Lorenz gauged vector potential formulations for the time-harmonic eddy-current problem with L^∞-regularity of material properties

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Abstract. In this paper we consider some Lorenz gauged vector potential formulations of the eddy-current problem for the time-harmonic Maxwell equations with material properties having only L^∞-regularity. We prove that there exists a unique solution of these problems, and we show the convergence of a suitable finite element approximation scheme. Moreover, we show that some previously proposed Lorenz gauged formulations are indeed formulations in terms of the modified magnetic vector potential, for which the electric scalar potential is vanishing.

1. Introduction

Let us consider a bounded connected open set $\Omega \subset \mathbf{R}^3$, with boundary $\partial\Omega$. The unit outward normal vector on $\partial\Omega$ will be denoted by $\mathbf{n}$. We assume that $\overline{\Omega}$ is split into two parts, $\overline{\Omega} = \overline{\Omega_C} \cup \overline{\Omega_I}$, where Ω_C (a non-homogeneous isotropic conductor) and Ω_I (a perfect insulator) are open disjoint subsets, such that $\overline{\Omega_C} \subset \Omega$. We denote by $\Gamma := \partial\Omega_I \cap \partial\Omega_C$ the interface between the two subdomains; note that, in the present situation, $\partial\Omega_C = \Gamma$ and $\partial\Omega_I = \partial\Omega \cup \Gamma$.

In this paper we study the time-harmonic *eddy-current problem*, which is derived from the full Maxwell system by neglecting the displacement current term $\frac{\partial\mathcal{D}}{\partial t}$, and by assuming that the electric field $\mathcal{E}$, the magnetic field $\mathcal{H}$ and the applied current density $\mathcal{J}_e$ are of the form

$$\begin{aligned} \mathcal{E}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{J}_e(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x})\exp(i\omega t)]\ , \end{aligned}$$

where $\omega \neq 0$ is a given angular frequency (see, e.g., Bossavit [6], p. 219).

The constitutive relation $\mathcal{B} = \mu\mathcal{H}$ (where μ is the magnetic permeability coefficient) is assumed to hold, as well as the (generalized) Ohm's law $\mathcal{J} = \sigma\mathcal{E} + \mathcal{J}_e$ (where σ is the electric conductivity).

The magnetic permeability μ is assumed to be a (real) symmetric matrix, uniformly positive definite in Ω, with entries in $L^\infty(\Omega)$. Since Ω_I is a perfect insulator, we require that $\sigma_{|\Omega_I} \equiv 0$; moreover, as Ω_C is a non-homogeneous isotropic conductor, $\sigma_{|\Omega_C}$ is assumed to be a (real) scalar $L^\infty(\Omega_C)$-function, uniformly positive in Ω_C.

Concerning the boundary condition, we consider the *magnetic* boundary value problem, namely, $\mathbf{H} \times \mathbf{n}$, representing the tangential component of the magnetic field, is assumed to vanish on $\partial\Omega$. The case of the *electric* boundary value problem, in which $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, can be treated following a similar approach, but in the sequel we will not dwell on it.

We make the following assumptions on the geometry of Ω:

$(H1)$ either $\partial\Omega \in C^{1,1}$, or else Ω is a Lipschitz polyhedron; the same assumption holds for Ω_C and Ω_I.

For the sake of simplicity, we also suppose that:

- $\partial\Omega$ and Γ are connected
- non-bounding cycles are not present neither on $\partial\Omega$, regarded as a part of the boundary of Ω_I, nor on Γ, regarded as the boundary of Ω_C.

The last assumption means that each cycle on $\partial\Omega$ (respectively, on Γ) can be represented as ∂S, S being a surface contained in Ω_I (respectively, in Ω_C).

From all this, we know in particular that the spaces of harmonic fields

$$\begin{aligned}\mathcal{H}_{\Gamma;\partial\Omega}(\Omega_I) \quad := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \operatorname{rot}\mathbf{v}_I = \mathbf{0}, \operatorname{div}\mathbf{v}_I = 0, \\ \mathbf{v}_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma, \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\end{aligned}$$

$$\mathcal{H}(m;\Omega_C) := \{\mathbf{v}_C \in (L^2(\Omega_C))^3 \mid \operatorname{rot}\mathbf{v}_C = \mathbf{0}, \operatorname{div}\mathbf{v}_C = 0, \mathbf{v}_C \cdot \mathbf{n}_C = 0 \text{ on } \Gamma\}$$

$$\mathcal{H}(e;\Omega_C) := \{\mathbf{v}_C \in (L^2(\Omega_C))^3 \mid \operatorname{rot}\mathbf{v}_C = \mathbf{0}, \operatorname{div}\mathbf{v}_C = 0, \mathbf{v}_C \times \mathbf{n}_C = \mathbf{0} \text{ on } \Gamma\}$$

$$\mathcal{H}(m;\Omega) := \{\mathbf{v} \in (L^2(\Omega))^3 \mid \operatorname{rot}\mathbf{v} = \mathbf{0}, \operatorname{div}\mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

are trivial.

The vector potential formulation of the eddy-current problem in a general geometrical configuration is more technical, and could be faced by adapting the procedure described in Bíró and Valli [4] (for the Coulomb gauged vector potential formulation).

Let us assume that the current density $\mathbf{J}_e \in (L^2(\Omega))^3$ satisfies the (necessary) conditions

$$(H2) \qquad \operatorname{div}\mathbf{J}_{e,I} = 0 \text{ in } \Omega_I \quad , \quad \mathbf{J}_e \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \ .$$

In Alonso Rodríguez, Fernandes and Valli [3] it has been proved that, in the present geometrical situation, the complete system of equations describing the eddy-current problem in terms of the magnetic field $\mathbf{H}$ and the electric field

$\mathbf{E}_C$ is:

$$
(1.1) \qquad \begin{cases}
\operatorname{rot} \mathbf{E}_C + i\omega\mu_C \mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\
\operatorname{rot} \mathbf{H}_C - \sigma_C \mathbf{E}_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\operatorname{rot} \mathbf{H}_I = \mathbf{J}_{e,I} & \text{in } \Omega_I \\
\operatorname{div}(\mu_I \mathbf{H}_I) = 0 & \text{in } \Omega_I \\
\mathbf{H}_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\
\mu_I \mathbf{H}_I \cdot \mathbf{n}_I + \mu_C \mathbf{H}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\mathbf{H}_I \times \mathbf{n}_I + \mathbf{H}_C \times \mathbf{n}_C = \mathbf{0} & \text{on } \Gamma \ ,
\end{cases}
$$

where $\mathbf{n}_C = -\mathbf{n}_I$ is the unit outward normal vector on $\partial\Omega_C = \Gamma$, and we have set $\mathbf{E}_C := \mathbf{E}_{|\Omega_C}$ (and similarly for Ω_I and any other restriction of function). In particular, in [3] it is proved that, under the assumptions $(H1)$–$(H2)$, problem (1.1) has a unique solution $(\mathbf{H}, \mathbf{E}_C) \in H(\mathrm{rot}; \Omega) \times H(\mathrm{rot}; \Omega_C)$.

In the following, we consider problem (1.1) in terms of the Lorenz gauged vector potential formulation, that we present in Section 2 in three alternative versions. In Section 4 we will prove that these formulations are well-posed, namely, there exists a solution for each one of them and this solution is unique. Moreover, in Sections 5 and 6 we will derive the three corresponding variational formulations, and finally establish a stability result and prove the convergence of a suitable finite element approximation by nodal elements for two of them.

In Section 3 we will also comment on other previously proposed Lorenz gauged vector potential formulations, showing in particular that in the one presented by Bossavit [7] a different gauge was indeed indirectly enforced.

2. The $(\mathbf{A}_C, V_C)$–$\mathbf{A}_I$ formulation

We are looking for a magnetic vector potential $\mathbf{A}$ and a scalar electric potential V_C such that

$$
(2.1) \qquad \mathbf{E}_C = -i\omega \mathbf{A}_C - \operatorname{grad} V_C \quad , \quad \mu_C \mathbf{H}_C = \operatorname{rot} \mathbf{A}_C \quad , \quad \mu_I \mathbf{H}_I = \operatorname{rot} \mathbf{A}_I \ .
$$

In this way one has $\operatorname{rot} \mathbf{E}_C = -i\omega \operatorname{rot} \mathbf{A}_C = -i\omega\mu_C \mathbf{H}_C$, and therefore the Faraday equation in Ω_C is satisfied. Moreover, $\mu_I \mathbf{H}_I$ is a solenoidal field in Ω_I.

The matching conditions for $\mu \mathbf{H} \cdot \mathbf{n}$ can be enforced by

$$
(2.2) \qquad \mathbf{A}_I \times \mathbf{n}_I + \mathbf{A}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on } \Gamma \ ,
$$

as taking the tangential divergence of this relation one finds

$$
\operatorname{rot} \mathbf{A}_I \cdot \mathbf{n}_I + \operatorname{rot} \mathbf{A}_C \cdot \mathbf{n}_C = 0 \quad \text{on } \Gamma \ .
$$

As a consequence, we have $\mu \mathbf{H} = \operatorname{rot} \mathbf{A}$ in the whole Ω.

In order to have a unique vector potential $\mathbf{A}$, it is necessary to impose some gauge conditions: instead of the most commonly used Coulomb gauge $\operatorname{div} \mathbf{A} = 0$ in Ω, we consider the Lorenz gauge

$$\operatorname{div} \mathbf{A}_C + \mu_{*,C} \sigma_C V_C = 0 \ \text{ in } \Omega_C \quad , \quad \operatorname{div} \mathbf{A}_I = 0 \ \text{ in } \Omega_I \ , \tag{2.3}$$

where the scalar function $\mu_* = \mu_*(\mathbf{x})$, defined in Ω and satisfying $\mu_* \in L^\infty(\Omega)$, $0 < \mu_{*,1} \leq \mu_*(\mathbf{x}) \leq \mu_{*,2}$ in Ω, will be chosen in the sequel. (For instance, one can think that $\mu_* = \frac{1}{3} \operatorname{trace}(\mu)$, so that $\mu_{*,1} = \mu_{\min}$, the minimum eigenvalue of μ in Ω, and $\mu_{*,2} = \mu_{\max}$, the maximum eigenvalue of μ in Ω.)

Moreover, we also assume the boundary condition

$$\mathbf{A} \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \ , \tag{2.4}$$

and an additional condition on the interface Γ. In this respect, we consider three possible alternatives: the first one is the "slip" condition

$$\mathbf{A}_C \cdot \mathbf{n}_C = 0 \qquad \text{on } \Gamma \ ; \tag{2.5}$$

the second one is the Dirichlet condition

$$V_C = 0 \qquad \text{on } \Gamma \ ; \tag{2.6}$$

the last one is the matching condition

$$\mathbf{A}_I \cdot \mathbf{n}_I + \mathbf{A}_C \cdot \mathbf{n}_C = 0 \qquad \text{on } \Gamma \tag{2.7}$$

(in Section 4 these three choices will be called case (i), case (ii) and case (iii), respectively).

Let us start specifying in detail the formulation associated to the matching condition (2.7). First, note that $(2.3)_2$, (2.4) and (2.7) imply that $\int_\Gamma \mathbf{A}_I \cdot \mathbf{n}_I = 0 = \int_\Gamma \mathbf{A}_C \cdot \mathbf{n}_C$, hence $\int_{\Omega_C} \operatorname{div} \mathbf{A}_C = 0$. As a consequence, we can also impose

$$\int_{\Omega_C} \mu_{*,C} \sigma_C V_C = 0 \tag{2.8}$$

without actually introducing any further constraint.

In conclusion, taking into account (1.1), we are left with the problem

$$
(2.9)\quad
\begin{cases}
\operatorname{rot}(\mu_C^{-1}\operatorname{rot}\mathbf{A}_C) + i\omega\sigma_C\mathbf{A}_C + \sigma_C \operatorname{grad} V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\operatorname{rot}(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I) = \mathbf{J}_{e,I} & \text{in } \Omega_I \\
\operatorname{div}\mathbf{A}_C + \mu_{*,C}\sigma_C V_C = 0 & \text{in } \Omega_C \\
\operatorname{div}\mathbf{A}_I = 0 & \text{in } \Omega_I \\
\mathbf{A}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\
(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\
\mathbf{A}_I \cdot \mathbf{n}_I + \mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\mathbf{A}_I \times \mathbf{n}_I + \mathbf{A}_C \times \mathbf{n}_C = \mathbf{0} & \text{on } \Gamma \\
(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I) \times \mathbf{n}_I + (\mu_C^{-1}\operatorname{rot}\mathbf{A}_C) \times \mathbf{n}_C = \mathbf{0} & \text{on } \Gamma \\
\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0 \ .
\end{cases}
$$

If we replace the interface condition $(2.9)_7$ (i.e. (2.7)) with (2.5), we obtain another problem, that will be denoted by (2.9)*. Moreover, if we replace $(2.9)_7$ with (2.6) and we drop the average condition $(2.9)_{10}$, we obtain a third problem, that will be denoted by (2.9)**.

Defining

$$
(2.10)\qquad \mathbf{J} := \begin{cases} \mathbf{J}_{e,C} - i\omega\sigma_C\mathbf{A}_C - \sigma_C \operatorname{grad} V_C & \text{in } \Omega_C \\ \mathbf{J}_{e,I} & \text{in } \Omega_I \end{cases} ,
$$

as a consequence of (2.9) (or (2.9)*, or (2.9)**) we also have $\operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{A}) = \mathbf{J}$ in Ω, therefore

$$
(2.11)\qquad \begin{cases} \operatorname{div}(i\omega\sigma_C\mathbf{A}_C + \sigma_C \operatorname{grad} V_C - \mathbf{J}_{e,C}) = 0 & \text{in } \Omega_C \\ (i\omega\sigma_C\mathbf{A}_C + \sigma_C \operatorname{grad} V_C - \mathbf{J}_{e,C}) \cdot \mathbf{n}_C = \mathbf{J}_{e,I} \cdot \mathbf{n}_I & \text{on } \Gamma \end{cases} .
$$

Remark 2.1. As we have already noted, the condition $\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$ follows from the gauge conditions $(2.9)_3$, $(2.9)_4$, $(2.9)_5$ and $(2.9)_7$. Therefore, we could omit it in (2.9). However, this vanishing average condition will be useful when we will analyze the variational formulation of the Lorenz gauged eddy-current problem in Section 5; hence we prefer to keep it in formulation (2.9). The same remark applies to the formulation (2.9)*. □

3. Lorenz gauge or something else?

As a starting point, with the aim of making clear the reasons of our choice in Section 5, let us discuss some of the variational formulations that have been previously proposed for problem (2.9) (or (2.9)*, or (2.9)**). Let us point out that we are not assuming that σ_C is smooth, but only that $\sigma_C \in L^\infty(\Omega_C)$.

In the following, in order to give a meaning to the integrals we are going to consider, we assume, as it will be proved in Section 4, that there exists a solution to (2.9) (or (2.9)*, or (2.9)**), satisfying $(\mathbf{A}, V_C) \in Q_0 \times H^1(\Omega_C)$, where

$$Q_0 := \{\mathbf{w} \in H(\mathrm{rot};\Omega) \quad | \ \mathrm{div}\,\mathbf{w}_C \in L^2(\Omega_C),\ \mathrm{div}\,\mathbf{w}_I \in L^2(\Omega_I),\\ \mathbf{w}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\ ,$$

and moreover we consider the space of test functions

$$W_0 := \{\mathbf{w} \in H(\mathrm{rot};\Omega) \quad | \ \mathrm{div}(\sigma_C \mathbf{w}_C) \in L^2(\Omega_C),\ \mathrm{div}\,\mathbf{w}_I \in L^2(\Omega_I),\\ \mathbf{w}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

(for a smooth conductivity σ_C we have $Q_0 = W_0$).

We also note that, for the ease of the reader, in the sequel we are always denoting the duality pairings as surface integrals (the interested reader can refer to Bossavit [6], Dautray and Lions [12], Girault and Raviart [14] for more details on these aspects related to functional analysis and to linear spaces of functions).

Multiply $(2.9)_1$ and $(2.9)_2$ by a test function $\mathbf{w} \in W_0$ and integrate in Ω. Integration by parts yields:

$$(3.1)\quad \begin{aligned} &\int_{\Omega_C} [\mu_C^{-1} \,\mathrm{rot}\,\mathbf{A}_C \cdot \mathrm{rot}\,\overline{\mathbf{w}_C} + i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} - V_C \,\mathrm{div}(\sigma_C \overline{\mathbf{w}_C})] \\ &\qquad - \int_\Gamma [(\mu_C^{-1} \,\mathrm{rot}\,\mathbf{A}_C) \times \mathbf{n}_C] \cdot \overline{\mathbf{w}_C} + \int_\Gamma V_C \sigma_C \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\ &\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\ &\int_{\Omega_I} \mu_I^{-1} \,\mathrm{rot}\,\mathbf{A}_I \cdot \mathrm{rot}\,\overline{\mathbf{w}_I} \\ &\qquad - \int_{\Gamma \cup \partial\Omega} [(\mu_I^{-1} \,\mathrm{rot}\,\mathbf{A}_I) \times \mathbf{n}_I] \cdot \overline{\mathbf{w}_I} \\ &\quad = \int_{\Omega_I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{w}_I}\ . \end{aligned}$$

Using the Lorenz gauge in Ω_C permits to replace the unknown V_C and gives

$$(3.2)\quad \begin{aligned} &\int_{\Omega_C} [\mu_C^{-1} \,\mathrm{rot}\,\mathbf{A}_C \cdot \mathrm{rot}\,\overline{\mathbf{w}_C} + \mu_{*,C}^{-1} \sigma_C^{-1} \,\mathrm{div}\,\mathbf{A}_C \,\mathrm{div}(\sigma_C \overline{\mathbf{w}_C})] \\ &\qquad + \int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} \\ &\qquad - \int_\Gamma [(\mu_C^{-1} \,\mathrm{rot}\,\mathbf{A}_C) \times \mathbf{n}_C] \cdot \overline{\mathbf{w}_C} + \int_\Gamma V_C \sigma_C \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\ &\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C}\ . \end{aligned}$$

For what is concerned with the other equation, as the Lorenz gauge in Ω_I is $\mathrm{div}\,\mathbf{A}_I = 0$, we can also write

$$(3.3)\quad \begin{aligned} &\int_{\Omega_I} (\mu_I^{-1} \,\mathrm{rot}\,\mathbf{A}_I \cdot \mathrm{rot}\,\overline{\mathbf{w}_I} + \mu_{*,I}^{-1} \,\mathrm{div}\,\mathbf{A}_I \,\mathrm{div}\,\overline{\mathbf{w}_I}) \\ &\qquad - \int_{\Gamma \cup \partial\Omega} [(\mu_I^{-1} \,\mathrm{rot}\,\mathbf{A}_I) \times \mathbf{n}_I] \cdot \overline{\mathbf{w}_I} \\ &\quad = \int_{\Omega_I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{w}_I}\ . \end{aligned}$$

Taking into account the interface and boundary conditions, we add (3.2) and (3.3) and we find that $\mathbf{A}$ satisfies

$$
\begin{aligned}
&\textstyle\int_\Omega \mu^{-1}\operatorname{rot}\mathbf{A}\cdot\operatorname{rot}\overline{\mathbf{w}} + \int_{\Omega_C}\mu_{*,C}^{-1}\sigma_C^{-1}\operatorname{div}\mathbf{A}_C\operatorname{div}(\sigma_C\overline{\mathbf{w}_C})\\
&\qquad\textstyle + \int_{\Omega_C} i\omega\sigma_C\mathbf{A}_C\cdot\overline{\mathbf{w}_C} + \int_{\Omega_I}\mu_{*,I}^{-1}\operatorname{div}\mathbf{A}_I\operatorname{div}\overline{\mathbf{w}_I}\\
&\qquad\textstyle + \int_\Gamma V_C\sigma_C\overline{\mathbf{w}_C}\cdot\mathbf{n}_C\\
&\quad\textstyle = \int_\Omega \mathbf{J}_e\cdot\overline{\mathbf{w}} \qquad\qquad \forall\ \mathbf{w}\in W_0\ .
\end{aligned}
\tag{3.4}
$$

To conclude, let us obtain the variational formulation for the scalar potential. From (2.11), we see that V_C satisfies

$$
\begin{aligned}
0 &\textstyle= -\int_{\Omega_C}\operatorname{div}(i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C - \mathbf{J}_{e,C})\,\overline{\psi_C}\\
&\textstyle= \int_{\Omega_C}(i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C - \mathbf{J}_{e,C})\cdot\operatorname{grad}\overline{\psi_C}\\
&\qquad\textstyle - \int_\Gamma(i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C - \mathbf{J}_{e,C})\cdot\mathbf{n}_C\,\overline{\psi_C}\\
&\textstyle= \int_{\Omega_C}(i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C - \mathbf{J}_{e,C})\cdot\operatorname{grad}\overline{\psi_C} - \int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n}_I\,\overline{\psi_C}\ ,
\end{aligned}
$$

namely

$$
\begin{aligned}
&\textstyle\int_{\Omega_C}\sigma_C\operatorname{grad}V_C\cdot\operatorname{grad}\overline{\psi_C} = -\int_{\Omega_C} i\omega\sigma_C\mathbf{A}_C\cdot\operatorname{grad}\overline{\psi_C}\\
&\qquad\textstyle + \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\operatorname{grad}\overline{\psi_C} + \int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n}_I\,\overline{\psi_C} \qquad \forall\ \psi_C\in H^1(\Omega_C)\ .
\end{aligned}
\tag{3.5}
$$

Now, in order to obtain a formulation which looks feasible and for which the unknowns $\mathbf{A}$ and V_C are decoupled, we have to eliminate the term containing V_C in (3.4). This can be done either assuming that the test function $\mathbf{w}$ belongs to W_{00}, where

$$
W_{00} := \{\mathbf{w}\in W_0 \mid \sigma_C\mathbf{w}_C\cdot\mathbf{n}_C = 0 \text{ on } \Gamma\}\ ,
$$

or else using the interface condition (2.6), i.e., $V_C = 0$ on Γ.

In the first case the final problem, associated to the interface conditions (2.5) or (2.7), is

$$
\begin{aligned}
\mathbf{A}\in Q_0\ :\ &\textstyle\int_\Omega \mu^{-1}\operatorname{rot}\mathbf{A}\cdot\operatorname{rot}\overline{\mathbf{w}} + \int_{\Omega_C}\mu_{*,C}^{-1}\sigma_C^{-1}\operatorname{div}\mathbf{A}_C\operatorname{div}(\sigma_C\overline{\mathbf{w}_C})\\
&\qquad\textstyle + \int_{\Omega_C} i\omega\sigma_C\mathbf{A}_C\cdot\overline{\mathbf{w}_C} + \int_{\Omega_I}\mu_{*,I}^{-1}\operatorname{div}\mathbf{A}_I\operatorname{div}\overline{\mathbf{w}_I}\\
&\quad\textstyle = \int_\Omega \mathbf{J}_e\cdot\overline{\mathbf{w}} \qquad\qquad \forall\ \mathbf{w}\in W_{00}\ ,
\end{aligned}
\tag{3.6}
$$

followed by

$$
\begin{aligned}
V_C\in H^1_*(\Omega_C)\ :\ &\textstyle\int_{\Omega_C}\sigma_C\operatorname{grad}V_C\cdot\operatorname{grad}\overline{\psi_C}\\
&\quad\textstyle = -\int_{\Omega_C} i\omega\sigma_C\mathbf{A}_C\cdot\operatorname{grad}\overline{\psi_C} + \int_{\Omega_C}\mathbf{J}_{e,C}\cdot\operatorname{grad}\overline{\psi_C}\\
&\qquad\textstyle + \int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n}_I\,\overline{\psi_C} \quad \forall\ \psi_C\in H^1_*(\Omega_C)\ ,
\end{aligned}
\tag{3.7}
$$

where

$$
H^1_*(\Omega_C) := \left\{\psi_C\in H^1(\Omega_C) \mid \int_{\Omega_C}\mu_{*,C}\sigma_C\psi_C = 0\right\}\ .
$$

In the latter case the problem, associated to the interface condition (2.6), is

$$
\begin{array}{rl}
\mathbf{A} \in Q_0 \ : & \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} + \int_{\Omega_C} \mu_{*,C}^{-1} \sigma_C^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div}(\sigma_C \overline{\mathbf{w}_C}) \\
& + \int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} \\
& = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad \forall\, \mathbf{w} \in W_0 \ ,
\end{array}
\tag{3.8}
$$

followed by

$$
\begin{array}{rl}
V_C \in H_0^1(\Omega_C) \ : & \int_{\Omega_C} \sigma_C \operatorname{grad} V_C \cdot \operatorname{grad} \overline{\psi_C} = -\int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \operatorname{grad} \overline{\psi_C} \\
& + \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{\psi_C} \qquad \forall\, \psi_C \in H_0^1(\Omega_C) \ .
\end{array}
\tag{3.9}
$$

While problems (3.7) and (3.9) are classical elliptic boundary value problems, without additional assumptions the formulations (3.6) or (3.8) are not easy to handle. A favourable situation appears when $\sigma_C = \text{const}$, as, first of all, in this case one has $Q_0 = W_0$ and, moreover, for the interface condition (2.5) we know that $\mathbf{A} \in W_{00}$. Therefore, in problems (3.6) (for the interface condition (2.5)) and (3.8) (for the interface condition (2.6)), the space of trial functions and the space of test functions are the same (on the contrary, even for $\sigma_C = \text{const}$ this is not the case for the interface condition (2.7)). Furthermore, one also has $\int_{\Omega_C} \mu_{*,C}^{-1} \sigma_C^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div}(\sigma_C \overline{\mathbf{w}_C}) = \int_{\Omega_C} \mu_{*,C}^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C}$, so that the first order terms in the sesquilinear forms at the left hand side of (3.6) and (3.8) are hermitian and positive definite.

An analysis of these two formulations for $\sigma_C = \text{const}$ is presented here below, for a slightly generalized form of the Lorenz gauge proposed by Bossavit [7], that indeed for $\sigma_C = \text{const}$ coincides with the usual one. However, in the general case of a non-constant σ_C, the formulations (3.6) and (3.8) are not suitable: for instance, it is not clear that an uniqueness result holds for them, even if in (3.6) we use the additional information that the solution satisfies (2.5) or (2.7).

A change of the point of view is thus in order. Bossavit [7] proposed to modify the Lorenz gauge in Ω_C in the following way:

$$
\operatorname{div}(\sigma_C \mathbf{A}_C) + \mu_{*,C} \sigma_C^2 V_C = 0 \qquad \text{in } \Omega_C \ , \tag{3.10}
$$

which, as we already noted, for a constant value of σ_C reduces to the usual Lorenz gauge. Accordingly, instead of the interface condition (2.5) one has to consider $\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ, while condition (2.6) is kept unchanged (in the following, the interface condition (2.7) will not be considered).

Let us suppose that there exists a solution $(\mathbf{A}, V_C) \in W_{00} \times H^1(\Omega_C)$ (for the interface condition $\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ) or $(\mathbf{A}, V_C) \in W_0 \times H_0^1(\Omega_C)$ (for the interface condition (2.6)) to these Bossavit–Lorenz gauged problems; without entering into details, we note that we could adapt the proofs reported in Section 4 to show that these existence results are in fact true.

Proceeding as before, for the interface condition $\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ the corresponding variational formulation now reads:

$$
(3.11)\qquad
\begin{aligned}
\mathbf{A} \in W_{00} \ : \ \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} & \\
+ \int_{\Omega_C} \mu_{*,C}^{-1} \sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C) \operatorname{div}(\sigma_C \overline{\mathbf{w}_C}) & \\
+ \int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} & \\
= \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad\qquad \forall\ \mathbf{w} \in W_{00}\ . &
\end{aligned}
$$

Similarly, for the interface condition (2.6) one can write:

$$
(3.12)\qquad
\begin{aligned}
\mathbf{A} \in W_{0} \ : \ \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} & \\
+ \int_{\Omega_C} \mu_{*,C}^{-1} \sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C) \operatorname{div}(\sigma_C \overline{\mathbf{w}_C}) & \\
+ \int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} & \\
= \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad\qquad \forall\ \mathbf{w} \in W_{0}\ . &
\end{aligned}
$$

These variational problems look indeed easier to handle. First of all, it is easy to prove that they are well-posed, namely, that uniqueness holds. In fact, for $\mathbf{J}_e = \mathbf{0}$ it follows at once that $\mathbf{A}_C = \mathbf{0}$ in Ω_C; consequently, $\mathbf{A}_I$ satisfies $\operatorname{rot} \mathbf{A}_I = \mathbf{0}$ in Ω_I, $\operatorname{div} \mathbf{A}_I = 0$ in Ω_I, $\mathbf{A}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ and finally $\mathbf{A}_I \times \mathbf{n}_I = -\mathbf{A}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ, hence $\mathbf{A}_I = \mathbf{0}$ in Ω_I.

Another important result is the following: the function $\mu_{*,C}^{-1}\sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C)$ has a distributional gradient belonging to $(L^2(\Omega_C))^3$ and, moreover, $\operatorname{div} \mathbf{A}_I = 0$ in Ω_I. (Since, as we have already noted, it is possible to show that there exists a solution to the Bossavit–Lorenz gauged vector potential problems, these results are indeed trivial, as $\mu_{*,C}^{-1}\sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C) = -V_C \in H^1(\Omega_C)$, and $\operatorname{div} \mathbf{A}_I = 0$ is the gauge condition in Ω_I; however, it is useful to show that they follow directly from the intrinsic structure of the variational problems (3.11) or (3.12).)

In fact, take $\mathbf{q}_C \in (C_0^\infty(\Omega_C))^3$ and let $\varphi_C \in H^1(\Omega_C)$ be the solution of the Neumann problem

$$
(3.13)\qquad
\begin{cases}
\operatorname{div}(\sigma_C \operatorname{grad} \varphi_C) + i\omega\mu_{*,C}\sigma_C^2\varphi_C = \operatorname{div} \mathbf{q}_C & \text{in } \ \Omega_C \\
\sigma_C \operatorname{grad} \varphi_C \cdot \mathbf{n}_C = 0 & \text{on } \ \Gamma
\end{cases}
$$

(for the interface condition $\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ) or of the Dirichlet problem

$$
(3.14)\qquad
\begin{cases}
\operatorname{div}(\sigma_C \operatorname{grad} \varphi_C) + i\omega\mu_{*,C}\sigma_C^2\varphi_C = \operatorname{div} \mathbf{q}_C & \text{in } \ \Omega_C \\
\varphi_C = 0 & \text{on } \ \Gamma
\end{cases}
$$

(for the interface condition (2.6)). Then, for $g_I \in L^2(\Omega_I)$ let $\varphi_I \in H^1(\Omega_I)$ be the solution of the mixed problem

$$
(3.15)\qquad
\begin{cases}
\Delta\varphi_I = \mu_{*,I}\, g_I & \text{in } \ \Omega_I \\
\varphi_I = \varphi_C & \text{on } \ \Gamma \\
\operatorname{grad} \varphi_I \cdot \mathbf{n} = 0 & \text{on } \ \partial\Omega\ .
\end{cases}
$$

Setting

$$\varphi := \begin{cases} \varphi_C & \text{in } \Omega_C \\ \varphi_I & \text{in } \Omega_I \end{cases} \quad , \tag{3.16}$$

we have $\operatorname{grad}\varphi \in W_{00}$ (respectively, $\operatorname{grad}\varphi \in W_0$) for the interface condition $\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ (respectively, for the interface condition (2.6)). Choosing the test function $\mathbf{w} = \operatorname{grad}\varphi$ in (3.11) or in (3.12) gives

$$\begin{aligned}\int_{\Omega_C} i\omega\sigma_C \mathbf{A}_C \cdot \operatorname{grad}\overline{\varphi_C} &= -\int_{\Omega_C} i\omega \operatorname{div}(\sigma_C \mathbf{A}_C)\,\overline{\varphi_C} + \int_\Gamma i\omega\sigma_C \mathbf{A}_C \cdot \mathbf{n}_C\,\overline{\varphi_C} \\ &= -\int_{\Omega_C} i\omega \operatorname{div}(\sigma_C \mathbf{A}_C)\,\overline{\varphi_C}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \mathbf{J}_e \cdot \operatorname{grad}\overline{\varphi} &= \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad}\overline{\varphi_C} + \int_{\Omega_I} \mathbf{J}_{e,I} \cdot \operatorname{grad}\overline{\varphi_I} \\ &= \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad}\overline{\varphi_C} + \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I\,\overline{\varphi_I} \ ,\end{aligned}$$

therefore

$$\begin{aligned}&\int_{\Omega_C} \mu_{*,C}^{-1}\sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C) \operatorname{div}\overline{\mathbf{q}_C} + \int_{\Omega_I} \operatorname{div}\mathbf{A}_I\,\overline{g_I} \\ &\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad}\overline{\varphi_C} + \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I\,\overline{\varphi_C} \ .\end{aligned} \tag{3.17}$$

Taking $\mathbf{q}_C = \mathbf{0}$ we have $\varphi_C = 0$ in Ω_C, hence the right hand side in (3.17) is vanishing, and we conclude that $\operatorname{div}\mathbf{A}_I = 0$ in Ω_I.

The map $\mathbf{q}_C \rightarrow \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad}\overline{\varphi_C} + \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I\,\overline{\varphi_C}$ is anti-linear and continuous with respect to the norm in $(L^2(\Omega_C))^3$. Therefore, it can be extended by density to $\mathbf{q}_C \in (L^2(\Omega_C))^3$, and then, by the Riesz theorem, represented as $\int_{\Omega_C} \mathbf{G}_C \cdot \overline{\mathbf{q}_C}$ for a suitable $\mathbf{G}_C \in (L^2(\Omega_C))^3$. In conclusion, we have $\operatorname{grad}[\mu_{*,C}^{-1}\sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C)] = -\mathbf{G}_C \in (L^2(\Omega_C))^3$ in Ω_C.

However, the most interesting property of the solution $\mathbf{A}$ to the variational problems (3.11) or (3.12) stems when the current density satisfies the assumption $\operatorname{div}\mathbf{J}_e = 0$ in Ω (namely, due to $(H2)$, $\operatorname{div}\mathbf{J}_{e,C} = 0$ in Ω_C and $\mathbf{J}_{e,C} \cdot \mathbf{n}_C + \mathbf{J}_{e,I} \cdot \mathbf{n}_I = 0$ on Γ). In this case the formulations (3.11) and (3.12) are not related to a "genuine" Lorenz gauged problem, as we find $\operatorname{div}(\sigma_C \mathbf{A}_C) = 0$ in Ω_C. In fact, by integrating by parts one easily sees that the right hand side of (3.17) is vanishing, and, repeating the arguments above by replacing $\operatorname{div}\mathbf{q}_C$ in (3.13) and (3.14) with $\mu_{*,C}\sigma_C^2 g_C$, where $g_C \in L^2(\Omega_C)$, we end up with

$$\int_{\Omega_C} \operatorname{div}(\sigma_C \mathbf{A}_C)\overline{g_C} = 0 \ ,$$

hence $\operatorname{div}(\sigma_C \mathbf{A}_C) = 0$ in Ω_C.

Furthermore, from the variational problems (3.7) (with $H^1_*(\Omega_C)$ replaced by $H^1_{**}(\Omega_C) := \{\psi_C \in H^1(\Omega_C) \mid \int_{\Omega_C} \mu_{*,C}\sigma_C^2\psi_C = 0\}$. in order to be consistent with the gauge condition (3.10)) or (3.9), it follows that $V_C = 0$ in Ω_C.

In conclusion, under the very common assumption $\operatorname{div}\mathbf{J}_e = 0$ in Ω the formulations (3.11) and (3.12) are not "genuine" Lorenz gauged formulations, since they both essentially reduce to the well-known formulation in terms of the modified magnetic vector potential $\mathbf{A}^* = i\omega^{-1}\mathbf{E}$, which, however, is more easily handled by setting $V_C = 0$ in Ω_C from the beginning.

Moreover, the results in Costabel, Dauge and Nicaise [11] show that the piecewise H^1-regularity of $\mathbf{A}$ that is required for the convergence of a finite element approximation when nodal elements with double degrees of freedom on the interfaces are used (a possibility considered in [7]) is not guaranteed except for very specific geometrical configurations.

Remark 3.1. The assumption $\operatorname{div} \mathbf{J}_e = 0$ in Ω is not needed to solve the eddy-current problem, as the necessary and sufficient condition for solving it is just $(H2)$. However, as the physically significant quantity to be found in Ω_C is $\mathbf{J}$, owing to (2.10) $\mathbf{J}_{e,C}$ is somehow arbitrary. Hence, the divergence-free condition on $\mathbf{J}_e$ is not particularly restrictive, and is very often imposed. As a matter of fact, it is automatically satisfied whenever the support of $\mathbf{J}_e$ is contained in Ω_I, and may be a convenient additional condition in more complicate situations (see, e.g., Bossavit [5], Section 5.2.1, Fig. 5.4). □

Remark 3.2. In Bossavit [7], a variant of (3.11) is also proposed, for a piecewise smooth conductivity σ_C. Denoting by Σ the interface between regions with different conductivities, the variational problem reads:

$$\begin{aligned} \mathbf{A} \in \widehat{W} \ : \ & \textstyle\int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} \\ & + \textstyle\int_{\Omega_C} \mu_{*,C}^{-1} \sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C) \operatorname{div}(\sigma_C \overline{\mathbf{w}_C}) \\ & + \textstyle\int_\Omega i\omega\sigma \mathbf{A} \cdot \overline{\mathbf{w}} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} \\ = & \textstyle\int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad \forall\, \mathbf{w} \in \widehat{W} \ , \end{aligned} \tag{3.18}$$

where

$$\widehat{W} := \{\mathbf{w} \in W_0 \,|\, \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \Gamma \cup \Sigma\} \subset W_{00} \ .$$

Since σ is piecewise smooth, the strong matching condition on $\Gamma \cup \Sigma$ gives $\widehat{W} \subset H(\operatorname{div};\Omega)$ (and therefore in finite element numerical approximations one can use standard nodal elements).

However, the fact that (3.18) is a correct formulation of the eddy-current problem is questionable. In fact, even in the simplest case of a smooth conductivity σ_C, so that $\Sigma = \emptyset$, in order to show that the distributional gradient of $\mu_{*,C}^{-1}\sigma_C^{-2} \operatorname{div}(\sigma_C \mathbf{A}_C)$ belongs to $(L^2(\Omega_C))^3$ one cannot repeat the arguments above, as the gradient of the test function φ defined in (3.16) does not satisfies $\operatorname{grad} \varphi_I \cdot \mathbf{n}_I = 0$ on Γ, therefore it does not belong to $\widehat{W}$. Similary, imposing on φ_I this Neumann condition instead of the Dirichlet one gives that the gradient of φ is an L^2-function only locally in Ω_C and Ω_I, and not globally in Ω.

The proofs reported in [7], Appendix 1 and Appendix 2, suffer of this inaccuracy. □

4. Well-posed formulations based on the Lorenz gauge

In this Section, following the approach proposed in Fernandes [13], we present some "genuine" Lorenz gauged formulations for which we are able to prove well-posedness. Let us recall that, concerning the smoothness of the conductivity, we are only assuming $\sigma_C \in L^\infty(\Omega_C)$.

We start from the unique solution $(\mathbf{H}, \mathbf{E}_C) \in H(\mathrm{rot}; \Omega) \times H(\mathrm{rot}; \Omega_C)$ of the eddy-current problem (1.1). The vector potential formulation we are considering is based on the conditions $\mathrm{rot}\, \mathbf{A} = \mu \mathbf{H}$ in Ω and $i\omega \mathbf{A}_C + \mathrm{grad}\, V_C = -\mathbf{E}_C$ in Ω_C. Gauging is just the determination of additional conditions in such a way that the vector potential $\mathbf{A}$ and the scalar potential V_C become unique.

(i) First case: $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ (condition (2.5))

Solve

$$\begin{cases} -\Delta V_C + i\omega \mu_{*,C} \sigma_C V_C = \mathrm{div}\, \mathbf{E}_C & \text{in } \Omega_C \\ \mathrm{grad}\, V_C \cdot \mathbf{n}_C = -\mathbf{E}_C \cdot \mathbf{n}_C & \text{on } \Gamma \quad , \end{cases}$$

to be intended in the following weak sense

$$\begin{aligned} V_C \in H^1(\Omega_C) \ : \ & \textstyle\int_{\Omega_C} \mathrm{grad}\, V_C \cdot \mathrm{grad}\, \overline{\psi_C} + i\omega \int_{\Omega_C} \mu_{*,C} \sigma_C V_C \overline{\psi_C} \\ & = - \textstyle\int_{\Omega_C} \mathbf{E}_C \cdot \mathrm{grad}\, \overline{\psi_C} \qquad \forall\, \psi_C \in H^1(\Omega_C) \ , \end{aligned} \tag{4.1}$$

then

$$\begin{cases} \mathrm{rot}\, \mathbf{A}_C = \mu_C \mathbf{H}_C & \text{in } \Omega_C \\ \mathrm{div}\, \mathbf{A}_C = -\mu_{*,C} \sigma_C V_C & \text{in } \Omega_C \\ \mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \quad , \end{cases} \tag{4.2}$$

and finally

$$\begin{cases} \mathrm{rot}\, \mathbf{A}_I = \mu_I \mathbf{H}_I & \text{in } \Omega_I \\ \mathrm{div}\, \mathbf{A}_I = 0 & \text{in } \Omega_I \\ \mathbf{A}_I \times \mathbf{n}_I = -\mathbf{A}_C \times \mathbf{n}_C & \text{on } \Gamma \\ \mathbf{A}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \quad . \end{cases} \tag{4.3}$$

As it is well-known, each one of the three problems above has a unique solution, provided that the following compatibility conditions are satisfied: for problem (4.2), $\mathrm{div}(\mu_C \mathbf{H}_C) = 0$ in Ω_C and $\int_{\Omega_C} \mu_{*,C} \sigma_C V_C = 0$; for problem (4.3), $\mathrm{div}(\mu_I \mathbf{H}_I) = 0$ in Ω_I and $-\mathrm{div}_\tau(\mathbf{A}_C \times \mathbf{n}_C) = \mu_I \mathbf{H}_I \cdot \mathbf{n}_I$ on Γ.

Indeed, taking $\psi_C = 1$ in (4.1) we know that V_C satisfies $\int_{\Omega_C} \mu_{*,C} \sigma_C V_C = 0$; moreover, from (1.1) we have $\mathrm{div}(\mu_C \mathbf{H}_C) = 0$ in Ω_C, $\mathrm{div}(\mu_I \mathbf{H}_I) = 0$ in Ω_I, and $\mu_I \mathbf{H}_I \cdot \mathbf{n}_I = -\mu_C \mathbf{H}_C \cdot \mathbf{n}_C$ on Γ, hence from $(4.2)_1$ $\mu_I \mathbf{H}_I \cdot \mathbf{n}_I = -\mathrm{rot}\, \mathbf{A}_C \cdot \mathbf{n}_C = -\mathrm{div}_\tau(\mathbf{A}_C \times \mathbf{n}_C)$ on Γ.

Now, we can easily check that:

Proposition 4.1. *There exists a unique solution* $(\mathbf{A}, V_C) \in H(\mathrm{rot};\Omega) \times H^1(\Omega_C)$ *to the problem*

$$
(4.4) \qquad \begin{cases}
\operatorname{rot} \mathbf{A} = \mu \mathbf{H} & \text{in } \Omega \\
i\omega \mathbf{A}_C + \operatorname{grad} V_C = -\mathbf{E}_C & \text{in } \Omega_C \\
\operatorname{div} \mathbf{A}_C + \mu_{*,C} \sigma_C V_C = 0 & \text{in } \Omega_C \\
\operatorname{div} \mathbf{A}_I = 0 & \text{in } \Omega_I \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\mathbf{A}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \quad .
\end{cases}
$$

and it is given by the solution to (4.1)–(4.3).

Proof. Concerning existence, the only point to verify is that $(4.4)_2$ is satisfied. Setting $\mathbf{Q}_C := i\omega \mathbf{A}_C + \operatorname{grad} V_C + \mathbf{E}_C$, from the Faraday equation $(1.1)_1$, (4.1) and (4.2) we have $\operatorname{rot} \mathbf{Q}_C = \mathbf{0}$ in Ω_C, $\operatorname{div} \mathbf{Q}_C = 0$ in Ω_C and $\mathbf{Q}_C \cdot \mathbf{n}_C = 0$ on Γ, therefore $\mathbf{Q}_C = \mathbf{0}$ in Ω_C.

Let us prove uniqueness. For $\mathbf{H} = \mathbf{0}$ and $\mathbf{E}_C = \mathbf{0}$, from $(4.4)_2$, $(4.4)_3$ and $(4.4)_5$ we have that V_C is the solution of

$$
\begin{cases}
-\Delta V_C + i\omega \mu_{*,C} \sigma_C V_C = 0 & \text{in } \Omega_C \\
\operatorname{grad} V_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \quad ,
\end{cases}
$$

therefore $V_C = 0$ in Ω_C. Then $(4.4)_2$ gives $\mathbf{A}_C = \mathbf{0}$, and finally, from $(4.4)_1$, $(4.4)_4$ and $(4.4)_6$, we have $\operatorname{rot} \mathbf{A}_I = \mathbf{0}$, $\operatorname{div} \mathbf{A}_I = 0$, $\mathbf{A}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$, and moreover $\mathbf{A}_I \times \mathbf{n}_I = -\mathbf{A}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ, hence $\mathbf{A}_I = \mathbf{0}$. □

As noted in Section 2, conditions $(4.4)_3$–$(4.4)_6$ are in this case the gauge conditions for the $(\mathbf{A}_C, V_C)$–$\mathbf{A}_I$ formulation.

We also have an existence and uniqueness result for the correspondent modified version of (2.9) (in Section 2 we called it (2.9)*):

Theorem 4.2. *There exists a unique solution* $(\mathbf{A}, V_C) \in H(\mathrm{rot};\Omega) \times$

$H^1(\Omega_C)$ to the Lorenz gauged problem

$$
(4.5)\qquad
\begin{cases}
\operatorname{rot}(\mu_C^{-1}\operatorname{rot}\mathbf{A}_C) + i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C\\
\operatorname{rot}(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I) = \mathbf{J}_{e,I} & \text{in } \Omega_I\\
\operatorname{div}\mathbf{A}_C + \mu_{*,C}\sigma_C V_C = 0 & \text{in } \Omega_C\\
\operatorname{div}\mathbf{A}_I = 0 & \text{in } \Omega_I\\
\mathbf{A}_I\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\\
(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I)\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\\
\mathbf{A}_C\cdot\mathbf{n}_C = 0 & \text{on } \Gamma\\
\mathbf{A}_I\times\mathbf{n}_I + \mathbf{A}_C\times\mathbf{n}_C = \mathbf{0} & \text{on } \Gamma\\
(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I)\times\mathbf{n}_I + (\mu_C^{-1}\operatorname{rot}\mathbf{A}_C)\times\mathbf{n}_C = \mathbf{0} & \text{on } \Gamma\\
\int_{\Omega_C}\mu_{*,C}\sigma_C V_C = 0\ .
\end{cases}
$$

and it is given by the solution to (4.4).

Proof. The proof of the existence is trivial, and follows from the arguments already presented in Section 2.

Concerning uniqueness, assume that $\mathbf{J}_e = \mathbf{0}$ in Ω, multiply $(4.5)_1$ by $\overline{\mathbf{A}_C}$, $(4.5)_2$ by $\overline{\mathbf{A}_I}$, integrate by parts and add the results: from the interface conditions $(4.5)_8$ and $(4.5)_9$ one obtains

$$\int_\Omega \mu^{-1}\operatorname{rot}\mathbf{A}\cdot\operatorname{rot}\overline{\mathbf{A}} + i\omega\int_{\Omega_C}\sigma_C|\mathbf{A}_C|^2 + \int_{\Omega_C}\sigma_C\operatorname{grad}V_C\cdot\overline{\mathbf{A}}_C = 0\ .$$

From the interface conditions $(4.5)_9$ one also has

$$\operatorname{rot}(\mu_C^{-1}\operatorname{rot}\mathbf{A}_C)\cdot\mathbf{n}_C = -\operatorname{rot}(\mu_I^{-1}\operatorname{rot}\mathbf{A}_I)\cdot\mathbf{n}_I = 0 \qquad \text{on } \Gamma\ ;$$

thus, multiplying $(4.5)_1$ by $(i\omega)^{-1}\operatorname{grad}\overline{V_C}$ and integrating by parts, one finds

$$\int_{\Omega_C}\sigma_C\mathbf{A}_C\cdot\operatorname{grad}\overline{V_C} + (i\omega)^{-1}\int_{\Omega_C}\sigma_C|\operatorname{grad}V_C|^2 = 0\ .$$

Therefore, $\operatorname{Re}(\int_{\Omega_C}\sigma_C\mathbf{A}_C\cdot\operatorname{grad}\overline{V_C}) = 0 = \operatorname{Re}(\int_{\Omega_C}\sigma_C\operatorname{grad}V_C\cdot\overline{\mathbf{A}}_C)$, hence $\int_\Omega\mu^{-1}\operatorname{rot}\mathbf{A}\cdot\operatorname{rot}\overline{\mathbf{A}} = 0$ and consequently $\operatorname{rot}\mathbf{A} = \mathbf{0}$ in Ω. In addition, inserting this result in $(4.5)_1$, we obtain $i\omega\sigma_C\mathbf{A}_C + \sigma_C\operatorname{grad}V_C = \mathbf{0}$ in Ω_C.

We have thus found a solution $(\mathbf{A}, V_C)$ of problem (4.4) with vanishing right hand sides, hence the uniqueness result for problem (4.4) gives $\mathbf{A} = \mathbf{0}$ in Ω and $V_C = 0$ in Ω_C. $\square$

Remark 4.3. Let us point out the final result: the unique solution $(\mathbf{A}, V_C)$ to (4.5), which is determined by the given current density $\mathbf{J}_e$ and not by the fields $\mathbf{H}$ and $\mathbf{E}_C$, as it was the case for the solution to the problems (4.1)–(4.4), is furnishing the solution to the eddy-current problem (1.1), through the definitions $\mathbf{H} := \mu^{-1} \operatorname{rot} \mathbf{A}$ and $\mathbf{E}_C := -i\omega\mathbf{A}_C - \operatorname{grad} V_C$. $\square$

(ii) Second case: $V_C = 0$ on Γ (condition (2.6))

The only modification in (4.1)–(4.3) concerns the gauge condition on Γ: this time, instead of $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ, we set $V_C = 0$ on Γ. This has as a consequence that the Neumann condition for the scalar potential V_C must be replaced by the homogeneous Dirichlet condition $V_C = 0$ (namely, in (4.1) the variational space $H^1(\Omega_C)$ must be replaced by $H_0^1(\Omega_C)$), and moreover $(4.2)_3$ has to be substituted by $\mathbf{A}_C \times \mathbf{n}_C = -(i\omega)^{-1}\mathbf{E}_C \times \mathbf{n}_C$. The existence and uniqueness of a solution V_C, $\mathbf{A}_C$ and $\mathbf{A}_I$ to the problems (4.1), (4.2) and (4.3) thus modified is again well-known, provided that the following compatibility conditions are satisfied: for problem (4.2), $\operatorname{div}(\mu_C \mathbf{H}_C) = 0$ in Ω_C and $-(i\omega)^{-1} \operatorname{div}_\tau(\mathbf{E}_C \times \mathbf{n}_C) = \mu_C \mathbf{H}_C \cdot \mathbf{n}_C$ on Γ; for problem (4.3), $\operatorname{div}(\mu_I \mathbf{H}_I) = 0$ in Ω_I and $-\operatorname{div}_\tau(\mathbf{A}_C \times \mathbf{n}_C) = \mu_I \mathbf{H}_I \cdot \mathbf{n}_I$ on Γ.

Remembering the compatibility conditions required and verified in the first case (i), we see that we have only to check the condition on Γ for problem (4.3): indeed, we have $-(i\omega)^{-1} \operatorname{div}_\tau(\mathbf{E}_C \times \mathbf{n}_C) = -(i\omega)^{-1} \operatorname{rot} \mathbf{E}_C \cdot \mathbf{n}_C = \mu_C \mathbf{H}_C \cdot \mathbf{n}_C$ on Γ, having used the Faraday equation $(1.1)_1$.

Moreover, we also have

Proposition 4.4. *There exists a unique solution* $(\mathbf{A}, V_C) \in H(\operatorname{rot}; \Omega) \times H^1(\Omega_C)$ *to the problem*

$$
\begin{cases}
\operatorname{rot} \mathbf{A} = \mu \mathbf{H} & \text{in } \Omega \\
i\omega \mathbf{A}_C + \operatorname{grad} V_C = -\mathbf{E}_C & \text{in } \Omega_C \\
\operatorname{div} \mathbf{A}_C + \mu_{*,C} \sigma_C V_C = 0 & \text{in } \Omega_C \\
\operatorname{div} \mathbf{A}_I = 0 & \text{in } \Omega_I \\
V_C = 0 & \text{on } \Gamma \\
\mathbf{A}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \ .
\end{cases}
\tag{4.6}
$$

and it is given by the solution to the modified problem (4.1)–(4.3)*, with the Neumann condition for* V_C *replaced by the homogeneous Dirichlet* $V_C = 0$ *on* Γ *and* $(4.2)_3$ *replaced by* $\mathbf{A}_C \times \mathbf{n}_C = -(i\omega)^{-1}\mathbf{E}_C \times \mathbf{n}_C$ *on* Γ.

Proof. Proceeding as in the proof of Proposition 4.1, for the existence of the solution we only need to show that $(4.6)_2$ is satisfied. Setting again $\mathbf{Q}_C := i\omega\mathbf{A}_C + \operatorname{grad} V_C + \mathbf{E}_C$, from the Faraday equation $(1.1)_1$, (4.1) with $H^1(\Omega_C)$ replaced by $H_0^1(\Omega_C)$, $(4.2)_1$ and $(4.2)_2$ we have $\operatorname{rot} \mathbf{Q}_C = \mathbf{0}$ in Ω_C and

$\operatorname{div} \mathbf{Q}_C = 0$ in Ω_C. Moreover, the modified interface conditions $V_C = 0$ and $[\mathbf{A}_C + (i\omega)^{-1}\mathbf{E}_C] \times \mathbf{n}_C = \mathbf{0}$ on Γ give $\mathbf{Q}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ, therefore $\mathbf{Q}_C = \mathbf{0}$ in Ω_C.

To prove the uniqueness it is enough to observe that the solution of

$$\begin{cases} -\Delta V_C + i\omega\mu_{*,C}\sigma_C V_C = 0 & \text{in } \Omega_C \\ V_C = 0 & \text{on } \Gamma \end{cases}$$

satisfies $V_C = 0$ in Ω_C. Then the proof follows as in Proposition 4.1. □

We also obtain at once an existence and uniqueness result for a Lorenz gauged problem similar to (4.5), where the only modifications are the substitution of the interface condition $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ with $V_C = 0$ on Γ, and the elimination of the vanishing average condition $\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$ (in the sequel, this problem will be called (4.5)*; in Section 2 it has been denoted by (2.9)**).

Remark 4.5. It can be noted that, if $\operatorname{div} \mathbf{J}_e = 0$ in Ω and $\sigma_C = \text{const}$, then either for the first case (i) or for the second case (ii) the solution V_C to (4.1) satisfies $V_C = 0$ in Ω_C and therefore $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C. In fact, we have $0 = \operatorname{div}(\sigma_C \mathbf{E}_C) + \operatorname{div} \mathbf{J}_{e,C} = \sigma_C \operatorname{div} \mathbf{E}_C$ in Ω_C, and $0 = \sigma_C \mathbf{E}_C \cdot \mathbf{n}_C + \mathbf{J}_{e,C} \cdot \mathbf{n}_C + \mathbf{J}_{e,I} \cdot \mathbf{n}_I = \sigma_C \mathbf{E}_C \cdot \mathbf{n}_C$ on Γ.

Indeed, the vector potential $\mathbf{A}$ turns out to be the solution to (3.11) (case (i)) or (3.12) (case (ii)). □

(iii) Third case: $\mathbf{A}_C \cdot \mathbf{n}_C + \mathbf{A}_I \cdot \mathbf{n}_I = 0$ on Γ (condition (2.7))

This time we assume that $\mathbf{H}$ and $\mathbf{E}$ (and not only $\mathbf{E}_C$) are the solution to the eddy-current problem, in particular we assume that the Faraday equation $\operatorname{rot} \mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0}$ is satisfied in the whole domain Ω (the existence of such a solution is proved, for instance, in Alonso Rodríguez, Fernandes and Valli [3]).

Solve

$$(4.7) \qquad \begin{cases} -\Delta V + i\omega\mu_*\sigma V = \operatorname{div} \mathbf{E} & \text{in } \Omega \\ \dfrac{\partial V}{\partial n} = -\mathbf{E} \cdot \mathbf{n} & \text{on } \partial\Omega \\ \int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0 \ , \end{cases}$$

to be intended in the weak sense made precise in Proposition 4.6, then

$$(4.8) \qquad \begin{cases} \operatorname{rot} \mathbf{A} = \mu\mathbf{H} & \text{in } \Omega \\ \operatorname{div} \mathbf{A} = -\mu_*\sigma V & \text{in } \Omega \\ \mathbf{A} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

(note that this implicitly says that $\mathbf{A}_C \cdot \mathbf{n}_C + \mathbf{A}_I \cdot \mathbf{n}_I = 0$ on Γ).

The solvability conditions for (4.8) are $\mathrm{div}(\mu\mathbf{H}) = 0$ in Ω, as usual following from the Faraday equation, and $\int_\Omega \mu_*\sigma V = 0$, namely, $\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$, that is satisfied due to $(4.7)_3$.

Hence, it remains to show that (4.7) has a unique solution.

Proposition 4.6. *There exists a unique solution of the Neumann problem* (4.7).

Proof. We start showing that the following variational problem has a unique solution: find $V \in H^1(\Omega)$ with $\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$ such that

$$\int_\Omega \mathrm{grad}\, V \cdot \mathrm{grad}\, \overline{\eta_0} + i\omega \int_\Omega \mu_*\sigma V\, \overline{\eta_0} = -\int_\Omega \mathbf{E} \cdot \mathrm{grad}\, \overline{\eta_0} \tag{4.9}$$

for all $\eta_0 \in H^1(\Omega)$ with $\int_{\Omega_C} \mu_{*,C}\sigma_C \eta_{0,C} = 0$.

The existence and uniqueness of the solution to (4.9) is a consequence of the Lax–Milgram lemma, as it is easy to prove that the Poincaré inequality holds for functions $\eta_0 \in H^1(\Omega)$ with $\int_{\Omega_C} \mu_{*,C}\sigma_C \eta_{0,C} = 0$ (one can adapt, for instance, the proof reported in Dautray and Lions [12], Volume 2, Chapter IV, Section 7, Proposition 2, where the function η_0 is assumed to satisfy $\int_\Omega \eta_0 = 0$ instead of $\int_{\Omega_C} \mu_{*,C}\sigma_C \eta_{0,C} = 0$). Taking now $\eta \in H^1(\Omega)$, we set

$$\eta_0 := \eta - \left(\int_{\Omega_C} \mu_{*,C}\sigma_C\right)^{-1} \left(\int_{\Omega_C} \mu_{*,C}\sigma_C \eta_C\right) ;$$

clearly, η_0 can be used as a test function in (4.9). Therefore we have

$$\int_\Omega \mathrm{grad}\, V \cdot \mathrm{grad}\, \overline{\eta} + i\omega \int_\Omega \mu_*\sigma V\, \overline{\eta} = -\int_\Omega \mathbf{E} \cdot \mathrm{grad}\, \overline{\eta} ,$$

as $\mathrm{grad}\, \eta_0 = \mathrm{grad}\, \eta$ and $\int_\Omega \mu_*\sigma V = \int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$.

Integrating by parts, one gets easily that $\mathrm{div}(\mathrm{grad}\, V + \mathbf{E}) = i\omega\mu_*\sigma V$ in Ω and $(\mathrm{grad}\, V + \mathbf{E}) \cdot \mathbf{n} = 0$ on $\partial\Omega$, namely, V is the solution to (4.7). □

We can thus obtain:

Proposition 4.7. *There exists a unique solution* $(\mathbf{A}, V) \in H(\mathrm{rot};\Omega) \times H^1(\Omega)$ *to the problem*

$$\begin{cases} \mathrm{rot}\, \mathbf{A} = \mu\mathbf{H} & \text{in } \Omega \\ i\omega\mathbf{A} + \mathrm{grad}\, V = -\mathbf{E} & \text{in } \Omega \\ \mathrm{div}\, \mathbf{A} + \mu_*\sigma V = 0 & \text{in } \Omega \\ \mathbf{A} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{4.10}$$

and it is given by the solution to (4.7)–(4.8).

Proof. For the existence of the solution we only need to show that $(4.10)_2$ is satisfied. Setting $\mathbf{Q} := i\omega\mathbf{A} + \operatorname{grad} V + \mathbf{E}$, from the Faraday equation, (4.7) and (4.8) we have $\operatorname{rot}\mathbf{Q} = \mathbf{0}$ in Ω, $\operatorname{div}\mathbf{Q} = 0$ in Ω and $\mathbf{Q}\cdot\mathbf{n} = 0$ on $\partial\Omega$, therefore $\mathbf{Q} = \mathbf{0}$ in Ω.

To prove the uniqueness it is enough to observe that, putting $\mathbf{H} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$ in (4.10), V satisfies

$$\begin{cases} -\Delta V + i\omega\mu_*\sigma V = 0 & \text{in } \Omega \\ \dfrac{\partial V}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0 \ , \end{cases}$$

hence $V = 0$ in Ω and $\mathbf{A} = \mathbf{0}$ in Ω. $\square$

We finally have

Theorem 4.8. *There exists a solution* $(\mathbf{A}, V) \in H(\mathrm{rot};\Omega) \times H^1(\Omega)$ *to the Lorenz gauged problem*

$$(4.11) \qquad \begin{cases} \operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{A}) + i\omega\sigma\mathbf{A} + \sigma\operatorname{grad} V = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{div}\mathbf{A} + \mu_*\sigma V = 0 & \text{in } \Omega \\ \mathbf{A}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega \\ (\mu^{-1}\operatorname{rot}\mathbf{A}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0 \ . \end{cases}$$

and it is given by the solution to (4.10)*. Moreover,* $\mathbf{A}$ *and* $V_C := V_{|\Omega_C}$ *are uniquely determined, hence they are the unique solution to the Lorenz gauged problem* (2.9)*.*

Proof. The proof of the existence follows as in Theorem 4.2. For uniqueness, as proved there, we find $\operatorname{rot}\mathbf{A} = \mathbf{0}$ in Ω and $i\omega\mathbf{A}_C + \operatorname{grad} V_C = \mathbf{0}$ in Ω_C. The irrotationality condition guarantees the existence of a function $W \in H^1(\Omega)$ such that $i\omega\mathbf{A} = -\operatorname{grad} W$ in Ω; moreover, it is not restrictive to suppose that $W_C = V_C$ in Ω_C, namely, $\sigma W = \sigma V$ in Ω. Hence we have

$$-\Delta W = i\omega \operatorname{div}\mathbf{A} = -i\omega\mu_*\sigma V = -i\omega\mu_*\sigma W \quad \text{in } \Omega$$

and $\operatorname{grad} W \cdot \mathbf{n} = -i\omega\mathbf{A}\cdot\mathbf{n} = 0$ on $\partial\Omega$, therefore W is a solution to the homogeneous Neumann problem (4.7). We thus have $W = 0$ in Ω, and consequently $\mathbf{A} = \mathbf{0}$ in Ω and $V_C = 0$ in Ω_C. $\square$

Remark 4.9. It is worthy to note that, in all cases (i), (ii) and (iii), after having solved the Lorenz gauged problems (4.5) or (4.5)* or (2.9), hence having determined $\mathbf{A}$ and V_C from the data of the problem, we are also in a condition to find the electric field $\mathbf{E}_I$ in Ω_I. In fact, first we solve the mixed problem

$$\begin{cases} -\operatorname{div}(\varepsilon_I \operatorname{grad} \psi_I) = i\omega \operatorname{div}(\varepsilon_I \mathbf{A}_I) & \text{in } \Omega_I \\ \psi_I = V_C & \text{on } \Gamma \\ \varepsilon_I \operatorname{grad} \psi_I \cdot \mathbf{n} = -i\omega\varepsilon_I \mathbf{A}_I \cdot \mathbf{n} & \text{on } \partial\Omega \end{cases}$$

(here ε_I is the dielectric coefficient, a symmetric matrix, uniformly positive definite in Ω_I, with entries in $L^\infty(\Omega_I)$; moreover, let us underline again that the problem has to be intended in the weak sense).

Then, setting $\mathbf{E}_I := -i\omega\mathbf{A}_I - \operatorname{grad} \psi_I$ in Ω_I and taking into account (2.1), it is easily checked that $\operatorname{rot} \mathbf{E}_I = -i\omega\mu_I \mathbf{H}_I$ in Ω_I, $\operatorname{div}(\varepsilon_I \mathbf{E}_I) = 0$ in Ω_I, $\mathbf{E}_I \times \mathbf{n}_I = -i\omega\mathbf{A}_I \times \mathbf{n}_I - \operatorname{grad} \psi_I \times \mathbf{n}_I = i\omega\mathbf{A}_C \times \mathbf{n}_C + \operatorname{grad} V_C \times \mathbf{n}_C = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ and $\varepsilon_I \mathbf{E}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$, therefore $\mathbf{E}_I$ is the electric field in Ω_I (see, e.g., Alonso Rodríguez, Fernandes and Valli [3]).

It can be noted that, for the case (iii), one has $\psi_I = V_{|\Omega_I}$, where V is the solution to (4.10).$\Box$

5. Variational formulations and positiveness

In order to devise a finite element approximation scheme, we are now interested in deriving the variational formulation of all the three Lorenz gauged vector potential problems we have proposed: namely, (2.9), (4.5) and (4.5)* (i.e., the one obtained from (4.5) by replacing $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ with $V_C = 0$ on Γ and eliminating the vanishing average condition $\int_{\Omega_C} \mu_{*,C}\sigma_C V_C = 0$).

We will see that, among the formulations we are going to present and analyze, there is that proposed by Bryant, Emson, Fernandes and Trowbridge [8] and Bossavit [7]. However, let us underline that here we are only assuming that the conductivity σ_C is a scalar $L^\infty(\Omega_C)$-function, uniformly positive in Ω_C.

Starting from (2.9), the usual integration by parts and the boundary and interface conditions $(2.9)_6$, $(2.9)_9$ give

$$\begin{aligned} &\textstyle\int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} + \int_{\Omega_C} (i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma_C \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\ &\qquad = \textstyle\int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad\qquad \forall\ \mathbf{w} \in H(\operatorname{rot}, \Omega)\ . \end{aligned}$$

Let us introduce the space

$$Q := \{\mathbf{w} \in H(\operatorname{rot}, \Omega) \,|\, \operatorname{div} \mathbf{w}_C \in L^2(\Omega_C), \operatorname{div} \mathbf{w}_I \in L^2(\Omega_I)\}\ ,$$

endowed with the norm

$$||\mathbf{w}||_Q^2 := \int_\Omega (|\mathbf{w}|^2 + |\operatorname{rot} \mathbf{w}|^2) + \int_{\Omega_C} |\operatorname{div} \mathbf{w}_C|^2 + \int_{\Omega_I} |\operatorname{div} \mathbf{w}_I|^2\ .$$

Due to the Lorenz gauge, one can add three other terms, finding

$$
\begin{aligned}
(5.1)\qquad & \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} \\
& \quad + \int_{\Omega_C} \mu_{*,C}^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} \\
& \quad + \int_{\Omega_C} (i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma_C \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C} + \sigma_C V_C \operatorname{div} \overline{\mathbf{w}_C}) \\
& = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \qquad \forall\, \mathbf{w} \in Q \ .
\end{aligned}
$$

On the other hand, using the Lorenz gauge equation in (3.5) and multiplying by $i\omega^{-1}$ yields

$$
\begin{aligned}
(5.2)\qquad & \int_{\Omega_C} (i\omega^{-1}\sigma_C \operatorname{grad} V_C - \sigma_C \mathbf{A}_C) \cdot \operatorname{grad} \overline{\psi_C} \\
& \quad + \int_{\Omega_C} \sigma_C (\operatorname{div} \mathbf{A}_C + \mu_{*,C}\sigma_C V_C)\, \overline{\psi_C} \\
& = i\omega^{-1} \Big(\int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{\psi_C} + \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I\, \overline{\psi_C} \Big) \quad \forall\, \psi_C \in H^1(\Omega_C) \ .
\end{aligned}
$$

Note that the same procedure can be followed also when starting from (4.5) or (4.5)*: we always obtain the problem (5.1)–(5.2).

Let us introduce the variational spaces

$$
(5.3)\qquad Q_\sharp := \begin{cases}
\{\mathbf{w} \in Q \,|\, \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \mathbf{w}_C \cdot \mathbf{n}_C = 0 \text{ on } \Gamma\} \\
\qquad\qquad\qquad\qquad\qquad \text{(case (i), problem (4.5))} \\
\{\mathbf{w} \in Q \,|\, \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\} \qquad \text{(case (ii), problem (4.5)*)} \\
\{\mathbf{w} \in Q \,|\, \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \mathbf{w}_C \cdot \mathbf{n}_C + \mathbf{w}_I \cdot \mathbf{n}_I = 0 \text{ on } \Gamma\} \\
\qquad = H(\operatorname{rot};\Omega) \cap H_0(\operatorname{div};\Omega) \qquad \text{(case (iii), problem (2.9))}
\end{cases}
$$

$$
(5.4)\qquad H_\sharp := \begin{cases}
\{\psi_C \in H^1(\Omega_C) \,|\, \int_{\Omega_C} \mu_{*,C}\sigma_C\psi_C = 0\} \\
\qquad = H^1_*(\Omega_C) \qquad\qquad \text{(case (i), problem (4.5))} \\
\{\psi_C \in H^1(\Omega_C) \,|\, \psi_C = 0 \text{ on } \Gamma\} \\
\qquad = H^1_0(\Omega_C) \qquad\qquad \text{(case (ii), problem (4.5)*)} \\
\{\psi_C \in H^1(\Omega_C) \,|\, \int_{\Omega_C} \mu_{*,C}\sigma_C\psi_C = 0\} \\
\qquad = H^1_*(\Omega_C) \qquad\qquad \text{(case (iii), problem (2.9))}\ ,
\end{cases}
$$

the sesquilinear form

$$
\begin{aligned}
(5.5)\qquad & \mathcal{B}((\mathbf{A}, V_C), (\mathbf{w}, \psi_C)) := \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} \\
& \quad + \int_{\Omega_C} \mu_{*,C}^{-1} (\operatorname{div} \mathbf{A}_C + \mu_{*,C}\sigma_C V_C)(\operatorname{div} \overline{\mathbf{w}_C} + \mu_{*,C}\sigma_C \overline{\psi_C}) \\
& \quad + i\omega^{-1} \int_{\Omega_C} \sigma_C (i\omega \mathbf{A}_C + \operatorname{grad} V_C)(-i\omega \overline{\mathbf{w}_C} + \operatorname{grad} \overline{\psi_C}) \\
& = \int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{w}} \\
& \quad + \int_{\Omega_I} \mu_{*,I}^{-1} \operatorname{div} \mathbf{A}_I \operatorname{div} \overline{\mathbf{w}_I} + \int_{\Omega_C} \mu_{*,C}^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C} \\
& \quad + \int_{\Omega_C} (i\omega\sigma_C \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma_C \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C} + \sigma_C V_C \operatorname{div} \overline{\mathbf{w}_C}) \\
& \quad + \int_{\Omega_C} (i\omega^{-1}\sigma_C \operatorname{grad} V_C \cdot \operatorname{grad} \overline{\psi_C} + \sigma_C^2 \mu_{*,C} V_C\, \overline{\psi_C}) \\
& \quad + \int_{\Omega_C} (\sigma_C \operatorname{div} \mathbf{A}_C\, \overline{\psi_C} - \sigma_C \mathbf{A}_C \cdot \operatorname{grad} \overline{\psi_C})
\end{aligned}
$$

defined (and continuous) in $Q \times H^1(\Omega_C)$, and the anti-linear functional

$$\mathcal{F}(\mathbf{w}, \psi_C) := \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} + i\omega^{-1} \left(\int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{\psi_C} + \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{\psi_C} \right)$$

defined (and continuous) in $L^2(\Omega) \times H^1(\Omega_C)$.

Theorem 5.1. *There exists a unique solution to the variational problem*

$$(\mathbf{A}, V_C) \in Q_\sharp \times H_\sharp \ : \ \mathcal{B}((\mathbf{A}, V_C), (\mathbf{w}, \psi_C)) = \mathcal{F}(\mathbf{w}, \psi_C) \quad \forall \ (\mathbf{w}, \psi_C) \in Q_\sharp \times H_\sharp \ . \tag{5.6}$$

Proof. For case (i), the existence is an easy consequence of (4.5), while for case (ii) comes from (4.5)*, and for case (iii) from (2.9).

Uniqueness follows from the fact that, in all cases, if $\mathcal{B}((\mathbf{A}, V_C), (\mathbf{w}, \psi_C)) = 0$ for each $(\mathbf{w}, \psi_C) \in Q_\sharp \times H_\sharp$, choosing $\mathbf{w} = \mathbf{A}$, $\psi_C = V_C$ one finds

$$\int_\Omega \mu^{-1} \operatorname{rot} \mathbf{A} \cdot \operatorname{rot} \overline{\mathbf{A}} + \int_{\Omega_I} \mu_{*,I}^{-1} |\operatorname{div} \mathbf{A}_I|^2 + \int_{\Omega_C} \mu_{*,C}^{-1} |\operatorname{div} \mathbf{A}_C + \mu_{*,C} \sigma_C V_C|^2 + i\omega^{-1} \int_{\Omega_C} \sigma_C |i\omega \mathbf{A}_C + \operatorname{grad} V_C|^2 = 0 \ .$$

Therefore, for case (i) one has obtained a solution to the homogeneous problem (4.5), while for case (ii) of the homogeneous problem (4.5)*, and for case (iii) of the homogeneous problem (2.9). Since all these problems have a unique solution (see Theorem 4.2, or the sentences after Proposition 4.4, or Theorem 4.8), the thesis follows. □

For numerical approximation, it is useful to check that the sesquilinear form $\mathcal{B}(\cdot, \cdot)$ is coercive in $Q_\sharp \times H_\sharp$. We will succeed in proving this result for the spaces $Q_\sharp$ and $H_\sharp$ associated to the cases (i) and (iii) in (5.3)–(5.4) (namely, not for the interface condition $\psi_C = 0$ on Γ).

In the sequel, therefore, we are always considering the spaces $Q_\sharp$ and $H_\sharp$ associated to the cases (i) and (iii) in (5.3)–(5.4). The following Poincaré-type inequalities will be useful:

$$\int_\Omega |\operatorname{rot} \mathbf{w}|^2 + \int_{\Omega_I} |\operatorname{div} \mathbf{w}_I|^2 + \int_{\Omega_C} |\operatorname{div} \mathbf{w}_C|^2 \geq \kappa_1 \int_\Omega |\mathbf{w}|^2 \qquad \forall \ \mathbf{w} \in Q_\sharp \tag{5.7}$$

$$\int_{\Omega_C} |\operatorname{grad} \psi_C|^2 \geq \kappa_2 \int_{\Omega_C} |\psi_C|^2 \qquad \forall \ \psi_C \in H_\sharp \ . \tag{5.8}$$

For case (iii) in (5.3)–(5.4), the proof of (5.7) can be found, for instance, in Girault and Raviart [14], Chapter I, Lemma 3.6. Instead, for case (i) in (5.3)–(5.4), the integral $\int_{\Omega_C} |\mathbf{w}_C|^2$ can be estimated by means of the same lemma applied to Ω_C, whereas the integral $\int_{\Omega_I} |\mathbf{w}_I|^2$ can be estimated as follows:

$$\begin{aligned} \int_{\Omega_I} |\mathbf{w}_I|^2 &\leq c_1 \Big(\int_{\Omega_I} |\operatorname{rot} \mathbf{w}_I|^2 + \int_{\Omega_I} |\operatorname{div} \mathbf{w}_I|^2 + ||\mathbf{w}_I \times \mathbf{n}_I||^2_{Y_\Gamma} \Big) \\ &= c_1 \Big(\int_{\Omega_I} |\operatorname{rot} \mathbf{w}_I|^2 + \int_{\Omega_I} |\operatorname{div} \mathbf{w}_I|^2 + ||\mathbf{w}_C \times \mathbf{n}_C||^2_{Y_\Gamma} \Big) \\ &\leq c_2 \Big(\int_{\Omega_I} |\operatorname{rot} \mathbf{w}_I|^2 + \int_{\Omega_I} |\operatorname{div} \mathbf{w}_I|^2 + \int_{\Omega_C} |\mathbf{w}_C|^2 + \int_{\Omega_C} |\operatorname{rot} \mathbf{w}_C|^2 \Big) \ , \end{aligned}$$

where Y_Γ is the space of tangential traces on Γ of $H(\mathrm{rot};\Omega_I)$ and $H(\mathrm{rot};\Omega_C)$ (see, e.g., Alonso and Valli [1, 2]).

On the other hand, for both cases (i) and (iii) in (5.3)–(5.4), estimate (5.8) follows easily by adapting the proof presented, for instance, in Dautray and Lions [12], Volume 2, Chapter IV, Section 7, Proposition 2, where the function ψ_C is assumed to satisfy $\int_{\Omega_C} \psi_C = 0$ instead of $\int_{\Omega_C} \mu_{*,C}\sigma_C\psi_C = 0$.

Proposition 5.2. *Let us consider cases* (i) *and* (iii) *in* (5.3)–(5.4). *The sesquilinear form* $\mathcal{B}(\cdot,\cdot)$ *is coercive in* $Q_\sharp \times H_\sharp$, *provided that the maximum value* $\mu^C_{*,2}$ *of the scalar function* $\mu_{*,C}$ *is small enough.*

Proof. Since $z + \overline{z} = 2\,\mathrm{Re}\,z$ and $z - \overline{z} = 2\,i\,\mathrm{Im}\,z$, we have

$$\begin{aligned}\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C)) &= \textstyle\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{w}\cdot\mathrm{rot}\,\overline{\mathbf{w}}\\ &+ \textstyle\int_{\Omega_I}\mu^{-1}_{*,I}|\,\mathrm{div}\,\mathbf{w}_I|^2 + \int_{\Omega_C}\mu^{-1}_{*,C}|\,\mathrm{div}\,\mathbf{w}_C|^2\\ &+ \textstyle\int_{\Omega_C}(i\omega\sigma_C|\mathbf{w}_C|^2 + i\omega^{-1}\sigma_C|\,\mathrm{grad}\,\psi_C|^2 + \sigma_C^2\mu_{*,C}|\psi_C|^2)\\ &+2\,\mathrm{Re}\textstyle\int_{\Omega_C}\sigma_C\psi_C\,\mathrm{div}\,\overline{\mathbf{w}_C} + 2\,i\,\mathrm{Im}\int_{\Omega_C}\sigma_C\,\mathrm{grad}\,\psi_C\cdot\overline{\mathbf{w}_C}\ .\end{aligned}$$

On the other hand,

$$\left|2\,\mathrm{Re}\int_{\Omega_C}\sigma_C\psi_C\,\mathrm{div}\,\overline{\mathbf{w}_C}\right| \le \sigma_{\max}\left(\alpha\int_{\Omega_C}|\psi_C|^2 + \alpha^{-1}\int_{\Omega_C}|\,\mathrm{div}\,\mathbf{w}_C|^2\right)$$

and

$$\left|2\,\mathrm{Im}\int_{\Omega_C}\sigma_C\,\mathrm{grad}\,\psi_C\cdot\overline{\mathbf{w}_C}\right| \le \sigma_{\max}\left(\beta\int_{\Omega_C}|\,\mathrm{grad}\,\psi_C|^2 + \beta^{-1}\int_{\Omega_C}|\mathbf{w}_C|^2\right)\ ,$$

for each possible choice of the numbers $\alpha > 0$ and $\beta > 0$.

Let us denote by $\mu_{\max}$ the maximum eigenvalue of $\mu(\mathbf{x})$ in Ω, and by $\sigma_{\min}$ and $\sigma_{\max}$ the minimum and the maximum of $\sigma_C(\mathbf{x})$ in Ω_C, respectively. Remember also that the auxiliary function μ_* is assumed to satisfy $0 < \mu^C_{*,1} \le \mu_{*,C}(\mathbf{x}) \le \mu^C_{*,2}$ in Ω_C and $0 < \mu^I_{*,1} \le \mu_{*,I}(\mathbf{x}) \le \mu^I_{*,2}$ in Ω_I. Here, we also assume that $\mu^C_{*,2} \le \mu^I_{*,2}$. Using the inequalities $a^2 + b^2 \ge a^2 + \gamma b^2$ for each

$\gamma \in (0,1]$ and $(c+d)^2 \geq c^2/2 - d^2$, we have

$$
\begin{aligned}
&|\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C))|^2 \\
&\quad = (\mathrm{Re}\,\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C)))^2 + (\mathrm{Im}\,\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C)))^2 \\
&\quad \geq (\mathrm{Re}\,\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C)))^2 + \gamma(\mathrm{Im}\,\mathcal{B}((\mathbf{w},\psi_C),(\mathbf{w},\psi_C)))^2 \\
&\quad = \Big(\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{w}\cdot\mathrm{rot}\,\overline{\mathbf{w}} + \int_{\Omega_I} \mu_{*,I}^{-1}|\,\mathrm{div}\,\mathbf{w}_I|^2 + \int_{\Omega_C} \mu_{*,C}^{-1}|\,\mathrm{div}\,\mathbf{w}_C|^2 \\
&\qquad\qquad + \int_{\Omega_C} \sigma_C^2 \mu_{*,C}|\psi_C|^2 + 2\,\mathrm{Re}\int_{\Omega_C}\sigma_C\psi_C\,\mathrm{div}\,\overline{\mathbf{w}_C}\Big)^2 \\
&\qquad +\gamma\Big(\omega^{-1}\int_{\Omega_C}(\omega^2\sigma_C|\mathbf{w}_C|^2 + \sigma_C|\,\mathrm{grad}\,\psi_C|^2) \\
&\qquad\qquad +2\,\mathrm{Im}\int_{\Omega_C}\sigma_C\,\mathrm{grad}\,\psi_C\cdot\overline{\mathbf{w}_C}\Big)^2 \\
&\quad \geq \tfrac{1}{8\max(\mu_{\max}^2,(\mu_{*,2}^I)^2)}\Big(\int_\Omega|\,\mathrm{rot}\,\mathbf{w}|^2 + \int_{\Omega_I}|\,\mathrm{div}\,\mathbf{w}_I|^2 + \int_{\Omega_C}|\,\mathrm{div}\,\mathbf{w}_C|^2\Big)^2 \\
&\qquad +\tfrac{1}{8(\mu_{*,2}^C)^2}\Big(\int_{\Omega_C}|\,\mathrm{div}\,\mathbf{w}_C|^2\Big)^2 + \gamma\tfrac{\sigma_{\min}^2}{2\omega^2}\Big(\int_{\Omega_C}|\,\mathrm{grad}\,\psi_C|^2\Big)^2 \\
&\qquad -2\sigma_{\max}^2\alpha^2\Big(\int_{\Omega_C}|\psi_C|^2\Big)^2 - 2\sigma_{\max}^2\alpha^{-2}\Big(\int_{\Omega_C}|\,\mathrm{div}\,\mathbf{w}_C|^2\Big)^2 \\
&\qquad -2\gamma\sigma_{\max}^2\beta^2\Big(\int_{\Omega_C}|\,\mathrm{grad}\,\psi_C|^2\Big)^2 - 2\gamma\sigma_{\max}^2\beta^{-2}\Big(\int_{\Omega_C}|\mathbf{w}_C|^2\Big)^2\ ,
\end{aligned}
$$

having split the term $\int_{\Omega_C}\mu_{*,C}^{-1}|\,\mathrm{div}\,\mathbf{w}_C|^2$ as $(\frac{1}{2}+\frac{1}{2})\int_{\Omega_C}\mu_{*,C}^{-1}|\,\mathrm{div}\,\mathbf{w}_C|^2$, and having dropped the non-negative terms $\int_{\Omega_C}\sigma_C^2\mu_{*,C}|\psi_C|^2$ and $\int_{\Omega_C}\omega^2\sigma_C|\mathbf{w}_C|^2$.

The negative term in $\mathrm{grad}\,\psi_C$ can be controlled by choosing β small enough (here, and in the sequel, this means: with respect to a certain combination of the data of the problem, $\mu_{*,2}^C$ *excluded*, that can be explicitly computed). Then the negative term in $\mathbf{w}_C$ can be controlled by using (5.7) and taking γ small enough. Further, the negative term in ψ_C can be controlled by using (5.8) and taking α small enough. Finally, the negative term in $\mathrm{div}\,\mathbf{w}_C$ can be controlled by taking $\mu_{*,2}^C$ small enough. □

Remark 5.3. We note that all the results presented in this Section still hold if, as in Bryant, Emson, Fernandes and Trowbridge [8] and Bossavit [7], the sign in front of the integral at the second line of (5.2) is the minus sign. With that choice, changing the sign to the whole equation (5.2) and choosing $\mathbf{w} = \mathbf{A}$, $\psi_C = V_C$, it turns out that the off-diagonal terms in system (5.1)–(5.2) are one the complex conjugate of the other. Instead, with the choice of the plus sign, presented above, one has renounced to this symmetry property, but has gained positivity.

When choosing the minus sign, due to the lack of positivity the proof of uniqueness in Theorem 5.1 needs some additional efforts, as firstly one has to prove that the gauge $\mathrm{div}\,\mathbf{A}_C + \mu_{*,C}\sigma_C V_C = 0$ is satisfied, and this can be done by choosing a suitable couple of test functions in (5.6) (this time, not simply $\mathbf{w} = \mathbf{A}$, $\psi_C = V_C$). Also the proof of Proposition 5.2 needs some modifications, but the result still holds unchanged.

It is also worthy to note that, when σ_C is piecewise constant, the case (iii) with the minus sign essentially reduces to the formulation proposed in [8], which, therefore, is well-posed. □

6. Numerical approximation

Assume that Ω, Ω_C and Ω_I are Lipschitz polyhedra, and that $\mathcal{T}_{I,h}$ and $\mathcal{T}_{C,h}$ are two regular families of triangulations of Ω_I and Ω_C, respectively. For the sake of simplicity, we suppose that each element K of $\mathcal{T}_{I,h}$ and $\mathcal{T}_{C,h}$ is a tetrahedron. We also assume that these triangulations match on Γ, so that they furnish a family of triangulations $\mathcal{T}_h$ of Ω.

Numerical approximation of problem (5.6) via conforming finite elements can be easily devised. In fact, it is enough to choose suitable finite element subspaces $W_h^r \subset Q_\sharp$ and $X_{C,h}^s \subset H_\sharp$, and rewrite (5.6) in $W_h^r \times X_{C,h}^s$. The uniqueness of the discrete solution follows as in Theorem 5.1 (this is not the case, however, for the modified version described in Remark 5.3); its existence is then a consequence of the uniqueness result.

Moreover, in the preceding Section we have proved that, when considering the gauge conditions described in case (i) and (iii) in Section 4 and assuming that the maximum value $\mu_{*,2}^C$ of the scalar function $\mu_{*,C}$ is small enough, the sesquilinear form $\mathcal{B}(\cdot,\cdot)$ is continuous and coercive (and this result is true also for the modified version described in Remark 5.3). Therefore, the convergence analysis is easily performed.

Let us focus only on case (iii) in (5.3)–(5.4), the case (i) being similar (but a little bit more difficult to implement, due to the constraint $\mathbf{w}_C \cdot \mathbf{n}_C = 0$ on Γ). Denoting by $\mathbf{P}_r$, $r \geq 1$, the space of polynomials of degree less than or equal to r, we choose the discrete spaces of nodal finite elements

$$W_h^r := \{\mathbf{w}_h \in (C^0(\overline{\Omega}))^3 \mid \mathbf{w}_{h|K} \in (\mathbf{P}_r)^3 \ \forall\, K \in \mathcal{T}_h \ , \ \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \ ,$$

and

$$X_{C,h}^s := \left\{\psi_{C,h} \in C^0(\overline{\Omega_C}) \mid \psi_{C,h|K} \in \mathbf{P}_s \ \forall\, K \in \mathcal{T}_{C,h} \ , \ \textstyle\int_{\Omega_C} \mu_{*,C}\sigma_C \psi_{C,h} = 0\right\}$$

(for implementation, one could replace the average condition $\int_{\Omega_C} \mu_{*,C}\sigma_C\psi_{C,h} = 0$ with $\psi_{C,h}(\mathbf{x}_0) = 0$ for a point $\mathbf{x}_0 \in \Omega_C$).

Via Céa lemma for each $\mathbf{w}_h \in W_h^r$ and $\psi_{C,h} \in X_{C,h}^s$ we have

$$\begin{aligned}
&\Big(\textstyle\int_\Omega (|\mathbf{A} - \mathbf{A}_h|^2 + |\operatorname{rot}(\mathbf{A} - \mathbf{A}_h)|^2 + |\operatorname{div}(\mathbf{A} - \mathbf{A}_h)|^2) \\
&\qquad\qquad + \textstyle\int_{\Omega_C} (|V_C - V_{C,h}|^2 + |\operatorname{grad}(V_C - V_{C,h})|^2)\Big)^{1/2} \\
&\quad \leq \tfrac{C_0}{\kappa_0}\Big(\textstyle\int_\Omega (|\mathbf{A} - \mathbf{v}_h|^2 + |\operatorname{rot}(\mathbf{A} - \mathbf{v}_h)|^2 + |\operatorname{div}(\mathbf{A} - \mathbf{v}_h)|^2) \\
&\qquad\qquad + \textstyle\int_{\Omega_C} (|V_C - \phi_{C,h}|^2 + |\operatorname{grad}(V_C - \phi_{C,h})|^2)\Big)^{1/2} \ ,
\end{aligned}$$

where $\kappa_0 > 0$ and $C_0 > 0$ are the coerciveness and the continuity constant of $\mathcal{B}(\cdot,\cdot)$, respectively. Therefore, provided that Ω has not reentrant corners or edges and the solutions $\mathbf{A}$ and V_C are regular enough, by means of well-known interpolation results we find the error estimate

$$
\begin{aligned}
(6.1)\qquad & \Big(\int_\Omega (|\mathbf{A}-\mathbf{A}_h|^2 + |\operatorname{rot}(\mathbf{A}-\mathbf{A}_h)|^2 + |\operatorname{div}(\mathbf{A}-\mathbf{A}_h)|^2) \\
& \quad + \int_{\Omega_C} (|V_C - V_{C,h}|^2 + |\operatorname{grad}(V_C - V_{C,h})|^2) \Big)^{1/2} \leq C h^{\min(r,s)} \ .
\end{aligned}
$$

Remark 6.1. It is worthy to note that the regularity of $\mathbf{A}$ is not assured if Ω has reentrant corners or edges (see Costabel and Dauge [10]). More important, in that case the space $H^1_\tau(\Omega) := (H^1(\Omega))^3 \cap H_0(\operatorname{div};\Omega)$ turns out to be a proper *closed* subspace of $H(\operatorname{rot};\Omega) \cap H_0(\operatorname{div};\Omega)$ (the two spaces coincide if and only if Ω is convex). Hence the discrete solution $\mathbf{A}_h \in W^r_h \subset H^1_\tau(\Omega)$ cannot approach an exact solution $\mathbf{A} \in H(\operatorname{rot};\Omega) \cap H_0(\operatorname{div};\Omega)$ with $\mathbf{A} \notin H^1_\tau(\Omega)$, and convergence is lost.

However, the assumption that Ω is convex is not a severe restriction, since in most of the real-life applications $\partial\Omega$ arises from a somehow arbitrary truncation of the whole space. Hence, reentrant corners and edges of Ω can be easily avoided.

We note instead that in case (i), owing to the condition $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ, also reentrant corners and edges of Ω_C disrupt the convergence. This is a real limitation, as non-convex conductors may occur.

Finally, even if Ω has not reentrant corners, the speed of convergence in (6.1) anyway depends on the smoothness of $\mathbf{A}$ and V_C: it should also be noted that the smoothness of $\mathbf{A}$ cannot be high, as, due to the particular structure of the Lorenz gauge, $\operatorname{div}\mathbf{A}$ has a jump on Γ. $\Box$

Remark 6.2. In order to reduce the number of degrees of freedom, the magnetic scalar potential ψ such that $\mathbf{H} = -\operatorname{grad}\psi$ is very often used (see, e.g., [8]) in the more external part of Ω_I. In that case, the condition $\mathbf{A}\cdot\mathbf{n} = 0$ is imposed on the interface between the regions $\Omega_{\mathbf{A}} \supset \Omega_C$ and $\Omega_\psi \subset \Omega_I$, where $\mathbf{A}$ and ψ are used, respectively. If this is done for case (iii), the region whose convexity is required, in order to assure convergence, is then $\Omega_{\mathbf{A}}$, which, again, can be rather arbitrarily chosen. $\Box$

Remark 6.3. In Jin [15], Chapter 5, Section 5.7.4, it is underlined that a finite element approximation based on a variational form in which the terms $\int_{\Omega_I} \mu^{-1}_{*,I} \operatorname{div}\mathbf{A}_I \operatorname{div}\overline{\mathbf{w}_I} + \int_{\Omega_C} \mu^{-1}_{*,C} \operatorname{div}\mathbf{A}_C \operatorname{div}\overline{\mathbf{w}_C}$ are present can be not efficient if the coefficients $\mu_{*,C}$ and $\mu_{*,I}$ have jumps. In this respect, it should be noted that in (5.5) $\mu_{*,C}$ and $\mu_{*,I}$ are auxiliary functions, and are not required to be equal to the physical magnetic permeability μ. Therefore, jumps can be avoided, choosing μ_* as smooth as one likes. $\Box$

7. Conclusions

Which type of conclusion can we reach from all this? The Lorenz gauge $\operatorname{div}\mathbf{A}_C + \mu_{*,C}\sigma_C V_C = 0$ in Ω_C has been originally proposed with the aim of decoupling the equation for $\mathbf{A}$ from the equation for V_C, substituting $\sigma_C \operatorname{grad} V_C$ with $-\sigma_C \operatorname{grad}(\mu_{*,C}^{-1}\sigma_C^{-1} \operatorname{div} \mathbf{A}_C)$ (in particular, an additional interesting feature of this approach is that, for a constant σ_C and a constant $\mu_{*,C} = \mu_C$, the latter term simplifies to $-\mu_C^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C$, which, added to $\mu_C^{-1} \operatorname{rot} \operatorname{rot} \mathbf{A}_C$, gives at last $-\mu_C^{-1}\Delta\mathbf{A}_C$).

However, this decoupling is difficult to handle for a non-constant conductivity σ_C, as one arrives to a problem which looks hard to solve. To overcome this difficulty, a modified Lorenz gauge (that reduces to the usual one for $\sigma_C = \text{const}$) has been proposed by Bossavit [7]: in this case, decoupling is achieved, and the resulting problem seems easier to tackle. However in Section 3 we show that, under the very common assumption $\operatorname{div} \mathbf{J}_e = 0$ in Ω, this approach leads indeed to $\operatorname{div}(\sigma_C \mathbf{A}_C) = 0$ and $V_C = 0$ in Ω_C, therefore to a (decoupled) modified vector potential formulation in Ω_C (essentially, an electric field formulation). For a constant (or smooth) conductivity σ_C this can be an interesting approach: however, not a Lorenz gauged formulation. On the other hand, for a non-smooth σ_C the implementation suffers the need of imposing the matching condition $[\sigma_C \mathbf{w}_C \cdot \mathbf{n}_C] = 0$, which is not straightforward on the interelements where σ_C is jumping. Moreover, the convergence of nodal finite element approximations is assured only for very particular geometrical configurations of the interfaces.

The results in Section 4 show that "genuine" Lorenz gauged formulations can indeed be introduced, even for a general non-smooth conductivity, but no decoupling of $\mathbf{A}$ and V_C is now present (it was already shown in Bryant, Emson, Fernandes and Trowbridge [8], [9] that, in spite of the original purpose of the Lorenz gauge formulations, $\mathbf{A}$ and V_C are coupled in the most appealing versions of them). In particular, we present three approaches, different one from the other for the interface gauge condition on Γ (cases (i), (ii) and (iii)).

Concerning nodal finite element numerical approximation, we have seen that, for cases (i) and (iii), a quasi-optimal error estimate holds, provided the parameter $\mu_{*,C}$ is small enough. Since for case (i) the implementation requires that the finite elements satisfy the interface constraint $\mathbf{w}_C \cdot \mathbf{n}_C = 0$ on Γ, which instead is not present in case (iii), and convergence is assured only if the conductor Ω_C is convex, we can conclude that, among the three we have proposed, case (iii) is the most suitable Lorenz gauged vector potential formulation to use.

Finally, though the smallness assumption on $\mu_{*,C}$ is not particularly restrictive, because μ_* is not the physical magnetic permeability μ, but an *auxiliary function* that we have inserted into the problem, we conclude pointing out that it could be interesting to find an approximation via nodal finite elements which is at the same time convergent and free of any restriction on the parameters of the problem.

Acknowledgement. The authors wish to thank Ana Alonso Rodríguez for many useful conversations about the subject of this paper.

References

[1] A. Alonso and A. Valli, *Some remarks on the characterization of the space of tan-*

gential traces of $H(\mathrm{rot};\Omega)$ and the construction of an extension operator, Manuscripta Mathematica, 89 (1996), pp. 159–178.

[2] ———, *A domain decomposition approach for heterogeneous time-harmonic Maxwell equations*, Comp. Meth. Appl. Mech. Engr., 143 (1997), pp. 97–112.

[3] A. Alonso Rodríguez, P. Fernandes, and A. Valli, *Weak and strong formulations for the time-harmonic eddy-current problem in general multi-connected domains*, European J. Appl. Math., 14 (2003), pp. 387–406.

[4] O. Bíró and A. Valli, *The Coulomb gauged vector potential formulation for the edd-current problem in general geometry: well-posedness and numerical approximation*, Research Report UTM 682, Department of Mathematics, University of Trento, May 2005. Submitted to Computer Methods in Applied Mechanics and Engineering.

[5] A. Bossavit, *Électromagnétisme, en Vue de la Modélisation*, Springer, Paris, 1993.

[6] ———, *Computational Electromagnetism. Variational Formulation, Complementarity, Edge Elements*, Academic Press, San Diego, 1998.

[7] ———, *On the Lorenz gauge*, COMPEL, 18 (1999), pp. 323–336.

[8] C. Bryant, C. Emson, P. Fernandes, and C. Trowbridge, *Lorentz gauge eddy current formulations for multiply connected piecewise homogeneous conductors*, COMPEL, 17 (1998), pp. 732–740.

[9] ———, *Lorentz gauge formulations for eddy current problems involving piecewise homogeneous conductors*, IEEE Trans. Magn., 34 (1998), pp. 2559–2562.

[10] M. Costabel and M. Dauge, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Ration. Mech. Anal., 151 (2000), pp. 221–276.

[11] M. Costabel, M. Dauge, and S. Nicaise, *Singularities of Maxwell interface problems*, Math. Model. Numer. Anal., 33 (1999), pp. 627–649.

[12] R. Dautray and J.-L. Lions, *Mathematical Analysis and NumericalMethods for Science and Technology. Volume 2: Functional and Variational Methods*, Springer, Berlin, 1992.

[13] P. Fernandes, *General approach to prove the existence and uniqueness of the solution in vector potential formulations of 3-d eddy current problems*, IEE Proc.-Sci. Meas. Technol., 142 (1995), pp. 299–306.

[14] V. Girault and P. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer, Berlin, 1986.

[15] J. Jin, *The Finite Element Method in Electromagnetics*, 2nd edition, John Wiley & Sons, New York, 2002.