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Ph.D thesis

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Ciclo XXXV

Paracausal deformations, geometric Møller operators and Hadamard states in CCR AQFT

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Abstract

In this thesis, we address several problems related to the bosonic classical and algebraic quantum field theories in curved spacetime. In particular, the main question is: how do the theories change under finite global variations of the spacetime metric tensor? To answer this question a new deformation tool, the paracausal deformation, is developed and studied on its own as a new approach to investigate the structure of the space of globally hyperbolic metric tensors associated with a smooth manifold M. Then the classical Møller maps are constructed to compare solutions of the hyperbolic PDEs defining the classical field theories and the quantum Møller $*$-isomorphisms follow to compare the CCR quantum algebras associated to the propagation of the quantum fields on the different background geometries. These maps possess the important property of preserving Hadamard states, providing a new way to implement the deformation argument used to prove the existence of such states in general globally hyperbolic spacetime. Moreover, the algebraic quantization of the Proca field, i.e the massive spin 1 field, on a general globally hyperbolic spacetime is for the first time studied in detail: by employing techniques coming from microlocal analysis and spectral theory a Hadamard state is constructed on ultrastatic spacetimes and then the Møller operator is used to prove the existence of such states in general globally hyperbolic spacetimes. A discussion about the definition of Hadamard states for the massive vector fields closes the work.

The thesis is based on two works on algebraic quantization of bosonic field theories and Hadamard states: [87], [88]. The papers are co-authored by my supervisor Prof. Valter Moretti (UniTN) and cosupervisor Simone Murro (UniGe). The first [90] has not been included since, at the time it was written, the paracausal deformation, the construction of Møller operators, the right approach to intertwine the causal propagators and all the other tools developed in the subsequent works were still at a rough stage.

Contents

Introduction

Algebraic quantum field theory [12, 18] (AQFT) is a mathematically rigorous framework where quantum field theory on curved spacetime can be defined, according to the Haag-Kastler axioms, assigning $*$-algebras or C^*-algebras of abstract operators to open regions of a spacetime manifold in such a way that they satisfy physical properties such as locality, causality and the time slice axiom. The choice of algebraic states leads through the GNS construction to representations of the quantum algebras via unbounded operators defined on a common dense subspace of a Hilbert space. Among all possible states these algebras admit, a prominent role is played by the ones satisfying the Hadamard condition, since they finely generalize to curved spacetime the Minkowski vacuum.

In this thesis we study how the behaviour of aforementioned classical and quantum field theoretical structure one can build on a Lorentzian globally hyperbolic manifold changes under variations of the background geometry, with special attention on Hadamard states.

The main novelties presented are:

- the introduction of the paracausal relation in order to investigate the structure of the space of globally hyperbolic metrics;
- the construction of geometric Møller operators to compare quasi-free classical and quantum field theories living on different, though paracausally related spacetimes;
- the use of the aforementioned construction to constructively prove the existence of Hadamard states for the (generalized) Klein-Gordon field, rewriting a standard deformation argument through explicit operators, [50, 51]:
- the application of such a technique to prove existence of Hadamard states for the Proca field in general spacetime, concluding a discussion started in [44] where the existence had been proved just in Cauchy compact spacetimes;
- the explicit construction of a Hadamard state for the Proca field on ultrastatic spacetimes exploiting techniques coming from microlocal analysis and elliptic Hilbert complexes.

The thesis is organized as follows. Chapter 1 is totally devoted to Lorentzian geometry and thought to be self-consistent: basic results of causality theory are introduced as tools to review the geometry of globally hyperbolic spacetimes, then the paracausal relation is introduced and a lot of results about it are proved. A brief discussion about the topology of the space of globally hyperbolic metrics concludes.

In chapter 2 we introduce Green hyperbolic operators and compare the spaces of solutions of partial differential equations describing classical fields propagating on curved spacetimes, under the assumption that the spacetimes are paracausally related. Again this chapter discusses problems in linear Green hyperbolic differential operators and is self contained, despite dependent on the geometric notions developed in chapter 1. Therefore Møller operators and Møller maps are defined and their properties are investigated.

Chapter 3 deals with problems in quantum field theory in curved spacetimes: we briefly introduce the algebraic approach to quantum field theory, focusing on CCR algebras and Hadamard states. Then the geometric Møller operators are promoted to algebra isomorphisms preserving the singularity structure of states and the construction is employed to prove the existence of

Hadamard states for Klein-Gordon and Proca quantum fields on curved spacetime. Such a construction, known in the literature as "deformation argument", is in this work implemented in a new fashion through explicit operators. This approach allows to prove rigorously the existence of Hadamard states for Green hyperbolic field theories on any globally hyperbolic spacetime.

Since the chapters are as independent as possible, each one has its own conclusive sections summarizing the results and discussing possible future research lines.

General notation and conventions

- $A \subset B$ permits the case $A = B$, otherwise we write $A \subsetneq B$.
- The symbol $\mathbb{K}$ denotes any element of $\{\mathbb{R}, \mathbb{C}\}$.
- (M, g) denotes a $(n+1)$-dimensional spacetime (*cf.* Definition 1.1.5) and we adopt the convention that g has the signature $(-,+\ldots,+)$. $g^\sharp$ denotes the associated dual metric on the cotangent bundle.
- $\mathcal{M}_\mathsf{M}$, $\mathcal{T}_\mathsf{M} \subset \mathcal{M}_\mathsf{M}$ and $\mathcal{GH}_\mathsf{M} \subset \mathcal{T}_\mathsf{M}$ denote respectively the sets of smooth Lorentzian metrics, **time-oriented** Lorentzian metrics and **globally hyperbolic** metrics on M;
- $J^\pm(A)$ and $I^\pm(A)$ are respectively the causal and chronological future/past of the set A;
- $V_p^{g^\pm}$ are the the future/past lightcones w.r.t the metric g at the point $p \in \mathsf{M}$;
- $g \leq g'$ denotes that $g, g' \in \mathcal{M}_\mathsf{M}$ and the open light cone V_p^g of g is a subset of the open lightcone $V_p^{g'}$ of g' at every point $p \in \mathsf{M}$;
- $g \simeq g'$ denotes that g and g' are **paracausally related** (*cf.* Definition 1.3.1);
- We denote by E a general $\mathbb{K}$-vector bundle over a manifold M with finite rank N;
- We denote by $\Gamma_c(\mathsf{E}), \Gamma_{pc}(\mathsf{E}), \Gamma_{fc}(\mathsf{E}), \Gamma_{tc}(\mathsf{E}), \Gamma_{sc}(\mathsf{E})$ resp. $\Gamma(\mathsf{E})$ the spaces of compactly supported, past compactly supported, future compactly supported, timelike compactly supported, spacelike compactly supported resp. smooth sections of a vector bundle E.
- Tensor fields and sections of $\mathbb{K}$-vector bundles on M are always supposed to be smooth.
- $\mathsf{E} \otimes \mathsf{E}'$ denotes the **tensor product** of the two $\mathbb{K}$-vector bundles over M. This $\mathbb{K}$-vector bundle has basis $\mathsf{M} \ni p$ and fibers given by the pointwise tensor product $\mathsf{E}_p \otimes \mathsf{E}'_p$ of the fibers of the two bundles.
- $\mathsf{E} \boxtimes \mathsf{E}'$ denotes the **external tensor product** of the two $\mathbb{K}$-vector bundles over M. This $\mathbb{K}$-vector bundle has basis $\mathsf{M} \times \mathsf{M} \ni (p,q)$ and fibers given by the pointwise tensor product $\mathsf{E}_p \otimes \mathsf{E}'_q$ of the fibers of the two bundles.
- Referring to $\Gamma(\mathsf{E} \boxtimes \mathsf{E}')$, if $\mathfrak{f} \in \mathsf{E}$ and $\mathfrak{f}' \in \mathsf{E}'$, then $\mathfrak{f} \otimes \mathfrak{f}' \in \Gamma(\mathsf{E} \boxtimes \mathsf{E}')$ denotes the section defined by $(\mathfrak{f} \otimes \mathfrak{f}')(p,q) := \mathfrak{f}(p) \otimes \mathfrak{f}'(q)$ where the tensor product on the right-hand side is the one of the fibers and $(p,q) \in \mathsf{M} \times \mathsf{M}$.
- $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{TM})$ and its inverse $\flat : \Gamma(\mathsf{TM}) \to \Gamma(\mathsf{T}^*\mathsf{M})$ denote the standard (fiberwise) **musical isomorphisms** (*cf.* Section 1.2.1) referred to a given metric g on M.
- Let X be a topological vector space, we indicate by X' its topological dual. For example $\Gamma'_c(\mathsf{E})$ represents the space of distributions acting on compactly supported test sections of the bundle E, and shall not be confused with the space of compactly supported distributions.

Chapter 1

Paracausal deformations of globally hyperbolic spacetimes

The aim of this chapter is twofold. On the one hand we shall investigate the properties of *convex interpolation of Lorentzian metrics* and, as a natural consequence, we introduce and discuss *paracausal deformations of globally hyperbolic metrics.*
As we shall see, these mathematical tools rely on a non strict preorder relation on the set of Lorentzian metrics on a given manifold, very similar to the strict preorder used by Geroch to define the interval topology with respect to which the globally hyperbolic metrics are stable, [65]. The equivalence relation we define later is quite interesting in its own right and will be exploited in the second part of this work to construct Møller operators and Møller $*$-isomorphisms of algebras of quantum fields and, in the end, to construct Hadamard states for different theories on general globally hyperbolic spacetimes. We refer to [4, 9, 92] for standard results in Lorentzian geometry. To the authors' knowledge this equivalence relation represents a complete novelty on the subject. Though the effective definition of paracausal equivalence relation on the set of globally hyperbolic metrics on M (Definition 1.3.1) is effective for the issues regarding Møller maps we will discuss later, a complete characterization of it can be stated as follows in terms of elementary Lorentzian geometry:

Theorem 1 (Theorem 1.3.7)**.** *The globally hyperbolic metric g on M is paracausally related to the globally hyperbolic metric g' on M if and only if there is a finite sequence $g_0 := g, g_1, \ldots, g_N := g'$ of globally hyperbolic metrics on M such that, at each step g_k, g_{k+1}, the future open light cones of these metrics have non-empty intersection $V_x^{g_k^+} \cap V_x^{g_{k+1}+} \neq \varnothing$ at every point $x \in \mathsf{M}$.*

Despite the class of paracausally related metrics on a given manifold M is very large, for example we will see that any class admits ultrastatic representatives, some elementary counterexamples of topological nature can be constructed and some of them are not trivial. This fact suggests that the paracausal classes may be related to topological properties of the Cauchy surfaces. Such a problem, possible future research lines and other counterexamples of differential topological nature will be sketched in section 1.4 concluding the chapter.

The chapter is organized as follows. Section 1.1 contains a recap on the relevant notions of Lorentzian geometry we exploit throughout, in section 1.2 we introduce some results about convex interpolations of globally hyperbolic metrics which are preparatory to section 1.3, where we present the definition of paracausal deformation and we give all the results we obtained about this equivalence relation. In section 1.4 some possible future research lines are discussed.

1.1 Lorentzian geometry

In this section we review the basic tools of Lorentzian geometry and causality theory, which are of fundamental importance in the mathematical formulation of general relativity and, consequently, in the formulation of quantum field theory in curved spacetime, with focus on the geometry of the so called globally hyperbolic spacetimes, i.e spacetimes where the hyperbolic PDEs describing the propagation of classical fields have a well posed Cauchy problem. We will therefore fix all the geometric setup of the thesis and then proceed in the rest of the chapter with the discussion of the original contribution given to such a field.

1.1.1 Lorentzian manifolds and causality

Let M be a smooth connected paracompact Hausdorff manifold and assume that M is noncompact or its Euler characteristic vanishes. Under these assumptions, M admits a Lorentzian metric and we denote the space of Lorentzian metrics on M by $\mathcal{M}_{\mathsf{M}}$ (see *e.g.* [9]). Once that a Lorentzian metric g is assigned to a smooth manifold $\mathsf{M} \ni p$, we can classify the vectors $v_p \in \mathsf{T}_p\mathsf{M}$ into three different types:

- **spacelike** *i.e.* $g(v_p, v_p) > 0$ or $v_p = 0$,
- **timelike** *i.e.* $g(v_p, v_p) < 0$,
- **lightlike** (also called **null**) *i.e.* $g(v_p, v_p) = 0$ *and* $v_p \neq 0$.

As usual, we denote as **causal vectors** any timelike or lightlike vector. Piecewise smooth curves are classified analogously according to the nature of their tangent vectors.

Remark 1.1.1. Notice that, with our convention, the tangent vector 0 is spacelike to prevent constant curves to fall in the class of causal curves.

Embedded codimension-1 submanifolds $\Sigma \subset \mathsf{M}$ of a Lorentzian manifold (M, g), also called **hypersurfaces**, are classified according to their normal covector n: They are **spacelike**, **timelike**, **null** if respectively n is timelike, spacelike, null everywhere in Σ. Notice that an embedded $n-1$ submanifold $\Sigma \subset \mathsf{M}$ is spacelike if and only if its tangent vectors are spacelike in (M, g). The restriction of g to the tangent vectors to a spacelike hypersurface Σ defines a Riemannian metric on it.

Remark 1.1.2. In the rest of this chapter we are going to deal with different metric tensors associated to the same manifold M. We remark that the normal n is metric dependent, so it will be better to define spacelike surfaces with respect to a metric tensor $g \in \mathcal{M}_{\mathsf{M}}$ according to their tangent space which is intrinsic and metric independent.

Keeping in mind this classification, the open **lightcone** of (M, g) at $p \in \mathsf{M}$ is the set

$$V_p^g := \{v_p \in \mathsf{T}_p\mathsf{M} \mid g(v_p, v_p) < 0\} .$$

It is not difficult to see that it is an open convex cone made of two disjoint open convex halves defining the two connected components of V_p^g.

The notion of *time orientation* is defined as in [4]: A smooth Lorentzian manifold (M, g) is said to be **time-orientable** if there is a continuous timelike vector field X on M.

If (M, g) is time orientable and a preferred continuous timelike vector field X has been chosen as above, the **future lightcone** $V_p^{g+} \subset V_p^g$ at $p \in \mathsf{M}$ is the connected component of V_p^g containing X_p. The other connected component V_p^{g-} is the **past lightcone** at p. V_p^{g+} and V_p^{g-} respectively include the **future-directed** and **past-directed** timelike vectors at p. The terminology extends to the causal (lightlike) vectors which belong to the closures of the said halves. A classification of (piecewise smooth) causal curves into past-directed and future-directed curves (see [4]) arises according to their tangent vectors.

If (M, g) is time orientable, the continuous choice of one of the two halves of V_p^g for all $p \in \mathsf{M}$ through a continuous timelike vector field as above defines a **time orientation** of (M, g). (M, g) with this choice of preferred halves of cones is said to be **time oriented**. If (M, g) is connected and time orientable, then it admits exactly two time orientations.

Notation 1.1.3. In the following, we denote with $\mathcal{M}_{\mathsf{M}}$, the set of smooth Lorentzian metrics on the smooth manifold M and with $\mathcal{T}_{\mathsf{M}}$ the set of time-oriented Lorentzian metrics on M.

We have an elementary fact whose proof is immediate if working in a g-orthonormal basis.

Proposition 1.1.4. *Assume that $g \in \mathcal{T}_{\mathsf{M}}$, $p \in \mathsf{M}$, and $Y_p, Z_p \in V_p^g$. Then*

(i) $Y_p \in V_p^{g\mp}$ and $Z_p \in V_p^{g\pm}$ if and only if $g(Y_p, Z_p) > 0$,

(ii) $Y_p, Z_p \in V_p^{g\pm}$ if and only if $g(Y_p, Z_p) < 0$.

If $g \in \mathcal{M}_{\mathsf{M}}$, the associated standard (fiberwise) **musical isomorphism** $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{TM})$ is pointwise defined by

$$g(\sharp\omega_p, v_p) = \omega_p(v_p) \quad \text{for every } v \in \Gamma(\mathsf{TM}) \text{ and } \omega \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M},$$

and we denote the (fiberwise) **inverse musical isomorphism** by $\flat : \Gamma(\mathsf{TM}) \to \Gamma(\mathsf{T}^*\mathsf{M})$. The notation $g^\sharp \in \Gamma(\mathsf{TM} \otimes \mathsf{TM})$ indicates the Lorentzian metric induced on 1-forms from $\sharp$ as

$$g^\sharp(\omega_{1p}, \omega_{2p}) = g(\sharp\omega_{1p}, \sharp\omega_{2p}) \quad \text{for every } \omega_1, \omega_2 \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M}.$$

Once that a Lorentzian metric is introduced on 1-forms, we can distinguish three different types of co-vectors: $\omega_p \in \mathsf{T}_p^*\mathsf{M}$ is **spacelike**, **timelike**, **null** and **causal** if, respectively, $\sharp\omega_p \in \mathsf{T}_p\mathsf{M}$ is spacelike, timelike, null or causal. With the definition, we can define the open **lightcone of 1-forms** at $p \in \mathsf{M}$ analogously to the case of vectors

$$V_p^{g^\sharp} := \{\omega_p \in \mathsf{T}_p^*\mathsf{M} \mid g^\sharp(\omega_p, \omega_p) < 0\}.$$

Analogously, if $g \in \mathcal{T}_{\mathsf{M}}$, the **future** and **past lightcones of 1-forms** at $p \in \mathsf{M}$ are defined as

$$V_p^{g^\sharp\pm} := \{\omega_p \in \mathsf{T}_p^*\mathsf{M} \mid \sharp\omega_p \in V_p^{g\pm}\}.$$

1.1.2 Spacetimes and causality

We are ready to give the precise definition of spacetime we will use throughout this work.

Definition 1.1.5. A **spacetime** is a $(n+1)$-dimensional $(n \geqslant 1)$, connected, oriented, time-oriented, smooth Lorentzian manifold (M, g)

Remark 1.1.6. Sometimes it not assumed that M is orientable and oriented, but we do adopt this hypothesis here since later on we will need to integrate over the manifold, see chapter 16 of [80]. However none of the results discussed in this chapter depends on the choice of an orientation. Conversely, the time orientation is crucial. So, when we write that (M, g) *is a spacetime*, we also mean that a *time-orientation* of (M, g) as Lorentzian manifold has been chosen. In this case, with a little misuse of language, we speak of the *time-orientation of the metric g*.

According to the amount of time-like symmetry, three important nested subclasses of spacetimes can be defined:

- **stationary** if it admits a globally defined time-like smooth Killing vector field K;
- **static** if it is stationary and the Killing vector field is also irrotational;
- **ultrastatic** if it is static and $g(K, K) = -1$.

Let now $A \subset \mathsf{M}$ for a spacetime (M, g). The **causal sets** $J_\pm(A)$ and the **chronological sets** $I_\pm(A)$ are defined according to [4]: $J_\pm(A)$ is made of the points of A itself and all $p \in M$ such that there is a smooth future-directed/past-directed causal curve $\gamma : [a, b] \to \mathsf{M}$ with $\gamma(a) \in A$ and $\gamma(b) = p$. *Notice that* $J_\pm(A) \supset A$ *by definition*, while $I_\pm(A)$ is made of the points $p \in M$ such that there is a smooth future-directed/past-directed timelike curve $\gamma : [a, b] \to \mathsf{M}$ with $\gamma(a) \in A$ and $\gamma(b) = p$. As usual we define $J(A) := J_+(A) \cup J_-(A)$.

A closed set $A \subset \mathsf{M}$, with (M, g) time-oriented, is **past compact** if $J_-(p) \cap A$ is compact for every $p \in \mathsf{M}$. The definition of **future compact** is analogous, just replacing J_- for J_+. A closet set $A \subset \mathsf{M}$ is called **space compact** or spatially compact if there exists a compact set $K \subset \mathsf{M}$ such that $A \subset J(K)$. Sections of a bundle are called **past/future/space compact** if their support is, respectively, past, future or space compact.

Let us recall that, on a spacetime (M, g), a smooth causal curve $\gamma : I \to \mathsf{M}$ with $I \subset \mathbb{R}$ open interval is said to be **future inextendible** [92] if there is no *continuous* curve $\gamma' : J \to \mathsf{M}$, defined on an open interval $J \subset \mathbb{R}$, such that $\sup J > \sup I$ and $\gamma'|_I = \gamma$. A **past inextendible** causal curve is defined analogously. A causal curve is said to be **inextendible** if it is both past and future inextendible.

We eventually define the **future Cauchy development** $D_+(A)$ of A to be the set of points $p \in \mathsf{M}$ such that every past inextendible future-directed smooth causal curve passing through p meets A in the past. Similarly, the **past Cauchy development** $D_-(A)$ is the set of points $p \in \mathsf{M}$ such that every future inextendible future-directed smooth causal curve passing through p meets A in the future.

On a generic Lorentzian manifold, the Cauchy problem for the differential operators we will deal with is in general ill-posed: This can be a consequence of the presence of closed timelike curves or the presence of naked singularities. Therefore, it is convenient to restrict ourselves to the class of *globally hyperbolic spacetimes.*

Definition 1.1.7. A **globally hyperbolic spacetime** is a spacetime (M, g) such that

(i) there are no closed causal curves;

(ii) for all points $p, q \in \mathsf{M}$, $J_+(p) \cap J_-(q)$ is compact.

Notation 1.1.8. If M is a smooth connected $(n+1)$-manifold, $\mathcal{GH}_\mathsf{M} \subset \mathcal{T}_\mathsf{M}$ denotes the class of Lorentzian metrics g such that (M, g) is globally hyperbolic for a time-orientation. Any $g \in \mathcal{GH}_\mathsf{M}$ is called **globally hyperbolic** metric on M.

Remark 1.1.9. The first condition in 1.1.7 is also known as **causality**. We remind the reader that this definition of global hyperbolicity is recent, see [17] for the proof of the equivalence of the two. In the standard definition, see for example [9], the first condition is replaced by the so called **strong causality**. It requires that at all points $p \in \mathsf{M}$ and for all neighbourhoods $U_p \subset \mathsf{M}$ there exists a smaller neighbourhood $U'_p \subset U$ which is causally convex, that is such that any causal curve with endpoints in U'_p does not intersect the complement of U'_p in a disconnected set.

However globally hyperbolic spacetimes can be characterized by more physically intuitive and practically useful conditions.
In his seminal paper [65], Geroch established the equivalence for a Lorentzian manifold being globally hyperbolic and the existence of a *Cauchy hypersurface.*

Definition 1.1.10. A subset $\Sigma \subset \mathsf{M}$ of a spacetime (M, g) is called **Cauchy hypersurface** if it intersects exactly once any inextendible future-directed smooth timelike curve.

In particular, a Cauchy hypersurface is **achronal**: it intersects at most once every future-directed smooth timelike curve.

Theorem 1.1.11 ([65, Theorem 11]). *A spacetime* (M, g) *is globally hyperbolic if and only if it contains a Cauchy hypersurface.*

It turns out that Cauchy hypersurfaces of (M, g) are closed co-dimension 1 topological submanifolds of M homeomorphic one to each other. As a byproduct of Geroch's theorem, it follows that the globally hyperbolic manifold (M, g) admits a continuous foliation with Cauchy hypersurfaces Σ as leaves, namely M is homeomorphic to $\mathbb{R} \times \Sigma$. The proof of these facts was carried out by finding a *Cauchy time function*, i.e., a continuous function $t : \mathsf{M} \to \mathbb{R}$ which is strictly increasing on any future-directed timelike curve and such that its level sets $t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, are Cauchy hypersurfaces homeomorphic to Σ. Geroch's splitting appears at a topological level, and the possibility to smooth them remained as open folk questions for many years. Only recently, in [15] Bernal and Sánchez "smoothened" the result of Geroch by introducing the notion of *Cauchy temporal function*.

Theorem 1.1.12 ([15, Theorems 1.1 and 1.2], [16, Theorem 1.2],)**.** *For every globally hyperbolic spacetime* (M, g) *there is an isometry* $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$*, where the latter spacetime is equipped with the smooth Lorentzian metric*

$$- \beta^2 d\tau \otimes d\tau \oplus h_\tau \,, \tag{1.1.1}$$

and the time-orientation induced from (M, g) *through* ψ*. Above* τ *is the canonical projection*

$$\mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$$

and the following facts are valid:

(i) $\nabla \tau := \sharp d\tau$ *is past-directed timelike,*

(ii) $\beta : \mathbb{R} \times \Sigma \to (0, +\infty)$ *(called* **lapse function***) is a smooth function,*

(iii) h_t *(called* **spatial metric***) is a smooth Riemannian metric on each leaf* $\{t\} \times \Sigma$*,* $t \in \mathbb{R}$*,*

*(iv) every embedded co-dimension-*1 *submanifold* $\{t_0\} \times \Sigma = \tau^{-1}(t_0)$ *is a spacelike (smooth) Cauchy hypersurface.*

Finally, if $S \subset \mathsf{M}$ *is a spacelike Cauchy hypersurface of* (M, g)*, then we can define an isometry* $\psi : \mathsf{M} \to \mathbb{R} \times S$*, and* τ*,* β*,* h *as above in order that* $S = \psi^{-1}(\{0\} \times S)$*.*

Remarks 1.1.13.

(1) As we will see later global hyperbolicity is a sufficient condition to guarantee such a smooth orthogonal splitting, but not a necessary one.

(2) A globally hyperbolic ultrastatic spacetime is always diffeomorphic to $\mathbb{R} \times \Sigma$ and isometric to a spacetime with metric of the form:

$$g = -dt \otimes dt \oplus h,$$

where h is a Riemannian metric on Σ and $dt = K^\flat$.

The characterization given by Bernal and Sánchez immediately allows us to give some relevant definitions.

Definition 1.1.14. Given a spacetime (M, g), a smooth surjective function $t : \mathsf{M} \to \mathbb{R}$ with dt past-directed timelike is

(a) a **Cauchy temporal function** if

  (i) (M, g) is isometric, through some isometry $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$, to a spacetime $(\mathbb{R} \times \Sigma, h)$ with the time-orientation induced from (M, g),

  (ii) $t = \tau \circ \psi$ (where $\tau : \mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$),

  (iii) h has the form (1.1.1) as in Theorem 1.1.12 satisfying (i)-(iv);

(b) a **smooth Cauchy time function** if

  (i) (M, g) is isometric, through some isometry $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$, to a spacetime $(\mathbb{R} \times \Sigma, h)$ with the time-orientation induced from (M, g),

  (ii) $t = \tau \circ \psi$ (where $\tau : \mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$),

  (iii) every $\Sigma_{t_0} := t^{-1}(t_0) = \psi^{-1}(\{t_0\} \times \Sigma)$ is a spacelike Cauchy hypersurface of (M, g) for $t_0 \in \mathbb{R}$.

Remarks 1.1.15.

(1) An intrinsic way to write (1.1.1) for a Cauchy temporal function t without making use to the splitting diffeomorphism ψ is, for $p \in \Sigma_s = t^{-1}(p)$

$$g_p(X, Y) = \frac{dt \otimes dt(X, Y)}{g^\sharp(dt, dt)} + h_s(\pi_{t,g} X, \pi_{t,g} Y)\,, \quad X, Y \in \mathsf{T}_p\mathsf{M} = L(\sharp_g dt) \oplus \Sigma_s$$

where

$$\mathsf{T}_p\mathsf{M} \ni X \mapsto \pi_{t,g} X := X - \frac{\langle dt, X\rangle\, \sharp_g dt}{g^\sharp(dt, dt)} \in \mathsf{T}_p\Sigma_s$$

defines the orthogonal projector onto $\mathsf{T}_p\Sigma_s$ associated to t and g, using $\sharp_g dt$ as normal (contravariant) vector to Σ_s.

(2) If an either smooth Cauchy time or temporal function t exists for (M, g), the level sets $\Sigma_{t_0} := t^{-1}(t_0)$ are smooth spacelike Cauchy surfaces diffeomorphic to each other and (M, g) is globally hyperbolic. Theorem 1.1.12 proves that Cauchy temporal functions – thus also smooth Cauchy time functions – exist for every globally hyperbolic spacetime. Furthermore, every smooth spacelike Cauchy hypersurface can be embedded in the foliation induced by a suitable Cauchy temporal function.

(3) A Cauchy temporal function is always a Cauchy time function, but even a smooth time function may not be a temporal one, since the manifold may be foliated in level sets of such a function, which are spacelike and Cauchy, but the metric tensor could not be in the orthogonal form.

(4) A Cauchy hypersurface may meet a causal curve in more than a point (say, a segment), but this is not the case for the spacelike Cauchy hypersurfaces since they are **acausal**: they intersect *at most once* every future-directed smooth causal curve, as easily arises from Theorem 1.1.12.

On a globally hyperbolic spacetime the past/future/space compact sets can be characterized in a very useful way by exploiting the notion of Cauchy surface. in fact:

Proposition 1.1.16. *Let* (M, g) *be a globally hyperbolic spacetime. Then we have that:*

- *A is past compact if and only if it is closed and there exists a Cauchy surface* $\Sigma \subset \mathsf{M}$ *such that* $A \subset J^+(\Sigma)$*;*

- *A is future compact if and only if it is closed and there exists a Cauchy surface* $\Sigma \subset \mathsf{M}$ *such that* $A \subset J^-(\Sigma)$*.*

- *A is space compact if and only if it is closed and for any Cauchy surface* $\Sigma \subset \mathsf{M}$ $A \cap \Sigma$ *is compact.*

Examples 1.1.17. We shall list a few globally hyperbolic spacetimes which appear commonly in general relativity and quantum field theory over curved backgrounds. As one can infer per direct inspection, they are all in the orthogonal form of Theorem 1.1.12:

- the prototype example is Minkowski spacetime which isometric to $\mathbb{R}^{n+1}$ with Cartesian coordinates $(t, x^1, \ldots, x^n)$ and equipped with the Minkowski metric

$$-dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i \,;$$

- de Sitter spacetime, that is the maximally symmetric solution of Einstein's equations with a positive cosmological constant Λ. As a manifold it is topologically $\mathbb{R} \times \mathbb{S}^3$ and the metric reads:

$$g = -dt^2 + \frac{3}{\Lambda}\cosh^2\left(\sqrt{\frac{\Lambda}{3}}t\right)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]$$

where $t \in \mathbb{R}$ while (χ, θ, φ) are the standard coordinates on $\mathbb{S}^3$;

- the Friedmann-Robertson-Walker (FRW) cosmological spacetimes, i.e., an isotropic and homogeneous manifold which is topologically $\mathbb{R} \times \Sigma$ and

$$g = -dt^2 + a(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

where k can be either 0 or ± 1 and function $a(t)$ is smooth and positive valued;

- The external Schwarzschild spacetime, i.e., a stationary spherically symmetric solution of vacuum Einstein's equations which is topologically $\mathbb{R}^2 \times \mathbb{S}^2$ with metric

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,.$$

Here $M > 0$ is interpreted as the mass of the spherically symmetric source (a black hole, a star,...) and the domain of definition of the coordinates is $t \in \mathbb{R}$, $r \in (2M, +\infty)$ and $(\theta, \varphi) \in \mathbb{S}^2$;

- finally, given any n-dimensional complete Riemannian manifold (Σ, h), an open interval $I \subseteq \mathbb{R}$ and a smooth function $f : I \to (0, +\infty)$, the Lorentzian *warped product* defined topologically by $I \times \Sigma$ with metric $g = -dt^2 + f(t)h$ is a globally hyperbolic spacetime, [9].

- A static spacetime $\mathbb{R} \times \Sigma$ with metric tensor $g = -\beta dt^2 + h$ where $h\backslash\beta$ is a complete Riemannian metric on the slice is globally hyperbolic, [96]. In particular any ultrastatic spacetime with complete slices is globally hyperbolic.

A non-trivial result about global hyperbolicity of a spacetime is the following lemma, which we will use a lot in the next sections. Essentially, it says that we can open-up or close-down the cones of any globally hyperbolic metric, uniformly and smoothly all over the manifold, without spoiling global hyperbolicity.

Lemma 1.1.18. *Let* (M, g) *and a globally hyperbolic spacetime,* $t : \mathsf{M} \to \mathbb{R}$ *a Cauchy temporal function according to Definition 1.1.14,* $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$ *a diffeomorphism mapping isometrically* (M, g) *to* $(\mathbb{R} \times \Sigma, -\beta^2 d\tau \otimes d\tau \oplus h_\tau)$ *and, moreover, let* (M, g') *be a time oriented spacetime with time orientation such that dt is past directed. If* ψ *maps* (M, g') *isometrically to* $(\mathbb{R} \times \Sigma, g_\alpha)$ *with*

$$g_{\alpha=} - d\tau \otimes d\tau \oplus \alpha^2(\tau)\beta^{-2}h_\tau$$

and $\alpha \in C^\infty(\mathbb{R}, (0, \infty))$*, then* (M, g') *is globally hyperbolic.*

Proof. We will henceforth omit to write the isometry ψ and consider, without loss of generality, $\mathsf{M} = \mathbb{R} \times \Sigma$, $t = \tau$, $g = -\beta^2 dt \otimes dt \oplus h_t$ and $g_\alpha = -dt \otimes dt \oplus \alpha^2(t)\beta^{-2}h_t$. We proceed to prove the global hyperbolicity of g_α implying global hyperbolicity of g'.
We want to prove that Σ, viewed as the $t = 0$ slice of the temporal function t, is a spacelike Cauchy hypersurface for g_α. Evidently Σ is a spacelike hypersurface for g_α so that it suffices to prove that it meets exactly once every inextendible future directed g_α-timelike curve $\gamma : I \ni s \mapsto \gamma(s) \in \mathsf{M}$. Since $\frac{dt}{ds} = g_\alpha(\partial_t, \dot\gamma) < 0$ by hypothesis, that γ can be re-parametrized by t itself as $\gamma' : J \ni t \mapsto \gamma'(t) \in M$ for some open interval $J \subset \mathbb{R}$. There must exist a finite $a > 0$ such that $(-a, a) \cap J \neq \varnothing$. Since $\gamma'|_{(-a,a)\cap J}$ is inextendible in the spacetime $(-a, a) \times \Sigma$ (otherwise it would not be inextendible in the whole spacetime), to conclude it is sufficient to prove that $(-a, a) \times \Sigma$ equipped with the metric g_α and the time-orientation induced by dt admits Σ as a Cauchy hypersurface. Indeed, in that case, γ' must meet Σ exactly once in $(-a, a)\times\Sigma$ and thus Σ would be a Cauchy hypersurface for $(\mathbb{R}\times\Sigma, g_\alpha)$. Moreover, notice that it cannot meet $\Sigma = t^{-1}(0)$ again outside $(-a, a)\times\Sigma$ because γ' is parametrized by t. Global hyperbolicity of $((-a, a)\times\Sigma, g_\alpha)$ can be proved as follows. If $a > 0$, there exists a positive constant α_0 such that $\alpha(t) \geqslant \alpha_0 > 0$ for all $t \in [-a, a]$. We therefore have $g_\alpha \preceq g_{\alpha_0}$ on $(-a, a)\times\Sigma$. In particular, with the time-orientation declared in the hypothesis, every future-directed causal tangent vector for g_α is a future-directed causal vector for g_{α_0}. Therefore, according to (2) in Lemma 1.2.3, it suffices to show that g_{α_0} is globally hyperbolic on $(-a, a) \times \Sigma$ and that Σ is a Cauchy hypersurface for g_{α_0}. To this end, consider an inextendible future-directed timelike curve $\gamma = (\gamma^0, \hat\gamma)$ in $((-a, a) \times \Sigma, g_{\alpha_0})$. The curve $\tilde\gamma := (\alpha_0^{-1}\gamma_0, \hat\gamma)$ is future directed timelike w.r.t. g and still inextendible , therefore it meets $\Sigma = t^{-1}(0)$ exactly once, but $\tilde\gamma$ and γ intersect in $\gamma_0 = t = 0$. Thus γ intersects Σ once. This shows g_{α_0} and therefore g_α to be globally hyperbolic on $(-a, a) \times \Sigma$. □

1.2 Convex interpolation of Lorentzian metrics

We are now interested in the structure of the set $\mathcal{M}_{\mathsf{M}}$ of Lorentzian metrics on a given manifold M. In particular, we are interested in the following problem:

Are there some natural operations which can be used to produce (globally hyperbolic) Lorentzian metrics starting from (globally hyperbolic) Lorentzian metrics?

Given two globally hyperbolic metrics g, g', a linear combination of them is in general not a Lorentzian metric and, when it is, it fails to be globally hyperbolic in general. However, as shown in [19, Appendix B], if g and g' coincide outside a compact set, then there exists a sequence of 5 globally hyperbolic metrics starting with g and ending with g', such that for each neighbouring pair all pointwise convex combinations are globally hyperbolic metrics. Therefore, this section aims to provide sufficient conditions for some kind of linear combination of globally hyperbolic metrics to be a globally hyperbolic Lorentzian metric. We shall see that convex combinations are an interesting case of study under suitable conditions. We point out the recent work [97] where, in addition to several related issues, the convex structure of the space of globally hyperbolic metrics on a given manifold is addressed with a number of results. In particular we had claimed that the set of globally hyperbolic metrics sharing a Cauchy temporal function was convex, while in that paper a counterexample is found.
Even if such a problem is of mathematical interest on its own, it was posed because of the problem of constructing Møller operators for symmetric hyperbolic systems [90], and normally hyperbolic operators.

1.2.1 A preorder relation of Lorentzian metrics

Definition 1.2.1. Let $g, g' \in \mathcal{M}_{\mathsf{M}}$ and denote

$$g \preceq g' \qquad \text{iff} \quad V^g_p \subset V^{g'}_p \text{ for all } p \in \mathsf{M}.$$

We say that $g, g' \in \mathcal{M}_{\mathsf{M}}$ are **$\preceq$-comparable** if either $g \preceq g'$ or $g' \preceq g$ (see e.g. Figure 1.1).

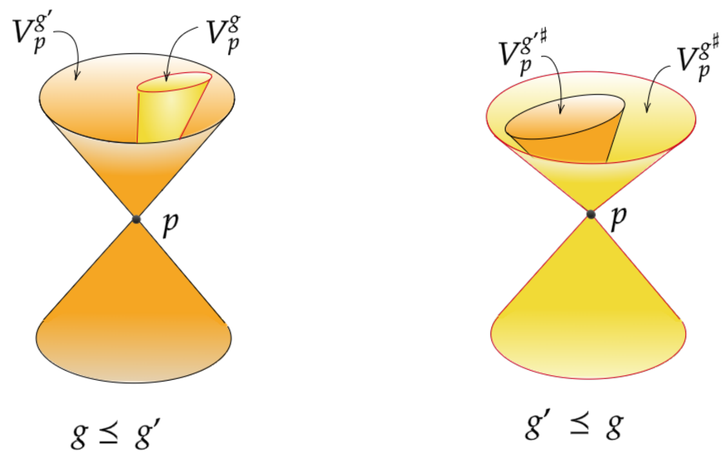


Figure 1.1: Lorentzian metrics $\preceq$-comparable

Remarks 1.2.2.

(1) Let us remark that the definition above can be generalized by considering the so-called *causal diffeomorphisms*, namely a time-orientation preserving diffeomorphism $\varphi : \mathsf{M} \to \mathsf{N}$ such that the open light cone V_p^g of g is included in the open light cone $V_p^{\varphi^* g'}$ of $\varphi^* g'$ for every $p \in \mathsf{M}$. For further details and properties we refer to [48, 52, 53].

(2) The preorder relation introduce in Definition 1.2.1 has a corresponding for the associated metrics in the cotangent space: If $g, g' \in \mathcal{M}_{\mathsf{M}}$,

$$g^\sharp \preceq g'^\sharp \qquad \text{iff} \quad V_p^{g^\sharp} \subset V_p^{g'^\sharp} \text{ for all } p \in \mathsf{M}.$$

We observe that if $g \preceq g'$ for $g, g' \in \mathcal{T}_{\mathsf{M}}$ and the two metrics share the same time-orientation – i.e., there is a continuous vector field on M which is timelike for both metrics and defines the same time-orientation for both of them – then $V_p^{g+} \subset V_p^{g'+}$ and $V_p^{g-} \subset V_p^{g'-}$ for every $p \in \mathsf{M}$. Similar inclusions hold when considering the closures of the considered half cones. As a consequence, we have both inclusions with obvious notations

$$I_\pm^g(A) \subset I_\pm^{g'}(A)\,, \quad J_\pm^g(A) \subset J_\pm^{g'}(A) \quad \text{for every } A \subset \mathsf{M}.$$

The relation $\preceq$ in $\mathcal{M}_{\mathsf{M}}$ has several consequences whose most elementary ones are established in the following proposition.

Lemma 1.2.3. *Let M be a smooth $(n+1)$-dimensional manifold and $g, g' \in \mathcal{M}_{\mathsf{M}}$. The following facts are valid for the preordering relation $\preceq$ in $\mathcal{M}_{\mathsf{M}}$.*

(1) For $p \in \mathsf{M}$ and $v \in \mathsf{T}_p\mathsf{M}$, if $g \preceq g'$ then

(i) $g(v,v) = 0$ implies $g'(v,v) \leqslant 0$.

(ii) $g'(v,v) > 0$ implies $g(v,v) > 0$.

(iii) $g'(v,v) = 0$ implies $g(v,v) \geqslant 0$.

(2) If $g \preceq g'$ with $g \in \mathcal{T}_{\mathsf{M}}$ and $g' \in \mathcal{GH}_{\mathsf{M}}$, then (M, g) is globally hyperbolic as well when, e.g., equipped with the same orientation and time-orientation of (M, g') and

(i) a spacelike Cauchy hypersurface for (M, g') *is also a spacelike Cauchy hypersurface for* (M, g)*;*

(ii) a smooth Cauchy time function for (M, g') *is also a smooth Cauchy time function for* (M, g)*;*

(iii) a closed set $A \subset \mathsf{M}$ *is past/future compact in* (M, g) *if it is respectively past/future compact in* (M, g')*.*

(3) $g \preceq g'$ *if and only if* $g'^{\sharp} \preceq g^{\sharp}$*.*

(4) If $g, g' \in \mathcal{T}_{\mathsf{M}}$*,* $g \preceq g'$ *and* $p \in \mathsf{M}$*, then* $V_p^{g+} \subset V_p^{g'+}$ *if and only if* $V_p^{g'^{\sharp}+} \subset V_p^{g^{\sharp}+}$*.*

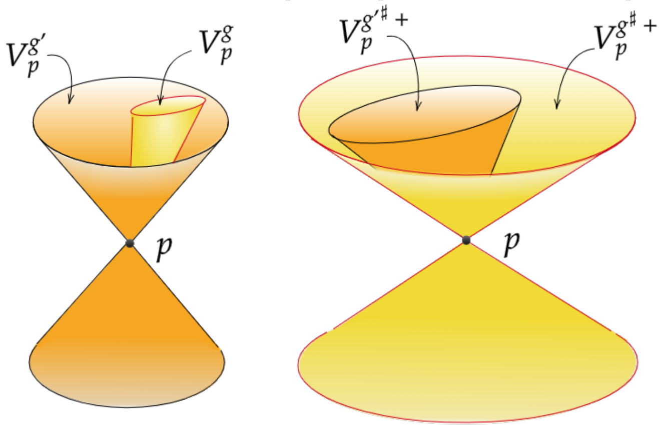


Figure 1.2: Inclusion-of-cones relations

Proof. (1) is trivial. Let us pass to (2). Notice that if $g \preceq g'$, then a smooth g-timelike vector of M is also a smooth g'-timelike vector of M, so (M, g) receives a time-orientation from a time-orientation of (M, g'). A spacelike Cauchy surface Σ of (M, g') (it exists in view of Theorem 1.1.12) is a smooth spacelike hypersurface of (M, g) since the tangent vectors to Σ are spacelike for g' and thus for g because $g \preceq g'$. Every inextendible timelike curve γ for (M, g) is a inextendible timelike curve for (M, g'), since (a) $g \preceq g'$ and (b) the notion of inextendibility is just topological. Hence γ meets Σ exactly once. Therefore Σ is a Cauchy surface of (M, g) as well, which is globally hyperbolic for Theorem 1.1.11. (i) has been proved above. Let us prove (ii). If $t : \mathsf{M} \to \mathbb{R}$ is a smooth Cauchy time function for g', then its level sets are smooth spacelike Cauchy surfaces for (M, g') and $g'^{\sharp}(dt,)$ is timelike (and past-directed). As seen above, the level sets of t are therefore also spacelike Cauchy surfaces for (M, g). Moreover, these submanifolds are spacelike also for g in view of (1)(ii). This is equivalent to saying that their normal vector $g^{\sharp}(dt,)$ is timelike (and past-directed if choosing the same time-orientation as for g'). All that proves that t is also a smooth Cauchy time function for (M, g). The proof of (iii) easily arises from (i).
(3) We take advantage of the following elementary fact.

Lemma 1.2.4. *Let* $g : V \times V \to \mathbb{R}$*, with* $\dim V = n+1$ *be a Lorentzian scalar product.* $\xi \in V^* \setminus \{0\}$ *is timelike if and only if there is a set of* n *linearly-independent elements* $e_1, \dots, e_n \in V$ *whose span is made of spacelike vectors, such that* $\xi(e_k) = 0$ *for* $k = 1, \dots, n$*.*

Proof. If $\xi \in V^*$ is timelike, $\sharp\xi \in V$ is timelike as well. Completing $e_0 := \sharp\xi/\sqrt{-g(\sharp\xi, \sharp\xi)}$ to a pseudo orthonormal basis $e_0, e_1, \dots, e_n$ of V, the vectors $e_1, \dots, e_n$ satisfy trivially the thesis. Suppose that $\xi \in V^* \setminus \{0\}$ admits a set of vectors $e_1, \dots, e_n \in V$ whose span S is made of spacelike vectors and such that $\xi(e_k) = 0$ for $k = 1, \dots, n$. Extract an orthonormal basis $f_1, \dots, f_n$ of S out of $e_1, \dots, e_n$, and complete it to a pseudo orthonormal basis $f_0, f_1, \dots f_n$ of V. It holds $\sharp\xi = c^k f_k$ and $g(\sharp\xi, f_k) = \sum_{h=1}^n b^h e_h(\xi) = 0$ for $k = 1, 2, \dots, n$, so that $\sharp\xi = c^0 f_0$ which is timelike. $c^0 = 0$ is not permitted since $\xi \neq 0$. This concludes our claim. □

Evidently, by duality, $v \in V\backslash\{0\}$ is timelike if and only if there is a set of n linearly-independent forms $\omega_1, \ldots, \omega_n \in V^*$ whose span is spacelike and such that $\omega_k(v) = 0$ for $k = 1, \ldots, n$.
Let us pass to the proof of (3). If $g \preceq g'$, let $\xi \in V_p^{g'^\sharp}$, then there is a set of n linearly-independent vectors $e_1, \ldots, e_n \in \mathsf{T}_p\mathsf{M}$ whose span is g'-spacelike and such that $\xi(e_k) = 0$ for $k = 1, \ldots, n$. These vectors span a g-spacelike subspace because $g \preceq g'$ and (1)(ii) is valid. Therefore $\xi \in V_p^{g*}$, i.e., $g \preceq g'$ implies $g'^\sharp \preceq g^\sharp$. The same argument stated for vectors instead of forms proves that $g'^\sharp \preceq g^\sharp$ implies $g \preceq g'$.
(4) Fix a basis of $T_x\mathsf{M}$, denote by $\widetilde{X} \in \mathbb{R}^{n+1}$ the ordered set of components of $X \in \mathsf{T}_x\mathsf{M}$ with respect to that basis and by G (resp. G') the invertible symmetric matrix representing g (resp. g') with respect to that basis. Let us assume $V_x^{g+} \subset V_x^{g'+}$ and we prove that $V_x^{g'^\sharp+} \subset V_x^{g^\sharp+}$. If $X \in V_x^{g+} \subset V_x^{g'+}$, then $\flat_g X \in V_x^{g^\sharp+}$ and $\flat_{g'} X \in V_x^{g'^\sharp+} \subset V_x^{g^\sharp}$, where the last inclusion is due to (3). From that we have that $\flat_{g'} X \in V_x^{g'^\sharp+}$ also satisfies $\flat_{g'} X \in V_x^{g^\sharp+}$ if $g^\sharp(\flat_g X, \flat_{g'} X) < 0$. This condition is actually satisfied because

$$g^\sharp(\flat_g X, \flat_{g'} X) = (G\widetilde{X})^t G^{-1} G' \widetilde{X} = \widetilde{X}^t G^t G^{-1} G' \widetilde{X} = \widetilde{X}^t G G^{-1} G' \widetilde{X} = \widetilde{X}^t G' \widetilde{X} = g'(X, X) < 0$$

and Proposition 1.1.4 holds. We have so far established that $V_x^{g'^\sharp+} \cap V_x^{g^\sharp+} \neq \varnothing$. Since $V_x^{g'^\sharp+} \subset V_x^{g^\sharp}$ may intersect only one of $V_x^{g^\sharp+}$ and $V_x^{g^\sharp-}$ (otherwise $V_x^{g'^\sharp}$ would have more than two connected components), it must be $V_x^{g'^\sharp+} \subset V_x^{g^\sharp+}$. The fact that $V_x^{g'^\sharp+} \subset V_x^{g^\sharp+}$ implies $V_x^{g+} \subset V_x^{g'+}$ can be proved with an analogous argument. □

Using the lemma above, we can immediately conclude that (pointwise) conformally equivalent metric tensors are obviously in relation.

Proposition 1.2.5. *If $g \in \mathcal{M}_\mathsf{M}$ and $\mu : \mathsf{M} \to (0, +\infty)$ is smooth, then*

(a) μg and $\mu^{-1} g$ are Lorentzian,

(b) $\mu g \preceq g \preceq \mu g$,

(c) $\mu^{-1} g \preceq g \preceq \mu^{-1} g$.

(d) μg and $\mu^{-1} g$ are globally hyperbolic if g is, and the spacelike Cauchy hypersurfaces of g are also spacelike Cauchy hypersurfaces for μg and $\mu^{-1} g$.

Proof. Properties (a)-(c) follow easily from the definitions and (d) is a direct consequence of (a),(b),(c) and Proposition 1.2.3 point (2). □

Remark 1.2.6. Since any two metric tensors having identical lightcones are (pointwise) conformally equivalent our preorder descends to an actual partial ordering on conformal classes.

1.2.2 Properties of convex combinations of Lorentzian metrics

A more interesting set of properties arises when focusing on smooth *convex combinations* of Lorentzian metrics. This is the first main result of this section.

Theorem 1.2.7. *Let M be a smooth $(n+1)$-dimensional manifold, $g, g' \in \mathcal{M}_\mathsf{M}$, and consider a smooth function $\chi : \mathsf{M} \to [0, 1]$. If $g \preceq g'$, the following facts are valid*

(1) $(1-\chi)g + \chi g'$ is a metric of Lorentzian type;

(2) $g \preceq (1-\chi)g + \chi g' \preceq g'$;

(3) if $g_\chi^\sharp := (1-\chi)g^\sharp + \chi g'^\sharp$, then $g_\chi^\sharp := (g_\chi)^\sharp$ for a (unique) metric g_χ of Lorentzian type;

(4) $g \preceq g_\chi \preceq g'$;

(5) If g' is globally hyperbolic and g time-orientable, then $(1-\chi)g + \chi g'$ and g_χ are globally hyperbolic.

Proof. (1) It is sufficient to prove the thesis point by point. Let q, q' be quadratic forms in a real $n+1$ dimensional linear space V of signature $(-,+,\dots,+)$ such that $q'(x) \leqslant 0$ implies $q(x) \leqslant 0$. We prove that the strict convex combination $q'' = cq + (1-c)q'$ for $c \in (0,1)$ has signature $(-,+,\dots,+)$. Indeed, there is a 1-dimensional subspace L on which $q'(x) < 0$ if $x \neq 0$. So $q(x) \leqslant 0$ on L and hence $q''(x) < 0$ on L for $x \neq 0$. There is also a n-dimensional subspace H on which $q(x) > 0$ if $x \neq 0$. Then $q'(x) > 0$ on H for $x \neq 0$ and hence $q''(x) > 0$ on H if $x \neq 0$. By construction, $L \cap H = \{0\}$ necessarily, so that $V = L \oplus H$. The bilinear form $Q'' : V \times V \to \mathbb{R}$ associated to q'', in a basis of V made of $0 \neq e_0 \in L$ and $\{e_k\}_{k=1,\dots,n} \in H$ with $Q''(e_k, e_h) = \delta_{kh}$, is represented by the $(n+1)\times(n+1)$ matrix $\begin{bmatrix} q''(e_0) & c^t \\ c & I \end{bmatrix}$. Since the determinant is $q''(e_0) - c^t c < 0$ and n eigenvalues are $+1$, its signature is $(-,+,\dots,+)$.

(2) Suppose that $g(v,v) < 0$, then $g'(v,v) < 0$ because $g \preceq g'$ and thus $(1-\chi)g(v,v) + \chi g'(v,v) < 0$ because $\chi, 1-\chi \geqslant 0$. We have obtained that $g \preceq (1-\chi)g + \chi g'$. Let us pass to the remaining inequality. If $(1-\chi)g(v,v) + \chi g'(v,v) < 0$ then $g'(v,v) < 0$ or $g(v,v) < 0$, in this second case also $g'(v,v) < 0$ because $g \preceq g'$. In both cases $g'(v,v) < 0$. Summing up, $(1-\chi)g + \chi g' \preceq g'$, concluding the proof of (2).

(3) $g^\sharp$ and $g'^\sharp$ are Lorentzian metric on $\mathsf{T}^*\mathsf{M}$ and $g'^\sharp \preceq g^\sharp$ due to Lemma 1.2.3, we can recast the same argument used to establish (1) with trivial re-arrangements, obtaining that $g^\sharp_\chi$ is Lorentzian and $g'^\sharp \preceq g^\sharp_\chi = (1-\chi')g'^\sharp + \chi' g^\sharp \preceq g^\sharp$ with $\chi' := 1-\chi$. Notice that $g_\chi(v,v) := g^\sharp_\chi(\flat v, \flat v)$ defines a Lorentzian metric as well, since it has the same signature of h by construction, and $g^\sharp = h$ trivially (and it is the unique metric with this property since $\flat$ is an isomorphism).

(4) It immediately arises from Lemma 1.2.3 by using $g'^\sharp \preceq g^\sharp_\chi = (1-\chi')g'^\sharp + \chi' g^\sharp \preceq g^\sharp$ with $\chi' := 1-\chi$.

(5) A smooth timelike vector field of (M, g) is also timelike for $(1-\chi)g + \chi g'$ and g_χ for (2) and (4) respectively. Hence these metrics are time-orientable and the thesis follows from Lemma 1.2.3 point (2). □

1.3 Paracausal deformation of Lorentzian metrics

The aim of this section is to provide a new definition that shall encode the idea to deform a Lorentzian metric equipped with a time-orientation into another Lorentzian metric with a corresponding time-orientation, taking advantage of a procedure consisting of a finite number of steps. At each step, the light cones of the final metric g_k are related to those of the initial one g_{k-1} through an inclusion relation, either $g_{k-1} \preceq g_k$ or $g_k \preceq g_{k-1}$ preserving the time-orientation at each step, i.e., the future cone of g_k, respectively, includes or is included in the future cone of g_{k-1}.

1.3.1 Paracausal relation

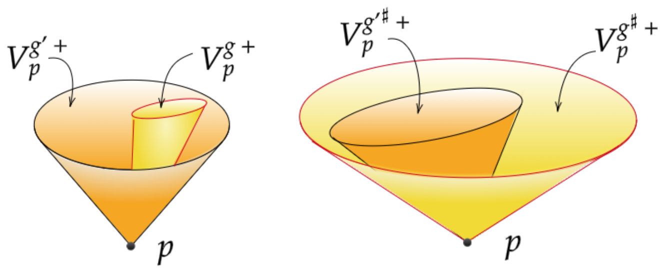
Definition 1.3.1. Consider a pair of globally hyperbolic spacetimes on the same manifold M with corresponding metrics $g, g' \in \mathcal{GH}_{\mathsf{M}}$ and corresponding time-orientations. We say that g is **paracausally related** to g' – and we denote it by $g \simeq g'$ – or equivalently g' is a **paracausal deformation** of g, if there is a finite sequence, said **paracausal chain**, $g_0 = g, g_1, \dots, g_N = g' \in \mathcal{GH}_{\mathsf{M}}$ with corresponding time-orientations, such that either

$$V_p^{g_k+} \subset V_p^{g_{k+1}+} \text{for all } p \in \mathsf{M}$$

or

$$V_p^{g_{k+1}+} \subset V_p^{g_k+} \quad \text{for all } p \in \mathsf{M},$$

where the choice may depend on $k = 0, \dots, N-1$.



Remarks 1.3.2.

(1) Let us remark that our notion of paracausally deformation implies in particular that g_k and g_{k+1} are always $\preceq$-comparable.

(2) Evidently, to be paracausally related is an *equivalence relation* in $\mathcal{GH}_{\mathsf{M}}$.

(3) We stress that paracausal deformations explicitly consider the time-orientations of the used sequences of globally hyperbolic spacetimes. So, even if we say that "metrics are paracausally related", the relation actually involves *the metrics equipped with corresponding time-orientations*.

(3) We shall show below a characterization of the paracausal relationship which seems more natural from a geometric and physical viewpoint. However, the definition above *as it stands* is much more directly suitable for the applications to Møller operators we shall introduce in the second part of this work.

Examples 1.3.3.

(1) There are two elementary cases of paracausally related (globally hyperbolic) metrics g_0, g_1 on M which are not directly $\preceq$-comparable:

   1. There is a globally hyperbolic metric g on M such that, simultaneously $g \preceq g_0$ and $g \preceq g_1$ and the future lightcones are correspondingly included.
   2. There is a globally hyperbolic metric g on M such that, simultaneously $g_0 \preceq g$ and $g_1 \preceq g$ and the future lightcones are correspondingly included.

   In both cases, the existence of sequence g_0, g, g_1 proves that $g_0 \simeq g_1$.

(2) Let us give an elementary concrete example of paracausally related metrics. Consider the following smooth manifold $\mathbb{R}^n$ endowed with the Minkowski metrics

$$\eta_0 = -dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i \qquad \eta_1 = -d\tau \otimes d\tau + \sum_{i=1}^{n} dy^i \otimes dy^i$$

where $(t, x_1, \ldots, x_n)$ and $(\tau, y_1, \ldots, y_n)$ are two different systems of Cartesian coordinates on $\mathbb{R}^{n+1}$. Here t and τ are Cauchy temporal functions associated to the respective Lorentzian metric and defining the time-orientation of the two metrics: dt and $d\tau$ are assumed to be past directed for the respective metric. More precisely, we assume that the two coordinate systems are related by means of a physically non-trivial permutation which interchanges space and time, as in Figure 1.3, $\tau = x_1$, $y_1 = t$, and $y_k = x_k$ for $k > 1$. It is not difficult to see that even if $\eta_0 \neq \eta_1$ evidently, we have $\eta_0 \simeq \eta_1$: they are paracausally related by the sequence of metrics $\eta_0, \overline{g}_1, \overline{g}_2, \eta_1$ whose future cones are given as in Figure 1.4. It is evident that by further implementing the procedure, it is possible to reverse the time-orientation of (M, η_0) through a sequence of paracausal deformations leaving the final metric identical to the initial one.

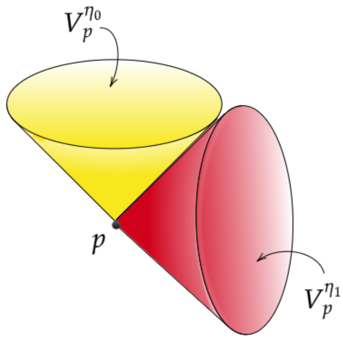


Figure 1.3: Future light cones of different Minkowski metrics on $\mathbb{R}^{n+1}$.

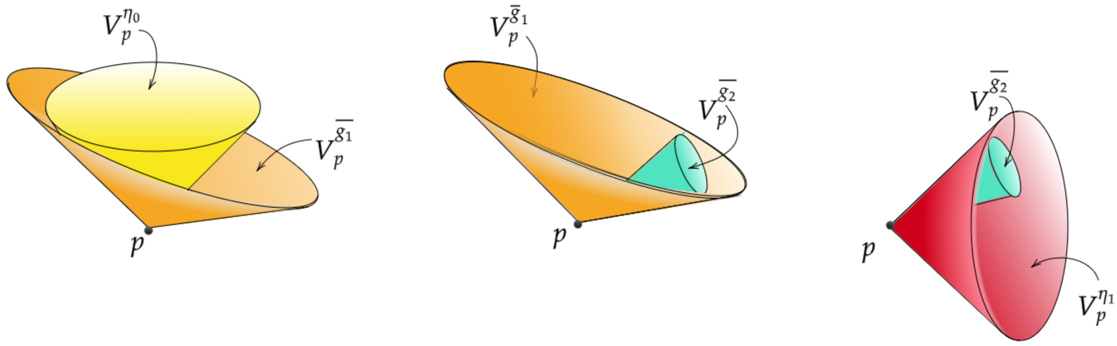


Figure 1.4: Auxiliary future light cones to prove $\eta_0 \simeq \eta_1$

(3) We pass to present a case where a pair of globally hyperbolic metrics are *not* paracausally related. Consider the $2D$ Minkowski cylinder M obtained by identifying x and $x + L$ in $\mathbb{R}^2$ with coordinates x, y. The first globally hyperbolic spacetime is (M, η_1) where $\eta_1 = -dy \otimes dy + dx \otimes dx$, taking the identification into account, and with time-orientation defined by assuming that ∂_y is future-directed. The second globally hyperbolic spacetime is (M, η_2) where again $\eta_2 = -dy \otimes dy + dx \otimes dx$, taking the identification into account, but with the opposite time-orientation, i.e., defined by $-\partial_y$. See also Figure 1.5.

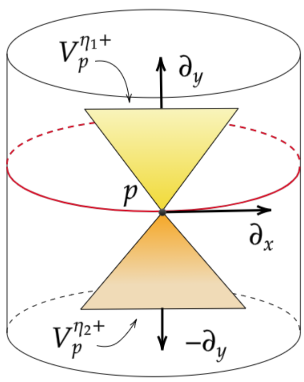


Figure 1.5: 2-D Minkowski cylinder.

These two metrics are not paracausally related. Any attempt to use the procedure as in the previous example to rotate the former into the latter faces the insurmountable obstruction that one of the auxiliary metrics would have Cauchy hypersurfaces given by the x-constant lines. This Lorentzian manifold is not globally hyperbolic because it admits closed temporal curves as in Figure 1.6.

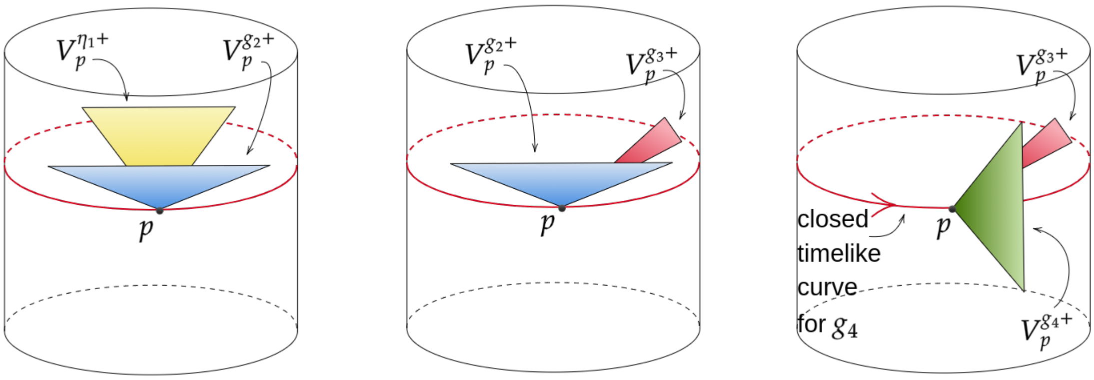


Figure 1.6: Sequence of metrics where g_4 is not globally hyperbolic

Notice that this obstruction does not take place without the identification $x \equiv x + L$.

1.3.2 Characterization of paracausal deformation in terms of future cones

There is a natural situation where two globally hyperbolic metrics g and g' on M are paracausally related. The generalization of the following result leads to a natural characterization of the paracausal relationship.

Proposition 1.3.4. *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes on the same manifold* M*. If* $V_x^{g+} \cap V_x^{g'+} \neq \varnothing$ *for every* $x \in \mathsf{M}$*, then the metrics* g *and* g' *are paracausally related.*

Proof. To prove the assertion it is sufficient to prove the existence of a Lorentzian metric $h \in \mathcal{T}_\mathsf{M}$ such that $h \preceq g$ and $h \preceq g'$. In this case, h would be globally hyperbolic according to (2) in Lemma 1.2.3 and the same argument as in (1) Examples 1.3.3 would prove the thesis.

Let us start by proving that a smooth vector field X on M exists such that $X_p \in V_p^{g+} \cap V_p^{g'+}$ for all $p \in \mathsf{M}$. Let us define the smooth functions

$$G : \mathsf{TM} \ni (p, v) \mapsto g_p(v, v) \in \mathbb{R}\,, \quad G_Y : \mathsf{TM} \ni (p, v) \mapsto g_p(v, Y) \in \mathbb{R}\,,$$

where Y is a smooth timelike future oriented vector field for g. By construction (with obvious notation) $\cup_{p\in\mathsf{M}} V_p^{g+} = G^{-1}(-\infty, 0) \cap G_Y^{-1}(-\infty, 0) \subset \mathsf{TM}$ is an open set. With the same argument, we have that also $\cup_{p\in\mathsf{M}} V_p^{g'+} \subset \mathsf{TM}$ is open. Finally, $\cup_{p\in\mathsf{M}} V_p^{g+} \cap \cup_{p\in\mathsf{M}} V_p^{g'+} = \cup_{p\in\mathsf{M}} V_p^{g+} \cap V_p^{g'+}$ is therefore open, non-empty by hypothesis, and projects onto the whole M by construction. As a consequence, given a local trivialization patch $\mathsf{T}U$ around $p \in U$, where (U, ψ) is a local chart on M (with $\dim(\mathsf{M}) = n + 1$), the set $\left(\cup_{p\in\mathsf{M}} V_p^{g+} \cap V_p^{g'+}\right) \cap \mathsf{T}U$ is diffeomorphic to an open subset $A \subset V \times \mathbb{R}^{n+1}$ with $V := \psi(U) \subset \mathbb{R}^{n+1}$ and $\pi_1(A) = V$ ($\pi_1 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ being the standard projection onto the first factor). Working in coordinates, it is then trivially possible first to pick out a smooth local section $X^{(U)}$ of $\mathsf{T}U$ such that $X_q^{(U)} \in V_q^{g+} \cap V_q^{g'+}$ if $q \in U$. To conclude, consider a partition of the unity $\{\chi_i\}_{i\in I}$ of M subordinated to a locally finite covering $\{U_i\}_{i\in I}$ of domains of local charts of M and let $X_p^{(U_i)} \in V_p^{g+} \cap V_p^{g'+}$ be constructed as above when $p \in U_i$ for every $i \in I$. The smooth vector field constructed as a locally finite convex linear combination $X := \sum_{i\in I} \chi_i X^{(U_i)}$ satisfies $X_p \in V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$ because the cones V_q^{g+}, $V_q^{g'+}$ are convex sets in a vector space and thus their intersection is also convex. X is the vector field we were searching for.

As the second step we construct a Lorentzian metric h, whose future cones V_p^{h+} satisfy $X_p \in V_p^{h+} \subset V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$. Notice that it means $h \in \mathcal{T}_\mathsf{M}$ since X is future directed for h (and also for the two metrics g and g') and thus it defines a time-orientation for (M, h). Since $h \preceq g, g'$, this would conclude our proof.

Let us construct h taking advantage of the vector field X. Consider $p \in M$ and define a g-pseudo orthonormal basis $e_0, \ldots, e_n$ where $e_0 = \frac{X_p}{\sqrt{-g(X_p,X_p)}}$ and the remaining vectors are g-spacelike. If $v, v' \in \mathsf{T}_p\mathsf{M}$,

$$g(v, v') = -g(e_0, v)g(e_0, v') + \sum_{k=1}^{n} g(e_k, v)g(e_k, v') \,.$$

If $a \in (0, 1)$, the new Lorentzian scalar product in $\mathsf{T}_p\mathsf{M} \ni v, v'$

$$g^a(v, v') := -ag(e_0, v)g(e_0, v') + \sum_{k=1}^{n} g(e_k, v)g(e_k, v') = g(v, v') + (a-1)\frac{g(X_p, v)g(X_p, v')}{g(X_p, X_p)} \quad (1.3.1)$$

trivially satisfies (the closure being taken in $\mathsf{T}_p\mathsf{M}\backslash\{0\}$)

$$X_p \in V_p^{g^a+} \subsetneq \overline{V_p^{g^a+}} \subsetneq V_p^{g+} \quad \text{for } a \in (0, 1).$$

The strong inclusions are due to the fact that the lightlike boundary of $V_p^{g^a+}$ is made of timelike vectors of g as it arises from the definition of g^a. Now note that $\partial V_p^{g^a+}$ becomes more and more concentrated around the set $\{\lambda X_p \,|\, \lambda > 0\}$ as a approaches 0 from above. (In particular, the limit and degenerate case $g_p^{a=0}(v, v) = 0$ implies v is parallel to X_p.) Since $X_p \in V_p^{g'+}$ which is also an open convex cone as $V_p^{g^a+}$, there must exist $a_p \in (0, 1)$ such that $V_p^{g^{a_p}+} \subset V_p^{g'+}$. This property is locally uniform in a as established in the following technical lemma:

Lemma.[1] *Within the hypotheses of the proposition, if $x \in \mathsf{M}$, there is a coordinate patch with domain $V \ni x$, an open set $U \ni x$ with compact closure $\overline{U} \subset V$, and a constant $a_U \in (0, 1)$ such that $V_p^{g^{a_U}+} \subset V_p^{g'+}$ for every $p \in U$.*

Proof. If $x \in \mathsf{M}$, there is a coordinate patch with domain $V \ni x$ and coordinates $V \ni p \mapsto \varphi(p) = (x^0(p), \ldots, x^n(p)) \in \mathbb{R}^{n+1}$ such that $U \ni x$ for some open subset $U \subset V$ such that $\overline{U}$ is compact. We will henceforth deal with U and the coordinates $(x^0, \ldots, x^n)$ restricted to thereon. We will also take advantage of the compact set $K := \varphi(\overline{U}) \subset \mathbb{R}^{n+1}$ and identify $\mathsf{T}\overline{U}$ with $K \times \mathbb{R}^{n+1}$ using the coordinates. Finally, we will equip both K and $\mathbb{R}^{n+1}$ (representing $\mathsf{T}_p\mathsf{M}$ at each $p \in \overline{U} \equiv K$) with the standard Euclidean metric of $\mathbb{R}^{n+1}$ whose norm will be denoted by $|| \cdot ||$.

Let us start the proof by proving that the family of cones $V_p^{g'+}$ of g' has a minimal width $m > 0$ when p ranges in K. We henceforth view the above future-directed timelike vector field X and g' as geometric objects on K using the coordinate system. In particular, if $p \in K$, let us indicate by $v_p \in \mathbb{R}^{n+1}$ the unique future-directed timelike vector parallel to X_p (now viewed as a vector in $\mathbb{R}^{n+1}$) such that $||v_p|| = 1$. Consider the set made of future-directed elements of $T\mathsf{M}$

$$C := \{(p, u) \in K \times \mathbb{S}^n \mid g'_p(u, v_p) \leqslant 0 \,, g'_p(u, u) = 0\}$$

(above $\mathbb{S}^n := \{z \in \mathbb{R}^{n+1} \mid ||z|| = 1\}$) and the continuous function

$$W : C \ni (p, u) \mapsto ||u - v_p|| \geqslant 0 \,,$$

which computes the width of $\partial V_p^{g'+}$ (that is of $V_p^{g'+}$ itself) around X_p along the direction u by using the Euclidean distance induced by $|| \cdot ||$. Observe that C is compact since it is the

[1] As noticed by the referee, a different strategy for proving this lemma would be showing that the function $M \ni p \mapsto a(p) = \sup\{a \in (0, 1) : V_p^{g^a+} \subset V_p^{g'+}\}$ is continuous. In that case, one can alternatively define $a_U := \min_{p \in \overline{U} a(p)}$. However the proof of continuity is not technically easy.

intersection of preimages of a pair of closed sets along two corresponding continuous maps and it is included in a compact set. Since this map is continuous and C is compact, there exists

$$m := \min_C W > 0\,.$$

In particular, $m > 0$, otherwise $u = v_q$ for some $(q,u) \in C$ and this is not possible since it would imply $g'(v_q,v_q) = g'(u,u) = 0$, but v_q is timelike ($g'_q(v_q,v_q) < 0$) since it does not vanish ($||v_q|| = 1$) and it is proportional to the timelike vector X_q.

An analogous width-cone function can be defined for the cones of g^a (including the degenerated case $a = 0$) on a set C' which also embodies the dependence on a:

$$C' := \{(a,p,u) \in [0,1/2] \times K \times \mathbb{S}^n \mid g^a_p(u,v_p) \leqslant 0\,, g^a_p(u,u) = 0\}\,.$$

We also define the continuous function

$$W' : C' \ni (a,p,u) \mapsto ||u - v_p|| \geqslant 0\,.$$

Observe that C' is again compact since it is the intersection of preimages of two closed sets along a pair of corresponding continuous maps of (a,p,u) and C' is included in a compact set.

We want to prove that there exists $a^m \in [0,1/2]$ such that $W'(a^m,p,u) < m$ for all $(p,u) \in C$. If this were not the case, then for every $a_n := 1/n$ there would be a pair $(p_n,u_n) \in C$ such that $W'(a_n,p_n,u_n) \geqslant m$. Since C' is a compact metric space, we could extract a subsequence of triples $(a_{n_k},p_{n_k},u_{n_k}) \to (0,p_\infty,u_\infty) \in [0,1/2] \times C$ for $k \to +\infty$ and some $(p_\infty,u_\infty) \in C$. By continuity $0 = g^{a_n}_{p_n}(u_n,u_n) \to g^0_{p_\infty}(u_\infty,u_\infty)$ where $||u_\infty|| = 1$. From (1.3.1), $g^0_{p_\infty}(u_\infty,u_\infty) = 0$ would entail that u_∞ is parallel to v_{p_∞} and thus $W'(0,p_\infty,u_{p_\infty}) = ||u_\infty - v_{p_\infty}|| = 0$. That is in contradiction with the requirement $W'(a_n,p_n,u_n) \geqslant m > 0$ for every $n = 1,2,\ldots$ in view of the continuity of W'.

We have therefore established that there exists $a^m \in [0,1/2]$ such that $W'(a^m,p,u) < m$ for all $(p,u) \in C$. From the definition of W and W', we have also obtained that $V_p^{g^{a^m}+} \subset V_p^{g'+}$ for all $p \in K$. It is enough to conclude that $V_p^{g^{a_U}+} \subset V_p^{g'+}$ for all $p \in U$ as wanted simply by taking $a_U := a^m$. This concludes our claim. □

Let us go on with the main proof. For every U as in the previous lemma, define the constant function $a(p) = a_U$ for $p \in U$. Since this can be done in a neighbourhood of every point $p \in \mathsf{M}$, using a partition of the unity $\{\chi_i\}_{i\in I}$ subordinated to a locally finite covering of charts $\{U_i\}_{i\in I}$, we can construct the metric h, where now every $a_i := a_{U_i} : U_i \to (0,1)$ is a constant in U; and thus it is a smooth function therein.

$$h_p(v,v') = \sum_i \chi_i(p) g_p^{a_i(p)}(v,v') = \sum_i \chi_i(p)\left(g_p(v,v') + (a_i(p)-1)\frac{g_p(X_p,v)g_p(X_p,v')}{g_p(X_p,X_p)}\right)$$

$$= g_p(v,v') + \left(\sum_i \chi_i(p)a_i(p) - 1\right)\frac{g_p(X_p,v)g_p(X_p,v')}{g_p(X_p,X_p)}$$

Since $\sum_i \chi_i(p)a_i(p) \in (0,1)$, this metric is still Lorentzian and of the form (1.3.1) point by point, where now $a(p) = \sum_i \chi_i(p)a_i(p)$. By construction $X_p \in V_p^{h+} \subset V_p^{g+}$ for every $p \in \mathsf{M}$, just because it happens point by point with the above choice of $a(p)$. In particular, we can endow h with the time-orientation induced by X as it happens for g, g' and all local metrics g^{a_i}. Finally, $V_p^{h+} \subset V_p^{g'+}$ because, if $h_p(v,v) < 0$, at least one of the values $g^{a_{i_0}(p)}(v,v)$ appearing in $\sum_i \chi_i(p) g_p^{a_i(p)}(v,v)$ must be negative and thus, if v is future-directed, $v \in V_+^{g^{a_i(p)}+} \subset V_p^{g'+}$. The proof is over because h satisfies all requirements $X_p \in V_p^{h+} \subset V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$. □

As an immediate by-product, it is easy to see that for any globally hyperbolic metric g, there exists a paracausal deformation g' of g which is ultrastatic.

Corollary 1.3.5. *Let (M, g) be a globally hyperbolic spacetime. Then there exists a paracausal deformation g' of g such that (M, g') is an ultrastatic spacetime.*

Proof. Let t be a Cauchy temporal function for the globally hyperbolic spacetime (M, g) so that M is isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$. We indicate by ∂_t the tangent vector to the submanifold $\mathbb{R}$. Let h be a complete Riemannian metric on Σ. Then the ultrastatic metric $g' := -dt^2 + h$ is globally hyperbolic [96] and the vector ∂_t is contained in the intersection of V_p^{g+} and $V_p^{g'+}$ for any $p \in \mathsf{M}$. Proposition 1.3.4 ends the proof. □

Remark 1.3.6. The complete Riemannian metric h on the slice can be chosen, in particular, of bounded geometry since any paracompact manifold admits one, [67]. This important fact will be exploited in the last chapter to construct a Hadamard state for the massive vector field.

The result established in Proposition 1.3.4 leads to a crucial characterization of paracausally related metrics, which represent the second main result of this section.

Theorem 1.3.7. *Let M be a smooth manifold. Two metrics $g, g' \in \mathcal{GH}_{\mathsf{M}}$ are paracausally related if and only if there exists a finite sequence of globally hyperbolic metrics $g_1 = g, g_2 \ldots, g_n = g'$ on M such that all pairs of consecutive metrics g_k, g_{k+1} satisfy $V_x^{g_k+} \cap V_x^{g_{k+1}+} \neq \varnothing$ for every $x \in \mathsf{M}$.*

Proof. If g, g' are paracausally related, then a sequence of metrics as in Definition 1.3.1 trivially satisfies the condition in the thesis. If that condition is *vice versa* satisfied, then the metrics of each pair g_k, g_{k+1} of the sequence are paracausally related in view of Proposition 1.3.4. Since paracausal relation is transitive, g and g' are paracausally related. □

We conclude this first analysis of the paracausal relation with a very important necessary condition coming as a corollary of the equivalence of the very first definition and the characterization above.

Corollary 1.3.8. *Let M be a smooth manifold and $g, g' \in \mathcal{GH}(\mathsf{M})$ be such that $g \simeq g'$. Then two Cauchy surfaces Σ and Σ', respectively for g and for g', are diffeomorphic.*

Proof. Let $g = g_1, g_2 \ldots, g_N = g'$ be the paracausal chain connecting the two metrics. We have proved that it exists if and only if there exists another sequence of globally hyperbolic metrics $g = \widetilde{g}_1, \ldots, \widetilde{g}_M = g'$ such that their future cones intersect pairwise. But this means that for each couple of consecutive metrics $\widetilde{g}_i$ and $\widetilde{g}_{i+1}$ their intersection is non-empty and a paracausal chain can be built by constructing a metric $\widetilde{g}_{i,i+1}$ such that (1) $\widetilde{g}_{i,i+1} \preceq \widetilde{g}_i$ and (2) $\widetilde{g}_{i,i+1} \preceq \widetilde{g}_{i+1}$.
But by 1.2.3, (1) implies that any Cauchy surface Σ_i of $\widetilde{g}_i$ is a Cauchy surface of $\widetilde{g}_{i,i+1}$ and (2) implies that any Cauchy surface Σ_{i+1} of $\widetilde{g}_{i+1}$ is a Cauchy surface of $\widetilde{g}_{i,i+1}$, but since all Cauchy surfaces associated to a globally hyperbolic metric are diffeomorphic we get that Σ_i and Σ_{i+1} are diffeomorphic. Iterating the procedure to the whole chain proves the claimed result. □

1.3.3 Paracausal deformation and Cauchy temporal functions

We now study the interplay of the notion of Cauchy temporal function and the one of paracausal deformation.
We first state and prove a result concerning Cauchy surfaces and the paracausal relation[2].

Proposition 1.3.9. *Let (M, g) and (M, g') be globally hyperbolic spacetimes on M which share a Cauchy temporal function $t : \mathsf{M} \to \mathbb{R}$ according to Definition 1.1.14. Then $g \simeq g'$.*

[2]The following proof is actually extracted by a result due to M. Sánchez who, with Theorem 3.4 of [97], improved a similar statement in a previous version of this work where we also assumed that the Cauchy surfaces were compact. We are grateful to M. Sánchez for providing this improved version of our result.

Proof. As before, we will henceforth omit to write the isometries identifying the various spacetimes. However we may have two different isometries from M to $\mathbb{R}\times\Sigma$ for g and g'. Proposition 1.2.5 yields $g \preceq \hat{g} \preceq g$, $g' \preceq \hat{g}' \preceq g'$ if

$$\hat{g} := \beta_0^{-2} g = -dt\otimes dt + \beta_0^{-2} h_t \qquad \text{and} \qquad \hat{g}' := \beta_1^{-2} g' = -dt\otimes dt + \beta_1^{-2} h'_t\,,$$

where β_0^2, β_1^2 are the lapse function we choose in accordance with Theorem 1.1.12. The metrics $\hat{g}$ and $\hat{g}'$ are globally hyperbolic for Lemma 1.1.18 (with $\alpha = 1$). The proof ends by proving that $\hat{g}$ and $\hat{g}'$ are paracausally related. Referring to the splitting of M as $\mathbb{R}\times\Sigma$ induced by the Cauchy temporal function t, define the globally hyperbolic metric $-dt\otimes dt + h$, where h is a complete Riemannian metric on Σ (see, e.g., [96]). For every $\lambda \in (0,1)$, direct inspection proves that,

$$g_\lambda := \lambda(-dt\otimes dt + h) + (1-\lambda)\hat{g} = -dt\otimes dt + \lambda h + (1-\lambda)\beta_0^{-2} h_t \preceq -dt\otimes dt + \lambda h$$

and

$$g_\lambda = \lambda(-dt\otimes dt + h) + (1-\lambda)\hat{g} = -dt\otimes dt + \lambda h + (1-\lambda)\beta_0^{-2} h_t \preceq -dt\otimes dt + (1-\lambda)\beta_0^{-2} h_t\,.$$

Since λh is complete, from the former line we conclude that the metric g_λ is globally hyperbolic due to (2) Lemma 1.2.3 and that it is paracausally related to $dt\otimes dt + \lambda h$. From the latter, since $-dt\otimes dt + (1-\lambda)\beta_0^{-2} h_t$ is globally hyperbolic in view of Lemma 1.1.18, we have that this metric and g_λ are paracausally related. Since $(1-\lambda) \in (0,1)$, the cones of $-dt\otimes dt + (1-\lambda)\beta_0^{-2} h_t$ include the cones of $-dt\otimes dt + \beta_0^{-2} h_t = \hat{g}$ so that these metrics are paracausally related as well. Transitivity implies that $-dt\otimes dt + \lambda h$ and $\hat{g}$ are paracausally related. The same argument proves that $-dt\otimes dt + \lambda h$ and $\hat{g}'$ are paracausally related so that $\hat{g} \simeq \hat{g}'$ and the thesis holds. □

Now we prove another non trivial result about paracausally related metrics for Cauchy compact spacetimes and conclude the chapter.

Proposition 1.3.10. *Let (M, g) and (M, g') be spacetimes such that $g, g' \in \mathcal{GH}_{\mathsf{M}}$. Suppose that g admits a Cauchy temporal function $t : \mathsf{M} \to \mathbb{R}$ whose spacelike Cauchy hypersurfaces are compact and are also g'-spacelike, then $g \simeq g'$ up to a change of the temporal orientation of g'.*

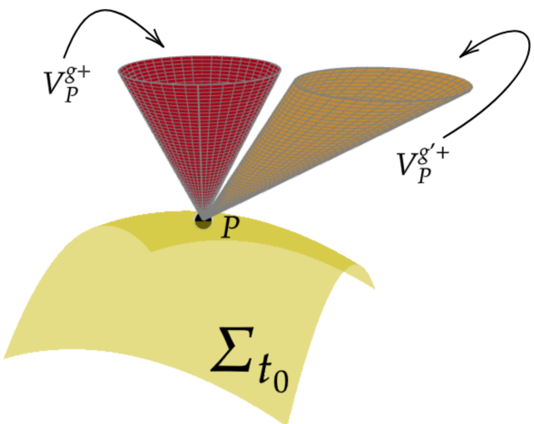


Figure 1.7: Over a point $P \in \Sigma_{t_0}$ which is g-Cauchy, we see a case in which the hypersurface is spacelike also for the metric g', nevertheless the lightcones do not intersect.

Proof. First of all, by defining the g-normal $n_g = \frac{\sharp_g dt}{\sqrt{-g^\sharp(dt,dt)}}$, any vector field X can be written as $X = X_n n_g + \pi_g(X)$, where $X_n = g(n_g, X)$ and $\pi_g(X) = Id - g(n_g, X) n_g$ projects on the Cauchy surface.
The metric tensor $g = -\frac{dt^2}{g^\sharp(dt,dt)} + h(\pi(\cdot), \pi(\cdot))$, following 1.1.12, under the action of a diffeomorphism ψ_g gets recast in the orthogonal form $g_{ort} = -\beta dt^2 + h_t$. This metric is obviously, by 1.2.5,

paracausally related to the conformal metric $g_c = -dt^2 + \frac{1}{\beta_t}h_t$, which is, in turn, paracausally related to a globally hyperbolic metric $\widetilde{g} = -dt^2 + h$ with h a complete Riemannian metric on the slice and if we choose coherently the time orientation, see corollary 1.3.5.
Then we look at the metric g' after the action of the isometric diffeomorphism ψ_g and define $\widetilde{g}' = \psi_g^* g'$. The proof ends if we are able to find a globally hyperbolic metric g'' such that $\widetilde{g} \simeq g'' \simeq \widetilde{g}'$.
If we choose a function $\alpha \in C^\infty(\mathbb{R}, (0, \infty))$, then, by lemma 1.1.18 the metric tensor $g_\alpha = -dt^2 + \alpha(t)h$ is globally hyperbolic and, by 1.3.4, paracausally related to $\widetilde{g}$. We want to tune the function α in order to have that the cones of g_α intersect the cones of $\widetilde{g}'$.
First we define pointwise n' the smooth vector field g'-normal to the Cauchy hypersurfaces of g of the foliation induced by the temporal function t and decompose it with respect to the splitting of the tangent space induced by the metric g_α through its normal n_α. We get $n' = Zn_\alpha + W$ where $Z = g_\alpha(n_\alpha, n')$ and $W = \pi_{g_\alpha}(n')$.
Since the Cauchy hypersurfaces of g and g_α are spacelike also for g', we have that $Z \neq 0$. The cones of the two metrics intersect if α is such that n' is g_α-timelike i.e. iff

$$||n'||_{g_\alpha} = -|Z|^2 + \alpha(t)||W||_h^2 < 0 \iff \frac{1}{\alpha(t)} > \frac{||W||_h^2}{|Z|^2}.$$

The manifold $\mathbb{R} \times \Sigma$ can be covered by the time-strips $\mathcal{TS}_n = \{[-n, n] \times \Sigma\}_{n \in \mathbb{N}}$ which are obviously compact since Σ is compact by hypothesis.
This means that for all $n \in \mathbb{N}$ the smooth function $f : \mathsf{M} \to \mathbb{R}^+$ defined by $f := \frac{||W||_h^2}{|Z|^2}$ attains a maximum M_n and a minimum m_n when restricted to the strip $\mathcal{TS}_n$. So we construct the required function $\frac{1}{\alpha(t)} : \mathbb{R} \to \mathbb{R}^+$ such that

$$\frac{1}{\alpha(t)} = M_n + 1 \quad t \in [-n-1, -n) \cup (n, n+1].$$

This function isn't even continuous, but it piecewise constant. The maximum has been increased by one to avoid the possibility that this function gets null: it could happen if the normal n' and n_α get aligned in the first time-strip and then depart.
The last thing to do is to smoothen the function $\alpha(t)$, something which can of course be done by standard gluing arguments.
Now that we know that the cones of g_α and $\widetilde{g}'$ intersect, if the temporal orientation of g' is such that $V^+_{g_\alpha} \cap V^+_{\widetilde{g}'} \neq \varnothing$ we define $g'' := g_\alpha \simeq \widetilde{g}'$ and the proof is concluded.
If $V^+_{g_\alpha} \cap V^+_{\widetilde{g}'} = \varnothing$ the metric g'', and therefore the metric $\widetilde{g}$, is paracausally related to $\widetilde{g}'$ with opposite time orientation. □

1.4 Conclusions

A lot of problems remain open about the paracausal relation, which definitely deserves to be studied separately from the problem of Møller operators. To conclude the chapter we list and briefly discuss some of the possible research lines that should be followed in the future.

Two equivalent characterizations of such a relation have been proved to be equivalent and some useful sufficient conditions and a necessary condition have been studied and are known. To summarize, it has been proved that:

(1) paracausally related metrics have diffeomorphic Cauchy surfaces;

(2) if $g, g' \in \mathcal{GH}_\mathsf{M}$ satisfy the condition that $\mathrm{supp}\,(g - g')$ is compact in M the two metrics are paracausally related, [19];

(3) globally hyperbolic metrics sharing Cauchy slicings are paracausally related;

(4) for Cauchy compact spacetimes if the Cauchy surfaces of g are spacelike for g', then they are paracausally related;

(5) all paracausal classes have a, of course non-unique, ultrastatic representative.

However in general almost nothing is known about the paracausal classes, i.e the quotient set $\mathcal{GH}_{\mathsf{M}}/\simeq$: the first thing to ask is if in 4 spacetime dimensions, in the case of $\mathbb{R}^4$, all globally hyperbolic metrics are paracausally related or how the classes are related to the topology of the spacetime manifold and of the (diffeomorphic) Cauchy surfaces associated to each metric in the chain.

Since we know that paracausally related metrics necessarily have diffeomorphic Cauchy surfaces the following proposition, using the interesting results described in [91], holds. In fact it is known that $\mathbb{R}^4$ admits infinite globally hyperbolic metrics whose Cauchy splittings are $\mathbb{R}^4 \cong \mathbb{R}\times\Sigma_1$ and $\mathbb{R}^4 \cong \mathbb{R} \times \Sigma_2$ with $\Sigma_1 \ncong \Sigma_2$:

Proposition 1.4.1. *$\mathbb{R}^4$ endowed with the standard smooth differentiable structure admits at least a paracausal class of globally hyperbolic metrics for each of the (uncountable) possible smooth 3-manifolds Σ such that $\mathbb{R}^4 \cong_{diff} \mathbb{R} \times \Sigma$.*

And such a result can be generalized for a general non compact smooth manifold as follows:

Proposition 1.4.2. *A smooth non-compact $d > 1$ dimensional manifold M, admitting Lorentzian metrics, admits at least a paracausal class of globally hyperbolic metrics for each of the possible smooth, non diffeomorphic $d-1$ dimensional manifolds Σ such that $\mathsf{M} \cong_{diff} \mathbb{R} \times \Sigma$.*

Outside of the real world 3 dimensional Cauchy surfaces more classes may arise whenever the differentiable structures compatible with the topology of Σ are more than one, up to diffeomorphisms.

Fortunately Cauchy-compact globally hyperbolic spacetime 4-manifolds would not suffer such pathologies, because if there exists a closed oriented smooth 3-manifold Σ such that $\mathsf{M} \cong_{diff} \mathbb{R}\times\Sigma$, then Σ is unique up to diffeomorphisms, [26].

However we still have no rigorous proofs regarding metrics that are *not* paracausally relates when the Cauchy surfaces are diffeomorphic, even thought in 1.3.3 we have conjectured that Cauchy compact metrics with equal cones, but opposite temporal orientation, should fall in different paracausal classes.

Conjecture 1.4.3. Let $(\mathbb{R} \times \Sigma, g)$ and $(\mathbb{R} \times \Sigma, g')$ spacetimes with $g, g' \in \mathcal{GH}_{\mathsf{M}}$ such that the hypersurface Σ is compact and Cauchy for both metrics. If $V^{\pm}g' = V_g^{\mp}$, then g and g' are not paracausally related.

The idea behind the conjecture is that g and g' in part (3) of the Example 1.3.3 have somehow 'different time-orientation'. Since the time-orientation depends on the metric on M, we have to provide criteria to translate the requirement that g and g' have in some sense a common 'future-direction'. Keeping in mind what said above, a conjecture which urges to be proved or disproved is the following one, maybe adding further hypothesis, for example concerning the dimension of the spacetime.

Conjecture 1.4.4. Let t and t' be Cauchy temporal functions for globally hyperbolic spacetimes (M, g) and (M, g'). Denote with $\langle\cdot,\cdot\rangle$ the natural paring between $\mathsf{T}^*\mathsf{M}$ and TM. Then

$$g \simeq g' \qquad \text{if and only if} \qquad \langle\partial_t, dt'\rangle > 0 \ \text{ and } \ \langle\partial_{t'}, dt\rangle > 0\,,$$

where ∂_t (resp. $\partial_{t'}$) is the dual of dt (resp. dt') with respect to g (resp. g').

Remark 1.4.5. The requirement $\langle\partial_t, dt'\rangle > 0$ implies that the integral curve $\gamma = \gamma(t)$ of ∂_t on (M, g') satisfies $t'(\gamma(t_2)) > t'(\gamma(t_1))$ if $t_2 > t_1$. This requirement is weaker than assuming the ∂_t is timelike and future-directed for g'. The reason why we also impose $\langle\partial_{t'}, dt\rangle > 0$ is that being paracausally related is an equivalence relation in $\mathcal{GH}_{\mathsf{M}}$.

Something interesting which has been observed (and is obvious to prove) is that paracausal chains are in one to one correspondence with specific curves joining the metrics in $\mathcal{GH}_{\mathsf{M}}$.

Proposition 1.4.6. *Let $g, g' \in \mathcal{GH}_{\mathsf{M}}$. Then $g \simeq g'$ with a chain $\{g_i\}_i = 1^N$ made of $N \in \mathbb{N}$ metrics, if and only if there exists a piecewise convex curve $\gamma : [0, N] \to \mathcal{GH}(\mathsf{M})$, $\gamma(0) = g$ and $\gamma(1) = g'$.*

The following question immediately arises: is this piece-wise convex curve continuous w.r.t some suitable topology? If that is the case one could prove that metrics such as the ones of the previous conjecture lie in different connected components on the space $\mathcal{GH}_{\mathsf{M}}$, then no paracausal chain joining them can exist. But of much more interest would be if all paracausal chains were necessary 'samplings' of a continuous curve joining the metrics in the space of globally hyperbolic metrics, seen as a subset of the space of all Lorentzian metrics.

Conjecture 1.4.7. There is a one-to-one correspondence between paracausal classes and connected components of $\mathcal{GH}_{\mathsf{M}} \subset \mathcal{M}_{\mathsf{M}}$ with respect to some suitable topology.

In such a case the existence of the Møller operators, that will be largely discussed in the next chapters, would depend on the topology of the space of globally hyperbolic metrics, whose structure is supposed to be related to the topology of the manifold M.
The possible topologies that may be considered are the Geroch interval topology, [65], which does not seem suitable because of its coarseness preventing the convex segments to be continuous, Whitney topologies and compact-open topology.

Chapter 2

The Møller map for Green hyperbolic operators

In this chapter we develop and apply to several examples a technique to compare solutions of Green hyperbolic differential operators, describing the propagation of free classical fields on curved spacetime, under finite global variations of the background metric tensor. Recently a great deal of progress has been made in this directions as well as in the comparison of the associated quantum field theories. More precisely, given a pair P and P' of Green hyperbolic differential operators on (possibly different) globally hyperbolic spacetimes (M, g) and (M, g'), a natural issue concerns the existence of a linear isomorphism $\mathsf{S} : \mathsf{Sol}_{\mathsf{P}} \to \mathsf{Sol}_{\mathsf{P}'}$ between the linear spaces of the solutions of the equations $\mathsf{P}\psi = 0$ and $\mathsf{P}'\psi' = 0$. Such an isomorphism, if it exists, is called a **Møller map**. These problems have been tackled in the past for special cases of metrics g, g' and several types of Green hyperbolic field operators which rule the dynamics of bosonic fields [27,35] or fermionic fields [37,90]. In *loc. cit.*, the pairs of Lorentzian metrics g, g' had to satisfy one of the following assumptions: they shared a common foliation of smooth spacelike Cauchy surfaces; they coincided outside a compact set.
In this work we exploit the notion of paracausal relation, described in detail in the previous chapter to show that, whenever $g \simeq g'$ a geometric Møller operator can be constructed. The procedure can be summarized as follows. The overall idea is inspired by the scattering theory in the special case of a pair of globally hyperbolic metrics g_0, g_1 over M such that the light cones of g_0 are included in the light cones of g_1 (this is the most elementary case of paracausal relation). We start with two "free theories", described by the space of solutions of Green hyperbolic operators P_0 and P_1 in corresponding spacetimes (M, g_0) and (M, g_1), respectively, and we intend to connect them through an "interaction spacetime" (M, g_χ) with a "temporally localized" interaction defined by interpolating the two metrics by means of a smoothing function χ. Here we need two Møller maps: Ω_+ connecting (M, g_0) and (M, g_χ) – which reduces to the identity in the past when χ is switched off – and a second Møller map connecting (M, g_χ) to (M, g_1) – which reduces to the identity in the future when χ constantly takes the value 1. The "S-matrix" given by the composition $\mathsf{S} := \Omega_-\Omega_+$ will be the Møller map connecting P_0 and P_1.

The above construction generalizes to the case of a pair of globally hyperbolic metrics g, g' on M which are paracausally related and this fact is denoted by $g \simeq g'$.

The chapter is structured as follows. In the very beginning we recap the basic properties of Green hyperbolic operators. In the first main section 2.1 a class of differential operators that we will analyse is discussed: normally hyperbolic operators. They generalize the d'Alembert wave operator, i.e the classical Klein-Gordon field, in a sense that will be immediately clarified; for sake of generality this operator will act on smooth sections of hermitian vector bundles with bundle metric that does not depend on the spacetime metric.

Later the problem of constructing the interpolating spacetime and the interpolating operator through convex combinations of differential operators is tackled and its Cauchy problem is studied.

For this reason we introduced the problem of the convex combination of globally hyperbolic metrics which led to the study of the paracausal relation, which was subject of the previous chapter.
Finally the Møller operator is introduced for the first time and its properties are discussed in detail, especially its adjoint and the feature that it intertwines the causal propagators of the compared theories, which will be crucial in the next chapter in order to construct Hadamard states. The definition of the adjoint operator $\mathsf{R}^{\dagger_{gg'}}$, mapping objects related to a spacetime to another spacetime, is completely new and all its useful properties are immediately investigated.

In the last main section 2.2 the Proca vector field is analyzed. Since it is a one form, the Proca field is a section of the cotangent bundle equipped with the dual of the spacetime metric, so it is not a hermitian bundle with a metric which does not depend on the spacetime. Moreover, in this case, the interpolating differential operator is not a convex combination of two Proca operators since the latter would fail to be Green hyperbolic in general. The Cauchy problem for this field is discussed in detail as one of a constrained normally hyperbolic PDE and, finally, the Møller operator an the definition of the adjoint are carefully modified to incorporate the change of the bundle metric.
We then conclude describing possible future research lines in such a field.

2.1 The normally hyperbolic Klein Gordon field

The main purpose of this section is to realize a geometric map to compare the space of solutions of *normally hyperbolic* operators defined on possibly different globally hyperbolic manifolds. Before starting to introduce our theory, we remind some general definitions and we fix the notation that will be used from now on. Let E be a vector bundle (always on $\mathbb{K}$ and of finite rank) over a spacetime (M, g), whose generic fiber (a $\mathbb{K}$ vector space isomorphic to a canonical fiber) is denoted by E_p for $p \in \mathsf{M}$. $\Gamma(\mathsf{E})$ is the $\mathbb{K}$-space of smooth sections of E. $\Gamma(\mathsf{E})$ has a number of useful subspaces we list below.

(i) $\Gamma_c(\mathsf{E}) \subset \Gamma(\mathsf{E})$ is the subspace of compactly supported smooth sections.

(ii) $\Gamma_{pc}(\mathsf{E})$ and $\Gamma_{fc}(\mathsf{E})$ denote the subspaces of $\Gamma(\mathsf{E})$ whose elements have respectively past compact support and future compact support.

(iii) If (M, g) is globally hyperbolic, $\Gamma_{sc}(\mathsf{E}) \subset \Gamma(\mathsf{E})$ is the subspace of **spatially compact** sections: the smooth sections whose support intersects every spacelike Cauchy hypersurface in a compact set.

These spaces are equipped with natural topologies as discussed in [4]. In case there are several metrics on a common spacetime M basis of E, the used metric g will be indicated as well, for instance $\Gamma^g_{pc}(\mathsf{E})$, if the nature of the space of sections depends on the chosen metric (this is not the case for $\Gamma_c(\mathsf{E})$).

A summary of the main results obtained in the case of normally hyperbolic operators is the following where also the special notion of adjoint operator $\mathsf{R}^{\dagger_{gg'}}$ is used.

Theorem 2 (Theorems 2.1.20, 2.1.21, and 2.1.27)**.** *Let E be a $\mathbb{K}$-vector bundle over the smooth manifold M with a non-degenerate, real or Hermitian depending on $\mathbb{K}$, fiber metric $\langle \cdot \mid \cdot \rangle$. Consider $g, g' \in \mathcal{GH}_M$ with respectively associated normally hyperbolic formally-selfadjoint operators N, N'. If the metrics are paracausally related $g \simeq g'$, then it is possible to define a (non-unique) $\mathbb{K}$-vector space isomorphism $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, called* **Møller operator** *of g, g' (with this order), such that the following facts are true.*

(1) The restrictions to the relevant subspaces of $\Gamma(\mathsf{E})$ respectively define symplectic Møller maps S^0 (see Definition 2.1.22) which preserve the symplectic forms σ^{N}_g, $\sigma^{\mathsf{N}'}_{g'}$ defined as in Equation (2.1.7)*, namely*

$$\sigma^{\mathsf{N}'}_{g'}(\mathsf{S}^0\Psi, \mathsf{S}^0\Phi) = \sigma^{\mathsf{N}}_g(\Psi, \Phi) \quad \textit{for every } \Psi, \Phi \in \mathsf{Ker}^g_{sc}(\mathsf{N}).$$

(2) *The causal propagators* $\mathsf{G}_{\mathsf{N}'}$ *and* G_{N}*, respectively of* N' *and* N*, satisfy* $\mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{N}'}$*, where* $\mathsf{R}^{\dagger_{gg'}}$ *is the adjoint of the Møller operator (see Definition 2.1.9).*

(3) *By denoting* c' *the smooth function such that* $\mathrm{vol}_{g'} = c' \,\mathrm{vol}_g$*, we have* $c'\mathsf{N}'\mathsf{R} = \mathsf{N}$ *.*

(4) *It holds* $\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'|_{\Gamma_c(\mathsf{E})} = \mathsf{N}|_{\Gamma_c(\mathsf{E})}$ *.*

2.1.1 Green hyperbolic operators

The reason why we focused on globally hyperbolic spacetimes comes from their very first reason of existence, [81]: linear partial differential operators defining linear field theory have a well posed Cauchy problem thereon.
As a consequence of the well-posedness of the Cauchy problem with "finite propagation speed of the solutions" stated in (2.1.2), one may establish the existence of Green operators. We review now briefly what Green hyperbolic operators are and the most important results we will use throughout the whole chapter.

Definition 2.1.1. Let E be a real or complex vector bundle over the spacetime (M, g). A linear differential operator $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is called **Green hyperbolic** if

(1) there exist linear maps, dubbed **advanced Green operator** $\mathsf{G}^+ : \Gamma_{pc}(\mathsf{E}) \to \Gamma(\mathsf{E})$ and **retarded Green operator** $\mathsf{G}^- : \Gamma_{fc}(\mathsf{E}) \to \Gamma(\mathsf{E})$, satisfying

- (i.a) $\mathsf{G}^+ \circ \mathsf{P}\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^+\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$,
- (ii.a) $\mathrm{supp}\,(\mathsf{G}^+\mathfrak{f}) \subset J^+(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$;
- (i.b) $\mathsf{G}^- \circ \mathsf{P}\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^-\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$,
- (ii.b) $\mathrm{supp}\,(\mathsf{G}^-\mathfrak{f}) \subset J^-(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$;

(2) the **formally dual operator** P^* admits advanced and retarded Green operators as well.

For sake of completeness, let us recall that the formally dual operator $\mathsf{P}^* : \Gamma(\mathsf{E}^*) \to \Gamma(\mathsf{E}^*)$ is the unique linear differential operator acting on the smooth sections of the dual bundle E^* satisfying

$$\int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{P}\mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathsf{P}^*\mathfrak{f}', \mathfrak{f} \rangle \,\mathrm{vol}_g$$

for every $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ and $\mathfrak{f}' \in \Gamma_c(\mathsf{E}^*)$ (which is equivalent to saying $\mathfrak{f} \in \Gamma(\mathsf{E})$ and $\mathfrak{f}' \in \Gamma(\mathsf{E}^*)$ such that $\mathrm{supp}(\mathfrak{f}) \cap \mathrm{supp}(\mathfrak{f})'$ is compact), vol_g being the volume form induced by g on M.

Remarks 2.1.2.

(1) The Green operators we define below are the extensions to $\Gamma_{pc/fc}(\mathsf{E})$ of the analogues defined in [3] and indicated by $\overline{G}_\pm$ therein.

(2) It is possible to prove that the Green operators are unique for a Green hyperbolic operator (cf. [3, Corollary 3.12]). Furthermore as a consequence of [3, Lemma 3.21], it arises that if $\mathfrak{f}' \in \Gamma_c(\mathsf{E}^*)$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$ or $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$ respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P*}\mathfrak{f}', \mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^+_P\mathfrak{f} \rangle \,\mathrm{vol}_g \,, \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P*}\mathfrak{f}', \mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^-_P\mathfrak{f} \rangle \,\mathrm{vol}_g \,, \tag{2.1.1}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P*}$ indicate the Green operators of P^*.

Proposition 2.1.3. *If* P *is a Green hyperbolic operator on a vector bundle* E *over the globally hyperbolic spacetime* (M, g) *and* $\rho : \mathsf{M} \to (0, +\infty)$ *is smooth, then* $\rho\mathsf{P}$ *is Green hyperbolic as well and* $\mathsf{G}^\pm_{\rho\mathsf{P}} = \mathsf{G}^\pm_{\mathsf{P}}\rho^{-1}$*.*

Proof. The thesis immediately follows form the fact that $\mathsf{G}^{\pm}_{\mathsf{P}}\rho^{-1}$ and $\rho^{-1}\mathsf{G}^{\pm}_{\mathsf{P}^*}$ satisfy the properties of the Green operators for $\rho\mathsf{P}$ and $(\rho\mathsf{P})^* = \mathsf{P}^*\rho$ respectively. □

Given a Green hyperbolic operator with Green operators $\mathsf{G}^{\pm}$, a relevant operator constructed out of $\mathsf{G}^{\pm}$ is the so-called **causal propagator**,

$$\mathsf{G} := \mathsf{G}^{+}|_{\Gamma_c(\mathsf{E})} - \mathsf{G}^{-}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})\,.$$

It satisfies remarkable properties we are going to discuss (see *e.g.* [4, Theorem 3.6.21]).

Theorem 2.1.4. *Let* G *be the causal propagator of a Green hyperbolic differential operator* $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *on the vector bundle* E *over a globally hyperbolic spacetime* (M, g)*. The following sequence is exact*

$$\{0\} \to \Gamma_c(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_c(\mathsf{E}) \xrightarrow{\mathsf{G}} \Gamma_{sc}(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_{sc}(\mathsf{E}) \to \{0\}\,.$$

Proof. Injectivity of $\Gamma_c(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_c(\mathsf{E})$ easily arises from $\mathsf{G}^{\pm}P\mathfrak{f} = \mathfrak{f}$. Let us pass to the other parts of the sequence. First of all notice that $\mathsf{G}^{\pm}(\Gamma_c(\mathsf{E})) \subset \Gamma_{sc}(\mathsf{E})$ since $\mathrm{supp}(\mathsf{G}^{\pm}(\mathfrak{f})) \subset J_{\pm}(\mathrm{supp}(\mathfrak{f}))$ and the first assertion then follows form known facts of globally hyperbolic spacetimes. Let us prove that $\Gamma_c(\mathsf{E}) \xrightarrow{\mathsf{G}} \Gamma_{sc}(\mathsf{E})$ is surjective when the image is restricted to the kernel of $\Gamma_{sc}(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_{sc}(\mathsf{E})$. Suppose that $\mathsf{P}\Psi = 0$ for $\Psi \in \Gamma_{sc}(\mathsf{E})$. If t is a smooth Cauchy time function of (M, g) and $\chi : \mathsf{M} \to [0, 1]$ is smooth, vanishes for $t < t_0$ and is constantly 1 for $t > t_1$, then

$$\mathfrak{f}_{\Psi} := \mathsf{P}(\chi\Psi) \in \Gamma_c(\mathsf{E})$$

is such that $\Psi = \mathsf{G}\mathfrak{f}_{\Psi}$. Notice that $\mathrm{supp}(\mathfrak{f}_{\Psi})$ is included between the Cauchy hypersurfaces $t^{-1}(t_0)$ and $t^{-1}(t_1)$. Indeed,

$$\mathsf{G}\mathfrak{f}_{\Psi} = \mathsf{G}^{+}\mathsf{P}(\chi\Psi) - \mathsf{G}^{-}\mathsf{P}(\chi\Psi) = \mathsf{G}^{+}\mathsf{P}(\chi\Psi) + \mathsf{G}^{-}\mathsf{P}((1-\chi)\Psi) = \chi\Psi + (1-\chi)\Psi = \Psi\,.$$

It is obvious that that $\mathfrak{f}_{\Psi}$ can be changed by adding a section of the form $\mathsf{P}\mathfrak{h}$ with $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ preserving the property $\mathsf{G}\mathfrak{f}_{\Psi} = \Psi$. This exhaust the kernel of $\Gamma_c(\mathsf{E}) \xrightarrow{\mathsf{G}} \Gamma_{sc}(\mathsf{E})$ as asserted in the thesis. Indeed, if $\mathsf{G}\mathfrak{f} = 0$, then $\mathsf{G}^{+}\mathfrak{f} = \mathsf{G}^{-}\mathfrak{f}$. From the properties of the supports of $\mathsf{G}^{\pm}\mathfrak{f}$, we conclude that $\mathsf{G}^{\pm}\mathfrak{f} = \mathfrak{h}_{\pm} \in \Gamma_c(\mathsf{E}) \subset \Gamma_{pc}(\mathsf{E}) \cap \Gamma_{fc}(\mathsf{E})$. Hence $\mathfrak{f} = \mathsf{P}\mathsf{G}^{\pm}\mathfrak{f} = \mathsf{P}\mathfrak{h}_{\pm}$. To conclude, we prove that $\Gamma_{sc}(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_{sc}(\mathsf{E})$ is surjective. If $\mathfrak{f} \in \Gamma_{sc}(\mathsf{E})$, with χ as above,

$$\mathfrak{f} = \chi\mathfrak{f} + (1-\chi)\mathfrak{f} = \mathsf{P}\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{P}\mathsf{G}^{-}((1-\chi)\mathfrak{f}) = \mathsf{P}[\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{G}^{-}((1-\chi)\mathfrak{f})]$$

and $\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{G}^{-}((1-\chi)\mathfrak{f}) \in \Gamma_{sc}(\mathsf{E})$. □

2.1.2 Normally hyperbolic operators

Let $g \in \mathcal{M}_{\mathsf{M}}$ and $g^{\sharp}$ be the induced metric on the cotangent bundle. If (M, g) is globally hyperbolic, by fixing a Cauchy temporal function $t : \mathsf{M} \to \mathbb{R}$ such that $g = -\beta^2 dt \otimes dt + h_t$, we have

$$g^{\sharp} = -\beta^{-2}\partial_t \otimes \partial_t + h_t^{\sharp}\,.$$

Definition 2.1.5. A linear second order differential operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is **normally hyperbolic** if its principal symbol σ_{N} satisfies

$$\sigma_{\mathsf{N}}(\xi) = -g^{\sharp}(\xi, \xi)\,\mathrm{Id}_{\mathsf{E}}$$

for all $\xi \in \mathsf{T}^*\mathsf{M}$, where Id_{E} is the identity automorphism of E.

Referring to a foliation of (M, g) as in Definition 1.1.14, in local coordinates (t, x) on M adapted to the foliation so that $x = (x_1, \ldots, x_n)$ are local coordinates on Σ_t, and using a local trivialization of E, any normally hyperbolic operator N in a point $p \in \mathsf{M}$ reads as

$$\mathsf{N} = \frac{1}{\beta^2}\partial_t^2 - \sum_{i,j=1}^{n} h_t^{\sharp ij}\partial_{x_i}\partial_{x_j} + A^0(t,x)\partial_t + \sum_{j=1} A^j(t,x)\partial_{x_j} + B(t,x)$$

where A_0, A_j and B are linear maps $\mathsf{E}_{(t,x)} \to \mathsf{E}_{(t,x)}$ depending smoothly on (t, x).

Examples 2.1.6. In the class of normally hyperbolic operators we can find many operators of interest in quantum field theory:

- Let E be the trivial real bundle, i.e. $\mathsf{E} = \mathsf{M} \times \mathbb{R}$, so that the space of smooth sections of E can be identified with the ring of smooth functions on M. The Klein-Gordon operator $\mathsf{N} = \Box + m^2$ is normally hyperbolic, where $\Box$ is the d'Alembert operator and m is a mass-term.
- Let now $\mathsf{E} = \Lambda^k \mathsf{T}^*\mathsf{M}$ be the bundle of k-forms and d (resp δ) the exterior derivative (resp. the codifferential). The operator $\mathsf{N} := d\delta + \delta d + m^2$ is normally hyperbolic and it is used to describe the dynamics of Proca fields, for further details we refer to [6, Example 2.17].
- Let SM be a spinor bundle over a globally hyperbolic spin manifold M_g and let ∇ be a spin connection. By denoting with $\gamma : \mathsf{TM} \to \mathrm{End}(\mathsf{SM})$ the Clifford multiplication, the classical Dirac operator reads as $\mathsf{D} = \gamma \circ \nabla : \Gamma(\mathsf{SM}) \to \Gamma(\mathsf{SM})$, see [37, 38, 68] for further details. By Lichnerowicz-Weitzenböck formula we get the spinorial wave operator $\mathsf{N} = \mathsf{D}^2 = \nabla^\dagger\nabla + \frac{1}{4}\mathrm{Scal}_g$, where Scal_g is the scalar curvature.

It is well-known that, once that the Cauchy data are suitably assigned, the Cauchy problem for N turns out to be well-posed, see *e.g.* [4, 66].

Theorem 2.1.7. *Let E be a vector bundle (of finite rank) over a globally hyperbolic manifold (M, g), let N be a normally hyperbolic operator with a N-compatible connection ∇ (see (2.1.3) below) and Σ_0 a (smooth) spacelike Cauchy hypersurface of (M, g). Then the Cauchy problem for N is well-posed, i.e. for any $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, $\mathfrak{h}_1, \mathfrak{h}_2 \in \Gamma_c(\mathsf{E}|_{\Sigma_0})$ there exists a unique solution $\Psi \in \Gamma_{sc}(\mathsf{E})$ to the initial value problem*

$$\begin{cases} \mathsf{N}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h}_1 \\ (\nabla_{\mathfrak{n}}\Psi)|_{\Sigma_0} = \mathfrak{h}_2 \end{cases}$$

being $\mathfrak{n}$ the future directed timelike unit normal field along Σ_0, and it depends continuously on the data $(\mathfrak{f}, \mathfrak{h}_1, \mathfrak{h}_2)$ w.r.to the standard topologies of smooth sections and satisfies

$$supp(\Psi) \subset J(supp(\mathfrak{f})) \cup J(supp(\mathfrak{h}_1)) \cup J(supp(\mathfrak{h}_2))\,. \tag{2.1.2}$$

2.1.3 Formal selfadjointness and the symplectic form

Let E be a $\mathbb{K}$-vector bundle on a globally hyperbolic spacetime (M, g). As shown in [8, Lemma 1.5.5], for any normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ there exists a unique covariant derivative ∇ on E such that

$$\mathsf{N} = -\mathrm{tr}_g(\widetilde{\nabla}\nabla) + c \tag{2.1.3}$$

for some some zero-order differential operator $c : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$. In the formula above the left $\widetilde{\nabla}$ is actually the connection induced on $\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}$ by the *Levi-Civita connection* associated to g and the original connection ∇ given on E Adopting the terminology of [4], we shall refer to ∇ as the **N-compatible connection** on E.

We stress that, if we suppose that E is equipped with a smooth assignment of Hermitian fiber metrics

$$\langle\cdot\,|\,\cdot\rangle_p : \mathsf{E}_p \times \mathsf{E}_p \to \mathbb{K}\,.$$

then the above ∇ is g-metric but not necessarily metric with respect to $\langle\cdot\,|\,\cdot\rangle$.

The physical relevance of the fiber metric is that it permits to equip $\mathsf{Ker}_{sc}(\mathsf{N})$ with a symplectic form with important properties in the formulation of QFT in curved spacetime. This symplectic form can be derived using the Green identity for a normally hyperbolic operator N and its formal adjoint. As anticipated, we have the following result:

Proposition 2.1.8 ([4, Corollary 3.4.3]). *A normally hyperbolic operator* N *on a vector bundle* E *(of finite rank) on a globally hyperbolic manifold* (M, g) *is Green hyperbolic.*

Definition 2.1.9. The **formal adjoint** of a differential operator $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is the unique differential operator $\mathsf{P}^\dagger : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ satisfying

$$\int_{\mathsf{M}} \langle \mathfrak{f}' \,|\, \mathsf{P}\mathfrak{f}\rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathsf{P}^\dagger \mathfrak{f}' \,|\, \mathfrak{f}\rangle \,\mathrm{vol}_g$$

for every $\mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathsf{E})$ (which is equivalent to saying $\mathfrak{f}, \mathfrak{f}' \in \Gamma(\mathsf{E})$ such that $\mathrm{supp}(\mathfrak{f}) \cap \mathrm{supp}(\mathfrak{f})'$ is compact). If $\mathsf{P} = \mathsf{P}^\dagger$ then N is said to be (formally) **selfadjoint**.

Remark 2.1.10. If $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is normally hyperbolic on the bundle E over (M, g), equipped with a non-degenerate, Hermitian fiber metric $\langle\cdot\,|\,\cdot\rangle$, P is Green hyperbolic as said above. In this case $\mathsf{P}^\dagger$ has the same principal symbol as P and thus it is Green hyperbolic as well. Taking advantage of the natural (antilinear if $\mathbb{K} = \mathbb{C}$) isomorphism $\Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}^*)$ induced by $\langle\cdot\,|\,\cdot\rangle$ and (2.2.33), it is not difficult to prove that, if $\mathfrak{f}' \in \Gamma_c(\mathsf{E})$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$ or $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$ respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P^\dagger} \mathfrak{f}' \,|\, \mathfrak{f}\rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}' \,|\, \mathsf{G}^+_P \mathfrak{f}\rangle \,\mathrm{vol}_g\,, \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P^\dagger} \mathfrak{f}' \,|\, \mathfrak{f}\rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}' \,|\, \mathsf{G}^-_P \mathfrak{f}\rangle \,\mathrm{vol}_g\,. \tag{2.1.4}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P^\dagger}$ indicate the Green operators of $\mathsf{P}^\dagger$.

Let us pass to introduce a Green-like identity where we explicitly exploit the N-compatible connection ∇.

Lemma 2.1.11 (Green identity). *Let* E *be a non-degenerate, Hermitian* $\mathbb{K}$ *vector bundle over a* $(n+1)$*-dimensional spacetime* (M, g) *and denote the fiber metric* $\langle\cdot\,|\,\cdot\rangle$*. Moreover, let* $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be a normally hyperbolic operator with* N*-compatible connection* ∇*. Let* $\mathsf{M}_0 \subset \mathsf{M}$ *be a submanifold with continuous piecewise smooth boundary. Then for every* $\Phi, \Psi \in \Gamma_c(\mathsf{E}))$

$$\int_{\mathsf{M}_0} \left(\langle \Psi \,|\, \mathsf{N}\Phi\rangle - \langle \mathsf{N}\Psi \,|\, \Phi\rangle\right) \mathrm{vol}\,_g = \int_{\partial\mathsf{M}_0} \Xi^{\mathsf{N}}_{\partial\mathsf{M}_0}(\Psi, \Phi)\,, \tag{2.1.5}$$

where $\Xi^{\mathsf{N}}_{\partial\mathsf{M}_0}$ *is the n-form in* $\partial\mathsf{M}_0$

$$\Xi^{\mathsf{N}}_{\partial\mathsf{M}_0}(\Psi, \Phi) := \imath^*_{\partial\mathsf{M}_0}\left[\sharp\Big(\langle \Psi \,|\, \nabla\Phi\rangle - \langle \nabla\Psi \,|\, \Phi\rangle\Big) \lrcorner \,\mathrm{vol}\,_g\right]$$

$\imath_{\partial\mathsf{M}_0} : \partial\mathsf{M}_0 \to \mathsf{M}$ *being the inclusion embedding. If the normal vectors to* $\partial\mathsf{M}_0$ *are either spacelike or timelike (up to zero-measure sets), then*

$$\Xi^{\mathsf{N}}_{\partial\mathsf{M}_0}(\Psi, \Phi) = \Big(\langle \Psi \,|\, \nabla_{\mathtt{n}}\Phi\rangle - \langle \nabla_{\mathtt{n}}\Psi \,|\, \Phi\rangle\Big) \mathrm{vol}\,_{\partial\mathsf{M}_0} \tag{2.1.6}$$

where $\mathtt{n}$ *is the outward unit normal vector to* $\partial\mathsf{M}_0$ *and* $\mathrm{vol}\,_{\partial\mathsf{M}_0} = \mathtt{n} \lrcorner \,\mathrm{vol}\,_g$ *is the volume form of* $\partial\mathsf{M}_0$ *induced by* g*.*

Proof. Consider the n-form in M

$$Z := \sharp\Big(\langle \Psi \,|\, \nabla\Phi\rangle - \langle \nabla\Psi \,|\, \Phi\rangle\Big) \lrcorner \operatorname{vol}_g .$$

If the normal vectors to $\partial\mathsf{M}_0$ are either spacelike or timelike, some computations with the exterior differential of forms yields (2.1.6). In all cases it is easy to prove that

$$dZ = \Big(\langle \Psi \,|\, g^{ij}\nabla_i\nabla_j\Phi\rangle - \langle g^{ij}\nabla_i\nabla_j\Psi \,|\, \Phi\rangle\Big)\operatorname{vol}_g = \Big(\langle \Psi \,|\, \mathsf{N}\Phi\rangle - \langle \mathsf{N}\Psi \,|\, \Phi\rangle\Big)\operatorname{vol}_g .$$

At this juncture, Stokes' theorem for $(n+1)$-forms,

$$\int_{\partial\mathsf{M}_0} Z = \int_{\mathsf{M}_0} dZ ,$$

produces (2.1.5). □

We have the following crucial result when applying the previous lemma to the theory on globally hyperbolic spacetimes.

Proposition 2.1.12. *Let $\Sigma \subset \mathsf{M}$ be a smooth spacelike Cauchy hypersurface with its future-oriented unit normal vector field $\mathtt{n}$ in the globally hyperbolic spacetime (M, g) and its induced volume element $\operatorname{vol}_\Sigma$. Furthermore, let N be a formally self-adjoint normally hyperbolic operator. Then*

$$\sigma^{\mathsf{N}}_{(\mathsf{M},g)} : \mathsf{Ker}_{sc}(\mathsf{N}) \times \mathsf{Ker}_{sc}(\mathsf{N}) \to \mathbb{C} \quad \textit{such that} \quad \sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi, \Phi) = i\int_\Sigma \Xi^{\mathsf{N}}_\Sigma(\Psi, \Phi) \tag{2.1.7}$$

where Ξ^{N}_Σ is defined in Equation (2.1.6), yields a non-degenerate symplectic form (Hermitian if $\mathbb{K} = \mathbb{C}$) which does not depend on the choice of Σ.

Proof. First note that, referring to a spacelike Cauchy hypersurface Σ, $\operatorname{supp}(\Psi) \cap \Sigma$ is compact since $\operatorname{supp}(\Psi)$ is spacelike compact, so that the integral is well-defined. The fact that σ_{N} is not degenerate can be proved as follows. If $\sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi, \Phi) = 0$ for all $\Phi \in \Gamma_{sc}(\mathsf{E})$, from the definition of σ_{N} and non-degenerateness of $\langle\cdot\,|\,\cdot\rangle_p$ (passing to local trivializations referred to local coordinates on Σ re-writing $\langle\cdot\,|\,\cdot\rangle_p$ in terms of the pairing with E^*_p), we have that the Cauchy data of Ψ vanishes on every local chart on Σ and thus they vanish on Σ. According to Theorem 2.1.7, $\Psi = 0$. The other entry can be worked out similarly.
Let $\Psi, \Phi \in \mathsf{Ker}_{sc}(\mathsf{N})$ and Σ'_t and $\Sigma_{t''}$ be a pair of smooth spacelike Cauchy hypersurfaces associated to a smooth time Cauchy function t with $t'' > t'$. Let us focus on the submanifold with boundary $\mathsf{M}_0 = t^{-1}((t', t''))$. Its boundary is $\partial\mathsf{M}_0 = \Sigma_{t'} \cup \Sigma_{t''}$. The supports of Ψ and Φ between the two Cauchy surfaces are included in the causal future of the compact supports of the Cauchy data on $\Sigma_{t'}$ of Ψ and Φ respectively, and these portions of causal sets are compact as (M, g) is globally hyperbolic (see *e.g.* [4, Proposition 1.2.56]). We end up with a pair of functions in $\Gamma_c(\mathsf{E})$ and we can apply the Green identity (see Lemma 2.1.11) to M_0. Using a smoothly vanishing function as a factor, we can make smoothly vanishing Ψ and Φ before $\Sigma_{t'}$ and after $\Sigma_{t''}$ without touching them between the two Cauchy surfaces. As a matter of fact the resulting sections constructed out Ψ and Φ by this way are smooth, compactly supported and coincide with Ψ and Φ between the two Cauchy surfaces. We can therefore apply Lemma 2.1.11, obtaining

$$\int_{\mathsf{M}_0} (\langle \Psi \,|\, \mathsf{N}\Phi\rangle - \langle \mathsf{N}\Psi \,|\, \Phi\rangle)\operatorname{vol}_g = \int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} - \int_\Sigma \Xi^{\mathsf{N}}_\Sigma .$$

Since N is assumed to be self-adjoint, $\langle \Psi \,|\, \mathsf{N}\Phi\rangle - \langle \mathsf{N}\Psi \,|\, \Phi\rangle = \langle \Psi \,|\, \mathsf{N}\Phi\rangle - \langle \Psi \,|\, \mathsf{N}^\dagger\Phi\rangle = 0$. Therefore we can conclude that $\int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} = \int_\Sigma \Xi^{\mathsf{N}}_\Sigma$. Finally consider the case of two spacelike Cauchy functions Σ and Σ' belonging to different foliations induced by different smooth Cauchy time functions (notice that a spacelike Cauchy hypersurface always belong to a foliation generated by a suitable smooth Cauchy time (actually temporal) function for Theorem 1.1.12). We sketch a proof of the identity

$$\int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} = \int_{\Sigma} \Xi^{\mathsf{N}}_{\Sigma} .$$

Let $K \subset \Sigma$ a compact set including the Cauchy data of Ψ and Φ. If t is the smooth Cauchy time function such that $\Sigma_{t_1} = \Sigma'$, let $T = \max_K t$. If $t_1 < T$ we can always take $t_2 > T$ and to consider the symplectic form evaluated on Σ_{t_2}. In view of the previous part of our proof the symplectic form on Σ_{t_1} and Σ_{t_2} coincide, so that our thesis can be re-written

$$\int_{\Sigma_2} \Xi^{\mathsf{N}}_{\Sigma'} = \int_{\Sigma} \Xi^{\mathsf{N}}_{\Sigma} .$$

As $t_2 > \max_K t$, we conclude that Σ_{t_2} does not intersect Σ in the set K. Therefore we can define the solid set L_K made of the portion of $J_+(K)$ between Σ and Σ_{t_2}. L is compact (see *e.g.* [4, Proposition 1.2.56]) and is a "truncated cone" whose "lateral surface" is part of the boundary of $J_+(K)$ and whose "non-parallel bases" are parts of Σ_2 and Σ. We can include L in the interior of a larger manifold with boundary M_0 whose part of the boundary are portions of Σ and Σ_{t_2} including the support of the Cauchy data of Ψ and Φ. Notice that M_0 includes the supports of Ψ and Φ between the two Cauchy surfaces according to Theorem 2.1.7 and these supports do not touch the "lateral surface" of M_0. We can now apply the Green identity 2.1.11 to M_0 proving the thesis. □

There is a nice interplay of the causal propagator G of $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ as above and the symplectic form $\sigma^{\mathsf{N}}_{(\mathsf{M},g)}$.

Proposition 2.1.13. *With the same hypotheses as of Proposition 2.1.12, if* $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$ *and* $\Psi_{\mathfrak{f}} := \mathsf{G}\mathfrak{f}$, $\Psi_{\mathfrak{h}} := \mathsf{G}\mathfrak{h}$, *it holds*

$$\sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi_{\mathfrak{f}}, \Psi_{\mathfrak{h}}) = \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{G}\mathfrak{h} \rangle \operatorname{vol}_g .$$

Proof. If $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$, consider a smooth Cauchy time function t and fix $t_0 < t_1$ such that the supports of $\mathfrak{f}$ and $\mathfrak{h}$ are included in the interior of the submanifold with boundary M_0 contained between the spacelike Cauchy hypersurfaces $\Sigma_{t_0} := t^{-1}(t_0)$ and $\Sigma_{t_1} := t^{-1}(t_1)$. It holds

$$\int_{\mathsf{M}} \langle \Psi_{\mathfrak{f}} \,|\, \mathfrak{h} \rangle \operatorname{vol}_g = \int_{\mathsf{M}_0} \langle \Psi_{\mathfrak{f}} \,|\, \mathfrak{h} \rangle \operatorname{vol}_g = \int_{\mathsf{M}_0} \langle \Psi_{\mathfrak{f}} \,|\, \mathsf{N}\mathsf{G}^+\mathfrak{h} \rangle \operatorname{vol}_g$$

Since $\mathsf{N}\Psi_{\mathfrak{f}} = 0$, we have found that

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \operatorname{vol}_g = \int_{\mathsf{M}_0} \left(\langle \Psi_{\mathfrak{f}} \,|\, \mathsf{N}\mathsf{G}^+\mathfrak{h} \rangle - \langle \mathsf{N}\Psi_{\mathfrak{f}} \,|\, \mathsf{G}^+\Psi_{\mathfrak{h}} \rangle \right) \operatorname{vol}_g .$$

Applying Lemma 2.1.11, we find

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \operatorname{vol}_g = - \int_{\partial \mathsf{M}_0} \Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}(\Psi_{\mathfrak{f}}, \mathsf{G}^+\mathfrak{h}) = \int_{\Sigma_{t_1}} \Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}(\Psi_{\mathfrak{f}}, \mathsf{G}^+\mathfrak{h}) ,$$

where we noticed that $\mathsf{G}^+\mathfrak{h}$ vanishes on the remaining part of the boundary Σ_{t_0}. On the other hand, we can replace $\mathsf{G}^+\mathfrak{h}$ for $\mathsf{G}^+\mathfrak{h} - \mathsf{G}^-\mathfrak{h} = \mathsf{G}\mathfrak{h}$ in the last integral, since $\mathsf{G}^-\mathfrak{h}$ gives no contribution to the integral on Σ_{t_1}. In summary,

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{G}\mathfrak{h} \rangle \operatorname{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \operatorname{vol}_g = \int_{\Sigma_{t_1}} \Xi^{\mathsf{N}}_{\Sigma_{t_1}}(\Psi_{\mathfrak{f}}, \mathsf{G}\mathfrak{h}) = \sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi_{\mathfrak{f}}, \Psi_{\mathfrak{h}}) .$$

and this is the thesis. □

2.1.4 Convex combinations of normally hyperbolic operators

Let now N_0, N_1 be normally hyperbolic operators with respect different Lorentzian metric g_0 and g_1 (the former time-orientable and the latter globally hyperbolic) on the same manifold M and assume that they are acting on the smooth sections of the same vector bundle E. It turns out, that a positive (and convex) combination $(1-\chi)\mathsf{N}_0 + \chi\mathsf{N}_1$ is also (a) normally hyperbolic with respect to the naturally associated metric g_χ – the unique Lorentzian metric in TM whose associated metric in $\mathsf{T^*M}$ is $(1-\chi)g_0^\sharp + \chi g_1^\sharp$ according to Theorem 1.2.7 – and (b) Green hyperbolic with respect to g_1, everything provided that $g_0 \preceq g_1$. This is the main result of this section.

Theorem 2.1.14. *Let E be a $\mathbb{K}$-vector bundle over a smooth manifold M, let be $g_0, g_1 \in \mathcal{GM}_{\mathsf{M}}$ with $g_0 \preceq g_1$, and let $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be normally hyperbolic operator with respect to g_0 and g_1 respectively. If $\chi \in C^\infty(\mathsf{M}, [0,1])$, define g_χ as the unique Lorentzian metric whose associated metric in $\mathsf{T^*M}$ is $(1-\chi)g_0^\sharp + \chi g_1^\sharp$ according to Theorem 1.2.7. Then the second order differential operator defined by*

$$\mathsf{N}_\chi := (1-\chi)\mathsf{N}_0 + \chi\mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.8}$$

satisfies the following properties:

(1) It is normally and Green hyperbolic over (M, g_χ);

(2) It is Green hyperbolic over (M, g_1) and, with obvious notation,

$$\Gamma^{g_1}_{pc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{pc}(\mathsf{E})\,, \quad \Gamma^{g_1}_{fc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{fc}(\mathsf{E})\,,$$

$$\mathsf{G}^{g_1+}_{\mathsf{N}_\chi} = \mathsf{G}^{g_\chi+}_{\mathsf{N}_\chi}|_{\Gamma^{g_1}_{pc}(\mathsf{E})}\,, \quad \mathsf{G}^{g_1-}_{\mathsf{N}_\chi} = \mathsf{G}^{g_\chi-}_{\mathsf{N}_\chi}|_{\Gamma^{g_1}_{fc}(\mathsf{E})}\,.$$

In particular, (2) is true for N_0 by choosing $\chi = 0$.

Proof. (1) Since N_0 is a normally hyperbolic operator for (M, g_0) and N_1 is a normally hyperbolic operator for (M, g_1), by linearity

$$\sigma_2(\mathsf{N}_\chi, \xi) = (1-\chi)\sigma_2(\mathsf{N}_0, \xi) + \chi\sigma_2(\mathsf{N}_1, \xi).$$

In particular, we have that N_χ is normally hyperbolic with respect to g_χ:

$$\sigma_2(\mathsf{N}_\chi, \xi) = -(1-\chi)g_0^\sharp(\xi,\xi)\mathrm{Id}_{\mathsf{E}} - \chi g_1^\sharp(\xi,\xi)\mathrm{Id}_{\mathsf{E}} = -g_\chi^\sharp(\xi,\xi)\mathrm{Id}_{\mathsf{E}}\,.$$

By Theorem 1.2.7, the metric g_χ is globally hyperbolic and, on account of Proposition 2.1.8 N_χ is Green-hyperbolic over (M, g_χ).
Regarding (2), and referring to the existence of Green operators of N_χ in (M, g_1) we can proceed as follows. Observe that, since $g_\chi \preceq g_1$, we have $J^{g_\chi}_\pm(A) \subset J^{g_1}_\pm(A)$ and, with obvious notation, $\Gamma^{g_1}_{pc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{pc}(\mathsf{E})$ together with $\Gamma^{g_1}_{fc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{fc}(\mathsf{E})$, in view of (iii) (2) Lemma 1.2.3. As a consequence, the Green operators of N_χ with respect to (M, g_χ) are also Green operators with respect to (M, g_1). Finally we pass to the existence of the Green operators of N^*_χ – *where* * *is here referred to the volume form of g_1 and not g_χ* – in (M, g_1). Since N^*_χ has the same principal symbol $g^\sharp_\chi(\xi,\xi)\mathrm{Id}_{\mathsf{E}}$ as N_χ it is normally hyperbolic in (M, g_χ) and hence Green hyperbolic thereon. With the same argument used above, we see that the Green operators of N^*_χ (with * always referred to g_1) in (M, g_χ) are also Green operators in (M, g_1). □

Remark 2.1.15. We stress that, when $g_0 \preceq g_1$ are globally hyperbolic, N_χ and N_0 are therefore Green-hyperbolic second-order differential operators on (M, g_1) though they are *not* normally hyperbolic thereon. These are examples of *second-order* linear differential operators which are Green hyperbolic but *not* normally hyperbolic in a given globally hyperbolic spacetime.

2.1.5 General approach to construct Møller maps when $g_0 \preceq g_1$

We are in the position to introduce the notion of so-called **Møller operator**, which we shall later specialize to the case of a **Møller map**, namely the claimed (geometric) map which compares the space of solutions of different normally hyperbolic operators. The novelty of this approach consists in defining the notion of Møller map in a more general fashion. More in detail, in [27,37,39,90] the Møller operator was constructed once that a foliation of M in Cauchy hypersurfaces was assigned and referring to the family of the metrics which are decomposed as in (1.1.1) with respect to *that* foliation. Here we shall see, that the construction of a Møller map still requires the choice of a foliation (associated to some smooth Cauchy time function), but the involved metrics do not have any particular relationship with the choice of the foliation. Instead they should enjoy some interplay concerning their light-cone structures which generalizes $g \preceq g'$ in the sense of paracausal deformations.

Let us consider a globally hyperbolic spacetime (M, g) equipped with a vector bundle $\mathsf{E} \to \mathsf{M}$ as before. If $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is a linear differential operator, a family of physically relevant solutions of the inhomogeneous equation $\mathsf{P}\mathfrak{f} = \mathfrak{h}$ is the linear vector space of spacelike compact smooth solutions with compactly supported source:

$$\mathsf{Sol}^g_{sc,c}(\mathsf{P}) := \{\mathfrak{f} \in \Gamma^g_{sc}(\mathsf{E}) \mid \mathsf{P}\mathfrak{f} \in \Gamma_c(\mathsf{E})\} .$$

Its subspace corresponding to the solutions of the homogeneous equation $\mathsf{P}\mathfrak{f} = 0$ is denoted by

$$\mathsf{Ker}^g_{sc}(\mathsf{P}) := \{\mathfrak{f} \in \Gamma^g_{sc}(\mathsf{E}) \mid \mathsf{P}\mathfrak{f} = 0\}$$

and it will play a pivotal role in the formulation of linear QFT.

We now specialize the operators P to 2nd-order normally-hyperbolic linear operators $\mathsf{N}_1, \mathsf{N}_0, \mathsf{N}_\chi$ (2.1.8) over $\Gamma(\mathsf{E})$ associated to globally hyperbolic metrics $g_0 \preceq g_1$ and g_χ on the common spacetime manifold M. Our goal is to construct several families of *Møller maps*, namely linear operators such that

(a) they are linear space isomorphisms between $\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$, $\mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)$, $\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)$;

(b) they restrict to isomorphisms to the subspaces $\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)$, $\mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)$, $\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)$.

For later convenience, we shall additionally require that the Møller maps preserve also the symplectic forms, which are of interest in applications to linear QFT.

The overall idea is inspired by the scattering theory. We start with two "free theories", described by the space of solutions of normally hyperbolic operators N_0 and N_1 in corresponding spacetimes (M, g_0) and (M, g_1), respectively, and we intend to connect them through an "interaction spacetime" (M, g_χ) with a "temporally localized" interaction defined by interpolating the two metrics by means of a smoothing function χ. Here we need two Møller maps: Ω_+ connecting (M, g_0) and (M, g_χ) – which reduces to the identity in the past when χ is switched off – and a second Møller map connecting (M, g_χ) to (M, g_1) – which reduces to the identity in the future when χ constantly takes the value 1. The "S-matrix" given by the composition $\mathsf{S} := \Omega_-\Omega_+$ will be the Møller map connecting N_0 and N_1.

The first step consists of comparing N_0 and N_1 with N_χ separately to construct the Møller map. As usual, we denote with E the $\mathbb{K}$-vector bundle over a spacetime (M, g).

We first start with operators denoted by $\mathsf{R}_\pm$ defined on the whole space of smooth sections $\Gamma(\mathsf{E})$ which is in common for the three metrics on M and next we will restrict these operators to the special spaces of solutions with spatially compact support and compactly supported sources, proving that these restrictions $\Omega_\pm$ are still linear space isomorphisms.

Proposition 2.1.16. *Let $g_0, g_1 \in \mathcal{GM}_\mathsf{M}$ be such that $g_0 \preceq g_1$ and $V^{g_0+}_x \subset V^{g_1+}_x$ for all $x \in M$. Let E be a vector bundle over M and $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be normally hyperbolic operators associated to g_0 and g_1 respectively. Choose*

(a) *a smooth Cauchy time g_1-function $t : \mathsf{M} \to \mathbb{R}$ and $\chi \in C^\infty(\mathsf{M}; [0,1])$ such that $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$;*

(b) *a pair of smooth functions $\rho, \rho' : \mathsf{M} \to (0, +\infty)$ such that $\rho(p) = 1$ for $t(p) < t_0$ and $\rho'(p) = \rho(p) = 1$ if $t(p) > t_1$. (Notice that $\rho = \rho' = 1$ constantly is allowed.)*

The following facts are true.

(1) *The operators*

$$\mathsf{R}_+ = \mathrm{Id} - \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.9}$$

$$\mathsf{R}_- = \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.10}$$

are linear space isomorphisms, whose inverses are given by

$$\mathsf{R}_+^{-1} = \mathrm{Id} + \mathsf{G}^+_{\mathsf{N}_0}(\rho\mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.11}$$

$$\mathsf{R}_-^{-1} = \mathrm{Id} + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})\,. \tag{2.1.12}$$

(2) *It holds*

$$\rho\mathsf{N}_\chi\mathsf{R}_+ = \mathsf{N}_0 \qquad \text{and} \qquad \rho'\mathsf{N}_1\mathsf{R}_- = \rho\mathsf{N}_\chi\,. \tag{2.1.13}$$

(3) *If $\mathfrak{f} \in \Gamma(\mathsf{E})$, then*

$$(\mathsf{R}_+\mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) < t_0, \tag{2.1.14}$$

$$(\mathsf{R}_-\mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) > t_1\,. \tag{2.1.15}$$

Proof. Observe that $\rho\mathsf{N}_\chi$ and $\rho'\mathsf{N}_1$ are Green hyperbolic with respect to g_χ (as in Theorem 2.1.14) and g_1 respectively according to Theorem 2.1.14 and 2.1.3, and thus they are with respect to g_1. Moreover $\mathsf{G}^\pm_{\rho\mathsf{N}_\chi} = \mathsf{G}^\pm_{\mathsf{N}_\chi}\rho^{-1}$ and $\mathsf{G}^\pm_{\rho'\mathsf{N}_1} = \mathsf{G}^\pm_{\mathsf{N}_1}\rho'^{-1}$.
(1) If $\mathfrak{f} \in \Gamma(\mathsf{E})$, in view of the hypotheses $((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(p) = 0$ and $((\mathsf{N}_1 - \mathsf{N}_\chi)\mathfrak{f})(p) = 0$ is respectively $t(p) < t_0$ and $t(p) > t_1$ where $t^{-1}(t_0)$ and $t^{-1}(t_1)$ are spacelike Cauchy hypersurfaces in common for the metrics g_0, g_χ, g_1. Therefore the operators R_- and R_+ are linear and well defined on the domain $\Gamma(\mathsf{E})$ because $(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f} \in \Gamma^{g_1}_{pc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{pc}(\mathsf{E}) \subset \mathsf{Dom}(\mathsf{G}^+_{\rho\mathsf{N}_\chi})$ and $(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathfrak{f} \in \Gamma^{g_1}_{fc}(\mathsf{E}) \subset \mathsf{Dom}(\mathsf{G}^-_{\rho'\mathsf{N}_1})$. A similar argument holds for $\mathsf{R}_\pm^{-1}$. To prove bijectivity of $\mathsf{R}_\pm$ it suffices to establish that R_-^{-1} in (2.1.12) is a two-sided inverse of R_- and that R_+^{-1} in (2.1.11) is a two-sided inverse of R_+ on $\Gamma(\mathsf{E})$:

$$\mathsf{R}_- \circ \mathsf{R}_-^{-1} = \mathsf{R}_-^{-1} \circ \mathsf{R}_- = \mathrm{Id} \quad \text{and} \quad \mathsf{R}_+ \circ \mathsf{R}_+^{-1} = \mathsf{R}_+^{-1} \circ \mathsf{R}_+ = \mathrm{Id}.$$

We prove that R_- defined as in (2.1.12) inverts R_- from the right by direct computation:

$$\mathsf{R}_- \circ \mathsf{R}_-^{-1} = (\mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)) \circ (\mathrm{Id} + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)) =$$
$$= \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi).$$

Now, by exploiting the identity

$$\mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathsf{G}^-_{\rho\mathsf{N}_\chi} = \mathsf{G}^-_{\rho\mathsf{N}_\chi} - \mathsf{G}^-_{\rho'\mathsf{N}_1} : \Gamma^{g_\chi}_{fc}(\mathsf{E}) \cap \Gamma^{g_1}_{fc}(\mathsf{E}) \to \Gamma(\mathsf{E}),$$

we can prove our claim

$$\mathsf{R}_- \circ \mathsf{R}_-^{-1} = \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) - (\mathsf{G}^-_{\rho\mathsf{N}_\chi} - \mathsf{G}^-_{\rho'\mathsf{N}_1})(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) = \mathrm{Id}\,.$$

The proof that R_-^{-1} is also a left inverse is the same with obvious changes and analogous calculations show that R_+^{-1} is a left and right inverse of R^+.

(2) Taking advantage of (ia)-(iib) in Definition 2.1.1 and the definition of N_χ and the one of $\mathsf{R}_\pm$, a direct computation establishes (2.1.13).
(3) Let us prove (2.1.14). Consider a compactly supported smooth section $\mathfrak{h}$ whose support is included in the set $t^{-1}((-\infty, t_0))$. Taking advantage of the former in (2.2.33), we obtain

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = \int_{\mathsf{M}} \langle \mathsf{G}^-_{(\rho\mathsf{N}_\chi)^*}\mathfrak{h}, (\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

since $\mathrm{supp}(\mathsf{G}^-_{(\rho\mathsf{N}_\chi)^*}\mathfrak{h}) \subset J^{g_\chi}_-(\mathrm{supp}(\mathfrak{h}))$ from Definition 2.1.1 and thus that support does not meet $\mathrm{supp}((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})$ because $((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(p)$ vanishes if $t(p) < t_0$. As $\mathfrak{h}$ is an arbitrary smooth section compactly supported in $t^{-1}((-\infty, t_0))$,

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

entails that $\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f} = 0$ on $t^{-1}((-\infty, t_0))$. Eventually, the very definition (2.1.9) of $\mathsf{G}^+_{\rho\mathsf{N}_\chi}$ implies (2.1.14). The proof of (2.1.15) is strictly analogous, so we leave it to the reader. □

We can now pass to the second step, namely we perform restrictions of $\mathsf{R}_\pm$ to the relevant subspaces of solutions.

Proposition 2.1.17. *With the same hypotheses as in Proposition 2.1.16 (in particular $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$), we have*

$$\mathsf{R}_+(\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)) = \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi) \quad \textit{and} \quad \mathsf{R}_-(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) = \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1) \tag{2.1.16}$$

and

$$\mathsf{R}_+(\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)) = \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi) \quad \textit{and} \quad \mathsf{R}_-(\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)) = \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,. \tag{2.1.17}$$

As a consequence, the restrictions

$$\begin{aligned}
\Omega_+ &:= \mathsf{R}_+|_{\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)} : \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \to \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)\,, \quad \Omega^0_+ : \mathsf{R}_+|_{\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)} : \mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)\,,\\
\Omega_- &:= \mathsf{R}_-|_{\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)} : \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi) \to \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)\,, \quad \Omega^0_- : \mathsf{R}_-|_{\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)} : \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi) \to \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,,
\end{aligned}$$

define linear space isomorphisms such that

$$\rho\mathsf{N}_\chi\Omega_+ = \mathsf{N}_0\,, \quad \rho'\mathsf{N}_1\Omega_- = \rho\mathsf{N}_\chi \tag{2.1.18}$$

and, for $\mathfrak{f}$ in the respective domains,

$$(\Omega_+\mathfrak{f})(p) = \mathfrak{f}(p)\,, \quad (\Omega^0_+\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) < t_0, \tag{2.1.19}$$

$$(\Omega_-\mathfrak{f})(p) = \mathfrak{f}(p)\,, \quad (\Omega^0_-\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) > t_1\,. \tag{2.1.20}$$

Before we prove our claim, we need a preparatory lemma.

Lemma 2.1.18. *Let $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be a 2nd order normally hyperbolic differential operator on the vector bundle $\mathsf{E} \to \mathsf{M}$ on the globally hyperbolic spacetime (M, g). Let $\Psi \in \Gamma(\mathsf{E})$ be such that $\mathsf{P}\Psi \in \Gamma_c(\mathsf{E})$. Then the following facts are equivalent.*

(a) $\Psi \in \Gamma^g_{sc}(\mathsf{E})$;

(b) there is a spacelike Cauchy hypersurface of (M, g) such that Ψ has compactly supported Cauchy data thereon.

Proof. If $\Psi \in \Gamma^g_{sc}(\mathsf{E})$ then, by definition, (b) is true. Suppose that (b) is true for Σ_0. According to Theorem 2.1.7, Ψ is the unique solution of the Cauchy problem whose equation is $\mathsf{P}\Psi = \mathfrak{f}$, where $\mathfrak{f} \in \Gamma_c(\mathsf{E})$. As a consequence the support of Ψ completely lies in $J(\mathrm{supp}(\mathfrak{f})) \cup J(\mathrm{supp}(\mathfrak{h}_0)) \cup J(\mathrm{supp}(\mathfrak{h}_1)) \subset J(K)$ where $\mathfrak{h}_0$ and $\mathfrak{h}_1$ are the Cauchy data of Ψ on Σ_0 and $K := \mathrm{supp}(\mathfrak{f}) \cup \mathrm{supp}(\mathfrak{h}_0) \cup J(\mathrm{supp}(\mathfrak{h}_1)$. In particular K is compact. In view of well known properties of globally hyperbolic spacetimes (see *e.g.* [4, Proposition 1.2.56]), since K is compact $J(K) \cap \Sigma$ is compact for every Cauchy hypersurface Σ of (M, g) so that $\Psi \in \Gamma^g_{sc}(\mathsf{E})$. □

Proof of Proposition 2.1.17. $\mathsf{R}_\pm$ and $\mathsf{R}_\pm^{-1}$ are bijective on $\Gamma(\mathsf{E})$. As a consequence (2.1.16) and thesis for $\Omega_\pm$, including (2.1.18) which is a specialization of (2.1.13), immediately arise when proving that

$$\mathsf{R}_+(\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)) \subset \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)\,, \quad \mathsf{R}_+^{-1}(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) \subset \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \tag{2.1.21}$$

and

$$\mathsf{R}_-(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) \subset \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)\,, \quad \mathsf{R}_-^{-1}(\mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)) \subset \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)$$

The identities in (2.1.17) and the thesis for $\Omega^0_\pm$ immediately arises by bijectivity of the linear maps $\Omega_\pm$ and (2.1.18) where we know that $\rho, \rho' > 0$. To conclude, let us establish the first inclusion in (2.1.21), the remaining three inclusions have a strictly analogous proof. Suppose that $\mathfrak{f} \in \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$. Hence $\rho\mathsf{N}_\chi\mathsf{R}_+\mathfrak{f} = \mathsf{N}_0\mathfrak{f} \in \Gamma_c(\mathsf{E})$ and $\mathsf{N}_\chi\mathsf{R}_+\mathfrak{f} = \rho^{-1}\mathsf{N}_0\mathfrak{f} \in \Gamma_c(\mathsf{E})$. Next pass to consider the Cauchy hypersurfaces of t which are in common with the three considered metrics g_0, g_1, g_χ and choose $t' < t_0$. (3) in Proposition 2.1.16 yields $(\mathsf{R}_+\mathfrak{f})(t', x) = \mathfrak{f}(t', x)$ where $x \in \Sigma_{t'}$. The Cauchy data of $\mathfrak{f}$ on $\Sigma_{t'}$ have compact support because $\mathfrak{f} \in \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$. On the ground of Lemma 2.1.18, noticing that N_χ is normally hyperbolic in (M, g_χ), referring to the Cauchy problem on $\Sigma_{t'}$ for the equation $\mathsf{N}_\chi\mathsf{R}_+\mathfrak{f} = \rho^{-1}\mathsf{N}_0\mathfrak{f} \in \Gamma^{g_\chi}_c(\mathsf{E})$ in the spacetime (M, g_χ), we conclude that $\mathsf{R}_+\mathfrak{f} \in \Gamma^{g_\chi}_{sc,c}(\mathsf{E})$ because its Cauchy data on $\Sigma_{t'}$ (now interpreted as a Cauchy hypersurface for g_χ) have compact support as they coincide with the ones of $\mathfrak{f}$ itself. □

2.1.6 General Møller maps for paracausally related metrics

We are now in a position to state a result regarding the existence of Møller maps between two normally hyperbolic operators N_0 and N_1 on respective globally hyperbolic spacetimes over the same manifold (and vector bundle) whose metrics are $\preceq$ comparable. The final goal is to extend the results to pairs of paracausally related metrics.

Proposition 2.1.19. *Let $g_0, g_1 \in \mathcal{GM}_{\mathsf{M}}$ be such that either $g_0 \preceq g_1$ or $g_1 \preceq g_0$ with, respectively, either $V^{g_0+}_x \subset V^{g_1+}_x$ for all $x \in M$ or $V^{g_1+}_x \subset V^{g_0+}_x$ for all $x \in M$. Let E be a vector bundle over M and $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be normally hyperbolic operators associated to g_0 and g_1 respectively. There exist (infinitely many) vector space isomorphisms,*

$$\mathsf{S} : \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \to \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)$$

such that, for some smooth function $\mu : \mathsf{M} \to (0, +\infty)$ depending on S (which can be chosen $\mu = 1$),

(1) referring to the said domains,

$$\mu\mathsf{N}_1\mathsf{S} = \mathsf{N}_0 \quad \textit{and} \quad \mu^{-1}\mathsf{N}_0\mathsf{S}^{-1} = \mathsf{N}_1$$

(2) the restriction $\mathsf{S}^0 := \mathsf{S}|_{\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)}$ defines a vector space isomorphism

$$\mathsf{S}^0 : \mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,.$$

Proof. First consider the case $g_0 \preceq g_1$. Referring to a smooth Cauchy time function t of (M, g_1) and a smoothing function χ, $\mathsf{S} := \Omega_-\Omega_+$ constructed as in Proposition 2.1.17 satisfies all the requirements trivially for $\mu := \rho'$. The previous result is also valid for $g_1 \preceq g_0$. It is sufficient to construct $\Omega_\pm$ as in Proposition 2.1.17, *but using g_1 as the initial metric and g_0 as the final one*, and eventually defining $\mu := \rho^{-1}$, $\mathsf{S} := (\Omega_-\Omega_+)^{-1} = \Omega_+^{-1}\Omega_-^{-1}$, and $\mathsf{S}^0 := (\Omega_-^0\Omega_+^0)^{-1} = (\Omega_+^0)^{-1}(\Omega_-^0)^{-1}$. □

We can pass to the generic case $g \simeq g'$, obtaining the first main result of this work.

Theorem 2.1.20. *Let (M, g) and (M, g') be globally hyperbolic spacetimes, E a vector bundle over M and $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ normally hyperbolic operators associated to g and g' respectively.*
If $g \simeq g'$, then there exist (infinitely many) vector space isomorphisms, called **Møller maps** *of g, g' (with this order),*

$$\mathsf{S} : \mathsf{Sol}^g_{sc,c}(\mathsf{N}) \to \mathsf{Sol}^{g'}_{sc,c}(\mathsf{N}')$$

such that

(1) referring to the said domains,

$$\mu\mathsf{N}'\mathsf{S} = \mathsf{N}$$

for some smooth $\mu : \mathsf{M} \to (0, +\infty)$ (which can be always taken $\mu = 1$ constantly in particular),

(2) the restriction $\mathsf{S}^0 := \mathsf{S}|_{\mathsf{Ker}^g_{sc}(\mathsf{N})}$ (also called **Møller map** *of g', g') defines a vector space isomorphism*

$$\mathsf{S}^0 : \mathsf{Ker}^g_{sc}(\mathsf{N}) \to \mathsf{Ker}^{g'}_{sc}(\mathsf{N}') .$$

Proof. First of all we notice that there always exists a normally hyperbolic operator N on E associated to every $g \in \mathcal{GM}_M$: For instance the *connection-d'Alembert operator* in [4, Example 2.1.5] referred to a generic connection ∇ on E, which always exists, and the Levi-Civita connection on (M, g). Let us consider a sequence $g_0 = g, g_1, \dots, g_N = g'$ of globally hyperbolic metrics on M satisfying Definition 1.3.1 and a corresponding sequence of formally selfadjoint normally hyperbolic operators N_k with $\mathsf{N}_0 := \mathsf{N}$ and $\mathsf{N}_N := \mathsf{N}'$. We can apply Proposition 2.1.19 for each pair g_k, g_{k+1} for $k = 0, 1, \dots, N-1$. It turns immediately out that, with an obvious notation,

$$\mathsf{S} := \mathsf{S}_{N-1}\mathsf{S}_{N-2}\cdots\mathsf{S}_0\,, \quad \mu := \mu_0\cdots\mu_{N-1}\,, \quad \text{where} \quad \mu_k\mathsf{N}_{k+1}\mathsf{S}_k = \mathsf{N}_k \quad k = 0, \dots N-1\,.$$

satisfies the thesis of the theorem, where either $S_k := \Omega_{k-}\Omega_{k+}$, $\mu_k := \rho_k$ or $S_k := (\Omega_{k+})^{-1}(\Omega_{k-})^{-1}$, $\mu_k := \rho_k^{-1}$ according to $g_k \preceq g_{k+1}$ or $g_{k+1} \preceq g_k$ respectively. With the same convention it results that $\mathsf{S}^0 = \mathsf{S}^0_{N-2}\mathsf{S}^0_{N-1}\cdots\mathsf{S}^0_0$ where either $S^0_k = \Omega^0_{k-}\Omega^0_{k+}$ or $S_k = (\Omega^0_{k+})^{-1}(\Omega^0_{k-})^{-1}$ according to the discussed cases. □

Moreover the Møller maps S^0 as in Theorem 2.1.20 preserve the symplectic forms of the normal operators they relate when these operators are formally selfadjoint.

Theorem 2.1.21. *Consider $g, g' \in \mathcal{GH}_M$ with respectively associated normally hyperbolic operators N, N' on the $\mathbb{K}$-vector bundle E over M. If $g' \simeq g$ and N and N' are formally selfadjoint with respect to a non-degenerate, Hermitian fiber metric $\langle\cdot|\cdot\rangle$, then there are Møller maps S^0 satisfying the thesis of Theorem 2.1.20 such that*

$$\sigma^{\mathsf{N}'}_{g'}(\mathsf{S}^0\Psi, \mathsf{S}^0\Phi) = \sigma^{\mathsf{N}}_g(\Psi, \Phi) \quad \textit{for every } \Psi, \Phi \in \mathsf{Ker}^g_{sc}(\mathsf{N}),$$

where we used the notation σ^{N}_g in place of $\sigma^{\mathsf{N}}_{(\mathsf{M},g)}$.

Proof. It is sufficient to prove the thesis for the maps $\Omega^0_\pm$ referred to two metrics $g_0 \preceq g_1$, which immediately implies the thesis also for the inverse maps $(\Omega^0_\pm)^{-1}$ they being isomorphisms. Indeed, according the proof of Theorem 2.1.20, the isomorphisms S^0 are compositions of various copies of $\Omega^0_\pm$ and their inverses. Let us consider $\Omega^0_+ : \mathsf{Ker}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}_{sc}(\mathsf{N}_\chi)$ and we prove the thesis for it, the other case being very similar. Consider a smooth Cauchy time function t for g_1 and the associated foliation made of spacelike Cauchy hypersurfaces Σ_t in common for g_0, g_1, and g_χ. If the smoothing function χ used to build up g_χ and N_χ vanishes before t_0 and we use Σ_t with $t < t_0$ to compute the relevant symplectic forms, due to (2.2.31),

$$\sigma^{\mathsf{N}_\chi}_{g_\chi}(\Omega^0_+\Psi, \Omega^0_+\Phi) = \sigma^{\mathsf{N}_0}_{g_0}(\Psi, \Phi) \quad \text{for every } \Psi, \Phi \in \mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0).$$

Above, we have used the definition of the symplectic form, we have noticed that $g_\chi = g_0$ around Σ_t and that the N_0 and N_χ compatible connections must coincide there as they are locally defined and uniquely determined by $\mathsf{N}_0\Psi = \mathsf{N}_\chi\Psi = (-\mathrm{tr}_g(\nabla\nabla) + c)\Psi$ for every smooth Ψ compactly supported around a point p with $t(p) < t_0$. Thinking of $\sigma^{\mathsf{N}_\chi}_{g_\chi}(\Omega^0_+\Psi, \Omega^0_+\Phi)$ as defined in (M, g_χ) and of $\sigma^{\mathsf{N}_0}_{g_0}(\Psi, \Phi)$ as defined in (M, g_0), though both computed on Σ_t with $t < t_0$, Proposition 2.1.12 concludes the proof. □

Definition 2.1.22. We call **symplectic Møller map** any linear isomorphism defined in accordance with Theorem 2.1.21.

2.1.7 Adjoint operators

We pass now to prove how it is possible to choose the functions ρ and ρ' affecting the definitions (2.1.9)-(2.1.10) of $\mathsf{R}_\pm$ in order to satisfy a further requirement with some crucial implications in QFT: the preservation of the causal propagator of two operators N and N' when the associated metrics are paracausally related. Essentially speaking, a Møller map satisfying this further requirement will be named *Møller operator*.
To study the relation between Møller maps and the causal propagator of normally hyperbolic operators defined on a vector bundle equipped with a non-degenerate (Hermitian) fiber metric, we need a suitable notion of *adjoint operator* which generalizes the notion of formal adjoint of differential operators.

Let E be a $\mathbb{K}$-vector bundle on the oriented manifold M equipped with a non-degenerate, symmetric if $\mathbb{K} = \mathbb{R}$ or Hermitian if $\mathbb{K} = \mathbb{C}$, fiber metric $\langle \cdot | \cdot \rangle$. Suppose that g and g' (possibly $g \neq g'$) are Lorentzian metrics on M. Consider a $\mathbb{K}$-linear operator

$$\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E}) \,,$$

where $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{E})$ is a $\mathbb{K}$-linear subspace and $\mathsf{Dom}(\mathsf{T}) \supset \Gamma_c(\mathsf{E})$.

Definition 2.1.23. An operator

$$\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$$

is said to be the **adjoint of T with respect to** g, g' (with the said order) if it satisfies

$$\int_{\mathsf{M}} \langle \mathfrak{h}(x) \,|\, (\mathsf{T}\mathfrak{f})(x) \rangle \mathrm{vol}_{g'}(x) = \int_{\mathsf{M}} \left\langle \left(\mathsf{T}^{\dagger_{gg'}}\mathfrak{h}\right)(x) \,|\, \mathfrak{f}(x) \right\rangle \mathrm{vol}_g(x) \quad \forall \mathfrak{f} \in \mathsf{Dom}(\mathsf{T}) \,,\ \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}).$$

Notation 2.1.24. If $g = g'$ then we shall denote the adjoint of T with respect to g simply as $\mathsf{T}^{\dagger_g}$.

We prove below that $\mathsf{T}^{\dagger_{gg'}}$ is unique if it exists so that calling it "the" adjoint operator of T is appropriate.

Remark 2.1.25. If $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ is defined as in Definition 2.1.23 and $\mathsf{T}^{\dagger_{gg'}}$ exists, then

$$\int_{\mathsf{M}} \langle \mathfrak{h} \,|\, \mathsf{T}\mathfrak{f}_n \rangle \mathrm{vol}_{g'} \to 0 \quad \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}) \text{ as } \Gamma_c(\mathsf{E}) \ni \mathfrak{f}_n \to 0 \text{ for } n \to +\infty \text{ in the } \textit{topology of test sections } [4].$$

Vice versa, this only condition is not sufficient to guarantee the existence of $\mathsf{T}^{\dagger_{gg'}}$ as a $\Gamma_c(\mathsf{E})$-valued operator. Using a straightforward extension of the Schwartz kernel theorem, the condition above just implies the existence of a weaker version of $\mathsf{T}^{\dagger_{gg'}}$ which is distribution-valued.

From now on if $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ and $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$, we define the **standard domains** of their compositions as follows, where $a \in \mathbb{K}$.

(a) $\mathsf{Dom}(a\mathsf{T}) := \mathsf{Dom}(\mathsf{T})$ – or $\mathsf{Dom}(a\mathsf{T}) := \Gamma(\mathsf{E})$ if $a = 0$ – is the domain of $a\mathsf{T}$ defined pointwise;

(b) $\mathsf{Dom}(\mathsf{T} + \mathsf{T}') := \mathsf{Dom}(\mathsf{T}) \cap \mathsf{Dom}(\mathsf{T}')$ is the domain of $\mathsf{T} + \mathsf{T}'$ defined pointwise;

(c) $\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) := \{\mathfrak{f} \in \mathsf{Dom}(\mathsf{T}) \,|\, \mathsf{T}(\mathfrak{f}) \in \mathsf{Dom}(\mathsf{T}')\}$ is the domain of $\mathsf{T}' \circ \mathsf{T}$.

Proposition 2.1.26. *Referring to the notion of adjoint in Definition 2.1.23, the following facts are valid.*

(1) If the adjoint $\mathsf{T}^{\dagger_{gg'}}$ *of* T *exists, then it is unique.*

(2) If $\mathsf{T} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *is a differential operator and* $g = g'$*, then* $\mathsf{T}^{\dagger_{gg}}$ *exists and is the restriction of the formal adjoint to* $\Gamma_c(\mathsf{E})$*. (In turn, the formal adjoint of* $\mathsf{T}^{\dagger}$ *is the unique extension to* $\Gamma(\mathsf{E})$ *of the differential operator* $\mathsf{T}^{\dagger}$ *as a differential operator)*

(3) Consider a pair of $\mathbb{K}$*-linear operators* $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$, $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$ *and* $a, b \in \mathbb{K}$*. Then*

$$(a\mathsf{T} + b\mathsf{T}')^{\dagger_{gg'}} = \overline{a}\mathsf{T}^{\dagger_{gg'}} + \overline{b}\mathsf{T}'^{\dagger_{gg'}}$$

provided $\mathsf{T}^{\dagger_{gg'}}$ *and* $\mathsf{T}'^{\dagger_{gg'}}$ *exist.*

(4) Consider a pair of $\mathbb{K}$*-linear operators* $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ *and* $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$ *such that*

(i) $\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) \supset \Gamma_c(\mathsf{E})$,

(ii) $\mathsf{T}^{\dagger_{gg'}}$ *and* $\mathsf{T}'^{\dagger_{g'g''}}$ *exist,*

then $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ *exists and*

$$(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}} = \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} .$$

(5) If $\mathsf{T}^{\dagger_{gg'}}$ *exists, then* $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}} = \mathsf{T}|_{\Gamma_c(\mathsf{E})}$.

(6) If $\mathsf{T} : \mathsf{Dom}(T) = \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *is bijective, admits* $\mathsf{T}^{\dagger_{gg'}}$*, and* T^{-1} *admits* $(\mathsf{T}^{-1})^{\dagger_{g'g}}$*, then* $\mathsf{T}^{\dagger_{gg'}}$ *is bijective and* $(\mathsf{T}^{-1})^{\dagger_{g'g}} = (\mathsf{T}^{\dagger_{gg'}})^{-1}$.

Proof. We write below $\dagger$ in place of $\dagger_{gg'}$ if it is not strictly necessary to specify the metrics. To prove (1) let's assume that, fixed an operator $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ there exist two different adjoints $\mathsf{T}_1^{\dagger}, \mathsf{T}_2^{\dagger} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ both satisfying definition 2.1.23, i.e.

$$\int_{\mathsf{M}} \langle \mathsf{T}_1^{\dagger}\mathfrak{h} \,|\, \mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathsf{T}_2^{\dagger}\mathfrak{h} \,|\, \mathfrak{f} \rangle \,\mathrm{vol}_g$$

for all $\mathfrak{f} \in \mathsf{Dom}(\mathsf{T})$ and all $\mathfrak{h} \in \Gamma_c(\mathsf{E})$. Then by linearity of the integration and (anti) linearity of the product, the former identity is equivalent to

$$\int_{\mathsf{M}} \langle \mathsf{T}_1^{\dagger}\mathfrak{h} - \mathsf{T}_2^{\dagger}\mathfrak{h} \,|\, \mathfrak{f} \rangle \,\mathrm{vol}_g = 0.$$

Since $\Gamma_c(\mathsf{E}) \subset \mathsf{Dom}(\mathsf{T})$, the thesis follows by reducing to every fixed local trivialization over every arbitrarily fixed coordinate patch U on M. Restricting to U, the equation above can be recast to

$$\int_U \sum_{a=1}^{N} (\mathsf{T}_1^\dagger \mathfrak{h} - \mathsf{T}_2^\dagger \mathfrak{h})^a(p) \mathfrak{f}_a(p) \operatorname{vol}_g(p) = 0.$$

where $\mathfrak{f}_a(p)$ is a fiber component of $\langle \cdot | \mathfrak{f} \rangle_p \in \mathsf{E}_p^*$ with $p \in U$. Since $U \ni p \mapsto (\mathsf{T}_1^\dagger \mathfrak{h} - \mathsf{T}_2^\dagger \mathfrak{h})^a(p)$ is continuous and $U \ni p \mapsto \mathfrak{f}_a(p)$ is smooth, compactly supported (with support in U) and arbitrary (because $\langle \cdot | \cdot \rangle$ is non-degenerate), the fundamental lemma of calculus of variations implies that $U \ni p \mapsto (\mathsf{T}_1^\dagger \mathfrak{h} - \mathsf{T}_2^\dagger \mathfrak{h})^a(p)$ is the zero function for $a = 1, \ldots, N$. Since U can be fixed as a neighbourhood of every point of M, (1) follows.

The proof of (2) and (3) is obvious: (2) follows by comparing definitions 2.1.23 and 2.1.9, while (3) follows by direct computation checking that $\overline{a}\mathsf{T}^\dagger + \overline{b}\mathsf{T}'^\dagger$ satisfies the definition of $(\overline{a}\mathsf{T} + \overline{b}\mathsf{T}')^\dagger$ (notice that $\Gamma_c(\mathsf{E}) \subset \mathsf{Dom}(\overline{a}\mathsf{T}^\dagger + \overline{b}\mathsf{T}'^\dagger)$ if $\mathsf{T}^\dagger$ and $\mathsf{T}'^\dagger$ exist).

To prove (4), since the composition is well defined on a suitable domain, we can just use twice the definition 2.1.23

$$\int_{\mathsf{M}} \langle \mathfrak{h} \,|\, \mathsf{T}' \circ \mathsf{T}\mathfrak{f} \rangle \operatorname{vol}_{g''} = \int_{\mathsf{M}} \langle \mathsf{T}'^{\dagger_{g'g''}} \mathfrak{h} \,|\, \mathsf{T}\mathfrak{f} \rangle \operatorname{vol}_{g'} = \int_{\mathsf{M}} \langle \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} \mathfrak{h} \,|\, \mathfrak{f} \rangle \operatorname{vol}_g$$

for all $\mathfrak{f} \in \mathsf{Dom}(\mathsf{T}' \circ \mathsf{T})$ and all $\mathfrak{h} \in \Gamma_c(\mathsf{E})$: notice that using the definition of the adjoint in the second equality is possible because $\mathsf{T}'^{\dagger_{g'g''}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$. The found identity proves that $\mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}}$ satisfies the definition of $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ ending the proof of (4).

(5) is true because, if $\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ exists, then $\mathsf{T}|_{\Gamma_c(\mathsf{E})}$ satisfies the definition of $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}}$.

Finally, (6) arises by taking the $^{\dagger_{gg}}$ adjoint of both sides of the identity $T \circ T^{-1} = I$ and the $^{\dagger_{g'g'}}$ adjoint of both sides of the identity $T^{-1} \circ T = I$ and taking (4) into account. □

2.1.8 Møller operators and causal propagator

We are in a position to state one of the most important results of this work by specializing the isomorphisms introduced in Theorem 2.1.20 by means of a suitable choice of the function μ. As a matter of fact (1) and (3) have been already established in Theorem 2.1.20.

Theorem 2.1.27. *Let* E *be* $\mathbb{K}$*-vector bundle over the smooth manifold* M *with a non-degenerate, real or Hermitian depending on* $\mathbb{K}$*, fiber metric* $\langle \cdot | \cdot \rangle$*. Consider* $g, g' \in \mathcal{GH}_M$ *with respectively associated normally hyperbolic formally-selfadjoint operators* N, N'.
If $g \simeq g'$*, then it is possible to define (in infinite ways) a* $\mathbb{K}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$*, called* **Møller operator** *of* g, g' *(with this order), such that the following facts are true.*

(1) The restrictions to the relevant subspaces of $\Gamma(\mathsf{E})$ *respectively define Møller maps (hence linear isomorphisms) as in Theorem 2.1.20.*

$$\mathsf{R}|_{\mathsf{Sol}^g_{sc,c}(\mathsf{N})} = \mathsf{S} : \mathsf{Sol}^g_{sc,c}(\mathsf{N}) \to \mathsf{Sol}^{g'}_{sc,c}(\mathsf{N}') \quad \text{and} \quad \mathsf{R}|_{\mathsf{Ker}^g_{sc}(\mathsf{N})} = \mathsf{S}^0 : \mathsf{Ker}^g_{sc}(\mathsf{N}) \to \mathsf{Ker}^{g'}_{sc}(\mathsf{N}') \,.$$

(2) The causal propagators $\mathsf{G}_{\mathsf{N}'}$ *and* G_{N}*, respectively of* N' *and* N*, satisfy*

$$\mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{N}'} \,. \tag{2.1.22}$$

(3) By denoting c' *the smooth function such that* $\operatorname{vol}_{g'} = c' \operatorname{vol}_g$*, we have*

$$c'\mathsf{N}'\mathsf{R} = \mathsf{N} \,. \tag{2.1.23}$$

(4) It holds

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'|_{\Gamma_c(\mathsf{E})} = \mathsf{N}|_{\Gamma_c(\mathsf{E})} \,.$$

(5) *The maps* $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *and* $(\mathsf{R}^{\dagger_{gg'}})^{-1} = (\mathsf{R}^{-1})^{\dagger_{g'g}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *are continuous with respect to the natural topologies of* $\Gamma_c(\mathsf{E})$ *in the domain and in the co-domain.*

Remarks 2.1.28. Before we prove our claim we want to underline the following:

(1) Any Møller operator defines a symplectic Møller map (*cf.* Definition 2.1.22). Indeed, the preservation of the causal propagator (*cf.* (2) in Theorem 2.1.27) implies that the symplectic forms are preserved in view of Proposition 2.1.13. However, the converse is false since the preservation of the causal propagator relies upon a suitable choice of the function ρ, whereas this choice is immaterial for the preservation of the symplectic forms.

(2) Møller operators can be explicitly constructed as follows. If $g' \simeq g$, and referring to a finite sequence of metrics $g_0 := g, g_1, \ldots, g_N := g' \in \mathcal{GH}_\mathsf{M}$ as in Definition 1.3.1, then there exists a corresponding sequence of formally selfadjoint g_k-normally hyperbolic operators $\mathsf{N}_0 := \mathsf{N}, \mathsf{N}_1, \ldots, \mathsf{N}_N := \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ such that

$$\mathsf{R} = \mathsf{R}_{N-1} \cdots \mathsf{R}_0 \,, \tag{2.1.24}$$

is a Møller operator of g, g' where

$$\mathsf{R}_k := \mathsf{R}_-^{(k)}\mathsf{R}_+^{(k)} \quad \text{if } g_k \preceq g_{k+1} \qquad \text{or} \qquad \mathsf{R}_k := (\mathsf{R}_+^{(k)})^{-1}(\mathsf{R}_-^{(k)})^{-1} \quad \text{if } g_{k+1} \preceq g_k. \tag{2.1.25}$$

Above, for every given k, $\mathsf{R}_\pm^{(k)}$ are defined as $\mathsf{R}_\pm$ as in Equations (2.1.9) and (2.1.10) where

(i) N_0 is replaced by N_k and N_1 is replaced by N_{k+1} if $g_k \preceq g_{k+1}$,

(ii) N_0 is replaced by N_{k+1} and N_1 is replaced by N_k if $g_{k+1} \preceq g_k$,

(iii) $\rho := c_0^\chi$, and $\rho' := c_0^1$ (assuming $\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0}$ and $\mathrm{vol}_{g_1} = c_\chi^1 \mathrm{vol}_{g_\chi}$).

The smooth Cauchy time function χ in (2.1.9) and (2.1.10) can be chosen arbitrarily and depending on k in general. The final Møller operator R of g, g' also depends on all the made choices.

Proof of Theorem 2.1.27. We divide the proof into several steps.

(1)-(3) Let us first prove the thesis for the special case of $g = g_0 \preceq g_1 = g'$, with $V_x^{g_0+} \subset V_x^{g_1+}$ for all $x \in \mathsf{M}$, and specialize the definition of the isomorphisms (2.1.9) and (2.1.10) to

$$\mathsf{R}_+ = \mathrm{Id} - \mathsf{G}^+_{c_0^\chi \mathsf{N}_\chi}(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.26}$$

$$\mathsf{R}_- = \mathrm{Id} - \mathsf{G}^-_{c_0^1 \mathsf{N}_1}(c_0^1 \mathsf{N}_1 - c_0^\chi \mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{2.1.27}$$

where

$$\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0} \quad \text{and} \quad \mathrm{vol}_{g_1} = c_0^1 \mathrm{vol}_{g_0}$$

It is easy to see that

$$(c_0^\chi \mathsf{N}_\chi)^{\dagger_{g_0}} = c_0^\chi \mathsf{N}_\chi \quad \text{and} \quad (c_0^1 \mathsf{N}_1)^{\dagger_{g_0}} = c_0^1 \mathsf{N}_1 \,. \tag{2.1.28}$$

Our goal is to prove that the isomorphism $\mathsf{R} := \mathsf{R}_-\mathsf{R}_+ : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ satisfies the thesis.
Per direct inspection, applying the definition of adjoint operator and taking advantage of (2.1.28), Proposition 2.1.3, and (2.1.4), we almost immediately have that

$$\mathsf{R}_+^{\dagger_{g_0}} = \mathrm{Id} - (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} \quad \text{and} \quad \mathsf{R}_-^{\dagger_{g_0}} = \mathrm{Id} - (c_0^1 \mathsf{N}_1 - c_0^\chi \mathsf{N}_\chi)\mathsf{G}^+_{c_0^1 \mathsf{N}_1}|_{\Gamma_c(\mathsf{E})} \,. \tag{2.1.29}$$

Again per direct inspection we see that

$$c_0^\chi \mathsf{N}_\chi \mathsf{R}_+ = \mathsf{N}_0 \quad \text{and} \quad c_0^1 \mathsf{N}_1 \mathsf{R}_- = c_0^\chi \mathsf{N}_\chi$$

and thus

$$c_0^1 \mathsf{N}_1 \mathsf{R} = c_0^1 \mathsf{N}_1 \mathsf{R}_- \mathsf{R}_+ = \mathsf{N}_0$$

as wanted.
As we prove below, the following identities are valid

$$\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}\mathsf{R}_+^{\dagger g_0} = \mathsf{G}_{c_0^\chi \mathsf{N}_\chi} \quad \text{and} \quad \mathsf{R}_-\mathsf{G}_{c_0^\chi \mathsf{N}_\chi}\mathsf{R}_-^{\dagger g_0} = \mathsf{G}_{c_0^1\mathsf{N}_1} = \mathsf{G}_{\mathsf{N}_1}(c_0^1)^{-1} \tag{2.1.30}$$

so that

$$\mathsf{R}_-\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}R_+^{\dagger g_0}R_-^{\dagger g_0} = \mathsf{G}_{\mathsf{N}_1}(c_0^1)^{-1}$$

which is equivalent to

$$\mathsf{R}_-\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}(\mathsf{R}_-\mathsf{R}_+)^{\dagger g_0}c_0^1 = \mathsf{G}_{\mathsf{N}_1}\,.$$

On the other hand, we have

$$A^{\dagger g_0}c_0^1 = A^{\dagger g_0 g_1}$$

so that

$$\mathsf{R}\mathsf{G}_{\mathsf{N}_0}\mathsf{R}^{\dagger g_0 g_1} = \mathsf{R}_-\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}(\mathsf{R}_-\mathsf{R}_+)^{\dagger g_0 g_1} = \mathsf{G}_{\mathsf{N}_1}\,.$$

To conclude the proof of (1)-(3) for the case $g = g_0 \preceq g_1 = g'$ we prove (2.1.30).
Since G_{N_0} is defined as the difference of the advanced and retarded Green operators restricted to compact sections, we perform the computation separately for the two operators.
We start from $\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}|_{\Gamma_c(\mathsf{E})}\mathsf{R}_+^{\dagger g_0}$: the adjoint of the Møller operator is defined over $\Gamma_c(\mathsf{E})$ and gives back compactly supported sections, then the advanced Green operator maps a compactly supported section $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ to a solution such that $\mathrm{supp}\,(\mathsf{G}_{\mathsf{N}_0}^+\mathfrak{f}) \subset J_0^+(\mathrm{supp}\,(\mathfrak{f}) \subset J_\chi^+(\mathrm{supp}\,(\mathfrak{f}))$, where the last inclusion is due to the crucial hypothesis $g_0 \preceq g_\chi \preceq g_1$. Now since $\mathrm{supp}\,(\mathfrak{f})$ is compact the smooth Cauchy time function t attains a minimum $t_0 \in \mathbb{R}$ therein, so we choose a common smooth Cauchy hypersurface Σ_{t_1} of the foliation induced by t such that $t_1 < t_0$ and deduce that $\mathrm{supp}\,(\mathsf{G}_{\mathsf{N}_0}^+\mathfrak{f}) \subset J_\chi^+(\mathrm{supp}\,(\mathfrak{f})) \subset J_\chi^+(\Sigma_{t_1})$ which implies by [4, Lemma 1.2.61] that $\mathsf{G}_{\mathsf{N}_0}^+\mathfrak{f} \in \Gamma_{pc}^\chi(\mathsf{E})$.
Omitting the restriction of the domain of the causal propagators from the notation for sake of clarity, but having in mind that it is crucial for the validity of the argument, we obtain:

$$\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}^+ = \mathsf{G}_{\mathsf{N}_0}^+ - G_{c_0^\chi\mathsf{N}_\chi}^+ c_0^\chi\mathsf{N}_\chi\mathsf{G}_{\mathsf{N}_0}^+ + G_{c_0^\chi\mathsf{N}_\chi}^+\mathsf{N}_0\mathsf{G}_{\mathsf{N}_0}^+ = \mathsf{G}_{c_0^\chi\mathsf{N}_\chi}^+.$$

A similar reasoning proves that

$$G_{\mathsf{N}_0}^-\mathsf{R}_+^{\dagger g_0} = \mathsf{G}_{c_0^\chi\mathsf{N}_\chi}^-.$$

where now the restriction of the domains of the causal propagators to compactly supported sections is assumed from the definition of the adjoint. Collecting together the two identities found, we have

$$\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}\mathsf{R}_+^{\dagger g_0} = (\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}^+ - G_{\mathsf{N}_0}^-\mathsf{R}_+^{\dagger g_0}) + M = \mathsf{G}_{c_0^\chi\mathsf{N}_\chi} + M\,,$$

with, where both sides have to be computed on compactly supported sections,

$$M := (\mathrm{Id} - \mathsf{R}_+)\mathsf{G}_{\mathsf{N}_0}^-\mathsf{R}_+^{\dagger g_0} - \mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}^+(\mathrm{Id} - \mathsf{R}_+^{\dagger g_0})\,.$$

A direct evaluation of M using (2.1.26) and the former in (2.1.29) shows that $M = 0$. All that establishes the first identity in (2.1.30), while the latter follows by almost identical facts.
Let us pass to prove (1)-(3) for the case $g_1 \preceq g_0$, with $V_x^{g_1+} \subset V_x^{g_0+}$ for all $x \in \mathsf{M}$. First of all we observe that from the previously treated case ($g_0 \preceq g_1$) we have $c_1^0\mathsf{N}_0\mathsf{R}^{-1} = \mathsf{N}_1$ where $c_1^0 = (c_1^0)^{-1}$ and $\mathrm{vol}_{g_0} = c_1^0\mathrm{vol}_{g_1}$. Interchanging the names of g_0 and g_1, this result implies that (2.1.23) is true for $g_1 \preceq g_0$ when using R^{-1} in place of R. An analogous procedure proves (2.1.22) for the case $g_1 \preceq g_0$ from the same equation, already established, valid when $g_0 \preceq g_1$. Also in this case the relevant Møller operator is R^{-1}. To this end, we have only to prove that $(\mathsf{R}^{-1})^{\dagger g_1 g_0}$ exists and coincides to $(\mathsf{R}^{\dagger g_0 g_1})^{-1}$. Indeed, under these assumptions (2.1.22) implies

$$\mathsf{N}_1 = \mathsf{R}^{-1}\mathsf{N}_0(\mathsf{R}^{\dagger g_0 g_1})^{-1} = \mathsf{R}^{-1}\mathsf{N}_0(\mathsf{R}^{-1})^{\dagger g_1 g_0}$$

which is our thesis when interchanging g_0 and g_1. This fact that $(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}} = (\mathsf{R}^{\dagger_{g_0 g_1}})^{-1}$ actually can be established exploiting (6) in 2.1.26: R is bijective over $\Gamma(\mathsf{E})$, and admits the adjoint $\mathsf{R}^{\dagger_{g_0 g_1}}$, so if the inverse R^{-1} admits the adjoint $(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$, then $\mathsf{R}^{\dagger_{g_0 g_1}}$ is bijective and its inverse is such that $(\mathsf{R}^{\dagger_{g_0 g_1}})^{-1} = (\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$. Let us prove that R^{-1} admits adjoint (with respect to any metric among g_0, g_χ, g_1 since the existence of the adjoint with respect one of them trivially implies the existence of the adjoint with respect to the other metrics) to end the proof for the case $g_1 \preceq g_0$. By recalling that $\mathsf{R}^{-1} = \mathsf{R}_+^{-1} \circ \mathsf{R}_-^{-1}$ it suffices to show that R_+^{-1} and R_-^{-1} both admit adjoints. We explicitly give the g_0-adjoint of R_+^{-1} the other case being analogous,

$$(\mathsf{R}_+^{-1})^{\dagger_{g_0}} = \mathrm{Id} + (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{\mathsf{N}_0}|_{\Gamma_c(\mathsf{M})}\,.$$

Let us pass to the proof of (1)-(3) for the general case $g \simeq g'$ also establishing the last part of the thesis. In this case there is a sequence $g_0 = g, g_1, \ldots, g_N = g'$ of globally hyperbolic metrics on M satisfying Definition 1.3.1 and a corresponding sequence of selfadjoint normally hyperbolic operators N_k with $\mathsf{N}_0 := \mathsf{N}$ and $\mathsf{N}_N := \mathsf{N}'$. (This sequence always exists because, for every globally hyperbolic metric g, there is a normally hyperbolic operator N as proved in the proof of Theorem 2.1.20. The operator $\widetilde{\mathsf{N}} := \frac{1}{2}(\mathsf{N} + \mathsf{N}^{\dagger_g})$ is simultaneously formally selfadjoint with respect to $\langle \cdot | \cdot \rangle$ and normally hyperbolic.) Taking advantage of the validity of the thesis in the cases $g \preceq g'$ and $g' \preceq g$, using in particular (4) and (6) in Proposition 2.1.26, one immediately shows that we can build a Møller map for a paracausal deformation of metrics just by defining R as the composition of the various similar operators defined for each copy g_k, g_{k+1} as in (2.1.24) and (2.1.25).

(4) If $\mathfrak{f} \in \Gamma_c(\mathsf{E})$,

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'\mathfrak{f} = \mathsf{R}^{\dagger_{gg'}}\mathsf{N}'^{\dagger_{g'}}\mathfrak{f} = (\mathsf{N}'\mathsf{R})^{\dagger_{gg'}}\mathfrak{f} = \left(\frac{1}{c'}\mathsf{N}\right)^{\dagger_{gg'}}\mathfrak{f} = \mathsf{N}^{\dagger_g}\mathfrak{f} = \mathsf{N}\mathfrak{f}\,.$$

(5) It is sufficient to prove the thesis for the case $g = g_0 \preceq g_1 = g'$ and for $\mathsf{R}_+^{\dagger_{g_0}}$. The case of $\mathsf{R}_-^{\dagger_{g_0}}$ is analogous. In the case $g_1 \preceq g_0$ one uses the inverses of the operators above, and all remaining cases are proved just by observing that the considered Møller operators are compositions of the elementary operators $\mathsf{R}_\pm^{\dagger_{g_0}}$ and/or their inverses and smooth functions used as multiplicative operators. We know that

$$\mathsf{R}_+^{\dagger_{g_0}} = \mathrm{Id} - (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})}\,.$$

The identity operator has already the requested continuity property so that we have only to focus on the second addend using the fact that a linear combination of continuous maps is continuous as well. The map $\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})$ is continuous with respect to the natural topologies of the domain and co-domain (see *e.g.* [4, Corollary 3.6.19]). Since $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)$ is a smooth differential operator $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})$ is still continuous. To conclude the proof it is sufficient to prove that if $\Gamma_c(\mathsf{E}) \ni \mathfrak{f}_n \to 0$ in the topology of $\Gamma_c(\mathsf{E})$ and $K \supset supp(\mathfrak{f}_n)$ for all $n \in \mathbb{N}$ is a compact set, then there is a compact set K' such that $K' \supset supp((c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}\mathfrak{f}_n)$ for all $n \in \mathbb{N}$. If $t : \mathsf{M} \to \mathbb{R}$ is the Cauchy temporal function of g_1 used to construct R_+ and R_-, whose level sets $\Sigma_\tau := t^{-1}(\tau)$ are Cauchy hypersurfaces for g_0, g_χ, g_1 and $g_\chi = g_0$ in the past of Σ_{t_0}, then the set $J_-^{(\mathsf{M},g_\chi)}(K) \cap D_+^{(\mathsf{M},g_\chi)}(\Sigma_{t_0})$, which is compact for known properties of globally hyperbolic spacetimes, includes all supports of $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}\mathfrak{f}_n$ from the very definition of retarded Green operator also using the fact that $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)$ vanishes in the past of Σ_{t_0}. □

As a by-product of Theorem 2.1.27 we get a technical, but important, corollary.

Corollary 2.1.29. *Consider $g, g', g'' \in \mathcal{GH}_\mathsf{M}$, corresponding formally selfadjoint and normally hyperbolic operators $\mathsf{N}, \mathsf{N}', \mathsf{N}''$ on the $\mathbb{K}$-vector bundle E on M equipped with a non-degenerate, Hermitian, fiberwise metric. Assume that $g \simeq g'$ and $g' \simeq g''$ and suppose that $\mathsf{R}_{gg'}$ is a Møller operator of g, g' and $\mathsf{R}_{g'g''}$ is a Møller operator of g', g'' according to (2.1.24). The following facts are true.*

(1) $\mathsf{R}_{gg'}^{-1}$ *is a Møller operator of* g', g.

(2) $\mathsf{R}_{gg'}\mathsf{R}_{g'g''}$ *is a Møller operator of* g', g''.

Proof. It is immediate form the construction of R described at the end of Theorem 2.1.27 relying on (2.1.24). □

Remark 2.1.30. Observe that the construction of the Møller operator R of g_0, g_1, for $g_0 \preceq g_1$, as $\mathsf{R} = \mathsf{R}_-\mathsf{R}_+$ we used several times in this work is nothing but an elementary case of (2). Indeed, in that case, $g_0 \preceq g_\chi \preceq g_1$ and R_+, R_- are, respectively, a Møller operator of g_0, g_χ and g_χ, g_1.

2.2 The Proca field

In this section we apply the same construction to the Proca operator. Below, $\mathsf{G}^\pm_\mathsf{P}$ denote the retarded and advanced Green operators of the Proca equation (2.2.3), we shall discuss in 2.2.6. The symbol $\kappa_{g'g}$ denotes a linear fiber-preserving isometry from the spaces of smooth sections $\Gamma(\mathsf{V}_g)$ to $\Gamma(\mathsf{V}_{g'})$ constructed in 2.2.5. Here, V_g indicates the vector bundle of real 1-forms over the spacetime (M, g) whose sections are the argument of the Proca operator P. The main result can be summarized as follows.

Theorem 3 (Theorems 2.1.20 and 2.2.14)**.** *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes, with associated real Proca bundles* V_g *and* $\mathsf{V}_{g'}$ *and Proca operators* P, P'.
If the metric are paracausally related $g \simeq g'$*, then there exists a* $\mathbb{R}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$*, called* **Møller operator** *of* g, g' *(with this order), such that the following facts are true.*

(1) The restriction, called **Møller map**

$$\mathsf{S}^0 := \mathsf{R}|_{\mathsf{Ker}_{sc}(\mathsf{P})} : \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathsf{Ker}_{sc}(\mathsf{P}')$$

is well-defined vector space isomorphism with inverse given by

$$(\mathsf{S}^0)^{-1} := \mathsf{R}^{-1}|_{\mathsf{Ker}_{sc}(\mathsf{P}')} : \mathsf{Ker}_{sc}(\mathsf{P}') \to \mathsf{Ker}_{sc}(\mathsf{P}) .$$

(2) It holds $\kappa_{gg'}\mathsf{P}'\mathsf{R} = \mathsf{P}$.

(3) The causal propagators $\mathsf{G}_\mathsf{P} := \mathsf{G}^+_\mathsf{P} - \mathsf{G}^-_\mathsf{P}$ *and* $\mathsf{G}_{\mathsf{P}'} := \mathsf{G}^+_{\mathsf{P}'} - \mathsf{G}^-_{\mathsf{P}'}$*, respectively of* P *and* P'*, satisfy* $\mathsf{R}\mathsf{G}_\mathsf{P}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{P}'}$.

(4) It holds $\mathsf{R}^{\dagger_{gg'}}\mathsf{P}'\kappa_{g'g}|_{\Gamma_c(\mathsf{V}_g)} = \mathsf{P}|_{\Gamma_c(\mathsf{V}_g)}$*, where the adjoint* $^{\dagger_{gg'}}$ *is defined in Definition 2.1.23.*

2.2.1 The Proca operator as a constrained Klein Gordon field

We will frequently deal with real smooth k-forms $\mathfrak{f}, \mathfrak{h} \in \Omega^k(\mathsf{M})$, where $k = 0, \ldots, n = \dim \mathsf{M}$ (and one usually adds $\Omega^{n+1}(\mathsf{M}) = \Omega^{-1}(\mathsf{M}) = \{0\}$). The **Hodge** *real* **inner product** can be computed by integrating the fiberwise product with respect to the volume form induced by g:

$$(\mathfrak{f}|\mathfrak{h})_{g,k} := \int_\mathsf{M} \mathfrak{f} \wedge *\mathfrak{h} = \int_\mathsf{M} g^\sharp_{(k)}(\mathfrak{f}, \mathfrak{h}) \,\mathrm{vol}_g ,$$

where at least one of the two forms has compact support and $g^\sharp_{(k)}$ is the natural inner product of k-forms induced by g. This symmetric real scalar product $(\cdot|\cdot)_{g,k}$ is always non-degenerate but it is not positive when g is Lorentzian as in the considered case. It is positive when g is Riemannian. If $k = 1$, we simply write

$$(\mathfrak{f}|\mathfrak{h})_g = \int_\mathsf{M} g^\sharp(\mathfrak{f}, \mathfrak{h}) \,\mathrm{vol}_g . \tag{2.2.1}$$

In this context, $d^{(k)} : \Omega^k(\mathsf{M}) \to \Omega^{k+1}(\mathsf{M})$ is the exterior derivative and $\delta_g^{(k)} : \Omega^k(\mathsf{M}) \to \Omega^{k-1}(\mathsf{M})$ is the codifferential operator acting on the relevant spaces of smooth k-forms $\Omega^k(\mathsf{M})$ on M depending on the metric g on M. $d^{(k)}$ and $\delta_g^{(k+1)}$ are the **formal adjoint** of one another with respect to the Hodge product (2.2.1) in the sense that

$$(d^{(k)}\mathfrak{f}|\mathfrak{h})_{g,k+1} = (\mathfrak{f}|\delta_g^{(k+1)}\mathfrak{h})_{g,k}\,, \quad \forall \mathfrak{f} \in \Omega^k(\mathsf{M})\,,\ \forall \mathfrak{h} \in \Omega^{k+1}(\mathsf{M}) \quad \text{if } \mathfrak{f} \text{ or } \mathfrak{h} \text{ is compactly supported.}$$

We will often omit the indices $_{g,k}$ and $^{(k)}$ referring to the metric and the order of the used forms, when the choice of the used metric and order will be obvious from the context.

If (M, g) is globally hyperbolic, we call **Proca bundle** the real vector bundle $\mathsf{V}_g := (\mathsf{T}^*\mathsf{M}, g^\sharp)$ obtained by endowing the cotangent bundle with the fiber metric given by the dual metric $g^\sharp$ (also appearing in (2.2.1)) defined by

$$g^\sharp(\omega_p, \omega'_p) := g(\sharp\omega_p, \sharp\omega'_p) \quad \text{for every } \omega, \omega' \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M},$$

where $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{T}\mathsf{M})$ is the standard musical isomorphism.

By construction $\Gamma(\mathsf{V}_g) = \Omega^1(\mathsf{M})$ and $\Gamma_c(\mathsf{V}_g) = \Omega^1_c(\mathsf{M})$. Here and henceforth $\Omega^k_c(\mathsf{M}) \subset \Omega^k(\mathsf{M})$ is the subspace of compactly supported real smooth k-forms on M.

The formally selfadjoint **Proca operator** P on (M, g) is defined by choosing a (mass) constant $m > 0$, *the same for all globally hyperbolic metrics we will consider on* M *in this work*,

$$\mathsf{P} = \delta d + m^2 : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g), \tag{2.2.2}$$

where $d := d^{(1)}$, $\delta := \delta_g^{(2)}$. Actually P depends also on g, but we shall not indicate those dependencies in the notation for the sake of shortness.

The **Proca equation** we shall consider reads

$$\mathsf{P}A = 0 \quad \text{for } A \in \Gamma_{sc}(\mathsf{V}_g)\,, \tag{2.2.3}$$

where, as said above, $\Gamma_{sc}(\mathsf{V}_g)$ is the space of real smooth 1-forms which have compact support on the Cauchy surfaces of the globally hyperbolic spacetime (M, g).

We pass to tackle the issue of normal hyperbolicity of P. As we shall see here, it is not really necessary to construct the Møller maps, and the weaker requirement of *Green hyperbolicity* is sufficient.

Let N be the **Klein-Gordon operator** associated to the Proca operator P (2.2.2) acting on 1-forms

$$\mathsf{N} := \delta d + d\delta + m^2 : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)\,. \tag{2.2.4}$$

Notice that this operator is **normally hyperbolic**: its principal symbol σ_{N} satisfies

$$\sigma_{\mathsf{N}}(\xi) = -g^\sharp(\xi, \xi)\,\mathrm{Id}_{\mathsf{V}_g} \quad \text{for all } \xi \in \mathsf{T}^*\mathsf{M}, \text{ where } \mathrm{Id}_{\mathsf{V}_g} \text{ is the identity automorphism of } \mathsf{V}_g. \tag{2.2.5}$$

Therefore the Cauchy problem for N is well-posed [3, 6]. Both N and P are formally selfadjoint with respect to the Hodge scalar product (2.2.1) on $\Omega^1_c(\mathsf{M}) = \Gamma_c(\mathsf{V}_g)$.

Since $m^2 > 0$ and $\delta_g^{(1)}\delta_g^{(2)} = 0$, it is easy to prove that the Proca equation (2.2.3) is *equivalent* to the pair of equations

$$\mathsf{N}A = 0\,, \quad \text{for } A \in \Gamma_{sc}(\mathsf{V}_g)\,, \tag{2.2.6}$$

$$\delta A = 0\,. \tag{2.2.7}$$

As already noticed, differently from N, the Proca operator is not normally hyperbolic. However, it is **Green hyperbolic** [3, 6, 12] as N, i.e. there exist linear maps, dubbed **advanced Green operator** $\mathsf{G}^+_{\mathsf{P}} : \Gamma_{pc}(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ and **retarded Green operator** $\mathsf{G}^-_{\mathsf{P}} : \Gamma_{fc}(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ uniquely defined by the requirements (i.a) $\mathsf{G}^+_{\mathsf{P}} \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^+_{\mathsf{P}}\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$,

(ii.a) $\mathrm{supp}\,(\mathsf{G}^+_{\mathsf{P}}\mathfrak{f}) \subset J^+(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$;

(i.b) $\mathsf{G}^-_{\mathsf{P}} \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^-_{\mathsf{P}}\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{V}_g)$,

(ii.b) $\mathrm{supp}\,(\mathsf{G}^-_{\mathsf{P}}\mathfrak{f}) \subset J^-(\mathrm{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{V}_g)$;

The **causal propagator** of P is defined as

$$\mathsf{G}_{\mathsf{P}} := \mathsf{G}^+_{\mathsf{P}} - \mathsf{G}^-_{\mathsf{P}} : \Gamma_c(\mathsf{V}_g) \to \Gamma_{sc}(\mathsf{V}_g)\,. \tag{2.2.8}$$

All these maps are also continuous with respect to the natural topologies of the definition spaces [12]. As a matter of fact (see [12, Proposition 3.19] and also [6]), the advanced and retarded Green operator $\mathsf{G}^{\pm}_{\mathsf{P}} : \Gamma_{pc/fc}(\mathsf{V}_g) \to \Gamma_{pc/fc}(\mathsf{V}_g)$ can be written as

$$\mathsf{G}^{\pm}_{\mathsf{P}} := \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{G}^{\pm}_{\mathsf{N}} = \mathsf{G}^{\pm}_{\mathsf{N}}\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)$$

where $\mathsf{G}^{\pm}_{\mathsf{N}}$ are the analogous Green operators for the Klein-Gordon operator N. Therefore

$$\mathsf{G}_{\mathsf{P}} := \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{G}_{\mathsf{N}} = \mathsf{G}_{\mathsf{N}}\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\,. \tag{2.2.9}$$

The fact that P is Green hyperbolic can be proved just by checking that the operators above satisfy the requirements which define the Green operators as stated above, using the analogous properties for $\mathsf{G}^{\pm}_{\mathsf{N}}$.

Eq. (2.2.9) and the analogous properties for G_{N} entail

$$\mathsf{G}_{\mathsf{P}}(\Gamma_c(\mathsf{V}_g)) = \{A \in \Gamma_{sc}(\mathsf{V}_g) \mid \mathsf{P}A = 0\}\,. \tag{2.2.10}$$

Indeed, if $\mathsf{P}A = 0$ then $\mathsf{N}A = 0$ and $\delta A = 0$. If $A \in \Gamma_{sc}(\mathsf{V}_g)$, [87, Theorem 3.8] implies $A = \mathsf{G}_{\mathsf{N}}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, so that $A = \left(\mathrm{Id} + \frac{d\delta}{m^2}\right)A = \mathsf{G}_{\mathsf{P}}\mathfrak{f}$ as said.
Furthermore,

$$\mathrm{Ker}\mathsf{G}_{\mathsf{P}} = \{\mathsf{P}\mathfrak{g} \mid \mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\}\,. \tag{2.2.11}$$

Indeed, if $\mathsf{P}A = 0$ then $m^2\left(\mathrm{Id} + \frac{d\delta}{m^2}\right)\mathsf{P}A = \mathsf{N}A = 0$. If $A \in \Gamma_{sc}(\mathsf{V}_g)$, again [87, Theorem 3.8] implies that $A = \mathsf{N}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$. Since we also know that $\delta A = 0$, the form (2.2.5) of N yields $A = \mathsf{P}\mathfrak{f}$. On the other hand, if $A = \mathsf{P}\mathfrak{f}$ for some $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, then $\mathsf{G}_{\mathsf{P}}A = \mathsf{G}^+_{\mathsf{P}}\mathfrak{f} - \mathsf{G}^-_{\mathsf{P}}\mathfrak{f} = \mathfrak{f} - \mathfrak{f} = 0$.

On account of [87, Proposition 3.6], for any smooth function $\rho : \mathsf{M} \to (0, +\infty)$ also $\rho\mathsf{P}$ is Green hyperbolic and $\mathsf{G}^{\pm}_{\rho\mathsf{P}} = \mathsf{G}^{\pm}_{\mathsf{P}}\rho^{-1}$.

2.2.2 The Cauchy problem in ultrastatic spacetimes

We study here the Cauchy problem for the Proca (real and complex) field in ultrastatic spacetimes $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, where (Σ, h) is complete. A more general treatise appears in [98] where the Cauchy problem is studied, also in the presence of a source of the Proca field, in a generic globally hyperbolic spacetime and the continuity of the solutions with respect to the initial data is focused.

Let us consider the Proca equation (2.2.3) (where $m^2 > 0$) on the above ultrastatic spacetime. As observed in [44], every smooth 1-form $A \in \Omega^1(\mathsf{M})$ naturally uniquely decomposes as

$$A(t,p) = A^{(0)}(t,p)dt + A^{(1)}(t,p) \tag{2.2.12}$$

where $A^{(i)}(t,\cdot) \in \Omega^i(\Sigma)$ for $i = 0, 1$ and $t \in \mathbb{R}$. By direct inspection and taking the equivalence of (2.2.3) and (2.2.6)-(2.2.7) into account, one sees that Eq. (2.2.3) is equivalent to the constrained double Klein-Gordon system

$$\begin{aligned} \partial_t^2 A^{(0)} &= -(\Delta_h^{(0)} + m^2)A^{(0)}\,, && (2.2.13)\\ \partial_t^2 A^{(1)} &= -(\Delta_h^{(1)} + m^2)A^{(1)}\,, && (2.2.14)\\ \partial_t A^{(0)} &= -\delta_h^{(1)} A^{(1)}\,. && (2.2.15) \end{aligned}$$

Above, $\Delta_h^{(k)} := \delta_h^{(k+1)} d^{(k)} + d^{(k-1)}\delta_h^{(k)}$ is the Hodge Laplacian on (Σ, h) for k-forms and the last condition (2.2.15) is nothing but the constraint $\delta_g^{(1)} A = 0$ arising from (2.2.3).

The theory for the fields $A^{(1)}$ and $A^{(0)}$ is a special case of the theory of *normally hyperbolic equations on corresponding vector bundles with positive inner product* over a globally hyperbolic spacetime [3, 6]. In our case,

(1) there is a real vector bundle $\mathsf{V}_g^{(1)}$ with basis M, canonical fiber isomorphic to $T_q^*\Sigma$, and equipped with a fiberwise real symmetric scalar product induced by $h_q^\sharp$. $A^{(1)} \in \Gamma(V_g^{(1)})$;

(2) there is another real vector bundle $\mathsf{V}_g^{(0)}$ with basis M, canonical fiber isomorphic to $\mathbb{R}$, and equipped with a positive fiberwise real symmetric scalar product given by the natural product in $\mathbb{R}$. $A^{(0)} \in \Gamma(V_g^{(0)})$.

Evidently

$$\mathsf{V}_g = \mathsf{V}_g^{(0)} \oplus \mathsf{V}_g^{(1)}\,. \tag{2.2.16}$$

Equations (2.2.13) and (2.2.14) admit existence and uniqueness theorems for smooth compactly supported Cauchy data and corresponding smooth spacelike compact solutions in $\Gamma_{sc}(\mathsf{V}_g^{(0)})$ and $\Gamma_{sc}(\mathsf{V}_g^{(1)})$ respectively, as a consequence of very well-known results in the theory of normally hyperbolic equations [3, 6, 66]. However, when viewing $A^{(0)}$ and $A^{(1)}$ as parts of the Proca field A, we have also to deal with the additional constraint (2.2.15). Notice that (2.2.15) imposes two constraints on the Cauchy data of $A^{(0)}$ and $A^{(1)}$ on Σ:

$$\partial_t A^{(0)}(0,p) = -\delta_h^{(1)} A^{(1)}(0,p) \qquad \partial_t^2 A^{(0)}(0,p) = -\delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

The second constraint is only apparently of the second order. Indeed, taking (2.2.13) into account, it can be re-written as a condition of the Cauchy data

$$(\Delta_h^{(0)} + m^2)A^{(0)}(0,p) = \delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

At this juncture we observe that, with some elementary computation (use $\Delta_h^{(0)}\delta_h^{(1)} = \delta_h^{(1)}\Delta_h^{(1)}$), Equations (2.2.13) and (2.2.14) imply also the crucial condition

$$(\partial_t^2 + \Delta_h^{(0)} - m^2)(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)}) = 0$$

which, in turn, implies Equation (2.2.15) , if the initial condition of that scalar Klein-Gordon equation for $(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)})$ are the zero initial conditions. This exactly amounts to

$$\partial_t A^{(0)}(0,p) = -\delta_h^{(1)} A^{(1)}(0,p) \qquad \text{and} \qquad (\Delta_h^{(0)} + m^2)A^{(0)} = \delta_h^{(1)} \partial_t A^{(1)}(0,p)\,.$$

In summary, we are naturally led to focus on this Cauchy problem

$$\begin{aligned} &\partial_t^2 A^{(0)} + (\Delta_h^{(0)} + m^2)A^{(0)} = 0\,, && (2.2.17)\\ &\partial_t^2 A^{(1)} + (\Delta_h^{(1)} + m^2)A^{(1)} = 0\,, && (2.2.18)\\ &(\partial_t^2 + \Delta_h^{(0)} - m^2)(\partial_t A^{(0)} + \delta_h^{(1)} A^{(1)}) = 0\,, && (2.2.19) \end{aligned}$$

with initial data

$$A^{(0)}(0,\cdot) = a^{(0)}(\cdot), \quad \partial_t A^{(0)}(0,\cdot) = \pi^{(0)}(\cdot), \quad A^{(1)}(0,\cdot) = a^{(1)}(\cdot), \quad \partial_t A^{(1)}(0,\cdot) = \pi^{(1)}(\cdot) \quad (2.2.20)$$

where $a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}$ are pairs of smooth compactly supported, respectively 0 and 1, forms on Σ, and the constraints are valid

$$\pi^{(0)} = -\delta_h^{(1)} a^{(1)}\,, \quad (\Delta_h^{(0)} + m^2) a^{(0)} = \delta_h^{(1)} \pi^{(1)}\,. \quad (2.2.21)$$

If A is a spacelike compact solution of the Proca equation (2.2.3), then it satisfies (2.2.13)-(2.2.15) and its Cauchy data (2.2.20) satisfy the constraints (2.2.21). On the other hand, if we have smooth compactly supported Cauchy data (2.2.20), then the two Klein-Gordon equations (2.2.13) and (2.2.14) admit unique spacelike compact smooth solutions which also satisfies (2.2.19) as a consequence. If the said Cauchy data satisfy the constraint (2.2.21), then also (2.2.15) is satisfied, because it is equivalent to the unique solution of (2.2.19) with zero Cauchy data. In that case, the two solutions $A^{(0)}$ and $A^{(1)}$ define a unique solution of the Proca equation with the said Cauchy data.

We have established the following result completely extracted from the theory of normally hyperbolic equations.

Proposition 2.2.1. *Let $(\mathsf{M}, g) = (\Sigma, -dt \otimes dt + h)$ be a smooth globally hyperbolic ultrastatic spacetime with dt past directed, where h is a smooth complete Riemannian metric on Σ. Consider the Cauchy problem on (M, g) for the smooth 1-form A satisfying the Proca equation (2.2.3) for $m^2 > 0$, with smooth compactly supported Cauchy data (2.2.20) on Σ viewed as the $t = 0$ time slice.*
The Proca Cauchy problem for A with constraints (2.2.21) is equivalent, regarding existence and uniqueness of spacelike compact smooth solutions , to the double normally hyperbolic Klein-Gordon constrained Cauchy problem (2.2.13)-(2.2.15), for the fields $A^{(0)} \in \Gamma_{sc}(\mathsf{V}_g^{(0)})$ and $A^{(1)} \in \Gamma_{sc}(\mathsf{V}_g^{(1)})$, with the same initial data (2.2.20) and constraints (2.2.21). As a consequence,

(1) every smooth spacelike compact solution of the Proca equation $A \in \Gamma_{sc}(\mathsf{V}_g)$ (2.2.3) defines compactly supported smooth Cauchy data on Σ which satisfy the constraints (2.2.21);

(2) if the Cauchy data are smooth, compactly supported and satisfy (2.2.21), then there is a unique smooth spacelike compact solution of the Proca equation $A \in \Gamma_{sc}(\mathsf{V}_g)$ (2.2.3) associated to them;

(3) the support of a solution $A \in \Gamma_{sc}(\mathsf{V}_g)$ with smooth compactly supported initial data satisfies supp$(A) \subset J^+(S) \cup J^-(S)$, where $S \subset \Sigma$ is the union of the supports of the Cauchy data.

Remark 2.2.2. (1) All the discussion above, and Proposition 2.2.1 in particular, extends to the case of a *complex* Proca field and corresponding associated complex Klein Gordon fields. The stated results can be extended easily to the case of the non-homogeneous Proca equation and also considering continuity properties of the solutions with respect to the source and the initial data referring to natural topologies. (See [98] for a general discussion.)

(2) A naive idea may be that we can freely fix smooth compactly supported Cauchy data for $A^{(1)}$ and then define associated Cauchy conditions for $A^{(0)}$ by solving the constraints (2.2.21). In this case the true degrees of freedom of the Proca field would be the vector part $A^{(1)}$, whereas $A^{(0)}$ would be a constrained degree of freedom. This viewpoint is incorrect, if we decide to deal with spacelike compact solutions, because the second constraint in Equation (2.2.21) in general does not produce a compactly supported function $a^{(0)}$ when the source $\delta_h^{(1)}\pi^{(1)}$ is smooth compactly supported (the smoothness of $a^{(0)}$ is however guaranteed by elliptic regularity from the smoothness of $\delta_h^{(1)}\pi^{(1)}$). $a^{(0)}$ is compactly supported only

for some smooth compactly supported initial conditions $\pi^{(1)}$. Therefore the linear subspace of initial data (2.2.20) compatible with the constraints (2.2.21) does not include *all* possible compactly supported initial conditions $\pi^{(1)}$ which, therefore, cannot be freely chosen.

(3) However this space of constrained Cauchy data is non-trivial, i.e., it does not contain only zero initial conditions and in particular there are couples $(a^{(0)}, \pi^{(1)})$ such that both elements do not vanish. This is because, for every smooth compactly supported 1-form $f^{(1)}$ (with $\delta^{(1)} f^{(1)} \neq 0$ in particular) and for every smooth compactly supported 2-form $f^{(2)}$,

$$a^{(0)} := \delta_h^{(1)} f^{(1)} \qquad \pi^{(1)} := \left(\Delta_h^{(1)} + m^2\right) f^{(1)} + \delta_h^{(2)} f^{(2)}$$

are smooth, and compactly supported, they solve the nontrivial constraint in (2.2.21) $\delta_h^{(1)} \pi^{(1)} = (\Delta_{(0)} + m^2) a^{(0)}$ and $f^{(1)}, f^{(2)}$ can be chosen in order that neither of $a^{(0)}$ and $\pi^{(1)}$ vanishes. The easier constraint $\pi^{(0)} = -\delta_h^{(1)} a^{(1)}$ is solved by every smooth compactly supported 1-form $a^{(1)}$ by defining the smooth compactly supported 0-form $\pi^{(0)}$ correspondingly.

2.2.3 The Proca symplectic form in ultrastatic spacetimes

Consider two solutions $A, A' \in \Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{Ker P}$ of the Proca equation in our ultrastatic spacetime, choose $t \in \mathbb{R}$ and consider the bilinear form

$$\sigma_t^{(\mathsf{P})}(A, A') := \int_\Sigma h^\sharp(a_t^{(1)}, \pi_t^{(1)'} - da_t^{(0)'}) - h^\sharp(a_t^{(1)'}, \pi_t^{(1)} - da_t^{(0)}) \,\mathrm{vol}_h\,, \tag{2.2.22}$$

where we are referring to the Cauchy data on Σ of the smooth spacelike compact solutions of the Proca equation. Σ is viewed as the time slice at time t. As is well known, it is possible to define a natural symplectic form for the Proca field in general globally hyperbolic spacetimes [12] with properties analogous to the ones we are going to discuss here. Here we stick to the ultrastatic spacetime case which is enough for our ends.

According to [12] (with an argument very similar to the proof of Propositions 3.12 and 3.13 in [87]) we have immediately that

$$\sigma_t^{(\mathsf{P})}(A, A') = \sigma_{t'}^{(\mathsf{P})}(A, A') \quad \forall t, t' \in \mathbb{R}\,,$$

and, omitting the index t as the symplectic form is independent of it,

$$\sigma^{(\mathsf{P})}(A, A') = \int_{\mathsf{M}} g^\sharp\left(\mathfrak{f}, \mathsf{G_P}\mathfrak{f}'\right) \,\mathrm{vol}_g \tag{2.2.23}$$

where A, f (resp. A, f') are related by $A := \mathsf{G_P}\mathfrak{f}$ (resp. $A' := \mathsf{G_P}\mathfrak{f}'$).

Remark 2.2.3. The important identity (2.2.23) is also valid in a generic globally hyperbolic spacetime when $\sigma^{(P)}$ is interpreted as the general symplectic form of the Proca field according to [12].

Let us suppose to deal with the Cauchy data of the real vector space $C_\Sigma \subset \Omega^0_c(\Sigma)^2 \times \Omega^1_c(\Sigma)^2$ of smooth compactly supported Cauchy data (a_0, π_0, a_1, π_1) subjected to the linear constraints (2.2.21),

$$C_\Sigma := \left\{(a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in \Omega^0_c(\Sigma)^2 \times \Omega^1_c(\Sigma)^2 \;\middle|\; \pi^{(0)} = -\delta_h^{(1)} a^{(1)}\,, \quad (\Delta_h^{(0)} + m^2) a^{(0)} = \delta_h^{(1)} \pi^{(1)} \right\}. \tag{2.2.24}$$

Not only the Cauchy problem is well behaved in that space as a consequence of Proposition 2.2.1, but we also have the following result.

Proposition 2.2.4. *The bilinear antisymmetric map $\sigma^{(P)} : C_\Sigma \times C_\Sigma \to \mathbb{R}$ defined in (2.2.22) is non-degenerate and therefore it is a symplectic form on C_Σ.*

Proof. Taking (2.2.10) into account, suppose that $\Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{KerP} \ni A' = \mathsf{G_P}\mathfrak{f}$ whose Cauchy data are $(a^{(0)'}, \pi^{(0)'}, a^{(1)'}, \pi^{(1)'}) \in C_\Sigma$ is such that $\sigma^{(\mathsf{P})}(A, A') = 0$ for all $A = \mathsf{G_P}\mathfrak{f} \in \Gamma_{sc}(\mathsf{V}_g) \cap \mathsf{KerP} \equiv C_\Sigma$, we want to prove that $A' = 0$ namely, its initial conditions are $(0,0,0,0)$. From (2.2.23), using the fact that $g^\sharp$ is non-degenerate, we have that $A' = \mathsf{G_P}\mathfrak{f}' = 0$ so that its Cauchy data are the zero data in view of the well-posedness of the Cauchy problem Proposition 2.2.1. □

To conclude we prove that, when using Cauchy data in C_Σ, the expression of $\sigma^{(\mathsf{P})}$ can be rearranged in order to make contact with the analogous symplectic forms of the two Klein-Gordon fields $A^{(0)}$ and $A^{(1)}$ the solution A is made of, as discussed in 2.2.2. Indeed, remembering the constraint $\pi^{(0)} = -\delta_h^{(1)} a^{(1)}$, and using the duality of δ and d, part of the integral in the right-hand side of (2.2.22) can be rearranged to

$$\int_\Sigma h^\sharp(a_t^{(1)}, da_t^{(0)'}) - h^\sharp(a_t^{(1)'}, da_t^{(0)}) \operatorname{vol}_h = \int_\Sigma h^\sharp(\delta_h^{(1)} a_t^{(1)}, a_t^{(0)'}) - h^\sharp(\delta_h^{(1)} a_t^{(1)'}, a_t^{(0)}) \operatorname{vol}_h$$
$$= -\int_\Sigma h^\sharp(\pi_t^{(0)}, a_t^{(0)'}) - h^\sharp(\pi_t^{(0)'}, a_t^{(0)}) \operatorname{vol}_h .$$

As a consequence, if $\eta_i = 1$ for $i = 1$ and $\eta_i = -1$ for $i = 0$ and $h^\sharp_{(i)}$ is $h^\sharp$ for $i = 1$ and the pointwise product for $i = 0$,

$$\sigma^{(\mathsf{P})}(A, A') = \sum_{i=0}^{1} \eta_i \int_{\Sigma_t} h^\sharp_{(i)}(a_t^{(i)}, \pi_t^{(i)'}) - h^\sharp_{(i)}(a_t^{(i)'}, \pi_t^{(i)}) \operatorname{vol}_h . \tag{2.2.25}$$

In other words, referring to the (Klein-Gordon) symplectic forms introduced in [87] for normally hyperbolic equations (2.2.13) and (2.2.14)

$$\sigma^{(\mathsf{P})}(A, A') = \sigma^{(1)}(A^{(1)}, A^{(1)'}) - \sigma^{(0)}(A^{(0)}, A^{(0)'})$$

where $\sigma^{(k)}$ is the symplectic form for a normally hyperbolic field operator on a real vector bundle defined, e.g., [87, Proposition 3.12].

A similar result is valid for the causal propagators. Decomposing $\mathfrak{f} = \mathfrak{f}^{(0)} dt + \mathfrak{f}^{(1)} \in \Gamma_c(\mathsf{V}_g)$ where $\mathfrak{f}^{(0)} \in \Gamma_c(\mathsf{V}_g^{(0)})$ and $\mathfrak{f}^{(1)} \in \Gamma_c(\mathsf{V}_g^{(1)})$, (2.2.23), the analogues for scalar and vector Klein Gordon fields [87] and (2.2.25) imply

$$\int_{\mathsf{M}} g^\sharp(\mathfrak{f}, \mathsf{G_P}\mathfrak{f}') \operatorname{vol}_g = \int_{\mathsf{M}} h^\sharp(\mathfrak{f}^{(1)}, \mathsf{G}^{(1)}\mathfrak{f}^{(1)'}) \operatorname{vol}_g - \int_{\mathsf{M}} \mathfrak{f}^{(0)} \mathsf{G}^{(0)} \mathfrak{f}^{(0)'} \operatorname{vol}_g$$

where $\mathsf{G}^{(i)}$, $i = 0, 1$ are the causal propagators for the normally hyperbolic operators

$$\mathsf{N}^{(i)} := \partial_t^2 + \Delta_h^{(i)} + m^2 I : \Gamma_{sc}(\mathsf{V}_g^{(i)}) \to \Gamma_{sc}(\mathsf{V}_g^{(i)}) \quad i = 0, 1$$

according to the theory of [87]. Here $\Delta_h^{(0)}$ coincides with the standard Laplace-Beltrami operator for scalar fields on Σ.

Remark 2.2.5. With the same argument, the found results immediately generalize to the case of *complex* k-forms. More precisely, if the Cauchy data belong to $C_\Sigma + iC_\Sigma$,

$$\sigma^{(\mathsf{P})}(\overline{A}, A') = \sigma^{(1)}(\overline{A^{(1)}}, A^{(1)'}) - \sigma^{(0)}(\overline{A^{(0)}}, A^{(0)'}) ,$$

where the left-hand side is again (2.2.22) evaluated for complex Proca fields, i.e., complex Cauchy data. Above, the bar denotes the complex conjugation and the Cauchy data of the considered complex Proca fields satisfy the constraints (2.2.21). Furthermore

$$\int_{\mathsf{M}} g^\sharp(\bar{\mathfrak{f}}, \mathsf{G_P}\mathfrak{f}') \operatorname{vol}_g = \int_{\mathsf{M}} h^\sharp(\overline{\mathfrak{f}^{(1)}}, \mathsf{G}^{(1)}\mathfrak{f}^{(1)'}) \operatorname{vol}_g - \int_{\mathsf{M}} \overline{\mathfrak{f}^{(0)}} \mathsf{G}^{(0)} \mathfrak{f}^{(0)'} \operatorname{vol}_g$$

where the smooth compactly supported sections are complex. We have used the same symbols as for the real case for the causal propagators since the associated operators commute with the complex conjugation. As a consequence, a standard argument about the uniqueness of Green operators implies that the causal propagators for the real case are nothing but the restriction of the causal propagator of the complex case which, in turn, are the trivial complexification of the real ones.

2.2.4 The Proca energy density in ultrastatic spacetimes

Starting from the Proca Lagrangian in every curved spacetime (see, e.g, [42])

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \quad \text{with} \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

and referring to local coordinates $(x^0, \ldots, x^{n-1})$ adapted to the split $\mathsf{M} = \mathbb{R} \times \Sigma$ of our ultrastatic spacetime, where $x^0 = t$ runs along the whole $\mathbb{R}$ and $x^1, \ldots, x^{n-1}$ are local coordinates on Σ, the energy density reads in terms of initial conditions on Σ of the considered Proca field

$$\begin{aligned} T_{00} = \frac{1}{2} h^\sharp(\pi^{(1)} - da^{(0)}, \pi^{(1)} - da^{(0)}) + \frac{1}{2} h^\sharp_{(2)}(da^{(1)}, da^{(1)}) \\ + \frac{m^2}{2} \left(h^\sharp(a^{(1)}, a^{(1)}) + a^{(0)} a^{(0)} \right) \geqslant 0 \,. \end{aligned} \tag{2.2.26}$$

Above $h^\sharp_{(2)}$ is the natural scalar product for the 2-forms on Σ induced by the metric tensor. It is evident that the energy density is non-negative since the metric h and its inverse $h^\sharp$ are positive by hypothesis. The total energy at time t is the integral of T_{00} on Σ, using the natural volume form, when replacing $A^{(0)}$ and $A^{(1)}$ for the respective Cauchy data. As ∂_t is a Killing vector and the solution is spacelike compact, the total energy is finite and constant in time.

$$\begin{aligned} E^{(P)} = \frac{1}{2} \int_\Sigma \Big(h^\sharp(\pi^{(1)} - da^{(0)}, \pi^{(1)} - da^{(0)}) + h^\sharp_{(2)}(da^{(1)}, da^{(1)}) \\ + m^2 \big(h^\sharp(a^{(1)}, a^{(1)}) + a^{(0)} a^{(0)} \big) \Big) \mathrm{vol}_h \,. \end{aligned} \tag{2.2.27}$$

Using Hodge duality of d and δ and the definition of the Hodge Laplacian, the expression of the total energy can be re-arranged to

$$\begin{aligned} E^{(P)} = \frac{1}{2} \int_\Sigma \Big(h^\sharp(\pi^{(1)}, \pi^{(1)}) + h^\sharp(da^{(0)}, da^{(0)}) - 2h^\sharp(\pi^{(1)}, da^{(0)}) - \delta^{(1)}_h a^{(1)} \delta^{(1)}_h a^{(1)} \\ + h^\sharp\big(a^{(1)}, \Delta^{(1)}_h a^{(1)}\big) + m^2 (a^{(0)} a^{(0)} + h^\sharp(a^{(1)}, a^{(1)})) \Big) \mathrm{vol}_h \,. \end{aligned}$$

Using again the Hodge duality of d and δ the third term in the integral can be rearranged to

$$-\int_\Sigma h^\sharp(\pi^{(1)}, da^{(0)}) \mathrm{vol}_h = -\int_\Sigma a^{(0)} \delta^{(1)}_h \pi^{(1)} \mathrm{vol}_h \,.$$

The term $\delta^{(1)} \pi^{(1)}$ above and the term $\delta^{(1)}_h a^{(1)} \delta^{(1)}_h a^{(1)}$ appearing in the expression for the total energy can be worked out exploiting the constraints (2.2.21). Inserting the results in the found formula for the total energy, we finally find, with the notation already used for the symplectic form,

$$E^{(P)} = \sum_{i=0}^{1} \eta_i \frac{1}{2} \int_\Sigma h^\sharp_{(i)}(\pi^{(i)}, \pi^{(i)}) + h^\sharp_{(i)}(a^{(i)}, (\Delta^{(i)}_h + m^2 I) a^{(i)}) \, \mathrm{vol}_h \,, \tag{2.2.28}$$

when the used Cauchy data belong to the constrained space C_Σ. It is now clear that the total energy of the Proca field is the difference between the total energies of the two Klein-Gordon fields composing it exactly as it happened for the symplectic form. This difference is however positive when working on smooth compactly supported initial conditions satisfying the constraints (2.2.21), because the found expression of the energy is the same as the one computed with the density (2.2.26).

Remark 2.2.6. We notice that the negative energy component of the field can be interpreted as a ghost, in this case however no issues arise since dynamical constraints covariantly remove such a state. A different approach to the problem by generalizing to curved spacetime the Stuckelberg Lagrangian, can be found in [10], where it is apparently argued the no Hadamard states exist for the Proca field, contrarily to the results of [44] and of this work.

Remark 2.2.7. With the same argument, the found result immediately generalizes to the case of *complex* k-forms and one finds

$$\begin{aligned}\sum_{i=0}^{1}\eta_i\frac{1}{2}\int_\Sigma h^\sharp_{(i)}(\overline{\pi^{(i)}},\pi^{(i)}) + h^\sharp_{(i)}(\overline{a^{(i)}},(\Delta_h^{(i)}+m^2 I)a^{(i)})\mathrm{vol}_h = \\ = \frac{1}{2}\int_\Sigma\Big(h^\sharp(\overline{\pi^{(1)}-da^{(0)}},\pi^{(1)}-da^{(0)}) + h^\sharp_{(2)}(\overline{da^{(1)}},da^{(1)}) \\ + m^2\big(h^\sharp(\overline{a^{(1)}},a^{(1)}) + \overline{a^{(0)}}a^{(0)}\big)\mathrm{vol}_h\Big) \geqslant 0\end{aligned} \tag{2.2.29}$$

where the bar over the forms denotes the complex conjugation and $(a^{(0)},\pi^{(0)},a^{(1)},\pi^{(1)})$ are *complex* forms of $C_\Sigma + iC_\Sigma$.

2.2.5 Linear fiber-preserving isometry

As said above, to construct Møller maps for the Proca field we should be able to compare different fiberwise metrics on $\mathsf{T}^*\mathsf{M}$ when we change the metric g on M. This will be done by defining suitable fiber preserving isometries.

If g and g' are globally hyperbolic on M and $g \simeq g'$, it is possible to define a linear fiber-preserving isometry from $\Gamma(\mathsf{V}_g)$ to $\Gamma(\mathsf{V}_{g'})$ we denote with $\kappa_{g'g}$ and we shall take advantage of it very frequently in the rest of this work. In other words, if $\mathfrak{f} \in \Gamma(\mathsf{V}_g)$, then $\kappa_{g'g}\mathfrak{f} \in \Gamma(\mathsf{V}_{g'})$, the map $\kappa_{g'g} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ is $\mathbb{R}$ linear, and

$$g'^\sharp((\kappa_{g'g}\mathfrak{f})(p),(\kappa_{g'g}\mathfrak{g})(p)) = g^\sharp(\mathfrak{f}(p),\mathfrak{g}(p)) \quad \forall p \in \mathsf{M}\,.$$

Let us describe the (highly non-unique) construction of $\kappa_{gg'}$. If $\chi \in C^\infty(\mathsf{M};[0,1])$ and $g_0 \preceq g_1$, then

$$g_\chi := (1-\chi)g_0 + \chi g_1 \tag{2.2.30}$$

is a Lorentzian metric globally hyperbolic on M (see chapter 1 for details) and satisfies

$$g_0 \preceq g_\chi \preceq g_1\,.$$

Now consider the product manifold $\mathsf{N} := \mathbb{R}\times\mathsf{M}$, equipped with the indefinite non-degenerate metric

$$h := -dt\otimes dt + g_t\,,$$

where $g_t = (1-f(t))g_0 + f(t)g_\chi$ and $f:\mathbb{R}\to[0,1]$ is smooth and $f(t)=0$ for $t\leqslant 0$, $f(t)=1$ for $t \geqslant 1$. Notice that g_t is Lorentzian according to [87] because $g_0 \preceq g_\chi$ and h is indefinite non-degenerate by construction. At this point $\widetilde{\kappa}_{\chi 0} : \mathsf{TM}\to\mathsf{TM}$ is the fiber preserving diffeomorphism such that $\widetilde{\kappa}_{\chi 0}(x,v)$ is the parallel transport form $(0,x)$ to $(1,x)$ of $v \in \mathsf{T}_x\mathsf{M} \subset \mathsf{T}_{(0,x)}\mathsf{N}$ along the complete h-geodesic $\mathbb{R} \ni t \mapsto (t,x) \in \mathsf{N}$. Standard theorems on joint smoothness of the flow of ODEs depending on parameters assure that $\widetilde{\kappa}_{\chi 0} : \mathsf{TM}\to\mathsf{TM}$ is smooth. Notice that $\widetilde{\kappa}_{\chi 0}|_{\mathsf{T}_x\mathsf{M}} : \mathsf{T}_x\mathsf{M}\to\mathsf{T}_x\mathsf{M}$ is also a h-isometry from known properties of the parallel transport and thus it is a g_0, g_χ-isometry by construction because $h_{(t,x)}(v,v) = g_t(v,v)$ if $v \in \mathsf{T}_x\mathsf{M} \subset \mathsf{T}_{(t,x)}\mathsf{N}$. Taking advantage of the musical isomorphisms, $\widetilde{\kappa}_{\chi 0}$ induces a fiber-bundle map $\kappa_{\chi 0} : T^*\mathsf{M} \to T^*\mathsf{M}$ which can be seen as a map on the sections of $\Gamma(\mathsf{V}_{g_0})$ and producing sections of $\Gamma(\mathsf{V}_{g_\chi})$, preserving the metrics $g_0^\sharp$, $g_\chi^\sharp$. Then the required Proca bundle isomorphism $\kappa_{g'g} = \kappa_{g_1 g_0}$ is defined by composition:

$$\kappa_{1,0} = \kappa_{1\chi}\kappa_{\chi 0}.$$

where $\kappa_{1\chi}$ from $\Gamma(\mathsf{V}_{g_\chi})$ to $\Gamma(\mathsf{V}_{g_1})$ is defined analogously to $\kappa_{\chi 0}$. The general case $g \simeq g'$ can be defined by composing the fiber preserving linear isometries $\kappa_{g_{k+1}g_k}$ or $\kappa_{g_k, g_{k+1}}$.

2.2.6 Møller Maps and Møller Operators

We recall that a smooth Cauchy time function 1.1.14 in a globally hyperbolic spacetime (M, g) relaxes the notion of temporal Cauchy function, it is a smooth map $t : \mathsf{M} \to \mathbb{R}$ such that dt is everywhere timelike and past directed, the level surfaces of t are smooth spacelike Cauchy surfaces and (M, g) is isometric to $(\mathbb{R} \times \Sigma, h)$. Here, t identifies with the natural coordinate on $\mathbb{R}$ and the Cauchy surfaces of (M, g) identify with the sets $\{t\} \times \Sigma$.

From now on we indicate by N_0, N_1, N_χ the Klein-Gordon operators (2.2.4) on M constructed out of g_0, g_1 and g_χ respectively, where the globally hyperbolic metric g_χ is defined as in (2.2.30) (and thus $g_0 \preceq g_\chi \preceq g_1$ [87, Theorem 2.18]) and depends on the choice of a function $\chi \in C_0^\infty(\mathsf{M}, [0,1])$. Similarly, P_0, P_1, P_χ denote the Proca operators (2.2.2) on M constructed out of g_0, g_1 and g_χ respectively.

We can state the first technical result.

Proposition 2.2.8. *Let g_0, g_1 be globally hyperbolic metrics satisfying $g_0 \preceq g_1$ and let be $\chi \in C^\infty(\mathsf{M}; [0,1])$. Choose*

(a) a smooth Cauchy time g_1-function $t : \mathsf{M} \to \mathbb{R}$ and $\chi \in C^\infty(\mathsf{M}; [0,1])$ such that $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$;

(b) a pair of smooth functions $\rho, \rho' : \mathsf{M} \to (0, +\infty)$ such that $\rho(p) = 1$ for $t(p) < t_0$ and $\rho'(p) = \rho(p) = 1$ if $t(p) > t_1$. (Notice that $\rho = \rho' = 1$ constantly is allowed.)

Then the following facts are true where g_χ is defined as in (2.2.30).

(1) The operators

$$\mathsf{R}_+ : \Gamma(\mathsf{V}_{g_0}) \to \Gamma(\mathsf{V}_{g_\chi}) \qquad \mathsf{R}_+ := \kappa_{\chi 0} - \mathsf{G}^+_{\rho\mathsf{P}_\chi}\left(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0\right),$$
$$\mathsf{R}_- : \Gamma(\mathsf{V}_{g_\chi}) \to \Gamma(\mathsf{V}_{g_1}) \qquad \mathsf{R}_- := \kappa_{1\chi} - \mathsf{G}^-_{\rho\mathsf{P}_1}\left(\rho'\mathsf{P}_1\kappa_{1\chi} - \rho\kappa_{1\chi}\mathsf{P}_\chi\right)$$

are linear space isomorphisms, whose inverses are given by

$$\mathsf{R}_+^{-1} : \Gamma(\mathsf{V}_{g_\chi}) \to \Gamma(\mathsf{V}_{g_0}) \qquad \mathsf{R}_+^{-1} = \kappa_{0\chi} + \mathsf{G}^+_{\mathsf{P}_0}(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\rho\kappa_{0\chi}),$$
$$\mathsf{R}_-^{-1} : \Gamma(\mathsf{V}_{g_1}) \to \Gamma(\mathsf{V}_{g_\chi}) \qquad \mathsf{R}_-^{-1} := \kappa_{\chi 1} + \mathsf{G}^-_{\rho\mathsf{P}_\chi}\left(\rho'\kappa_{\chi 1}\mathsf{P}_1 - \rho\kappa_{1\chi}\mathsf{P}_\chi\right).$$

*By composition we define the **Møller operator**:*

$$\mathsf{R} : \Gamma(\mathsf{V}_{g_0}) \to \Gamma(\mathsf{V}_{g_1}) \qquad \mathsf{R} := \mathsf{R}_- \circ \mathsf{R}_+,$$

which is also a linear space isomorphism.

(2) It holds

$$\rho\kappa_{0\chi}\mathsf{P}_\chi\mathsf{R}_+ = \mathsf{P}_0 \qquad \textit{and} \qquad \rho'\kappa_{\chi 1}\mathsf{P}_1\mathsf{R}_- = \rho\mathsf{P}_\chi.$$

and also

$$\rho\kappa_{0\chi}\mathsf{P}_\chi = \mathsf{P}_0\mathsf{R}_+^{-1} \qquad \textit{and} \qquad \rho'\kappa_{\chi 1}\mathsf{P}_1 = \mathsf{P}_\chi\mathsf{R}_-^{-1}.$$

(3) If $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_0})$ or $\Gamma(\mathsf{V}_{g_\chi})$ respectively, then

$$(\mathsf{R}_+\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) < t_0, \tag{2.2.31}$$
$$(\mathsf{R}_-\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) > t_1\,. \tag{2.2.32}$$

Proof. First of all, we notice that the operator R_+ is well defined on the whole space $\Gamma(\mathsf{V}_{g_0})$ since for all sections $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_0})$ we have that $(\mathsf{P}_\chi \frac{\kappa_{\chi 0}}{\rho} - \frac{\kappa_{\chi 0}}{\rho}\mathsf{P}_0)\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_{g_1})$: indeed by definition, there exists a $t_0 \in \mathbb{R}$ such that on $t^{-1}(-\infty, t_0)$ and we have that $\mathsf{P}_\chi = \mathsf{P}_0$, $\kappa_{\chi 0} = \mathrm{Id}$ and t is a smooth g_1-Cauchy time function. Moreover, since $g_\chi \preceq g_1$ it follows that $\Gamma_{pc}(\mathsf{V}_{g_1}) \subset \Gamma_{pc}(\mathsf{V}_{g_\chi}) = \mathsf{Dom}(G_{\mathsf{P}_\chi})$.

To prove (1), we can first notice that

$$\begin{aligned}
\mathsf{R}_+^{-1} \circ \mathsf{R}_+ &= \Big(\kappa_{0\chi} + \mathsf{G}^+_{\mathsf{P}_0}(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\Big) \circ \Big(\kappa_{\chi 0} - \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\Big) \\
&= \mathrm{Id} - \kappa_{0\chi}\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0) + \mathsf{G}^+_{\mathsf{P}_0}(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\kappa_{\chi 0} \\
&\quad - \mathsf{G}^+_{\mathsf{P}_0}(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\,.
\end{aligned}$$

To conclude it is enough to show that everything cancels out except the identity operator, but that just follows by using basic properties of Green operators and straightforward algebraic steps. We easily see that the last addend can be recast as:

$$\begin{aligned}
&\mathsf{G}^+_{\mathsf{P}_0}(\rho\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\,\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0) \\
&= \mathsf{G}^+_{\mathsf{P}_0}\rho\kappa_{0\chi}\mathsf{P}_\chi\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0) - \mathsf{G}^+_{\mathsf{P}_0}\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0) \\
&= \mathsf{G}^+_{\mathsf{P}_0}\kappa_{0\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0) - \kappa_{0\chi}\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0),
\end{aligned}$$

which fulfils its purpose.
A specular computation proves that R_+^{-1} is also a right inverse. Almost identical reasoning prove that R_-^{-1} is a two sided inverse of R_- which is also well defined, then bijectivity of R is obvious.

(2) follows by the following direct computation

$$\begin{aligned}
\rho\kappa_{0\chi}\mathsf{P}_\chi\mathsf{R}_+ &= \rho\kappa_{0\chi}\mathsf{P}_\chi\Big(\kappa_{\chi 0} - \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi\kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\Big) \\
&= \kappa_{0\chi}\kappa_{\chi 0}\mathsf{P}_0 = \mathsf{P}_0.
\end{aligned}$$

(3) Let us prove (2.2.31). In the following P^* denotes the formal dual operator of P acting on the sections of the dual bundle $\Gamma_c(V^*_g)$. It is known that it is Green hyperbolic if P is (e.g., see [3]) and, if $\mathfrak{f}' \in \Gamma_c(\mathsf{V}^*_g)$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{V}_g)$ or $\mathfrak{f} \in \Gamma_{fc}(V_g)$ respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P^*}\mathfrak{f}', \mathfrak{f}\rangle\,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^+_P\mathfrak{f}\rangle\,\mathrm{vol}_g\,, \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P^*}\mathfrak{f}', \mathfrak{f}\rangle\,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^-_P\mathfrak{f}\rangle\,\mathrm{vol}_g\,, \tag{2.2.33}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P^*}$ indicate the Green operators of P^*. Consider now a compactly supported smooth section $\mathfrak{h}$ whose support is included in the set $t^{-1}((-\infty, t_0))$. Taking advantage of the Equation (2.2.33), we obtain

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle\,\mathrm{vol}_{g_\chi} = \int_{\mathsf{M}} \langle \mathsf{G}^-_{(\rho\mathsf{P}_\chi)^*}\mathfrak{h}, (\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle\,\mathrm{vol}_{g_\chi} = 0$$

since $\mathrm{supp}(\mathsf{G}^-_{(\rho\mathsf{P}_\chi)^*}\mathfrak{h}) \subset J^{g_\chi}_-(\mathrm{supp}(\mathfrak{h}))$ and thus that support does not meet $\mathrm{supp}((\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f})$ because $((\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f})(p)$ vanishes if $t(p) < t_0$. As $\mathfrak{h}$ is an arbitrary smooth section compactly supported in $t^{-1}((-\infty, t_0))$,

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f}\rangle\,\mathrm{vol}_{g_\chi} = 0$$

entails that $\mathsf{G}^+_{\rho\mathsf{P}_\chi}(\rho\mathsf{P}_\chi - \mathsf{P}_0)\mathfrak{f} = 0$ on $t^{-1}((-\infty, t_0))$. The proof of (2.1.15) is strictly analogous, so we leave it to the reader. □

Using Proposition 2.2.8, we can pass to the generic case $g \simeq g'$.

Theorem 2.2.9. *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes, with associated Proca bundles* V_g *and* $\mathsf{V}_{g'}$ *and Proca operators* P, P'.
If $g \simeq g'$*, then there exist (infinitely many) vector space isomorphisms,*

$$\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$$

such that

(1) referring to the said domains,

$$\mu\kappa_{gg'}\mathsf{P}'\mathsf{R} = \mathsf{P}$$

for some smooth $\mu : \mathsf{M} \to (0, +\infty)$ *(which can always be chosen* $\mu = 1$ *constantly in particular), and a smooth fiberwise isometry* $\kappa_{gg'} : \Gamma(\mathsf{V}_{g'}) \to \Gamma(\mathsf{V}_g)$.

(2) The restriction, called **Møller map**

$$\mathsf{S}^0 := \mathsf{R}|_{\mathsf{Ker}_{sc}(\mathsf{P})} : \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathsf{Ker}_{sc}(\mathsf{P}')$$

is well-defined vector space isomorphism with inverse given by

$$(\mathsf{S}^0)^{-1} := \mathsf{R}^{-1}|_{\mathsf{Ker}_{sc}(\mathsf{P}')} : \mathsf{Ker}_{sc}(\mathsf{P}') \to \mathsf{Ker}_{sc}(\mathsf{P})\,.$$

Proof. Since $g \simeq g'$, there exists a finite sequence of globally hyperbolic metrics $g_0 = g, g_1, .., g_N = g$ such that at each step $g_k \preceq g_{k+1}$ or $g_{k+1} \preceq g_k$. For all $k \in \{0, .., N\}$ we can associate to the metric a Proca operator P_k.
At each step the hypotheses of Proposition 2.2.8 are verified, in fact we can choose functions ρ_k and ρ'_k and the Møller map is given by $\mathsf{R}_k = \mathsf{R}_{k-} \circ \mathsf{R}_{k+}$. The general map is then built straightforwardly by composing the N maps constructed step by step:

$$\mathsf{R} = \mathsf{R}_N \circ ... \circ \mathsf{R}_1.$$

Regarding (1), by direct calculation we get that $\mu = \prod_{k=1}^{N} \rho'_k$, while $\kappa_{gg'} = \kappa_{g_0 g_1} \circ ... \circ \kappa_{g_{N-1} g_N}$. The proof of (2) is trivial. □

2.2.7 Møller operators and the causal propagator

We now study the interplay between Møller maps and the causal propagator of Proca operators. To this end, we use a natural extension of the notion of *adjoint operator* introduced in the previous section.

Let g and g' (possibly $g \neq g'$) globally hyperbolic metric and let V_g and $\mathsf{V}_{g'}$ be a Proca bundle on the manifold M. Consider a $\mathbb{R}$-linear operator

$$\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})\,,$$

where $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{V}_g)$ is a $\mathbb{R}$-linear subspace and $\mathsf{Dom}(\mathsf{T}) \supset \Gamma_c(\mathsf{V}_g)$.

Definition 2.2.10. An operator

$$\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{V}_{g'}) \to \Gamma_c(\mathsf{V}_g)$$

is said to be the **adjoint of** T **with respect to** g, g' (with the said order) if it satisfies

$$\int_{\mathsf{M}} g'^{\sharp}\left(\mathfrak{h}, \mathsf{T}\mathfrak{f}\right)(x)\,\mathrm{vol}_{g'}(x) = \int_{\mathsf{M}} g^{\sharp}\left(\mathsf{T}^{\dagger_{gg'}}\mathfrak{h}, \mathfrak{f}\right)(x)\,\mathrm{vol}_{g}(x) \quad \forall \mathfrak{f} \in \mathsf{Dom}(\mathsf{T})\,,\ \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}).$$

When $g = g'$, we use the simplified notation $\mathsf{T}^{\dagger} := \mathsf{T}^{\dagger_{gg}}$.

As in [87], the adjoint operator satisfies a lot of useful properties which we summarize in the following proposition. Since the proof is analogous to the one of [87, Proposition 4.11], we leave it to the reader. We will focus on the the real case only, but now we state the theorem encompassing the case where the sections are complex and the fiber scalar product is made Hermitian by adding a complex conjugation of the left entry in the usual fiberwise real $g^\sharp$ inner product, which becomes $g^\sharp(\bar{\mathfrak{f}}, \mathfrak{g})$, where the bar denotes the complex conjugation. Definition 2.2.10 extends accordingly. For this reason $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$, and the complex conjugate $\bar{c}$ reduces to c itself when $\mathbb{K} = \mathbb{R}$. We keep the notation V_g for indicating either the real or complex vector bundle $\mathsf{T}^*\mathsf{M}$ or $\mathsf{T}^*\mathsf{M} + i\mathsf{T}^*\mathsf{M}$ corresponding to two possible choices of $\mathbb{K}$.

Proposition 2.2.11. *Referring to the notion of adjoint in Definition 2.2.10, the following facts are valid.*

(1) *If the adjoint* $\mathsf{T}^{\dagger_{gg'}}$ *of* T *exists, then it is unique.*

(2) *If* $\mathsf{T} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ *is a differential operator and* $g = g'$*, then* $\mathsf{T}^{\dagger_{gg}}$ *exists and is the restriction of the formal adjoint to* $\Gamma_c(\mathsf{E})$*. (In turn, the formal adjoint of* T *is the unique extension to* $\Gamma(\mathsf{E})$ *of the differential operator* $\mathsf{T}^\dagger$ *as a differential operator.)*

(3) *Consider a pair of* $\mathbb{K}$*-linear operators* $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})$*,* $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{V}_{g'})$ *with* $\mathsf{Dom}(\mathsf{T}), \mathsf{Dom}(\mathsf{T}') \subset \Gamma(\mathsf{V}_g)$ *and* $a, b \in \mathbb{K}$*. Then*

$$(a\mathsf{T} + b\mathsf{T}')^{\dagger_{gg'}} = \bar{a}\mathsf{T}^{\dagger_{gg'}} + \bar{b}\mathsf{T}'^{\dagger_{gg'}}$$

provided $\mathsf{T}^{\dagger_{gg'}}$ *and* $\mathsf{T}'^{\dagger_{gg'}}$ *exist.*

(4) *Consider a pair of* $\mathbb{K}$*-linear operators* $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{V}_{g'})$*,* $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{V}_{g''})$ *with* $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{V}_g)$ *and* $\mathsf{Dom}(\mathsf{T}') \subset \Gamma(\mathsf{V}_{g'})$ *such that*

(i) $\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) \supset \Gamma_c(\mathsf{V}_g)$*,*

(ii) $\mathsf{T}^{\dagger_{gg'}}$ *and* $\mathsf{T}'^{\dagger_{g'g''}}$ *exist,*

then $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ *exists and*

$$(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}} = \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} .$$

(5) *If* $\mathsf{T}^{\dagger_{gg'}}$ *exists, then* $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}} = \mathsf{T}|_{\Gamma_c(\mathsf{V}_g)}$*.*

(6) *If* $\mathsf{T} : \mathsf{Dom}(T) = \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ *is bijective, admits* $\mathsf{T}^{\dagger_{gg'}}$*, and* T^{-1} *admits* $(\mathsf{T}^{-1})^{\dagger_{g'g}}$*, then* $\mathsf{T}^{\dagger_{gg'}}$ *is bijective and* $(\mathsf{T}^{-1})^{\dagger_{g'g}} = (\mathsf{T}^{\dagger_{gg'}})^{-1}$*.*

Now we are ready to prove that the operators R admit adjoints and we explicitly compute them.

Proposition 2.2.12. *Let* g_0, g_1 *be globally hyperbolic metrics satisfying* $g_0 \preceq g_1$*. Let* R_+*,* R_- *and* R *be the operators defined in Proposition 2.2.8 and fix, once and for all,* $\rho = c_0^\chi$ *and* $\rho' = c_0^1$ *where* c_0^χ*,* c_0^1 *are the unique smooth functions on* M *such that:*

$$\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0} \qquad \mathrm{vol}_{g_1} = c_0^1 \mathrm{vol}_{g_0}. \tag{2.2.34}$$

Then we have:

(1) $\mathsf{R}_+^{\dagger_{g_0 g_\chi}} : \Gamma_c(\mathsf{V}_{g_\chi}) \to \Gamma_c(\mathsf{V}_{g_0})$ *satisfies:*

$$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \left(c_0^\chi \kappa_{0\chi} - (c_0^\chi \kappa_{0\chi} \mathsf{P}_\chi - \mathsf{P}_0 \kappa_{0\chi})\, \mathsf{G}^-_{\mathsf{P}_\chi} \right) |_{\Gamma_c(\mathsf{V}_\chi)}$$

and can be recast in the form

$$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} = \mathsf{P}_0 \kappa_{0\chi} \mathsf{G}^-_{\mathsf{P}_\chi} |_{\Gamma_c(\mathsf{V}_\chi)}.$$

(2) $\mathsf{R}_-^{\dagger g_\chi g_1} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_\chi})$ *satisfies*

$$\mathsf{R}_-^{\dagger g_\chi g_1} = \left(c_1^\chi \kappa_{\chi 1} - (c_1^\chi \kappa_{\chi 1}\mathsf{P}_1 - \mathsf{P}_\chi \kappa_{\chi 1})\, \mathsf{G}^+_{\mathsf{P}_1}\right)|_{\Gamma_c(\mathsf{V}_1)},$$

and can be recast in the form

$$\mathsf{R}_-^{\dagger g_\chi g_1} = P_\chi \kappa_{\chi 1} \mathsf{G}^+_{\mathsf{P}_1}|_{\Gamma_c(\mathsf{V}_1)}.$$

(3) The map $\mathsf{R}^{\dagger g_0 g_1} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_0})$ *defined by* $\mathsf{R}^{\dagger g_0 g_1} := \mathsf{R}_+^{\dagger g_0 g_\chi} \circ \mathsf{R}_-^{\dagger g_\chi g_1}$ *is invertible and*

$$(\mathsf{R}^{\dagger g_0 g_1})^{-1} = (\mathsf{R}^{-1})^{\dagger g_1 g_0} : \Gamma_c(\mathsf{V}_{g_1}) \to \Gamma_c(\mathsf{V}_{g_0}) .$$

We call it ***adjoint Møller operator****.*
Moreover $\mathsf{R}^{\dagger g_0 g_1}$ *is a homeomorphism with respect to the natural (test section) topologies of the domain and of the co-domain.*

Proof. We start by proving points (1) and (2). For any $\mathfrak{f} \in \mathsf{Dom}(\mathsf{R}_+) = \Gamma(\mathsf{V}_{g_0})$ and $\mathfrak{h} \in \Gamma_c(\mathsf{V}_{g_\chi})$ we have

$$\int_\mathsf{M} g^\sharp_\chi\,(\mathfrak{h}, \mathsf{R}_+\mathfrak{f})\,\mathrm{vol}_{g_\chi} = \int_\mathsf{M} g^\sharp_\chi \left(\mathfrak{h}, \left(\kappa_{\chi 0} - \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\,(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\right)\mathfrak{f}\right)\mathrm{vol}_{g_\chi} =$$
$$\int_\mathsf{M} g^\sharp_\chi\,(\mathfrak{h}, \kappa_{\chi 0}\mathfrak{f})\,\mathrm{vol}_{g_\chi} - \int_\mathsf{M} g^\sharp_\chi \left(\mathfrak{h}, \left(\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\,(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\right)\mathfrak{f}\right)\mathrm{vol}_{g_\chi}.$$

We now split the problem and compute the adjoint of the two summands separately.
The adjoint of the first one follows immediately by exploiting the properties of the existing isometry and Equations (2.2.34)

$$\int_\mathsf{M} g^\sharp_\chi\,(\mathfrak{h}, \kappa_{\chi 0}\mathfrak{f})\,\mathrm{vol}_{g_\chi} = \int_\mathsf{M} g^\sharp_0\,(c_0^\chi \kappa_{0\chi}\mathfrak{h}, \mathfrak{f})\,\mathrm{vol}_{g_0}.$$

For the second summand the situation is trickier and we cannot split the calculation in two more summands since it is crucial that the whole difference $(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)$ acts on a general $\mathfrak{f} \in \Gamma(\mathsf{V}_{g_\chi})$ before we apply the Green operator whose domain is $\Gamma_{pc}(\mathsf{V}_{g_\chi})$.
So we first rewrite $\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi} = \mathsf{G}^+_{\mathsf{P}_\chi}\frac{1}{c_0^\chi}$ and use the properties of standard adjoints of Green operators for formally self-adjoint Green hyperbolic differential operators to get

$$\int_\mathsf{M} g^\sharp_\chi \left(\mathfrak{h}, \left(\mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi}\,(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0}\mathsf{P}_0)\right)\mathfrak{f}\right)\mathrm{vol}_{g_\chi} = \int_\mathsf{M} g^\sharp_\chi \left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \left(\mathsf{P}_\chi \kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\right)\mathfrak{f}\right)\mathrm{vol}_{g_\chi}.$$

Now we are tempted to exploit the linearity of the integral and of the fiber product, but first, to ensure that the two integrals individually converge, we need to introduce a cutoff function:

- We notice again that there is a Cauchy surface of the foliation Σ_{t_0} such that for all leaves with $t < t_0$ the operator $\left(\mathsf{P}_\chi \kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c_0^\chi}\mathsf{P}_0\right) = 0$;
- So take a $t' < t_0$ and define a cutoff smooth function $s : \mathsf{M} \to [0, 1]$ such that $s = 0$ on all leaves with $t < t'$ and $s = 1$ for $t > t_0$.

In this way we are allowed to rewrite our last integral and split it in two convergent summands without modifying its numerical value.

$$\int_{\mathsf{M}} g^\sharp_\chi\Big(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, (\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c^\chi_0}\mathsf{P}_0)s\mathfrak{f}\Big)\mathrm{vol}\,_{g_\chi} =$$

$$= \int_{\mathsf{M}} g^\sharp_\chi\left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \mathsf{P}_\chi\kappa_{\chi 0}s\mathfrak{f}\right)\mathrm{vol}\,_{g_\chi} - \int_{\mathsf{M}} g^\sharp_\chi\left(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \frac{\kappa_{\chi 0}}{c^\chi_0}\mathsf{P}_0 s\mathfrak{f}\right)\mathrm{vol}\,_{g_\chi}$$
$$= \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left(c^\chi_0\kappa_{0\chi}\mathsf{P}_\chi\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f}\right)\mathrm{vol}\,_{g_0} - \int_{\mathsf{M}} \mathsf{g}^\sharp_0(\mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f})\mathrm{vol}\,_{g_0}$$
$$= \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left((c^\chi_0\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, s\mathfrak{f}\right)\mathrm{vol}\,_{g_0}$$
$$= \int_{\mathsf{M}} \mathsf{g}^\sharp_0\left((c^\chi_0\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{h}, \mathfrak{f}\right)\mathrm{vol}\,_{g_0}.$$

where in the last identities we have used properties of the standard adjoints of the formally self-adjoint operators, of the isometries and of the cutoff function.
Since the domain of the operator is just made up of compactly supported sections, we may exploit the inverse property of the Green operators to immediately obtain that

$$c^\chi_0\kappa_{0\chi} - (c^\chi_0\kappa_{0\chi}\mathsf{P}_\chi - \mathsf{P}_0\kappa_{0\chi})\,\mathsf{G}^-_{\mathsf{P}_\chi}|_{\Gamma_c(\mathsf{V}_\chi)} = \mathsf{P}_0\kappa_{0\chi}\mathsf{G}^-_{\mathsf{P}_\chi}|_{\Gamma_c(\mathsf{V}_\chi)}.$$

To see that the image of the operators is indeed compactly supported we can focus on $\mathsf{R}^{\dagger g_0 g_\chi}$, the rest follows straightforwardly. The first summand $c^\chi_0\kappa_{0\chi}$ does not modify the support of the sections, whereas the second does. Let us fix $\mathfrak{f} \in \Gamma_c(\mathsf{V}_{g_\chi})$, then $\mathrm{supp}\,(\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f}) \subset J^-_{g_\chi}(\mathrm{supp}\,\mathfrak{f})$ which means that $\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f} \in \Gamma_{sfc}$, i.e it is space-like and future compact. The thesis follows by again observing that there is a Cauchy surface such that in its past $\left(\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c^\chi_0}\mathsf{P}_0\right)\mathsf{G}^-_{\mathsf{P}_\chi}\mathfrak{f} = 0$.
The computation of the adjoint of R_- is almost identical to the one just performed.

The first part of (3) is an immediate consequence of property (4) in Proposition 2.1.26, while the invertibility of the adjoint can be proved by explicitly showing that the operator

$$(\mathsf{R}^{\dagger g_0 g_\chi}_+)^{-1} = \left(\frac{\kappa_{\chi 0}}{c^\chi_0} + \left(\mathsf{P}_\chi\kappa_{\chi 0} - \frac{\kappa_{\chi 0}}{c^\chi_0}\mathsf{G}^-_{\mathsf{P}_0}\right)\right)\Big|_{\Gamma_c(\mathsf{V}_{g_0})}$$

serves as a left and right inverse of $\mathsf{R}^{\dagger g_0 g_\chi}_+$. An analogous argument can be used for $\mathsf{R}^{\dagger g_\chi g_1}_-$.
The continuity of both the adjoint and its inverse comes by the same arguments used in the proof of [87, Theorem 4.12] (with the only immaterial difference that this time the smooth isometry $\kappa_{\chi 0}$ is included in the definition of the Møller operator.) □

Remark 2.2.13. An interesting fact to remark is that having defined the adjoints over compactly supported sections makes the dependence on the auxiliary volume fixing functions disappear.

We conclude by proving the second part of Theorem 3.

Theorem 2.2.14. *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes, with associated Proca bundles* V_g *and* $\mathsf{V}_{g'}$ *and Proca operators* P, P'.
If $g \simeq g'$, *it is possible to specialize the* $\mathbb{R}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$ *of Proposition 2.1.20 such that the following further facts are true.*

(1) The causal propagators G_P *and* $\mathsf{G}_{\mathsf{P}'}$ *(2.2.8), respectively of* P *and* P', *satisfy*

$$\mathsf{R}\mathsf{G}_\mathsf{P}\mathsf{R}^{\dagger g g'} = \mathsf{G}_{\mathsf{P}'}\,.$$

(2) It holds

$$\mathsf{R}^{\dagger g g'}\mathsf{P}'\kappa_{g' g}|_{\Gamma_c(\mathsf{V}_g)} = \mathsf{P}|_{\Gamma_c(\mathsf{V}_g)}\,.$$

R *as above is called* **Møller operator** *of* g, g' *(with this order).*

Proof. Since $g \simeq g'$ and the Møller map is defined as the composition $\mathsf{R} = \mathsf{R}_N \circ ... \circ \mathsf{R}_1$, we can use properties (4) in Proposition 2.1.26 and reduce to the case where $g = g_0 \preceq g_1 = g'$. With this assumption, (2) can be obtained following the proof of Proposition 2.2.8. So we leave it to the reader.

It remains to prove (1). Decomposing R as above, we define the maps $\mathsf{R}_\pm^{g_0 g_\chi}$, $\mathsf{R}_\pm^{g_\chi g_1}$ by choosing the various arbitrary functions as in Proposition 2.2.12. We first notice

$$\mathsf{R}_+ \mathsf{G}^+_{\mathsf{P}_0} \mathsf{R}_+^{\dagger g_0 g_\chi} = \left(\kappa_{\chi 0} - \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi} \left(c_0^\chi \mathsf{P}_\chi \kappa_{\chi 0} - \kappa_{\chi 0} \mathsf{P}_0\right)\right) \mathsf{G}^+_{\mathsf{P}_0} \left(\mathsf{P}_0 \kappa_{0\chi} \mathsf{G}^-_{\mathsf{P}_\chi}\right)|_{\Gamma_c(\mathsf{V}_\chi)}$$

$$= \mathsf{G}^+_{c_0^\chi \mathsf{P}_\chi} \kappa_{\chi 0} \left(\mathsf{P}_0 \kappa_{0\chi} \mathsf{G}^-_{\mathsf{P}_\chi}\right)|_{\Gamma_c(\mathsf{V}_\chi)} = \mathsf{G}^+_{\mathsf{P}_\chi} - \mathsf{G}^+_{\mathsf{P}_\chi} \left(\mathsf{P}_\chi - \frac{\kappa_{\chi 0}}{c_0^\chi} \mathsf{P}_0 \kappa_{0\chi}\right) \mathsf{G}^-_{\mathsf{P}_\chi}.$$

where the first equality follows by definition, in the second one we have used the properties of Green operators, while in the third one we have just equated the two expressions for the adjoint operator according to (1) in Proposition 2.2.12 and performed some trivial algebraic manipulations.
Another analogous computation can be performed for the retarded Green operator yielding

$$\mathsf{R}_+ \mathsf{G}^-_{\mathsf{P}_0} \mathsf{R}_+^{\dagger g_0 g_\chi} = \mathsf{G}^-_{\mathsf{P}_\chi} - \mathsf{G}^+_{\mathsf{P}_\chi} \left(\mathsf{P}_\chi - \frac{\kappa_{\chi 0}}{c_0^\chi} \mathsf{P}_0 \kappa_{0\chi}\right) \mathsf{G}^-_{\mathsf{P}_\chi}.$$

Therefore, subtracting the two terms we get

$$\mathsf{R}_+ \mathsf{G}_{\mathsf{P}_0} \mathsf{R}_+^{\dagger g_0 g_\chi} = \mathsf{R}_+ (\mathsf{G}^+_{\mathsf{P}_0} - \mathsf{G}^-_{\mathsf{P}_0}) \mathsf{R}_+^{\dagger g_0 g_\chi} = \mathsf{G}_{\mathsf{P}_\chi}.$$

Applying now R_- and its adjoint we get the claimed result. □

2.3 Conclusions

In this chapter we have seen that an interesting family of (infinitely many) geometric isomorphisms can be constructed to relate the solution spaces of three classes of Green hyperbolic operators under variations of the background geometry, namely normally hyperbolic and Proca operators. For the latter also the Cauchy problem has been studied in detail and the energy density has been presented, ready to be used in the next chapter to construct Hadamard states.
For the two classes of operators the Møller maps depend on various elements: the paracusal chain chosen, the interpolating spacetimes and the chosen interpolating operators.
The strategies only differ in some aspects related to the chosen interpolating operators:

- for normally hyperbolic operators, at each step, the interpolating operator was a convex combination of the two starting operators, since it was proved to be Green hyperbolic on an interpolating spacetime;
- for the Proca operator every strategy based on convex combinations is doomed to fail, so we have developed one which is probably the most general and would work also for the previous cases: to associate to the interpolating spacetime a Proca operator built out of its geometry.

The last approach suggests that this strategy may work to study the solution space of any Green hyperbolic operator under variations of the background geometry, even though the problem of building interpolating operators preserving Green hyperbolicity by convex combinations is still interesting on its own, since such a space of operators is not stable under linear combinations.
More specifically whenever we have smooth spacetime manifolds (M, g) and (M, g') with $g \preceq g'$ and the associated interpolating spacetime (M, g_χ), if there is a rule to map globally hyperbolic

spacetime metrics to Green hyperbolic operators to $g \to \mathsf{P}_g$ in a way that P_{g_χ} equals P_g in the past of some Cauchy surface Σ_t and equals $\mathsf{P}_{g'}$ in the future of another one $\Sigma_{t'}$ with $t' > t$, then a Møller operator and a related Møller map are supposed to exist and the kernels of these operators can be compared in the ways discussed throughout all this section.
Some possible operators whose solutions may be compared through Møller maps are Dirac type operators, twisted Dirac operators, Buchdahl operators and the Rarita-Schwinger operator [6].

Another interesting problem comes from the possibility to extend such a procedure to non-Green hyperbolic operators like the ones describing gauge theories: for example to the abelian Maxwell field. The problem has not been tackled because it does not seem to be compatible with the quantization of the Møller operators and in the quantum realm the studied procedure finds its main applications. However a Møller map for the classical electromagnetic field is supposed to exist and could be constructed in Lorentz gauge exploiting the Møller operator for normally hyperbolic field theories.

Chapter 3

The Møller *-isomorphism and Hadamard states

Since the spaces of solutions compared in the previous chapter are the first step in the construction of corresponding (algebraic) free quantum field theories, a natural related issue concerns the possibility to promote the Møller map R to a $*$-isomorphism between the associated abstract operator algebras $\mathcal{A}$ and $\mathcal{A}'$ constructed out of the Green hyperbolic operators P and P' respectively on (M, g) and (M, g'), in terms of corresponding generators given by *abstract field operators* $\Phi(\mathfrak{h})$ and $\Phi'(\mathfrak{h}')$ and the associated *causal propagators* G_P, $\mathsf{G}_{\mathsf{P}'}$. Actually, *off-shell* linear QFT can be used to build up a perturbative approach to interacting QFT, a final problem would concern the possibility to extend the Møller isomorphism of algebras to an isomorphism of more physically interesting algebras, for instance including Wick powers or time-ordered powers.
Therefore one of the aims of this chapter is to investigate the role of the Møller operator at the quantum level. In order to achieve our goal, we will follow the so-called algebraic approach to quantum field theory, see *e.g.* [6, 7, 12, 47, 76]. In *loc. cit.* the quantization of a free field theory on a (curved) spacetime is interpreted as a two-step procedure:

1. The first consists of the assignment to a physical system of a $*$-algebra of observables which encodes structural properties such as causality, dynamics and canonical commutation relations.

2. The second step calls for the identification of an algebraic state, which is a positive, linear and normalized functional on the algebra of observables.

Using this framework, in this chapter we shall lift the action of the Møller operators on the algebras of the free quantum fields and then we will pull-back the action of the Møller operators on quantum states, showing that the maps preserve the Hadamard condition, which will be discussed precisely later, with quite weak hypotheses which, in principle, permit an extension of the theory to a perturbative approach. Existence of Hadamard states in general globally hyperbolic spacetimes is then a consequence of the fact that any spacetime is paracausally related to an ultrastatic one, where Hadamard states are known to exist.
For a more detailed introduction to the algebraic approach to quantum field theory we refer to [18, 54] for textbook and to [11–13, 20, 23, 24, 27–33, 35, 56–63] for some recent applications.

The second aim of this chapter is to characterize Hadamard states for the Proca field. In [44] Proca Hadamard states are defined and, just for Cauchy compact spacetimes, a state is constructed in ultrastatic spacetimes and a standard deformation argument is employed to prove their existence on general globally hyperbolic spacetimes. In this work we aim to prove that the ad hoc definition given in that work is equivalent to the standard one in term of wavefront sets, then we employ techniques coming from microlocal analysis and elliptic Hilbert complexes to construct a state on a general ultrastatic globally hyperbolic spacetime without topological assumptions on its Cauchy surfaces and prove that it satisfies the Hadamard condition. By properties of Møller operators we obtain the existence on general globally hyperbolic spacetimes.

The chapter is structured as follows. We begin section 3.1 by discussing the CCR algebraic quantization of the normally hyperbolic Klein-Gordon field, reviewing in particular states and Hadamard states and then perform the construction outlined above.
Secondly 3.2 for the Proca field we describe in detail its CCR quantization, the construction of the Hadamard states first in ultrastatic, and then in general globally hyperbolic spacetime. In the end we compare the standard definition of Hadamard states with the Fewster-Pfenning one. We close the chapter and the whole work in 3.3 by discussing possible future research lines related to the topic treated in this chapter.

3.1 The normally hyperbolic quantum field

In this section the on-shell and off-shell CCR $*$-algebras describing the quantization of the normally hyperbolic Klein Gordon field is introduced along with the procedure to promote the Møller operators studied in the previous chapter to $*$-isomorphisms of $*$-algebras. Then states and Hadamard states over these algebras are introduced in generality and their importance and physical relevance is discussed. Finally the most important feature of the Møller operator is revealed: the pullback of a Hadamard state of the spacetime (M, g) is a Hadamard state of the spacetime (M, g'), if g and d' are paracausally related. In this sense Møller operators implement via explicit operators the standard deformation argument known to prove the existence of Hadamard states in general globally hyperbolic spacetimes, [50, 51].

In fact, Theorem 2 allows us to promote R to a $*$-isomorphism of the algebras of field operators $\mathcal{A}$, $\mathcal{A}'$ respectively associated to the paracausally related metrics g and g' (and the associated N, N') and generated by respective field operators $\Phi(\mathfrak{f})$ and $\Phi'(\mathfrak{f}')$ with $\mathfrak{f}, \mathfrak{f}'$ compactly supported smooth sections of E. These field operators satisfy respective CCRs

$$[\Phi(\mathfrak{f}), \Phi(\mathfrak{h})] = i\mathsf{G}_{\mathsf{N}}(\mathfrak{f}, \mathfrak{h})\mathbb{I}\,, \quad [\Phi'(\mathfrak{f}'), \Phi'(\mathfrak{h}')] = i\mathsf{G}_{\mathsf{N}'}(\mathfrak{f}', \mathfrak{h}')\mathbb{I}'$$

and the said unital $*$-algebra isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ is determined by the requirement (Proposition 3.1.5)

$$\mathcal{R}(\Phi'(\mathfrak{f}')) = \Phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f})\,.$$

The final important result regards the properties of $\mathcal{R}$ for the algebras of a pair of paracausally related metrics g, g' when it acts on the states $\omega : \mathcal{A} \to \mathbb{C}$, $\omega' : \mathcal{A}' \to \mathbb{C}$ of the algebras in terms of pull-back.

$$\omega' = \omega \circ \mathcal{R}\,.$$

As is known, the most relevant (quasifree) states in algebraic QFT are *Hadamard states* characterized by a certain wavefront set of their two-point function. To this regard, we prove that the pull-back through $\mathcal{R}$ of a Hadamard state $\omega : \mathcal{A} \to \mathbb{C}$ is a Hadamard state of the off-shell algebra $\mathcal{A}'$, provided the metrics g, g' be paracausally related. The result is extended to a generic bidistribution ν (corresponding to the two-point function of ω, dropping the remaining requirements included in the definition of state). The proof of the theorem below is both of geometrical and microlocal analytic nature (see also Theorem 3.1.13).

Theorem 4 (Theorem 3.2.8)**.** *Let* E *be an* $\mathbb{R}$*-vector bundle on a smooth manifold* M *equipped with a non-degenerate, symmetric, fiberwise metric* $\langle \cdot | \cdot \rangle$*. Let* $g, g' \in \mathcal{GH}_{\mathsf{M}}$*, consider the corresponding formally-selfadjoint normally hyperbolic operators* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *and refer to the associated CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$*.*
Let $\nu \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$ *be of Hadamard type and satisfy*

$$\nu(x, y) - \nu(y, x) = i\mathsf{G}_{\mathsf{N}}(x, y) \quad mod \quad C^\infty\,,$$

$\mathsf{G}_\mathsf{N}(x,y)$ *being the distributional Kernel of* G_N.
Assuming $g \simeq g'$, *let us define*

$$\nu' := \nu \circ \mathsf{R}^{\dagger_{gg'}} \otimes \mathsf{R}^{\dagger_{gg'}} \,,$$

for a Møller operator $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *of* g, g'. *Then the following facts are true.*

(i) ν *and* ν' *are bisolutions mod* C^∞ *of the field equations defined by* N *and* N' *respectively,*

(ii) $\nu' \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$,

(iii) $\nu'(x,y) - \nu'(y,x) = i\mathsf{G}_{\mathsf{N}'}(x,y)$ *mod* C^∞,

(iv) ν' *is of Hadamard type.*

As this crucial result concerns off-shell algebras, in principle, it could be exploited in perturbative constructions of interacting theories. Indeed the preservation of the Hadamard singularity structure plays a crucial role in the development of the perturbative theory [35].

3.1.1 The CCR algebra of observables and the Møller $*$-isomorphism

Given a formally-selfadjoint normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ and its causal propagator G, we first define the unital complex $*$-algebra $\mathcal{A}_f$ as the *free complex unital* $*$*-algebra* with abstract (distinct) generators $\phi(\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, identity 1, and involution * as discussed in [76]. (As a matter of fact $\mathcal{A}_f$ is made of finite linear complex combinations of 1 and finite products of generic elements $\phi(\mathfrak{f})$ and $\phi(\mathfrak{h})^*$). Then we define a refined complex unital $*$-algebra by imposing the following relations by the quotient $\mathcal{A} = \mathcal{A}_f/\mathcal{I}$ where $\mathcal{I}$ is the two sided $*$-ideal generated by the following elements of $\mathcal{A}$:

- $\phi(a\mathfrak{f} + b\mathfrak{h}) - a\phi(\mathfrak{f}) - b\phi(\mathfrak{h})\,, \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$
- $\phi(\mathfrak{f})^* - \phi(\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$
- $\phi(\mathfrak{f})\phi(\mathfrak{h}) - \phi(\mathfrak{h})\phi(\mathfrak{f}) - i\mathsf{G}_N(\mathfrak{f},\mathfrak{h})1\,, \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$,

where we have used the notation

$$\mathsf{G}_\mathsf{N}(\mathfrak{f},\mathfrak{h}) := \int_\mathsf{M} \langle \mathfrak{f}(x) \,|\, (\mathsf{G}_\mathsf{N}\mathfrak{h})(x)\rangle \mathrm{vol}_g(x)\,.$$

We have the further possibility to enrich the ideal with the generators:

- $\phi(\mathsf{N}\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Notation 3.1.1. The equivalence classes $[\phi(\mathfrak{f})]$ will be denoted by $\Phi(\mathfrak{f})$ and they will be called **field operators** (**on-shell** if the ideal is enlarged by including the generators $\phi(\mathsf{N}\mathfrak{f})$), and we use the notation $\mathbb{I}$ for the identity [1] of $\mathcal{A}_f/\mathcal{I}$.

Definition 3.1.2. Given a formally-selfadjoint normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ and its causal propagator G, we call **CCR algebra** of the quantum fields Φ, the unital $*$-algebra defined by $\mathcal{A} := \mathcal{A}_f/\mathcal{I}$. The algebra is said to be **on-shell** in case the ideal is enlarged by including the generators $\phi(\mathsf{N}\mathfrak{f})$. Furthermore, we call **observables** of $\mathcal{A}$ any Hermitian element of it.

With the above notation, the following properties are valid

- **$\mathbb{R}$-Linearity**. $\Phi(a\mathfrak{f} + b\mathfrak{h}) = a\Phi(\mathfrak{f}) + b\Phi(\mathfrak{h})\,, \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$
- **Hermiticity**. $\Phi(\mathfrak{f})^* = \Phi(\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$
- **CCR**. $\Phi(\mathfrak{f})\Phi(\mathfrak{h}) - \Phi(\mathfrak{h})\Phi(\mathfrak{f}) = i\mathsf{G}_N(\mathfrak{f},\mathfrak{h})\mathbb{I}\,, \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$.

The on-shell field operators also satisfy

- **Equation of motion**. $\Phi(\mathsf{N}\mathfrak{f}) = 0\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Remark 3.1.3. The idea behind the notation $\Phi(\mathfrak{f})$ is a formal smearing procedure which uses the scalar product

$$\Phi(\mathfrak{f}) = \int_{\mathsf{M}} \langle \Phi(x) \,|\, \mathfrak{f}(x) \rangle \mathrm{vol}_g(x)\,.$$

From this perspective, since N is formally selfadjoint, the identity $\Phi(\mathsf{N}\mathfrak{f}) = 0$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ has the distributional meaning $\mathsf{N}\Phi = 0$. Alternatively, as explained in [95], one may use a different representation where Φ is viewed as a "generalized section" of the dual bundle E^*. In that case the formal identity $\mathsf{N}\Phi = 0$ corresponding to the equation of motion has to be replaced by $\mathsf{N}^*\Phi = 0$.

Given different normally hyperbolic operators N, N' all the information about causality and dynamics is encoded in the ideal $\mathcal{I}, \mathcal{I}'$. In that case we have two corresponding initial unital $*$-algebras $\mathcal{A}_f$ and $\mathcal{A}'_f$ with respective generators $\phi(\mathfrak{f})$ and $\phi'(\mathfrak{f})$. Though the freely generated algebras are canonically isomorphic, under the unique unital $*$-isomorphism such that $\phi(\mathfrak{f}) \to \phi'(\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, the quotient algebras are intrinsically different because the CCR are different depending on the choice of the causal propagator G_{N} or $\mathsf{G}_{\mathsf{N}'}$. However there is an isomorphism between them as soon as a Møller operator exists. Indeed, the existence of the Møller operator discussed in the previous sections can be exploited to define first an isomorphism of the free algebras $\mathcal{A}_f$ and $\mathcal{A}'_f$ since the operator $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ is an isomorphism.

Definition 3.1.4. Let $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be two formally-selfadjoint (with respect to a fiber metric $\langle \cdot \,|\, \cdot \rangle$) normally hyperbolic operators on globally hyperbolic spacetimes (M, g) and (M, g'). If $g \simeq g'$, we define an isomorphism $\mathcal{R}_f : \mathcal{A}'_f \to \mathcal{A}_f$ as the unique unital $*$-algebra isomorphism between the said free unital $*$-algebras such that $\mathcal{R}_f(\phi'(\mathfrak{f})) = \phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$. where R is a Møller operator of g, g' (in this order) satisfying Theorem 2.1.27 and Equation (2.1.24).

As we shall see in the next proposition, the isomorphism between freely generated algebras induces an isomorphism of the quotient algebras.

Proposition 3.1.5. *Let* N *and* N' *be two formally-sefadjoint normally hyperbolic operators acting on the sections of the* $\mathbb{R}$*-vector bundle* E *over* M*, and referred to respective* $g, g' \in \mathcal{GM}_{\mathsf{M}}$*.*
If $g \simeq g'$ *and* R *is a Møller operator of* g,g' *in the sense of Theorem 2.1.27 and Equation* (2.1.24)*, then the CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$ *(possibly both on-shell) respectively associated to* N *and* N' *are isomorphic under the quotient isomorphism* $\mathcal{R} : \mathcal{A}'_f/\mathcal{I}' \to \mathcal{A}_f/\mathcal{I}$ *constructed out of* $\mathcal{R}_f$*, the unique unital* $*$*-algebra isomorphism satisfying* $\mathcal{R}(\Phi'(\mathfrak{f})) = \Phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Proof. To prove the statement it suffices to show that the operator $\mathcal{R}_f$ maps the ideal $\mathcal{I}'$ to the ideal $\mathcal{I}$. Each ideal $\mathcal{I}$ and $\mathcal{I}'$ is the intersection of three (four) ideals corresponding to the requirements of linearity, Hermiticity, CCR (and equation of motion). The fact that $\mathcal{R}_f$ preserves the ideals due to linearity and Hermiticity is an immediate consequence of the fact that $\mathcal{R}_f$ is a $*$-algebra homomorphism of the involved freely generates algebras. The ideal arising from the equation of motion condition is preserved due to the first statements of Theorem 2.1.27 and item (4) therein.
The situation is more delicate regarding the ideal generated by the CCR. Preservation of that ideal is actually an immediate consequence of $\mathcal{R}_f(\mathbb{I}') = \mathbb{I}$ ($\mathcal{R}_f$ is unital by hypothesis) and the

structure of CCR together with (2.1.22):

$$\begin{aligned}\mathsf{G}_{\mathsf{N}}(\mathfrak{f}', \mathfrak{h}') &= \mathsf{G}_{\mathsf{N}_0}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}, \mathsf{R}^{\dagger_{gg'}}\mathfrak{h})\\ &= \int_{\mathsf{M}} \langle \mathsf{R}^{\dagger_{gg'}}\mathfrak{f} \,|\, \mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}\rangle \mathrm{vol}_{g}\\ &= \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}\rangle \mathrm{vol}_{g'}\\ &= \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{G}_{\mathsf{N}'}\mathfrak{h}\rangle \mathrm{vol}_{g'}\\ &= \mathsf{G}_{\mathsf{N}'}(\mathfrak{f}, \mathfrak{h})\,.\end{aligned}$$

This concludes our proof. □

Definition 3.1.6. A unital $*$-isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ defined in Proposition 3.1.5 out of the Møller operator R of g, g' as in Theorem 2.1.27 and (2.1.24) is called **Møller $*$-isomorphism** of the CCR algebras $\mathcal{A}$,$\mathcal{A}'$ (in this order)

3.1.2 Pull-back of algebraic states through the Møller $*$-isomorphism

As explained in the beginning of this section, the subsequent step in the quantization of a field theory consists in identifying a distinguished state on the $*$-algebra of the quantum fields. The *GNS construction* then guarantees the existence of a representation of the quantum field algebra through, in general unbounded, operators defined over a common dense subspace of some Hilbert space. We will not care about the explicit representation and recall some definitions (see [40] for a general discussion also pointing out several not completely solved standing issues).

Definition 3.1.7. We call an **(algebraic) state** over a unital $*$-algebra $\mathcal{B}$ a $\mathbb{C}$-linear functional $\omega : \mathcal{B} \to \mathbb{C}$ which is

(i) **Positive** $\omega(a^*a) \geqslant 0 \quad \forall a \in \mathcal{B}$,

(ii) **Normalized** $\omega(\mathbb{I}) = 1$

A generic element of the CCR algebras $\mathcal{A}$ of a quantum field Φ associated to the normally hyperbolic operators discussed before can be written as a finite polynomial of the generators $\Phi(f)$, where the zero grade term is proportional to $\mathbb{I}$, to specify the action of a state it's sufficient to know its action on the monomials, i.e its **n-point functions**

$$\omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n) := \omega(\Phi(\mathfrak{f}_1)...\Phi(\mathfrak{f}_n)) \tag{3.1.1}$$

The map $\Gamma_c(\mathsf{E}) \times \cdots \times \Gamma_c(\mathsf{E}) \ni (\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \mapsto \omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n)$ can be extended by linearity to the space of finite linear combinations of sections $\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_n \in \Gamma_c(\mathsf{E}^{n\boxtimes})$, where $\mathsf{E}^{n\boxtimes}$ is n-times exterior tensor product of the vector bundle E with itself. If we impose continuity with respect to the usual topology on the space of compactly supported test sections, since the said linear combinations are dense, we can uniquely extend the n-point functions to distributions in $\Gamma'_c(\mathsf{E}^{n\boxtimes})$ we shall hereafter indicate by the same symbol ω_n. It has a formal integral kernel,

$$\omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n) = \int_{\mathsf{M}^n} \widetilde{\omega}_n(x_1, ..., x_n)\mathfrak{f}_1(x_1)...\mathfrak{f}_n(x_n)\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n),$$

where

$$\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n) := \mathrm{vol}_g(x_1) \otimes \cdots (n \text{ times}) \cdots \otimes \mathrm{vol}_g(x_n)$$

henceforth. Notice that if more strongly $\omega_n \in \Gamma'_c(\mathsf{E}^{n\boxtimes})$, then

$$\omega_n(\mathfrak{h}) = \int_{\mathsf{M}^n} \widetilde{\omega}_n(x_1, ..., x_n)\mathfrak{h}(x_1, \ldots, x_n)\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n)$$

is also defined for $\mathfrak{h} \in \Gamma_c(\mathsf{E}^{n\boxtimes})$. The case $n = 2$ is the easiest one. The Schwartz kernel theorem implies $\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega_2(\mathfrak{h}, \mathfrak{f})$ is (sequentially) continuous at $\mathfrak{f} = 0$ for every fixed $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ if and only if ω_2 continuously extends to a unique distribution we hereafter indicate with the same symbol $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$.

An important fact (see the comment after [95, Proposition 5.6]) is that, if the CCR algebra $\mathcal{A}$ admits states, then the fiberwise metric $\langle\cdot|\cdot\rangle$ must be positive. In other words, $\langle\cdot|\cdot\rangle$ is a real symmetric positive scalar product. We shall assume it henceforth.

Differently from a free quantum field theory on Minkowski spacetime, where the Poincaré invariant state – known as Minkowski vacuum – might be a natural choice, on a general curved spacetime there might be no choice of a natural state. However there is a class of states, known as quasifree (or Gaussian) states, whose GNS representation mimics the Fock representation of Minkowski vacuum (see *e.g.* [76]).

Definition 3.1.8. Let $\mathcal{A}$ be the CCR algebra. A state $\omega : \mathcal{A} \to \mathbb{C}$ is called **quasifree**, or equivalently **Gaussian**, if the following properties for its n-point functions hold

(i) $\omega_n(\mathfrak{f}_1, ..., \mathfrak{f}_n) = 0$, if $n \in \mathbb{N}$ is odd,

(ii) $\omega_{2n}(\mathfrak{f}_1, ..., \mathfrak{f}_{2n}) = \sum_{partitions} \omega(f_{i_1}, f_{i_2}) \cdots \omega(f_{i_{n-1}}, f_{i_n})$, if $n \in \mathbb{N}$ is even,

where "partitions" for even n refers to the class of all possible decompositions of the set $\{1, 2, \ldots, n\}$ into $n/2$ pairwise disjoint subsets of 2 elements $\{i_1, i_2\}$, $\{i_3, i_4\}$, ..., $\{i_n - 1, i_n\}$ with $i_{2k-1} < i_{2k}$ for $k = 1, 2, \ldots, n/2$.

For these states all the information is encoded in the two-point distribution, as one can expect in a free theory. It is not difficult to prove that, for a quasifree state in view of the definition above, $\omega_2 \in \Gamma'_c(\mathsf{E})$ entails that ω_n continuously extends to $\omega_n \in \Gamma'_c(\mathsf{E}^{n\boxtimes})$ obtained, for $n = 2k$, as a linear combination of tensor products of distributions ω_2 and trivial if $n = 2k + 1$.

Remark 3.1.9. If $\mathcal{A}$ is on-shell, then the n-point function satisfies trivially

$$\omega_n(\mathfrak{f}_1, \ldots, \mathsf{N}\mathfrak{f}_k, \ldots, \mathfrak{f}_n) = 0 \quad \text{for every } k = 1, \ldots, n \text{ and } f_k \in \Gamma_c(\mathsf{M}).$$

as a consequence of (3.1.1) and $\Phi(\mathsf{N}\mathfrak{f}) = 0$. However it may happen that these identities are valid (for some n) even if the algebra is not on-shell.

In the next proposition, we shall see that the action of the Møller isomorphism $\mathcal{R}$ between CCR-algebras can be pull-backed on the quantum states. Furthermore, the pull-back of a quasifree state is again a quasifree state.

Proposition 3.1.10. *Let be $g, g' \in \mathcal{GH}_{\mathsf{M}}$, consider the algebras $\mathcal{A}$, $\mathcal{A}'$ respectively associated to formally-selfadjoint normally hyperbolic operators $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ constructed out of g and g' and let $\omega : \mathcal{A} \to \mathbb{C}$ be a state. Assuming that $g \simeq g'$, we define a functional $\omega' : \mathcal{A}' \to \mathbb{C}$ by pull-back through a Møller $*$-isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ of $\mathcal{A}$,$\mathcal{A}'$ as in Definition 3.1.6, i.e.*

$$\omega' = \omega \circ \mathcal{R}.$$

Then the following statements hold true:

(1) ω' is a state on $\mathcal{A}'$;

(2) $\omega'_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ if and only if $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$;

(3) ω' is quasifree if and only if ω is.

Proof. (1) Linearity is obvious since we are composing linear maps. Normalization follows from 1 in 3.1.5 and from the fact that ω is normalized. Positivity follows from positivity of ω and the fact that $\mathcal{R}$ preserves the involutions, the products, and is surjective. (2) Since $\omega_2 \in \Gamma'_c(\mathsf{E} \times \mathsf{E})$, then it is $\Gamma_c(\mathsf{E})$-continuous in the right entry (taking values in $\Gamma'_c(\mathsf{E})$ and with respect to the corresponding topology). As a consequence, by composition of continuous functions, if $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ is given,

$$\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega'_2(\mathfrak{h}, \mathfrak{f}) = \omega_2(\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}, \mathsf{R}^{\dagger_{gg'}}\mathfrak{f})$$

is $\Gamma_c(\mathsf{E})$-continuous as well because $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ is continuous in the $\Gamma_c(\mathsf{E})$ topology in domain and co-domain for (5) of Theorem 2.1.27. In other words $\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega'_2(\cdot, \mathfrak{f}) \in \Gamma'_c(\mathsf{E})$ is continuous. We conclude that $\omega'_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ due to the Schwartz kernel theorem. The result can be reversed swapping the role of the states and the metrics, noticing that $\omega = \omega' \circ \mathcal{R}^{-1}$ where $\mathcal{R}^{-1}$ is also a Møller $*$-isomorphism, the one constructed out of R^{-1} which is, in turn, a Møller operator associated to the pair g', g in this order in view of Corollary 2.1.29.
(3) The proof is immediate and follows by construction. □

3.1.3 States and Hadamard states

It is widely accepted that, among all possible (quasifree) states, the physical ones are required to satisfy the so-called the Hadamard condition. The reasons for this choice are manifold: For example, it implies the finiteness of the quantum fluctuations of the expectation value of every observable and it allows us to construct Wick polynomials [71, 75] and other observables, as the stress energy tensor, relevant in semi-classical quantum gravity following a covariant scheme [70, 82], encompassing a locally covariant ultraviolet renormalization [72] (see also [76] for a recent pedagogical review). These states have been also employed, e.g. (the following list is far from being exhaustive) in the study of the Black hole radiation [31, 55, 77, 89], in cosmological models [28, 30] and other applications to spacetime models [45, 46, 83], and to study energy quantum inequalities [43]. For later convenience, we decided to present the Hadamard condition as a microlocal condition on the wave-front set of the two-point distribution [93,94] instead of the equivalent geometric version based on the Hadamard parametrix [2, 78, 84]. Let's briefly sketch what they are and why they are useful.

From now on we adopt the definitions of wave-front set $WF(\psi)$ of distribution ψ on $\mathbb{R}$-vector bundles equipped with a non-degenerate, symmetric, fiberwise metric[1] as in [95].

We shall use some very known definitions and results of microlocal analysis applied to distributions of $\Gamma'_c(\mathsf{F})$ where F is a $\mathbb{K}$-vector bundle, $\mathsf{F} = \mathsf{E} \boxtimes \mathsf{E}$ for instance (see [95] for details). In particular,

- $\psi \in \Gamma'_c(\mathsf{F})$ is a smooth section of the dual bundle F^*, indicated with the same symbol $\psi \in \Gamma(\mathsf{F}^*)$, if and only if $WF(\psi) = \emptyset$.
- We say that $\psi, \psi' \in \Gamma'_c(\mathsf{F})$ are **equal mod** C^∞, if $\psi - \psi' \in \Gamma(\mathsf{F}^*)$.
- Let us assume that $\mathsf{F} = \mathsf{E} \boxtimes \mathsf{E}$ where E is equipped with a non-degenerate, symmetric (Hermitian if $\mathbb{K} = \mathbb{C}$), fiberwise metric and let $\mathsf{P} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ be a formally selfadjoint smooth differential operator. We say that $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is a **bi-solution** $\mathsf{P}\mathfrak{f} = 0$ **mod** C^∞, if there exist $\psi, \psi' \in \Gamma(\mathsf{F}^*)$ such that

$$\nu(\mathsf{P}\mathfrak{f} \otimes \mathfrak{h}) = \int_{\mathsf{M}} \langle \psi, \mathfrak{f} \otimes \mathfrak{h} \rangle \operatorname{vol}_g \otimes \operatorname{vol}_g\,, \quad \nu(\mathfrak{f} \otimes \mathsf{P}\mathfrak{h}) = \int_{\mathsf{M}} \langle \psi', \mathfrak{f} \otimes \mathfrak{h} \rangle \operatorname{vol}_g \otimes \operatorname{vol}_g \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})\,.$$

We are in a position to state the definition of *micro local spectrum condition* and *Hadamard state*. Below, $\sim_\parallel$ is the relation in $T^*\mathsf{M}^2 \backslash \{0\}$ such that $(x, k_x) \sim_\parallel (y, k_y)$ if there is a *null geodesic*

[1] The authors of [95] more generally study the case of a complex Hermitian vector bundle endowed with an antilinear involution (here the identity bundle map) there indicated by Γ.

passing through $x, y \in \mathsf{M}$ and the geodesic parallely transports the co-tangent vector to that geodesic $k_x \in T^*_x\mathsf{M}$ into the co-tangent vector to that geodesic $k_y \in T^*_y\mathsf{M}$. Finally, $k_x \rhd 0$ means that the covector k_x is *future directed*.

Definition 3.1.11. With $\mathcal{A}$ as in Definition 3.1.8, a state $\omega : \mathcal{A} \to \mathbb{C}$ is called a **Hadamard state** if $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ and the following **microlocal spectrum condition** is valid

$$WF(\omega_2) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\} . \tag{3.1.2}$$

More generally, a distribution $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is said to be of **Hadamard type** if its wave-front set $WF(\nu)$ is the right-hand side of (3.2.26).

Remark 3.1.12.

(1) Notice that $(x, k_x; x, -k_x) \in WF(\nu)$ for every future directed lightlike covector $k_x \in T^*_x\mathsf{M}$ if $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is of Hadamard type.

(2) It is possible to prove that a fiberwise scalar product $\langle\cdot|\cdot\rangle$ must be necessarily positive if $\mathcal{A}$ admits quasifree Hadamard states as proved in the comment after [95, Proposition 5.6]. We henceforth assume that $\langle\cdot|\cdot\rangle$ is positive.

3.1.4 Møller preservation of the microlocal spectrum condition for off-shell algebras

The theorem below shows that the Hadamard condition is preserved under the pull-back along the Møller isomorphism.

Theorem 3.1.13. *Let E be an $\mathbb{R}$-vector bundle over the smooth manifold M and denote with $\langle\cdot\,|\,\cdot\rangle$ positive, symmetric, fiberwise metric. Let be $g, g' \in \mathcal{GH}_{\mathsf{M}}$, consider the corresponding formally-selfadjoint normally hyperbolic operators $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ and refer to the associated CCR algebras $\mathcal{A}$ and $\mathcal{A}'$ (off-shell in general). Finally, suppose that $g \simeq g'$.*
$\omega : \mathcal{A} \to \mathbb{C}$ is a quasifree Hadamard state, if and only if

$$\omega' := \omega \circ \mathcal{R} : \mathcal{A}' \to \mathbb{C}\,,$$

constructed out of a Møller $*$-isomorphism $\mathcal{R}$ of $\mathcal{A}, \mathcal{A}'$, is a quasifree Hadamard state of $\mathcal{A}'$.

Remark 3.1.14. We stress that it is not required that the algebras are on-shell nor that the relevant two-point functions satisfy the equation of motion with respect to the corresponding normally hyperbolic operators.

The rest of this section is devoted to prove Theorem 3.1.13, a refinement of it stated in the last Theorem 3.2.8, and a proof of existence of Hadamard states based on our formalism.

Our first observation is the following.

Lemma 3.1.15. *Let $\mathsf{S} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be any of the four operators R_+, R_-, R_+^{-1}, R_-^{-1}, defined as in (2.1.9), (2.1.10), (2.1.11), (2.1.12), and $U \subset \mathbb{R}^m$ an open set.*
If $\{\mathfrak{f}_z\}_{z \in U} \subset \Gamma(\mathsf{E})$ is such that $\mathsf{M} \times U \ni (x, z) \mapsto \mathfrak{f}_z(x)$ is jointly smooth, then

$$\mathsf{M} \times U \ni (x, z) \mapsto (\mathsf{S}\mathfrak{f}_z)(x)$$

is jointly smooth as well.

Proof. We consider the case of R_+, the remaining three instances having a similar proof. What we have to prove is that $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is smooth under the said hypotheses. Let us first consider the case where there is compact $K \subset \mathsf{M}$ such that $\mathrm{supp}(\mathfrak{f}_z) \subset K$ for all $z \in U$. In this case, defining $F(x, z) := (\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z(x)$, the projection $\pi : \mathrm{supp}(F) \ni$

$(x, z) \mapsto z \in U$ is proper[2] and this fact will be used shortly. Interpreting $\mathsf{G}^+_{\rho\mathsf{N}_\chi} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ and thus as a Schwartz kernel, we can compute the wavefront set of the map $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ viewed as the distributional kernel of the composition of the kernel $\mathsf{G}^+_{\rho\mathsf{N}_\chi}(x, y)$ and the smooth kernel $F(y, z)$. We know that (see, e.g. [76] for the scalar case, the vector case being analogous)

$$WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi}) = \left\{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x) \sim_\parallel (y, k_y), x \in J_+(y) \text{ or } k_x = k_y\,, x = y\right\}$$

whereas, since F is jointly smooth,

$$WF(F) = \varnothing\,.$$

The known composition rules of wavefront sets of Schwartz kernels, which use in particular the fact that the projection π above is proper (in [76, Theorem 5.3.14] which is valid also in the vector field case), immediately yields

$$WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F) \subset \varnothing\,.$$

It being $WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F) = \varnothing$, we conclude that $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is a smooth function as desired.

Let us pass to consider the generic jointly smooth family $\{\mathfrak{f}_z\}_{z\in U} \subset \Gamma(\mathsf{E})$ without restrictions on the supports. First of all, we observe that $\mathfrak{f}'_z(x) := ((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(x)$ is past compact by construction for every $z \in U$, because its support is contained in the future of Σ_{t_0} referring to the construction of N_χ. According to the proof of [4, Theorem 3.6.7], if $\mathfrak{h}$ is past compact, $x_0 \in \mathsf{M}$, and $A \supset \mathrm{supp}(\mathfrak{h}) \cap J_-(x_0)$ is an open relatively compact set, for every compactly supported smooth function $s_A \in C^\infty_c(\mathsf{M}; [0, 1])$ such that $s_A(x) = 1$ if $x \in A$, it holds

$$(\mathsf{G}^+_{\rho\mathsf{N}_\chi}\mathfrak{h})(x_0) = (\mathsf{G}^+_{\rho\mathsf{N}_\chi} s_A\mathfrak{h})(x_0)\,.$$

We want to apply this identity for $\mathfrak{h} = \mathfrak{f}_z$. Take $t' < t_0$. Given $x_0 \in \mathsf{M}$ we can always define $A := I_-(\widetilde{x}_0) \cap I^+(\Sigma_{t'})$ where $\widetilde{x}_0 \in I_+(x_0)$ [3]. With this choice, A does not depend on $z \in U$ and the same A can be used for x varying in an open neighbourhood A' of x_0, since $I_-(\widetilde{x}_0)$ is open. We conclude that, if $(x, z) \in A' \times U$, then

$$\left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x) = (\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F)(x, z) \quad \text{where } F(x, z) = s_A(x)(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z(x)\,. \quad (3.1.3)$$

In this case $K := \mathrm{supp}(s_A)$ includes all the supports of the maps $\mathsf{M} \ni x \mapsto F(x, z)$ for every $z \in U$. The first part of the proof is therefore valid for the map $\mathsf{M} \times U \ni (x, z) \mapsto (\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F)$ which must be jointly smooth as a consequence. In particular, its restriction $A' \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is jointly smooth as well. Since A' can be taken as a neighbourhood of every point in M and $z \in U$ is arbitrary, from (3.1.3) the whole function $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is jointly smooth. □

Relying on Lemma 3.1.15, we can notice the following.

Lemma 3.1.16. *Consider a pair of globally hyperbolic metrics g_0 and g_χ on M as in Proposition 2.1.16 and corresponding normally hyperbolic operators $\mathsf{N}_0, \mathsf{N}_\chi : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ for the $\mathbb{R}$-vector bundle on M equipped with the positive symmetric fiberwise metric $\langle\cdot|\cdot\rangle$.*
Then, $\nu_0 \in \Gamma'_c(\mathsf{E}\boxtimes\mathsf{E})$ is a bisolution of $\mathsf{N}_0\mathfrak{f} = 0$ mod C^∞ if and only if $\nu_\chi := \nu \circ \mathsf{R}_+^{\dagger g_0 g_\chi} \otimes \mathsf{R}_+^{\dagger g_0 g_\chi}$ is a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0$ mod C^∞, where R_+ is defined in (2.1.26).

[2]If $C \subset U$ is compact and thus closed, then $\pi^{-1}(C)$ is a closed set, π being continuous, contained in the compact $K \times C$, so that $\pi^{-1}(C)$ is compact as well.

[3]Notice that since the spacetime is globally hyperbolic, $\overline{I_\pm(x)} = J_\pm(x)$ and $\overline{I_-(\widetilde{x}_0) \cap I^+(\Sigma_{t'})} = J_-(\widetilde{x}_0) \cap J^+(\Sigma_{t'})$ which is compact because $\Sigma_{t'}$ is a smooth spacelike Cauchy surface.

Proof. We start by stressing that, as already noticed, in view of the known continuity properties of $\mathsf{R}_+^{\dagger g_0 g_\chi}$ and its inverse and using Schwartz' kernel theorem, $\nu_0 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ if and only if $\nu_\chi \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$.
We pass to prove that if ν_0 is a bisolution mod C^∞, then ν_χ is a bisolution mod C^∞, referring to the corresponding operators. Let us hence suppose that $\nu_0(\mathsf{N}_0\mathfrak{f}, \mathfrak{h}) = \psi(\mathfrak{f}\otimes\mathfrak{h})$ and $\nu_0(\mathfrak{f}, \mathsf{N}_0\mathfrak{h}) = \psi'(\mathfrak{f}\otimes\mathfrak{h})$ for some smooth sections $\psi, \psi' \in \Gamma((\mathsf{E} \boxtimes \mathsf{E})^*)$ and all $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$. The identity

$$\mathsf{R}_+^{\dagger g_0 g_\chi} \mathsf{N}_\chi|_{\Gamma_c(\mathsf{E})} = \mathsf{N}_0|_{\Gamma_c(\mathsf{E})} \,,$$

immediately implies that, if $\varphi(x, y) := c_0^\chi(x)c_0^\chi(y)\psi(x, y)$, $\varphi'(x, y) := c_0^\chi(x)c_0^\chi(y)\psi'(x, y)$,

$$\nu_\chi(\mathsf{N}_\chi\mathfrak{f}, \mathfrak{h}) = \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi(x, y), (\mathrm{Id} \otimes \mathsf{R}_+^{\dagger g_0 g_\chi}(\mathfrak{f} \otimes \mathfrak{h}))(x, y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

and

$$\nu_\chi(\mathfrak{f}, \mathsf{N}_\chi\mathfrak{h}) = \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi'(x, y), (\mathsf{R}_+^{\dagger g_0 g_\chi} \otimes \mathrm{Id}(\mathfrak{f} \otimes \mathfrak{h}))(x, y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

The proof ends if proving that there are smooth sections $\varphi_1, \varphi'_1 \in \Gamma((\mathsf{E} \boxtimes \mathsf{E})^*)$, such that

$$\int_{\mathsf{M}\times\mathsf{M}} \langle\varphi, \mathrm{Id} \otimes \mathsf{R}_+^{\dagger g_0 g_\chi}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi} = \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi_1(x, y), \mathfrak{f}(x)\mathfrak{h}(y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

and

$$\int_{\mathsf{M}\times\mathsf{M}} \langle\varphi, \mathsf{R}_+^{\dagger g_0 g_\chi} \otimes \mathrm{Id}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi} = \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi'_1(x, y), \mathfrak{f}(x)\mathfrak{h}(y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

for every pair $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$. We prove the former identity only, the second one having an identical proof. To this end we pass to the index notation (also assuming Einstein's summing convention), the indices being referred to the fiber in the local trivialization,

$$\int_{\mathsf{M}\times\mathsf{M}} \langle\varphi, \mathsf{R}_+^{\dagger g_0 g_\chi} \otimes \mathrm{Id}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi}$$

$$= \sum_{j,k} \int_{\mathsf{M}\times\mathsf{M}} \chi_j(x)\chi'_k(y)\varphi_{ab}(x, y)(\mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f})^a(x)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

Above $\{\chi_j\}_{j\in J}$ and $\{\chi'_k\}_{k\in K}$ are partitions of the unity of M subordinated to corresponding locally finite coverings of M supporting local trivializations, whose fiber coordinates are labelled by ^{a} and ^{b}. Moreover, only a finite number of indices $(j, k) \in J \times K$ give a contribution to the sum, uniformly in x, y, in view of the compactness of the supports of $\mathfrak{f}$ and $\mathfrak{h}$ and the local finiteness of the used coverings. The right-hand side can be rearranged to

$$= \sum_{k\in K} \int_{\mathsf{M}} \chi'_k(y) \left(\sum_{j\in J} \int_{\mathsf{M}} \chi_j(x)\varphi_{ab}(x, y)(\mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f})^a(x) \right) \mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)$$

$$= \sum_{k\in K} \int_{\mathsf{M}} \chi'_k(y) \left(\int_{\mathsf{M}} \langle\varphi'_{yb}(x) \,|\, (\mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f})(x)\rangle \mathrm{vol}_{g_\chi}(x) \right) \mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)$$

$$= \int_{\mathsf{M}} \sum_{k\in K} \chi'_k(y) \left(\int_{\mathsf{M}} \langle(\mathsf{R}_+\varphi'_{yb})(x) \,|\, \mathfrak{f}(x)\rangle \mathrm{vol}_{g_0}(x) \right) \mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)$$

$$= \sum_{j,k} \int_{\mathsf{M}\times\mathsf{M}} \chi_j(x)\chi'_k(y)c_0^\chi(x)(\mathsf{R}_+\varphi'_{yb})_a(x)\mathfrak{f}^a(x)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

$$= \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi_1(x, y), \mathfrak{f} \otimes \mathfrak{h}(x, y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y) \,,$$

where we have locally defined ${\varphi'}^a_{yb}(x) := \xi^{ac}(x)\varphi_{cb}(x,y)$, with $\xi^{ab}(x)$ being the inverse fiber metric at $x \in M$ in any considered local trivialization. Above, $\varphi_{1\,ab}(x,y) := c_0^\chi(x)(\mathsf{R}_+\varphi'_{yb})_a(x)$ is the candidate section of $(\mathsf{E}\boxtimes\mathsf{E})^*$ we were looking for, represented in local coordinates of the atlas of the said trivialization. That section is smooth, i.e., $\varphi_1 \in \Gamma((\mathsf{E}\boxtimes\mathsf{E})^*)$ as desired. Indeed, the maps $\mathsf{M} \times U_k \ni (x,y) \mapsto \varphi'_{yb}(x)$ define a family of sections of $\Gamma(\mathsf{E})$ parametrized by $y \in U_k$ for every given $b \in \{1,\dots,N\}$, where $U_k \subset \mathsf{M}$ is the projection onto M of the domain of the considered local trivialization. This family is jointly smooth in x,y as established in Lemma 3.1.15.
The converse statement, that ν_0 is a bisolution mod C^∞ if ν_χ is, can be proved with the same procedure simply replacing R_+ with $(\mathsf{R}_+)^{-1}$ and using Lemma 3.1.15 again. □

Before giving the proof of Theorem 3.1.13, we need a final lemma, which shows that any Hadamard distribution whose antisymmetric part is given by the causal propagator of a normally hyperbolic system N is actually a bisolution of N itself modulo smooth errors.

Lemma 3.1.17. *Let $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be a formally selfadjoint normally hyperbolic operator and suppose that $\nu \in \Gamma'_c(\mathsf{E}\boxtimes\mathsf{E})$ is of Hadamard type and satisfies*

$$\nu(x,y) - \nu(y,x) = i\mathsf{G}_\mathsf{N}(x,y) \quad mod \quad C^\infty$$

where $G_\mathsf{N}(x,y)$ is the distributional kernel of the causal propagator G_N. In this case ν is a bisolution of $\mathsf{N}\mathfrak{f} = 0$ mod C^∞.

Proof. The proof is a straightforward re-adaptation of the proof appearing in the *Note added in proof* of [93]. □

We are finally in a position to prove Theorem 3.1.13.

Proof of Theorem 3.1.13. We have only to prove that ω' is Hadamard if and only if ω is, since the other preservation property has been already proved in (4) of Proposition 3.1.10. If $g_0 \simeq g_1$, there is a sequence of globally hyperbolic metrics $g'_0 = g_0, g'_1, \dots, g'_N = g_1$ such that either $g'_k \preceq g'_{k+1}$ or $g'_{k+1} \preceq g'_k$ and the future cones satisfy a corresponding inclusion. The Møller operator $\mathcal{R}$ of $\mathcal{A}, \mathcal{A}'$ is obtained as the composition of the Møller operators $\mathcal{R}_k$ of the formally-selfadjoint normally hyperbolic operators $\mathsf{N}'_k, \mathsf{N}'_{k+1}$ associated to the pairs g'_k, g'_{k+1}:

$$\mathcal{R} := \mathcal{R}'_0\mathcal{R}'_1 \cdots \mathcal{R}'_{N-1}$$

as in the proof of Theorems 2.1.20, 2.1.27 and (2.1.24). The thesis is demonstrated if we prove that, with obvious notation, ω^{k+1} is Hadamard if and only if ω^k is. So in principle we have to prove the thesis only for a pair of metrics g_0, g_1 with the two cases $g_0 \preceq g_1$ and $g_1 \preceq g_0$. Actually the latter is a consequence of the former, using the fact that Møller $*$-isomorphisms are bijective and that a Møller operator of the second case is the inverse operator of a Møller operator of the first case in accordance to Corollary 2.1.29. In summary, the proof is over by establishing the thesis for the case $g = g_0 \preceq g_1 = g'$ and we shall concentrate on that case only in the rest of the proof.

Recalling by (2.1.27) and (2.1.26) that $\mathsf{R}^{\dagger g_0 g_1} = \mathsf{R}_+^{\dagger g_0 g_\chi}\mathsf{R}_-^{\dagger g_\chi g_1}$, we write

$$\omega_2^1(\mathfrak{f}_1,\mathfrak{f}_2) = \omega_2^0(\mathsf{R}^{\dagger g_0 g_1}\mathfrak{f}_1, \mathsf{R}^{\dagger g_0 g_1}\mathfrak{f}_2) = \omega_2^0(\mathsf{R}_+^{\dagger g_0 g_\chi}\mathsf{R}_-^{\dagger g_\chi g_1}\mathfrak{f}_1, \mathsf{R}_+^{\dagger g_0 g_\chi}\mathsf{R}_-^{\dagger g_\chi g_1}\mathfrak{f}_2).$$

To analyze the wave-front set of this bidistribution, we split again the operation in two steps. First we define a pull-back state on the algebra $\mathcal{A}_\chi$ of quantum fields defined for the formally-selfadjoint normally hyperbolic operator N_χ, i.e a normally hyperbolic operator on (M, g_χ). This intermediate pull-back states reads

$$\omega_2^\chi(\mathfrak{f}_1,\mathfrak{f}_2) = \omega_2^0(\mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f}_1, \mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f}_2). \tag{3.1.4}$$

We intend to prove that $\omega_2^\chi \in \Gamma_c'(\mathsf{E})$ is of Hadamard type if and only ω_2^0 is. Notice that both two-point functions have antisymmetric parts that coincide with $i\mathsf{G}_{\mathsf{N}_\chi}$ and $i\mathsf{G}_{\mathsf{N}_0}$, respectively, in view of the CCRs of the respective algebras. If $\omega_2^0 \in \Gamma_c'(\mathsf{E})$ is of Hadamard type, then it is a a bisolution of $\mathsf{N}_0\mathfrak{f} = 0 \mod C^\infty$ in view of Lemma 3.1.17. The same argument proves that, if $\omega_2^\chi \in \Gamma_c'(\mathsf{E})$ is of Hadamard type, then it is a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0 \mod C^\infty$ due to 3.1.17. Applying Lemma 3.1.16 to both cases we have that,

(a) $\omega_2^0 \in \Gamma_c'(\mathsf{E})$ of Hadamard type implies that ω_2^0 is a bisolution $\mathsf{N}_0\mathfrak{f} = 0 \mod C^\infty$ and ω_2^χ is a a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0 \mod C^\infty$;

(b) $\omega_2^\chi \in \Gamma_c'(\mathsf{E})$ of Hadamard type implies that ω_2^χ is a bisolution $\mathsf{N}_\chi\mathfrak{f} = 0 \mod C^\infty$ and ω_2^0 is a a bisolution of $\mathsf{N}_0\mathfrak{f} = 0 \mod C^\infty$.

We are now in a position to apply the Hadamard singularity propagation theorem. Consider the smooth Cauchy time function t in common with g_0 and g_χ, such that $\chi(x) = 0$ if $t(x) < t_0$. As a preparatory remark we notice that $\mathsf{R}_+^{\dagger g_0 g_\chi}\mathfrak{f} = \mathfrak{f}$ from (2.1.29) when the support of $\mathfrak{f}$ stays in the past of the Cauchy surface $\Sigma_{t_0} = t^{-1}(t_0)$. In that region $g_0 = g_\chi$ by definition of g_χ. Finally due to (3.1.4),

$$\omega_2^\chi(\mathfrak{f}, \mathfrak{h}) = \omega_2^0(\mathfrak{f}, \mathfrak{h}) \quad \text{if } t(\mathrm{supp}(\mathfrak{f})) < t_0,\ t(\mathrm{supp}(\mathfrak{h})) < t_0$$

Hence, in particular, ω_2^χ is of Hadamard type when the supports of the test functions are taken in that region if and only if ω_2^0 is of Hadamard type when the supports of the test functions are taken there. More precisely, it happens when the supports of the arguments $\mathfrak{f}, \mathfrak{h}$ are taken in a (globally hyperbolic) neighbourhood of a Cauchy surface (for both metrics!) $\Sigma_\tau := t^{-1}(\tau)$ with $\tau < t_0$ between two similar slices. Since both distributions are bisolutions of the respective equation of motion mod C^∞ and the operators are normally hyperbolic, the theorem of propagation of Hadamard singularity (see, e.g., Theorem 5.3.17 in [76][4]) implies that ω_2^χ and ω_2^0 are of Hadamard type everywhere in (M, g_χ) and (M, g_0), respectively.
A similar reasoning shows that $\omega_2^1 \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$, with

$$\omega_2^1(\mathfrak{f}_1, \mathfrak{f}_2) = \omega_2^\chi(\mathsf{R}_-^{\dagger g_\chi g_1}\mathfrak{f}_1, \mathsf{R}_-^{\dagger g_\chi g_1}\mathfrak{f}_2)\,,$$

is Hadamard on (M, g_1) if and only if ω_2^χ is on (M, g_χ). Combining the two results we have that $\omega' = \omega^1$ is Hadamard on $(\mathsf{M}, g' = g_1)$ if and only if $\omega = \omega^0$ is Hadamard on $(\mathsf{M}, g = g_0)$ concluding the proof. □

We are now in the position to prove our last result.

Theorem 3.1.18. *Let* E *be an* $\mathbb{R}$*-vector bundle on a smooth manifold* M *equipped with a positive, symmetric, fiberwise metric* $\langle\cdot|\cdot\rangle$*. Let* $g, g' \in \mathcal{GH}_\mathsf{M}$*, consider the corresponding formally-selfadjoint normally hyperbolic operators* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *and refer to the associated CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$*.*
Let $\nu \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$ *be of Hadamard type and satisfy*

$$\nu(x,y) - \nu(y,x) = i\mathsf{G}_\mathsf{N}(x,y) \quad mod \quad C^\infty\,,$$

$\mathsf{G}_\mathsf{N}(x,y)$ *being the distributional Kernel of* G_N*.*
Assuming $g \simeq g'$*, let us define*

$$\nu' := \nu \circ \mathsf{R}^{\dagger gg'} \otimes \mathsf{R}^{\dagger gg'}\,,$$

for a Møller operator $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *of* g, g'*. Then the following facts are true.*

[4]The proof which appears there is valid for the on-shell algebra of the scalar real Klein-Gordon field, but the passage to normally hyperbolic operators also weakening the bisolution requirement to bisolution mod C^∞ is immediate, since it is based on standard Hörmander theorems about singularity propagation which works mod C^∞. See the comments in Remark 5.3.18 in [76]

(i) ν *and* ν' *are bisolutions mod* C^∞ *of the field equations defined by* N *and* N' *respectively,*

(ii) $\nu' \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$*,*

(iii) $\nu'(x,y) - \nu'(y,x) = i\mathsf{G}_{\mathsf{N}'}(x,y)$ *mod* C^∞*,*

(iv) ν' *is of Hadamard type.*

Proof. Since we never exploited the fact that ω is positive, nor the fact that the antisymmetric part of its two points function is *exactly* the causal propagator, nor the fact that the relevant algebras of fields are *on-shell* (i.e., the equation of motion are satisfied by the two-point functions), we can use the same arguments as in the proof of the previous theorem to conclude. □

We conclude this section with the following straightforward result of existence of Hadamard quasifree states which apparently does not use the Hadamard singularity propagation argument (actually this argument was used in the proof of Theorem 3.1.13).

Corollary 3.1.19. *Let* (M, g) *be a globally hyperbolic spacetime,* N *be a formally-selfadjoint normally hyperbolic operator acting on the sections of the* $\mathbb{R}$*-vector bundle* E *over* M *and refer to the associated CCR algebras* $\mathcal{A}$*. Then there exists an Hadamard state on* $\mathcal{A}$*.*

Proof. It is well-known [49] that, in a globally hyperbolic ultrastatic spacetime, the (unique) CCR quasifree ground state which is invariant under the preferred Killing time is Hadamard. Hence, combining Corollary 1.3.5 with Theorem 3.1.13 we can conclude. □

3.2 The Proca quantum field

Most of the quantum theories are described by *Green hyperbolic operators* [3], as Klein-Gordon operators N discussed above or the *Proca operator* [44, 98], studied in this section,

$$\mathsf{P} = \delta d + m^2$$

acting on smooth 1-forms $A \in \Omega^1(\mathsf{M})$ and where $m^2 > 0$ is a constant. These operators are formally self-adjoint w.r.t. a (Hermitian or real symmetric) scalar product induced by the analogue γ on the fibers of the relevant vector bundle. In general γ is *not positive definite.* Very common and physical examples are: the *standard vector* Klein-Gordon field, the Proca field, the Maxwell field, more generally, the Yang-Mills field and also the linearized gravity. Referring to the Proca, and in general all 1-form fields, we have that $\gamma = g^\sharp$ is the inverse (indefinite!) Lorentzian metric of the spacetime (M, g).

Unfortunately, in those situations, the Hadamard condition is in conflict with the positivity of states, respectively. It is known that for a vectorial Klein-Gordon operator that is formally self-adjoint w.r.t. an *indefinite Hermitian/real symmetric scalar product*, the existence of quasifree Hadamard states is forbidden (see the comment after [95, Proposition 5.6] and [59, Section 6.3]).

The case of a (real) Proca field seems to be even more complicated at first glance. In fact, on the one hand differently from the Klein-Gordon operator, the Proca operator is not even *normally hyperbolic* and this makes more difficult (but not impossible) the proof of the well-posedness of the Cauchy problem, in particular. On the other hand, similarly to the case of the vectorial Klein-Gordon theory, the Proca theory deals with an indefinite fiberwise scalar product. Actually, as we shall see in the rest of the work, *these two apparent drawbacks cooperate to permit the existence of quasifree Hadamard states.* Positivity of the two-point function ω_2 is restored when dealing with a *constrained* space of Cauchy conditions that make well-posed the Cauchy problem.

In the present section, we study the existence of quasifree Hadamard states for the real Proca field on a general globally hyperbolic spacetime. A definition of Hadamard states for the Proca field was introduced by Fewster and Pfenning in [44], to study *quantum energy inequalities*, with a definition more involved than the one based on the microlocal spectrum condition. They also managed to prove that such states exist in globally hyperbolic spacetimes whose Cauchy surfaces are compact.

Differently from Fewster-Pfenning's definition, here we adopt a standard definition of Hadamard state and we consider a generic globally hyperbolic spacetime. At the end of the work, we actually prove that the two definitions of Hadamard states are substantially equivalent.

Before establishing that equivalence, using the technology of the Møller operators we introduced in [87] for normally hyperbolic operators, and here extended to the Proca field, we prove the existence of quasifree Hadamard states in every globally hyperbolic spacetime, also in the case in which their Cauchy hypersurfaces are not compact.

As a matter of fact, it is enough to focus our attention on *ultrastatic* spacetimes of *bounded geometry*. In this class of spacetimes, we directly work at the level of initial data for the Proca equation and we establish the following, also by taking advantage of some technical results of spectral theory applied to *elliptic Hilbert complexes* [21].

1. The initial data of the Proca equations are a subspace C_Σ of the initial data of *a pair* of Klein-Gordon equations, one scalar and the other vectorial, however both defined on bundles with fiberwise *positive* real symmetric scalar product;

2. The difference of a pair of certain Hadamard two-point functions for two above-mentioned Klein-Gordon fields becomes positive once that its arguments are restricted to C_Σ. There, it defines a two-point function ω_2 for a quasifree state ω of the Proca field;

3. ω is also Hadamard since it is the difference of two two-point functions of Klein-Gordon fields which are Hadamard. They are Hadamard in view of known results of microlocal analysis of pseudodifferential operators on Cauchy surfaces of bounded geometry, for more details the interested reader can refer to [54].

Every field theory defined on a globally hyperbolic spacetime (M, g) is connected to one defined on an ultrastatic spacetime of bounded geometry $(\mathbb{R} \times \Sigma, -dt^2 + h)$ through a Møller operator: the associated Møller $*$-isomorphism between the algebras of Proca observables preserves the Hadamard condition. We therefore conclude that every globally hyperbolic spacetime (M, g) admits a Hadamard state for the Proca field. This state is nothing but the Hadamard state on $(\mathbb{R} \times \Sigma, -dt^2 + h)$ pulled back to (M, g) by the Møller $*$-isomorphism.

One novelty of this work is in particular a direct control of the positivity of the two-point functions, obtained by spectral calculus of elliptic Hilbert complexes. Some microlocal property of the Møller operators then guarantees the validity of the Hadamard condition as in the case of normally hyperbolic field theories.

3.2.1 The CCR algebra of observables of the Proca field

We now introduce the algebraic formalism to quantize the Proca field [44, 98].

Let (M, g) be a globally hyperbolic spacetime, V_g be a Proca bundle and denote by $\mathsf{P} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ the Proca operator. Following [76], we call **on-shell Proca** CCR **$*$-algebra**, the $*$-algebra defined as

$$\mathcal{A}_g = \mathfrak{A}_g / \mathfrak{I}_g$$

where:

- $\mathfrak{A}_g$ is the free complex unital algebra generated by the set of abstract elements $\mathbb{I}$ (the unit element), $\mathfrak{a}(\mathfrak{f})$ and $\mathfrak{a}(\mathfrak{f})^*$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, and endowed with the unique (antilinear) $*$-involution which associates $\mathfrak{a}(\mathfrak{f})$ to $\mathfrak{a}(\mathfrak{f})^*$and satisfies $\mathbb{I}^* = \mathbb{I}$ and $(ab)^* = b^* a^*$.

- $\mathfrak{I}_g$ is the two-sided $*$-ideal generated by the following elements of $\mathfrak{A}_f$:

    1. $\mathfrak{a}(a\mathfrak{f} + b\mathfrak{h}) - a\mathfrak{a}(\mathfrak{f}) - b\mathfrak{a}(\mathfrak{h})\,, \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$;

2. $\mathfrak{a}(\mathfrak{f})^* - \mathfrak{a}(\mathfrak{f})$, $\quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$;
3. $\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{h}) - \mathfrak{a}(\mathfrak{h})\mathfrak{a}(\mathfrak{f}) - i\mathsf{G}_\mathsf{P}(\mathfrak{f},\mathfrak{h})\mathbb{I}$, $\quad \forall \mathfrak{f},\mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$;
4. $\mathfrak{a}(\mathsf{P}\mathfrak{f})$, $\quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$.

The four entries of the list respectively implement linearity, Hermiticity of the generators, canonical commutation relations and the equations of motion for the quantum field.

Remark 3.2.1. As in [44], we adopt the interpretation of $\mathfrak{a}(\mathfrak{f})$ as $(\mathfrak{a}|\mathfrak{f})$, where the pairing is the Hodge inner product of 1-forms (2.2.1).
An equivalence class in $\mathcal{A}_g$ is denoted by $[\mathfrak{a}(\mathfrak{f})] = \hat{\mathfrak{a}}(\mathfrak{f})$, the equivalence class corresponding to the identity is denoted by $[\mathbb{I}] = \mathrm{Id}$. The hermitian elements of the algebra $\mathcal{A}_g$ are called **observables**.

Remark 3.2.2. Requirement 4, when we pass to the quotient algebra corresponds to the distributional relation $\mathsf{P}\hat{\mathfrak{a}} = 0$, when we take Remark 3.2.1 into account and the fact that P is formally selfadjoint. Since every solution of the Proca equation is a co-closed solution of the Klein-Gordon equation and *vice versa*, we conclude that $\delta\hat{\mathfrak{a}} = 0$, i.e. $\hat{\mathfrak{a}}(d\mathfrak{f}) = 0$ for every $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, must be valid.
If, moreover, we deprive the ideal $\mathfrak{I}_g$ of the generators in 4, the quotient algebra is said to be **off-shell**, however it would still be convenient to assume the closedness constraint when defining the off-shell algebra. That is when defining the relevant ideal of the free off-shell algebra, we should keep 1-3, we should drop 4, and we should replace it with the weaker condition

4'. $\hat{\mathfrak{a}}(d\mathfrak{f})$, $\quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$.

This work however deals with the on-shell algebra only, we shall indicate by $\mathcal{A}_g$ throughout. A study of the off-shell algebra, which may result in some relevance in perturbative renormalization procedure will be done elsewhere.

3.2.2 Pull-back of Proca algebraic states through the Møller $*$-isomorphism

Having built the *CCR*-algebra, the subsequent step in quantization consists in finding a way to associate numbers to the abstract operators in $\mathcal{A}_g$ by identifying a distinguished state.

Regarding the notion of Hadamard state for the Proca field, which is a vector field, we adopt the notions of microlocal analysis for vector-valued distributions introduced in [95].

Remark 3.2.3. The interpretation of the action of a distribution on test sections is formalized in the sense of the Hodge product (2.2.1). This interpretation is necessary in order to agree with the interpretation of the symbol $\hat{\mathfrak{a}}(\mathfrak{f})$ stated in Remark 3.2.1, since some of the distributions we shall consider arise from field operators, as the two-point functions $\omega_2(\mathfrak{f},\mathfrak{g}) := \omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{g}))$. If

$$\Gamma_c(\mathsf{V}_g) \ni \mathfrak{g} \mapsto \omega_2(\cdot,\mathfrak{g}) \in \Gamma'_c(\mathsf{V}_g)$$

is well-defined and continuous, ω_2 actually defines a distribution of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ and *vice versa*, as a consequence of the *Schwartz kernel theorem* as clarified below.
From now on, if $F \in \Gamma'_c(\mathsf{V}_g)$ and $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$, the action of the former on the latter is therefore interpreted as the Hodge product (2.2.1)

$$F(\mathfrak{f}) = (F|\mathfrak{f}) = (\mathfrak{f}|F) = \int_\mathsf{M} \mathsf{g}^\sharp(F,\mathfrak{f})\mathrm{vol}_g \,.$$

With a straightforward extension of the Definition 2.2.10, operators working on a generic space of k test-forms $\mathsf{T} : \Omega^k_c(\mathsf{M}) \to \Omega^k_c(\mathsf{M})$ can be extended to the topological duals, i.e the associated distributions, in terms of the action $\mathsf{T}^\dagger$ on the argument of the distribution:

$$(\mathsf{T}F)(\mathfrak{f}) := F(\mathsf{T}^\dagger\mathfrak{f}) \,.$$

For instance, if $F \in \Omega^{2\prime}_c(\mathsf{M})$ and $H \in \Omega^{0\prime}_c(\mathsf{M})$,

$$(\delta F)(\mathfrak{f}) := F(d\mathfrak{f}) \,, \quad (dH)(\mathfrak{f}) := H(\delta\mathfrak{f}) \,, \quad \mathfrak{f} \in \Omega^1_c(\mathsf{M}) \,.$$

If $\mathsf{S} : \Gamma_c(\mathsf{V}_g) \to \Gamma'_c(\mathsf{V}_g)$ is continuous (in particular if $\mathsf{S} : \Gamma_c(\mathsf{V}_g) \to \Gamma_c(\mathsf{V}_g)$ is continuous), the standard Schwartz kernel theorem permits to introduce the distribution indicated with the same symbol $\mathsf{S} \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$, which is the unique distribution such that

$$\mathsf{S}(\mathfrak{f} \otimes \mathfrak{g}) := \mathsf{S}(\mathfrak{f}, \mathfrak{g}) := (\mathsf{S}\mathfrak{g})(\mathfrak{f}) \text{ ``} = (\mathfrak{f}|\mathsf{S}\mathfrak{g})\text{''} \,.$$

Conversely, a distribution of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ defines a unique map $\Gamma_c(\mathsf{V}_g) \to \Gamma'_c(\mathsf{V}_g)$ that fulfils the identity above. In the rest of the work we shall take advantage of these facts and notations above. Furthermore, we adopt the notion of *wavefront set* of a distribution on test sections of a vector bundle on M as defined in [95].

Definition 3.2.4. Consider the globally hyperbolic spacetime (M, g) and a state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g). ω is called **Hadamard** if it is quasifree and its two-point function $\omega_2 \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ satisfies the **microlocal spectrum condition**[5], i.e.

$$WF(\omega_2) = \mathcal{H} := \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x) \sim_\parallel (y, k_y), k_x \rhd 0\} \,. \tag{3.2.1}$$

Above, $(x, k_x) \sim_\parallel (y, k_y)$ means that x and y are connected by a lightlike geodesic and k_y is the co-parallel transport of k_x from x to y along said geodesic, whereas $k_x \rhd 0$ means that the covector k_x is future pointing.

As for Klein-Gordon scalar field theory, Hadamard states for Proca fields have two important properties which were also established in [44] for the notion of Hadamard state adopted there. We present here independent proofs only based on Definition 3.2.4. Indeed, [44] uses a definition of Hadamard states which is apparently different from our definition. A comparison of the two definitions and an equivalence result appear in Section 3.2.5.
The first property of Hadamard states is the fact that the difference between the two-point functions of a pair of Hadamard states is a smooth function. This fact plays a crucial role in the point-splitting renormalization procedure (for instance of Wick polynomials and time-ordered polynomials [71, 72, 74, 75] and of the stress-energy tensor [70, 82, 99]) and is, in fact, one of the reasons for assuming that Hadamard states are the physically relevant ones.

Proposition 3.2.5. *Let $\omega, \omega' \in \Gamma'_c(V_g \boxtimes V_g)$ be a pair of two-point functions of Hadamard states on the algebra $\mathcal{A}_g$ of the Proca field according to Definition 3.2.4. Then, $\omega - \omega' \in \Gamma(V_g \boxtimes V_g)$, i.e., $\omega - \omega'$ is smooth.*
More generally, $\omega - \omega'$ is smooth if ω, ω' are distributions satisfying (3.2.1) such that their anti-symmetric parts coincide mod. C^∞.

Proof. Let us first prove the second statement. Let us define $\omega_2^+(\mathfrak{f}, \mathfrak{g}) := \omega_2(\mathfrak{f}, \mathfrak{g})$ and $\omega_2^-(\mathfrak{f}, \mathfrak{g}) := \omega_2(\mathfrak{g}, \mathfrak{f})$,

$$N^+ := \{(x, k) \in T^*\mathsf{M}\backslash\{0\} \mid k_a k^a = 0 \,,\ k \rhd 0\} \,, \quad N^- := \{(x, k) \in T^*\mathsf{M}\backslash\{0\} \mid k_a k^a = 0 \,,\ k \lhd 0\} \,,$$

$$\Gamma' := \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x; y, k_y) \in \Gamma\} \,. \tag{3.2.2}$$

for every $\Gamma \subset \mathsf{T}^*\mathsf{M}^2\backslash\{0\}$. If both distributions satisfy (3.2.1), then

$$WF(\omega_2^\pm)' \subset N^\pm \times N^\pm \,. \tag{3.2.3}$$

[5]The notion of wavefront set refers to distributions acting on *complex* valued test sections in view of the pervasive use of the Fourier transform. For this reason, when dealing with these notions we consider the natural complex extension of the involved distributions, by imposing that they are also $\mathbb{C}$-linear.

With the hypotheses of the proposition define $R^\pm := \omega_2^\pm - \omega_2'^\pm$. Since $\omega_2^+ - \omega_2^- = \omega_2'^+ - \omega_2'^- + F$ where F is a smooth function, we have that $R^+ = R^-$ mod. C^∞. At this juncture, (3.2.3) yields $WF(R^+)' \cap WF(R^-)' = \varnothing$ because $N^+ \cap N^- = \varnothing$. Since $WF(R^+) = WF(-R^- + F) = WF(-R^-) = WF(R^-)$, we conclude that the wavefront set of the distributions $R^\pm$ is empty and thus they are smooth functions. This is the thesis of the second statement. The latter statement implies the former because, since both ω and ω' are states on the Proca $*$-algebra, their antisymmetric part must be identical and it amounts to $i\mathsf{G}_P$, furthermore ω and ω' satisfy (3.2.1) in view of Definition 3.2.4. □

The second property regards the so called propagation property of the Hadamard singularity or also the local-global feature of Hadamard states. It has a long history which can be traced back to [51] passing through [78], [93, 94] and [95] (and the recent [84]) at least.

Proposition 3.2.6. *Consider a globally hyperbolic spacetime* (M, g) *and a globally hyperbolic neighbourhood* $\mathcal{N}$ *of a smooth spacelike Cauchy surface* Σ *of* (M, g). *Finally, let* $\omega_\mathcal{N}$ *be a quasifree state for the on-shell algebra of the Proca field in* $(\mathcal{N}, g|_\mathcal{N})$. *The following facts are valid.*

(a) There exists a unique a quasifree state $\omega : \mathcal{A}_g \to \mathbb{C}$ *for the Proca field on the whole* (M, g) *which restricts to* $\omega_\mathcal{N}$ *on the Proca algebra on* $\mathcal{N}$.

(b) If $\omega_\mathcal{N}$ *is Hadamard according to Definition 3.2.4, then* ω *is.*

Proof. (a) According to (2.2.11), $\mathsf{G}_\mathsf{P}\mathfrak{f} = 0$ for $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$ if and only if $\mathfrak{f} = \mathsf{P}\mathfrak{g}$ for some $\mathfrak{g} \in \Gamma_c(\mathsf{V}_g)$. We will use this fact to construct ω out of $\omega_\mathcal{N}$. Consider two other smooth spacelike surfaces (for both M and $\mathcal{N}$) Σ_+ in the future of Σ and Σ_- in the past of Σ. Let $\chi^+, \chi^- : \mathsf{M} \to [0, 1]$ be smooth maps such that $\chi^+(p) = 0$ if p stays in the past of Σ_- and $\chi^+(p) = 1$ if p stays in the future of Σ_+ and $\chi^+ + \chi^- = 1$. Then, defining

$$\mathsf{T}\mathfrak{f} := \mathsf{P}\chi^+\mathsf{G}_\mathsf{P}\mathfrak{f}\,, \quad \mathfrak{f} \in \Gamma_c(\mathsf{V}_g) \tag{3.2.4}$$

we have that $\mathsf{T}\mathfrak{f} \in \Gamma_c(\mathsf{V}_g|_\mathcal{N})$ (more precisely *supp*$(\mathsf{T}\mathfrak{f})$ stays between Σ_- and Σ_+), and

$$\mathsf{T}\mathfrak{f} - \mathfrak{f} = \mathsf{P}\mathfrak{g} \quad \text{for some } \mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\,, \tag{3.2.5}$$

because by standard properties of Green operators:

$$\mathsf{G}_\mathsf{P}\mathsf{T}\mathfrak{f} = \mathsf{G}_\mathsf{P}^+\mathsf{T}\mathfrak{f} - \mathsf{G}_\mathsf{P}^-\mathsf{T}\mathfrak{f} = \left(\mathsf{G}_\mathsf{P}^+\mathsf{P}\right)\chi^+\mathsf{G}_\mathsf{P}\mathfrak{f} - \mathsf{G}_\mathsf{P}^-\mathsf{P}(1-\chi^-)\mathsf{G}_\mathsf{P}\mathfrak{f} =$$
$$\chi^+\mathsf{G}_\mathsf{P}\mathfrak{f} - \mathsf{G}_\mathsf{P}^-\left(\mathsf{P}\mathsf{G}_\mathsf{P}\mathfrak{f}\right) + \mathsf{G}_\mathsf{P}^-\mathsf{P}\chi^-\mathsf{G}_\mathsf{P}\mathfrak{f} = \chi^+\mathsf{G}_\mathsf{P}\mathfrak{f} + \chi^-\mathsf{G}_\mathsf{P}\mathfrak{f} = \mathsf{G}_\mathsf{P}\mathfrak{f}.$$

Therefore we can apply (2.2.11) obtaining (3.2.5).
With these results, let us define

$$\omega_2(\mathfrak{f}, \mathfrak{g}) := \omega_{\mathcal{N}2}(\mathsf{T}\mathfrak{f}, \mathsf{T}\mathfrak{g})\,, \quad \mathfrak{f}, \mathfrak{g} \in \Gamma_c(\mathsf{V}_g)\,. \tag{3.2.6}$$

Taking the continuity properties of G_P into account, we leave to the reader the elementary proof of the fact that there is a unique distribution $\Gamma_c'(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ such that its value on $\mathfrak{f} \otimes \mathfrak{g}$ coincides with[6] $\omega_2(\mathfrak{f}, \mathfrak{g})$. (We will indicate that distribution by ω_2 with the usual misuse of language.) Furthermore, in view of the definition of T, it is obvious that ω_2 is also a bisolution of the Proca equation, since $\mathsf{G}_\mathsf{P}\mathsf{P} = \mathsf{P}\mathsf{G}_\mathsf{P} = 0$. To construct a candidate quasifree state ω on $\mathcal{A}_g$ out of its two-point function ω_2, it is clear that the positivity requirement is guaranteed because $\omega_\mathcal{N}$ satisfies it. We conclude that there is a quasifree state ω on $\mathcal{A}_g$, whose two point function is (3.2.6), and this two point function is a distribution which is also bisolution of the Proca equation. Finally, observe that ω extends to the whole $\mathcal{A}_g$ the state $\omega_\mathcal{N}$ since the states are quasifree and the two-point function of the former extends the two point function of the latter. Indeed,

$$\omega_2(\mathfrak{f}, \mathfrak{g}) = \omega_{\mathcal{N}2}(\mathsf{T}\mathfrak{f}, \mathsf{T}\mathfrak{g}) = \omega_{\mathcal{N}2}(\mathfrak{f}, \mathfrak{g}) \quad \text{if } \mathfrak{f}, \mathfrak{g} \in \Gamma_c(\mathsf{V}_g|_\mathcal{N})\,.$$

[6]If ω_2 indicates the distribution associated to the two-point function: $\omega_2 = \omega_{\mathcal{N}2} \circ \mathsf{T} \otimes \mathsf{T}$.

This is because, specializing (2.2.11) and (3.2.4)-(3.2.5) to the globally hyperbolic spacetime $(\mathcal{N}, g|_{\mathcal{N}})$ since $\mathfrak{f} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})$, we have that $\mathsf{T}\mathfrak{f} - \mathfrak{f} = \mathsf{P}\mathfrak{g}$ with $\mathfrak{g} \in \Gamma_c(\mathsf{V}_g|_{\mathcal{N}})$ and $\omega_{\mathcal{N}2}$ vanishes when one argument has the form $\mathsf{P}\mathfrak{g}$, because it is a bisolution of the Proca equation in $\mathcal{N}$.
There is only one such quasifree state which is an extension of $\omega_{\mathcal{N}}$ to the whole algebra $\mathcal{A}_g$, and such that its two-point function is a bisolution of the Proca equation. In fact, another such extension ω' would satisfy

$$\omega'_2(\mathfrak{f}, \mathfrak{g}) = \omega'_2(\mathsf{T}\mathfrak{f}, \mathsf{T}\mathfrak{g}) = \omega_{\mathcal{N}}(\mathsf{T}\mathfrak{f}, \mathsf{T}\mathfrak{g}) = \omega_2(\mathsf{T}\mathfrak{f}, \mathsf{T}\mathfrak{g}) = \omega_2(\mathfrak{f}, \mathfrak{g}) , \quad \text{for all } \mathfrak{f}, \mathfrak{g} \in \Gamma_c(\mathsf{V}_g).$$

(b) We pass to the proof that ω is Hadamard if $\omega_{\mathcal{N}}$ is. We have to prove that (3.2.1) is valid if it is valid for $\omega_{\mathcal{N}}$ in $(\mathcal{N}, g|_{\mathcal{N}})$. Interpreting the two-point functions as distributions of $\Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$,

$$\omega_2 = \omega_{\mathcal{N}2} \circ \mathsf{P}\chi^+ \mathsf{G}_\mathsf{P} \otimes \mathsf{P}\chi^+ \mathsf{G}_\mathsf{P} . \tag{3.2.7}$$

The wavefront sets of G_P and $\mathsf{P}\chi^+\mathsf{G}_\mathsf{P}$ can be computed as follows. First of all, let N be the normally hyperbolic operator associated to P from (2.2.9),

$$\mathsf{G}_\mathsf{P} = \mathsf{Q}\mathsf{G}_\mathsf{N} = \mathsf{G}_\mathsf{N}\mathsf{Q} \tag{3.2.8}$$

where $\mathsf{Q} = I + m^{-2} d\delta_g$. It is known that

$$WF(\mathsf{G}_\mathsf{N}) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y)\}$$

Notice that, in particular $k_x \neq 0$ and $k_y \neq 0$ nor simultaneously by definition, nor separately since they are connected by a coparallel transport.
So, since Q is a differential operator we immediately deduce by 3.2.8 that $WF(\mathsf{G}_\mathsf{P}) \subset WF(\mathsf{G}_N)$. Then we associate to the two operator their distributional kernels $\mathsf{G}_\mathsf{P}(x,y)$ and $\mathsf{G}_\mathsf{N}(x,y)$ and recast equation 3.2.8 in the form:

$$\mathsf{G}_\mathsf{P}(x,y) = (\mathrm{Id}_x \otimes \mathsf{Q}_y)\, \mathsf{G}_\mathsf{N}(x,y),$$

which, by standard microlocal analysis results, implies that

$$WF(\mathsf{G}_\mathsf{N}) \subset Char(\mathrm{Id}_x \otimes \mathsf{Q}_y) \cup WF(\mathsf{G}_\mathsf{P}).$$

However explicit computations give that $Char(\mathrm{Id}_x \otimes \mathsf{Q}_y) = \{(x, k_x; y, 0) | (x, k_x) \in \mathsf{T}^*\mathsf{M}, y \in \mathsf{M}\}$ which does not intersect $WF(\mathsf{G}_\mathsf{N})$ at any point, implying

$$WF(\mathsf{G}_\mathsf{N}) \subset WF(\mathsf{G}_\mathsf{P}) \subset WF(\mathsf{G}_\mathsf{N}).$$

So G_P and G_N have the same wavefront set. Therefore, since $\mathsf{P}\chi^+$ is a smooth differential operator,

$$WF(\mathsf{P}\chi\mathsf{G}_\mathsf{N}) \subset \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y)\}$$

A this point, a standard estimate of composition of wavefront sets in (3.2.7) yields (see, e.g., [76])

$$WF(\omega_2) \subset \mathcal{H}$$

where the Hadamard wavefront set $\mathcal{H}$ is the one in (3.2.1). To conclude the proof, it is sufficient to establish the converse inclusion. To this end, observe that, since the antisymmetric part of ω_2 is $\omega_2^+ - \omega_2^- = i\mathsf{G}_\mathsf{P}$,

$$WF(\mathsf{G}_\mathsf{P}) \subset WF(\omega_2^+) \cup WF(\omega_2^-) ,$$

where we adopted the same notation as at the beginning of the proof of Proposition 3.2.5: $\omega_2^+ = \omega_2$, $\omega_2^-(\mathfrak{f}, \mathfrak{g}) = \omega_2(\mathfrak{g}, \mathfrak{f})$. If, according to that notation, the prime applied to wavefront sets is defined as in (3.2.2), the above inclusion can be re-phrased to

$$\{(x, k_x; y, k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y)\} = WF(\mathsf{G}_\mathsf{P})' \subset WF(\omega_2^+)' \cup WF(\omega_2^-)' \tag{3.2.9}$$

Above

$$WF(\omega_2^+)' \subset \mathcal{H}' = \{(x, k_x; y, k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\}$$

and, with a trivial computation,

$$WF(\omega_2^-)' \subset \{(x, -k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_y \rhd 0\} ,$$

Now suppose that $(x, k_x; y, k_y) \in \mathcal{H}'$ does not belong to $WF(\omega_2^+)'$. According to (3.2.9), $(x, k_x; y, k_y) \notin WF(\mathsf{G_P})'$ (notice that $\mathcal{H}' \ni (x, k_x; y, k_y) \notin WF(\omega_2^-)'$ since the two sets are disjoint). This is impossible because every $(x, k_x; y, k_y) \in \mathcal{H}'$ belongs to $WF(\mathsf{G_P})'$ as it immediately arises by comparing the explicit expressions of these two sets written above. In summary $\mathcal{H}' \subset WF(\omega_2)'$, that is $\mathcal{H} \subset WF(\omega_2)$, concluding the proof. □

We are finally ready to extend the Møller operator to the quantum algebras, proving that they are indeed isomorphic. To this end, for any paracausally related metric $g \simeq g'$, we define an isomorphism of the free algebras $\mathcal{R}_{gg'} : \mathfrak{A}_{g'} \to \mathfrak{A}_g$ as the unique unital $*$-algebra isomorphism between the said free unital $*$-algebras such that

$$\mathcal{R}_{gg'}(\mathfrak{a}'(\mathfrak{f})) = \mathfrak{a}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_{g'}) ,$$

where R is a Møller operator of g, g' and the adjoint $\mathsf{R}^{\dagger_{gg'}}$ is defined as in Proposition 2.2.12.

When we pass to the quotient algebras, the preservation of the causal propagators discussed in the previous sections, immediately implies that the induced map on the quotient algebras is an isomorphism, that we call **Møller $*$-isomorphism.**

Proposition 3.2.7. *Let now $\mathcal{R}_{gg'} : \mathcal{A}_{g'} = \mathfrak{A}_{g'}/\mathfrak{I}_{g'} \to \mathcal{A}_g = \mathfrak{A}_g/\mathfrak{I}_g$ be the quotient morphism constructed out of $\mathcal{R}_{gg'}$. Then $\mathcal{R}_{gg'}$ is well defined and is indeed a $*$-algebra isomorphism.*

Proof. The proof of this statement is identical to the one performed in [87, Proposition 5.6]. Indeed it just relies on the preservation of the causal propagators proved in Theorem 2.2.14, which implies that the associated CCR-ideals are $*$-isomorphic. □

The final step in our construction is to define a pullback of the Møller $*$-isomorphism to the states and then to prove that the Hadamard condition is preserved, as done in [87, Theorem 5.14] for normally hyperbolic field theories.

Theorem 3.2.8. *Let $\mathcal{R}_{gg'}$ be the Møller *-isomorphism and let $\omega : \mathcal{A}_g \to \mathbb{C}$ be a quasifree Hadamard state, we define the pull-back state $\omega' : \mathcal{A}_{g'} \to \mathbb{C}$ by $\omega' = \omega \circ \mathcal{R}_{gg'}$. The following facts are true:*

1 ω' is a well-defined state;

2 ω' is quasifree;

3 ω' is a Hadamard state.

Proof. The proof of 1-2 is trivial and discussed in [87, Proposition 5.11]. The proof of 3 follows from the Hadamard propagation property stated in Proposition 3.2.6. To prove the statement we can just focus on the case in which the Møller operator is constructed out of two spacetimes such that $g \preceq g'$, the reasoning can then be iterated at each step of the paracausal chain.
The two-point function of the pullback state can be written as

$$\omega'_2(\mathfrak{f}, \mathfrak{h}) = \omega'(\hat{\mathfrak{a}}'(\mathfrak{f})\hat{\mathfrak{a}}'(\mathfrak{h})) = \omega(\mathcal{R}_{gg'}(\hat{\mathfrak{a}}'(\mathfrak{f})\hat{\mathfrak{a}}'(\mathfrak{h}))) = \omega(\hat{\mathfrak{a}}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f})\hat{\mathfrak{a}}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{h})) = \omega_2(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}, \mathsf{R}^{\dagger_{gg'}}\mathfrak{h}).$$

We recall that the operator is the composition of two pieces $\mathsf{R}^{\dagger_{gg'}} = \mathsf{R}_+^{\dagger_{gg_\chi}} \circ \mathsf{R}_-^{\dagger_{g_\chi g'}}$ and split the proof in two steps.
First we focus on the bidistribution $\omega_2^\chi(\mathfrak{f}, \mathfrak{h}) := \omega_2(\mathsf{R}_+^{\dagger_{gg_\chi}}\mathfrak{f}, \mathsf{R}_+^{\dagger_{gg_\chi}}\mathfrak{h})$ on (M, g_χ) defining a quasifree

state therein. By the property 2.1.14, in the region in which $g_\chi = g$, there is, for some t_0, there is a Cauchy surface Σ_{t_0} in common for g and g_χ, a common globally hyperbolic neighbourhood $\mathcal{N}$ of that Cauchy surface such that $\omega_2^\chi(\mathfrak{f},\mathfrak{h}) = \omega_2(\mathfrak{f},\mathfrak{h})$ when the supports of $\mathfrak{f}$ and $\mathfrak{g}$ are chosen in $\mathcal{N}$ and thus the corresponding state is Hadamard in $(\mathcal{N}, g_\chi)$. Now Proposition 3.2.15 implies that ω_2^χ is Hadamard in the whole (M, g_χ). Secondly, the same argument can be used once again for the operator $\mathsf{R}_-^{\dagger_{g_\chi g'}}$ on the Hadamard state ω^χ on (M, g_χ), proving that the state induced by $\omega_2(\mathsf{R}^{\dagger_{gg'}}\cdot, \mathsf{R}^{\dagger_{gg'}}\cdot) = \omega_2^\chi(\mathsf{R}_-^{\dagger_{g_\chi g'}}\cdot, \mathsf{R}_-^{\dagger_{g_\chi g'}}\cdot)$ is Hadamard as well on (M, g'). In other words the full Møller operator preserves the Hadamard form. □

3.2.3 Existence of Proca Hadamard states in globally hyperbolic spacetimes

The next subsections are devoted to the construction of Hadamard states for the real Proca field in a generic globally hyperbolic spacetime. Actually, the technology of Møller operators, in particular Theorem 3.2.8, allows us to reduce the construction of Hadamard states for the Proca equation to the special case of an ultrastatic spacetime with Cauchy hypersurfaces of bounded geometry. Indeed, as shown in the first chapter, for any globally hyperbolic spacetime (M, g), there exists a paracausally related globally hyperbolic spacetime (M, g_0) which is ultrastatic. In other words, first of all (M, g_0) is isometric to $\mathbb{R} \times \Sigma$ where (Σ, h_0) is a t-independent complete Riemannian manifold and $g_0 = -dt \otimes dt + h_0$, where t is the natural coordinate on $\mathbb{R}$ and dt is past directed. We also denote by ∂_t the tangent vector to the submanifold $\mathbb{R}$ of $\mathbb{R} \times \Sigma$. In view of the completeness of h, these spacetimes are globally hyperbolic (see e.g. [49]) and Σ is a Cauchy surface of this spacetime. In turn, it is possible to change the metric on Σ in order that the final metric, indicated by h is both complete and *of bounded geometry* [67]. By construction, the final ultrastatic spacetime $(\mathsf{M}, -dt\otimes dt + h)$ is paracausally related to (M, g_0) because the intersection of the corresponding open cones is non-empty as it always contains ∂_t. By transitivity (M, g) is paracausally related with $(\mathbb{R} \times \Sigma, -dt \otimes dt + h)$.

Hence, we assume without loss of generalities, that $(\mathsf{M}, g) = (\mathbb{R}\times\Sigma, -dt\otimes dt + h)$ is a globally hyperbolic ultrastatic spacetime, with dt past directed, whose spatial metric h is complete. When dealing with the construction of Hadamard states we also assume that the spatial manifold (Σ, h) is also of bounded geometry. In the final part of the section, we will come back to consider a generic globally hyperbolic spacetime (M, g)

We can proceed to the construction of quasifree states. As we shall see shortly, this construction for the Proca field uses some consequences of the spectral theory applied to the theory of *elliptic Hilbert complexes* [21] defined in terms of the closure of Hodge operators in natural L^2 spaces of forms.

Some of the following ideas were inspired by [44]. However we now work in the space of Cauchy data instead of in the space of smooth compactly supported forms and/or modes. Furthermore we do not assume restrictions on the topology of the Cauchy surfaces used in [44] to impose a pure point spectrum to the Hodge Laplacians.

To define quasifree states for the Proca field we observe that, as P is Green hyperbolic, the CCR algebra $\mathcal{A}_g$ is isomorphic to the analogous unital $*$-algebra $\mathcal{A}_g^{(symp)}$ generated by the **solution-smeared field operators** $\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A)$, for $A \in \mathsf{Ker}_{sc}(P)$, which are $\mathbb{R}$-linear in A, Hermitian, and satisfy the commutation relations[7]

$$\left[\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A), \sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A')\right] = i\sigma^{(\mathsf{P})}(A, A')I \, . \tag{3.2.10}$$

The said unital $*$-algebra isomorphism $F : \mathcal{A}_g \to \mathcal{A}_g^{(symp)}$ is completely defined as the unique homomorphism of unital $*$-algebras that satisfies

$$F : \hat{\mathfrak{a}}(\mathfrak{f}) \mapsto \sigma^{(\mathsf{P})}\left(\hat{\mathfrak{a}}, \mathsf{G}_\mathsf{P}\mathfrak{f}\right) \quad \text{with } A = \mathsf{G}_\mathsf{P}\mathfrak{f}, \quad \mathfrak{f} \in \Gamma_c(\mathsf{V}_g) \, .$$

[7]Notice that, as $\sigma^{(\mathsf{P})}(A, A')$ is non degenerate, we have that $\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A) = 0$ only if $A = 0$.

The definition is well-posed in view of (2.2.23), (2.2.10), (2.2.11), and the definition of $\mathcal{A}_g$. Within this framework, the two point function ω_2 is interpreted as the integral kernel of

$$\omega\left(\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A)\sigma^{(\mathsf{P})}(\hat{\mathfrak{a}}, A')\right) .$$

In particular, its antisymmetric part is universally given by $\frac{i}{2}\sigma^{(P)}(A, A')$ due to (3.2.10). The specific part of the two point function is therefore completely embodied in its symmetric part $\mu(A, A')$.

According to this observation, a general recipe for real (bosonic) CCR in generic globally hyperbolic spacetimes to define a quasifree state on the *-algebra $\mathcal{A}_g$ (e.g., see [76, 78, 99] for the scalar case and [54, Chapter 4, Proposition 4.9] for the generic case of real bosonic CCRs) is to assign a real scalar product on the space of spacelike compact solutions

$$\mu : \mathsf{Ker}_{sc}(\mathsf{P}) \times \mathsf{Ker}_{sc}(\mathsf{P}) \to \mathbb{R}$$

satisfying

(a) the strict positivity requirement $\mu(A, A) \geqslant 0$ where $\mu(A, A) = 0$ implies $A = 0$;

(b) the continuity requirement with respect to the relevant symplectic form $\sigma^{(P)}$ (see, e.g., [54, Proposition 4.9]),
$$\sigma^{(\mathsf{P})}(A, A')^2 \leqslant 4\mu(A, A)\mu(A', A') . \tag{3.2.11}$$

The continuity requirement directly arises from the fact that the quasifree state induced by μ on the whole $*$-algebra $\mathcal{A}_g \equiv \mathcal{A}_g^{symp}$ is a positive functional. The converse implication, though true, is less trivial [54, 78]. The two mentioned requirements are nothing but the direct translation of (2)' and (3)' stated in the introduction. (Regarding the latter, observe that $\sigma^{(P)}$ corresponds to the causal propagator at the level of solutions – Eq. (2.2.23) in our case – as discussed in Section 2.2.3.) At this point, it should be clear that the quasifree state defined by μ has two-point function, viewed as bilinear map on $\Gamma_c(\mathsf{V}_g) \times \Gamma_c(\mathsf{V}_g)$,

$$\omega_\mu(\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{f}')) = \omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') := \mu(\mathsf{G}_\mathsf{P}\mathfrak{f}, \mathsf{G}_\mathsf{P}\mathfrak{f}') + \frac{i}{2}\sigma^{(P)}(\mathsf{G}_\mathsf{P}\mathfrak{f}, \mathsf{G}_\mathsf{P}\mathfrak{f}') .$$

However, since the Cauchy problem is well posed on the time slices Σ of an ultrastatic spacetime $(\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, as proved in Proposition 2.2.1, we can directly define μ (and $\sigma^{(\mathsf{P})}$) in the space of Cauchy data C_Σ on Σ, for smooth spacelike compact solutions, viewed as the time slice at $t = 0$,

$$\mu : C_\Sigma \times C_\Sigma \to \mathbb{R} .$$

In view of the peculiarity of the Cauchy problem for the Proca field as discussed in Section 2.2.2, the real vector space of the Cauchy data C_Σ is *constrained*. We underline that working at the level of constrained initial data does not affect the construction of quasifree states. Indeed, it is sufficient that the space of constrained initial conditions is a real (or complex) vector space and that the constrained Cauchy problem is well posed. With this in mind, referring to the canonical decomposition $A = A^{(0)}dt + A^{(1)}$ of a real smooth spacelike compact solution A of the Proca equation, we remember that

$$C_\Sigma := \left\{(a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in \Omega^0_c(\Sigma)^2 \times \Omega^1_c(\Sigma)^2 \;\middle|\; \pi^{(0)} = -\delta^{(1)}_h a^{(1)} , \quad (\Delta^{(0)}_h + m^2)a^{(0)} = \delta^{(1)}_h \pi^{(1)} \right\} .$$

Above $(a^{(0)}, \pi^{(0)}) := (A^{(0)}, \partial_t A^{(0)})|_{t=0}$ and $(a^{(1)}, \pi^{(1)}) := (A^{(1)}, \partial_t A^{(1)})|_{t=0}$.

With the said definitions and *where A denotes both a solution of Proca equation and its Cauchy data on* Σ, we have the first result.

Proposition 3.2.9. *Consider the $*$-algebra $\mathcal{A}_g$ of the real Proca field on the ultrastatic spacetime $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, with dt past directed and (Σ, h) a smooth complete Riemannian manifold. Let $\eta_0 := -1$, $\eta_1 := 1$ and $h^\sharp_{(j)}$ denote the standard inner product of j-forms on Σ induced by h. The bilinear map on the space C_Σ of real smooth compactly supported Cauchy data (2.2.24)*

$$\mu(A, A') := \sum_{j=0}^{1} \frac{\eta_j}{2} \int_\Sigma h^\sharp_{(j)}(\pi^{(j)}, (\overline{\Delta^{(j)} + m^2})^{-1/2}\pi^{(j)\prime}) + h^\sharp_{(j)}(a^{(j)}, (\overline{\Delta^{(j)} + m^2})^{1/2} a^{(j)\prime}) \mathrm{vol}_h \quad (3.2.12)$$

is a well defined symmetric positive inner product which satisfies (3.2.11) and thus it defines a quasifree state ω_μ on $\mathcal{A}_g$ completely defined by its two-point function

$$\omega_\mu \left(\mathfrak{a}(\mathfrak{f})\mathfrak{a}(\mathfrak{f}')\right) = \omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') := \mu\left(\mathsf{G_P}\mathfrak{f}, \mathsf{G_P}\mathfrak{f}'\right) + \frac{i}{2}\sigma^{(P)}\left(\mathsf{G_P}\mathfrak{f}, \mathsf{G_P}\mathfrak{f}'\right) \quad (3.2.13)$$

where $\mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathsf{V}_g)$ satisfy

$$\sigma^{(P)}\left(\mathsf{G_P}\mathfrak{f}, \mathsf{G_P}\mathfrak{f}'\right) = \int_{\mathsf{M}} g^\sharp(\mathfrak{f}, \mathsf{G_P}\mathfrak{f}') \,\mathrm{vol}_g\,.$$

The bar over the operators in (3.2.12) denotes the closure in suitable Hilbert spaces of the operators originally defined on domains of compactly supported smooth functions. To explain this formalism, before starting with the proof we have to permit some technical facts about the properties of the Hodge operators at the level of L^2 spaces. Given the complete Riemannian manifold (Σ, h), with $n := \dim(\Sigma)$ consider the Hilbert space $\mathcal{H}_h := \bigoplus_{k=0}^n L^2_k(\Sigma, \mathrm{vol}_h)$, where the sum is orthogonal and $L^2_k(\Sigma, \mathrm{vol}_h)$ is the complex Hilbert space of the square-integrable k-forms with respect to the relevant *Hermitian* Hodge inner product:

$$(a|b)_k := \int_\Sigma h^\sharp_{(k)}(\overline{a}, b)\,\mathrm{vol}_h\,, \quad a, b \in L^2_k(\Sigma, \mathrm{vol}_h)\,,$$

where $\overline{a}$ denotes the pointwise complex conjugation of the complex form a. The overall inner product on $\mathcal{H}_h$ will be indicated by $(\cdot|\cdot)$ and the Hilbert space adjoint of a densely-defined operator $A : D(A) \to \mathcal{H}_h$, with $D(A) \subset \mathcal{H}_h$, will be denoted by $A^* : D(A^*) \to \mathcal{H}_h$. The closure of A will be denoted by the bar: $\overline{A} : D(\overline{A}) \to \mathcal{H}_h$.

If $\Omega_c(\Sigma)_\mathbb{C} := \bigoplus_{k=0}^n \Omega^k_c(\Sigma)_\mathbb{C}$ denotes the dense subspace of complex compactly supported smooth forms $\Omega^k_c(\Sigma)_\mathbb{C} := \Omega^k_c(\Sigma) + i\Omega^k_c(\Sigma)$, define the two operators (we omit the index h for shortness)

$$d := \oplus_{k=0}^n d^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C}\,, \quad \delta := \oplus_{k=0}^n \delta^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C}$$

with $d^{(n)} := 0$ and $\delta^{(0)} := 0$. Finally, introduce the Hodge Laplacian as

$$\Delta := \sum_{k=0}^n \Delta^{(k)} : \Omega_c(\Sigma)_\mathbb{C} \to \Omega_c(\Sigma)_\mathbb{C} \quad \text{with } \Delta^{(k)} := \delta^{(k+1)} d^{(k)} + d^{(k-1)}\delta^{(k)}.$$

Since (Σ, h) is complete, Δ can be proved to be essentially selfadjoint, for instance exploiting the well-known argument by Chernoff [25] (or directly referring to [1]). Since Δ is essentially selfadjoint, if $c \in \mathbb{R}$, also $\Delta + cI$ is essentially selfadjoint. In particular, its unique selfadjoint extension is its closure $\overline{\Delta + cI}$.

Referring to the theory of *elliptic Hilbert complexes* developed in [21, Section 3] and focusing in particular on [21, Lemma 3.3] based on previous achievements established in [1], we can conclude that the following couple of facts are true. (The compositions of operators are henceforth defined with their natural domains: $D(A + B) := D(A) \cap D(B)$, $D(AB) = \{x \in D(B) \mid Bx \in D(A)\}$, $D(aA) := D(A)$ for $a \neq 0$, $D(0A) := \mathcal{H}_h$, and $A \subset B$ means $D(A) \subset D(B)$ with $B|_{D(A)} = A$.)

(a) The identities hold

$$\overline{d}^* = \overline{\delta}\,, \quad \overline{\delta}^* = \overline{d} \tag{3.2.14}$$

where * denotes the adjoint in the Hilbert space $\mathcal{H}_h$.

(b) The unique selfadjoint extension $\overline{\Delta}$ of Δ satisfies

$$\overline{\Delta} = \overline{d}\,\overline{\delta} + \overline{\delta}\overline{d} = \sum_{k=0}^{n} \overline{\Delta^{(k)}} \quad \text{with } \overline{\Delta^{(k)}} := \overline{\delta^{(k+1)}}\,\overline{d^{(k)}} + \overline{d^{k-1}}\,\overline{\delta^{(k)}}. \tag{3.2.15}$$

A trivial generalization of the decomposition as in (3.2.15) holds for $\overline{\Delta + cI} = \overline{\Delta} + cI$ with $c \in \mathbb{R}$.

We are now prompt to prove a preparatory technical lemma – necessary to establish Proposition 3.2.9 – that will be fundamental for showing that the bilinear map μ is positive on the space C_Σ.

Lemma 3.2.10. *For every given $k = 0, 1, \ldots, n$, $c > 0$, and $\alpha \in \mathbb{R}$, the identities hold*

$$(\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}} x = \overline{d^{(k)}} (\overline{\Delta^{(k)} + cI})^\alpha x\,, \quad \forall x \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}})$$
$$(\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}} y = \overline{\delta^{(k-1)}} (\overline{\Delta^{(k)} + cI})^\alpha y\,, \quad \forall y \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}})\,.$$

Proof. Since $dd = 0$ and $\delta\delta = 0$, from (3.2.14), we also have $\overline{d}\,\overline{d}x = 0$ if $x \in D(\overline{d})$ and $\overline{\delta}\,\overline{\delta}y = 0$ if $y \in D(\overline{\delta})$, and thus (3.2.15) yields[8]

$$\overline{d}\,\overline{\Delta} \supset \overline{d}\,\overline{\delta}\,\overline{d} = \overline{\Delta}\,\overline{d}\,.$$

However, if $D(\overline{d}\,\overline{\Delta}) \supsetneq D(\overline{d}\,\overline{\delta}\,\overline{d})$, we would have $x \in D(\overline{\Delta}) = D(\overline{\delta}\,\overline{d}) \cap D(\overline{d}\,\overline{\delta})$ such that $\overline{\Delta}x = \overline{\delta}\overline{d}x + \overline{d}\overline{\delta}x \in D(\overline{d})$, but $x \notin D(\overline{d}\overline{\delta}\overline{d})$, namely $\overline{\delta}\overline{d}x \notin D(\overline{d})$. This is impossible since $\overline{\delta}\overline{d}x + \overline{d}\overline{\delta}x \in D(\overline{d})$, $D(\overline{d})$ is a subspace and $\overline{d}\,\overline{\delta}x \in D(\overline{d})$ (and more precisely $\overline{d}\,\overline{d}\,\overline{\delta}x = 0$ as stated above). Therefore

$$\overline{d}\,\overline{\Delta} = \overline{d}\,\overline{\delta}\,\overline{d} = \overline{\Delta}\,\overline{d}$$

and the same result is valid with δ in place of d. Evidently, in both cases $\overline{\Delta}$ can be replaced by the selfadjoint operator $\overline{\Delta} + cI = \overline{\Delta + cI}$ for every $c \in \mathbb{R}$:

$$\overline{d}\,\overline{\Delta + cI} = \overline{\Delta + cI}\,\overline{d}\,, \quad \overline{\delta}\,\overline{\Delta + cI} = \overline{\Delta + cI}\,\overline{\delta}\,. \tag{3.2.16}$$

We henceforth assume $c > 0$. In that case, as Δ is already positive on its domain, the spectrum of the selfadjoint operator $\overline{\Delta + cI}$ is strictly positive and thus $\overline{\Delta + cI}^{-1} : \mathcal{H}_h \to D(\overline{\Delta + cI})$ is well defined, selfadjoint and bounded. The former identity in (3.2.16) also implies that $D(\overline{d}\,\overline{\Delta + cI}) = D(\overline{\Delta + cI}\,\overline{d})$, so that

$$\overline{\Delta + cI}^{-1}\overline{d}\,\overline{\Delta + cI}|_{D(\overline{d}\,\overline{\Delta + cI})} x = \overline{d}|_{D(\overline{d}\,\overline{\Delta + cI})} x\,.$$

By construction, we can choose $x = \overline{\Delta + cI}^{-1} y$ with $y \in D(\overline{d})$ in view of the definition of the natural domain of the composition $\overline{d}\,\overline{\Delta + cI}$). In summary

$$\overline{\Delta + cI}^{-1}\overline{d}y = \overline{d}\,\overline{\Delta + cI}^{-1} y\,, \quad \forall y \in D(\overline{d})\,.$$

Since the argument is also valid for δ, we have established that

$$\overline{\Delta + cI}^{-1}\overline{d} \subset \overline{d}\,\overline{\Delta + cI}^{-1}\,, \quad \overline{\Delta + cI}^{-1}\overline{\delta} \subset \overline{\delta}\,\overline{\Delta + cI}^{-1}$$

Iterating the argument, for every $n = 0, 1, \ldots$,

$$(\overline{\Delta + cI}^{-1})^n \overline{d} \subset \overline{d}\,(\overline{\Delta + cI}^{-1})^n\,, \quad (\overline{\Delta + cI}^{-1})^n \overline{\delta} \subset \overline{\delta}\,(\overline{\Delta + cI}^{-1})^n\,.$$

[8] It holds $(B + C)A = BC + BA$, but $AB + AC \subset A(B + C)$.

This result extends to complex polynomials of $\overline{\Delta + cI}^{-1}$ in place of powers by linearity. Using the spectral calculus (see e.g. [85]) where $\mu_{xy}(E) = (x|P_E y)$ and P is the projector-valued spectral measure of $\overline{\Delta + cI}^{-1}$, the found result for $\overline{d}$ can be written

$$\int_{[0,b]} p(\lambda) d\mu_{x,\overline{d}y}(\lambda) = \int_{[0,b]} p(\lambda) d\mu_{\overline{\delta}x,y}(\lambda) \tag{3.2.17}$$

for every complex polynomial p, where $[0,b]$ is a sufficiently large interval to include the (bounded positive) spectrum of $\overline{\Delta + cI}^{-1}$, $x \in D(\overline{\delta})$, $y \in D(\overline{d})$, and where we have used $\overline{\delta} = \overline{d}^*$. Since the considered regular Borel complex measures are finite and supported on the compact $[0,b]$, we can pass in (3.2.17) from polynomials p to generic continuous functions f in view of the Stone-Weierstrass theorem. At this juncture, $P_E^* = P_E$ and the uniqueness part of Riesz' representation theorem for regular complex Borel measures, implies that

$$(P_E\overline{\delta}y|x) = (P_E y|\overline{d}x) \quad \text{for all } x \in D(\overline{\delta}),\ y \in D(\overline{d}), \text{ and every Borel set } E \subset \mathbb{R}.$$

which means $P_E\overline{\delta} \subset \overline{d}^* P_E$, namely $P_E\overline{\delta} \subset \overline{\delta}P_E$. Analogously, we also have $P_E\overline{d} \subset \overline{d}P_E$.

If $f : \mathbb{R} \to \mathbb{C}$ is measurable and *bounded*, the standard spectral calculus and (3.2.14), with a procedure similar to the one used to prove $P_E\overline{\delta} \subset \overline{\delta}P_E$ and taking into account the fact that $D(f(\overline{\Delta + cI}^{-1})) = \mathcal{H}_h$, yields

$$f(\overline{\Delta + cI}^{-1})\overline{\delta} \subset \overline{\delta} f(\overline{\Delta + cI}^{-1})\,, \quad f(\overline{\Delta + cI}^{-1})\overline{d} \subset \overline{d} f(\overline{\Delta + cI}^{-1}) \tag{3.2.18}$$

If f is unbounded, we can choose a sequence of bounded measurable functions f_n such that $f_n \to f$ pointwise. It is easy to prove that (see, e.g. [85]) $x \in D(\int_{\mathbb{R}} f dP)$ entails $\int_{\mathbb{R}} f_n dPx \to \int_{\mathbb{R}} f dPx$. This is the case for instance for $f(\lambda) = \lambda^\beta$ with $\beta < 0$ restricted to $[0,b]$. Referring to this function and the pointed out result for some sequence of bounded functions with $f_n \to f$ pointwise, the latter of (3.2.18) implies that[9] ,

$$(\overline{\Delta + cI})^\alpha \overline{d}x = \overline{d}(\overline{\Delta + cI})^\alpha x \quad \text{if } x \in D((\overline{\Delta + cI})^\alpha) \cap D(\overline{d}) \text{ and } \overline{d}x \in D((\overline{\Delta + cI})^\alpha),$$

where we used also the fact that $\overline{d}$ is closed. The case of δ can be worked out similarly. Summing up, we have proved that, if $\alpha \in \mathbb{R}$,

$$\begin{aligned}
(\overline{\Delta + cI})^\alpha \overline{d}x &= \overline{d}(\overline{\Delta + cI})^\alpha x\,, \quad \forall x \in D((\overline{\Delta + cI})^\alpha) \cap D((\overline{\Delta + cI})^\alpha \overline{d}) \\
(\overline{\Delta + cI})^\alpha \overline{\delta}y &= \overline{\delta}(\overline{\Delta + cI})^\alpha y\,, \quad \forall y \in D((\overline{\Delta + cI})^\alpha) \cap D((\overline{\Delta + cI})^\alpha \overline{\delta})\,.
\end{aligned}$$

Let us remark that for $\alpha \leqslant 0$ it is sufficient to choose $x \in D(\overline{d})$ and $y \in D(\overline{\delta})$. For every given $k = 0, 1, \ldots, n$, $c > 0$, and $\alpha \in \mathbb{R}$, taking the decomposition of $\mathcal{H}_h$ into account the above formulae imply

$$\begin{aligned}
(\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}}x &= \overline{d^{(k)}}(\overline{\Delta^{(k)} + cI})^\alpha x\,, \quad \forall x \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k+1)} + cI})^\alpha \overline{d^{(k)}}) \\
(\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}}y &= \overline{\delta^{(k-1)}}(\overline{\Delta^{(k)} + cI})^\alpha y\,, \quad \forall y \in D((\overline{\Delta^{(k)} + cI})^\alpha) \cap D((\overline{\Delta^{(k-1)} + cI})^\alpha \overline{\delta^{(k)}})\,.
\end{aligned}$$

That is the thesis. □

We are now prompted to prove that the bilinear map defined by Equation (3.2.12) defines a quasifree state defined by the two-point function given by (3.2.13) establishing the thesis of Proposition 3.2.9.

[9] Below, $\alpha > 0$ otherwise $(\overline{\Delta + cI})^\alpha$ is bounded in view of its spectral properties and (3.2.18) is enough to conclude the proof.

Proof of Proposition 3.2.9. To continue with the proof of the proposition, we now demonstrate that μ is well-defined and positive. That bilinear form is well-defined because $\Omega^{(j)}_c(\Sigma) \subset D(\overline{\Delta^{(j)} + m^2 I}^{\alpha})$ for $\alpha \leqslant 1$ as one immediately proves from spectral calculus. Furthermore, the integrand in the right-hand side of Equation (3.2.12) is the linear combination of products of L^2 functions (of which one of the two has also compact support). Let us pass to the positivity issue. Our strategy is to re-write $\mu(A, A)$, where $A = (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \in C_\Sigma$, as the quadratic form of the energy $\mu(A, A) = E^{(P)}(A_o)$, where the right-hand side is defined in Equation (2.2.27), for a new set of initial data A_o which are not necessarily smooth and compactly supported but such that $E^{(P)}(A_o)$ is well defined. If $A \in C_\Sigma$, define for $j = 0, 1$

$$\begin{aligned} A_o &= (a^{(0)}_o, \pi^{(0)}_o, a^{(1)}_o, \pi^{(1)}_o) \\ a^{(j)}_o &:= (\overline{\Delta^{(j)} + m^2 I})^{-1/4} a^{(j)} \\ \pi^{(j)}_o &:= (\overline{\Delta^{(j)} + m^2 I})^{-1/4} \pi^{(j)} \end{aligned} \tag{3.2.19}$$

Notice that the definition is well posed and the forms $a^{(j)}_o$ and $\pi^{(j)}_o$ belong to the respective Hilbert spaces of j-forms, because $\Omega^{(j)}_c(\Sigma) \subset D(\overline{\Delta^{(j)} + m^2 I}^{\alpha})$ for $\alpha \leqslant 1$ as said above. Furthermore the new forms are real since the initial ones are real and $\overline{\Delta^{(j)} + m^2 I}^{\alpha}$ commutes with the complex conjugation[10]. At this juncture, we have from (3.2.12)

$$\mu(A, A) = \sum_{j=0}^{1} \eta_j \int_\Sigma h^\sharp_{(j)}(\pi^{(j)}_o, \pi^{(j)}_o) + h^\sharp_{(j)}(a^{(j)}_o, (\overline{\Delta^{(j)} + m^2 I}) a^{(j)}_o) \mathrm{vol}_h \tag{3.2.20}$$

Furthermore, the new Cauchy data, though they stay outside C_Σ in general, they however satisfy the natural generalization of the constraints defining C_Σ in view of Lemma 3.2.10:

$$\pi^{(0)}_o = -\overline{\delta^{(1)}_h} a^{(1)}_o \,, \quad (\overline{\Delta^{(0)}_h + m^2}) a^{(0)}_o = \overline{\delta^{(1)}_h} \pi^{(1)}_o \,. \tag{3.2.21}$$

These identities arise immediately from Definitions (3.2.19), the constraints (2.2.21), and by applying Lemma 3.2.10 and paying attention to the fact that $\Omega^{(j)}_c(\Sigma) \subset D((\overline{\Delta^{(j-1)} + cI})^{\alpha} \overline{\delta^{(j)}})$ for every $\alpha \leqslant 1$ and also using $(\overline{\Delta^{(j)} + m^2 I})(\overline{\Delta^{(j)} + m^2 I})^{-1/4} = (\overline{\Delta^{(j)} + m^2 I})^{-1/4} \overline{\Delta^{(j)} + m^2 I}$ (for, e.g., [85, (f) in Proposition 3.60]). Using (3.2.14) and (3.2.21) in the right-hand side of (3.2.20), we can proceed backwardly as in the proof that (2.2.27) is equivalent to (2.2.28). Indeed, the only ingredients we used in that proof were the constraint equations which are valid also for A_o and the duality of δ and d with respect to the Hodge inner product, which extends to $\overline{\delta}$ and $\overline{d}$. In summary,

$$\begin{aligned} \mu(A, A) = \frac{1}{2} \int_\Sigma \Big(& h^\sharp_{(1)}(\pi^{(1)}_o - \overline{d^{(0)}} a^{(0)}_o, \pi^{(1)}_o - \overline{d^{(0)}} a^{(0)}_o) + h^\sharp_{(2)}(\overline{d^{(1)}} a^{(1)}_o, \overline{d^{(1)}} a^{(1)}_o) \\ & + m^2 \big(h^\sharp_{(1)}(a^{(1)}_o, a^{(1)}_o) + a^{(0)}_o a^{(0)}_o \big) \Big) \mathrm{vol}_h \,. \end{aligned}$$

From that identity, it is clear that $\mu(A, A) \geqslant 0$ and $\mu(A, A) = 0$ implies $A_o = 0$, which in turn yields $A = 0$ because the operators $\overline{\Delta^{(j)} + m^2 I}^{1/4}$ are injective. We have established that $\mu : C_\Sigma \times C_\Sigma \to \mathbb{R}$ is a positive real symmetric inner product.

Let us pass to prove (3.2.11). First of all, we change the notation concerning the scalar product μ making explicit the decomposition of A, and we work with *complex* valued forms. We use

$$A = (a, \pi) = (a^{(0)}, \pi^{(0)}, a^{(1)}, \pi^{(1)}) \,, \quad a := (a^{(0)}, a^{(1)}) \,, \quad \pi := (\pi^{(0)}, \pi^{(1)})$$

[10] It easily arises from spectral calculus using the fact that the complex conjugation is bijective from $\mathcal{H}_h$ to $\mathcal{H}_h$, continuous, and commutes with $\overline{\Delta^{(j)} + m^2 I}$.

so that, if $(a,\pi),(a',\pi') \in (L^2_0(\Sigma,\mathrm{vol}_h)\oplus L^2_1(\Sigma,\mathrm{vol}_h))\times(L^2_0(\Sigma,\mathrm{vol}_h)\oplus L^2_1(\Sigma,\mathrm{vol}_h))$ are such that the right-hand side below is defined, we can write

$$\mu((\overline{a},\overline{\pi}),(a',\pi')) := \sum_{j=0}^{1}\frac{\eta_j}{2}\int_\Sigma h^\sharp_{(j)}(\overline{\pi^{(j)}}, H^{-1}_{(j)}\pi^{(j)\prime}) + h^\sharp_{(j)}(\overline{a^{(j)}}, H_{(j)}a^{(j)\prime})\mathrm{vol}_h$$

where $H_{(j)} := \overline{\Delta^{(j)}+m^2 I}^{1/2}$, and the bar on forms denotes the complex conjugation. Finally, for $\alpha=\pm1$, we defined

$$H^\alpha a := (H^\alpha_{(0)}a^{(0)}, H^\alpha_{(1)}a^{(1)})\,, \quad H^\alpha\pi := (H^\alpha_{(0)}\pi^{(0)}, H^\alpha_{(1)}\pi^{(1)})\,.$$

By direct inspection one sees that, if $(a,\pi),(a',\pi')\in C_\Sigma + iC_\Sigma$, then the right-hand side of the first identity below is well-defined and

$$\begin{aligned}\Lambda((a,\pi),(a',\pi')) :=& \frac{1}{2}\mu\left((\overline{\pi}+iH^{-1}\overline{a},\overline{a}-iH\overline{\pi}),(\pi'-iH^{-1}a',a'+iH\pi')\right)\\ =&\mu((\overline{a},\overline{\pi}),(a',\pi'))+\frac{i}{2}\sigma^{(\mathsf{P})}((\overline{a},\overline{\pi}),(a',\pi'))\end{aligned}$$

where $\sigma^{(P)}$ is the right-hand side of (2.2.25), which however coincides with the original symplectic form (2.2.22) evaluated on complex Cauchy data because $(a,\pi),(a',\pi')\in C_\Sigma+iC_\Sigma$ and Remark 2.2.5 holds. Finally notice that if $(a,\pi)\in C_\Sigma+iC_\Sigma$ then $a_o := \pi - iHa$ and $\pi_o := a+iH^{-1}\pi$ satisfy the constraints (though they do not belong to $C_\Sigma+iC_\Sigma$ in general)

$$\pi^{(0)}_o = -\overline{\delta^{(1)}_h}a^{(1)}_o\,,\quad H_{(0)}a^{(0)}_o = \overline{\delta^{(1)}_h}\pi^{(1)}_o\,.$$

The proof is direct, using Lemma 3.2.10 once more. As a consequence, exploiting the same argument to prove (2.2.29) and observing that H^α commutes with the complex conjugation – so that it holds $\overline{\pi}-iH^{-1}\overline{a} = \overline{\pi+iH^{-1}a}$ for instance – we have that

$$\begin{aligned}2\Lambda((a,\pi),(a,\pi)) &= \mu\left((\overline{\pi}+iH^{-1}\overline{a},\overline{a}-iH\overline{\pi}),(\pi-iH^{-1}a,a+iH\pi)\right)\\ &= \mu\left((\overline{\pi-iH^{-1}a,a+iH\pi}),(\pi-iH^{-1}a,a+iH\pi)\right)\geqslant 0\,.\end{aligned}$$

The final inequality is due to the fact that μ is (the complexification of) a *real* positive bilinear symmetric form. All that means in particular that the *Hermitian* form Λ on $(C_\Sigma+iC_\Sigma)\times(C_\Sigma+iC_\Sigma)$ is (semi)positively defined and thus it satisfies the Cauchy-Schwartz inequality. In particular,

$$(Im\Lambda((a,\pi),(a',\pi')))^2 \leqslant |\Lambda((a,\pi),(a',\pi'))|^2 \leqslant \Lambda((a,\pi),(a,\pi))\,\Lambda((a',\pi'),(a',\pi'))\,.$$

If choosing $(a,\pi),(a',\pi')\in C_\Sigma$ (thus *real* forms), the above inequality specialises to

$$\sigma^{(\mathsf{P})}((a,\pi),(a',\pi'))^2 \leqslant 4\mu((a,\pi),(a,\pi))\,\mu((a',\pi'),(a',\pi'))$$

which is the inequality (3.2.11) we wanted to prove. □

3.2.4 Hadamard states in ultrastatic and generic globally hyperbolic spacetimes

With the next proposition, we show that the quasifree states defined in Proposition 3.2.9 is a Hadamard state when (Σ,h) is of bounded geometry. To prove the assertion we will take advantage of the general formalism developed in [54] and [57]. An alternative proof, which does not assume that the manifold is of bounded geometry (however we here take advantage of [67]), could be constructed along the procedure developed in [50] and extending it to the vectorial Klein-Gordon field.

Proposition 3.2.11. *If the metric h on the time slice Σ is of bounded geometry, then the quasifree state $\omega_\mu : \mathcal{A}_g \to \mathbb{C}$ defined in Proposition 3.2.9 is Hadamard according to Definition 3.2.4.*

Proof. Consider a pair of *complex* Klein-Gordon fields $A^{(0)}$ and $A^{(1)}$ in the ultrastatic spacetime $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$, with (Σ, h) a smooth complete Riemannian manifold of bounded geometry obeying the normally hyperbolic equations (2.2.13) and (2.2.14) in the respective vector bundles on M, according to Section 2.2.2. We stress that we now assume that the two fields are complex. Referring to [54, Chapter 4], we define the *covariances*, for $j = 0, 1$

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)\prime}) := \frac{1}{2} \int_\Sigma h^\sharp_{(j)}(\overline{\pi^{(j)}}, H^{-1}_{(j)} \pi^{(j)\prime}) + h^\sharp_{(j)}(\overline{a^{(j)}}, H_{(j)} a^{(j)\prime}) \operatorname{vol}_h + \frac{i}{2} \sigma^{(j)}(\overline{A^{(j)}}, A^{(j)\prime}) \tag{3.2.22}$$

$$\lambda^-_{(j)}(A^{(j)}, A^{(j)\prime}) := \frac{1}{2} \int_\Sigma h^\sharp_{(j)}(\pi^{(j)\prime}, H^{-1}_{(j)} \overline{\pi^{(j)}}) + h^\sharp_{(j)}(a^{(j)\prime}, H_{(j)} \overline{a^{(j)}}) \operatorname{vol}_h + \frac{i}{2} \sigma^{(j)}(A^{(j)\prime}, \overline{A^{(j)}}) \tag{3.2.23}$$

where $H_{(j)} := \overline{\Delta^{(j)} + m^2}^{1/2}$, $\sigma^{(j)}$ are the symplectic forms of the corresponding Klein-Gordon fields taking place in the right-hand side of (2.2.25), now evaluated on complex fields. Above, $a^{(j)}, \pi^{(j)} \in \Omega^j_c(\Sigma)_\mathbb{C}$ are the Cauchy data on Σ of $A^{(j)}$ respectively and $a^{(j)\prime}, \pi^{(j)\prime} \in \Omega^j_c(\Sigma)_\mathbb{C}$ are the Cauchy data on Σ of $A^{(j)\prime}$ respectively. Notice that we are not imposing constraints on these initial data since we are dealing with independent Klein-Gordon fields. $\lambda^\pm_{(j)}$ are evidently positive because, if all involved forms in the right-hand side are smooth and compactly supported, then the right-hand side of the identity above is well-defined and

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)\prime}) := \frac{1}{2} \int_\Sigma h^\sharp_{(j)}(\overline{H^{1/2} a^{(j)} + iH^{-1/2} \pi^{(j)}}, H^{1/2}_{(j)} a^{(j)\prime} + iH^{-1/2} \pi^{(j)\prime}) \operatorname{vol}_h .$$

The case of $\lambda^-_{(j)}$ is strictly analogous. Furthermore

$$\lambda^+_{(j)}(A^{(j)}, A^{(j)\prime}) - \lambda^-_{(j)}(A^{(j)}, A^{(j)\prime}) = i\sigma^{(j)}(\overline{A^{(j)}}, A^{(j)\prime}) .$$

Therefore $\lambda^\pm_{(j)}$ satisfy the hypotheses of [54, Proposition 4.14][11] so that they define a pair, for $j = 0, 1$, of quasifree states for the complex Klein-Gordon fields respectively associated to Equations (2.2.13) and (2.2.14). We pass to prove that both states are Hadamard exploiting the fact that (Σ, h) is of bounded geometry. By rewriting the covariances $\lambda^\pm_{(j)}$ as $\lambda^\pm_{(j)} = \pm q c^\pm_{(j)}$ ($q = i\sigma^{(j)}$) a quick computation shows that

$$c^\pm_{(j)} = \frac{1}{2} \begin{bmatrix} I & \pm H^{-1}_{(j)} \\ \pm H_{(j)} & I \end{bmatrix} .$$

We can immediately realize that the operator $c^\pm_{(j)}$ is the same Hadamard projector obtained in [57, Section 5.2][12] – see also [54, Section 11] for a more introductory explanation for the scalar case. This operator belongs to the necessary class of pseudodifferential operators $C^\infty_b(\mathbb{R}; \Psi^1_b(\Sigma))$ because (Σ, h) is of bounded geometry. Therefore, on account of [57, Proposition 5.4], the two quasifree states associated to $\lambda^+_{(j)}$, for $A^{(j)}$ and $j = 0, 1$, are Hadamard. In other words, the Schwartz kernels provided by the two-point functions $\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)$, viewed as distributions of $\Gamma(\mathsf{V}^{(j)}_g \boxtimes \mathsf{V}^{(j)}_g)'$, satisfy

$$WF(\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = \mathcal{H} ,$$

[11] The reader should pay attention to the fact that the Cauchy data used in [54], *in the complex case*, are defined as $(f_0, f_1) := (a, -i\pi)$ *instead of our* (a, π)! This is evident by comparing (2.4) and (2.20) in [54]. With the choice of [54], $i\overline{(f_0, f_1)}^t \cdot q(f_0', f_1') = \int \overline{f_0} f_1' + \overline{f_1} f_0' \operatorname{vol}_h = i\sigma(\overline{(a, \pi)}, (a', \pi'))$, where $\cdot q \equiv \sigma_1$ (the Pauli matrix) according to [54].

[12] It follows immediately since $b^+(t) = -b^-(t) = H := \overline{\Delta^{(j)} + m^2 I}^{1/2}$.

where $\mathcal{H}$ is defined in (3.2.1) and $\mathsf{G}^{(i)}$, $i = 0, 1$ are the causal propagators for the normally hyperbolic operators

$$\mathsf{N}^{(i)} := \partial_t^2 + \Delta_h^{(i)} + m^2 I : \Gamma_{sc}(\mathsf{V}_g^{(i)}) \to \Gamma_{sc}(\mathsf{V}_g^{(i)}) \quad i = 0, 1\,.$$

Above and from now on we use the same notation to indicate a bidistribution and the associated Schwartz kernel. Notice that we have used the same symbol $\mathsf{G}^{(j)}$ of the causal propagator we used for the real vector field case. This is because the causal propagators for the complex fields are the direct complexification of the scalar case (see Remark 2.2.5). We pass now to focus on the expression of $\omega_{\mu 2}$ provided in (3.2.13) taking the usual decomposition $\Omega^1_c(\mathsf{M})_{\mathbb{C}} \ni \mathfrak{f} = \mathfrak{f}^{(0)}dt + \mathfrak{f}^{(1)}$ into account. It can be written

$$\omega_{\mu 2}(\mathfrak{f}, \mathfrak{f}') = \omega_{\mu 2}^{(1)}(\mathfrak{f}^{(1)}, \mathfrak{f}^{(1)\prime}) - \omega_{\mu 2}^{(0)}(\mathfrak{f}^{(0)}, \mathfrak{f}^{(0)\prime})$$

where, comparing (3.2.12) and (3.2.13) with (3.2.22) for *real* arguments $\mathfrak{f}, \mathfrak{f}' \in \Gamma(\mathsf{V}_g)$, we find

$$\omega_{\mu 2}^{(j)}(\mathfrak{f}^{(j)}, \mathfrak{f}^{(j)\prime}) = \lambda^+_{(j)}(\mathsf{G}^{(j)}\mathfrak{f}^{(j)}, \mathsf{G}^{(j)}\mathfrak{f}^{(j)\prime})\,.$$

We have

$$WF(\pm\omega_{\mu 2}^{(j)}) = WF(\pm\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = WF(\lambda^+_{(j)}(\mathsf{G}^{(j)}\cdot, \mathsf{G}^{(j)}\cdot)) = \mathcal{H} \quad \text{for } j = 0, 1.$$

Taking (2.2.16) into account, we now observe that $\omega_{\mu 2} \in \Gamma(\mathsf{V}_g \boxtimes \mathsf{V}_g)' = \Gamma((\mathsf{V}_g^{(0)} \oplus V_g^{(1)}) \boxtimes (\mathsf{V}_g^{(0)} \oplus V_g^{(1)}))'$. As a matter of fact, however, $\omega_{\mu 2}$ does not have mixed components acting on sections of $V_g^{(1)} \boxtimes \mathsf{V}_g^{(0)}$ and $V_g^{(0)} \boxtimes \mathsf{V}_g^{(1)}$ and the only components of that distribution are those which act on sections of $V_g^{(0)} \boxtimes \mathsf{V}_g^{(0)}$ and $V_g^{(1)} \boxtimes \mathsf{V}_g^{(1)}$. These are respectively represented by $-\omega_{\mu 2}^{(0)}$ and $\omega_{\mu 2}^{(1)}$ whose wavefront set is $\mathcal{H}$ in both cases. The remaining two components have empty wavefront set since they are the zero distributions. Applying the definition of wavefront set of a vector-valued distribution [95], we conclude that

$$WF(\omega_{\mu 2}) = WF(-\omega_{\mu 2}^{(0)}) \cup WF(\omega_{\mu 2}^{(1)}) \cup \varnothing \cup \varnothing = \mathcal{H} \cup \mathcal{H} \cup \varnothing \cup \varnothing = \mathcal{H}\,,$$

concluding the proof. □

Combining the results obtained so far, we get the main result of this section.

Theorem 3.2.12. *Let* (M, g) *be a globally hyperbolic spacetime and refer to the CCR-algebra* $\mathcal{A}_g$ *of the real Proca field. Then there exists a quasifree Hadamard state on* $\mathcal{A}_g$*.*

Proof. As already explained in the beginning of Section 3.2.3, for any globally hyperbolic spacetime (M, g), there exists a paracausally related globally hyperbolic spacetime (M, g_0) which is ultrastatic and whose spatial metric is of bounded geometry. In particular, in this class of spacetimes, the quasifree states defined in Proposition 3.2.9 satisfy the microlocal spectrum condition, as proved in Proposition 3.2.11. Therefore, since the pull-back along a Møller $*$-isomorphism preserves the Hadamard condition on account of Theorem 3.2.8, we can conclude. □

3.2.5 Comparison with Fewster-Pfenning's definition of Hadamard states

Though the paper [44] by Fewster and Pfenning concerns *quantum energy inequalities*, it also offers a general theoretical discussion about the algebraic quantization of the Proca and the Maxwell fields in curved spacetime. In particular, the authors propose a definition of a Hadamard state which appears to be technically different from ours at first glance, even if it shares a number of important features with ours.

The definition of Hadamard state stated in [44, Equation (35)] is formulated in terms of causal normal neighbourhood of smooth spacelike Cauchy surfaces (see also below) and the *global*

Hadamard parametrix for distributions which are bisolutions of the vectorial Klein-Gordon equation. Our final goal is to prove an equivalence theorem of our definition of Hadamard state Definition 3.2.4 and the one adopted in [44].

As a first step, we translate the original Fewster-Pfenning's definition of a Hadamard state into an equivalent form which will turn out to be more useful for our comparison. The equivalence of the version stated below of Fewster-Pfenning's definition and the original one was established in [44, Section III C] (see also the comments under Definition 3.2.13).

Definition 3.2.13. [**Fewster-Pfenning's definition of Proca Hadamard state**] Consider the globally hyperbolic spacetime (M, g) and a state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the Proca algebra of observables on (M, g). ω is called **Hadamard** if it is quasifree and its two-point function has the form

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{h})) = W_g(\mathfrak{f}, Q\mathfrak{h}) \tag{3.2.24}$$

$\forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$, where $Q : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_g)$ in the differential operator $Q = \mathrm{Id} + m^{-2}(d\delta_g)$. Above $W_g \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ is a Klein-Gordon distributional bisolution such that

$$W_g(\mathfrak{f}, \mathfrak{g}) - W_g(\mathfrak{g}, \mathfrak{f}) = i\mathsf{G}_{\mathsf{N}}(\mathfrak{f}, \mathfrak{g}) \mod C^\infty , \tag{3.2.25}$$

G_{N} being the causal propagator of the Klein-Gordon operator (2.2.4) and which satisfies the microlocal spectrum condition

$$WF(W_g) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\} . \tag{3.2.26}$$

Remark 3.2.14. The equivalence of Definition 3.2.13 and the original one stated in [44] relies on Sahlmann -Verch's [95] generalization to vector (and spinor) fields of some classic Radzikowski results originally formulated for scalar fields. In practice, (a) if a distribution which is a bisolution of the vectorial Klein-Gordon equation and it is of Hadamard form in a normal causal neighbourhoods of a smooth spacelike Cauchy surface, then it necessarily has the wavefront set of the form (3.2.26) ((a) [95, Theorem 5.8]) and its antisymmetric part satisfies (3.2.25) directly from the definition of parametrix; (b) if a distribution which is a bisolution of the vectorial Klein-Gordon equation satisfies (3.2.26) and (3.2.25), then it is of Hadamard form in some normal causal neighbourhoods of a smooth spacelike Cauchy surface (see [95, Remark 5.9. (i)]).

For the Proca field in [44] established the property of propagation of the Hadamard condition stated in the next proposition. That result was already established for the Hadamard states of scalar and vector (including spinor) fields in [51, 78, 95] (see [76, 86] for a general recap for the KG scalar field). The pivotal tool is the already mentioned notion of *causal normal neighbourhood $\mathcal{N}$ of a smooth spacelike Cauchy surface* Σ in a globally hyperbolic spacetime $(\mathsf{M}; g)$. The notion introduced in [78] has been recently improved (closing a gap in the geometric definition of Hadamard states) in [86][13]. The propagation results established in [78, 95] and [44] are valid with the improved notion of causal normal neighbourhoods and Hadamard states of [86].

Proposition 3.2.15. *Let $\omega : \mathcal{A}_g \to \mathbb{C}$ be a quasifree state for the Proca field in the globally hyperbolic spacetime* (M, g). *Let $\mathcal{N}$ be a causal normal neighbourhood of a Cauchy surface Σ of* (M, g). *Suppose that the restriction of ω to $(\mathcal{N}, g|_{\mathcal{N}})$ is Hadamard according to Definition 3.2.13. Then ω is Hadamard in* (M, g) *according to the same definition.*

Remark 3.2.16. In order to compare Proposition 3.2.6 and Proposition 3.2.15 we stress that the requirement that the neighbourhood $\mathcal{N}$ of a Cauchy surface is causal normal can be relaxed also in Proposition 3.2.15 to make contact with our Proposition 3.2.6. One may only assume that $(\mathcal{N}, g|_{\mathcal{N}})$ is globally hyperbolic also therein. That is a consequence of the following facts.

[13] Where these open sets are named normal neighbourhoods of smooth spacelike Cauchy surfaces, omitting "causal".

(a) Every causal normal neighbourhood $\mathcal{N} \subset \mathsf{M}$ of a Cauchy surface Σ of (M, g) is, by definition [78, 84], a globally hyperbolic spacetime with respect to the restriction of the metric and Σ is also a Cauchy surface in $(\mathcal{N}, g|_{\mathcal{N}})$.

(b) Every smooth spacelike Cauchy surface admits a causal normal neighbourhood [78, 84].

(c) According to the proof of [78, Lemma 2.2] whose validity extends to [84], every neighbourhood of a smooth spacelike Cauchy surface includes a causal normal neighbourhood of that Cauchy surface[14].

The smoothness property corresponding to our Proposition 3.2.5 also holds for Hadamard bisolutions in the sense of Fewster-Pfenning. In [44], it is an immediate consequence of (3.2.24) and the analogous feature of Klein-Gordon bisolutions (see the discussion on p. 4488 in [44]).

Proposition 3.2.17. *Let $\omega, \omega' \in \Gamma'_c(V_g \boxtimes V_g)$ be a pair of bisolutions of the Proca equation satisfying the Hadamard condition (3.2.24) for corresponding Klein-Gordon bisolutions which, in turn, satisfy (3.2.25). Then, the differences between the two bisolutions is smooth: $\omega - \omega' \in \Gamma(V_g \boxtimes V_g)$.*

Finally, [44] also contains a proof of the existence of Hadamard states for the Proca (and the Maxwell) field in globally hyperbolic spacetimes with compact Cauchy surfaces (whose first homology group is trivial when treating the Maxwell field). This proof establishes first the existence in ultrastatic spacetimes and next it exploits a standard deformation argument [99].

We are in a position to state and prove our equivalence result.

Theorem 3.2.18. *Consider the globally hyperbolic spacetime (M, g) and a quasifree state $\omega : \mathcal{A}_g \to \mathbb{C}$ for the $*$-algebra of observables on (M, g) of the real Proca field. Let $\omega_2 \in \Gamma'_c(\mathsf{V}_g \boxtimes \mathsf{V}_g)$ be the two-point function of ω. The following facts are true.*

(a) If ω is Hadamard according to Definition 3.2.13, then it is also Hadamard according to Definition 3.2.4.

(b) If (M, g) admits a Proca quasifree Hadamard state according to Definition 3.2.13 and ω is Hadamard according to Definition 3.2.4, then ω is Hadamard in the sense of Definition 3.2.13.

Proof. The following argument is identical to the one used in 3.2.6 to prove $WF(\mathsf{G_P}) = WF(\mathsf{G_N})$, but we repeat it here to keep this section self-contained.
First of all notice that, since $\omega_2(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g})$, then viewing ω_2 and W_g as bidistributions, we have $\omega(x, y) = (Id_x \otimes Q_y)\, W_g(x, y)$ (where we have used the fact that Q is formally selfadjoint) taking Remark 3.2.3 into account).
Now suppose that ω is Hadamard according to Definition 3.2.13. Since W_g satisfies the microlocal spectrum condition and the differential operator $I \otimes Q$ is smooth, we have

$$WF(\omega_2) \subset WF(W_g) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \backslash \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\} .$$

Notice that, in particular, k_x and k_y cannot vanish (simultaneously or separately) if they take part of $WF(W_g)$. Let us prove the converse inclusion to complete the proof of (a). Again from known results, from $\omega_2(x, y) = (Id_x \otimes Q_y) W_g(x, y)$, we have

$$WF(W_g) \subset Char(I \otimes Q) \cup WF(\omega_2) .$$

However, by direct inspection, one sees that

$$Char(I \otimes Q) = \{(x, k_x; y, 0) \mid (x, k_x) \in \mathsf{T}^*\mathsf{M} , y \in \mathsf{M}\} ,$$

[14] Essentially because convex normal neighbourhoods of points form a topological basis of any spacetime and in view of [84, Proposition 9]

so that

$$WF(\omega_2) \subset WF(W_g) \subset WF(\omega_2) \cup \{(x, k_x; y, 0) \,|\, (x, k_x) \in \mathsf{T}^*\mathsf{M}\,, y \in \mathsf{M}\}\,. \tag{3.2.27}$$

However $WF(W_g) \cap \{(x, k_x; y, 0) \,|\, (x, k_x) \in \mathsf{T}^*\mathsf{M}\,, y \in \mathsf{M}\} = \varnothing$ and thus we can re-write the chain of inclusions (3.2.27) as

$$WF(\omega_2) \subset WF(W_g) \subset WF(\omega_2) \quad \text{so that} \quad WF(\omega_2) = WF(W_g)\,.$$

This is the thesis of (a) because we have established that Definition 3.2.4 is satisfied by ω.
To prove (b), let us assume that ω satisfies Definition 3.2.4. By hypotheses the antisymmetric part of ω_2 is $-iG_{\mathsf{P}}$. Let Ω be another quasifree state of the Proca field which satisfies Definition 3.2.13. Also the antisymmetric part of Ω_2 is $-iG_{\mathsf{P}}$.

Due to Proposition 3.2.5,

$$F(x, y) := \omega_2(x, y) - \Omega_2(x, y)\,.$$

is a smooth function. Furthermore it is a symmetric bisolution of the Proca equation. In particular it therefore satisfies[15] $F(\mathfrak{f}, d\mathfrak{h}^{(0)}) = 0$, where $\mathfrak{h}^{(0)} \in \Omega^0_c(\mathsf{M})$, so that

$$F(\mathfrak{f}, Q\mathfrak{g}) = F(\mathfrak{f}, \mathfrak{g}) + \frac{1}{m^2} F(\mathfrak{f}, d(\delta_g \mathfrak{g})) = F(\mathfrak{f}, \mathfrak{g})\,.$$

Collecting everything together, we can assert that, for some distributional bisolution of the Klein-Gordon equation W_g which satisfies (3.2.25), (3.2.26), and is associated to the Hadamard state Ω, it holds

$$\omega_2(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, \mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, Q\mathfrak{g})\,.$$

If we re-absorb F in the definition of W_g,

$$W'_g(\mathfrak{f}, \mathsf{Q}\mathfrak{g}) = W_g(\mathfrak{f}, Q\mathfrak{g}) + F(\mathfrak{f}, Q\mathfrak{g})\,.$$

the new W'_g is again a distributional bisolution of the Klein-Gordon equation which satisfies (3.2.25), (3.2.26) and

$$\omega_2(f, g) = W'_g(\mathfrak{f}, Q\mathfrak{g})\,.$$

In other words, the Hadamard state ω according to Definition 3.2.4 is also Hadamard in the sense of Definition 3.2.13 concluding the proof of (b). □

Remark 3.2.19. Regarding (b), the existence of Hadamard states in the sense of Definition 3.2.13 has been established in [44] for globally hyperbolic spacetimes whose Cauchy surfaces are compact: in those types of spacetimes at least, the two definitions are completely equivalent. We expect that actually the equivalence is complete, even dropping the compactness hypothesis (see the conclusion section). This issue will be investigated elsewhere.

3.3 Conclusions

In this chapter a lot of non trivial results have been obtained: the Møller operators construction has exhibited the important feature of preserving the Hadamard condition and this property has been exploited to construct Hadamard states in general spacetimes for Klein Gordon and Proca fields. Moreover the CCR algebras defining the aforementioned theories have been shown to be isomorphic for paracausally related spacetimes implying that a lot of structure is preserved in a CCR quantum field theory under finite global variations of the background geometry. Moreover Hadamard states have been constructed in general for the Proca field for the first time and the

[15] We are grateful to C. Fewster for this observation.

Fewster-Pfenning and the standard definition of Hadamard state for the Proca fields have been revealed to be (almost) equivalent.

However an issue we have faced for all the considered theories is the lack of control on the action of the group of $*$-automorphism induced by the isometry group of the spacetime M on ω. Indeed, the type of factor can be inferred by analyzing which and how many states are invariant. From a more physical perspective instead, invariant states can represent equilibrium states in statistical mechanics *e.g.* KMS-states or ground states. The previous remark leads us to the following open question:

Question 3.3.1. Under which conditions it is possible to perform an adiabatic limit, namely when is $\lim_{\chi\to 1}\omega_1$ well-defined?

A priori we expect that there is no positive answer in all possible scenarios, since it is known that certain free-field theories, *e.g.*, the massless and minimally coupled (scalar or Dirac) field on four-dimensional de Sitter spacetime, do not possess a ground state, even though their massive counterpart does. (Notice that this is not a no-go Theorem, but at least an indication that, in these situations, the map $\omega \to \omega \circ \mathcal{R}$ cannot be expected to preserve the ground state property.)

A partial investigation in this direction has been carried on in [27,36] for the case of a scalar field theory on globally hyperbolic spacetimes with empty boundary. In this situation it has been shown that, under suitable hypotheses the adiabatic limit can be performed preserving the invariance property under time translation but spoiling in general the ground state or KMS property.

Moreover, the results are valid also for off-shell algebras as well as for distribution of Hadamard type. Therefore, it could be possible to extend the action of the Møller operator also on the algebra of extended observables in a perspective of *deformation quantization* (see for instance Section 2 of [35]), which include, e.g., the Wick polynomials of the underlying fields. Wick polynomials and time-ordered products of Wick polynomial are the building blocks for perturbative renormalization of quantum fields, both in Minkowski spacetime and in curved spacetime, where the metric is considered as a given external classical field. Although of utmost physical relevance, these formal operators as the *stress energy operator* do not belong to the algebra of observables generated by the smoothly smeared field operators (operator-valued distributions). This is because they correspond to products of distributions at a given point and this notion is not well-defined in general. The popular and perhaps most effective procedure to eliminate the short-distance divergences consists of simply subtracting a suitable Hadamard distribution. This procedure is systematically embodied in a product deformation quantization procedure which relies on a suitable set of functionals with a specific wavefront set. The following observation leads to the following conjecture:

Conjecture 3.3.2. Let $\mathcal{A}_0, \mathcal{A}'_0$ be the algebra of observables of the globally hyperbolic spacetimes (M, g) and (M, g') and $\mathcal{R}_0$ a Møller $*$-isomorphism of them. If $\mathcal{A}, \mathcal{A}'$ are corresponding extended algebras of observable (which include the Wick polynomials etc.) and $g \simeq g'$, then $\mathcal{R}_0$ extends to a (Møller) $*$-isomorphism $\mathcal{R} : \mathcal{A} \to \mathcal{A}'$.

A detailed study of the Møller $*$-isomorphism in the case of the off-shell Proca algebra has not been carried out yet.
However, regarding the Proca field, much more can be done, especially in the study of the Hadamard state we explicitly constructed on ultrastatic spacetime $\mathsf{M} = \mathbb{R} \times \Sigma$. Thereon the one-parameter group of isometries given by time-translations has an associated action on $\mathcal{A}_g$ in terms of $*$-algebras isomorphisms α_u completely induced by

$$\alpha_u(\hat{\mathfrak{a}}(\mathfrak{f})) := \hat{\mathfrak{a}}(\mathfrak{f}_u)$$

for every $\mathfrak{f} \in \Gamma_c(\mathsf{M})$, where $\mathfrak{f}_u(t,p) := \mathfrak{f}(t-u,p)$ for every $t, u \in \mathbb{R}$ and $p \in \Sigma$. It shall not be difficult prove that the Hadamard state constructed is invariant under the action of α_u

$$\omega_\mu(\alpha_u(a)) = \omega_\mu(a) \quad \forall u \in \mathbb{R} \quad \forall a \in \mathcal{A}_g$$

It should be also true that the map

$$\mathbb{R} \ni u \mapsto \omega_\mu(b\alpha_u(a)) \in \mathbb{C}$$

is continuous for every $a, b \in \mathcal{A}_g$ which would assure (see, e.g. [85]) that $\alpha := \{\alpha_h\}_{h\in\mathbb{R}}$ is unitarily implementable by a strongly continuous unitary representation of $\mathbb{R}$ in the GNS representation of ω_μ and that the vacuum vector of the Fock-GNS representation is left invariant under the said unitary representation. We expect that the selfadjoint generator of that unitary group has a positive spectrum where, necessarily, the vacuum state is an eigenvector with eigenvalue 0. In other words ω_μ should be a *ground state* of α. We finally expect that ω_μ is *pure* and it is the *unique quasifree algebraic state which is invariant under* α. We can summarize the previous discussion in the following question.

Question 3.3.3. Is the Hadamard state defined on ultrastatic spacetimes a *ground state* for the time-translation? More precisely, is it the unique, pure, quasifree algebraic state which is invariant under the action of α?

Last, but not least, we have seen in Section 3.2.5 that if a globally hyperbolic manifold admits a Proca quasifree Hadamard state according to the definition of Fewster-Pfenning, then Definition 3.2.4 and 3.2.13 are equivalent. This is the case for example for globally hyperbolic spacetimes whose Cauchy surfaces are compact. We do expect to extend this result for the whole class of globally hyperbolic spacetime.

Conjecture 3.3.4. Definition 3.2.4 and 3.2.13 are equivalent on any globally hyperbolic spacetime.

As is evident from our quasi equivalence theorem, a complete equivalence would take place if a Hadamard state according to [44] is proven to exist for every globally hyperbolic spacetime. As a matter of fact, we expect that every globally hyperbolic spacetime (M, g) admits a quasifree Proca Hadamard state ω according to Fewster and Pfenning. This state should exist in every paracausally related ultrastatic spacetime $(\mathbb{R} \times \Sigma, -dt^2 + h)$ with complete Cauchy surfaces of bounded geometry. With the same argument used for our existence proof of Hadamard states or the deformation argument exploited in [44], it should be possible to export this state to the original space (M, g). We expect that the Hadamard Klein-Gordon bisolution for the real Proca field on $(\mathbb{R} \times \Sigma, -dt^2 + h)$ used to define ω according to (3.2.24) in Definition 3.2.13 should have this form.

$$W_g(\mathfrak{f}, \mathfrak{f}') := \mu(\mathsf{G}_\mathsf{N}\mathfrak{f}, \mathsf{G}_\mathsf{N}\mathfrak{f}') + \frac{i}{2}\sigma^{(N)}(\mathsf{G}_\mathsf{N}\mathfrak{f}, \mathsf{G}_\mathsf{N}\mathfrak{f}') , \quad \mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathbb{R} \times \Sigma) ,$$

where N is the Klein-Gordon operator (2.2.4) associated to P and G_N its causal propagator. The bilinear symmetric form $\mu : \big((\Omega^0_c(\Sigma))^2 \times (\Omega^1_c(\Sigma))^2\big) \times \big((\Omega^0_c(\Sigma))^2 \times (\Omega^1_c(\Sigma))^2\big) \to \mathbb{R}$ is defined as in (3.2.12), but with the crucial difference that here its arguments are not restricted to $C_\Sigma \times C_\Sigma$.

Another problem which has not been tackled is the extension of this formalism to gauge theories, as well as in the classical case, because we have not been able to prove that any claimed form of Møller operator for the Maxwell field can produce an isomorphism of the Maxwell algebras on paracausally related spacetimes. However, in principle, nothing forbids the procedure to be recast for gauge theories in a fruitful less trivial way, so this topic remains left to future investigation.

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