NONCOMMUTATIVE CAUCHY INTEGRAL FORMULA

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Abstract. The aim of this paper is to provide and prove the most general Cauchy integral formula for slice regular functions on a real alternative *-algebras, which represent a generalization of the classical concept of holomorphic function of a complex variable in the noncommutative and nonassociative settings.

1. Introduction

One of the main tasks in noncommutative complex analysis is the determination of the class of functions admitting a local power series expansion at every point of their domain of definition.

Let $\mathbb{A}$ denote the noncommutative structure we are working with: it may be, for instance, the skew field $\mathbb{H}$ of quaternions, the nonassociative division algebra $\mathbb{O}$ of octonions, the Clifford algebras $\mathbb{R}_{p,q}$, or any real alternative *-algebra. Then the noncommutative setting requires a distinction between polynomials (and series) with left and right coefficients in $\mathbb{A}$. Indeed if we consider for instance polynomials with coefficients on the right of the indeterminate x (the left case yields an analogous theory), then it is well known that the proper way to perform the multiplication consists in imposing commutativity of x with the coefficients (cf. [19]). Thus if $p(x) = \sum_n x^n c_n$ and $q(x) = \sum_n x^n d_n$, then their product is defined by

$$(p * q)(x) := \sum_n x^n \left(\sum_{k+h=n} c_k d_h \right). \tag{1.1}$$

Note that this product is different from the pointwise product of p and q. This happens even when one of the two polynomials is constant: indeed if $p(x) = c_0$ the pointwise product is $p(x)q(x) = \sum_n c_0(x^n d_n)$, while $(p * q)(x) = \sum_n x^n(c_0 d_n)$. In the case in which $q(x) = d_0$ and $p(x) = \sum_n x^n d_n$ then $p(x)q(x) = \sum_n (x^n c_n) d_0$ and $(p * q)(x) = \sum_n x^n(c_n d_0)$: nonassociativity also plays a role.

The problem of the power series representation was solved in the quaternionic case in [8]: the class of functions admitting a power series expansion is given by the *slice regular* functions. The theory of slice regularity on the quaternionic space was introduced in [10, 11, 9] and then it was extended to Clifford algebras and octonions in [5, 12], and to any real alternative *-algebras in [14, 15].

The notion of slice regular function is a generalization of the classical concept of holomorphic function of a complex variable. Let us briefly describe this notion in the simpler case in which $\mathbb{A}$ is $\mathbb{H}$ or $\mathbb{O}$. Let $\mathbb{S}$ be the subset of square roots of -1 and, for each $J \in \mathbb{S}$, let $\mathbb{C}_J$ be the plane generated by 1 and J. Observe that each $\mathbb{C}_J$ is a copy the complex plane. The quaternions and the octonions have a "slice complex" nature, described by the following two properties: $\mathbb{A} = \bigcup_{J \in \mathbb{S}} \mathbb{C}_J$ and $\mathbb{C}_J \cap \mathbb{C}_K = \mathbb{R}$ for every $J, K \in \mathbb{S}$ with $J \neq \pm K$. Let D be a open subset of $\mathbb{C}$ invariant under complex conjugation and let Ω_D be the open subset of $\mathbb{A}$ obtained by rotating D around $\mathbb{R}$, i.e. $\Omega_D = \bigcup_{J \in \mathbb{S}} D_J$, where $D_J := \{\rho + \sigma J \in \mathbb{C}_J \ : \ \rho, \sigma \in \mathbb{R}, \rho + \sigma i \in D\}$. A function $f : \Omega_D \longrightarrow \mathbb{A}$ of class C^1 is called *slice regular* if, for every $J \in \mathbb{S}$, its restriction f_J to

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D_J is holomorphic with respect to the complex structures on D_J and on $\mathbb{A}$ defined by the left multiplication by J, i.e. if $\partial f_J/\partial\rho + J\,\partial f_J/\partial\sigma = 0$ on D_J. The precise definition of slice regular function in the most general setting of real alternative *-algebras is recalled in Section 2 below.

One of the main achievement of the theory of slice regular functions is a Cauchy-type integral formula (see [11, 1, 15, 7, 2]), which has many consequences also in noncommutative functional analysis (cf. [4, 6, 3, 13, 18]). Let us show it again in the case $\mathbb{A} = \mathbb{H}$ or $\mathbb{A} = \mathbb{O}$. If D is bounded and its boundary is piecewise of class C^1, and f is C^1 and extends to a continuous function on the closure of Ω_D in $\mathbb{A}$, then for every $J \in \mathbb{S}$ it holds:

$$f(x) = \frac{1}{2\pi}\int_{\partial D_J} C_y(x)\, J^{-1}\,\mathrm{d}y\ f(y) - \frac{1}{2\pi}\int_{D_J} C_y(x)\, J^{-1}\,\mathrm{d}\overline{y}\wedge\mathrm{d}y\ \frac{\partial f}{\partial \overline{y}}(y) \qquad \forall x \in D_J, \tag{1.2}$$

where the function $C_y : \Omega_D \longrightarrow \mathbb{A}$ denotes the *(noncommutative) Cauchy kernel* defined by

$$C_y(x) := (x^2 - 2\,\mathrm{Re}(y)x + |y|^2)^{-1}(\overline{y} - x), \qquad x \in \Omega_D.$$

The two integrals in (1.2) are defined in a natural way:

$$\int_{\partial D_J} C_y(x)\, J^{-1}\,\mathrm{d}y\ f(y) := \int_0^1 C_{\alpha(t)}(x)\, J^{-1}\alpha'(t)\ f(\alpha(t))\,\mathrm{d}t$$

and

$$\int_{D_J} C_y(x)\, J^{-1}\,\mathrm{d}y^c\wedge\mathrm{d}y\ \frac{\partial f}{\partial \overline{y}}(y) := 2\int_{D_J} C_{\rho+\sigma J}(x)\,\frac{\partial f}{\partial \overline{y}}(\rho+\sigma J)\,\mathrm{d}\rho\,\mathrm{d}\sigma,$$

$\alpha : [0,1] \longrightarrow \mathbb{C}_J$ being a Jordan curve parametrizing ∂D_J, and (ρ,σ) being the real coordinates in $\mathbb{C}_J$. As usual $\partial f/\partial\overline{y} := \frac{1}{2}(\partial f_J/\partial\rho + J\partial f_J/\partial\sigma)$ and the fact that the differential $\mathrm{d}y$ appears on the left of $f(y)$ depends on the noncommutativity of $\mathbb{A}$. Notice that, if x and y belong to the same $\mathbb{C}_J$, and hence commute, then it turns out that $C_y(x) = (y-x)^{-1}$ and we find again the form of the classical Cauchy formula for holomorphic functions.

A drawback of formula (1.2) is that it is not a representation formula: indeed in the nonassociative case it holds only for $x \in D_J \subseteq \mathbb{C}_J$ and not on the whole domain Ω_D. The aim of the present paper is to find a Cauchy integral formula, proved in Theorem 3.1 for general real alternative *-algebras $\mathbb{A}$, allowing to represent the values $f(x)$ when x belongs to the whole domain of f. In order to do this, we exploit the notion of *slice product* between two slice regular functions f and g, which is recalled in Definition 2.4 below and will be denoted simply by $f \cdot g$. This product is the natural generalization to functions of the product (1.1) of polynomials and allows us to provide the following Cauchy integral representation formula:

$$f(x) = \frac{1}{2\pi}\int_{\partial D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}y\ f(y)\right)\right](x) - \frac{1}{2\pi}\int_{D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}\overline{y}\wedge\mathrm{d}y\ \frac{\partial f}{\partial\overline{y}}(y)\right)\right](x) \tag{1.3}$$

holding for every $J \in \mathbb{S}$ and *for every* $x \in \Omega_D$, where the parentheses are omitted in the term $J^{-1}\,\mathrm{d}y\,f(y)$, because this product is proved to be associative. Observe that the slice product in the integrand function of (1.3) is computed in the variable x between the function C_y and the constant function $J^{-1}\,\mathrm{d}y\,f(y)$, y being the fixed integration variable. Therefore, denoting by $\cdot_x$ the slice product performed with respect to the variable x, we can rewrite formula (1.3) in the following more explicit way:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\partial D_J} C_y(x)\cdot_x \left(J^{-1}\,\mathrm{d}y\ f(y)\right) - \frac{1}{2\pi}\int_{D_J} C_y(x)\cdot_x \left(J^{-1}\,\mathrm{d}\overline{y}\wedge\mathrm{d}y\ \frac{\partial f}{\partial\overline{y}}(y)\right) \\ &:= \frac{1}{2\pi}\int_0^1 \left[C_{\alpha(t)} \cdot \left(J^{-1}\alpha'(t)\ f(\alpha(t))\right)\right](x)\,\mathrm{d}t - \frac{1}{\pi}\int_{D_J}\left[C_{\rho+\sigma J}\cdot \frac{\partial f}{\partial\overline{y}}(\rho+\sigma J)\right](x)\ \mathrm{d}\rho\,\mathrm{d}\sigma \end{aligned}$$

for each $x \in \Omega_D$.

We observe that the Cauchy formula for slice regular functions (Corollary 3.1) can be applied to obtain in a straightforward way the series expansion at y of a slice regular function f with respect to slice powers $g_n(x) = (x-y)^{\cdot n}$ or to spherical polynomials $g_n(x) = \mathscr{S}_{y,n}(x)$. This result was achieved in [17] by a different method. The Cauchy formula (1.3) shows that, in order to cover also the nonassociative case, in the expansion $f(x) = \sum_n g_n(x) \cdot a_n$, it is necessary to consider the coefficients $a_n \in \mathbb{A}$ as constant functions and to take their slice product with the functions g_n. In the associative case, this product coincides with the pointwise product with the coefficient a_n. Finally, we remark that the proof of Cauchy formula (1.3) we will give in Section 3 is new also in the associative case.

2. Preliminaries

2.1. **Real alternative *-algebras.** Let us assume that

$$\mathbb{A} \text{ is a finite dimensional real } \textit{alternative algebra} \text{ with unity,} \tag{2.1}$$

i.e. $\mathbb{A}$ is a finite dimensional real algebra with unity $1_{\mathbb{A}}$ such that the mapping

$$(x, y, z) \longmapsto (xy)z - x(yz) \quad \text{is alternating.} \tag{2.2}$$

Observe that we are not assuming that $\mathbb{A}$ is associative, nevertheless by Artin's Theorem (cf. [20]), condition (2.2) implies that

$$\text{the subalgebra generated by two elements of } \mathbb{A} \text{ is associative.} \tag{2.3}$$

Here we assume that the real dimension of $\mathbb{A}$ is strictly greater than 1:

$$\dim_{\mathbb{R}}(\mathbb{A}) > 1$$

so that $\mathbb{A} \neq \{0\}$, i.e. $1_{\mathbb{A}} \neq 0$. A consequence of the bilinearity of the product in $\mathbb{A}$ is the formula

$$r(xy) = (rx)y = x(ry) \qquad \forall r \in \mathbb{R}, \quad \forall x, y \in \mathbb{A}. \tag{2.4}$$

Therefore if we identify $\mathbb{R}$ with the subalgebra generated by $1_{\mathbb{A}}$, the notation rx is not ambiguous if $r \in \mathbb{R}$ and $x \in \mathbb{A}$. Notice that

$$rx = xr \qquad \forall r \in \mathbb{R}, \quad \forall x \in \mathbb{A}. \tag{2.5}$$

We also assume that $\mathbb{A}$ is a **-algebra*, that is

$$\mathbb{A} \text{ is endowed with a } \textit{*-involution } \mathbb{A} \longrightarrow \mathbb{A} : x \longmapsto x^c, \tag{2.6}$$

i.e. a real linear mapping such that

$$\begin{aligned} &(x^c)^c = x \qquad \forall x \in \mathbb{A}, \\ &(xy)^c = y^c x^c \qquad \forall x, y \in \mathbb{A}, \\ &x^c = x \qquad \forall x \in \mathbb{R}. \end{aligned} \tag{2.7}$$

Summarizing the previous assumptions (2.1), (2.2), and (2.6), we say that

$$\mathbb{A} \text{ is a } \textit{finite dimensional real alternative *-algebra with unity.} \tag{2.8}$$

We will assume (2.8) in the remainder of the paper. We will endow $\mathbb{A}$ with the topology induced by any norm on it as a real vector space.

2.2. **The quadratic cone.**

Definition 2.1. *The* trace $t(x)$ *and the* squared norm $n(x)$ *of any* $x \in \mathbb{A}$ *are defined as follows*

$$t(x) := x + x^c, \quad n(x) := xx^c, \qquad x \in \mathbb{A}.$$

Moreover, we define $Q_{\mathbb{A}}$*, the* quadratic cone of $\mathbb{A}$*, and the set* $\mathbb{S}_{\mathbb{A}}$ *of* square roots of -1 *by:*

$$Q_{\mathbb{A}} := \mathbb{R} \cup \{x \in \mathbb{A} \ : \ t(x) \in \mathbb{R}, \ n(x) \in \mathbb{R}, \ t(x)^2 - 4n(x) < 0\},$$

$$\mathbb{S}_{\mathbb{A}} := \{J \in Q_{\mathbb{A}} \ : \ J^2 = -1\}.$$

For each $J \in \mathbb{S}_{\mathbb{A}}$*, we denote by* $\mathbb{C}_J := \langle 1, J \rangle$ *the subalgebra of* $\mathbb{A}$ *generated by* J*. Finally, the* real part $\mathrm{Re}(x)$ *and the* imaginary part $\mathrm{Im}(x)$ *of an element* x *of* $Q_{\mathbb{A}}$ *are given by*

$$\mathrm{Re}(x) := (x + x^c)/2, \quad \mathrm{Im}(x) := (x - x^c)/2, \qquad x \in Q_{\mathbb{A}}.$$

Since $\mathbb{A}$ is assumed to be alternative, one can prove (cf. [15, Proposition 3]) that the quadratic cone $Q_{\mathbb{A}}$ has the following two properties, which describe its "slice complex" nature:

$$Q_{\mathbb{A}} = \bigcup_{J \in \mathbb{S}_{\mathbb{A}}} \mathbb{C}_J, \tag{2.9}$$

$$\mathbb{C}_J \cap \mathbb{C}_K = \mathbb{R} \qquad \forall J, K \in \mathbb{S}_{\mathbb{A}}, \ J \neq \pm K. \tag{2.10}$$

Two easy consequences of (2.9) are the following:

$$\exists x^{-1} = n(x)^{-1} x^c \qquad \forall x \in Q_{\mathbb{A}} \setminus \{0\}, \tag{2.11}$$

$$x^n \in Q_{\mathbb{A}} \qquad \forall x \in Q_{\mathbb{A}}, \quad \forall n \in \mathbb{N}.$$

Observe that if $J \in \mathbb{S}_{\mathbb{A}}$ is fixed then

$$x = \alpha + \beta J \in \mathbb{C}_J, \ \alpha, \beta \in \mathbb{R} \quad \Longrightarrow \quad x^c = \alpha - \beta J.$$

In the following lemma we introduce a useful linear isomorphism between $\mathbb{C}$ and the subalgebra $\mathbb{C}_J$.

Lemma 2.1. *If* $J \in \mathbb{S}_{\mathbb{A}}$*, then the mapping* $\phi_J : \mathbb{C} \longrightarrow \mathbb{C}_J$ *defined by*

$$\phi_J(r + si) := r + sJ, \qquad r, s \in \mathbb{R},$$

is a complex algebra isomorphism, i.e. $\phi_J(1) = 1$ *and*

$$\phi_J(z_1 + z_2) = \phi_J(z_1) + \phi_J(z_2), \quad \phi_J(z_1 z_2) = \phi_J(z_1)\phi_J(z_2) \qquad \forall z_1, z_2 \in \mathbb{C}. \tag{2.12}$$

Moreover the product $\mathbb{C} \times \mathbb{A} \longrightarrow \mathbb{A} : (z, x) \longmapsto zx$ *defined by*

$$zx := \phi_J(z)x, \qquad z \in \mathbb{C}, \ x \in \mathbb{A}, \tag{2.13}$$

makes $\mathbb{A}$ *a complex vector space that will be denoted by* $\mathbb{A}_J$*.*

Proof. The fact that ϕ_J is an isomorphism is an easy consequence of (2.4) and (2.5). In order to prove that (2.13) makes $\mathbb{A}$ a complex vector space, we need to invoke Artin's theorem (2.3): indeed if $x \in \mathbb{A}$, then $J(Jx) = (JJ)x = -x$, and this implies, together with a straightforward calculation, that $z_1(z_2 x) = (z_1 z_2)x$ for every $z_1, z_2 \in \mathbb{C}$. The remaining axioms are trivially satisfied. □

From Lemma 2.1 it follows that $\phi_J(z^{-1}) = (\phi_J(z))^{-1}$ for any $z \neq 0$, and the product of $\mathbb{A}$ is commutative and associative in $\mathbb{C}_J$:

$$x_1 x_2 = x_2 x_1, \quad x_1(x_2 x_3) = (x_1 x_2)x_3 \qquad \forall x_1, x_2, x_3 \in \mathbb{C}_J,$$

thus the parentheses can be omitted.

2.3. **Complexification of $\mathbb{A}$.** Consider now the complexification of $\mathbb{A}$, i.e. the real vector space given by the tensor product

$$\mathbb{A}_{\mathbb{C}} := \mathbb{A} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{A}^2,$$

that can be described by setting $1 := (1,0) \in \mathbb{A}^2$ and $\mathbf{i} := (0,1) \in \mathbb{A}^2$, so that every $v = (x,y) \in \mathbb{A}^2$ can be uniquely written in the form $v = x1 + y\mathbf{i} = x + y\mathbf{i}$, and $\mathbf{i}$ is an *imaginary unit*. Thus the sum in $\mathbb{A}_{\mathbb{C}}$ reads

$$(x + y\mathbf{i}) + (x' + y'\mathbf{i}) = (x + x') + (y + y')\mathbf{i}.$$

and the product defined by

$$(x + y\mathbf{i})(x' + y'\mathbf{i}) := (xx' - yy) + (xy' + yy')\mathbf{i},$$

makes $\mathbb{A}_{\mathbb{C}}$ a complex alternative algebra as well, therefore $\mathbb{A}_{\mathbb{C}} = \mathbb{A} + \mathbb{A}\mathbf{i} = \{w = x + y\mathbf{i} \,:\, x, y \in \mathbb{A}\}$ and $\mathbf{i}^2 = -1$. The *complex conjugation* of $v = x + y\mathbf{i} \in \mathbb{A}_{\mathbb{C}}$ is defined by $\bar{v} := x - y\mathbf{i}$. We will also consider $\mathbb{A}_{\mathbb{C}}$ as a complex vector space, indeed one can easily infer the following lemma.

Lemma 2.2. *The product $\mathbb{C} \times \mathbb{A}_{\mathbb{C}} \longrightarrow \mathbb{A}_{\mathbb{C}} : (z, v) \longmapsto zv$:*

$$(r + si)(x + y\mathbf{i}) := (rx - sy) + (ry + sx)\mathbf{i} \tag{2.14}$$

for $z = r + si$, $v = x + y\mathbf{i}$, $r, s \in \mathbb{R}$, $x, y \in \mathbb{A}$, makes $\mathbb{A}_{\mathbb{C}}$ a complex vector space.

Lemma 2.3. *If $J \in \mathbb{S}_{\mathbb{A}}$, then the mapping $\Phi_J : \mathbb{A}_{\mathbb{C}} \longrightarrow \mathbb{A}$ defined by*

$$\Phi_J(a + b\mathbf{i}) := a + Jb, \qquad a, b \in \mathbb{A}.$$

is a continuous complex vector space linear map when $\mathbb{A}_{\mathbb{C}}$ and $\mathbb{A}$ are endowed with the complex structures defined by (2.14) *and* (2.13)*, respectively.*

Proof. Let $z = r + si \in \mathbb{C}$, $r, s \in \mathbb{R}$, and $v = x + y\mathbf{i} \in \mathbb{A}_{\mathbb{C}}$, $x, y \in \mathbb{A}$. Recalling definitions (2.14) and (2.13), and using (2.4), (2.5), and Artin's theorem (2.3), we get

$$\begin{aligned} z\Phi_J(v) &= (r + sJ)(x + Jy) \\ &= rx + r(Jy) + (sJ)x + (sJ)(Jy) \\ &= rx + J(ry + sx) + s((JJ)y) \\ &= (rx - sy) + J(ry + sx) \\ &= \Phi_J((rx - sy) + (ry + sx)\mathbf{i}) \\ &= \Phi_J((r + si)(x + y\mathbf{i})) = \Phi_J(zv). \end{aligned}$$

Thus Φ_J is homogeneous. The additivity and the continuity are clear. □

2.4. **Left slice functions.** We are now in position to recall the notion of slice function.

Let D be a subset of $\mathbb{C}$, invariant under the complex conjugation $z = r + si \longmapsto \overline{z} = r - si$, $r, s \in \mathbb{R}$. Define

$$\Omega_D := \{r + sJ \in Q_{\mathbb{A}} \,:\, r, s \in \mathbb{R}, r + si \in D, J \in \mathbb{S}_{\mathbb{A}}\}.$$

A subset of $Q_{\mathbb{A}}$ is said to be *circular* if it is equal to Ω_D for some set D as above.

Suppose now that D is open in $\mathbb{C}$, not necessarily connected. Thanks to (2.9) and (2.10), Ω_D is a relatively open subset of $Q_{\mathbb{A}}$.

Definition 2.2. *A function $F = F_1 + F_2\mathbf{i} : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ is called* stem function *if $F(\bar{z}) = \overline{F(z)}$ for every $z \in D$. The stem function $F = F_1 + F_2\mathbf{i}$ on D induces a* left slice function *$\mathcal{I}(F) : \Omega_D \longrightarrow \mathbb{A}$ on Ω_D as follows. Let $x \in \Omega_D$. By* (2.9)*, there exist $r, s \in \mathbb{R}$ and $J \in \mathbb{S}_{\mathbb{A}}$ such that $x = r + sJ$. Then we set:*

$$\mathcal{I}(F)(x) := F_1(z) + J\,F_2(z), \quad \textit{where } z = r + si \in D. \tag{2.15}$$

The reader observes that the definition of $\mathcal{I}(F)$ is well-posed. In fact, if $x \in \Omega_D \cap \mathbb{R}$, then $\alpha = x$, $y = 0$ and J can be arbitrarily chosen in $\mathbb{S}_{\mathbb{A}}$. However, $F_2(z) = 0$ and hence $\mathcal{I}(F)(x) = F_1(x)$, independently from the choice of J. If $x \in \Omega_{\mathbb{A}} \setminus \mathbb{R}$, then x has the following two expressions: $x = r + sJ = r + (-s)(-J)$, where $r = \mathrm{Re}(x)$, $s = |\operatorname{Im}(x)|$ and $J = \operatorname{Im}(x)/|\operatorname{Im}(x)|$. Anyway, if $z := r + si$, we have: $\mathcal{I}(F)(r + (-s)(-J)) = F_1(\overline{z}) + (-J)F_2(\overline{z}) = F_1(z) + (-J)(-F_2(z)) = F_1(z) + J\,F_2(z) = \mathcal{I}(F)(r + sJ)$.

It is important to observe that every left slice function $f : \Omega_D \longrightarrow \mathbb{A}$ is induced by a unique stem function $F = F_1 + F_2\mathbf{i}$. In fact, it is easy to verify that, if $x = r + sJ \in \Omega_D$ and $z = r + si \in D$, then

$$F_1(z) = (f(x) + f(x^c))/2 \quad \text{and} \quad F_2(z) = -J\,(f(x) - f(x^c))/2. \tag{2.16}$$

The proof of the following proposition can be found in [15, Proposition 5].

Proposition 2.1. *Every left slice function is uniquely determined by its values on a plane $\mathbb{C}_J$.*

Let us introduce a relevant subclass of left slice functions.

Definition 2.3. *Let $F = F_1 + F_2\mathbf{i} : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ be a stem function on D. The left slice function $f = \mathcal{I}(F)$ induced by F is said to be* real *if F_1 and F_2 are real-valued.*

One can prove the following (cf. [15, Proposition 10])

Proposition 2.2. *A left slice function f is real if and only if $f(\Omega_D \cap \mathbb{C}_J) \subseteq \mathbb{C}_J$ for every $J \in \mathbb{S}_{\mathbb{A}}$.*

In general, the pointwise product of slice functions is not a slice function. However, if $F = F_1 + F_2\mathbf{i}$ and $G = G_1 + G_2\mathbf{i}$ are stem functions, then it is immediate to see that their pointwise product

$$FG = (F_1G_1 - F_2G_2) + (F_1G_2 + F_2G_1)\mathbf{i}$$

is again a stem function. In this way, we give the following definition.

Definition 2.4. *Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ be two left slice functions on Ω_D. We define the* slice product *$f \cdot g$ as the left slice function $\mathcal{I}(FG)$ on Ω_D.*

It is easy to prove the following lemma.

Lemma 2.4. *Let F, G, H be stem functions such that $f = \mathcal{I}(F)$, $g = \mathcal{I}(G)$, and $h = \mathcal{I}(H)$. If f is real then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ and $f \cdot g = fg$.*

In the situation of previous Lemma 2.4, we will omit the parentheses.

In the remainder of the paper, a constant function will be denote by its value: if $f(x) = a \in \mathbb{A}$ for every $x \in \Omega_D$, we will write $f = a$.

2.5. **Slice regular functions.** Our next aim is to recall the concept of left slice regular function, which generalizes the notion of holomorphic function from $\mathbb{C}$ to any real alternative *-algebra like $\mathbb{A}$.

Let $F : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ be a stem function with components $F_1, F_2 : D \longrightarrow \mathbb{A}$. Since we endow $\mathbb{A}$ with the topology induced by any norm on it as a finite dimensional real vector space, if $z = r + si$, $r, s \in \mathbb{R}$, denotes the complex variable in $\mathbb{C}$, it makes sense to consider the partial derivatives $\partial F/\partial r$, $\partial F/\partial s\,\mathbf{i}$, which are also stem functions.

Definition 2.5. *Let $F = F_1 + F_2\mathbf{i} : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ be a stem function belonging to $C^1(D; \mathbb{A}_{\mathbb{C}})$ (i.e. $F_1, F_2 \in C^1(D; \mathbb{A})$). Let us denote by $z = r + si$, $r, s \in \mathbb{R}$, the complex variable in $\mathbb{C}$. We define the continuous stem functions $\partial F/\partial z : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ and $\partial F/\partial \overline{z} : D \longrightarrow \mathbb{A}_{\mathbb{C}}$ by*

$$\frac{\partial F}{\partial z} := \frac{1}{2}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\mathbf{i}\right), \qquad \frac{\partial F}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\mathbf{i}\right).$$

If $f = \mathcal{I}(F) : \Omega_D \longrightarrow \mathbb{A}$ is a left slice function we define the continuous slice functions

$$\frac{\partial f}{\partial x} := \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \qquad \frac{\partial f}{\partial x^c} := \mathcal{I}\left(\frac{\partial F}{\partial \overline{z}}\right).$$

We say that $f = \mathcal{I}(F)$ is left slice regular *if $\partial f/\partial x^c = 0$, i.e. if*

$$\frac{\partial F_1}{\partial r} = \frac{\partial F_2}{\partial s} \quad \text{and} \quad \frac{\partial F_1}{\partial s} = -\frac{\partial F_2}{\partial r}.$$

2.6. **The characteristic polynomial and the Cauchy kernel.** Significant examples of slice regular functions are the *polynomials with right coefficients in* $\mathbb{A}$, i.e. functions $p : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$ of the form $p(x) = \sum_{k=0}^{n} x^k c_k$ with $c_k \in \mathbb{A}$, $n \in \mathbb{N}$. We have that $p = \mathcal{I}(P)$ where $P : \mathbb{C} \longrightarrow \mathbb{A}_{\mathbb{C}}$ is defined by $P(z) = \sum_{k=0}^{n} z^k c_k$. Given two polynomials $p(x) := \sum_{k=0}^{n} x^k c_k$ and $q(x) := \sum_{k=0}^{m} x^k d_k$, their *star product* $p * q : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$ is defined by setting

$$(p * q)(x) := \sum_{j=0}^{n+m} x^j \bigg(\sum_{k+h=j} c_k d_h \bigg),$$

i.e. we impose the commutativity for the product of the variable x with the coefficients. Note that $p * q$ is different from the pointwise product pq. Nevertheless we have the following result (cf. [15, Proposition 12]).

Proposition 2.3. *If p and q are polynomials with right coefficients in $\mathbb{A}$, then*

$$p * q = p \cdot q,$$

i.e. the star product is equal to the slice product.

For any $y \in Q_{\mathbb{A}}$, the *characteristic polynomial of y* is the left slice regular function $\Delta_y : Q_{\mathbb{A}} \longrightarrow Q_{\mathbb{A}}$ defined by

$$\Delta_y(x) := (x - y) \cdot (x - y^c) = x^2 - xt(y) + n(y), \qquad x \in Q_{\mathbb{A}}.$$

Observe that $y \in \mathbb{C}_J$ for some $J \in \mathbb{S}_{\mathbb{A}}$, therefore the set of zeroes of Δ_y is

$$\mathbb{S}_y := \{\xi + \eta K \in Q_{\mathbb{A}} \,:\, K \in \mathbb{S}_{\mathbb{A}},\ y = \xi + \eta J\}$$

and, thanks to (2.16), $(\Delta_y(\cdot))^{-1} : Q_{\mathbb{A}} \setminus \mathbb{S}_y \longrightarrow Q_{\mathbb{A}}$ is also a left slice function. Moreover if $x, y \in Q_{\mathbb{A}}$, then $x \in \mathbb{C}_I$ for some $I \in \mathbb{S}_{\mathbb{A}}$, thus $\Delta_y(x) \in \mathbb{C}_I$ and we infer that

$$I \in \mathbb{S}_{\mathbb{A}},\ x \in \mathbb{C}_I \quad \Longrightarrow \quad \Delta_y(x) \in \mathbb{C}_I. \tag{2.17}$$

This fact, together with Proposition 2.2 and (2.11), yields

$$\Delta_y : Q_{\mathbb{A}} \longrightarrow Q_{\mathbb{A}} \quad \text{and} \quad (\Delta_y(\cdot))^{-1} : Q_{\mathbb{A}} \setminus \mathbb{S}_y \longrightarrow Q_{\mathbb{A}} \quad \text{are real.}$$

Hence we can define the left slice function $C_y : Q_{\mathbb{A}} \setminus \mathbb{S}_y \longrightarrow \mathbb{A}$ by setting

$$C_y(x) := \Delta_y(x)^{-1}(y^c - x). \tag{2.18}$$

Observe that, thanks to Lemma 2.4, the product (2.18) is a slice product in the variable x and

$$C_y(x) = (y - x)^{-1} \qquad \forall y, x \in \mathbb{C}_J,\ y \neq x,\ y \neq x^c.$$

We say that $C_y(x)$ is the *Cauchy kernel for left slice regular functions on* $\mathbb{A}$.

3. Cauchy integral formula

We are now in position to prove the general Cauchy integral representation formula for slice functions, the natural noncommutative and nonassociative generalization of the classical complex Cauchy integral formula:

$$F(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{D} \frac{(\partial F/\partial\overline{\zeta})(\zeta)}{\zeta - z}\,\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta \tag{3.1}$$

holding for $F \in C(\overline{D};\mathbb{C}) \cap C^1(D;\mathbb{C})$, where $D \subseteq \mathbb{C}$ is a bounded domain with piecewise C^1 boundary. Here $\frac{1}{2i}\,\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta$ is the 2-dimensional Lebesgue measure on $\mathbb{C}$.

Theorem 3.1 (Cauchy integral formula)**.** *Let $D \subseteq \mathbb{C}$ be a bounded domain, $J \in \mathbb{S}_{\mathbb{A}}$ and $D_J := \Omega_D \cap \mathbb{C}_J$. Let ∂D_J denote the boundary of D_J in $\mathbb{C}_J$ and assume that it is piecewise C^1. If $f = \mathcal{I}(F) : \Omega_D \longrightarrow \mathbb{A}$ is a left slice function and $F \in C(\overline{D};\mathbb{A}_{\mathbb{C}}) \cap C^1(D;\mathbb{A}_{\mathbb{C}})$, then*

$$f(x) = \frac{1}{2\pi}\int_{\partial D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}y\ f(y)\right)\right](x) - \frac{1}{2\pi}\int_{D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}y^c \wedge \mathrm{d}y\ \frac{\partial f}{\partial y^c}(y)\right)\right](x)$$

for every $x \in \Omega_D$.

Before showing the proof, some remarks are in order.

Remark 3.1.

(i) As we mentioned in the introduction, the position of the "differentials" inside integrals of $\mathbb{A}$-valued functions is important, so a rigorous definition is in order. We limit ourselves to the integrals involved in the Cauchy formula. If $a, b \in \mathbb{R}$, $a < b$, and $\alpha : [a,b] \longrightarrow \mathbb{C}_J$ is a piecewise C^1 parametrization of the (counterclockwise oriented) Jordan curve ∂D_J in the plane $\mathbb{C}_J$, then

$$\int_{\partial D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}y\ f(y)\right)\right](x) := \int_a^b \left[C_{\alpha(t)} \cdot \left(J^{-1}\alpha'(t)\ f(\alpha(t))\right)\right](x)\,\mathrm{d}t,$$

α' being the derivative of α. The second integral is simply

$$\int_{D_J}\left[C_y \cdot \left(J^{-1}\,\mathrm{d}y^c \wedge \mathrm{d}y\ \frac{\partial f}{\partial y^c}(y)\right)\right](x) := \int_{D_J}\left[C_{(\rho+\sigma J)} \cdot \left(J^{-1}\frac{\partial f}{\partial y^c}(\rho+\sigma J)\right)\right](x)\,\mathrm{d}\rho\,\mathrm{d}\sigma,$$

(ρ,σ) being the coordinates of y in $\mathbb{C}_J = \{y = \rho + \sigma J \ :\ \rho, \sigma \in \mathbb{R}\}$. Hence $\frac{J^{-1}}{2} dy^c \wedge dy$ may be considered the 2-dimensional Lebesgue measure on $\mathbb{C}_J \simeq \mathbb{R}^2$.

(ii) In the two integrand functions, the slice product $\cdot$ is computed with respect to the variable x: $J^{-1}\,\mathrm{d}y\ f(y)$ and $J^{-1}\,\mathrm{d}y^c \wedge \mathrm{d}y\ (\partial f/\partial y^c)(y)$ are here constant functions (w.r.t. x), y being the (fixed) integration variable. Using the notation $\cdot_x$ for the slice product w.r.t. x, the Cauchy formula can be written in the following way:

$$f(x) = \frac{1}{2\pi}\int_{\partial D_J} C_y(x) \cdot_x \left(J^{-1}\,\mathrm{d}y\ f(y)\right) - \frac{1}{2\pi}\int_{D_J} C_y(x) \cdot_x \left(J^{-1}\,\mathrm{d}y^c \wedge \mathrm{d}y\ \frac{\partial f}{\partial y^c}(y)\right).$$

(iii) Finally there are no parentheses in the term $J^{-1}\,\mathrm{d}y\ f(y) = J^{-1}\alpha'(t)\ f(\alpha(t))$, because it belongs to the subalgebra generated by J and $f(y)$, thus, by Artin's theorem (2.3), this product is associative.

Proof of Theorem 3.1. Let us first prove the theorem under the assumption

$$x \in D_J, \qquad z := \phi_J^{-1}(x). \tag{3.2}$$

Observe that from (2.15) we get

$$f(\phi_J(w)) = \Phi_J(F(w)) \qquad \forall w \in \overline{D}, \tag{3.3}$$

$$\frac{\partial f}{\partial y^c}(\phi_J(w)) = \Phi_J\Big(\frac{\partial F}{\partial \overline{w}}(w)\Big) \qquad \forall w \in D. \tag{3.4}$$

Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be a Jordan curve whose trace is ∂D (counterclockwise oriented) and let (ρ,σ) denote the real coordinates of $\zeta = \rho + \sigma i \in \mathbb{C}$. Since $F \in C(\overline{D};\mathbb{A}_{\mathbb{C}}) \cap C^1(D;\mathbb{A}_{\mathbb{C}})$, we can apply the classical vector complex Cauchy formula, which can be easily deduced from (3.1) by means of the Hahn-Banach theorem, or simply using coordinates. We get

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\int_{\partial D}\frac{F(\zeta)}{(\zeta - z)}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_D \frac{(\partial F/\partial\overline{\zeta})(\zeta)}{(\zeta - z)}\,\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_0^1 \gamma'(t)\frac{F(\gamma(t))}{\gamma(t) - z}\,\mathrm{d}t - \frac{1}{\pi}\int_D \frac{(\partial F/\partial\overline{\zeta})(\zeta)}{(\zeta - z)}\,\mathrm{d}\rho\,\mathrm{d}\sigma, \end{aligned}$$

where the product by a complex scalar in the integrand functions is defined by (2.14). Thus, recalling from Lemma 2.3 that $\Phi_J : \mathbb{A}_{\mathbb{C}} \longrightarrow \mathbb{A}$ is $\mathbb{C}$-linear and continuous when $\mathbb{A}_{\mathbb{C}}$ and $\mathbb{A}$ are endowed with the complex vector structures defined by (2.14) and (2.13), we get

$$\begin{aligned} \Phi_J(F(z)) &= \frac{1}{2\pi}\int_0^1 \Phi_J\left(\frac{\gamma'(t)}{(\gamma(t) - z)i}F(\gamma(t))\right)\mathrm{d}t - \frac{1}{\pi}\int_D \Phi_J\left(\frac{(\partial F/\partial\overline{\zeta})(\zeta)}{(\zeta - z)}\right)\mathrm{d}\rho\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi}\int_0^1 \frac{\gamma'(t)}{(\gamma(t) - z)i}\Phi_J(F(\gamma(t)))\,\mathrm{d}t - \frac{1}{\pi}\int_D \frac{1}{\zeta - z}\Phi_J\Big(\frac{\partial F}{\partial\overline{\zeta}}(\zeta)\Big)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi}\int_0^1 \phi_J\left(\frac{\gamma'(t)}{(\gamma(t) - z)i}\right)\Phi_J(F(\gamma(t)))\,\mathrm{d}t - \frac{1}{\pi}\int_D \phi_J\left(\frac{1}{\zeta - z}\right)\Phi_J\Big(\frac{\partial F}{\partial\overline{\zeta}}(\zeta)\Big)\,\mathrm{d}\rho\,\mathrm{d}\sigma. \end{aligned}$$

Now observe that $\gamma_J := \phi_J \circ \gamma : [0,1] \longrightarrow \mathbb{C}_J$ is a (counterclockwise oriented) parametrization of ∂D_J and that $\gamma_J' = \phi_J \circ \gamma'$. Hence, using (3.2)–(3.4), (2.12), (2.19), and Artin's theorem (2.3), we deduce that

$$\begin{aligned} f(x) &= \Phi_J(F(z)) \\ &= \frac{1}{2\pi}\int_0^1 \big((\gamma_J(t) - x)^{-1}J^{-1}\gamma_J'(t)\big)\,f(\gamma_J(t))\,\mathrm{d}t - \frac{1}{\pi}\int_{D_J}(y - x)^{-1}\frac{\partial f}{\partial y^c}(y)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi}\int_0^1 \big(C_{\gamma_J(t)}(x)J^{-1}\gamma_J'(t)\big)\,f(\gamma_J(t))\,\mathrm{d}t - \frac{1}{\pi}\int_{D_J} C_y(x)\frac{\partial f}{\partial y^c}(y)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi}\int_{\partial D_J} C_{\gamma_J(t)}(x)\,\big(J^{-1}\gamma_J'(t)f(\gamma_J(t))\big)\,\mathrm{d}t - \frac{1}{\pi}\int_{D_J} C_y(x)\frac{\partial f}{\partial y^c}(y)\,\mathrm{d}\rho\,\mathrm{d}\sigma. \end{aligned} \tag{3.5}$$

Now let us observe that if $a \in \mathbb{A}$ and $y \in D_J$, then, thanks to (2.17) and Artin's theorem (2.3), it follows that

$$(C_y \cdot a)(x) = C_y(x)a \qquad \forall x \in \mathbb{C}_J,$$

where a denotes the constant function taking on the value a. Therefore from (3.5) we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^1 \left[C_{\gamma_J(t)} \cdot \big(J^{-1}\gamma_J'(t)\ f(\gamma_J(t))\big)\right](x)\,\mathrm{d}t - \frac{1}{\pi}\int_{D_J}\Big(C_y \cdot \frac{\partial f}{\partial y^c}(y)\Big)(x)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi}\int_{\partial D_J}\left[C_y \cdot \big(J^{-1}\,\mathrm{d}y\ f(y)\big)\right](x) - \frac{1}{2\pi}\int_{D_J}\Big[C_y \cdot \Big(J^{-1}\,\mathrm{d}y^c \wedge \mathrm{d}y\ \frac{\partial f}{\partial y^c}(y)\Big)\Big](x), \end{aligned}$$

which proves the theorem in the case $x \in D_J$. In order to conclude, it is enough to invoke Proposition 2.1, since f and the function on the right hand side of the previous formula are slice functions on Ω_D. $\square$

Corollary 3.1. *Under the same assumptions of Theorem 3.1, if f is slice regular on Ω_D, then*

$$f(x) = \frac{1}{2\pi}\int_{\partial D_J} \left[C_y \cdot \left(J^{-1}\,\mathrm{d}y\ f(y)\right)\right](x) = \frac{1}{2\pi}\int_{\partial D_J} C_y(x) \cdot_x \left(J^{-1}\,\mathrm{d}y\ f(y)\right).$$

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