ON REGULAR HARMONICS OF ONE QUATERNIONIC VARIABLE

Alessandro Perotti

Abstract. We prove some results about the Fueter-regular homogeneous polynomials, which appear as components in the power series of any quaternionic regular function. Let B denote the unit ball in $\mathbb{C}^2 \simeq \mathbb{H}$ and $S = \partial B$ the group of unit quaternions. In §2.1 we obtain a differential condition that characterize the homogeneous polynomials whose restrictions to S extend as a regular polynomial. This result generalizes a similar characterization for holomorphic extensions of polynomials proved by Kytmanov.

In §2.2 we show how to define an injective linear operator $R : \mathcal{H}_k(S) \to U_k^\psi$ between the space of complex-valued spherical harmonics and the $\mathbb{H}$-module of regular homogeneous polynomials of degree k. In particular, we show how to construct bases of the module of regular homogeneous polynomials of a fixed degree starting from any choice of $\mathbb{C}$-bases of the spaces of complex harmonic homogeneous polynomials.

1. Notations and preliminaries

1.1. Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^2$ with smooth boundary. Let ν denote the outer unit normal to $\partial\Omega$ and $\tau = i\nu$. For every $F \in C^1(\overline{\Omega})$, let $\overline{\partial}_n F = \frac{1}{2}\left(\frac{\partial F}{\partial \nu} + i\frac{\partial F}{\partial \tau}\right)$ be the normal component of $\overline{\partial} F$ (see [K]§§3.3 and 14.2). It can be expressed by means of the Hodge $*$-operator and the Lebesgue surface measure as $\overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$. In a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} F$ in the tangential and the normal parts: $\overline{\partial} F = \overline{\partial}_b F + \overline{\partial}_n F \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}$. We denote by L the tangential Cauchy-Riemann operator $L = \frac{1}{|\overline{\partial}\rho|}\left(\frac{\partial \rho}{\partial \bar{z}_2}\frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho}{\partial \bar{z}_1}\frac{\partial}{\partial \bar{z}_2}\right)$.

Let $\mathbb{H}$ be the algebra of quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k denote the basic quaternions. We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ to $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We refer to [S] for the basic facts of quaternionic analysis. We will denote by $\mathcal{D}$ the left Cauchy-Riemann-Fueter operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is *(left-)regular* on a domain $\Omega \subseteq \mathbb{H}$ if $\mathcal{D}f = 0$ on Ω. We prefer to work with another class of regular functions, which

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is more explicitely connected with the hyperkähler structure of $\mathbb{H}$. It is defined by the Cauchy-Riemann-Fueter operator associated to the structural vector $\psi = \{1, i, j, -k\}$:

$$\mathcal{D}' = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right).$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is called *(left-)ψ-regular* on a domain Ω, if $\mathcal{D}'f = 0$ on Ω. This condition is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \qquad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1}.$$

The identity mapping is ψ-regular, and any holomorphic mapping (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$. This is no more true if we replace ψ-regularity with regularity. Moreover, the complex components of a ψ-regular function are either both holomorphic or both not-holomorphic (cf. [VS], [MS] and [P]). Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if, and only if, $f \circ \gamma$ is ψ-regular on $\gamma^{-1}(\Omega)$.

1.2. The two-dimensional Bochner-Martinelli form $U(\zeta, z)$ is the first complex component of the Cauchy-Fueter kernel $G'(p-q)$ associated to ψ-regular functions (cf. [F], [VS], [MS]). Let $q = z_1 + z_2 j$, $p = \zeta_1 + \zeta_2 j$, $\sigma(q) = dx[0] - i dx[1] + j dx[2] + k dx[3]$, where $dx[k]$ denotes the product of dx_0, dx_1, dx_2, dx_3 with dx_k deleted. Then $G'(p-q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j$, where $\omega(\zeta, z)$ is the complex $(1,2)$-form

$$\omega(\zeta, z) = -\frac{1}{4\pi^2}|\zeta - z|^{-4}((\bar{\zeta}_1 - \bar{z}_1)d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2)d\zeta_2) \wedge \overline{d\zeta}.$$

Here $\overline{d\zeta} = \overline{d\zeta_1} \wedge \overline{d\zeta_2}$ and we choose the orientation of $\mathbb{C}^2$ given by the volume form $\frac{1}{4} dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$. Given $g(\zeta, z) = \frac{1}{4\pi^2}|\zeta - z|^{-2}$, we can also write $U(\zeta, z) = -2 * \partial_\zeta g(\zeta, z)$ and $\omega(\zeta, z) = -\partial_\zeta(g(\zeta, z)\overline{d\zeta})$.

2. Regular polynomials

2.1. In this section we will obtain a differential condition that characterizes the homogeneous polynomials whose restrictions to the unit sphere extend regularly or ψ-regularly. We will use a computation made by Kytmanov in [K1] (cf. also [K] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let Ω be the unit ball B in $\mathbb{C}^2$, $S = \partial B$ the unit sphere. In this case the operators $\overline{\partial}_n$ and L have the following forms:

$$\overline{\partial}_n = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}, \qquad L = z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}$$

and they preserve harmonicity. Let $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$ be the Laplacian in $\mathbb{C}^2$ and D_k the differential operator

$$D_k = \sum_{0 \le l \le k/2-1} \frac{(k-2l-1)!(2l-1)!!}{k!(l+1)!} 2^l \Delta^{l+1}.$$

Theorem 1. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a ψ-regular function into B if, and only if,*

$$(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0 \quad \text{on } S.$$

Proof. In the first part we can proceed as in [K1]. The harmonic extension $\tilde{f}_1$ of $f_{1|S}$ into B is given by Gauss's formula: $\tilde{f}_1 = \sum_{s \geq 0} g_{k-2s}$, where g_{k-2s} is the homogeneous harmonic polynomial of degree $k - 2s$ defined by

$$(*) \qquad g_{k-2s} = \frac{k-2s+1}{s!(k-s+1)!} \sum_{j \geq 0} \frac{(-1)^j (k-j-2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1.$$

Then $\overline{\partial}_n \tilde{f}_1 = \overline{\partial}_n f_1 - D_k f_1$ on S (cf. [K]§23). Let $\tilde{f}_2$ be the harmonic extension of f_2 into B and $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$. Then $(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0$ on S is equivalent to $\overline{\partial}_n \tilde{f}_1 + \overline{L(f_2)} = 0$ on S. We now show that this implies the ψ-regularity of $\tilde{f}$. Let F^+ and F^- be the ψ-regular functions defined respectively on B and on $\mathbb{C}^2 \setminus \overline{B}$ by the Cauchy-Fueter integral of $\tilde{f}$:

$$F^{\pm}(z) = \int_S U(\zeta, z) \tilde{f}(\zeta) + \int_S \omega(\zeta, z) j \tilde{f}(\zeta).$$

From the equalities $U(\zeta, z) = -2 {*} \partial_\zeta g(\zeta, z)$, $\omega(\zeta, z) = -\partial_\zeta (g(\zeta, z) d\bar{\zeta})$, we get that

$$F^-(z) = -2 \int_S (\tilde{f}_1(\zeta) + f_2(\zeta) j) {*} \partial_\zeta g(\zeta, z) - \int_S \partial_\zeta (g(\zeta, z) \overline{d\zeta}) (\overline{\tilde{f}_1} j - \overline{\tilde{f}_2})$$

for every $z \notin \overline{B}$. From the complex Green formula and Stokes' Theorem and from the equality $\overline{\partial} \tilde{f}_2 \wedge d\zeta_{|S} = 2L(f_2) d\sigma$ on S, we get that the first complex component of $F^-(z)$ is

$$-2 \int_S \tilde{f}_1 \partial_n g d\sigma + \int_S \overline{\tilde{f}_2} \partial_\zeta g \wedge \overline{d\zeta} = -2 \int_S g \overline{\partial}_n \tilde{f}_1 d\sigma - \int_S g \partial_\zeta \overline{\tilde{f}_2} \wedge \overline{d\zeta} = -2 \int_S g (\overline{\partial}_n \tilde{f}_1 + \overline{L(f_2)}) d\sigma$$

and then it vanishes on $\mathbb{C}^2 \setminus \overline{B}$. Therefore, $F^- = F_2 j$, with F_2 a holomorphic function that can be holomorphically continued to the whole space. Let $\tilde{F}^- = \tilde{F}_2 j$ be such extension. Then $F = F^+ - \tilde{F}^-_{|B}$ is a ψ-regular function on B (indeed a polynomial of the same degree k), continuous on $\overline{B}$, such that $F_{|S} = f_{|S}$. The converse is immediate from the equations of ψ-regularity.

Let N and T be the differential operators

$$N = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2}, \qquad T = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}.$$

T is a tangential operator w.r.t. S, while N is non-tangential, such that $N(\rho) = |\overline{\partial} \rho|^2$, $\mathrm{Re}(N) = |\overline{\partial} \rho| \mathrm{Re}(\overline{\partial}_n)$, where $\rho = |z_1|^2 + |z_2|^2 - 1$. Let γ be the reflection introduced at the end of §1.1. The operator D_k is γ-invariant, i.e. $D_k(f \circ \gamma) = D_k(f) \circ \gamma$, since Δ is invariant. It follows a criterion for regularity of homogeneous polynomials.

Corollary 1. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a regular function into B if, and only if,*

$$(N - D_k)f_1 + \overline{T(f_2)} = 0 \quad \text{on } S.$$

Remark. Let $g = \sum_k g^k$ be the homogeneous decomposition of a polynomial g. After replacing $D_k g$ by $\sum_k D_k g^k$, we can extend the preceding results also to non-homogeneous polynomials.

2.2. Let $\mathcal{P}_k$ denote the space of homogeneous complex-valued polynomials of degree k on $\mathbb{C}^2$, and $\mathcal{H}_k$ the space of harmonic polynomials in $\mathcal{P}_k$. The space $\mathcal{H}_k$ is the sum of the pairwise $L^2(S)$-orthogonal spaces $\mathcal{H}_{p,q}$ $(p+q=k)$, whose elements are the harmonic homogeneous polynomials of degree p in z_1, z_2 and q in $\bar{z}_1, \bar{z}_2$ (cf. for example [R]§12.2). The spaces $\mathcal{H}_k$ and $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*). These spaces will be denoted by $\mathcal{H}_k(S)$ and $\mathcal{H}_{p,q}(S)$ respectively.

Let U_k^ψ be the right $\mathbb{H}$-module of (left)ψ-regular homogeneous polynomials of degree k. The elements of the modules U_k^ψ can be identified with their restrictions to S, which we will call *regular harmonics*.

Theorem 2. *For every $f_1 \in \mathcal{P}_k$, there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_2 j$ on S extends as a ψ-regular polynomial of degree at most k on $\mathbb{H}$. If $f_1 \in \mathcal{H}_k$, then $f_2 \in \mathcal{H}_k$ and $f = f_1 + f_2 j \in U_k^\psi$.*

Proof. We can suppose that f_1 has degree p in z and q in $\bar{z}$, $p+q=k$, and then extend by linearity. Let $\tilde{f}_1 = \sum_{s\geq 0} g_{p-s,q-s}$ be the harmonic extension of f_1 into B, where $g_{p-s,q-s} \in \mathcal{H}_{p-s,q-s}$ is given by formula (*). Then $\overline{\partial}_n \overline{L(g_{p-s,q-s})} = (p-s+1)\overline{L(g_{p-s,q-s})}$. We set

$$\tilde{f}_2 = \sum_{s\geq 0} \frac{1}{p-s+1} \overline{L(g_{p-s,q-s})} \in \bigoplus_{s\geq 0} \mathcal{H}_{k-2s}.$$

Then $\overline{\partial}_n \tilde{f}_2 = \overline{L(f_1)}$ on S and we can conclude as in the proof of Theorem 1 that $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$ is a ψ-regular polynomial of degree at most k. Now it suffices to define

$$f_2 = \sum_{s\geq 0} \frac{|z|^{2s}}{p-s+1} \overline{L(g_{p-s,q-s})} \in \mathcal{P}_k$$

to get a homogeneous polynomial $f = f_1 + f_2 j$, of degree k, that has the same restriction to S as $\tilde{f}$. If $f_1 \in \mathcal{H}_k$, then $\tilde{f}_1 = f_1$, $\tilde{f}_2 = f_2$ and therefore $f \in U_k^\psi$.

Let $C : U_k^\psi \to \mathcal{H}_k(S)$ be the complex-linear operator that associates to $f = f_1 + f_2 j$ the restriction to S of its first complex component f_1. The function $\tilde{f}$ in the preceding proof gives a right inverse $R : \mathcal{H}_k(S) \to U_k^\psi$ of the operator C. The function $R(f_1)$ is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of $\overline{B}$:

$$\int_S (R(f_1) - f_1)\overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{B}).$$

Corollary 2. *(i) The restriction operator C defined on U_k^ψ induces isomorphisms of real vector spaces*

$$\frac{U_k^\psi}{\mathcal{H}_{k,0}j} \simeq \mathcal{H}_k(S), \qquad \frac{U_k^\psi}{\mathcal{H}_{k,0}+\mathcal{H}_{k,0}j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

(ii) U_k^ψ has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

Proof. The first part follows from $\ker C = \{f = f_1 + f_2 j \in U_k^\psi \ : \ f_1 = 0 \text{ on } S\} = \mathcal{H}_{k,0}j$. Part (ii) can be obtained from any of the above isomorphisms, since $\mathcal{H}_{k,0}$ (as every space $\mathcal{H}_{p,q}, p+q=k$) and $\mathcal{H}_k(S)$ have real dimensions respectively $2(k+1)$ and $2(k+1)^2$.

As an application of Corollary 2, we have another proof of the known result (see [S] Theorem 7) that the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

2.3. The operator $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \to U_k^\psi$ can also be used to obtain $\mathbb{H}$-bases for U_k^ψ starting from bases of the complex spaces $\mathcal{H}_{p,q}(S)$. On $\mathcal{H}_{p,q}(S)$, R acts in the following way:

$$R(h) = h + M(h)j, \quad \text{where } M(h) = \frac{1}{p+1}\overline{L(h)} \in \mathcal{H}_{q-1,p+1} \quad (h \in \mathcal{H}_{p,q})$$

Note that $M \equiv 0$ on $\mathcal{H}_{k,0}(S)$. If $q > 0$, $M^2 = -Id$ on $\mathcal{H}_{p,q}(S)$, since $qh = \overline{\partial}_n h = -\overline{L(M(h))}$ on S, and therefore

$$h = -\frac{1}{q}\overline{L(M(h))} = -\frac{1}{q(p+1)}\overline{L}L(h) = -M^2(h).$$

If $k = 2m+1$ is odd, then M is a complex conjugate isomorphism of $\mathcal{H}_{m,m+1}(S)$. Then M induces a quaternionic structure on this space, which has real dimension $4(m+1)$. We can find complex bases of $\mathcal{H}_{m,m+1}(S)$ of the form

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}.$$

Theorem 3. *Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ $(p+q=k)$. Then:*
(i) if $k = 2m$ is even, a basis of U_k^ψ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q \le p \le k\}.$$

(ii) if $k = 2m+1$ is odd, a basis of U_k^ψ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q < p \le k\} \cup \{R(h_1), \dots, R(h_{m+1})\},$$

where $h_1, \dots, h_{m+1}$ are chosen such that the set

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

If the bases $\mathcal{B}_{p,q}$ are orthogonal in $L^2(S)$ and $h_1,\dots,h_{m+1}\in\mathcal{H}_{m,m+1}(S)$ are mutually orthogonal, then $\mathcal{B}_k$ is orthogonal, with norms

$$\|R(h)\|_{L^2(S,\mathbb{H})}=\left(\frac{p+q+1}{p+1}\right)^{1/2}\|h\|_{L^2(S)}\quad (h\in\mathcal{B}_{p,q})$$

w.r.t. the scalar product of $L^2(S,\mathbb{H})$.

Proof. From dimension count, it suffices to prove that the sets $\mathcal{B}_k$ are linearly independent. When $q\le p$, $q'\le p'$, $p+q=p'+q'=k$, the spaces $\mathcal{H}_{p,q}$ and $\mathcal{H}_{q'-1,p'+1}$ are distinct. Since $R(h)=h+M(h)j\in\mathcal{H}_{p,q}\oplus\mathcal{H}_{q-1,p+1}j$, this implies the independence over $\mathbb{H}$ of the images $\{R(h):h\in\mathcal{B}_{p,q}\}$. It remains to consider the case when $k=2m+1$ is odd. If $h\in\mathcal{H}_{m,m+1}(S)$, the complex components h and $M(h)$ of $R(h)$ belong to the same space. The independence of $\{R(h_1),\dots,R(h_{m+1})\}$ over $\mathbb{H}$ follows from the particular form of the complex basis chosen in $\mathcal{H}_{m,m+1}(S)$.

The scalar product of $L(h)$ and $L(h')$ in $\mathcal{H}_{p,q}(S)$ is

$$(L(h),L(h'))=(h,L^*L(h'))=-(h,\overline{L}L(h'))=q(p+1)(h,h'),$$

since the adjoint L^* is equal to $-\overline{L}$ (cf. [R]§18.2.2) and $\overline{L}L=q(p+1)M^2=-q(p+1)Id$. Therefore, if h,h' are orthogonal, $M(h)$ and $M(h')$ are orthogonal in $\mathcal{H}_{q-1,p+1}$ and then also $R(h)$ and $R(h')$. Finally, the norm of $R(h)$, $h\in\mathcal{H}_{p,q}(S)$, is

$$\|R(h)\|^2=\|h\|^2+\|M(h)\|^2=\|h\|^2+\frac{1}{(p+1)^2}\|L(h)\|^2=\frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof.

Remark. From Theorem 3 it is immediate to obtain also bases of the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k.

Examples. (i) The case $k=2$. Starting from the orthogonal bases $\mathcal{B}_{2,0}=\{z_1^2,2z_1z_2,z_2^2\}$ of $\mathcal{H}_{2,0}$ and $\mathcal{B}_{1,1}=\{z_1\bar z_2,|z_1|^2-|z_2|^2,z_2\bar z_1\}$ of $\mathcal{H}_{1,1}$ we get the orthogonal basis of regular harmonics

$$\mathcal{B}_2=\{z_1^2,2z_1z_2,z_2^2,z_1\bar z_2-\frac{1}{2}\bar z_1^2j,|z_1|^2-|z_2|^2+\bar z_1\bar z_2j,z_2\bar z_1+\frac{1}{2}\bar z_2^2j\}$$

of the six-dimensional right $\mathbb{H}$-module U_2^ψ.

(ii) The case $k=3$. From the orthogonal bases

$$\mathcal{B}_{3,0}=\{z_1^3,3z_1^2z_2,3z_1z_2^2,z_2^3\},\quad \mathcal{B}_{2,1}=\{z_1^2\bar z_2,2z_1|z_2|^2-z_1|z_1|^2,2z_2|z_1|^2-z_2|z_2|^2,z_2^2\bar z_1\},$$

$$\mathcal{B}_{1,2}=\{h_1=z_1\bar z_2^2,M(h_1)=-z_2\bar z_1^2,h_2=-2\bar z_2|z_1|^2+\bar z_2|z_2|^2,M(h_2)=-2\bar z_1|z_2|^2+\bar z_1|z_1|^2\},$$

we get the orthogonal basis of regular harmonics

$$\mathcal{B}_3=\{z_1^3,3z_1^2z_2,3z_1z_2^2,z_2^3,z_1^2\bar z_2-\frac{1}{3}\bar z_1^3j,2z_1|z_2|^2-z_1|z_1|^2-\bar z_1^2\bar z_2j,2z_2|z_1|^2-z_2|z_2|^2+\bar z_1\bar z_2^2j,$$
$$z_2^2\bar z_1+\frac{1}{3}\bar z_2^3j,z_1\bar z_2^2-z_2\bar z_1^2j,-2\bar z_2|z_1|^2+\bar z_2|z_2|^2+(\bar z_1|z_1|^2-2\bar z_1|z_2|^2+)j\}.$$

of the ten-dimensional right $\mathbb{H}$-module U_3^ψ.

In general, for any k, an orthogonal basis of $\mathcal{H}_{p,q}$ $(p+q=k)$ is given by the polynomials $\{P_{q,l}^k\}_{l=0,\dots,k}$ defined by formula (6.14) in [S]. The basis of U_k obtained from these bases by means of Theorem 3 and applying the reflection γ is essentially the same given in Proposition 8 of [S].

Another spanning set of the space $\mathcal{H}_{p,q}$ is given by the functions

$$g_\alpha^{p,q}(z_1,z_2)=(z_1+\alpha z_2)^p(\bar z_2-\alpha\bar z_1)^q \quad (\alpha\in\mathbb{C})$$

(cf. [R]§12.5.1). Since $M(g_\alpha^{p,q})=\frac{(-1)^q q\bar\alpha^{p+q}}{p+1}g_{-1/\bar\alpha}^{q-1,p+1}$ for $\alpha\neq 0$ and $M(g_0^{p,q})=-\frac{q}{p+1}z_2^{q-1}\bar z_1^{p+1}$, where we set $g_\alpha^{p,q}\equiv 0$ if $p<0$, from Theorem 3 we get that U_k^ψ is spanned over $\mathbb{H}$ by the polynomials

$$R(g_\alpha^{p,q})=\begin{cases} g_\alpha^{p,q}+\frac{(-1)^q q\bar\alpha^{p+q}}{p+1}g_{-1/\bar\alpha}^{q-1,p+1}j & \text{for } \alpha\neq 0\\ z_1^p\bar z_2^q-\frac{q}{p+1}z_2^{q-1}\bar z_1^{p+1}j & \text{for } \alpha=0\end{cases} \qquad (\alpha\in\mathbb{C},\ p+q=k)$$

Any choice of $k+1$ distinct numbers $\alpha_0,\alpha_1,\dots,\alpha_k$ gives rise to a basis of U_k^ψ.

The results obtained in this paper enabled the writing of a *Mathematica* package [P1], named `RegularHarmonics`, which implements efficient computations with regular and ψ-regular functions and with harmonic and holomorphic functions of two complex variables.

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Dipartimento di Matematica Università degli Studi di Trento Via Sommarive,14 I-38050 Povo Trento ITALY

E-mail address: `perotti@science.unitn.it`