Optimal control of finite horizon type for a multidimensional delayed switching system

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Abstract

We consider a finite horizon optimal control problem for an ODE system, with trajectories presenting a delayed two-values switching along a fixed direction. In particular the system exhibits hysteresis. Due to the presence of the switching component of the trajectories, several definitions of value functions are possible. No one of these value functions is in general continuous. We prove that, under general hypotheses, the "least value function", i.e. the value function of the more relaxed problem, is the unique lower semicontinuous viscosity solution of two suitably coupled Hamilton-Jacobi-Bellman equations. Such a coupling involves boundary conditions in the viscosity sense.

1 Introduction.

In the recent paper [1], the author applied the dynamic programming technique and the viscosity solutions theory to an optimal control problem of infinite horizon type for a "scalar" switching system. This means that the

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controlled state variable is the pair (y, z) where y is a scalar quantity (directly controlled by the controlled state equation), and z is a switching scalar quantity, which may assume only the values -1 and 1, and whose switchings are subject to the evolution of y by means of a so-called delayed relay hysteresis relationship. The motivation was to study an optimal control problem for a system presenting discontinuous hysteresis of delayed relay type, which appears in particular in the behavior of thermostats. Moreover, applications were given to an optimal control problem with a "discrete" version of the Preisach hysteresis operator (see [2] for the "continuous" version case).

In this paper, we study a problem of optimal control of finite horizon type, for a switching system where the directly controlled quantity is the multidimensional $y \in I\!\!R^n$, and the switchings of the scalar quantity z are subject to the evolution of a fixed component of y, let us say $y \cdot S$, where S is a fixed unit vector of $I\!\!R^n$. More precisely, if we represent by $z(\cdot) = h[y(\cdot) \cdot S, w](\cdot)$ the evolutive switching relationship between z and $y \cdot S$ (where w is a suitable initial state for z), we then consider the controlled system

$$\begin{cases} y'(t) = f(y(t), z(t), \alpha(t)) & \text{if } t > 0, \\ z(t) = h[y(t) \cdot S, w](t) & \text{if } t \geq 0, \\ y(0) = x, \end{cases} \tag{1.1}$$

where $\alpha : [0, +\infty[\to A$ is a measurable control, and $x \in I\!\!R^n$ is a suitable initial state for y. The delayed switching hysteresis relationship $z = h[y \cdot S, w]$ may be naively explained as follows. We have a pair of thresholds $\rho_1 < \rho_2$ which are respectively the threshold for switching down (from 1 to -1) and for switching up (from -1 to 1). This means the following: if $y(t) \cdot S > \rho_2$, then $z(t) = 1$; if $y(t) \cdot S < \rho_1$, then $z(t) = -1$; if $\rho_1 < y(t) \cdot S < \rho_2$ then the values of $z(t)$ depends on the past evolution of $y \cdot S$ by a suitable hysteresis rule; if $y(t) \cdot S$ is on a threshold, then the value of $z(t)$ depends on the particular switching rule we are considering (and many of them may be considered). In this contest, hysteresis is due to the fact that the thresholds for switching up and for switching down are different. Hence, in the input-output relationship between $y \cdot S$ and z, we experience a delay in the occurrence of the switchings, in the sense that, after a possible switching we cannot immediately have another switching, but a delay appears. In particular such a delay is not in time (i.e. waiting a sufficiently long period we have another switching), but instead it is in "space", in the sense that another switching may occur only if the input $y \cdot S$ reaches a suitable values (i.e. the other threshold). A possible evolution of the pair (y, z) solution of (1.1) is shown in Figure 1.

The finite horizon optimal control problem consists in minimizing, over all the measurable controls, a cost functional of the form

$$J(x, w, t, \alpha) = \int_0^t e^{-\lambda s}\ell(y(s), z(s), \alpha(s))ds + e^{-\lambda t}g(y(t), z(t)), \tag{1.2}$$

where (y, z) is solution of (1.1) with control α and initial state (x, w). We are interested in characterizing the value function

$$V(x, w, t) := \inf_{\alpha} J(x, w, t, \alpha), \tag{1.3}$$

as the unique viscosity solution of a suitable Hamilton-Jacobi problem. Actually, as in [1], the definition of the value function depends on the particular switching rule we are considering (for instance: z switches exactly when $y \cdot S$ reaches the threshold, or z switches when $y \cdot S$ is going to get over the threshold, or z may switch at any instants when $y \cdot S$ is on the threshold). To any possible switching rule, different trajectories are corresponding, and hence we get different value functions (this is also a typical feature of the so-called exit-time problems, and indeed we are going to regard our switching problem as a suitable coupling of exit-time problems).

The present problem has new complicate features with respect to the one in [1]. Indeed, since it is a finite horizon problem, there is the presence of the time-variable which, in correspondence of the switchings, may cause some irregularities; moreover there is the presence of the final cost g which depends on the discontinuous variable z; and finally the discount factor λ may be zero. However, a more critical aspect is the fact that y is multidimensional, and hence it may happen that no one of the possible value functions is continuous, even not lower or upper semicontinuous (in [1], without assuming any controllability condition, it is proved that some suitable value functions are continuous on the set of admissible initial states, and this fact of course is used in the proof of the main result).

Our goal is to derive a suitable problem for Hamilton-Jacobi equations, such that it is uniquely satisfied in the viscosity sense by a significant one of the possible value functions. Since the Hamiltonian of our control problem is

$$H(x, w, p) = \sup_{a \in A} \left\{ -f(x, w, a) \cdot p - \ell(x, w, a) \right\}, \tag{1.4}$$

which is convex in the p-variable, then that suitable value function should be at least lower semicontinuous in $(x, w, t) \in I\!\!R^n \times I\!\!R \times [0, +\infty[$. In order to have lower semicontinuity we have to relax our problem. A first relaxation is to consider the so-called "complete delayed switching rule", which in some sense may z free whether to switch or not when $y \cdot S$ is on the threshold (but it must switch if $y \cdot S$ gets over the threshold). This relaxed switching rule of course may have different outputs z for the same input $y \cdot S$ (for instance if $y \cdot S$ reaches the threshold and then it comes back without getting over it, then we may have two different evolutions for z: the constant evolution without switchings, and the evolution with one switching). This fact implies that, for any choice of initial state (x, w) and measurable control α, the system (1.1) may have more than one trajectory. Let us denote by $\mathcal{T}(x, w, \alpha)$ the set of possible trajectories corresponding to (x, w, α). We may then define the value function as

$$V(x, w, t) = \inf_{\alpha} \left(\inf_{(y(\cdot), z(\cdot)) \in \mathcal{T}(x, w, \alpha)} J(x, w, t, y(\cdot), z(\cdot), \alpha) \right), \tag{1.5}$$

where $J(x, w, t, y(\cdot), z(\cdot), \alpha)$ is the cost functional as in (1.2) evaluated along the trajectory $(y(\cdot), z(\cdot))$. The second relaxation is the use of the so-called "relaxed controls" which are probability measures on the set of controls (see for instance Warga [16]). With these two relaxations, and without any controllability assumption, the relaxed value function $\hat{V}$ as defined in (1.5) (when the relaxed controls are used) turns out to be lower semicontinuous on its domain of definition. In particular, such domain of definition is

$$\left(\overline{\mathcal{H}}_1 \cup \overline{\mathcal{H}}_{-1} \right) \times [0, +\infty[,$$

where, for instance, $\overline{\mathcal{H}}_1$ is the closure in $I\!\!R^n \times I\!\!R$ of the open horizontal semi-hyperplane (see Figure 1)

$$\mathcal{H}_1 = \left\{ (x, 1) \in I\!\!R^n \times I\!\!R \middle| x \cdot S > \rho_1 \right\}.$$

Note that, in some sense, the relaxed value function is the least value function among all the possible ones.

By using dynamic programming techniques, on every branch $\overline{\mathcal{H}}_w \times [0, +\infty[$ with $w \in \{1, -1\}$, we can regard the value function $\hat{V}$ as the value function of a finite horizon control problem with exit time. In particular, the final cost is $g(\cdot, w)$ and the exit cost is $\hat{V}(\cdot, -w, \cdot)$. In other words, the value function itself, evaluated on the other branch $\overline{\mathcal{H}}_{-w} \times [0,+\infty[$, plays the role of an exit cost. Hence, $\hat{V}$ solves the following Hamilton-Jacobi problem

$$\left\{\begin{array}{l}\forall w \in \{1,-1\},\ \hat{V} \text{ is a discontinuous viscosity solution of} \\ \left\{\begin{array}{ll}\hat{V}_t(\cdot,w,\cdot)+\lambda\hat{V}(\cdot,w,\cdot)+H(x,w,D\hat{V}(\cdot,w,\cdot))=0 & \text{in } \mathcal{H}_w\times]0,+\infty[, \\ \hat{V}(\cdot,w,\cdot)=\hat{V}(\cdot,-w,\cdot) & \text{on } \partial\mathcal{H}_w\times]0,+\infty[, \\ \hat{V}(\cdot,w,\cdot)=g(\cdot,w) & \text{on } \mathcal{H}_w\times\{0\}, \\ \hat{V}(\cdot,w,\cdot)=\min\left\{\hat{V}(\cdot,-w,\cdot),g(\cdot,w)\right\} & \text{on } \partial\mathcal{H}_w\times\{0\},\end{array}\right.\end{array}\right. \tag{1.6}$$

where also the boundary conditions are to be interpreted in the viscosity sense. By discontinuous viscosity solution, we mean that the lower semicontinuous envelop is a supersolution and the upper semicontinuous envelop is subsolution.

The boundary-initial value problems in (1.6) (one per $w \in \{1,-1\}$), have discontinuous boundary data, (the value function $\hat{V}$ itself evaluated on suitable points of the other branch). Usually, such problems, even with continuous data, do not have a unique solution. What is possible to prove is that they have a unique lower semicontinuous solution, provided that the boundary datum (let us say φ) satisfies

$$(\varphi^*)_* = \varphi_*, \tag{1.7}$$

where "*" and "$_*$" stay respectively for upper and lower semicontinuous envelop. However, note that the uniqueness of the solution for both Hamilton-Jacobi problems in (1.6) does not immediately imply the uniqueness of the solution of (1.6).

We introduce some controllability hypotheses in order to guarantee that the value function satisfies (1.7). With such hypothesis and other controllability ones (standard for exit time problems), we are able to prove that, among a rather general class of functions, the relaxed value function $\hat{V}$ is the unique lower semicontinuous solution of (1.6). Moreover, all the possible solutions of (1.6) have the same lower semicontinuous envelop, which coincides with $\hat{V}$. In particular, such uniqueness result is possible by the fact that the switching rule is delayed (i.e. presents hysteresis).

The theory of discontinuous viscosity solutions for Hamilton-Jacobi equations, with convex Hamiltonians, and with continuous data, goes back to the works of Barles-Perthame [6], [7], Ishii [11], and Barron-Jensen [8], where different approaches are considered. For the case of discontinuous data we refer to the works of Blanc [9], [10], where the results of Barles-Perthame are suitably extended. In particular, up to the knowledge of the author, [10] is the unique work concerning final horizon-exit time problems, which lead to an initial-boundary value problem for Hamilton-Jacobi equations with discontinuous data. Our proof of the comparison result for the initial-boundary value problem in (1.6) follows the one in [10]. However, our situation is different for the fact that the state-space is unbounded (i.e. $\overline{\mathcal{H}}_w$), and also for the possibility of non zero discount factor λ. Hence, we need to introduce some non obvious modifications in the proof. In [10], the notion of Barron-Jensen solution is also treated (see [8]), which again gives the uniqueness of the lower semicontinuous solution. A similar approach seems to be possible also in our case, and the results partially cover each other. An interesting approach to discontinuous viscosity solutions is also given in Soravia [14]. However, it does not seem to be possible in our case. Indeed, it requires a particular hypothesis satisfied by the boundary datum which is not obviously adapted to our case, where the boundary datum is given by the unknown function itself (see (1.6)), which is the value function of a problem starting with different dynamics and costs $(f(\cdot,-w,\cdot),\ell(\cdot,-w,\cdot),g(\cdot,-w)$ in place of $f(\cdot,w,\cdot),\ell(\cdot,w,\cdot),g(\cdot,w))$.

For a comprehensive account of the applications of viscosity solutions for Hamilton-Jacobi equations to optimal control problems, we refer to the book by Bardi-Capuzzo Dolcetta [4]. For the theory of hysteresis operator, and in particular of the delayed relay and its generalizations, we refer to the book by Visintin [15].

Optimal control problems for delayed switching systems are of course of very importance for the applications, when thermostatic components are considered, and in general when discontinuous hysteresis appears in the system under control. For instance, in the recent paper [12], Lenhart, Seidman, and Yong studied an optimal control problem for a bioreactor, where a bacterial population is suitably activated in order to metabolize some pollutant. The activity of the bacteria has two modes: "dormant" and "active" respectively. The transition from one mode to the other occurs with a delayed switching rule subject to the evolution of the nutrient amount. The state of the problem is the 4-upla given by the nutrient amount, the bacterial population, the pollutant amount, and the mode of activity. The control is the rate of introduction of nutrient and the cost to be minimized takes account of the pollutant remaining. Hence, such a problem enters the class of problems for multidimensional delayed switching system we address in this paper. However, in that paper, the authors are mainly concerned in existence of (non relaxed) optimal controls, and not in dynamic programming technique and in the viscosity solutions theory.

The plan of the paper is as follows. In Section 2, we describe the complete delayed switching rule, presenting it as a hysteresis operator between spaces of time dependent functions; we then give some results for ordinary differential equations with such switching rule. In Section 3, we introduce the control problem and its relaxation. In Section 4, we represent the problem as an exit time problem. In Section 5, we prove the uniqueness result. In Section 6, we give general results on Hamilton-Jacobi problems for finite horizon-exit time problems with discontinuous data; we also give the definition of discontinuous viscosity solutions, and of boundary conditions in the viscosity sense.

2 The complete delayed switching rule.

Let us consider a continuous input $u \in C^0([0,+\infty[)$, a discontinuous output $z : [0,+\infty[\to \{-1,1\}$, and two different thresholds for values of u, let us say ρ_1 and ρ_2, with $\rho_1 < \rho_2$, for which z respectively switches "down" from $+1$ to -1, and "up" from -1 to $+1$. We define the set

$$\overline{\mathcal{O}} := (]-\infty,\rho_2] \times \{-1\}) \cup ([\rho_1,+\infty[\times\{1\}) \subset I\!R^2,$$

and we can think to the switching evolution as a "continuous" evolution of the couple $(u(\cdot),z(\cdot))$ on the set $\overline{\mathcal{O}}$. The quotation marks on the word "continuous" mean that such evolution is certainly continuous at any instant t such that $(u(t),z(t)) \in \mathcal{O}$, where

$$\mathcal{O} := (]-\infty,\rho_2[\times\{-1\}) \cup (]\rho_1,+\infty[\times\{1\}) \subset I\!R^2,$$

but discontinuities may occur at any instant t such that $(u(t),z(t)) = (\rho_1,1)$ or $(u(t),z(t)) = (\rho_2,-1)$. How such switchings (discontinuities) occur is now addressed. In principle, we may have several rules for switching (see for instance the two rules respectively called "exact switching rule" and "getting over rule" in Bagagiolo [1]). Here, we are going to consider the most relaxed one, which leaves the output z free whether to switch or not when (u,z) is on one of the "switching points" $(\rho_1,1)$ or $(\rho_2,-1)$. Obviously, such freedom implies that, for a single continuous input u , we may have more than one output z. We give the following definition

Definition 2.1 *Let $u : [0, +\infty[\to I\!R$ be a continuous function, $\rho_1 < \rho_2$ be two thresholds, and $w \in \{-1, 1\}$ be an initial state for the output such that $(u(0), w) \in \overline{\mathcal{O}}$. A function $z : [0, +\infty[\to \{-1, 1\}$ is an output of the complete delayed switching rule with couple of thresholds $\rho = (\rho_1, \rho_2)$, input u, and initial state w, if and only if*

$$\begin{array}{ll} i)\ (u(t), z(t)) \in \overline{\mathcal{O}} & \forall t \geq 0, \\ ii)\ (u(\cdot), z(\cdot)) & \textit{is continuous at every } t \geq 0 \textit{ such that} \\ & (u(t), z(t)) \notin \{(\rho_1, 1), (\rho_2, -1)\}, \\ iii)\ z(0) = w & \textit{if } (u(0), w) \notin \{(\rho_1, 1), (\rho_2, -1)\} \end{array}$$

We equivalently say that z is an output of the "complete delayed relay" with couple of thresholds ρ, input u, and initial state w. We also write

$$z(t) \in h_\rho[u, w](t)\ \forall t \geq 0.$$

Note that the condition iii) in Definition 2.1 means that, at the initial time $t = 0$, the output z must be equal to the initial state w if $(u(0), w)$ is not a switching point, otherwise we allow both $z(0) = w$ and $z(0) = -w$. In other words, if we start from a switching point, then z can immediately switch, but this means that it switches at $t = 0$ or that it has already switched at $t = 0$. More generally, this consideration applies to any switching instant $t > 0$. A switching instant $t > 0$ is a time such that $z \equiv w \in \{1, -1\}$ immediately before t (for instance in $]t - \delta, t[$), and $z \equiv -w$ immediately after t (for instance in $]t, t+\delta[$). Hence, for every switching time we have the output $z_1 \equiv w$ in $]t - \delta, t]$, $z_1 \equiv -w$ in $]t, t + \delta]$, as well the output $z_2 \equiv w$ in $]t - \delta, t[$, $z_2 \equiv -w$ in $[t, t + \delta[$.

Since the input u is continuous, and hence locally uniformly continuous, after any switching instant t, z cannot immediately switch back. Indeed, at a switching time, the value of the input coincides with one of the two switching thresholds ρ_1 or ρ_2, and hence z may switch back only when the input u will possibly reach the other switching threshold. For the local uniformly continuities, this cannot happen before a sufficiently large interval of time, whose size may be chosen locally constant. This implies the existence of at least one output, even for fast oscillating inputs (see also Visintin [15]).

We define the set of admissible entries for h_ρ by

$$\mathcal{D} := \left\{(u,w) \in C^0([0,+\infty[) \times \{-1,1\} \Big| (u(0),w) \in \overline{\mathcal{O}}\right\},$$

We can then assert that h_ρ operates as

$$h_\rho : \mathcal{D} \to \mathcal{P}\left(BV_{loc}(0,+\infty)\right),$$

where $\mathcal{P}(A)$ is the set of the parts of the set A. Hence, $h_\rho[u,w]$ is the set of admissible outputs, and moreover $h_\rho[u,w](t)$ is the set of values at time t achieved by all the outputs $z \in h_\rho[u,w]$. The fact that the outputs $z \in h_\rho[u,w]$ are locally of total bounded variation, comes from the fact that their total variations change of a quantity equal to 2 exactly at every time when they switch, but for any compact time interval, they cannot switch more than a finite number of times.

The completed delayed relay has the following three properties: respectively "causality", "hysteresis", and "semigroup" property (see Visintin [15]):

$$\begin{array}{l}
i)\ (u,w),(v,w) \in \mathcal{D},\ u = v \text{ in } [0,t] \Longrightarrow h_\rho[u,w](t) = h_\rho[v,w](t),\\
ii)\ h_\rho[u \circ \varphi, w] = h_\rho[u,w] \circ \varphi\ \forall (u,w) \in \mathcal{D}\\
\quad \forall \text{ continuous, increasing, and surjective } \varphi : [0,+\infty[\to [0,+\infty[\\
iii)\ \forall z \in h_\rho[u,w]\ \forall t_1 \geq 0\ \forall z_1 \in h_\rho[u(\cdot + t_1), z(t_1)],\ \text{the function}\\
\tilde{z}(t) = z(t) \text{ in } [0,t_1],\ \tilde{z}(t) = z_1(t - t_1) \text{ in }]t_1,+\infty[\text{ belongs to } h_\rho[u,w].
\end{array} \tag{2.1}$$

We consider an ordinary differential system with a switching component. Let $f : I\!R^n \times \{-1,1\} \times [0,+\infty[\to I\!R^n$, $(x,w,t) \mapsto f(x,w,t)$, satisfy:

$$\begin{array}{l}
\forall w \in \{-1,1\}\ \forall x \in I\!R^n\ t \mapsto f(x,w,t) \text{ is measurable},\\
\exists L > 0 \text{ such that } \forall x,y \in I\!R^n,\ \forall w \in \{-1,1\},\ \forall t \geq 0 :\\
|f(x,w,t) - f(y,w,t)| \leq L|x-y|,\quad |f(x,w,t)| \leq L.
\end{array} \tag{2.2}$$

We consider the following system

$$\left\{\begin{array}{ll}
y'(t) = f(y(t),z(t),t) & t > 0,\\
z(t) \in h_\rho[y(\cdot)\cdot S, w](t) & t \geq 0,\\
y(0) = x,\\
(x \cdot S, w) \in \overline{\mathcal{H}}_w,
\end{array}\right. \tag{2.3}$$

where S is a fixed unit vector of $I\!R^n$, and, for $w \in \{1,-1\}$, $\mathcal{H}_w$ is respectively

$$\begin{aligned} \mathcal{H}_1 &:= \{(x,1) \in I\!R^n \times \{1\} | x \cdot S > \rho_1\}, \\ \mathcal{H}_{-1} &:= \{(x,-1) \in I\!R^n \times \{-1\} | x \cdot S < \rho_2\}. \end{aligned} \tag{2.4}$$

In the sequel, for every $w \in \{-1,1\}$, we will use respectively the notations: $\partial\mathcal{H}_1 := \{(x,1) \in I\!R^n \times \{1\} | x \cdot S = \rho_1\}$, and $\partial\mathcal{H}_{-1} := \{(x,-1) \in I\!R^n \times \{-1\} | x \cdot S = \rho_2\}$. With this notation, the closure of $\mathcal{H}_w$ is given by $\overline{\mathcal{H}}_w = \mathcal{H}_w \cup \partial\mathcal{H}_w$, and we refer to $\partial\mathcal{H}_w$ as the boundary of $\mathcal{H}_w$. The points of $\partial\mathcal{H}_w$ will also be called "switching points", since the switching may occur only on that points. Moreover, if (x,w) is a switching point, then its "conjugate point" is $(x,-w)$.

Definition 2.2 *A solution of (2.3) is a couple of functions $t \mapsto (y(t), z(t))$ belonging to $C^0([0,+\infty[; I\!R^n) \times (BV_{loc}(0,+\infty) \cup L^\infty(0,+\infty))$ such that,*

$$y(t) = x + \int_0^t f(y(s), z(s), s) ds, \quad z(t) \in h_\rho[y(\cdot) \cdot S; w](t) \quad \forall t \geq 0.$$

The fact that the switching rule presents hysteresis (which is essentially the fact that the "switching down threshold" ρ_1 is strictly less than the "switching up threshold" ρ_2), immediately leads to the existence of at least one solution for (2.3). Indeed, since f is bounded, after any possible switching we have to wait a strictly positive time $\delta > 0$ in order to possibly have a further switching (we have to wait that the input $y(\cdot) \cdot S$ passes from one threshold to the other). Hence, since δ is independent from (x,w,t), we have a sequence of time intervals of length δ where z (the switching variable) is constant, and hence the solution exists. We can then construct a solution defined in $[0,+\infty[$, by gluing together such solutions on the δ-intervals (see Bagagiolo [1] for similar constructions). By the fact that the h_ρ may admit several outputs for the same input, the system (2.3) is lacking uniqueness.

We regard the switching as a switching between the two parallel horizontal closed semi-hyperplanes in $I\!R^n \times I\!R \ni (x,w)$, $\overline{\mathcal{H}}_1$ and $\overline{\mathcal{H}}_{-1}$ (see Figure 1).

By the considerations made after Definition 2.1, if the initial datum (x,w) is a switching point, then we have trajectories for (2.3) starting from that point as well as starting from its conjugate point $(x,-w)$. Similarly, if we consider trajectories defined in the time interval $[0,T]$, and if at the final time T we are in a switching point, then we have two different trajectories to consider: the one ending on the switching point and the other one ending on the conjugate point. For switchings occurring on intermediate instants, the reasoning is the same. Roughly speaking, whenever, at the time t, a trajectory (y,z) reaches a switching point, this fact may generate for the

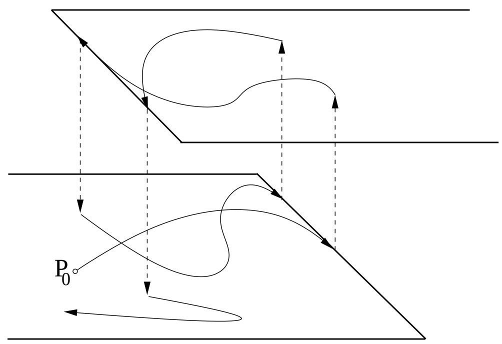

Figure 1: Delayed switching evolution, starting from P_0.

remaining time interval several different trajectories: a) the trajectory that at t has not yet switched, but switches exactly after t, i.e. $z(\tau)$ is constant for $\tau \in [t-\delta, t]$, and $z(\tau) = -z(t)$ for $\tau \in]t, t+\delta]$, for a suitable $\delta > 0$ (this trajectory always exists); b) the trajectory that switches exactly at time t, i.e $z(\tau)$ is constant in $[t, t+\delta]$, and $z(\tau) = -z(t)$ in $[t-\delta, t[$, for a suitable $\delta > 0$ (this trajectory always exists); c) any possible other trajectory which does not switch at t, i.e. $z(\tau)$ is constant (let us say equal to w) in $[t-\delta, t+\delta]$ for a suitable $\delta > 0$ (such trajectories exist only if, at time t, they are not forced to switch, i.e. if $y(\cdot) \cdot S$, whose evolution in $[t-\delta, t+\delta]$ is governed by $y' \cdot S = f(y, w, s) \cdot S$, in that time interval is not forced to cross the corresponding switching threshold (ρ_1 if $w = 1$, ρ_2 if $w = -1$)).

3 The control problem and its relaxation.

Let A be compact, $f : I\!R^n \times \{-1, 1\} \times A \to I\!R^n$ continuous, and

$$\begin{array}{l} \exists L > 0 \text{ such that } \forall x, y \in I\!R^n,\ \forall w \in \{-1,1\},\ \forall a \in A : \\ |f(x,w,a) - f(y,w,a)| \le L|x-y|, \quad |f(x,w,a)| \le L. \end{array} \tag{3.1}$$

We consider the following system

$$\begin{cases} y'(t) = f(y(t), z(t), \alpha(t)) & t > 0, \\ z(t) \in h_\rho[y(\cdot) \cdot S, w](t) & t \ge 0, \\ y(0) = x, & \\ (x \cdot S, w) \in \overline{\mathcal{H}}_w. & \end{cases} \tag{3.2}$$

In (3.2), h_ρ is the complete delayed relay with threshold $\rho = (\rho_1, \rho_2)$, $\rho_1 < \rho_2$; S is a fixed unit vector of $I\!R^n$; $\alpha(\cdot)$ is a measurable control belonging in

$$\mathcal{A} := \{\alpha : [0, +\infty[\to A | \text{such that } \alpha \text{ is measurable}\} \tag{3.3}$$

The existence of possible multiple solutions for (3.2) is discussed in the previous section. Let $\mathcal{T}(x, w, \alpha)$ be the set of all trajectories of (3.2) corresponding to the choices of $(x, w) \in \overline{\mathcal{H}}_w$ as initial point, and of $\alpha \in \mathcal{A}$ as control.

Then we consider a running cost $\ell : I\!R^n \times \{-1, 1\} \times A$, and a final cost $g : I\!R^n \times \{-1, 1\}$ such that

$$\begin{array}{l} \ell \text{ is bounded and continuous in } I\!R^n \times \{w\} \times A \ \forall w \in \{-1,1\}, \\ \exists \text{ a modulus of continuity } \omega_\ell \text{ such that } \forall w \in \{-1,1\},\ \forall a \in A : \\ |\ell(x,w,a) - \ell(y,w,a)| \le \omega_\ell(|x-y|)\ \forall x, y \in I\!R^n, \\ g \text{ is bounded and uniformly continuous in } I\!R^n \times \{w\}\ \forall w \in \{-1,1\}. \end{array} \tag{3.4}$$

Let $\lambda \ge 0$ be a discount factor. We consider the cost functional J, defined for every $(x, w) \in \overline{\mathcal{H}}_w$, for every $\alpha \in \mathcal{A}$, for every $(y, z) \in \mathcal{T}(x, w, \alpha)$, and for every $t \ge 0$, by

$$J(x, w, t, \alpha, y, z) := \int_0^t e^{-\lambda s} \ell(y(s), z(s), \alpha(s)) ds + e^{-\lambda t} g(y(t), z(t)). \tag{3.5}$$

Then the value function is defined by

$$V(x, w, t) := \inf_{\alpha \in \mathcal{A}} \left(\inf_{(y,z) \in \mathcal{T}(x,w,\alpha)} J(x, w, t, y, z) \right). \tag{3.6}$$

If (x, w) is a switching point, since all trajectories starting from $(x, -w)$ are also trajectories starting from (x, w), then

$$V(x, w, 0) = \min\{g(x, w), g(x, -w)\}, \quad V(x, w, t) \le V(x, -w, t)\ \forall t \ge 0, \tag{3.7}$$

Because of the discontinuity of the switching variable z, the value function is discontinuous. For our purpose, we prefer that the value function would be at least lower semicontinuous. The use of the complete delayed switching rule certainly helps to have such property. However, it is not enough. To show that, one can easily adapt a counterexample given in Barles-Perthame

[6]. Hence, we need a further relaxation (the first one is the use of the complete delayed switching rule) which should ensure that the uniform limit of trajectories is still an admissible trajectory. This is important to avoid that from a switching point we are forced to switch, whereas from a converging sequence of points to it, we may not switch in a time interval $[0,\delta]$ with $\delta > 0$. In particular, the non lower semicontinuity still holds if we also suppose that we have an outer field property, i.e. a further choice of a control for which the field is outward with respect to $\overline{\mathcal{H}}_w$ (such controllability condition on the boundary is the one that we will assume in the next sections).

We then introduce the relaxed controls. Let us suppose that (3.1) and (3.4) hold. We define the set of relaxed measurable controls

$$\hat{\mathcal{A}} = \{\hat{\alpha} : [0,+\infty[\to \mathcal{M}(A), t \mapsto \hat{\alpha}_t \ \text{ measurable}\}, \tag{3.8}$$

where $\mathcal{M}(A)$ is the set of Radon probability measures on A. If we suppose w fixed (i.e. not switching), and we take a relaxed control $\hat{\alpha}$, then the relaxed trajectory $\hat{y}(\cdot)$ with relaxed dynamics $\hat{f}(\cdot, w, \hat{\alpha}.)$, i.e. the solution of the Cauchy problem

$$\begin{cases} \hat{y}'(t) = \hat{f}(\hat{y}(t), w, \hat{\alpha}_t) & t > 0, \\ \hat{y}(0) = x, \end{cases} \tag{3.9}$$

is given in integral form by

$$\hat{y}(t) = x + \int_0^t \hat{f}(\hat{y}(s), w, \hat{\alpha}_s)ds = x + \int_0^t \left(\int_A f(\hat{y}(s), w, a)d\hat{\alpha}_s\right) ds. \tag{3.10}$$

Classical results (see Warga [16]), ensure that the solution of (3.9) exists, is unique, is continuous and suitably depends on data. We consider the relaxed switching controlled system (compare with (3.2))

$$\begin{cases} \hat{y}'(t) = \hat{f}(\hat{y}(t), \hat{z}(t), \hat{\alpha}_t) & t > 0, \\ \hat{z}(t) \in h_\rho[\hat{y}(\cdot) \cdot S, w](t) & t \geq 0, \\ \hat{y}(0) = x, \quad (x,w) \in \overline{\mathcal{H}}_w. \end{cases} \tag{3.11}$$

Because of the continuity of the relaxed (non switching) trajectory of (3.9), the relaxed trajectories $(\hat{y}, \hat{z})$ of (3.11) is defined just using the complete delayed switching rule of h_ρ, as we did for the solutions of (3.2). The regularity $(\hat{y}, \hat{z}) \in C^0([0,+\infty[, I\!R^n) \times BV_{loc}(0,+\infty)$ still holds. Moreover, for every initial point (x, w) and every relaxed control $\hat{\alpha}$, there are possible many trajectories. Let $\hat{\mathcal{T}}(x, w, \hat{\alpha})$ be the set of such trajectories.

For every point (x, w), final time $t \geq 0$, relaxed control $\hat{\alpha}$, and relaxed trajectory $(\hat{y}, \hat{z}) \in \hat{\mathcal{T}}(x, w, \hat{\alpha})$, we define the relaxed cost and value function

$$\begin{aligned}
&\hat{J}(x, w, t, \hat{\alpha}, \hat{y}, \hat{z}) := \int_0^t e^{-\lambda s}\hat{\ell}(\hat{y}(s), \hat{z}(s), \hat{\alpha}(s))ds + e^{-\lambda t}g(\hat{y}(t), \hat{z}(t)) \\
&= \int_0^t e^{-\lambda s}\left(\int_A \ell(\hat{y}(s), \hat{z}(s), a)d\hat{\alpha}_s\right)ds + e^{-\lambda t}g(\hat{y}(t), \hat{z}(t)); \\
&\hat{V}(x, w, t) := \inf_{\hat{\alpha}\in\hat{\mathcal{A}}}\left(\inf_{(\hat{y},\hat{z})\in\hat{\mathcal{T}}(x,w,\hat{\alpha})} \hat{J}(x, w, t, \hat{\alpha}, \hat{y}, \hat{z})\right).
\end{aligned} \tag{3.12}$$

Here are the main properties of the relaxed controls (see Warga [16]).

Proposition 3.1 *If we endow $\mathcal{M}(A)$ by the weak topology for measures, and $\hat{\mathcal{A}}$ by the weak star topology associated, we then have the following results: i) $\hat{\mathcal{A}}$ is compact; ii) every classical control $\alpha \in \mathcal{A}$ is uniquely associated to the relaxed control $t \mapsto \delta_{\alpha(t)}$, where δ_a is the Dirac mass at the point $a \in A$. Moreover, by such identification, the classical controls are dense in the set of relaxed controls; iii) if $\hat{\alpha}_n$ is a sequence of relaxed controls weakly star converging to $\hat{\alpha}$, then, regarding the (non switching) system (3.9), the sequence of relaxed trajectories $\hat{y}_n$ starting from x with control $\hat{\alpha}_n$, uniformly converges over the compact sets of $[0, +\infty[$ to the relaxed trajectory $\hat{y}$ starting from x with control $\hat{\alpha}$. Moreover, for every $t \geq 0$, and for every $w \in \{-1, 1\}$*

$$\lim_{n\to+\infty}\int_0^t e^{-\lambda s}\hat{\ell}(\hat{y}_n(s), w, \hat{\alpha}_n(s))ds = \int_0^t e^{-\lambda s}\hat{\ell}(\hat{y}(s), w, \hat{\alpha}(s))ds. \tag{3.13}$$

Proposition 3.2 *For every $(x, w, t) \in \overline{\mathcal{H}}_w \times [0, +\infty[$, there exist an optimal relaxed control $\hat{\alpha} \in \hat{\mathcal{A}}$ and an optimal relaxed trajectory $(\hat{y}, \hat{z}) \in \hat{\mathcal{T}}(x, w, \hat{\alpha})$.*

Proof. If $t = 0$ then the conclusion is obvious by definition (see (3.7)).

Let $\hat{\alpha}_n \in \hat{\mathcal{A}}$ and $(\hat{y}_n, \hat{z}_n) \in \hat{\mathcal{T}}(x, w, \hat{\alpha}_n)$ be respectively a sequence of minimizing relaxed controls and of minimizing trajectories for $V(x, w, t)$. Let $\hat{\alpha}$ be the weakly star limit of a subsequence of $\hat{\alpha}_n$ (still labelled by n). Moreover, for every n let r_n be the number of switchings of $\hat{z}_n$ during the time interval $[0, t]$, and let $\{t_n^i\}_{i=1,\ldots,r_n}$ be the sequence of switching times ($0 \leq t_n^1 < t_n^2 < \ldots < t_n^{r_n} \leq t$). Since the quantities r_n are equibounded with respect to n (this is true since f is bounded and the switching is delayed, i.e. $\rho_2 > \rho_1$, and hence there exists $c > 0$ independent from n such that $t_n^{i+1} - t_n^i \geq c$)), then there exists a subsequence, still labelled by n such that r_n is equal to a constant r. We analyze two cases: a) $r = 0$, b) $r > 0$.

a) $r = 0$. This means that $\hat{z}_n \equiv w$ for every n, that is there are no switchings in the time interval $[0, t]$. By Proposition 3.1, we classically can prove that $\hat{\alpha}$, the constant trajectory $\hat{z} \equiv w$, and the corresponding trajectory $\hat{y}$ solution of (3.11) are optimal. Note that, since $(\hat{y}, \hat{z}_n) = (\hat{y}_n, w) \in \overline{\mathcal{H}}_w$ in $[0, t]$ for all n, the same also holds for $(\hat{y}, w)$.

b) $r > 0$. We have $\hat{z}_n = w$ in $]0, t_n^1[$ ($= \emptyset$ if $t_n^1 = 0$), $\hat{z}_n = -w$ in $]t_n^1, t_n^2]$, $\hat{z} = w$ in $]t_n^2, t_n^3[$, and so on. At least for a subsequence, we can suppose that, as $n \to +\infty$, $t_n^i \to t^i \in [0, t]$, for every $i = 1, \ldots, r$. Note that, we certainly have $t^i < t^{i+1}$. Moreover we define $t^0 = 0$ and $t^{r+1} = t$. In the interval $]t^0, t^1[$, we define $\hat{z} \equiv w$, and solve the equation $\hat{y}' = \hat{f}(\hat{y}, w, \hat{\alpha}), \hat{y}(0) = x$, and we also define $x_1 = \hat{y}(t^1)$. By Proposition 3.1, $\hat{y}_n$ uniformly converge to $\hat{y}$, on every compact sub-interval of $]t^0, t^1[$ (if $t^1 = 0$, then we do not consider this first step). Moreover,

$$\int_{t^0}^{t^1} e^{-\lambda s} \hat{\ell}(\hat{y}_n(s), w, \hat{\alpha}_n(s)) ds \to \int_{t^0}^{t^1} e^{-\lambda s} \hat{\ell}(\hat{y}(s), w, \hat{\alpha}(s)) ds.$$

In the interval $]t^1, t^2[$, we define $\hat{z} \equiv -w$, we solve $\hat{y}' = \hat{f}(\hat{y}, -w, \hat{\alpha}))$ $\hat{y}(t^1) = x_1$, and we define $x_2 = \hat{y}(t^2)$. Again, $\hat{y}_n$ uniformly converge to $\hat{y}$ on every compact set of $]t^1, t^2[$, and

$$\int_{t^1}^{t^2} e^{-\lambda s} \hat{\ell}(\hat{y}_n(s), -w, \hat{\alpha}_n(s)) ds \to \int_{t^1}^{t^2} e^{-\lambda s} \hat{\ell}(\hat{y}(s), -w, \hat{\alpha}(s)) ds.$$

In general, in $]t^i, t^{i+1}[$, we define $\hat{z} \equiv (-1)^i w$, we solve $\hat{y}' = \hat{f}(\hat{y}, \hat{z}, \hat{\alpha})$, $\hat{y}(t^i) = x_i$, and we have similar convergences as before. Then we have constructed a continuous function $\hat{y}$ on $[0, t]$. When $t^r < t$, then the definition of $\hat{z}$ on $t^1, \ldots, t^r$ is irrelevant (note, that we are eventually interested to the convergence of the costs). On the other hand, if $t^r = t$, then $\hat{z}_n$ has the last switching in $t_n^r \leq t$, with $|t_n^r - t|$ infinitesimal. Hence, $g(\hat{y}_n(t), \hat{z}_n(t)) \to g(\hat{y}(t), \hat{z}(t))$. We then conclude that $(\hat{y}, \hat{z})$, which solves the system (3.2), in $[0, t]$ with control $\hat{\alpha}$, is an optimal trajectory (and $\hat{\alpha}$ is an optimal control). □

Proposition 3.3 *The relaxed value function $\hat{V}$ is lower semicontinuous in $\overline{\mathcal{H}}_w \times [0, +\infty[$, for every $w \in \{-1, 1\}$.*

Proof. Let (x_n, w, t_n) be a sequence converging to (x, w, t) in $\overline{\mathcal{H}}_w \times [0, +\infty[$. For every n, let $\hat{\alpha}_n \in \hat{\mathcal{A}}$ and $(\hat{y}_n, \hat{z}_n) \in \hat{\mathcal{T}}(x_n, w, \hat{\alpha}_n)$ be optimal for (x_n, t_n). Moreover, let $\hat{\alpha}$ be a relaxed control such that (a subsequence of) $\hat{\alpha}_n$ weakly star converges to it. Our aim is to select $(\hat{y}, \hat{z}) \in \hat{\mathcal{T}}(x, w, \hat{\alpha})$ such that

$$\liminf_{n \to +\infty} \hat{V}(x_n, w, t_n) \geq \hat{J}(x, w, t, \hat{\alpha}, \hat{y}, \hat{z}) \geq \hat{V}(x, w, t). \tag{3.14}$$

As in the proof of Proposition 3.2, there exists a subsequence (still labelled by n) such that, for every n, in the time interval $[0, t_n]$, $(\hat{y}_n, \hat{z}_n)$ has a constant number of switchings, let's say r. We have some cases.

a) $r = 0$. Then no trajectory $\hat{z}_n$ switches. Hence, by Proposition 3.1, we get (3.14).

b) $r = 1$. For every n, let $\tau_n \in [0, t_n]$ be the switching instant. Hence, for every n, we have

$$\begin{aligned} &\hat{V}(x_n, w, t_n) = \int_0^{\tau_n} e^{-\lambda s} \hat{\ell}(\hat{y}_n(s), w, \hat{\alpha}_n(s)) ds \\ &+ \int_{\tau_n}^{t_n} e^{-\lambda s} \hat{\ell}(\hat{y}_n(s), -w, \hat{\alpha}_n(s)) ds + e^{-\lambda t_n} g(\hat{y}(t_n) - w). \end{aligned}$$

Let $\tau \in [0, t]$ be such that (at least for a subsequence) $\tau_n \to \tau$. By Proposition 3.1, we get two limit trajectories $(\hat{y}, \hat{z})$ in $[0, \tau[$ $(= \emptyset$ if $\tau = 0)$ and in $]\tau, t]$ $(= \{t\}$ if $\tau = t)$, with $\hat{z} \equiv w$ and $\hat{z} \equiv -w$ respectively. Moreover we define $\hat{z}(\tau) = -w$. Hence, $(\hat{y}, \hat{z}) \in \hat{\mathcal{T}}(x, w, \hat{\alpha})$, and we get (3.14).

c) $r > 1$. We divide the analysis in r steps as the previous ones. □

4 Dynamic programming.

Warning 4.1 *In this section, unless differently advised, we are always concerned with the relaxed control problem and with the relaxed value function. However, to simplify notations, we drop the use of the word "relaxed" and of the symbol "^" to indicate the relaxed quantities. Hence, for instance, we will speak of value function, of cost, of controls, and of trajectories, but we will always mean relaxed value function, relaxed cost, relaxed controls, and relaxed trajectories, as defined in Section 3. Finally, even if we do not display the "hat" on our quantities, we will refer to their definitions and properties by using the numbers of the equations of Section 3, where the "hat" is displayed.*

Let $w \in \{-1,1\}$ and $(x,w,t) \in \overline{\mathcal{H}}_w \times [0,+\infty[$. For every control $\alpha \in \mathcal{A}$ and for every trajectory $(y,z) \in \mathcal{T}(x,w,\alpha)$ we define the set $X_{(y,z)} := \{\tau \in [0,+\infty[\big| z(s) = w \text{ for } s \in [0,\tau]\}$, and then the "first switching time" by

$$\begin{cases} t_{(y,z)} := \sup X_{(y,z)} & \text{if } X_{(y,z)} \neq \emptyset, \\ t_{(y,z)} = 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Proposition 4.2 *For every $w \in \{-1,1\}$, and for every $(x,w,t) \in \overline{\mathcal{H}}_w \times [0,+\infty[$, the following holds*

$$\begin{array}{l} V(x,w,t) = \inf_{\alpha\in\mathcal{A}} \left(\inf_{(y,z)\in\mathcal{T}(x,w,\alpha)} \left\{ \int_0^{t\wedge t_{(y,z)}} e^{-\lambda s}\ell(y(s),w,\alpha(s))ds \right.\right. \\ +e^{-\lambda t\wedge t_{(y,z)}} \left[\chi_{[0,t_{(y,z)}[}(t) g(y(t),w) \right. \\ +\chi_{\{t_{(y,z)}\}}(t) \min \Big(g(y(t),w), V(y(t),-w,0) \Big) \\ \left.\left.\left. +\chi_{]t_{(y,z)},+\infty[}(t) V(y(t_{(y,z)}),-w,t-t_{(y,z)}) \right] \right\} \right), \end{array} \tag{4.2}$$

where, χ_B is the characteristic function of the set B.

Proof. Let us take $\alpha \in \mathcal{A}$, and $(y,z) \in \mathcal{T}(x,w,\alpha)$. For $\tau \geq 0$, we define the shifted control $\alpha_\tau : s \mapsto \alpha(\tau+s)$ and the shifted trajectory $(y_\tau, z_\tau) : s \mapsto (y(\tau+s), z(\tau+s))$. Note that $(y_\tau,z_\tau) \in \mathcal{T}(y(\tau),z(\tau),\alpha_\tau)$. This is true by the semigroup property of the complete delayed relay (2.1). Moreover, if $\tau < t_{(y,z)}$, then $t_{(y,z)} = \tau + t_{(y_\tau,z_\tau)}$. We use the notations $I_1 := \int_0^{t\wedge t_{(y,z)}} e^{-\lambda s}\ell(y(s),w,\alpha(s))ds$. By the definition of the cost (3.12), and by standard techniques, we get

$$\begin{array}{l} J(x,w,t,\alpha,y,z) = I_1 + \chi_{[0,t_{(y,z)}[}(t)e^{-\lambda t}g(y(t),w) \\ +\chi_{[t_{(y,z)},+\infty[}(t)e^{-\lambda t\wedge t_{(y,z)}} J(y(t_{(y,z)}), z(t_{(y,z)}), t-t_{(y,z)}, \alpha_{t_{(y,z)}}, y_{t_{(y,z)}}, z_{t_{(y,z)}}). \end{array} \tag{4.3}$$

Let us prove the "$\geq$" inequality in (4.2). We prove that $J(x,w,t,\alpha,y,z)$ is not smaller than the right-hand side of (4.2). We analyze some cases.

a) $t < t_{(y,z)}$. By (4.3), we have that $J(x,w,t,\alpha,y,z)$ is equal to the right-hand side of (4.2).

b) $t = t_{(y,z)}$. We have two possibilities

$$\begin{array}{l} J(y(t_{(y,z)}), z(t_{(y,z)}), 0, \alpha_{t_{(y,z)}}, y_{t_{(y,z)}}, z_{t_{(y,z)}}) \\ = \begin{cases} \min\{g(y(t_{(y,z)}),w), g(y(t_{(y,z)}),-w)\} & \text{if } z(t_{(y,z)}) = w, \\ g(y(t_{(y,z)}),-w) & \text{if } z(t_{(y,z)}) = -w. \end{cases} \end{array}$$

In both cases, by (4.3), we get

$$J(x,w,t,\alpha,y,z) \geq I_1 + e^{-\lambda t_{(y,z)}} \min\{g(y(t_{(y,z)}),w), g(y(t_{(y,z)}),-w)\},$$

and we conclude observing that $g(y(t_{(y,z)}),-w) = V(y(t_{(y,z)}),-w,0)$.

c) $t > t_{(y,z)}$. In this case we obtain the conclusion by

$$\begin{aligned}&J(y(t_{(y,z)}), z(t_{(y,z)}), t - t_{(y,z)}, \alpha_{t_{(y,z)}}, y_{t_{(y,z)}}, z_{t_{(y,z)}})\\ &= J(y(t_{(y,z)}), -w, t - t_{(y,z)}, \alpha_{t_{(y,z)}}, y_{t_{(y,z)}}, z_{t_{(y,z)}}) \geq V(y(t_{(y,z)}), -w, t - t_{(y,z)}),\end{aligned}$$

Now, we prove the opposite inequality in (4.2). Let us fix $\alpha \in \mathcal{A}$ and $(y,z) \in \mathcal{T}(x,w,\alpha)$. Let us suppose that $t < t_{(y,z)}$. Then

$$V(x,w,t) \leq J(x,w,t,\alpha,y,z) = \int_0^t e^{-\lambda s}\ell(y(s),w,\alpha(s))ds + e^{-\lambda t}g(y(t),w)$$

Otherwise, if $t \geq t_{(y,z)}$, we take $\alpha_1 \in \mathcal{A}$ and $(y_1,z_1) \in \mathcal{T}(y(t_{(y,z)}), z(t_{(y,z)}), \alpha_1)$ optimal (for $(y(t_{(y,z)}), z(t_{(y,z))}, t - t_{(y,z)}))$ As usual, we define the control $\overline{\alpha}$ gluing together α and α_1 in $t_{(y,z)}$. It is clear that, gluing together (y,z) and (y_1,z_1) we obtain a trajectory $(\overline{y},\overline{z}) \in \mathcal{T}(x,w,\overline{\alpha})$. Hence, we get

$$V(x,w,t) \leq J(x,w,t,\overline{\alpha},\overline{y},\overline{z}) = I_1 + e^{-\lambda t_{(y,z)}}V(y(t_{(y,z)}), z(t_{(y,z)}), t - t_{(y,z)}),$$

where I_1 is defined as above. We may conclude, since, by (3.7),

$$V(y(t_{(y,z)}), z(t_{(y,z)}), t - t_{(y,z)}) \leq V(y(t_{(y,z)}), -w, t - t_{(y,z)}).$$

□

The equation (4.2) suggests that, on every $\overline{\mathcal{H}}_w \times [0,+\infty[$, we can view the value function V as the value of a finite horizon-exit time problem, with $V(\cdot,-w,\cdot)$ as exit cost, and g as final cost. In particular, such problem has x as the only state variable (note that w is fixed in (4.2)), and the exit time is from the set $\overline{\mathcal{H}}$, where $\mathcal{H}$ is the projection of $\mathcal{H}_w$ on $\mathbb{R}^n$. Moreover, the "exit rule" is given by the complete delayed switching rule, and then the exit time coincides with what is called the "best exit time" in Barles-Perthame [6] (see also Barles-Perthame [7], and Blanc [9], [10]). Indeed, if for every $x \in \overline{\mathcal{H}}$, and for every $\alpha \in \mathcal{A}$ we define the exit time from the closed set $\overline{\mathcal{H}}$ as

$$\tau(x,\alpha) := \inf\left\{s \geq 0 \middle| y_x(s) \notin \overline{\mathcal{H}}\right\}, \tag{4.4}$$

where $y_x(\cdot)$ is the trajectory of the non switching system (3.9), with w fixed, then (4.2) is equivalent to

$$\begin{array}{l} V(x,t) = \inf_{\alpha\in\mathcal{A}} \inf \Big\{ \int_0^{t\wedge\theta} e^{-\lambda s}\ell(y(s),w,\alpha(s))ds \\ +e^{-\lambda t\wedge\theta} \Big[\chi_{[0,\theta[}(t) g(y(t),w) \\ +\chi_{\{\theta\}}(t) \min\Big(g(y(t),w), V(y(t),-w,0)\Big) \\ +\chi_{]\theta,+\infty[}(t) V(y(\theta),-w,t-\theta)\Big] \Big| 0 \le \theta \le \tau(x,\alpha),\ y(\theta)\in\partial\mathcal{H}\Big\}, \end{array} \tag{4.5}$$

To understand the equivalence between (4.2) and (4.5), just note that, for example, in the case of one switching only, the choice of the switching trajectory $(y,z)\in\mathcal{T}(x,w,\alpha)$ is equivalent to the choice of the exit instant θ satisfying $0\le\theta\le\tau(x,\alpha)$, and $y_x(\theta)\in\partial\mathcal{H}$.

5 Uniqueness.

For this section, we again launch Warning 4.1.

By (4.5), we know that, for every $w\in\{-1,1\}$, in $\overline{\mathcal{H}}_w\times[0,+\infty[$ the value function $V(\cdot,w,\cdot)$ (i.e. with w fixed) is the value function of the finite horizon-exit time problem with final cost $g(\cdot,w)$ and exit cost $V(\cdot,-w,\cdot)$. Hence, by Proposition 6.2, we can say that $V(\cdot,w,\cdot)$ is a viscosity solution of

$$\left\{\begin{array}{ll} V_t(\cdot,w,\cdot)+\lambda V(\cdot,w,\cdot)+H(\cdot,w,DV(\cdot,w,\cdot))=0 & \text{in } \mathcal{H}_w\times]0,+\infty[, \\ V(\cdot,w,\cdot)=V(\cdot,-w,\cdot) & \text{on } \partial\mathcal{H}_w\times]0,+\infty[, \\ V(\cdot,w,0)=g(\cdot,w) & \text{on } \mathcal{H}_w\times\{0\}, \\ V(\cdot,w,0)=\min\left(g(\cdot,w),V(\cdot,-w,0)\right) & \text{on } \partial\mathcal{H}_w\times\{0\}, \end{array}\right. \tag{5.1}$$

where $H(\cdot,w,\cdot)$ is defined for $(x,p)\in I\!R^n\times I\!R^n$, $(x,w)\in\overline{\mathcal{H}}_w$, by

$$H(x,w,p) := \sup_{a\in A}\left\{-f(x,w,a)\cdot p-\ell(x,w,a)\right\}. \tag{5.2}$$

We point out again that, in (5.1), the function $V(\cdot,-w,\cdot)$ plays the role of a boundary datum: we are only considering a Hamilton-Jacobi problem in $\overline{\mathcal{H}}_w\times[0,+\infty[$ (i.e. with w fixed).

Remark 5.1 *The Hamiltonian (5.2) is the Hamiltonian of the non relaxed problem. Indeed, in its definition, the (non relaxed) constant controls $a \in A$ are used. In the same way, the next controllability assumptions (5.3), (5.4) refer to classical constant controls $a \in A$.*

We assume a controllability hypothesis on the boundaries $\partial\mathcal{H}_1$ and $\partial\mathcal{H}_{-1}$:

$$\begin{array}{l} \text{there exists } c > 0 \text{ such that} \\ i)\ \forall (x,1) \in \partial\mathcal{H}_1\ \exists a \in A \text{ such that } f(x,1,a)\cdot S \le -c < 0, \\ ii)\ \forall (x,-1) \in \partial\mathcal{H}_{-1}\ \exists a \in A \text{ such that } f(x,-1,a)\cdot S \ge c > 0. \end{array} \tag{5.3}$$

Condition (5.3) means that at every point of the boundaries $\partial\mathcal{H}_1$ and $\partial\mathcal{H}_{-1}$, there exists a (classical) constant control which gives an outward admissible field with respect to $\overline{\mathcal{H}}_1$ and $\overline{\mathcal{H}}_{-1}$ respectively.

For every $w \in \{-1,1\}$, let us define the set

$$(\partial\mathcal{H})^w := \left\{(x,-w)\Big|(x,w) \in \partial\mathcal{H}_w\right\}.$$

Roughly speaking, $(\partial\mathcal{H})^w$ is the line on $\mathcal{H}_{-w}$ where the state "drops" after a possible switching from $\overline{\mathcal{H}}_w$. We assume the following controllability hypothesis on $(\partial\mathcal{H})^w$

$$\begin{array}{l} \forall w \in \{-1,1\}\ \ \forall (x,-w) \in (\partial\mathcal{H})^w\ \ \exists a', a'' \in A \ \text{ such that} \\ f(x,-w,a')\cdot S < 0 < f(x,-w,a'')\cdot S. \end{array} \tag{5.4}$$

We want to give conditions in order to guarantee that for every $w \in \{-1,1\}$, the restriction to $(\partial\mathcal{H})^w \times [0,+\infty[$ of the value function satisfies

$$(V^*)_*(x,-w,t) = V(x,-w,t)\ \forall (x,-w,t) \in (\partial\mathcal{H})^w \times [0,+\infty[, \tag{5.5}$$

where the lower semicontinuous envelop "$_*$", and the upper semicontinuous envelop "*" are performed with respect to $(\partial\mathcal{H})^w \times [0,+\infty[$ only.

Proposition 5.2 *If (5.3), (5.4) hold, then (5.5) also holds.*

Proof. Since V is lower semicontinuous, it is sufficient to prove that

$$(V^*)_* \le V. \tag{5.6}$$

To prove (5.6), we have to prove that, for every $(x,-w,t)\in(\partial\mathcal{H})^w\times[0,+\infty[$, for every $\mu>0$, and for every $\varepsilon>0$ there exist $(\zeta,-w,\tau)\in(\partial\mathcal{H}^w)\times[0,+\infty[$, and a ball B around $(\zeta,-w,\tau)$, such that $|(\zeta,\tau)-(x,t)|\le\varepsilon$, and

$$\sup_{B\cap(\partial\mathcal{H})^w\times[0,+\infty[} V\le V(x,-w,t)+\mu.$$

Let $\alpha\in\mathcal{A}$ and $(y,z)\in\mathcal{T}(x,-w,\alpha)$ be optimal for $(x,-w,t)$. As example, we only analyze the case where z has exactly two switches, and the last one occurs at the end time t. The other cases are similarly treated. In particular, recall that the numbers of switchings cannot be large of a fixed integer k, and hence, in the general case we have only to repeat the procedure we are going to explain for a bounded finite number of steps.

Let $t_1\in]0,t[$ be the first switching time. Hence, we have: $z(\tau)=-w$ for $\tau\in[0,t_1[$, $z(\tau)=w$ for $\tau\in]t_1,t[$, and $z(t)=-w$. For any measurable control $\beta\in\mathcal{A}$, let us consider the (non switching) backward controlled system

$$\begin{cases}\tilde{y}'(s)=-f(\tilde{y}(s),w,\beta(t-s)) & s\in[0,t[,\\ \tilde{y}(0)=y(t).\end{cases}\tag{5.7}$$

By (5.3), for the system (5.7), on the boundary of $\overline{\mathcal{H}}_w$, there is an admissible inward field. Moreover, the property of (backward) inner field is also satisfied by the closed semi-hyperplane

$$\overline{\mathcal{H}}_{w,\delta}:=\left\{(x,w)\in\overline{\mathcal{H}}_w\middle|dist((x,w),\partial\mathcal{H}_w\ge\delta\right\},$$

for a suitably small $\delta>0$, at least locally around $(y(t),w)\in\partial\mathcal{H}_w$ (but this is not a problem since in $[0,t]$, $\tilde{y}(\cdot)$ does not exit from a compact set). Let $a_1\in A$ be such that $f(y(t),w,a_1)$ is outward with respect to $\overline{\mathcal{H}}_w$. If in (5.7) we firstly use the constant control a_1, $\tilde{y}$ enters in $\overline{\mathcal{H}}_{w,\delta}$ in a small time t_δ, with $t_\delta\to0$ as $\delta\to0$. Hence, starting from $(\tilde{y}(t_\delta),w)\in\overline{\mathcal{H}}_{w,\delta}$, using a result of Soner [13] (see also Bagagiolo-Bardi [3] for the time depending case), and using the controllability condition (5.4), we can construct a control $\overline{\alpha}\in\mathcal{A}$ such that the corresponding trajectory $\tilde{y}$ of the backward system (5.7) satisfies in $[0,t-t_1+\eta_\delta]$ (with $\eta_\delta>0$, $\eta_\delta\to0$ as $\delta\to0$)

$$\begin{aligned}&(\tilde{y}(\tau),w)\in\overline{\mathcal{H}}_w\ \forall\tau\in[0,t_\delta],\ (\tilde{y}(\tau),w)\in\overline{\mathcal{H}}_{w,\delta}\ \forall\tau\in[t_\delta,t-t_1+\eta_\delta[,\\ &(\tilde{y}(t-t_1+\eta_\delta),w)\in(\partial\mathcal{H})^{-w},\\ &\left|\int_0^{t-t_1+\eta_\delta}e^{\lambda s}\Big(\ell(\tilde{y}(s),w,\overline{\alpha}(t-s))-\ell(y(t-s),w,\alpha(t-s))\Big)ds\right|\in\mathcal{O}(\delta),\end{aligned}\tag{5.8}$$

where α and (y, z) are the optimal control and the optimal trajectory for $(x, -w, t)$. Now, we let the backward trajectory (backward) switch from $(\tilde{y}(t - t_1 + \eta_\delta), w)$ to $(\tilde{y}(t - t_1 + \eta_\delta, -w)$. Starting from the latter point, we then perform a similar procedure as before. We eventually get four positive numbers $t_\delta, \tau_\delta, \eta_\delta, \sigma_\delta$ which go to zero as $\delta \to 0$, and we construct a measurable control $\overline{\alpha}$, and a trajectory $(\tilde{y}, \tilde{z})$ in $[0, t + \eta_\delta + \sigma_\delta[$ such that

$$\tilde{z}(0) = -w,\ \tilde{z}(\tau) = w \text{ in }]0, t - t_1 + \eta_\delta[,\ \tilde{z}(\tau) = -w \text{ in }]t - t_1 + \eta_\delta, t + \eta_\delta + \sigma_\delta],$$

$$\begin{cases} \tilde{y}'(s) = -f(\tilde{y}(s), \tilde{z}(s), \overline{\alpha}(t + \eta_\delta + \sigma_\delta - s)) & 0 < s \le t + \eta_\delta + \sigma_\delta, \\ \tilde{y}(0) = y(t), \end{cases}$$

$$\begin{array}{l} (\tilde{y}(\tau), w) \in \overline{\mathcal{H}}_w \ \forall \tau \in [0, t - t_1 + \eta_\delta[, \\ (\tilde{y}(\tau), w) \in \overline{\mathcal{H}}_{w,\delta} \ \forall \tau \in [t_\delta, t - t_1 + \eta_\delta[, \\ (\tilde{y}(\tau), -w) \in \overline{\mathcal{H}}_{-w} \ \forall \tau \in]t - t_1 + \eta_\delta, t + \eta_\delta + \sigma_\delta], \\ (\tilde{y}(\tau), -w) \in \overline{\mathcal{H}}_{-w,\delta} \ \forall \tau \in [t - t_1 + \eta_\delta + \tau_\delta, t + \eta_\delta + \sigma_\delta], \\ (\tilde{y}(t + \eta_\delta + \sigma_\delta), -w) \in (\partial\mathcal{H})^w, \quad |(\tilde{y}(t + \eta_\delta + \sigma_\delta), -w) - (x, -w)| \in \mathcal{O}(\delta), \end{array}$$

$$\left| \int_0^{t+\eta_\delta+\sigma_\delta} e^{\lambda s} \Big(\ell(\tilde{y}(s), \tilde{z}(s), \overline{\alpha}(\overline{t} - s)) - \ell(y(\overline{t} - s), z(\overline{t} - s), \alpha(\overline{t} - s)) \Big) ds \right| \in \mathcal{O}(\delta).$$

In other words, we have constructed a control $\overline{\alpha}$ and a (forward) trajectory $(\overline{y}(\cdot), \overline{z}(\cdot)) = (\tilde{y}(t + \eta_\delta + \sigma_\delta - \cdot), \tilde{z}(t + \eta_\delta + \sigma_\delta - \cdot)) \in \mathcal{T}(\tilde{y}(t + \eta_\delta + \sigma_\delta), -w, \overline{\alpha})$ such that (recall that we are supposing z switching at the final time t).

$$J(\tilde{y}(t + \eta_\delta + \sigma_\delta), -w, t + \eta_\delta + \sigma_\delta, \overline{\alpha}, \overline{y}, \overline{z}) \le V(x, -w, t) + \mathcal{O}(\delta).$$

Now, we further modify the trajectory $(\overline{y}, \overline{z})$. Let $a_1, a_2 \in A$ be constant (classical) which respectively gives an outward field to $\overline{\mathcal{H}}_{-w}$ in $(\overline{y}(t_1), -w)$, and a outward field to $\overline{\mathcal{H}}_w$ in $(\overline{y}(t + \eta_\delta + \sigma_\delta, w$. We fix $\varepsilon > 0$, define

$$\underline{\alpha}(s) = \begin{cases} \overline{\alpha}(s) & \text{if } 0 \le s \le t_1 + \sigma_\delta, \\ a_1 & \text{if } t_1 + \sigma_\delta \le s \le t_1 + \sigma_\delta + \varepsilon, \\ \overline{\alpha}(s - \varepsilon) & \text{if } t_1 + \sigma_\delta + \varepsilon \le s \le t + \eta_\delta + \sigma_\delta + \varepsilon, \\ a_2 & \text{if } t + \eta_\delta + \sigma_\delta + \varepsilon \le s \le t + \eta_\delta + \sigma_\delta + 2\varepsilon, \end{cases}$$

and, in $[0, t+\eta_\delta+\sigma_\delta+2\varepsilon]$, we consider a trajectory $(\underline{y}, \underline{z})$ solution of (3.2) with control $\underline{\alpha}$ and initial state $\overline{y}(0), -w)$. Hence, we still get

$$J(\overline{y}(0), -w, t+\eta_\delta+\sigma_\delta+2\varepsilon, \underline{\alpha}, \underline{y}, \underline{z}) \leq V(x, -w, t) + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon).$$

Observe that, by the definition of $\underline{\alpha}$, the trajectory $(\underline{y}, \underline{z})$ has exactly two switches, but it is "forced to switch", otherwise it should cross the switching boundary. Moreover, for times preceding the first switching time for at least the quantity σ_δ, it is "uniformly bounded away" from the switching boundary. Similarly happens for the second switching time. We also note that the second switching time is not the final time $t+\eta_\delta+\sigma_\delta+2\varepsilon$. Hence, if we start from point sufficiently near to $(\overline{y}(0), -w)$ with control $\underline{\alpha}$, we can choose an admissible trajectory for (3.2) which is still close to $(\underline{y}, \underline{z})$ (in particular it has exactly two switchings, at two switching times close to the ones of $(\underline{y}, \underline{z})$). Hence, we conclude the proof observing that, if $(\xi, -w) \in (\partial\mathcal{H})^w$ is close to $(\overline{y}(0), -w)$, and if τ is also close to $t+\eta_\delta+\sigma_\delta+2\varepsilon$, we then get

$$\begin{array}{l} V(\xi, -w, \tau) \leq J(\overline{y}(0), -w, t+\eta_\delta+\sigma_\delta+2\varepsilon, \underline{\alpha}, \underline{y}, \underline{z}) \\ +\mathcal{O}(|(\xi, \tau) - (\overline{y}(0), t+\eta_\delta+\sigma_\delta+2\varepsilon)|), \end{array}$$

□

Remark 5.3 *Other controllability hypotheses are suitable in order to have property (5.5). For instance, we may assume a sort of "global" controllability inside $(\partial\mathcal{H})^w$ for every $w \in \{1, -1\}$. However, since condition (5.3) must hold for the comparison result, then (5.3)–(5.4) seem in some sense the weaker ones. Examples may be given where, even failing one inequality only in (5.4), with (5.3) still holding, (5.5) is not more true.*

Now we want to prove the uniqueness result. For every $w \in \{-1, 1\}$, and for every function u defined on $\overline{\mathcal{H}}_{-w} \times [0, +\infty[$, let us define

$$u^w : (\partial\mathcal{H})^w \times [0, +\infty[\to I\!R, \quad u^w := \left(\left(u_{*_g}\right)^*\right)_*, \tag{5.9}$$

where the nested lower envelop is performed in $\overline{\mathcal{H}}_{-w} \times [0, +\infty[$ (indeed it is denoted by the index g which stays for "global"); the other two exterior envelopes are performed with respect only to $(\partial\mathcal{H})^w \times [0, +\infty[$. For the definition of t-linearly bounded function, see Theorem 6.3.

Theorem 5.4 *Let us suppose that (5.3), and (5.5) hold (see Proposition 5.2, and Remark 5.5). Then, the relaxed value function V is the unique t-linearly bounded, lower semicontinuous function u defined on $\left(\overline{\mathcal{H}}_1 \cup \overline{\mathcal{H}}_{-1}\right) \times [0, +\infty[$ which solves*

$$\begin{cases} \text{for every } w \in \{-1, 1\} \ u \text{ is a viscosity solution of} \\ \begin{cases} u_t(\cdot, w, \cdot) + \lambda u(\cdot, w, \cdot) + H(\cdot, w, Du(\cdot, w, \cdot)) = 0 & \text{in } \mathcal{H}_w \times]0, +\infty[, \\ u(\cdot, w, \cdot) = u^w(\cdot, -w, \cdot) & \text{on } \partial\mathcal{H}_w \times]0, +\infty[, \\ u(\cdot, w, 0) = g(\cdot, w) & \text{on } \mathcal{H}_w \times \{0\}, \\ u(\cdot, w, 0) = \min\left(g(\cdot, w), u^w(\cdot, -w, 0)\right) & \text{on } \partial\mathcal{H}_w \times \{0\}, \end{cases} \end{cases} \tag{5.10}$$

where the Hamiltonian H is defined in (5.2). In particular, every t-linearly bounded solution of (5.10) has the lower semicontinuous envelope coincident with V. Finally, V has also the property $(V^*)_* = V$ in $\left(\overline{\mathcal{H}}_1 \cup \overline{\mathcal{H}}_{-1}\right) \times [0, +\infty[$.

Proof. Since V is lower semicontinuous, and it satisfies (5.5) by hypothesis, then it satisfies (5.10). Moreover, it is t-linearly bounded. A simple calculation shows that, for every function u, u^w has the stability property (5.5). For every $w \in \{-1, 1\}$, let us denote by $(5.10)_w$ the initial-boundary value problem in (5.10).

Let u be a t-linearly bounded solution of (5.10). We are going to prove that its lower semicontinuous envelope coincides with V. Let us fix $w \in \{-1, 1\}$. By the uniqueness result Theorem 6.4, we know that, on $\overline{\mathcal{H}}_w \times [0, +\infty[$, the lower semicontinuous envelope $u_*(\cdot, w, \cdot)$ of $u(\cdot, w, \cdot)$ is the relaxed value function of the relaxed problem with dynamics $f(\cdot, w, \cdot)$, running cost $\ell(\cdot, w, \cdot)$, final cost $g(\cdot, w)$ and exit cost $u^w(\cdot, -w, \cdot)$. Such a control problem differs from the one solved by V only for the exit cost.

Since the switchings are delayed (i.e. $\rho_1 > \rho_2$), and the dynamics f is bounded, there exists a positive time $\delta > 0$ such that every trajectory in $\overline{\mathcal{H}}_w$ starting from $(\partial\mathcal{H})^{-w}$ does not exit from $\overline{\mathcal{H}}_w$ in the time interval $[0, \delta]$. Hence, in the time interval $[0, \delta]$ the exit cost (whichever it is) does not play any role. It follows that $u_*(\cdot, w, \cdot) = V(\cdot, w, \cdot)$ on $(\partial\mathcal{H})^{-w} \times [0, \delta]$. From (5.5) we then get

$$u^{-w}(\cdot, w, \cdot) = V(\cdot, w, \cdot) \ \text{ on } \ (\partial\mathcal{H})^{-w} \times [0, \delta]. \tag{5.11}$$

Now, we use the fact that u is solution on $\overline{\mathcal{H}}_{-w} \times [0, +\infty[$ with boundary datum u^{-w}. By Theorem 6.4, this implies that, in $\overline{\mathcal{H}}_{-w} \times [0, +\infty[$, u_* coincides with the relaxed value function of the relaxed control problem with dynamics $f(\cdot, -w, \cdot)$, running cost $\ell(\cdot, -w, \cdot)$, final cost $g(\cdot, -w, \cdot)$, and exit cost $u^{-w}(\cdot, w, \cdot)$. Again, every trajectory on $\overline{\mathcal{H}}_{-w}$ starting from $(\partial\mathcal{H})^w$ does not exit from $\overline{\mathcal{H}}_{-w}$ in the time interval $[0, \delta]$. Moreover if the trajectory exits at $\theta \in [\delta, 2\delta]$, and if the selected final horizon is $t \in [\delta, 2\delta]$ with $t \geq \theta$, then the (discounted) cost payed is $u^{-w}(\cdot, w, t - \theta)$. By (5.11), this is equal to $V(\cdot, w, t - \theta)$, and we then get $u_* = V$ in $\overline{\mathcal{H}}_{-w} \times [0, 2\delta]$. Hence, we also get

$$u^w(\cdot, -w, \cdot) = V(\cdot, -w, \cdot) \text{ on } (\partial\mathcal{H})^w \times [0, 2\delta].$$

Applying again the same argument, we obtain $u_* = V$ in $\overline{\mathcal{H}}_w \times [0, 3\delta]$, and $u^{-w} = V$ on $(\partial\mathcal{H})^{-w} \times [0, 3\delta]$. Repeating this procedure we conclude that $u_* = V$ in $\left(\overline{\mathcal{H}}_1 \cup \overline{\mathcal{H}}_{-1}\right) \times [0, +\infty[$.

The other properties come from Theorem 6.4. □

6 On finite horizon-exit time problems.

Let us consider the following open semi-space of $I\!R^n$

$$\mathcal{H} := \left\{ x \in I\!R^n \middle| x_1 > 0 \right\},$$

where x_1 is the first coordinate of $x = (x_1, \ldots, x_n)$. We consider the set of measurable controls as in (3.3); a dynamics $f : I\!R^n \times A \to I\!R^n$, a running cost $\ell : I\!R^n \times A \to [0, +\infty[$, and a final cost $g : I\!R^n \to [0, +\infty[$ satisfying the usual regularity properties (compare respectively with (3.1) and (3.4)); and a lower semicontinuous, t-linearly bounded (see Theorem 6.3) exit cost $\varphi : \partial\mathcal{H} \times [0, +\infty[\to [0, +\infty[$. Let $y(\cdot)$ be the solution of the controlled dynamical system given by f, by a measurable control $\alpha \in \mathcal{A}$, and by an initial point $x \in \overline{\mathcal{H}}$ (compare with (3.9)). Moreover, for the trajectory y, let $t_x(\alpha)$ be the first exit time from $\overline{\mathcal{H}}$

$$t_x(\alpha) = \inf \left\{ t \geq 0 \middle| y(t) \notin \overline{\mathcal{H}} \right\}.$$

For a discount factor $\lambda \geq 0$, we consider the following cost, which is defined on $\overline{\mathcal{H}} \times [0, +\infty[\times \mathcal{A} \times \{\theta \geq 0 | y(\theta) \in \partial\mathcal{H}\}$ (see the discussion in Section 4)

$$\begin{aligned} J(x, t, \alpha, \theta) = & \int_0^{t\wedge\theta} e^{-\lambda s} \ell(y(s), \alpha(s)) ds \\ & + \chi_{[0,\theta[}(t) e^{-\lambda t} g(y(t)) + \chi_{\{\theta\}}(t) e^{-\lambda t} \min\{g(y(t)), \varphi(y(t), 0)\} \\ & + \chi_{]\theta,+\infty[}(t) e^{-\lambda\theta} \varphi(y(\theta), t - \theta). \end{aligned}$$

We then have the value function

$$V(x,t) = \inf \Big\{ J(x,t,\alpha,\theta) \Big| \alpha \in \mathcal{A}, \theta \in [0, t_x(\alpha)], y(\theta) \in \partial\mathcal{H} \Big\}. \qquad (6.1)$$

Of course, this is a classical (non relaxed controls) version of the optimal control problem with finite horizon and exit time. The value function we want to study is the one of the problem with relaxed controls as in Section 3, which is defined as the previous one, except for the use of relaxed controls. We denote it by an "hat", $\hat{V}$, as well as for all the other relaxed quantities (see Section 3).

Proposition 6.1 is standard (see similar results in Sections 3 and 4).

Proposition 6.1 *The relaxed value function $\hat{V}$ is lower semicontinuous in $\overline{\mathcal{H}} \times [0, +\infty[$, and satisfies the Dynamic Programming Principle: for every $(x,t) \in \overline{\mathcal{H}} \times [0, +\infty[$, and for every $0 \le \tau \le t$, we have*

$$\begin{aligned} \hat{V}(x,t) = & \inf_{\hat{\alpha} \in \hat{\mathcal{A}}} \inf \Big\{ \int_0^{\tau \wedge \theta} e^{-\lambda s} \hat{\ell}(\hat{y}(s), \hat{\alpha}(s)) ds + \chi_{\tau \le \theta}(\tau) e^{-\lambda \tau} \hat{V}(\hat{y}(\tau), t - \tau) \\ & + \chi_{\tau > \theta}(\tau) e^{-\lambda \theta} \varphi(\hat{y}(\theta), t - \theta) \Big| \theta \in [0, \hat{t}_x(\hat{\alpha})], \hat{y}(\theta) \in \partial\mathcal{H} \Big\}. \end{aligned} \qquad (6.2)$$

We consider the following Hamiltonian $H : I\!R^n \times I\!R^n \to I\!R$

$$H(x,p) := \sup_{a \in A} \Big\{ - f(x,a) \cdot p - \ell(x,a) \Big\},$$

and we denote by $D\hat{V}$ the "spatial gradient", and by $\hat{V}_t$ the time derivative.

Proposition 6.2 *The relaxed value function $\hat{V}$ is viscosity solution of the initial-boundary value problem*

$$\begin{cases} \hat{V}_t + \lambda \hat{V} + H(x, D\hat{V}) = 0 & \text{in } \mathcal{H} \times]0, +\infty[, \\ \hat{V}(x,t) = \varphi(x,t) & \text{on } \partial\mathcal{H} \times]0, +\infty[, \\ \hat{V}(x,0) = g(x) & \text{on } \mathcal{H} \times \{0\}, \\ \hat{V}(x,w,0) = \min(g(x), \varphi(x,0)) & \text{on } \partial\mathcal{H} \times \{0\}. \end{cases} \qquad (6.3)$$

Let us denote by u^* and by u_* respectively the upper and the lower semicontinuous envelop of a function u. A function $u : \overline{\mathcal{H}} \times [0,+\infty[\to I\!R$ which is bounded on $\overline{\mathcal{H}} \times [0,T]$ for every $T > 0$, is a viscosity solution of (6.3) (with boundary conditions in the viscosity sense), if for every function $\psi \in C^1(\overline{\mathcal{H}} \times [0,+\infty[)$, and for every point $(x,t) \in \overline{\mathcal{H}} \times [0,+\infty[$ of local null minimum for $u_* - \psi$ (respectively: local null maximum for $u^* - \psi$) with respect to $\overline{\mathcal{H}} \times [0,+\infty[$, we respectively have

$$\begin{cases} (i)\ \psi_t(x,t) + \lambda u_*(x,t) + H(x, D\psi(x,t)) \geq 0 & \text{if } (x,t) \in \mathcal{H}\times]0,+\infty[, \\ (ii) \text{ if } u_*(x,t) < \varphi(x,t), & \\ \text{then } \psi_t(x,t) + \lambda u_*(x,t) + H(x, D\psi(x,t)) \geq 0 & \text{if } x \in \partial\mathcal{H},\ t > 0, \\ (iii) \text{ if } u_*(x,0) < g(x), & \\ \text{then } \psi_t(x,0) + \lambda u_*(x,t) + H(x, D\psi(x,0)) \geq 0 & \text{if } x \in \mathcal{H},\ t = 0, \\ (iv) \text{ if } u_*(x,0) < \min\{g(x), \varphi(x,0)\} & \\ \text{then } \psi_t(x,0) + \lambda u_*(x,0) + H(x, D\psi(x,0)) \geq 0 & \text{if } x \in \partial\mathcal{H},\ t = 0; \end{cases} \tag{6.4}$$

$$\begin{cases} (i)\ \psi_t(x,t) + \lambda u^*(x,t) + H(x, D\psi(x,t)) \leq 0 & \text{if } (x,t) \in \mathcal{H}\times]0,+\infty[, \\ (ii) \text{ if } u^*(x,t) > \varphi^*(x,t) & \\ \text{then } \psi_t(x,t) + \lambda u^*(x,t) + H(x, D\psi(x,t)) \leq 0 & \text{if } x \in \partial\mathcal{H},\ t > 0, \\ (iii) \text{ if } u^*(x,0) > g(x) & \\ \text{then } \psi_t(x,0) + \lambda u^*(x,t) + H(x, D\psi(x,0)) \leq 0 & \text{if } x \in \mathcal{H},\ t = 0, \\ (iv) \text{ if } u^*(x,0) > \max\{g(x), \varphi^*(x,0)\} & \\ \text{then } \psi_t(x,0) + \lambda u^*(x,0) + H(x, D\psi(x,0)) \leq 0 & \text{if } x \in \partial\mathcal{H},\ t = 0. \end{cases} \tag{6.5}$$

If a function satisfies (6.4), then it is said a supersolution. If it satisfies (6.5), then it is said to be a subsolution. Note that the boundary of $\overline{\mathcal{H}} \times [0,+\infty[$ is $\partial\mathcal{H} \times [0,+\infty[\cup \mathcal{H} \times \{0\}$. If on such boundary, we define

$$h(x,t) := \begin{cases} g(x) & \text{if } x \in \mathcal{H},\ t = 0, \\ \min\{g(x), \varphi(x,0)\} & \text{if } x \in \partial\mathcal{H},\ t = 0, \\ \varphi(x,t) & \text{if } x \in \mathcal{H},\ t > 0, \end{cases}$$

then the boundary data inequalities in (6.4), (6.5), are just $u_* < h_*$, $u^* > h^*$.

Proof of Proposition (6.2). Starting from the dynamic programming principle (6.2), this proof is almost standard, and we refer the reader to the one in Ishii [11]. Here, we only say something about the third and the fourth boundary conditions of (6.4), and (6.5). First of all, recall that $\hat{V}$ is lower semicontinuous. By the definition of $\hat{V}$ (and by the fact that the dynamics f is bounded), $\hat{V}$ is continuous on the points of $\mathcal{H} \times \{0\}$, where we have $\hat{V}(x,0) = g(x)$. Hence, the third boundary conditions of (6.4) and (6.5) are point-wise satisfied by $\hat{V}(x,0) = \hat{V}_*(x,0) = \hat{V}^*(x,0) = g(x,0)$. Moreover, also the fourth boundary conditions are point-wise satisfied by $\hat{V}$ in (6.4) and by $\hat{V}^*$ in (6.5). Indeed, $\hat{V}(x,0) = \min\{g(x), \varphi(x,0)\}$ by definition; we claim that $\hat{V}^*(x,0) \leq \max\{g(x), \varphi^*(x,0)\}$. Let M be a bound for ℓ. By definition of $\hat{V}$, for every $(\xi, s) \in \overline{\mathcal{H}} \times [0,+\infty[$, end for every relaxed trajectory $\hat{y}$ starting from x, we have: i) $\hat{V}(\xi,s) \leq Ms + g(y(s))$ if $\hat{y}$ does not exit before s; ii) $\hat{V}(\xi,s) \leq Ms + \varphi(\hat{y}(\theta), s-\theta)$ if $\hat{y}$ exits in a time $\theta < s$; iii) $\hat{V}(\xi,s) \leq Ms + \min\{g(\hat{y}(s)), \varphi(\hat{y}(s),0)\}$ if $\hat{y}$ exists at time s. Taking the superior limit as (ξ,s) goes to $(x,0)$, we then get the claim. □

We now assume the following hypothesis

$$(\varphi^*)_* = \varphi \quad \text{on } \partial\mathcal{H} \times [0,+\infty[. \tag{6.6}$$

We recall that the semicontinuous envelopes in (6.6) are performed with respect to $\partial\mathcal{H} \times [0,+\infty[$. Moreover we assume the controllability hypothesis

$$\exists c > 0 : \ \forall x \in \partial\mathcal{H} \ \exists a \in A \text{ such that } f(x,a) \cdot e_1 < -c < 0, \tag{6.7}$$

where $e_1 = (1,0,\ldots,0) \in I\!R^n$. That is, there is a "uniform" outward field.

Theorem 6.3 *Let (6.6)–(6.7) hold. Let $u, v : \overline{\mathcal{H}} \times [0,+\infty[\to I\!R$ be two functions which have linear growth at infinity with respect to t (briefly t-linearly bounded), i.e. $\exists B_1, B_2 > 0$ such that*

$$|w(x,t)| \leq B_1 t + B_2 \ \forall (x,t) \in \overline{\mathcal{H}} \times [0,+\infty[, \ \forall w = u, v. \tag{6.8}$$

If u is a subsolution and v is a supersolution of (6.3), then

$$(u^*)_* (x,t) \leq v_*(x,t) \quad \text{in } \overline{\mathcal{H}} \times [0,+\infty[. \tag{6.9}$$

From Theorem 6.3, we immediately get the following uniqueness result.

Theorem 6.4 *If (6.6) and (6.7) hold, then the relaxed value function $\hat{V}$ is the unique t-linearly bounded, lower semicontinuous viscosity solution of (6.3). In other words, every t-linearly bounded viscosity solution u of (6.3) has the same lower semicontinuous envelop, which coincides with the relaxed value function $\hat{V}$, and moreover it has the regularity property $(u^*)_* = u_*$.*

Proof of Theorem 6.4 The relaxed value function is t- linearly bounded by its definition. We already know that it is lower semicontinuous and that it is a viscosity solution of (6.3). The conclusion follows from the fact that, by Theorem 6.3, if u and v are t-linearly bounded viscosity solutions

$$(u^*)_* \leq v_* \leq (v^*)_* \leq u_* \leq (u^*)_* .$$

□

Proof of Theorem 6.3. This proof will be split in several steps. Let u be a t-linearly bounded subsolution of (6.3). Let us take four constants $C_i > 0$, $i = 1, 2, 3, 4$, to be fixed later on. For the moment we require that

$$C_2 > \frac{C_1^4}{4}. \tag{6.10}$$

Let us take $\alpha > 0$, and define the function ϕ_α on $\overline{\mathcal{H}} \times \overline{\mathcal{H}}$ by

$$\phi_\alpha(x, y) = \frac{|x - y|^4 + C_1 |x - y|^3 (x_1 - y_1) + C_2 |x_1 - y_1|^4}{\alpha}.$$

Applying the Young inequality to the second addendum in the definition of ϕ_α, and using (6.10), we get

$$\phi_\alpha(x, y) > 0 \quad \forall (x, y) \in \overline{\mathcal{H}} \times \overline{\mathcal{H}}. \tag{6.11}$$

Let us take $\beta > 0$ and $\gamma > 0$, and define, on $\overline{\mathcal{H}} \times [0, +\infty[$ (compare with [10])

$$\begin{aligned} & u^{\alpha,\beta,\gamma}(x, t) = \inf_{y \in \overline{\mathcal{H}}, s \geq 0} \Big\{ (u^*)_* (y, s) \\ & + e^{-C_3 t} \left(\phi_\alpha(x, y) + \frac{\gamma}{s} + \frac{\gamma}{y_1} + \frac{|t - s|^2}{\beta} \right) + C_4 (x_1 - y_1) \Big\} . \end{aligned} \tag{6.12}$$

Lemma 6.5 *For $T > 0$, $K := K[T, \alpha, \beta, \gamma] > 0$ exists such that for every $(x, t) \in \overline{\mathcal{H}} \times [0, T]$, the infimum in the definition of $u^{\alpha,\beta,\gamma}(x, t)$ is achieved in a point $(\tilde{y}, \tilde{s})$ such that $\tilde{y}_1 > 0$, $\tilde{s} > 0$ and $|(x, t) - (\tilde{y}, \tilde{s})| \leq K$.*

Proof of Lemma 6.5. We define $\overline{y} := (\overline{y}_1, \ldots, \overline{y}_n)$, by $\overline{y}_1 = \max(x_1, \sqrt{\gamma})$ $\overline{y}_i = x_i$ for $i = 2, \ldots, n$, and $\overline{s} := \max(t, \sqrt{\gamma})$. We then have

$$\begin{aligned} & u^{\alpha,\beta,\gamma}(x, t) \\ & \leq (u^*)_* (\overline{y}, \overline{s}) + e^{-C_3 t} \left(\phi_\alpha(x, \overline{y}) + \frac{\gamma}{\overline{s}} + \frac{\gamma}{\overline{y}_1} + + \frac{|t - \overline{s}|^2}{\beta} \right) + C_4 (x_1 - \overline{y}_1). \end{aligned}$$

By the linear growth of u, the definitions of ϕ_α, $\overline{y}$ and $\overline{s}$, we get

$$u^{\alpha,\beta,\gamma}(x,t) \leq B_1 T + B_2 + \frac{1+C_2}{\alpha}\gamma^2 + \frac{\gamma}{\beta} + 2\sqrt{\gamma} =: \mathcal{C}[T,\alpha,\beta,\gamma].$$

On the other hand, for $(x,t) \in \overline{\mathcal{H}} \times [0,T]$, using again the Young inequality, in general we have

$$(u^*)_*(y,s) + e^{-C_3 t}\left(\phi_\alpha(x,y) + \frac{\gamma}{s} + \frac{\gamma}{y_1} + \frac{|t-s|^2}{\beta}\right) + C_4(x_1 - y_1) \geq$$
$$-B_1 s - B_2 + e^{-C_3 T}\left(\frac{|x-y|^4}{4\alpha} + \frac{|t-s|^2}{\beta}\right) - C_4(|x_1 - y_1|).$$

When $|x-y|$ and $|t-s|$ tend to $+\infty$, the right-hand side tends to infinity also, depending only on the distances $|x-y|$ and $|t-s|$ and not on $(x,t) \in \overline{\mathcal{H}} \times [0,T]$. Hence, we find a constant $K = K[T,\alpha,\beta,\gamma] > 0$ such that if $|(x,t)-(y,s)| > K$ then

$$(u^*)_*(y,s) + e^{-C_3 t}\left(\phi_\alpha(x,y) + \frac{\gamma}{s} + \frac{\gamma}{y_1} + \frac{|t-s|^2}{\beta}\right) + C_4(x_1 - y_1) > \mathcal{C}[T,\alpha,\beta,\gamma]$$

and hence (y,s) cannot be a point of minimum. We then get

$$u^{\alpha,\beta,\gamma}(x,t) = \inf_{|(x,t)-(y,s)|\leq K}\Big\{ (u^*)_*(y,s) +$$
$$e^{-C_3 t}\left(\phi_\alpha(x,y) + \frac{\gamma}{s} + \frac{\gamma}{y_1} + \frac{|t-s|^2}{\beta}\right) + C_4(x_1 - y_1)\Big\},$$

and we conclude by lower semicontinuity. Finally, we have $\tilde{y}_1 > 0, \tilde{s} > 0$. □

Remark 6.6 *We point out that $K[T,\alpha,\beta,\gamma]$ does not depend on $(x,t) \in \overline{\mathcal{H}} \times [0,T]$. Moreover, if we perform the following order of limits: $\gamma \to 0$, $(\alpha,\beta) \to (0,0)$, we have $K[T,\alpha,\beta,\gamma] \to 0$, for every $T > 0$.*

We also have

$$\tilde{s}, \tilde{y}_1 \geq \min\left(1, \frac{e^{-C_3 T}\gamma}{B_1 + B_2 + \mathcal{C}[T,\alpha,\beta,\gamma] + C_4 K[T,\alpha,\beta,\gamma]}\right). \qquad (6.13)$$

Indeed, by definitions, we cannot have (recall that $\phi_\alpha > 0$)

$$e^{-C_3T}\left(\frac{\gamma}{\tilde{s}}+\frac{\gamma}{\tilde{y}_1}\right) > B_1\tilde{s}+B_2+C_4K[T,\alpha,\beta,\gamma]+\mathcal{C}[T,\alpha,\beta,\gamma].$$

If $0<\tilde{s}\leq 1$, then we cannot have

$$e^{-C_3T}\frac{\gamma}{\tilde{s}} \geq B_1+B_2+C_4K[T,\alpha,\beta,\gamma]+\mathcal{C}[T,\alpha,\beta,\gamma],$$

from which we obtain (6.13). We argue in a similar way for $\tilde{y}_1$.

Lemma 6.7 *The function $u^{\alpha,\beta,\gamma}$ is locally Lipschitz in $\overline{\mathcal{H}}\times[0,+\infty[$.*

Proof of Lemma 6.7. Let us consider $R>0$, $T>0$, and $(x,t),(z,\tau)\in \overline{\mathcal{H}}\cap B_R(x)\times[0,T]$. Let $(y,s)\in\mathcal{H}\times]0,+\infty[$ be a point where the infimum in the definition of $u^{\alpha,\beta,\gamma}(z,\tau)$ is achieved and such that $|(z,\tau)-(y,s)|\leq K[T,\alpha,\beta,\gamma]$. Then we have

$$\begin{aligned}
&u^{\alpha,\beta,\gamma}(x,t)-u^{\alpha,\beta,\gamma}(z,\tau)\leq (u^*)_*(y,s)\\
&+e^{-C_3t}\left(\phi_\alpha(x,y)+\frac{\gamma}{s}+\frac{\gamma}{y_1}+\frac{|t-s|^2}{\beta}\right)+C_4(x_1-y_1)-(u^*)_*(y,s)\\
&-e^{-C_3\tau}\left(\phi_\alpha(z,y)+\frac{\gamma}{s}+\frac{\gamma}{y_1}+\frac{|\tau-s|^2}{\beta}\right)-C_4(z_1-y_1).
\end{aligned}$$

Note that we have (also recall (6.13))

$$\begin{aligned}
&|x-z|,|x-y|,|z-y|\leq K[T,\alpha,\beta,\gamma]+R,\\
&|t-\tau|,|t-s|,|\tau-s|\leq K[T,\alpha,\beta,\gamma]+T,\\
&\frac{\gamma}{s},\frac{\gamma}{y_1}\leq \max\left(\gamma,\frac{B_1+B_2+C_4K+\mathcal{C}}{e^{-C_3T}}\right)=:\mathcal{L}[T,\alpha,\beta,\gamma].
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&u^{\alpha,\beta,\gamma}(x,t)-u^{\alpha,\beta,\gamma}(z-\tau)\leq\left(e^{-C_3t}\phi_\alpha(x,y)-e^{-C_3\tau}\phi_\alpha(z,y)\right)\\
&+\left(C_4(x_1-y_1)-C_4(z_1-y_1)\right)+\left(e^{-C_3t}\frac{|t-s|^2}{\beta}-e^{C_3\tau}\frac{|\tau-s|^2}{\beta}\right)\\
&+\left(2\mathcal{L}[T,\alpha,\beta,\gamma]|e^{-C_3t}-e^{-C_3\tau}|\right).
\end{aligned}$$

Every term inside the parentheses of the right-hand side may be respectively seen as the difference of a suitable C^1 function ψ, defined on $I\!R\times I\!R\times I\!R^n$, $\psi(t,t-s,x-y)-\psi(\tau,\tau-s,z-y)$. Since such every ψ is locally Lipschitz, we then conclude.

□

Remark 6.8 *Let us note that the Lipschitz constant depends only on the radius of the compact set (i.e. on $|x-y|$ and $|t-s|$).*

By just a calculation, (6.7) implies the following

$$\begin{array}{l}\exists A>0 \text{ such that the function } \mu\mapsto H(x,p+\mu e_1)\\ \text{is nondecreasing in } [A(1+|p|),+\infty[,\ \forall\ (x,p)\in\partial\mathcal{H}\times I\!R^n.\end{array} \tag{6.14}$$

Moreover, by the uniform continuity of f and ℓ with respect to the space variable, there exists $\nu>0$ such that the controllability hypothesis (6.7), and (6.14) uniformly extend to a ν-neighborhood of $\partial\mathcal{H}$.

Also the following estimate, as usual, holds: there exists $C>0$ such that

$$|H(x,p)-H(y,p)|\le C(1+|p|)|x-y|\quad \forall x,y\in I\!R^n,\ \forall p\in I\!R^n, \tag{6.15}$$

Lemma 6.9 *Let us fix $T>0$, $(x,t)\in\mathcal{H}\times]0,T[$ a point of differentiability for $u^{\alpha,\beta,\gamma}$, and suppose that*

$$C_4=A,\ \ and\ C_1\ge 6C_3, \tag{6.16}$$

then there exist $\overline{K}>0$, independent from α,β,γ,C_3, and $\mathcal{C}>0$ which linearly depends on $K[T,\alpha,\beta,\gamma]$ such that

$$\begin{array}{l}H(\tilde{y},Du^{\alpha,\beta,\gamma}(x,t)+e^{-C_3t}\dfrac{\gamma}{\tilde{y}_1^2}e_1)\ge H(\tilde{y},Du^{\alpha,\beta,\gamma}(x,t))-\mathcal{C}\dfrac{\gamma}{x_1^2},\\ |Du^{\alpha,\beta,\gamma}(x,t)||x-\tilde{y}|\le\overline{K}(|x-\tilde{y}|+e^{-C_3t}\phi_\alpha(x,\tilde{y})),\end{array} \tag{6.17}$$

Proof of Lemma 6.9. See the analogous ones for Lemma 3.6 in [9] and Lemma A.3 in [10]. In particular, for the first inequalities of (6.17), due to the unboundedness of $\mathcal{H}$, a modification of the reasoning in [10] is needed. Referring to that proof, if $(n-1)K[T,\alpha,\beta,\gamma]\le x_1\le nK[T,\alpha,\beta,\gamma]$ with $n\ge3$, then $\tilde{y}_1\ge(n-2)K[T,\alpha,\beta,\gamma]$ and hence

$$\frac{1}{\tilde{y}_1}\le\frac{1}{(n-2)K[T,\alpha,\beta,\gamma]}\le\frac{n}{(n-2)x_1}\le\frac{3}{x_1}.$$

Hence, for $x_1\ge 2K[T,\alpha,\beta,\gamma]$ we are done, and for the other case we argue as in [10]. □

Proposition 6.10 *If u is an upper semicontinuous subsolution of (6.3) in $\mathcal{H}\times]0,+\infty[$, then $w = u_*$ is a viscosity supersolution of*

$$-w_t - \lambda w - H(x, Dw) = 0 \quad \text{in } \mathcal{H}\times]0,+\infty[. \tag{6.18}$$

Proof of Proposition 6.10. See [8]. We only recall that a fundamental hypothesis for this result is that the Hamiltonian is convex with respect to the variable p (the gradient). □

Lemma 6.11 *Let $C > 0$ be as in (6.15), and $\overline{K}$ be as in (6.17). We fix*

$$C_3 \geq C\overline{K} + \lambda. \tag{6.19}$$

Then, for every $T > 0$, $u^{\alpha,\beta,\gamma}$ is a viscosity subsolution in $\mathcal{H}\times]0,T[$ of

$$w_t + \lambda w + H(x, Dw) + \overline{K}(C + 1 + \lambda C_4)K[T, \alpha, \beta, \gamma] - \mathcal{C}\frac{\gamma}{x_1^2} = 0.$$

Proof of Lemma 6.11. Since $u^{\alpha,\beta,\gamma}$ is Lipschitz and the Hamiltonian is convex with respect the gradient, then it is sufficient to show that $u^{\alpha,\beta,\gamma}$ is subsolution almost everywhere (see for instance Bardi-Capuzzo Dolcetta [4]). Let $(x,t) \in \mathcal{H}\times]0,T[$ be a point of differentiability, then, by classical results on inf-convolution (see Bardi-capuzzo Dolcetta [4]), we have

$$\begin{aligned}
&Du^{\alpha,\beta,\gamma}(x,t) = e^{-C_3 t} D_x\phi_\alpha(x,\tilde{y}) + C_4 e_1, \\
&\frac{\partial u^{\alpha,\beta,\gamma}}{\partial t}(x,t) = -C_3 e^{-C_3 t}\left(\phi_\alpha(x,\tilde{y}) + \frac{\gamma}{\tilde{s}} + \frac{\gamma}{\tilde{y}_1} + \frac{|t-\tilde{s}|^2}{\beta}\right) + e^{-C_3 t}\frac{2(t-\tilde{s})}{\beta},
\end{aligned}$$

where, as previously, $(\tilde{y},\tilde{s})$ is a point where the infimum is achieved in the definition of $u^{\alpha,\beta,\gamma}(x,t)$. Then, for every $(y,s) \in \mathcal{H}\times]0,+\infty[$, we define

$$\psi(y,s) := -e^{-C_3 t}\left(\phi_\alpha(x,y) + \frac{\gamma}{s} + \frac{\gamma}{y_1} + \frac{|t-s|^2}{\beta}\right) - C_4(x_1 - y_1).$$

Noting that, for every $x, y \in \mathcal{H}$, it is $D_x\phi_\alpha(x,y) = -D_y\phi_\alpha(x,y)$, we obtain

$$\begin{aligned}
&D\psi(\tilde{y},\tilde{s}) = Du^{\alpha,\beta,\gamma}(x,t) + e^{-C_3 t}\frac{\gamma}{\tilde{y}_1^2}e_1, \\
&\frac{\partial\psi}{\partial s}(\tilde{y},\tilde{s}) = \frac{\partial u^{\alpha,\beta,\gamma}}{\partial t}(x,t) + e^{-C_3 t}\frac{\gamma}{\tilde{s}^2} \\
&+C_3 e^{-C_3 t}\left(\phi_\alpha(x,\tilde{y}) + \frac{\gamma}{\tilde{s}} + \frac{\gamma}{\tilde{y}_1} + \frac{|t-\tilde{s}|^2}{\beta}\right).
\end{aligned}$$

Since $(\tilde{y}, \tilde{s})$ is of local minimum for $(u^*)_* - \psi$, by Proposition 6.10, we get

$$\frac{\partial \psi}{\partial s}(\tilde{y}, \tilde{s}) + \lambda (u^*)_* (\tilde{y}, \tilde{s}) + H(\tilde{y}, D\psi(\tilde{y}, \tilde{s})) \leq 0.$$

Since $u^{\alpha,\beta,\gamma}(x,t) = (u^*)_* (\tilde{y}, \tilde{s}) - \psi(\tilde{y}, \tilde{s})$, $\phi_\alpha \geq 0$, by (6.19) we have

$$\begin{aligned} &\frac{\partial u^{\alpha,\beta,\gamma}}{\partial t}(x,t) + \lambda u^{\alpha,\beta,\gamma}(x,t) - \lambda C_4(x_1 - \tilde{y}_1) \\ &+(C_3 - \lambda)e^{-C_3 t}\phi_\alpha(x, \tilde{y}) + H\left(\tilde{y}, Du^{\alpha,\beta,\gamma}(x,t) + e^{-C_3 t}\frac{\gamma}{\tilde{y}_1^2}e_1\right) \leq 0. \end{aligned}$$

We then conclude by the Lemma 6.9, by the estimate of $|x - \tilde{y}|$ in Lemma 6.5, and by (6.15). □

Remark 6.12 *By a classical result, $u^{\alpha,\beta,\gamma}$ is also a viscosity subsolution in $\mathcal{H}\times]0,T]$ (in the same way, if v is a viscosity supersolution of $(6.3)_1$ in $\mathcal{H}\times]0,T[$, then it is also a supersolution in $\mathcal{H}\times]0,T]$).*

Now, we denote by $u^{\alpha,\beta,0}$, the function defined in (6.12), when we take $\gamma = 0$. Similarly, we define the positive constant $K[T,\alpha,\beta,0]$.

Lemma 6.13 *When γ goes to 0^+, then $u^{\alpha,\beta,\gamma}$ converges to $u^{\alpha,\beta,0}$ uniformly over the compact sets of $\overline{\mathcal{H}} \times [0,+\infty[$. Moreover, $u^{\alpha,\beta,0}$ is locally Lipschitz in $\overline{\mathcal{H}} \times [0,+\infty[$, and, for every $T > 0$, it is a viscosity subsolution of*

$$w_t + \lambda w + H(x, Dw) + \overline{K}(C + 1 + \lambda C_4)K[T,\alpha,\beta,0] = 0 \quad \text{in } \mathcal{H}\times]0,T]. \quad (6.20)$$

Proof of Lemma 6.13. By classical stability properties for viscosity solutions, the result comes from the fact that $u^{\alpha,\beta,\gamma}$ uniformly converges to $u^{\alpha,\beta,0}$ on the compact sets of $\overline{\mathcal{H}} \times [0,T[$, and, on every compact subset of $\mathcal{H}\times]0,+\infty[$, there is the uniform convergence of the Hamiltonians. The uniform convergence of $u^{\alpha,\beta,\gamma}$ comes from the fact that they are equi-lipschitzian, and that they point-wise converge to $u^{\alpha,\beta,0}$ in $\overline{\mathcal{H}} \times [0,+\infty[$. To obtain the latter we adapt the proof in [10]. Note that, for every $(x,t) \in \overline{\mathcal{H}} \times [0,+\infty[$ there exists a sequence of points $(x_n,t_n) \in \mathcal{H}\times]0,+\infty[$, such that $(x_n,t_n) \to (x,t)$ and $(u^*)_*(x_n,t_n) \to (u^*)_*(x,t)$ (note that, we cannot say that the infimum in the definition of $u^{\alpha,\beta,0}$ is achieved in the interior $\mathcal{H}\times]0,+\infty[$). Hence, for every $(x,t) \in \overline{\mathcal{H}} \times [0,+\infty[$, and $\varepsilon > 0$, there exists $(y,s) \in \mathcal{H} \times]0,+\infty[$ such that, (taking also γ small)

$$u^{\alpha,\beta,0}(x,y) + \varepsilon \geq (u^*)_*(y,s) + e^{-C_3 t}\left(\phi_\alpha(x,y) + \frac{|t-s|^2}{\beta}\right) + C_4(x_1 - y_1)$$
$$\geq u^{\alpha,\beta,\gamma}(x,t) - \varepsilon,$$

and we conclude since $u^{\alpha,\beta,0} \leq u^{\alpha,\beta,\gamma}$ in $\overline{\mathcal{H}} \times [0,+\infty[$. □

Note that Remark 6.8 still holds for $u^{\alpha,\beta,0}$.

Lemma 6.14 *The following boundary inequalities hold*

$$u^{\alpha,\beta,0} \leq \varphi \text{ on } \partial\mathcal{H} \times [0,+\infty[, \quad u^{\alpha,\beta,0} \leq g \text{ on } \overline{\mathcal{H}} \times \{0\}. \tag{6.21}$$

Proof of Lemma 6.14. Let us prove only the first condition. Condition (6.7) implies $u^* \leq \varphi^*$ on $\partial\mathcal{H} \times]0,+\infty[$ (see Barles [5]). Hence, for $(x,t) \in \partial\mathcal{H} \times [0,+\infty[$, by (6.6), we get

$$\liminf_{(y,s)\to(x,t)} u^*(y,s) \leq \liminf_{(y,s)\to(x,t), y\in\partial\mathcal{H}} \varphi^*(y,s) = (\varphi^*)_*(x,t) = \varphi(x,t).$$

The first equality of the last row is certainly true if $t > 0$, but, by virtue of the regularity property (6.6), it is true even if $t = 0$. Then, by the continuity of $u^{\alpha,\beta,0}$, by the fact that $u^{\alpha,\beta,0} \leq (u^*)_*$ in $\overline{\mathcal{H}} \times [0,+\infty[$, for every $(x,t) \in \partial\mathcal{H} \times]0,+\infty[$ we get

$$\varphi(x,t) \geq \liminf_{(y,s)\to(x,t)} (u^*)_*(y,s) \geq \liminf_{(y,s)\to(x,t)} u^{\alpha,\beta,0}(y,s) = u^{\alpha,\beta,0}(x,t).$$

□

Lemma 6.15 *For every* $(x,t) \in \overline{\mathcal{H}} \times [0,+\infty[$ *we have*

$$\liminf_{(\alpha,\beta)\to(0,0)} u^{\alpha,\beta,0}(x,t) = (u^*)_*(x,t).$$

Proof of Lemma 6.15. Let us take $(x,t) \in \overline{\mathcal{H}} \times [0,+\infty[$ and $(\tilde{y},\tilde{s}) \in \mathcal{H} \times]0,+\infty[$ which realize the minimum in $u^{\alpha,\beta,\gamma}(x,t)$. We have

$$(u^*)_*(\tilde{y},\tilde{s}) + C_4(x_1 - \tilde{y}_1) \leq u^{\alpha,\beta,\gamma}(x,t).$$

The minimum point $(\tilde{y}, \tilde{s})$ depends on α, β, and γ, and it belongs to a compact set around (x, t). Hence, letting γ go to 0, $(\tilde{y}, \tilde{s})$ converges, at least for a subsequence, to a point $(\overline{y}, \overline{s})$. By the Lemma 6.13, passing to the inferior limit as $\gamma \to 0$, noting that $u^{\alpha,\beta,0}(x, t) \leq (u^*)_*(x, t)$, we obtain

$$(u^*)_*(\overline{y}, \overline{s}) + C_4(x_1 - \overline{y}_1) \leq u^{\alpha,\beta,0}(x, t) \leq (u^*)_*(x, t).$$

Then we perform the inferior limit as $(\alpha, \beta) \to 0$, and we obtain the conclusion, recalling also that $(\overline{y}, \overline{s}) \to (x, t)$ (see Remark 6.6). □

End of the proof of Theorem 6.3. We claim that, for every $T > 0$, $u^{\alpha,\beta,0} - v_* \leq \overline{K}(C + 1 + \lambda C_4)K[T, \alpha, \beta, 0]$ in $\overline{\mathcal{H}} \times [0, T]$, from which we conclude by Remark 6.6 and by Lemma 6.15. Since $u^{\alpha,\beta,0}$ is subsolution of (6.20), the claim is proved by a classical comparison technique, where one needs to use suitable penalization terms as in Soner [13] (see also Bagagiolo-Bardi [3] for the evolutive Hamilton-Jacobi equation in unbounded domains), to use the fact that $u^{\alpha,\beta,0}$ is continuous, to use the boundary properties (6.21), and to use Remark 6.12 (see also Bardi-Capuzzo Dolcetta [4], Section V.4.2). □

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