Isochronous sections via normalizers

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Abstract

We give some sufficient conditions for the existence of isochronous sections of plane differential systems. We consider both isochronous sections at a critical point and at a cycle.

1 Introduction

Let us consider an autonomous differential system

$$\dot{z} = X(z), \tag{1}$$

where $z \equiv (x, y) \in \Omega$, open connected subset of the real plane, $X(z) = (P(z), Q(z))$ C^2 vector field defined on Ω. Given a second vector field $U \in C^2(\Omega, \mathbb{R}^2)$ we write $[X, U] := \partial_X U - \partial_U X$ for the Lie brackets of X and U. If $[X, U] \equiv 0$ on Ω, we say that X and U *commute*. Denoting by $X \wedge U$ the wedge product of X and U, we say that U is a *normalizer* of X if $[X, U] \wedge X \equiv 0$. The dynamics of commuting systems was studied in [7]. Some of the results obtained in [7] do not actually require commutativity, but can be proved by only assuming some normalizing property. For instance, the absence of limit cycles of X can be proved by only assuming X to have a non-trivial normalizer U [10].

In this paper we study the existence of isochronous sections under the assumption that X is the normalizer of a trasversal vector field U.

Looking for normalizers allows to study isochronicity phenomena in a simpler way than looking for commutators. In fact, given a vector field X, looking for a commutator is equivalent to look for a solution to a system of *two* PDE's, obtained imposing the vector condition $[X, U] = 0$. On the other hand, looking for a normalizer is equivalent to look for a solution to *one* PDE, given by $[X, U] \wedge U = 0$. As an example, consider the class of Loud quadratic systems,

$$x' = -y + Bxy, \qquad y' = x + By^2, \qquad B \in \mathbb{R}. \tag{2}$$

Such systems have the following systems as commutators,

$$x' = x(1 - Bx), \qquad y' = y(1 - Bx). \tag{3}$$

On the other hand, the systems (2) normalize every system of the type

$$x' = x\sigma(x, y), \qquad y' = y\sigma(x, y), \tag{4}$$

where σ is a non-vanishing scalar function of class C^2.

In what follows we consider first isochronous sections at critical points, then at cycles. For critical points of analytic systems, similar results have been recently proved in [4]. A result related to the presence of isochronous sections of limit cycles has been recently presented in [2].

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2 Isochronous sections at critical points

For every $z \in \Omega$, we denote by $\phi_X(t,z)$ the solution of (1) such that $\phi_X(0,z) = z$. We denote by $-\phi_X$ the negative local flow defined by (1), $-\phi_X(t,z) = \phi_X(-t,z)$. Let (ρ,θ) be the polar coordinates of a point in the plane. We denote by $\rho(\phi_X(t,z))$, resp. $\theta(\phi_X(t,z))$, the radius and the argument of $\phi_X(t,z)$.

Throughout this section, we assume $O = (0,0)$ to be the *unique critical point of (1)*. We say that O is a *center-focus* if there exists a neighbourhood Ω_1 of O such that for every non-critical orbit $\phi_X(t,z)$ starting at a point of Ω_1, the function $\theta(\phi_X(t,z))$ is increasing (decreasing) and diverging to $+\infty$ $(-\infty)$. We say that O is a *focus* if O is a center-focus and has a neighbourhood free of nontrivial cycles.

From now on, Ω_1 will denote a neighbourhood of the critical point O, and we shall write $\Omega_1^* := \Omega_1 \setminus \{O\}$.

We say that O is a *center* if O has a neighbourhood filled with nontrivial cycles. O is said to be *isochronous* if every cycle has the same period T. We write N_O for the largest connected region surrounding O, covered with non-trivial cycles. N_O is a punctured neighbourhood of O.

Given a second vector field $U \in C^2(\Omega, \mathbb{R}^2)$, $U(z) \equiv (R(z), S(z))$, we say that X is a *normalizer* of U, or normalizes U on Ω, if $[X,U] \wedge U \equiv 0$ on Ω. We say that X is a *non-trivial normalizer* of U if X and U are transversal at non-critical points. Let us set $X \wedge U := PS - QR$. If X normalizes U, setting $Z := \{z \in \Omega : U(z) = (0,0)\}$, there exists a scalar function $\nu \in C^2(\Omega \setminus Z, \mathbb{R})$ such that $[X,U] = \nu U$. The regularity of ν at non-critical points of U comes from the equality $\nu = \frac{U \cdot [X,U]}{|U|^2}$, where $U \cdot [X,U]$ is the scalar product of U and $[X,U]$. The regularity of ν at O is not relevant in what follows.

X is a normalizer of U if and only if [6] $\phi_X(t,\cdot)$ takes arcs of orbits of

$$\dot{z} = U(z), \tag{5}$$

into arcs of orbits of (5). This is usually expressed by saying that X is an *infinitesimal generator of a Lie symmetry for* U, or for ϕ_U, the local flow defined by (5). When X and U are non-trivial normalizers of each other, we say that they are *commutators*. A center of (1) is isochronous if and only if X has a non-trivial commutator [8]. Such a result can be improved by requesting X to be only a normalizer of a transversal vector field U.

Theorem 1 *Let O be a center of (1). Then O is isochronous if and only if there exists Ω_1 and $U \in C^2(\Omega_1)$ such that X is a non-trivial normalizer of U on Ω_1.*

Proof. If O is isochronous, then by theorem 2 in [8], X has a non-trivial commutator, which is also a normalizer.

Vice-versa, if X is a normalizer of U, then $\phi_X(t,\cdot)$ takes arcs of U-orbits into arcs of U-orbits. Let T be the period of $\phi_X(t,z_0)$, for $z_0 \in N_O$. Let $\epsilon > 0$ be such that $\eta(s) := \phi_U(s,z_0) \in N_O$, for all $s \in (-\epsilon,\epsilon)$. Then $\phi_X(T,\eta) \subset \eta$. Moreover, since every X-orbit in N_O is a cycle, one has $\phi_X(T,\eta(s)) = \eta(s)$, so that every X-cycle meeting $\eta(s)$, for $s \in (-\epsilon,\epsilon)$, has period T. This argument is independent of the particular $z_0 \in N_O$, hence the period of every cycle in N_O is T. ♣

In [3], assuming O to be a center and $[X,U] = \mu X$, it was proved that the monotonicity of the period function T depends on the sign of $\int_0^T \mu$, where the integration is performed along a cycle of the period annulus. Here we cannot prove anything similar, because if $[X,U] = \nu U$, then a center is isochronous. In fact, ν provides information about the character of a critical point, rather than about the period function.

Corollary 1 *Under the hypotheses of theorem 1, for every T-periodic non-trivial cycle γ contained in N_O one has*

$$\int_0^T \nu(\gamma(t))dt = 0.$$

Proof. The system (1) has an isochronous center at O. By [8], theorem 2, X has a non-trivial commutator $W \in C^2(N_O, \mathbb{R}^2)$. By the trasversality of X and U, there exist scalar functions σ, β defined in a punctured neighbourhood of O, such that $W = \sigma X + \beta U$, with $\beta \neq 0$. Without loss of generality, we can assume $\beta > 0$ in a punctured neighbourhood of O. The equalities

$$\sigma = \frac{W \wedge U}{X \wedge U}, \qquad \beta = \frac{W \wedge X}{U \wedge X}$$

show that σ and β are of class C^2 for $z \neq O$. Computing the commutator $[X, W]$ one has

$$\begin{array}{l} 0 \equiv [X, \sigma X + \beta U] = \partial_X(\sigma X) - \partial_{\sigma X} X + \partial_X(\beta U) - \partial_{\beta U} X = \\ (\partial_X \sigma) X + (\partial_X \beta) U + \beta \partial_X U - \beta \partial_U X = \\ (\partial_X \sigma) X + (\partial_X \beta) U + \beta [X, U] = \\ (\partial_X \sigma) X + (\partial_X \beta + \beta \nu) U. \end{array} \tag{6}$$

By the transversality of X and U one has, for $z \neq O$,

$$\nu = -\frac{(\partial_X \beta)}{\beta} = -\partial_X \log \beta.$$

Hence, integrating along $\gamma(t)$,

$$\int_0^T \nu(\gamma(t))dt = -\int_0^T \partial_X \log \beta(\gamma(t))dt = \log \beta(\gamma(T)) - \log \beta(\gamma(0)) = 0.$$

As an immediate consequence, one has the following corollary.

Corollary 2 *Let O be a center-focus of (1). Assume $[X, U] = \nu U$ for $U \in C^2(\Omega_1, \mathbb{R}^2)$, transversal at $z \neq O$. If $\nu(0) \neq 0$ in a punctured neighbourhood of O, then O is not a center.*

In particular, corollary 2 applies when ν has a continuous extension $\bar{\nu}$ to O and $\bar{\nu}(0) \neq 0$.

The following definition has been introduced in [9] in order to study isochronicity at center-foci.

Definition 1 *Let O be an isolated critical point of (1). Let $\eta : [0, +\infty) \to \mathbb{R}^2$ be a C^1 curve trasversal to X such that $\lim_{s \to +\infty} \eta(s) = O$. Then we say that η is an isochronous section of (1) at O if either ϕ_X or $-\phi_X$ has the following property:*

There exists $T > 0$ such that $\forall\ z \in \eta$, one has

(i) $\phi_X(nT, z) \in \eta$, for every positive integer n;

(ii) $\phi_X(t, z) \notin \eta$, for $t > 0, t \neq nT$.

If a system has an isochronous section η in a neighbourhood of O, then every curve $s \mapsto \phi_X(t, \eta(s))$, with $0 < t < T$, is an isochronous section of the system at O. Hence, if a system has an isochronous section, it has infinitely many isochronous sections.

In general, the existence of an isochronous section η does not immediately imply that every orbit in a punctured neighbourhood Ω_1^* of O meet η. Hence we say that η is a *complete* section on Ω_1 if every non-trivial X-orbit passing through Ω_1 meets η. The completeness of a section is an non-trivial issue when η is choosen as an orbit of a differential system. For instance, we could consider the system

$$x' = y + x(x^2 + y^2), \qquad x' = y + y(x^2 + y^2),$$

which commutes with

$$x' = y - x(x^2 + y^2), \qquad x' = y - y(x^2 + y^2).$$

It is not immediately evident whether the orbits of the first system meet every orbit of the second one.

In next lemma we give a simple sufficient condition for the existence of a complete isochronous section. We denote by $\eta'(s)$ the tangent vector of $\eta(s)$.

Lemma 1 *Let η be an isochronous section of X at a critical point O. If O has a neighbourhood without homoclinic orbits, then η is a complete section of (1) at O.*

Proof. If O is a center, then η crosses every cycle in a neighbourhood of O, since η tends to O.

If O is not a center, there exists a point $z_0 \in \eta$ which is not on a cycle. By possibly changing η's parametrization, we may assume that $z_0 = \eta(0)$. One has $\phi_X(T, z_0) = \eta(s_T)$, for some $s_T > 0$. Since $X \wedge \eta'(s) \neq 0$, the simple closed curve Γ consisting of the arc $\eta(s), s \in [0, s_T]$ and of the arc $\phi_X(t, z_0), t \in [0, T]$, is positively (negatively) invariant for (1).

By Poincaré-Bendixson theory, this implies that Γ surrounds a critical point, that coincides with O. Let us consider the family of curves $\eta_t(s) := \phi(t, \eta(s))$ for $t \in [0, +\infty)$. We claim that the set $A := \{O\} \cup \{\eta_t(s), s \in [0, +\infty), t \in [0, +\infty)\}$ coincides with the bounded region Γ_i defined by the Jordan curve Γ.

In fact, if by absurd $A \neq \Gamma_i$, then there would exist a point $z_1 \in \Gamma_i$, $z_1 \neq O$, $z_1 \neq \phi(t, \eta(s))$ $\forall t \in [0, +\infty)$ $\forall s \in [0, +\infty)$. Let us set $\delta(t) := \phi_X(t, z_1)$. By construction, $\delta \cap \Gamma = \emptyset$, hence $\delta \subset \Gamma_i$.

The positive an negative limit sets $\Lambda^+(\delta)$, $\Lambda^-(\delta)$ of δ are contained in $\Gamma_i \cup \Gamma$. None of them is a cycle, because in that case η would cross it, since it tends to O, and by transversality and continuity η would cross also δ, contradicting the definition of z_1. Hence $\Lambda^+(\delta) = \Lambda^-(\delta) = \{O\}$, contradicting the assumption about the absence of homoclinic orbits. ♣

As a consequence of lemma 1, every isochronous section of a center-focus is a complete section. The above argument also shows that in absence of homoclinic orbits, Γ_i is a neighbourhood of O.

Lemma 2 *Let $\bar{U} \in C^2(\Omega_1^*, \mathbb{R}^2)$ be a non-vanishing vector field. Then there exists $\lambda \in C^2(\Omega_1, \mathbb{R})$, $\lambda(z) \neq 0$ for $z \neq O$, such that the vector field*

$$U(O) = O, \qquad U(z) = \lambda(z)\bar{U}(z), \quad z \neq O,$$

is in $C^2(\Omega_1, \mathbb{R}^2)$.

Proof. Let us set $\bar{U} = (\bar{R}, \bar{S})$. We have to show the existence of λ such that both $\lambda\bar{R}$ and $\lambda\bar{S}$ are of class C^2 in Ω_1, when extended to O by setting $\bar{R}(O) = O$, $\bar{S}(O) = O$.

We do not report this proof in full detail. It is an adaptation of the proof of lemma 1.5 in [5], where a function $F : \mathbb{R}^+ \to \mathbb{R}$ was constructed so that, for every multi-index α,

$$\lim_{|z|\to 0} |D^\alpha (F(I(z))| = 0, \qquad |\alpha| = 1, \ldots, k$$

for a given $I(z)$, not regular at O. . In this lemma, we need a function $\lambda(z)$ such that

$$\lim_{|z|\to 0} |D^\alpha \left(\lambda(z)\bar{R}(z)\right)| = 0, \qquad \lim_{|z|\to 0} |D^\alpha \left(\lambda(z)\bar{S}(z)\right)| = 0, \qquad |\alpha| = 0, 1, 2.$$

this is possible by choosing $\lambda(z)$ of the form $F(|z|^2)$, and imposing that in suitable neighbourhoods of O the following inequalities hold

$$|D^\alpha \left(\lambda(z)\bar{R}(z)\right)| \le |z|^2, \qquad |D^\alpha \left(\lambda(z)\bar{S}(z)\right)| \le |z|^2.$$

In fact, all reduces to finding F such that in suitable neighbourhoods of O a finite set of inequalities of the form

$$\sum_{i=1}^{|\alpha|} \lambda^{(i)}(z) A_i^\alpha \le |z|^2,$$

hold, where the A_i^α are polynomials in x, y, $\bar{R}$, $\bar{S}$ and their derivatives. This can be done just as in lemma 1.5 of [5]. ♣

Theorem 2 *Let O be a center-focus of (1). If (1) has an isochronous section η at O, then there exists Ω_1 and $U \in C^2(\Omega_1, R^2)$, such that X is a non-trivial normalizer of U on Ω_1.*

Proof. Let us define the vector field $\bar{U}$ on Ω_1 as follows

$$\bar{U}(O) = O, \qquad \bar{U}(\phi_X(t, \eta(s))) := \frac{\partial}{\partial s}\phi_X(t, \eta(s)).$$

X is trivially a normalizer of $\bar{U}$ at O. $\bar{U}$ is not necessarily differentiable at O.

By construction, every $s \mapsto \eta_t(s) := \phi_X(t, \eta(s))$ is an integral curve of $z' = \bar{U}(z)$. Since every time-advancement map $\phi_X(\tau, \cdot)$ takes the curve $\eta_t(s)$ into the curve $\eta_{t+\tau}(s)$, X is a normalizer of $\bar{U}$. Also, X and $\bar{U}$ are transversal for $z \neq O$, because $\phi_X(\tau, \cdot)$ is a diffeomorphism. Let us set $\bar{\nu} := \frac{\bar{U}\cdot[X,\bar{U}]}{|\bar{U}|^2}$. By lemma **??**, there exists λ such that $U(z) := \lambda(z)\bar{U}(z)$, $U(O) := O$, is of class C^2, and vanishes at O with its derivatives. Then, for $z \neq O$,

$$[X, U] = [X, \lambda\bar{U}] = (\partial_X\lambda)\bar{U} + \lambda[X, \bar{U}] = \frac{\partial_X\lambda + \lambda\bar{\nu}}{\lambda}\lambda\bar{U} = \frac{\partial_X\lambda + \lambda\bar{\nu}}{\lambda}U$$

so that X normalizes also $U = \lambda\bar{U}$. ♣

The following theorem is a partial converse of theorem 2. We say that a C^1 curve $\eta(s)$ tends to O *with limit tangent* if there exists $\lim_{s\to+\infty}\theta(\eta'(s)) \neq \pm\infty$.

Theorem 3 *Let O be a center-focus of (1). If X is a non-trivial normalizer of $U \in C^2(\Omega_1, R^2)$ on Ω_1, and (5) has an orbit $\phi_U(s, z_0)$ tending to O with limit tangent, then X has a complete isochronous section.*

Proof. If O is a center, then one can apply theorem 1 in order to prove that it is isochronous. Then it is easy to prove that every orbit of U (in fact, every curve transversal to X) is a complete isochronous section of X.

If O is not a center, possibly considering $-\phi_X$, we may assume O to be stable, so that $\phi_X(t, z)$ exists for all $t > 0$.

Assume that $\lim_{s\to+\infty} \phi_U(s, z_0) = O$. Let us set $\eta(s) := \phi_U(s, z_0)$. By possibly rotating the axes, we may assume that $\lim_{s\to+\infty} \eta'(s) = (1, 0)$. Then, for every positive ε, there exists $\bar{s}$ such that for $s > \bar{s}$, the point $\eta(s)$ is in the angle $\{(x, y) \in \mathbb{R}^2 : -\varepsilon x < y < \varepsilon x\}$. O is a center-focus of (1), hence every X-orbit $\phi_X(t, z)$ close enough to O cuts both the line $y = -\varepsilon x$ and the line $y = \varepsilon x$ infinitely many times. By continuity, $\phi_X(t, z)$ has to meet η infinitely many times. Hence $\phi_X(t, z_1)$, where $z_1 := \eta(2s)$ meets η infinitely many times. Let us set $T := \min\{t > 0 : \phi_X(t, z_1) \in \eta\}$.

Since X is a normalizer of U, $\phi_x(T, \cdot)$ takes arcs of U-orbits into arcs of U-orbits. Hence, $\phi_X(T, \eta) \subset \eta$. By absurd, let us assume that for some $z \in \eta$ and $t \neq nT$, where n is a positive integer, one has $\phi_X(t, z) \in \eta$. Writing $t = \bar{n}T + \bar{t}$, $\bar{n}$ non-negative integer, $0 < \bar{t} < T$, one has $\phi_X(\bar{n}T + \bar{t}, z)) = \phi_X(\bar{n}T, \phi_X(\bar{t}, z)) \in \eta$. Also $-X$ is a normalizer of U, taking as well arcs of U-orbits into arcs of U-orbits. The $-X$-solution starting at $\phi_X(t, z)$ exists in the interval $[0, t]$. Applying $-\phi_X(\bar{n}T, \cdot)$ to $\phi_X(t, z)$ one has $-\phi_X(\bar{n}T, \phi_X(t, z)) = \phi_X(-\bar{n}T + t, z) = \phi_X(\bar{t}, z) \in \eta$. Since X normalizes U, one has $\phi_X(\bar{t}, \eta) \subset \eta$, with $\bar{t} \in (0, T)$, contradicting the definition of T. ♣

3 Isochronous sections at a cycle

We adapt to cycles the definition of isochronous section given previously. We do not restrict to limit cycles. Since a cycle γ is a closed Jordan curve, it separates the real plane into two disjoint connected regions. We denote the bounded one by Γ_i and the unbounded one by Γ_e.

Definition 2 *Let $\gamma(t) := \phi_X(t, z_0)$ be a T-periodic non-trivial cycle of (1). Let $\eta : (-\epsilon, \epsilon) \to \Omega$ be a C^1 curve transversal to X, such that $\eta(0) = z_0$. Then we say that η is a one-sided isochronous section of (1) at z_0 if (i) and (ii) of definition 1 hold for ϕ_X or $-\phi_X$ on Γ_i (Γ_e).*

Moreover, η is said to be an isochronous section of (1) at z_0 if it is a one-sided isochronous section both on Γ_i and on Γ_e.

In the above definition, it is possible that (i) and (ii) be satisfied as $t \to +\infty$ on Γ_i, and as $t \to -\infty$ on Γ_e, or vice-versa. This may occur when γ is a semistable limit cycle, with neighbouring orbits approaching γ as $t \to +\infty$ from the stable side, as $t \to -\infty$ from the unstable side.

If the system is not analytic, then cycles may accumulate on a cycle. For instance, is possible that a cycle be the boundary of a period annulus, or that a cycle have infinitely many limit cycles in every neighbourhood. Even in such cases, isochronous sections are possible.

Theorem 4 *X has a transversal isochronous section at a non-trivial cycle γ if and only if X is a non-trivial normalizer of a vector field U in a neighbourhood of γ.*

Proof. Let us assume X to have an isochronous section $\eta : (-\epsilon, \epsilon) \mapsto \mathbb{R}^2$ at $\gamma(t) = \phi_X(t, z_0)$. We work on Γ_e. Possibly considering the negative flow $-\phi_X$, we may assume Γ_e to be externally stable. Then there exists $\epsilon_1 < \epsilon$ such that $\forall z \in \eta([0, \epsilon_1)) \ : \ \phi_X(T, z) \in \eta([0, \epsilon_1))$.

As in section 2, one can prove that for $t \in [0, T)$, every curve $\eta_t(s) := \phi_X(t, \eta(s))$, defined on $[0, \epsilon_1)$ is an isochronous section of X. Since $\forall t \in [0, T)$ the map $\phi_X(t, \cdot)$ is a diffeomorphism, every $\eta_t(s)$ is trasversal to the orbits of X.

Let N_e be the half-neighbourhood of γ defined as $N_e := \{\phi_X(t, \eta(s)) : s \in [0, \epsilon_1), t \in [0, +\infty)\}$. Let us define the vector field U on N_e as follows:

$$U(\phi_X(t, \eta(s))) := \frac{\partial}{\partial s} \phi_X(t, \eta(s)).$$

By construction, the curves $\eta_t(s)$ are solutions to the differential system (5). Since by hypothesis every time-advancement map $\phi_X(\tau, \cdot)$ takes a curve $\eta_t(s)$ into the curve $\eta_{t+\tau}(s)$, X is a normalizer of U on N_e.

Now let us consider Γ_i. If also Γ_i is stable, we repeat the above procedure so to define a normalizer U on a full neighbourhood of γ. If Γ_i is negatively stable, we repeat the above procedure working on $-\phi_X$. In both cases we obtain a vector field U whose integral curves are the curves $\eta_t(s)$, so that X is a normalizer of U.

Vice-versa, if X is the normalizer of a transversal vector field U, one can work as in the proof of theorem 3. ♣

If X is a normalizer of U, then for every function $\zeta \in C^2(\Omega, \mathbb{R})$, $X + \zeta U$ is a normalizer of U. In fact

$$[X + \zeta U, U] = [X, U] + [\zeta U, U] = (\nu - \partial_U \zeta) U.$$

This suggests a simple procedure to construct examples of limit cycles with isochronous sections. It is sufficient to take X and U such that $[X, U] \wedge U \equiv 0$, and a function ζ taking opposite signs in suitable regions, then consider the vector field $X + \zeta U$. For instance, one can consider the following quadratic systems

$$x' = -y + Bxy, \qquad y' = x - \frac{Bx^2}{2} + \frac{By^2}{2} \tag{7}$$

with $B \in \mathbb{R}$, having as commutators, hence normalizers, the systems

$$x' = x - \frac{Bx^2}{2} + \frac{By^2}{2}, \qquad y' = y - Bxy. \tag{8}$$

Assume $B > 0$. The origin is an isochronous center of (7), with period annulus $\{(x, y) : x < \frac{1}{B}\}$. Denoting by X the vector field of (7) and by U that one of (8), let us consider the field $X + \zeta U$, with $\zeta(x, y) = x^2 + y^2 - 4B^2(x^2 + y^2)^2$. Since $\zeta(x, y) > 0$ for $0 < x^2 + y^2 < \frac{1}{4B^2}$, for small values of ζ the orbits of $X + \zeta U$ cross outwards every X-cycle contained in the circle $\{(x, y) : x^2 + y^2 < \frac{1}{4B^2}\}$. On the other hand, every X-cycle of N_O contained in $\{(x, y) : x^2 + y^2 > \frac{1}{4B^2}\}$ is crossed inwards by the orbits of $X + \zeta U$. This implies that $X + \zeta U$ has at least a limit cycle γ in N_O, since it has no critical points in N_O. The orbits of (8) contained in $\{(x, y) : x < \frac{1}{B}\}$ are isochronous sections of γ.

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