion ($0 < \dot{a}(t)$).

Another bounce solution may be obtained by making the following choice,

$$V(\phi) = \frac{c_1 b^2 (2 + 3\sinh^2 b(\phi - \phi_0))}{\cosh^2 b(\phi - \phi_0)} \,, \tag{4.75}$$

with b and ϕ_0 positive parameters. It means,

$$V(t) = \frac{c_1 b^2 (2 + 3\sinh^2 b(t - t_0))}{\cosh^2 b(t - t_0)} - \beta \,, \tag{4.76}$$

where we posed $\phi_0 = t_0$. The solution of (4.70) results to be

$$y(t) = y_0 \left(\cosh b(t - t_0)\right)^{3/2} \,. \tag{4.77}$$

The correponding bounce solution is given by

$$a(t) = a_0 \cosh b(t - t_0) \,, \quad H(t) = b \frac{\sinh b(t - t_0)}{\cosh b(t - t_0)} \,. \tag{4.78}$$

Here, y_0 and a_0 are again constants related to each others. The bounce is located at $t = t_0$. When $t = t_0$ the Hubble parameter vanishes, while for $t < t_0$ is negative and for $t_0 < t$ is positive.

Alternatively, one may use another approach, the so called adiabatic invariant method [133], and for the sake of completeness we report it in what follows.

We would like to find the solution of a differential equation of the kind

$$\ddot{y} + Q(t) y = 0 \,, \tag{4.79}$$

where, in the case of (4.70),

$$Q(t) = -\frac{3}{4c_1}(V(t) + \beta) \,. \tag{4.80}$$

The above linear homogeneous differential equation can be associated with the following non-linear differential equation respect to $u \equiv u(t)$, dubbed “Ermakov-Pinney” equation, namely

$$\ddot{u} + Q(t) u = \frac{h^2}{u^3} \,. \tag{4.81}$$

In fact, the following result holds true: the solutions y and u are related by

$$y = u \sin\theta \,, \quad \theta = \int \frac{h}{u^2} \,, \tag{4.82}$$

and the constant h is given by the so called Lewis adiabatic invariant

$$h^2 = \frac{h^2 y^2}{u^2} + (u\dot{y} - \dot{u}y)^2 \,. \tag{4.83}$$

As an example, we consider

$$Q(t) = \frac{q}{t^4}\,, \tag{4.84}$$

where q is a positive constant, which corresponds to the potential

$$V(t) = -\frac{4c_1 q}{3t^4} - \beta\,. \tag{4.85}$$

The solution of (4.81) reads

$$u(t) = \left(\frac{h^2}{q}\right)^{1/4} t\,, \tag{4.86}$$

and from (4.82) one has

$$\theta(t) = \theta_0 - \frac{q^{1/2}}{t}\,, \tag{4.87}$$

such that the solution for y is given by (up to a multiplicative constant)

$$y(t) = \left(\frac{h^2}{q}\right)^{1/4} t \sin\left[\theta_0 - \frac{q^{1/2}}{t}\right]\,, \tag{4.88}$$

which is a non trivial result which corresponds to the scale factor

$$a(t) = \left[\left(\frac{h^2}{q}\right)^{1/4} t \sin\left[\theta_0 - \frac{q^{1/2}}{t}\right]\right]^{2/3} \tag{4.89}$$

Note Eq. (4.83) is satisfied for any generic constant value of h.

4.3.2 Presence of matter

The presence of matter/radiation must be considered at the late-time, during the matter dominated epoch and the current accelerated expansion.

If matter is present, one may introduce the e-fold time (see also (4.19)),

$$N = \ln a\,. \tag{4.90}$$

As a result, the continuity equation of matter becomes

$$\frac{d\rho}{dN} = -3(1+\omega)\rho\,, \tag{4.91}$$

with solution

$$\rho(N) = \rho_0 e^{-3(1+\omega)N}\,, \tag{4.92}$$

ρ_0 being a positive constant. Furthermore, the equation of motion for H reads

$$c_1 \frac{dH(N)^2}{dN} + 3c_1 H(N)^2 = V(N) + \beta - \omega\rho(N) \,. \tag{4.93}$$

If

$$c_1 = (1 + \gamma - \alpha) \neq 0 \,, \tag{4.94}$$

the implicit solution is given by

$$H(N)^2 = e^{-3N} \left(C + \int dN e^{3N} \frac{V(N) + \beta - \omega\rho(N)}{c_1} \right) , \tag{4.95}$$

where C is an integration constant. Furthermore, by using (4.92) one has

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} e^{-3(\omega+1)N} + e^{-3N} \int dN e^{3N} \frac{V(N)}{c_1} \,, \quad \omega \neq -1 \,. \tag{4.96}$$

In the specific case $\omega = -1$ (dark energy), one obtains

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} + e^{-3N} \int dN e^{3N} \frac{V(N)}{c_1} \,, \quad \omega = -1 \,. \tag{4.97}$$

Some comments are in order. Since

$$t(N) = \int \frac{dN}{H(N)} \tag{4.98}$$

we may obtain $N = N(t)$ and $a(t) = e^{N(t)}$. The contribution depending on C is the contribution associated with the mimetic dark matter. The term depending on β acts an effective cosmological constant. Another constant contribution may be obtained by the simplest choice for the potential $V = V_0$, namely a constant potential. In this case, we get

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} e^{-3(1+\omega)N} + \frac{V_0}{3c_1} \,, \quad \omega \neq -1 \,, \tag{4.99}$$

and

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} + \frac{V_0}{3c_1} \,, \quad \omega = -1 \,. \tag{4.100}$$

For consistency, we have to assume $0 < \beta + V_0$. Thus, these solutions contain the dark matter contribute and tend at the late time (when $0 \ll N$) to the de Sitter space-time.

Another interesting example of potential is given by

$$V(N) = V_0 (N - N_0)^{b-1} \left(3(N - N_0) + b\right) \,, \tag{4.101}$$

where $V_0 \,, N_0 \,, b$ are constants with $b \neq 2$. In this case,

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} e^{-3(1+\omega)N} + \frac{V_0}{c_1} (N - N_0)^b \,, \quad \omega \neq -1 \,, \tag{4.102}$$

$$H^2(N) = Ce^{-3N} + \frac{\beta}{3c_1} + \frac{\rho_0}{3c_1} + \frac{V_0}{c_1}(N - N_0)^b \,, \quad \omega = -1 \,. \tag{4.103}$$

The nature of the solution depends on the sign of b. To simplify the discussion of this point, we will assume $C = \beta = \rho_0 = 0$ (absence of matter and dark matter), and $0 < V_0/c_1$. Thus,

$$H(N) = \left(\frac{V_0}{c_1}\right)^{1/2} (N - N_0)^{b/2} \,. \tag{4.104}$$

As a consequence one has

$$(t - t_0) = \left(\frac{V_0}{c_1}\right)^{-1/2} \frac{2}{2-b}(N - N_0)^{1-b/2} \,, \tag{4.105}$$

with $t_0 = t(N_0)$, and

$$(N - N_0) = \left(\frac{V_0}{c_1}\right)^{1/(2-b)} \left(\frac{2-b}{2}(t - t_0)\right)^{2/(2-b)} \,. \tag{4.106}$$

Therefore,

$$H(t) = A_0(t - t_0)^{b/(2-b)} \,, \tag{4.107}$$

where

$$A_0 = \left(\frac{V_0}{c_1}\right)^{\frac{1}{(2-b)}} \left(\frac{2-b}{2}\right)^{\frac{b}{(2-b)}} \,. \tag{4.108}$$

Thus, when $0 < b < 2$ we recover a bounce solution. When $2 < b$, as well as for $b < 0$, future singularities where Hubble parameter diverges appear (see also Eq. (3.14) and the related discussion).

For instance, for $b = 1$, one has the bounce solution

$$H(t) = \frac{V_0}{2c_1}(t - t_0) \,. \tag{4.109}$$

In this case, the mimetic field potential results to be

$$V(\phi) = V_0 \left(1 + \frac{3V_0}{4c_1}(\phi - \phi_0)^2\right) \,, \tag{4.110}$$

in agreement with the result discussed in the previous subsection after the redefinition $V_0 \to 2V_0$.

4.4 Cosmological perturbations in mimetic Horndeski gravity

To study the cosmological perturbations in our mimetic Horndeski model, we come back to the formalism presented in §3.4. If one considers the Newton's gauge for perturbed FLRW metric in (3.61) and (3.63), it is easy to find that relation (3.64) is still valid. A general derivation of the result in mimetic Horndeski theories has been presented in Ref. [134], and here we limit to reduce it to our particular case of Horndeski Lagrangian in (4.62). The perturbed equations of motion (in absence of matter) lead to[1]

$$\Psi\left[1-(\gamma-\alpha)\right]=\left[1+(\gamma-\alpha)\right]\delta\dot{\phi}+2(\gamma-\alpha)H\delta\phi\,, \tag{4.111}$$

$$\dot{\Psi}+H\delta\dot{\phi}+\dot{H}\delta\phi=0\,. \tag{4.112}$$

Now, by recasting (4.111) in (4.112), one easily obtains again (3.65), where the Laplacian of the perturbation is absent and the sound speed vanishes. Thus, we confirm the result according to which perturbations cannot propagate in a mimetic model with second order differenial equations. To obtain a correct picture of the cosmological perturbations, one has to follow the same approach of Ref. [51] that we illustrated in §3.4, namely we must brake the Horndeski structure of the model. In other words, since the mimetic constraint kills in fact the additional degree of freedom associated to the field, one has to implement the theory with higher derivative terms. In our case, we can rewrite the contribution I_H in the action (4.62) as

$$\begin{aligned} I_H &= \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\left(-ag^{\mu\nu}\partial_\mu\phi\partial_\nu\phi R+\frac{c}{2}(\Box\phi)^2-\frac{b}{2}\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right)\right. \\ &\quad \left.+\gamma\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\beta\phi\Box\phi\right]\,. \end{aligned} \tag{4.113}$$

If we take into account (4.10), we immediately see that we can recover (4.8) when $a=\alpha/2$ and $b=c=2\alpha$.

After the identification of the field with the cosmological time, the Friedmann-like equation (4.66) reads

$$\tilde{c}_1(2\dot{H}+3H^2)=V+\beta-\omega\rho\,, \tag{4.114}$$

where

$$\tilde{c}_1=(1-\tilde{\alpha}+\gamma)\,, \tag{4.115}$$

with

$$\tilde{\alpha}=-2a-\frac{b}{2}+\frac{3c}{2}\,. \tag{4.116}$$

When $a=\alpha/2$ and $b=c=2$ we find (4.66) with $\tilde{\alpha}=\alpha$. Thus, the model brings to all the FLRW solutions discussed in the preceding chapter with $\alpha\to\tilde{\alpha}$. On the other side, if one looks for the cosmological perturbations, the relation (3.64) is still valid and Equation (3.65) turns out to be

$$\delta\ddot{\phi}+H\delta\dot{\phi}-\frac{c_s^2}{a(t)^2}\triangle\delta\phi+\dot{H}\delta\phi=0\,, \tag{4.117}$$

where the sound speed reads

$$c_s^2\equiv\frac{b-c}{2c_2}\,, \tag{4.118}$$

[1]This result can be compared with the one obtained in Ref. [135] by substituting $\gamma\to 0\,,\alpha\to-\alpha/2$.

with

$$c_2 \equiv \frac{(2-b+2a)(4-3c+4a+b)}{4(1-a)} \,. \tag{4.119}$$

If the Horndeski constraint $b = c$ is implemented, we find a vanishing sound speed again, but in general now the cosmological perturbations can propagate and lead to a spectral index $n_s \neq 1$ and to a non-zero tensor-to-scalar ratio r. In other words, the mimetic model is able to describe the inhomogeneities observed in our Universe.

4.4.1 Stability of cosmological bounce

Equation (4.117) may be used to study the stability of any FRLW solution of the model. For example, it could be interesting to study the stability of the bounce solution (4.74). We remind that for the action (4.113) we must replace $c_1 \to \tilde{c}_1$ with $\tilde{c}_1 = (1 - \tilde{\alpha} + \gamma)$ in the solution. Moreover, for the sake of simplicity, we will pose $t_0 = 0$, namely $(0 < V_0/c_1)$,

$$a(t) = a_0 \mathrm{e}^{\frac{V_0}{2\tilde{c}_1}t^2} \,, \quad H(t) = \frac{V_0}{\tilde{c}_1} t \,. \tag{4.120}$$

The potential which supports the bounce solution above is given by (4.72) with $c_1 \to \tilde{c}_1$. If we pose

$$\delta\phi(t, \mathbf{x}) = \delta\phi(t) \mathrm{e}^{\pm i A \mathbf{x}} \,, \tag{4.121}$$

with A real number, we can use Eq. (4.117) to study the stability of the solution around the time of the bounce at $t = 0$, namely

$$\delta\ddot{\phi}(t) + \frac{c_s^2}{a_0^2} A^2 \delta\phi(t) + \frac{V_0}{c_1} \delta\phi \simeq 0 \,. \tag{4.122}$$

Thus, we get

$$\delta\phi(t, \mathbf{x}) = \phi_0 \mathrm{e}^{i \pm A \mathbf{x}} \mathrm{e}^{\pm i B t} \,, \tag{4.123}$$

with ϕ_0 constant and

$$B = \frac{\sqrt{c_1 A^2 c_s^2 + a_0^2 V_0}}{a_0 \sqrt{c_1}} \,. \tag{4.124}$$

Since from (3.64) we have

$$\Phi(t, \mathbf{x}) = \delta\dot{\phi}(t, \mathbf{x}) \propto \mathrm{e}^{\pm i B t} \mathrm{e}^{\pm i A \mathbf{x}} \,, \tag{4.125}$$

we see that if $0 < c_1 \,, V_0$ or $c_1 \,, V_0 < 0$, the solution never diverges and the cosmological bounce under consideration is stable.

Chapter 5

Cosmological bounce

The cosmological bounce is an alternative scenario with respect to the Big Bang theory. Instead from an initial singularity, the Universe emerges from a bounce which follows a contracting phase. The Friedmann equations may lead to the cosmological bounce if one considers the loop quantum gravity effects or a non-flat spatial topology. One model of Chaplygin gas for the cosmological bounce is also derived.

5.1 Bounce solutions in loop quantum gravity

The quantum gravity corrections take place at high curvature, during the early-time expansion of the Universe. If we take into account such corrections, the Friedmann equation for the Hubble parameter, once a generic FLRW metric,

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) , \tag{5.1}$$

has been choosen, reads

$$H^2 + \frac{k}{a^2} = \frac{1}{3}(\rho - \mu\rho^2) , \tag{5.2}$$

where $k = 0, \pm 1$, as usually, is the spatial curvature, ρ is the matter energy density and $1/\mu$, with $0 < \mu$, is the critical energy density at which the quantum effects appear. In the limit $\rho \ll 1/\mu$ we recover the usual Friedmann equation. Moreover, as a second equation, we will use the matter conservation law in (3.7).

Let us take $k = 0$ (flat Universe),

$$3H^2 = \rho(1 - \mu\rho) . \tag{5.3}$$

The derivative of this equation leads to

$$6H\dot{H} = \dot{\rho}(1 - 2\mu\rho) . \tag{5.4}$$

If we deal with a perfect fluid with $p = \omega\rho$, ω being a constant EoS parameter, Eq. (3.7) reads

$$\dot{\rho} = -3H\rho(1 + \omega) . \tag{5.5}$$

Thus, we obtain:

$$\dot{H} = -\frac{1}{2}(1+\omega)\rho(1-2\mu\rho)\,. \tag{5.6}$$

When $\rho = 1/\mu$, the Hubble parameter in (5.3) vanishes. At that time $t = t_0$, from last equation we get

$$\dot{H}(t=t_0) = -\frac{1}{2}(1+\omega)\rho(1-2)\,. \tag{5.7}$$

Such a quantity is positive only if

$$-1 < \omega\,, \tag{5.8}$$

and we can describe a cosmological bounce in the presence of repulsive matter [57]. Since $\rho \propto a^{-3(1+\omega)}$, we have in this case that the energy density of matter increases or decreases when Universe decreases or increases, respectively. The maximum value for the energy density is reached at $\rho = 1/\mu$.

Let us check for an explicit solution [136]. We may take the following Ansatz for ρ,

$$\rho(t) = \frac{1}{\mu(1+A(t-t_0)^2)}\,, \tag{5.9}$$

A being a positive constant. From the Friedmann equation (5.3) we have

$$a(t) = a_0(1+A(t-t_0)^2)^{\frac{1}{2\sqrt{3A\mu}}}\,, \tag{5.10}$$

where $a_0 = a(t_0)$, $t = t_0$ being the time of the bounce when $\rho = 1/\mu$. This solution is consistent with (5.5) if we set

$$A = \frac{3}{4\mu}(1+\omega)^2\,, \tag{5.11}$$

such that

$$a(t) = a_0(1+A(t-t_0)^2)^{\frac{1}{3(1+\omega)}}\,, \quad H = \frac{2A(t-t_0)}{3(1+\omega)(1+A(t-t_0)^2)}\,, \tag{5.12}$$

inside (5.9). Here, we clearly see that for $-1 < \omega$ the contraction phase takes place when $t < t_0$, while when $t_0 < t$ the Universe expands.

5.2 Bounce solutions in non-flat FLRW Universe

We may use the result of the preceding section to find the cosmological bounce in spherical FLRW space-time of General Relativity. In this case, the Equation (5.2) with $k = 1$ and $\mu \to 0$ reads

$$3H^2 = \rho - \frac{1}{a^2}\,, \tag{5.13}$$

whose derivative is given by

$$6H\dot{H} = \dot{\rho} + \frac{2H}{a^2}\,, \tag{5.14}$$

or, by using (5.5),

$$\dot{H} = -\frac{1}{2}(\rho + p) + \frac{1}{3a^2} = 0\,. \tag{5.15}$$

In order to obtain the same class of solutions of (5.3), we must require (on shell),

$$\rho = \frac{1}{\sqrt{\mu} a}\,, \tag{5.16}$$

such that

$$\dot{\rho} = -\frac{H}{\sqrt{\mu}\rho}\,. \tag{5.17}$$

Thus, if we take into account that,

$$a = \frac{1}{\sqrt{\mu}\rho}\,, \tag{5.18}$$

from (5.5) we have

$$p = -\frac{2}{3}\rho\,, \tag{5.19}$$

which is the case of a perfect fluid with $\omega = -2/3$. Moreover, it is easy to verify that (5.6) turns out to be (5.15) when $\omega = -2/3$. It means that with this kind of perfect fluid we obtain the same spherical FLRW solutions (i.e. scale factor and Hubble parameter) of flat FLRW loop quantum gravity. In particular, the cosmological bounce can be realized as in Eq. (5.12).

Let us consider now the Chaplygin gas [137] with

$$p = -A_0\rho - \gamma\sqrt{\rho} = 0\,, \tag{5.20}$$

where γ and A_0 are positive constants, in spherical FLRW Universe. If we plug (5.20) in (5.15) we get

$$\dot{H} = -\frac{1}{2}(\rho - A_0\rho - \gamma\sqrt{\rho}) + \frac{1}{3a^2}\,. \tag{5.21}$$

In the precence of a cosmological bounce at the time $t = t_0$ with $H(t = t_0) = 0$, it must be

$$\rho(t = t_0) = \frac{1}{a^2}\,. \tag{5.22}$$

For the sake of simplicity, we may pose

$$A_0 = \frac{1}{3}\,, \tag{5.23}$$

such that

$$\dot{H} = -\frac{1}{3}\rho + \frac{\gamma}{2}\sqrt{\rho} + \frac{1}{3a^2}\,, \tag{5.24}$$

and

$$\dot{H}(t = t_0) = \frac{1}{2}\gamma\sqrt{\rho}\,. \tag{5.25}$$

Thus, at the time of the bounce, $0 < \dot{H}$ and the Hubble parameter increases when $0 < \gamma$. Let us check now for the explicit solution. From the conservation law of the Chaplygin gas,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}\left(\frac{2}{3} - \frac{\gamma}{\sqrt{\rho}}\right), \tag{5.26}$$

one has

$$\rho = \frac{(1 - 3C_0\gamma a)^2}{4C_0^2 a^2}, \tag{5.27}$$

where C_0 is a constant. With the suitable choice

$$C_0 = \frac{1}{2}, \tag{5.28}$$

from (5.13) we get

$$a(t) = \frac{\mathrm{e}^{-\frac{\sqrt{3}\gamma}{2}(t-t_0)}}{36\gamma^2}\left(\mathrm{e}^{\frac{\sqrt{3}\gamma}{2}(t-t_0)+12\gamma}\right)^2,$$

$$H = \frac{\sqrt{3}\gamma}{2}\left(1 - \frac{24\gamma}{\mathrm{e}^{\frac{\sqrt{3}\gamma}{2}(t-t_0)} + 12\gamma}\right). \tag{5.29}$$

Thus, if

$$\gamma = \frac{1}{12}, \tag{5.30}$$

the bounce with $H = 0$ is realized when $t = t_0$. Moreover, when t is close to t_0,

$$H(t \simeq t_0) \simeq \dot{H}(t_0)(t - t_0), \tag{5.31}$$

and, since from (5.25) we have that $0 < \dot{H}(t_0)$, it is easy to see that $0 < H$ when $t_0 < t$, and $H < 0$ when $t < t_0$.

In conclusion, we investigated different approaches that bring to the cosmological bounce scenario by working with Friedmann equations. We showed that the account of loop quantum gravity leads to the same bounce solution that can be obtained with non-flat topology (at the early-time, the spatial curvature of the Universe could be different to zero). As an alternative, we may use non perfect fluids, like Chaplygin gas, to reproduce the bounce. For other studies concerning the bounce cosmology, see also the interesting works in Refs. [138, 139, 140, 141].

Chapter 6

Conclusions

In the first part of this work we have analyzed the gauge coordinate invariance of the black hole formalism. In the specific, we have investigated the dynamical patch of the de Sitter space-time provided with cosmological horizon, and we tried to better understand the temperature-versus-surface gravity paradigm. The asymptotic results obtained by tunneling semiclassical method in the case of trapping horizon of the black holes have been tested with quantum field theory techniques like the Unruh-DeWitt detector and the computation of the quantum fluctuations associated with a massless conformally coupled scalar field.

The formalism for calculating the Wightman function in FLRW space-time has been furnished for general topological cases, namely for flat, spherical and hyperbolic space manifolds. The Wightman function is the basic element for any investigation of quantum field theory in curved space-time. Thanks to the Wightman function, by carefully considering some processes of renormalization which avoid the divergences, it is possible to compute the response function of Unruh-De Witt detector and obtain informations about the temperature. In Refs. [89]–[94] the Unruh-De Witt detector has been applied to flat de Sitter universe and the result showed how in addition to the Unruh effect (namely, the fact that the quantum thermometer sees a temeperature proportional to the acceleration of the thermometer itself) one obtaines a contribution given by the temperaure of the gravitational field proportional to the surface gravity of the horizon. In our work, we derived the result for very generic non-flat spherical (and hyperbolic) topology. We must stress that in this case the Unruh-De Witt effects can be computed only when the acceleration of the detector vanishes, and the relation between surface gravity and Gibbons-Hawking temperature holds true.

Another approach to the problem is given by the direct investigation of the vacuum fluctuations of a massless scalar field conformally coupled with gravity in its ground state. Thus, from the vacuum fluctuations in FLRW space-time, it is possible to read off the temperature. This investigation allows to find more general results respect to the previous ones. Vacuum fluctuations can be computed for flat, spherical and hyperbolic topologies. When we pass to the De Sitter space-time, the temperature takes into account also the Unruh effect given by the acceleration of the observer, either in the flat and in the non-flat case. Moreover, we also are able to show that the intrinsic temperature of the Milne Universe is equal to zero (pure Unruh effect), being the Milne Universe a Minkowski space-time in expanding coordinates.

With regard to possible generalizations of our analysis (at least in the flat case), we may consider the Maxwell field. In Ref. [83], the quantization of the Maxwell field in the flat FLRW space-time by using the so called "W gauge" (a conformal lifting of the Lorenz gauge in Minkowski space-time) leads to the Maxwell Wightman function,

$$W_{\mu\nu}(x,x') = -\frac{1}{2}\left(\frac{g_{\mu\nu}(x)a(x')}{a(x)} + \frac{g_{\mu\nu}(x')a(x)}{a(x')}\right) W(x,x')\,, \tag{6.1}$$

where $W(x,x')$ is the Whightman function associated with a massless conformally coupled scalar field in flat FLRW space-time with metric $g_{\mu\nu}(x)$ and scale factor $a(x)$. This fact strongly suggests that our conformally coupled scalar probe may mimic quite well the quantum Maxwell field. We should note that, working with the Maxwell field, the gauge invariant has to be implemented (for instance, the relevant quantity may be the Wightman function associated to the magnetic field). This is a very interesting issue with important cosmological applications (see for example Refs. [84, 85]).

In the second part of our work, a specific Horndeski scalar-tensor mimetic model has been investigated within a FLRW space-time. The mimetic scalar field has been implemented making use of a Lagrangian multiplier, such that in the Friedmann Universe it can be identified with the time and, as a general feature of mimetic gravity, leads to a dark matter contribution in the field equations.

Several exact solutions describing inflation, bounce, future-time singularities and dark energy in the presence or absence of ordinary matter have been found by using different methods. Since the Horndeski mimetic gravitational models do not allow the propagation of the cosmological perturbations left from inflation [56], we showed a way to break the Horndeski structure of the theory without destroying its behavior in FLRW Universe.

As a possible development, one may try to reproduce inside the framework of mimetic gravity the correct inferred flat rotation curves of spiral galaxies, which is the manifestation of dark matter at the galactic scale. In doing it, it is necessary to work with spherical symmetric metrics and potentials of the mimetic field along the line of Ref. [116]. The goal is to apporting suitable modifications to the Newton's potential emerging from the solution in order to find a flat rotation velocity spectrum $v_{\rm rot} \simeq$ const in contrast with the expected value $v_{\rm rot} - 1/\sqrt{r}$ of Keplerian mechanics.

Appendix A

Wightman function in non flat FLRW space-time

In this Appendix, an elementary derivation of Wightman function $W(x,x')$ for a massless conformally coupled scalar field in FLRW space-time is presented. Due to the homogeneity and isotropy of FLRW space-times, we will take $W(x,x') = W(x',x) = W(\eta-\eta', r-r')$, fixing the angular part (radial trajectories).

For our aim, it is convenient to introduce the auxiliary quantity

$$Y(x,x') = a(\eta)a(\eta')W(x,x')\,,$$

where the scale factor $a(\eta)$ is a function of the conformal time $d\eta = dt/a(t)$. As a result, from the Friedmann equations of motion we obtain

$$-\frac{d^2Y}{d\eta^2} - kH_0^2 Y + \Delta_h Y = 0\,,$$

where Δ_h is the Laplace operator associated with the spatial part of the metric $d\Sigma_3^2$ depending on the topology.

We may take $x' = 0$. Let us start with the flat case $k = 0$. One has

$$-\frac{\partial^2 Y}{\partial\eta^2} + \frac{\partial^2 Y}{\partial r^2} + 2\frac{\partial Y}{\partial r} = 0\,.$$

The solution is given by,

$$Y = \frac{1}{-\eta^2 + r^2}\,.$$

As a result, making use of the homogeneity and isotropy, and dealing with the distribution nature of $W(x,x')$, we get

$$W(x,x') = \frac{1}{4\pi^2 a(\eta)a(\eta')}\,\frac{1}{(r-r')^2 - (\eta-\eta'-i\epsilon)^2}\,,\quad k=0\,.$$

As it is usual in the distribution theory, we shall leave understood the limit as $\epsilon \to 0^+$.

We may rewrite it in covariant form, according to Takagi [86] and Schlicht [87]. Thus, we adapt Schlicht's proposal to our conformally flat case, namely

$$W(x,x') = \frac{1}{4\pi^2 a(\eta)a(\eta')}\,\frac{1}{[(x-x') - i\epsilon(dx/d\tau + dx'/d\tau)]^2}\,,\quad k=0\,.$$

Coming back to the non-flat case, it is sufficient to consider the positive curvature case with $h_0 > 0$. The negative one may be obtained with the replacement $h_0 \to ih_0$.

For the sake of simplicity, here we may take $h_0 = 1$, then dimensional analysis will give the complete expression. Let us start with

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\chi^2 + \sin^2\chi dS^2\right)\,,\quad k=1\,.$$

The Ricci scalar reads

$$R = 6\left(\frac{d^2a/d\eta^2 + a}{a^3}\right),$$

and the equation for Y is given by

$$-\frac{\partial^2 Y}{\partial\eta^2} - Y + \frac{\partial^2 Y}{\partial\chi^2} + 2\frac{\cos\chi}{\sin\chi}\frac{\partial Y}{\partial\chi} = 0\,.$$

The solution of this partial differential equation reads

$$Y = \frac{1}{\cos\eta - \cos\chi}\,.$$

Making use of dimensional analysis one arrives at

$$W(x,x') = \frac{1}{8\pi^2 a(\eta)a(\eta')}\,\frac{H_0^2}{\cos H_0(\chi-\chi') - \cos H_0(\eta-\eta'-i\epsilon)}\,,\quad k=1\,.$$

In the case of negative spatial curvature one recovers

$$W(x,x') = -\frac{1}{8\pi^2 a(\eta)a(\eta')}\,\frac{H_0^2}{\cosh H_0(\chi-\chi') - \cosh H_0(\eta-\eta'-i\epsilon)}\,,\quad k=-1\,.$$

These results are in agreement with the ones obtained in **Chapter 2**.

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