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Asymptotic behaviour of a class of nonlinear heat conduction problems with memory effects

Stefano BONACCORSI[a], Giuseppe DA PRATO[b], Luciano TUBARO[a]

[a] *Dipartimento di Matematica, Università di Trento, via Sommarive 14, 38123 Povo (TN), Italia*

[b] *Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, 56126 Pisa, Italia*

stefano.bonaccorsi@unitn.it, daprato@sns.it, luciano.tubaro@unitn.it

In this paper, we investigate the existence and uniqueness problem for the solutions to a class of semilinear stochastic Volterra equations which arise in the theory of heat conduction with memory effects, where the heat source depends on the solution via a dissipative term. Further, we analyse the asymptotic behaviour of the solution and we prove the existence of a ergodic invariant measure.

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1 Introduction

There is a large literature where delay equations serve as an abstract formulation of problems of heat flow in material with memory, viscoelasticity, and many other physical phenomena; in this paper, we concentrate our attention to the case where the heat flux depends on the current value *and* the temporal history of the temperature gradient.

Let us briefly explain the problem we are concened with. Let $\mathcal{O} \subset \mathbb{R}^N$ represents an N-dimensional body that we assume to be rigid, isotropic and homogeneous. We set $Y = L^2(\mathcal{O})$ and let $A : D(A) \subset Y \to Y$ be the realization of the Laplace operator with Dirichlet boundary condition. The body's temperature $u(t)$ at time t is subject to an external, randomly perturbed, heat source that is dependent on the temperature in a nonlinear way; in order to handle the contribution of temperature values taken in the past, we introduce the new variable

$$u_t(s) = u(t-s), \qquad s \geq 0.$$

In a series of papers, Clément et al. [7, 8, 9] introduced white noise perturbations in order to model the presence of rapidly varying forces. In [2, 3] this approach was extended to cover the case of non-linear equations, both for Lipschitz non-linearities and for dissipative ones. However, this last result requires a special form of the equation, as well as stronger assumptions on the scalar kernel, with respect to what we impose in this paper. Instead, here we are able to get some a-priori estimates on the solution, which allows to treat directly the dissipative nonlinearity.

We are concerned with the following class of integral Volterra equations perturbed by a additive Wiener noise

$$\mathrm{d}u(t) = \left[k_0 A u(t) - \int_0^\infty k(s)\, A u_t(s)\,\mathrm{d}s + g(u(t))\right]\mathrm{d}t + Q\,\mathrm{d}W(t) \quad t \in [0,T]. \quad (1.1)$$

W is a cylindrical Wiener noise and we assume that g is a nonlinear, dissipative operator defined on a subset of the Hilbert space Y.

It is convenient to reduce problem (1.1) to an abstract Cauchy problem on an appropriate product space, which contains the whole history of the solution, where the system has a suitable representation as evolutionary dynamical system. On this space, we can characterize the generation properties of the leading (matrix) operator and prove that, in our setting, this operator is strictly dissipative. Similar approaches, which dates back at least to Miller [18], are widely used in the literature: see [14, Section VI.6] for further references. Let us denote $\phi(t) = (u(t), u_t(\cdot))$ and $\bar\phi = (\bar u, \bar u(\cdot))$; then, setting

$$\mathcal{A}(\theta,\eta) = (k_0\, A\theta - \int_0^\infty k(s) A\eta(s)\,\mathrm{d}s, -\partial_s \eta), \quad (1.2)$$

we can write problem (1.1) in the form

$$\mathrm{d}\phi(t) = [\mathcal{A}\phi(t) + (f(t), 0)]\,\mathrm{d}t + (Q\,\mathrm{d}W(t), 0) \quad t \in [0,T]. \quad (1.3)$$

with initial condition $\phi(0) = \bar\phi$ on an Hilbert space $\mathcal{H}$.

The *linear* dynamics of the system is well described by the semigroup $\mathcal{T}(t) = e^{t\,\mathcal{A}}$ generated by the operator $\mathcal{A}$. Moreover, the representation of the solution by means of the semigroup $e^{t\mathcal{A}}$ allows a precise characterization of the (unique) invariant measure for the stochastic system, as well as the convergence of the solution, seen as a vector of the state $u(t)$ *and* the history u_t.

Our main result states the existence of a unique mild solution for problem (1.3). However, we cannot handle the dissipative terms only within the semigroup setting. It happens, actually, that the dissipativity of the leading operator $\mathcal{A}$ only holds in L^2 setting; therefore, we cannot apply the standard theory of dissipative stochastic systems as in [13]. The way to circumvent this problem is to analize directly the first component of the system (1.3), i.e., give estimates starting from the equation (1.1).

1.1 Invariant measures

In the last part of the paper we study the asymptotic behaviour of the solutions of (1.1). For this, we employ the dissipativity of $\mathcal{A}$ and of the reaction-diffusion term g. Then, in particular by using the general theory of Da Prato & Zabczyk [12, Section 11.2], we obtain in Corollary 3.5 the characterization of invariant measures for equation (1.1) in the linear case $f \equiv 0$. In particular, we prove that there exists a unique invariant measure $\eta \sim \mathcal{N}(0, Q_\infty)$ for equation (1.1) when $g \equiv 0$, that is a centered Gaussian measure on the space $\mathcal{H}$; further, the system is ergodic and strongly mixing.

We shall also consider the problem of determining existence and uniqueness of the invariant measure concerning problem (1.1) when $g \neq 0$. We can exploit the dissipativity of the operator $\mathcal{A}$ proved in Theorem 3.1, as well as that of g, which is stated in hypothesis 2.2, in order to get the result (see Theorem 5.1 for the details): there exists a unique invariant measure π for equation (1.3) and also in this case the system is ergodic and strongly mixing.

We next consider the projections $\eta_s = \pi_s \# \eta$ and $\eta_h = \pi_h \# \eta$ (here, the subscript s and h stand for state and history, respectively); η_s and η_h are the first and second marginal of η and they are probability measures on Y and $L^2_\rho(\mathbb{R}_+; D(B))$, respectively. Let us take $\bar{\phi} = \binom{x}{y} \in L^2(\Omega; \mathcal{F}_0; \mathcal{H})$ be a random variable with the law η. Then, if we consider problem (1.1), we get that $u(t)$ is a stationary solution, with law $\mathcal{L}(u(t)) = \mathcal{L}(x) = \eta_s$.

The papers in the literature concerning stochastic functional evolution equations (which, in particular, includes the case of stochastic evolution equations with variable and distributed delays) usually discuss stability results for the solution. We refer, for instance, to [5, 6] where exponential and almost sure stability of stochastic partial functional differential equations are obtained by an argument based on Lyapunov functions.

As compared with our result, we see that the results in the quoted papers (see also [4]) do not allow delay in the highest order term, nor they discuss the existence of a stationary solution.

Delay equations can be considered within the framework of Volterra equations by considering the initial history as an inhomogeneity of the evolution system. For a complete account on this approach we refer to the monograph [21]. Stochastic Volterra equations in infinite dimensions was studied in a series of papers by Clément and Da Prato (see for instance [9] and the references therein). The authors discuss the existence of *stationary* solutions to equations with stochastic additive perturbations.

Further, let us consider the process $(u_t)_{t \geq 0}$. As opposite to the solution process $(u(t))_{t \geq 0}$, the history process is a Markov process. Concerning delay equation in the finite dimensional setting, both for the finite or the infinite delay case, various authors (see for instance [1, 19]) address their interest on this process, since it is a Markov and, under certain assumptions, also a Feller process. In this context, they prove also the existence of a stationary solution and discuss mixing properties of the system: see also the recent paper [15]. Notice that the same results are obtained in our setting for an infinite dimensional system (with memory entering in the system in the highest order term).

2 Main setting

Constant k_0 and the kernel $k : \mathbb{R}_- \to \mathbb{R}$ have an important role in the theory of thermal viscoelasticity, as they represent the *instantaneous conductivity* and the *heat flux memory kernel*, respectively. Concerning the sign of the heat flux memory term, the literature is quite controversial. In this paper, we consider the case $a(t) = k_0 - \int_0^t k(s)\,\mathrm{d}s$, where k is a positive function. The sign is therefore taken in accordance with Clément and Nohel [10], where the choice is made in order to handle with completely monotone kernels. On the other hand, in Nunziato [20] the kernel is taken to be $a(t) = k_0 + \int_0^t k(s)\,\mathrm{d}s$ that is nondecreasing and positive. Positivity of the kernel $a(t)$ is also necessary in our construction and will be imposed in the following assumption.

Hypothesis 2.1 $\{k(t),\ t \ge 0\}$ *is a completely monotone kernel, integrable on* $\mathbb{R}_+$, *which further verifies*

1. *there exists* $\eta > 0$ *such that* $k(t) + \frac{1}{\eta}\,k'(t) \le 0$;
2. *the instantaneous conductivity is larger than the heat flux memory*

$$k_0 > \int_{\mathbb{R}_+} k(s)\,\mathrm{d}s. \tag{2.1}$$

The physical interpretation of (2.1) becomes clear if one considers the static problem associated with (1.1). In fact, a static solution at equilibrium, subject to the exogenous heat supply $r(x)$, will satisfy the equation

$$\left(k_0 - \int_{\mathbb{R}_+} k(s)\,\mathrm{d}s\right) Au(x) = r(x)$$

and equation (2.1) simply sates that the *equilibrium conductivity* is positive (compare with the discussion in [21, Section 5.3] or the assumption (iii) in [8, Hypothesis 2.1]).

The nonlinear part of the heat supply: $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function on $\mathbb{R}$ which shows some dissipativeness, even if it can exhibit an antidissipative behaviour for low temperatures.

Hypothesis 2.2 *We assume that* $g(u) = \lambda u - p(u)$, *where* $\lambda \ge 0$ *and* $p(u)$ *is a strictly decreasing polynomial of order* $2d + 1$.

It follows that there exists positive constants c_j *such that*

(i) $|g(u)| \le c_1\,(1 + |u|^{2d+1})$
(ii) $u \cdot g(u) \le c_2 - c_3\,|u|^{2d+2}$
(iii) $g'(u) \le c_4$.

2.1 The construction of the state space

We let $Y = L^2(\mathcal{O})$ be the space of square integrable, real valued functions defined on $\mathcal{O}$ with scalar product $\langle u, v\rangle = \int_{\mathcal{O}} u(\xi)v(\xi)\,\mathrm{d}\xi$.

Sobolev spaces $H^1(\mathcal{O})$ and $H^2(\mathcal{O})$ are the spaces of functions whose first (resp. first and second) derivative are in $L^2(\mathcal{O})$. We set moreover $H^1_0(\mathcal{O})$ the subspace of $H^1(\mathcal{O})$ of functions which vanish (a.e.) on the boundary $\partial\mathcal{O}$.

We let $X = H^{-1}(\mathcal{O})$ the topological dual of $H^1_0(\mathcal{O})$. The Laplace operator Δ can be defined as an isomorphism of $H^1_0(\mathcal{O})$ onto $H^{-1}(\mathcal{O})$ and we shall consider the space $H^{-1}(\mathcal{O})$ endowed with the inner product

$$\langle u, v\rangle_X = \langle \Delta^{-1} u, v\rangle, \qquad u, v \in H^{-1}(\mathcal{O}).$$

We shall denote the Laplace operator Δ by $A : D(A) \subset X \to X$:

$$\begin{aligned} &D(A) = H^1_0(\mathcal{O}), \\ &Au = \Delta u \quad \text{for each } u \in D(A). \end{aligned} \tag{2.2}$$

We recall some well known facts from operator theory. The operator A defined in (2.2) is the generator of a C_0-semigroup of contractions $(e^{t\,A})_{t\geq 0}$; since A is self adjoint, the semigroup is analytic: see for instance [22, Theorem 1.5.7, Corollary 7.1.1]. If $\mathcal{O}$ is bounded with C^1 boundary, the injections $D(A) \subset L^2(\mathcal{O}) \subset X$ are compact.

We notice that there exists an orthonormal basis in X of eigenvalues of A, that we denote $\{e_k\}$, associated to eigenvalues $\{-\mu_k\}$ with $0 < \mu_1 < \mu_2 < \dots$. It shall be noticed that the spectrum of A in X is the same as that of the Laplace operator Δ endowed with Dirichlet boundary condition on $Y = L^2(\mathcal{O})$, compare [14, Proposition IV.2.17].

In order to control the unbounded delay interval, we shall consider L^2 weighted spaces. Let $\rho(t) = \int_t^\infty (\sigma)\,\mathrm{d}\sigma$. Then we set $\mathcal{X} = L^2_\rho(\mathbb{R}_+; H^1_0(\mathcal{O}))$ be the space of functions $y : \mathbb{R}_+ \to D(A) = H^1_0(\mathcal{O})$ endowed with the inner product

$$\langle y_1, y_2\rangle_{\mathcal{X}} = \int_0^\infty \rho(\sigma)\,\langle \nabla y_1(\sigma), \nabla y_2(\sigma)\rangle\,\mathrm{d}\sigma$$

and $\|y\|_{\mathcal{X}}$ the corresponding norm. On this space, we introduce the delay operator Φ with domain $D(\Phi) = \mathcal{X}$ by setting

$$\Phi y = \int_0^\infty k(\sigma) A y(\sigma)\,\mathrm{d}\sigma.$$

Finally, we define the Hilbert space $\mathcal{H} = Y \times L^2_\rho(\mathbb{R}_+; D(A))$ endowed with the *energy norm* $\|\phi\|^2_{\mathcal{H}} = \|x\|^2_Y + \|y\|^2_{\mathcal{X}}$, $\phi = \binom{x}{y} \in \mathcal{H}$.

2.2 The spatial regularity of the eigenvectors of A

In 1 space dimension, the eigenvalues and eigevfunctions of A can be explicitly computed; in particular, it holds that the asymptotic behaviour of the eigenvalues is $\mu_k \sim k^2$, while the eigenfunctions are uniformly bounded: there exists a constant M such that $|e_k(\xi)| \leq M$ for $\xi \in \mathcal{O}$.

However, the extension to the n dimensional case is not harmless. Actually, we have the following theorem by Daniel Grieser [16], which shows that in a general eigenvalue problem, the eigenfunctions are not uniformly bounded.

let $\mathcal{O}$ be a smooth, compact Riemannian manifold of dimension $n \geq 2$, with smooth boundary $\partial\mathcal{O}$. Let Δ denote the Laplace-Beltrami operator on functions on $\mathcal{O}$. Consider a solution of the eigenvalue problem, with Dirichlet or Neumann boundary conditions,

$$(\Delta + \lambda^2)u = 0, \quad u_{|\partial\mathcal{O}} = 0 \quad \text{or} \quad \partial_n u_{|\partial\mathcal{O}} = 0 \tag{2.3}$$

(∂_n denotes the normal derivate) normalized by the condition

$$\|u\|_{L^2(\mathcal{O})} = 1.$$

Theorem 2.3 *Let M be a n-dimensional compact Riemannian manifold with boundary. There is a constant $M = M(\mathcal{O})$ such that any solution of (2.3) satisfies*

$$\|u\|_\infty \le M\, \lambda^{\frac{n-1}{2}}. \tag{2.4}$$

In the case of dimension 1, we have that the eigenfunctions are uniformly bounded (see for instance [17]). In general, we shall use the following notation: for a constant $M > 0$ and (increasing) coefficients c_k it holds

$$|e_k(\xi)| \le M\, c_k, \qquad |\nabla e_k(\xi)| \le M\, \mu_k^{1/2}\, c_k, \qquad \xi \in \mathcal{O},\ k \in \mathbb{N}. \tag{2.5}$$

2.3 The semigroup setting

On the Hilbert space $\mathcal{H}$ then we can consider the linear operator $\mathcal{A}$ defined in (1.2) with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y(\cdot) \end{pmatrix} \in D(A) \times W^{1,2}_\rho(\mathbb{R}_-; D(A)) \mid y(0) = x,\ \Phi y + Ax \in Y \right\}.$$

Existence and uniqueness for the linear problem

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} u(t) &= k_0\, Au(t) - \int_0^\infty k(s) A u_t(s)\, \mathrm{d}s \\ \begin{pmatrix} u(0) \\ u_0(\cdot) \end{pmatrix} &= \begin{pmatrix} \bar u \\ \bar u_0(\cdot) \end{pmatrix} \in \mathcal{H} \end{aligned} \tag{2.6}$$

follows if we prove that the operator $\mathcal{A}$ generates a linear strongly continuous semigroup of contractions on $\mathcal{H}$. It deserves some attention the property that the operator $\mathcal{A}$ is, further, strictly dissipative on $\mathcal{H}$, see Theorem 3.1 below.

Theorem 2.4 *Problem (2.6) generates a strongly continuous semigroup of contractions $(\mathcal{T}(t))_{t\ge 0}$*

$$\mathcal{T}(t) \begin{pmatrix} \bar u \\ \bar u_0 \end{pmatrix} = \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{H}.$$

We shall prove this result, in Section 3.1, by an application of Lumer-Phillips's theorem, once we complete the following steps:

(i) the operator $\mathcal{A}$ is dissipative: there exists $\delta > 0$ such that for any $\phi = \binom{x}{y} \in D(\mathcal{A})$ it holds

$$\langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} \le -\delta\, \|\phi\|^2_{\mathcal{H}};$$

(ii) the resolvent $R(\mu; \mathcal{A})$ is surjective for some (and hence for all) $\mu > 0$.

2.4 The resolvent family

In the approach of [8], the resolvent family is used to provide the existence and uniqueness of the solution, on the one hand, and the regularity of the stochastic convolution process on the other. We refer to the monograph [21] for further details.

Definition 2.5 *A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in X is called a resolvent for the equation*

$$\dot{u} = Au - k * Au \tag{2.7}$$

if the following conditions are satisfied:

(S1) $S(0) = I$ and, for all $x \in X$, $t \to S(t)x$ is continuous on $\mathbb{R}_+$;

(S2) $S(t)$ commutes with A, that is for a.e. $t \geq 0$, $S(t)D(A) \subset D(A)$ and

$$AS(t)\bar{u} = S(t)A\bar{u} \qquad \forall\, \bar{u} \in D(A);$$

(S3) for any $\bar{u} \in D(A)$, $S(\cdot)\bar{u}$ is a strong solution of (2.7) on $[0,T]$, for any $T > 0$.

Let $\{S(t)\}$ be the resolvent for (2.7); if $g \in L^1(0,T;X)$, then the function $u \in C([0,T];X)$ given by

$$u(t) = S(t)\bar{u} + \int_0^t S(t-s)g(s)\,\mathrm{d}s$$

is a mild solution of the Cauchy problem

$$\begin{aligned} \dot{u} &= Au - k * Au + g,\\ u(0) &= \bar{u}. \end{aligned} \tag{2.8}$$

Set $w = \dot{u}$; then w satisfies the integral equation

$$w + 1 * (-A)w = 1 * k * Aw + \bar{g}, \tag{2.9}$$

where $\bar{g} = g + A\bar{u} + k * A\bar{u}$.

The following result comes from [21, Theorem 8.7].

Theorem 2.6 *For each $g \in L^2(0,T;X)$ there exists a unique solution of (2.9) $w \in L^2(0,T;X)$ with $1 * w \in L^2(0,T;D(A)) \cap H^{1,2}(0,T;X)$ and*

$$\begin{aligned} \|w\|_{L^2(0,T;X)} + \|1 * w\|_{H^{1,2}(0,T;X)} + \|A(1 * w)\|_{L^2(0,T;X)} \\ \leq C(T)\, \left(\|g\|_{L^2(0,T;X)} + \|A\bar{u}\|\right). \end{aligned}$$

2.5 The scalar resolvent family

In this section we write $a(t) = k_0 - \int_0^t k(s)\,\mathrm{d}s$.

It is possible to show that the resolvent family $S(t)$ admits a decomposition in the basis $\{e_k\}$ of $Y = L^2(\mathcal{O})$. We introduce, for every $\mu > 0$, the solution $s(\mu;\cdot)$ of the scalar integral equation

$$\dot{s}(\mu;t) + \mu \int_0^t s(\mu;t-\tau)\,\mathrm{d}a(\tau) = 0, \quad t \geq 0; \qquad s(0) = 1. \tag{2.10}$$

Given the sequence $\{-\mu_k\}$ of eigenvalues of A with respect to the basis $\{e_k\}$, we get

$$S(t)e_k = s(\mu_k;t)e_k, \qquad t \ge 0. \tag{2.11}$$

In the remaining of this section, we state some useful estimates on the scalar resolvent functions $s(\mu;t)$.

It is a plain remark, following to [21], to get

$$|s(\mu;t)| \le 1, \qquad t \in \mathbb{R}_+, \quad \mu > 0. \tag{2.12}$$

Lemma 2.7 *For every $\mu > 0$ and a constant C independent of μ it holds*

$$\int_{\mathbb{R}_+} |s(\mu;t)| \,\mathrm{d}t \le C\,\mu^{-1}. \tag{2.13}$$

Proof. Since $k(t)$ is completely monotonic, the same holds for $a(t)$; it is proved in [7] that (2.13) holds. In this case, it follows from [21, Lemma 4.1], see also [2], that $s(\mu;\cdot)$ is completely monotonic. □

Remark 2.1 *As a consequence of previous results, in both cases it holds that the following limit exists: $\lim\limits_{t\to\infty} s(\mu;t) = 0$.*

2.6 The representation of the semigroup

The solution of problem (2.6) can be written in terms of the *resolvent family* $\{S(t),\ t \ge 0\}$.

The variation of parameters formula for Volterra equations applies to Equation (2.6), see [21, Proposition 1.2], and we can write

$$\begin{aligned}
u(t) &= S(t)x + \int_0^t S(t-\tau)f_y(\tau)\,\mathrm{d}\tau \\
u_t(s) &= \begin{cases} u(t-s) = S(t-s)x + \int_0^{t-s} S(t-s-\tau)f_y(\tau)\,\mathrm{d}\tau & -t \le s \in (0,t) \\ y(t-s) & s > t \end{cases} \\
f_y(t) &= \int_{\mathbb{R}_+} k(t+\sigma)Ay(\sigma)\,\mathrm{d}\sigma.
\end{aligned} \tag{2.14}$$

If we set

$$\mathcal{T}(t) = \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{pmatrix}$$

then $u(t) = T_{11}(t)x + T_{12}(t)y$, hence

$$\begin{aligned}
T_{11}(t) &= S(t), \\
T_{12}(t)\phi &= \int_0^t S(t-r)f_y(r)\,\mathrm{d}r;
\end{aligned}$$

also, from the second part of formula (2.14) we have, for $\tau \geq 0$:

$$(T_{21}(t)x)(\tau) = S(t-\tau)x\mathbf{1}_{[0,t]}(\tau)$$
$$(T_{22}(t)y)(\tau) = \begin{cases} \int_0^{t-\tau} S(t-\tau-r) f_y(r)\,\mathrm{d}r & \tau \in (0,t) \\ y(t-\tau) & \tau > t. \end{cases}$$

2.7 The stochastic convolution process

In this section we consider the stochastic part of the heat source. We are given a cylindrical Wiener process $\{W(t),\ t \geq 0\}$ of the form

$$\langle W(t), x\rangle = \sum_{k=0}^{\infty} \langle e_k, x\rangle\, \beta_k(t), \qquad t \geq 0, \quad x \in Y,$$

where $\{\beta_k\}$ is a sequence of real, standard, independent Bronwian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$.

Hypothesis 2.8 *We let $Q : Y \to Y$ be a linear bounded operator; with no loss of generality, we shall assume in the sequel that A and Q diagonalizes on the same basis of Y (this is required only to the sake of simplicity).*

If $\{-\mu_k\}$ and $\{\lambda_k\}$ are the eigenvalues of A and Q, respectively, then we require

$$\sum_{k\geq 1} \lambda_k^2 < +\infty, \qquad \sum_{k\geq 1} \lambda_k^2\, \mu_k^{-1}\, c_k^2 < +\infty, \tag{2.15}$$

where c_k is the sequence from assumption (2.5).

Let us consider the following form of equation (1.1):

$$\tfrac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) - \int_0^\infty k(s) A u_t(s)\,\mathrm{d}s + Q\,\dot{W}(t) \tag{2.16}$$

with zero initial condition. The semigroup setting introduced in previous section allows us to write the solution of equation (2.16) in the following mild form

$$W_T(t) = \int_0^t \mathcal{T}(t-s) \begin{pmatrix} Q \\ 0 \end{pmatrix} \mathrm{d}W(s), \qquad t \geq 0.$$

We introduce, for arbitrary $T > 0$, the space $C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{H})) = C_{\mathcal{F}}([0,T]; \mathcal{H})$ of mean square continuous, adapted processes with values in $\mathcal{H}$ endowed with the norm

$$\|\phi\|^2 = \sup_{t\in[0,T]} \mathbb{E}\|\phi(t)\|_{\mathcal{H}}^2 < +\infty.$$

$C_{\mathcal{F}}([0,T]; \mathcal{H})$ is a Banach space. We shall exploit below the fact that this norm is equivalent to the following

$$\|X\|_\beta^2 = \sup_{t\in[0,T]} \mathbb{E}e^{-\beta t} |X(t)|_{\mathcal{H}}^2$$

for any $\beta > 0$.

In order to proceed with the nonlinear equation, we shall prove that the stochastic convolution process belongs to the space $C_{\mathcal{F}}([0,T];\mathcal{H})$. However, since it requires no additional effort, our statement will be slightly stronger. Let $L^2_{\mathcal{F}}(C([0,T];\mathcal{H}))$ denote the space of square integrable, adapted processes with continuous trajectories in $\mathcal{H}$. This is a subspace of $C_{\mathcal{F}}([0,T];\mathcal{H})$. Then we see that the stochastic convolution process belongs to this space.

Lemma 2.9 *In the above setting, the stochastic convolution process belongs to the space* $L^2_{\mathcal{F}}(C([0,T];\mathcal{H}))$ *for every* $T > 0$.

Theorem 2.10 *In our assumptions, the following properties hold for the stochastic convolution process:*

1. $W_T(t)$ *is a Gaussian random variable with mean 0 and covariance operator* Q_t*:*
$$Q_t = \int_0^t \mathcal{T}(\tau)\mathcal{Q}\mathcal{Q}^\star\mathcal{T}(\tau)^\star\,\mathrm{d}\tau, \qquad t > 0.$$
2. *The linear operator* $Q_\infty = \int_0^{+\infty} \mathcal{T}(\tau)\mathcal{Q}\mathcal{Q}^\star\mathcal{T}(\tau)^\star\,\mathrm{d}\tau$ *is a trace class operator.*
3. $Q_t \to Q_\infty$ *in the sense that*
$$\mathrm{Tr}(Q_\infty - Q_t) \to 0 \qquad \text{as } t \to \infty. \tag{2.17}$$
In particular, $W_T(t)$ *converges in law to a Gaussian distribution* $\mathcal{N}(0, Q_\infty)$.

As a consequence of Theorem 2.10 and the dissipativity of the semigroup $\mathcal{T}(t)$ stated in Theorem 2.4, by using the general theory of Da Prato & Zabczyk [12, Section 11.2], we obtain in Corollary 3.5 the characterization of invariant measures for equation (1.3) in the linear case $f \equiv 0$. In particular, we prove that there exists a unique invariant measure $\eta \sim \mathcal{N}(0, Q_\infty)$ for equation (1.3) when $f \equiv 0$, that is a centered Gaussian measure on the space $\mathcal{H}$; further, the system is ergodic and strongly mixing.

We conclude by refining the regularity for the stochastic convolution process. We consider only the first component $W_S(t)$ of the process $W_T(t)$. It is possible to prove that this process possesses a certain degree of regularity in both space and time; it is worth mentioning that the time regularity is affected by the kernel $k(t)$.

Theorem 2.11 *The stochastic convolution process* $W_S(t,\xi)$ *is almost surely continuous on* $[0,T] \times \mathcal{O}$.

3 The linear equation

In the first part of this section, we prove the generation theorem 2.4. Then we consider the stochastic convolution process $W_T(t)$ and we prove that the linear system with additive noise is ergodic and strongly mixing.

3.1 The semigroup setting

Theorem 3.1 *The operator $\mathcal{A}$ is dissipative: for any $\phi = \binom{x}{y} \in D(\mathcal{A})$ it holds*

$$\langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} \le -\delta \|\phi\|^2_{\mathcal{H}}$$

for some positive constant δ.

Proof. We compute the scalar product

$$\langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} = k_0 \, \langle Ax, x \rangle_Y - \int_{\mathbb{R}_+} k(\sigma) \langle x, Ay(\sigma) \rangle_Y \, \mathrm{d}\sigma - \int_{\mathbb{R}_+} \rho(\sigma) \langle \nabla y(\sigma), \nabla y'(\sigma) \rangle \, \mathrm{d}\sigma$$

and we get

$$\begin{aligned} &= -k_0 \, \|x\|^2_{H^1_0} + \int_{\mathbb{R}_+} k(\sigma) \langle x, y(\sigma) \rangle_{H^1_0} \, \mathrm{d}\sigma - \int_{\mathbb{R}_-} \rho(\sigma) \frac{1}{2} \frac{d}{d\sigma} \|y(\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \\ &\le -k_0 \, \|x\|^2_{H^1_0} + \int_{\mathbb{R}_+} k(\sigma) \, \|x\|_{H^1_0} \, \|y(\sigma)\|_{H^1_0} \, \mathrm{d}\sigma - \frac{1}{2} \rho(\sigma) \|y(\sigma)\|^2_{H^1_0} \Big|_{\mathbb{R}_+} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+} \rho'(\sigma) \|y(\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \end{aligned}$$

recall that $\rho'(s) = -k(s)$; choose some $\varepsilon > 0$ and use the bound $ab - \frac{1}{2}(1-\varepsilon) b^2 \le \frac{1}{2(1-\varepsilon)} a^2$ to get

$$\le \left(-k_0 + \frac{1 - \varepsilon/2}{1 - \varepsilon} \rho(0) \right) \|x\|^2_{H^1_0} + \frac{\varepsilon}{2} \int_{\mathbb{R}_+} \rho'(\sigma) \|y(\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma.$$

Assumption 2.1(2) implies that for ε small enough, the quantity in the first bracket is negative; further, from assumption 2.1(1) it follows that $\rho'(t) \le -\eta \rho(t)$, hence we write

$$\langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} \le \left(-k_0 + \frac{1 - \varepsilon/2}{1 - \varepsilon} \rho(0) \right) \|x\|^2_{H^1_0} - \frac{\varepsilon \, \eta}{2} \|y\|^2_{\mathcal{X}} \le -\delta \, \|\phi\|^2_{\mathcal{H}}$$

for some positive δ and we get that $\mathcal{A}$ is dissipative on $\mathcal{H}$. □

The fact that $\mathcal{A}$ generates a C_0-semigroup of contractions, as stated in Theorem 2.4, will follow from the celebrated Lumer-Phillips theorem once we prove that the resolvent $R(\mu; \mathcal{A})$ is surjective for some (and hence for all) $\mu > 0$.

Proposition 3.2 *For every $\mu > 0$ the equation*

$$(\mu - \mathcal{A})\phi = \psi, \qquad \psi \in \mathcal{H}, \tag{3.1}$$

has a unique solution $\phi \in D(\mathcal{A})$.

Proof. Let $\phi = \binom{x}{y}$, $\psi = \binom{u}{v}$. We write (3.1) in the form

$$\mu x - k_0 A x + \Phi y = u \tag{3.2}$$

$$\mu y(s) + \tfrac{\mathrm{d}}{\mathrm{d}s} y(s) = v(s). \tag{3.3}$$

We first solve equation (3.3). The variation of parameters formula applies and we get

$$y(s) = e^{-\mu s} y(0) + \int_0^s e^{-\mu(s-\sigma)} v(\sigma)\,\mathrm{d}\sigma; \tag{3.4}$$

we shall prove that $y \in W^{1,2}_\rho(\mathbb{R}_+; H^1_0(\mathcal{O}))$: we first prove that $y \in L^2_\rho(\mathbb{R}_+; H^1_0(\mathcal{O}))$ then we notice that $\frac{\mathrm{d}}{\mathrm{d}s} y(s) = -\mu y(s) + v(s)$, therefore it follows that also $\frac{\mathrm{d}}{\mathrm{d}s} y \in L^2_\rho(\mathbb{R}_+; H^1_0(\mathcal{O}))$, which implies the claim.

We compute

$$\begin{aligned}
\|y\|^2_{L^2_\rho(\mathbb{R}_+;H^1_0(\mathcal{O}))} &= \int_{\mathbb{R}_+} \rho(s) \left\| e^{-\mu s} y(0) + \int_0^s e^{-\mu(s-\sigma)} v(\sigma)\,\mathrm{d}\sigma \right\|^2_{H^1_0} \mathrm{d}s \\
&\le 2\|\rho\|_{L^1(\mathbb{R}_+)} \|y(0)\|^2_{H^1_0} + 2\,\frac{1}{\mu} \int_{\mathbb{R}_+} \int_\sigma^\infty \rho(s) e^{\mu(s-\sigma)} \|v(\sigma)\|^2_{H^1_0}\,\mathrm{d}s\,\mathrm{d}\sigma
\end{aligned}$$

recall that $\rho(s)$ is monotonically decreasing, hence $\rho(s) \le \rho(\sigma)$ for $s > \sigma$

$$\begin{aligned}
&\le 2\|\rho\|_{L^1(\mathbb{R}_+)} \|y(0)\|^2_{H^1_0} + 2\,\frac{1}{\mu} \int_{\mathbb{R}_+} \rho(\sigma) \int_\sigma^\infty e^{\mu(s-\sigma)} \|v(\sigma)\|^2_{H^1_0}\,\mathrm{d}s\,\mathrm{d}\sigma \\
&\le 2\|\rho\|_{L^1(\mathbb{R}_-)} \|y(0)\|^2_{H^1_0} + 2\,\frac{1}{\mu^2}\,\|v\|^2_{L^2_\rho(\mathbb{R}_+;H^1_0(\mathcal{O}))}.
\end{aligned} \tag{3.5}$$

We notice that it is still necessary to estimate $\|y(0)\|^2_{H^1_0}$ in order to conclude. By (3.2) and (3.4) we get

$$\begin{aligned}
&\mu x - k_0 A x + \left(\int_{\mathbb{R}_+} k(\sigma) A e^{-\mu\sigma} x\,\mathrm{d}\sigma \right) = u + \int_{\mathbb{R}_+} k(s) A \int_0^s e^{-\mu(s-\sigma)} v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\mu x - \left(k_0 - \int_{\mathbb{R}_+} k(\sigma) e^{-\mu\sigma}\,\mathrm{d}\sigma \right) A x = \tilde{u}
\end{aligned}$$

We notice that, for each $\mu \in \mathbb{R}_+$,

$$c_{k,\mu} = k_0 - \int_{\mathbb{R}_+} k(\sigma) e^{-\mu\sigma}\,\mathrm{d}\sigma \ge k_0 - \int_{\mathbb{R}_+} k(\sigma)\,\mathrm{d}\sigma$$

is a strictly positive constant (the bound being independent of μ), hence previous expression becomes

$$\mu x - c_{k,\mu} A x = \tilde{u}.$$

Since A is invertible, this equation is uniquely solvable, hence

$$x = \frac{1}{c_{k,\mu}} R(\tfrac{\mu}{c_{a,\mu}}, A)\tilde{u} \in D(A).$$

This guarantees, by (3.5), that $y \in W^{1,2}_\rho(\mathbb{R}_+; H^1_0(\mathcal{O}))$. We also have (again by (3.2))

$$k_0 Ax - \Phi y = \mu x - u \in L^2(\mathcal{O})$$

and the proof is complete. □

3.2 The stochastic convolution process

The first part of the section is devoted to prove the statements of Theorem 2.10. We are concerned with the stochastic convolution process

$$W_T(t) = \int_0^t \mathcal{T}(t-s)\begin{pmatrix} Q \\ 0 \end{pmatrix} \mathrm{d}W(s).$$

We divide the proof in several steps.

Lemma 3.3 *$W_T(t)$ is a Gaussian random variable with mean 0 and covariance operator Q_t:*

$$Q_t = \int_0^t \mathcal{T}(\tau)\mathcal{Q}\mathcal{Q}^\star\mathcal{T}(\tau)^\star \,\mathrm{d}\tau, \qquad t > 0.$$

The linear operator $Q_\infty = \int_0^{+\infty} \mathcal{T}(\tau)\mathcal{Q}\mathcal{Q}^\star\mathcal{T}(\tau)^\star \,\mathrm{d}\tau$ is a trace class operator.

Proof. Itô's isometry implies that

$$\mathbb{E}|W_T(t)|^2_{\mathcal{H}} = \int_0^t \|\mathcal{T}(s)\mathcal{Q}\|^2_{HS}\,\mathrm{d}s = \sum_{k=1}^\infty \int_0^t \left|\mathcal{T}(s)\begin{pmatrix} Qe_k \\ 0 \end{pmatrix}\right|^2_{\mathcal{H}} \mathrm{d}s \tag{3.6}$$

and we employ the definition of $\mathcal{T}(t)$ in order to get

$$\mathcal{T}(s)\begin{pmatrix} Qe_k \\ 0 \end{pmatrix} = \begin{pmatrix} S(s)\lambda_k e_k \\ S(s-\cdot\,)\lambda_k e_k\,\mathbf{1}_{[0,s]}(\cdot) \end{pmatrix}$$

therefore

$$\left|\mathcal{T}(s)\begin{pmatrix} Qe_k \\ 0 \end{pmatrix}\right|^2_{\mathcal{H}} = \lambda_k^2\,\|S(s)e_k\|^2_{L^2(\mathcal{O})} + \lambda_k^2 \int_0^\infty \rho(\sigma)\,\|S(s-\sigma)e_k\|^2_{H^1_0(\mathcal{O})}\,\mathbf{1}_{[0,s]}(\sigma)\,\mathrm{d}\sigma.$$

It follows that

$$\mathbb{E}|W_T(t)|^2_{\mathcal{H}} = \sum_{k=1}^\infty \lambda_k^2 \int_0^t \|S(s)e_k\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s + \sum_{k=1}^\infty \lambda_k^2 \int_0^t \int_{-\infty}^0 \rho(\sigma)\,\|S(s+\sigma)e_k\|^2_{H^1_0(\mathcal{O})}\,\mathbf{1}_{[-s,0]}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \tag{3.7}$$

We consider separately the two series in previous formula. As far as the first is concerned, we notice that, given the orthonormal basis $\{e_k\}$ in $Y = L^2(\mathcal{O})$ and the realization A of the Laplacian with Dirichlet boundary conditions on Y, it holds that $S(t)e_k = s(\mu_k; t)e_k$; therefore, we get

$$\sum_{k=1}^{\infty} \lambda_k^2 \int_0^t \|S(t-s)e_k\|_Y^2 \, \mathrm{d}s = \sum_{k=1}^{\infty} \lambda_k^2 \int_0^t |s(\mu_k;\sigma)|^2 \, \mathrm{d}\sigma$$

and since $\int_0^\infty |s(\mu_k;t)| \, \mathrm{d}t < C\,\mu_k^{-1}$, it follows that

$$\sum_{k=1}^{\infty} \lambda_k^2 \int_0^t \|S(t-s)e_k\|_Y^2 \, \mathrm{d}s \le C \sum_{k=1}^{\infty} \lambda_k^2 \mu_k^{-1}.$$

We consider next the second series in (3.7). We use Fubini's theorem to get

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k^2\, \mu_k \int_0^t \int_0^{t-\sigma} \rho(\sigma)|s(\mu_k;\tau)|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma &= \sum_{k=1}^{\infty} \lambda_k^2\, \mu_k \int_0^t \int_0^{t-\tau} \rho(\sigma)|s(\mu_k;\tau)|^2 \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &\le C\, \|\rho\|_{L^1(\mathbb{R}_+)} \sum_{k=1}^{\infty} \lambda_k^2.\end{aligned}$$

This proves at once both claims by using Hypothesis 2.8. □

Lemma 3.4 *$Q_t \to Q_\infty$ in the sense that*

$$\mathrm{Tr}(Q_\infty - Q_t) \to 0 \qquad \text{as } t \to \infty. \tag{3.8}$$

In particular, $W_T(t)$ converges in law to a Gaussian distribution $\mathcal{N}(0, Q_\infty)$.

Proof. Using the same computation as above we have

$$\begin{aligned}\mathrm{Tr}(Q_\infty - Q_t) \le &\sum_{k=1}^{\infty} \lambda_k^2 \int_t^\infty |s(\mu_k;\tau)|^2 \, \mathrm{d}\tau \\ &+ \sum_{k=1}^{\infty} \lambda_k^2\, \mu_k \int_t^\infty \int_0^\infty \rho(\sigma)\, |s(\mu_k;\tau-\sigma)|^2 \mathbf{1}_{[0,\tau]}(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}\tau\end{aligned} \tag{3.9}$$

and, in particular, $\mathrm{Tr}(Q_\infty - Q_t) \le \mathrm{Tr}(Q_\infty) < +\infty$.

We introduce the quantity

$$\Lambda(\mu;t) = \int_t^\infty \left(|s(\mu;\tau)|^2 + \int_0^\tau \rho(\sigma)\, \mu |s(\mu;\tau-\sigma)|^2 \, \mathrm{d}\sigma \right) \mathrm{d}\tau;$$

the thesis is clearly proved by the claim

$$\lim_{t\to\infty} \sum_k \lambda_k^2 \Lambda(\mu_k, t) = 0. \tag{3.10}$$

We use a direct reasoning to conclude. For the sake of clarity, we briefly sketch it in the following remark before we conclude.

Remark 3.1 *We are given a (continuous parameter) family of positive functions $k \mapsto \Lambda(\mu_k, t)$; we assume the convergence (for every k)*

$$\lim_{t\to\infty} \Lambda(\mu_k, t) = 0 \tag{3.11}$$

and we also assume that for some $\delta > 0$ it holds

$$\sup_{t>0} \sum_k \lambda_k^2 |\Lambda(\mu_k, t)|^{1+\delta} < +\infty. \tag{3.12}$$

Since $\{\lambda_k^2\}$ is a summable sequence, for every ϵ there exists ν_ϵ such that $\sum_{k>\nu_\epsilon} \lambda_k^2 < \epsilon$. Then we claim that

$$\sum_{k=1}^{\infty} \lambda_k^2\, \Lambda(\mu_k, t) \longrightarrow 0 \quad \textit{as } t \to \infty.$$

Proof. For any $\epsilon > 0$ we have

$$\left|\sum_{k=1}^{\infty} \lambda_k^2\, \Lambda(\mu_k, t)\right| \le \left|\sum_{k=1}^{\nu_\epsilon} \lambda_k^2\, \Lambda(\mu_k, t)\right| + \sum_{k>\nu_\epsilon} \left|\lambda_k^2\, \Lambda(\mu_k, t)\right|.$$

Applying Cauchy-Scwarz's inequality to the last term we have

$$\sum_{k>\nu_\epsilon} \left|\lambda_k^2\, \Lambda(\mu_k, t)\right| \le \left(\sum_{k>\nu_\epsilon} \lambda_k^2\right)^{\frac{\delta}{1+\delta}} \left(\sum_{k>\nu_\epsilon} \lambda_k^2\, \left|\Lambda(\mu_k, t)\right|^{1+\delta}\right)^{\frac{1}{1+\delta}} < \epsilon^{\frac{\delta}{1+\delta}}\, C^{\frac{1}{1+\delta}}$$

that is

$$\left|\sum_{k=1}^{\infty} \lambda_k^2\, \Lambda(\mu_k, t)\right| \le \left|\sum_{k=1}^{\nu_\epsilon} \lambda_k^2\, \Lambda(\mu_k, t)\right| + \epsilon^{\frac{\delta}{1+\delta}}\, C^{\frac{1}{1+\delta}}$$

from which we can easily deduce the statement. □

Let us return to prove claim (3.10). With some algebra we obtain

$$\begin{aligned}\Lambda(\mu; t) = \int_t^{\infty} |s(\mu;\tau)|^2\, \mathrm{d}\tau + \int_t^{\infty} \mu |s(\mu;\tau)|^2 \left(\int_0^{\tau/2} \rho(\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}\tau \\ + \int_{t/2}^{\infty} \left(\int_{0\vee(t-\sigma)}^{\sigma} \mu |s(\mu;t)|^2\, \mathrm{d}\tau\right) \rho(\sigma)\, \mathrm{d}\sigma\end{aligned}$$

We therefore estimate

$$\begin{aligned}\Lambda(\mu; t) \le \left[\sup_{\tau\ge t} |s(\mu;\tau)|\right] \left(\|s(\mu;\cdot)\|_{L^1(\mathbb{R}_+)} + \|\rho\|_{L^1(\mathbb{R}_+)}\, \|\mu\, s(\mu;\cdot)\|_{L^1(\mathbb{R}_+)}\right) \\ + \|\mu\, s(\mu;\cdot)\|_{L^1(\mathbb{R}_+)} \int_{t/2}^{\infty} \rho(\sigma)\, \mathrm{d}\sigma.\end{aligned}$$

It follows from Remark 2.1 and the eponential decay of ρ as $t \to \infty$ that both terms in previous estimate converges to 0 as t diverges: $\Lambda(\mu_k, t) \to 0$ for every $k \in \mathbb{N}$ as required in (3.11). Moreover, we can also write previous estimate in the following form

$$|\Lambda(\mu_k, t)| \le (C_1 + R) + C_1\, R$$

where $C_1 = C$ is the constant from Lemma 2.7 and $R = \|\rho\|_{L^1(\mathbb{R}_+)}$, which implies

$$|\Lambda(\mu_k, t)|^2 \le C\, \Lambda(\mu_k, t)$$

and condition (3.12) is verified:

$$\sup_{t>0} \sum_k \lambda_k^2 |\Lambda(\mu_k, t)|^2 < +\infty.$$

This concludes the proof of the lemma. □

By means of the estimate in Theorem 3.1 and appealing to [12, Theorem 11.11] we immediately obtain the following.

Corollary 3.5 *The invariant measure μ is unique and it is ergodic and strongly mixing.*

As a consequence of the strong mixing property, for any initial condition $\bar{\phi} = \binom{x}{y}$ we can write

$$\lim_{t\to\infty} \mathbb{E}[\Phi(u(t; x, y), u_t(x, y))] = \int_{\mathcal{H}} \Phi(x', y')\, \mu(\mathrm{d}x', \mathrm{d}y').$$

If we take the mapping Φ to be a function depending only on the first component, we can integrate on the second variable in the right-hand side and obtain

$$\lim_{t\to\infty} \mathbb{E}[\Phi(u(t; x, y))] = \int_H \Phi(x')\, \mu_s(\mathrm{d}x'). \tag{3.13}$$

We fix an history y (for simplicity, one can take $y \equiv 0$) and consider the Volterra integral equation

$$u(t) = x + \int_0^t k(t - s) A u(s)\, \mathrm{d}s + C\, W(t)$$

as in [8]. Introduce the mapping P_t on $B_b(Y)$ by setting

$$P_t \varphi(x) = \mathbb{E}[\varphi(u(t; x))]$$

and its dual $P_t^\star$ acting on the space $\mathcal{M}(Y)$ of bounded positive measures on H by

$$\langle P_t^\star \mu, \varphi\rangle = \langle \mu, P_t \varphi\rangle \qquad \text{for all } \varphi \in B_b(Y),\ \mu \in \mathcal{M}(Y).$$

We thus obtain from (3.13) that the first marginal μ_s of the invariant measure μ satisfies

$$P_t^\star \nu \to \mu_s \qquad \text{for all probability measures } \nu \in \mathcal{M}_1(Y). \tag{3.14}$$

3.3 Space-time continuity of the stochastic convolution process

We begin with the regularity of the stochastic convolution process stated in Lemma 2.9.

Proof. The same computation as in the first part of the proof for Lemma 3.3 applies, except that we shall use Doob's inequality instead of Itô isometry in (3.6) to get

$$\mathbb{E} \sup_{s \le t} \|W_T(s)\|^2_{\mathcal{H}} \le 4 \int_0^t \|\mathcal{T}(s)\mathcal{Q}\|^2_{L_2} \,\mathrm{d}s = 4 \sum_{k=1}^{\infty} \int_0^t \left\| \mathcal{T}(t-s) \begin{pmatrix} Qe_k \\ 0 \end{pmatrix} \right\|^2_{\mathcal{H}} \mathrm{d}s.$$

□

Let us consider the stochastic process

$$W_S(t) = \int_0^t S(t-s) Q \,\mathrm{d}W(s)$$

where $S(t)$ is the resolvent operator for the delay equation (2.7). Notice that W_S is the mild solution of the problem

$$\mathrm{d}u(t) = \left[k_0 A u(t) - \int_0^\infty k(s)\, A u_t(s) \,\mathrm{d}s \right] \mathrm{d}t + Q \,\mathrm{d}W(t) \quad t \in [0, T] \tag{3.15}$$

with zero initial condition $u(0) = 0$, $u_0(\cdot\,) = 0$. According to the representation of the semigroup $\mathcal{T}(t)$ provided in section 2.6, we remark that $W_S(t)$ is the first component of the vector $W_T(t)$.

In view of the spectral decompositions of A and Q we can write

$$W_S(t) = \sum_{k=1}^{\infty} \lambda_k \int_0^t s(\mu_k; t-\sigma) e_k \,\mathrm{d}\beta_k(\sigma), \qquad t \ge 0. \tag{3.16}$$

Then we can apply the reasonings of [8] in order to prove the continuity of the stochastic convolution process. We impose the following condition on the operator A. Notice that in most applications it is verified, since the Laplace operator on $\mathcal{O}$ does satisfy it.

Lemma 3.6 *Under the additional assumption*

$$\sum_{k=1}^{\infty} \lambda_k^2 \, c_k^2 < +\infty, \tag{3.17}$$

$W_S(t, \xi)$ is $\mathbb{P}$-a.s. continuous in $t \in [0, T]$ and $\xi \in \mathcal{O}$.

Proof. We apply Kolmogorov's criterium to get the thesis. Let us write, for $t > \tau$

and $\xi, \eta \in \mathcal{O}$,

$$\begin{aligned}
W_S(t,\xi) - W_S(\tau,\eta) =& W_S(t,\xi) - W_S(t,\eta) + W_S(t,\eta) - W_S(\tau,\eta) \\
=& \sum_{k=1}^\infty \lambda_k \int_0^t s(\mu_k; t-\sigma)\,\mathrm{d}\beta_k(\sigma)\,[e_k(\xi) - e_k(\eta)] \\
&+ \sum_{k=1}^\infty \lambda_k \int_0^\tau [s(\mu_k; t-\sigma) - s(\mu_k; \tau-\sigma)]\,\mathrm{d}\beta_k(\sigma)\,e_k(\eta) \\
&+ \sum_{k=1}^\infty \lambda_k \int_\tau^t s(\mu_k; t-\sigma)\,\mathrm{d}\beta_k(\sigma)\,e_k(\eta)
\end{aligned}$$

hence

$$\begin{aligned}
\mathbb{E}|W_S(t,\xi) - W_S(\tau,\eta)|^2 \le& C \sum_{k=1}^\infty \lambda_k^2 \int_0^t |s(\mu_k;\sigma)|^2\,\mathrm{d}\sigma\,[e_k(\xi) - e_k(\eta)]^2 \\
&+ C \sum_{k=1}^\infty \lambda_k^2 \int_0^\tau [s(\mu_k; t-\sigma) - s(\mu_k; \tau-\sigma)]^2\,\mathrm{d}\sigma\,|e_k(\eta)|^2 \\
&+ C \sum_{k=1}^\infty \lambda_k^2 \int_\tau^t |s(\mu_k;\sigma)|^2\,\mathrm{d}\sigma\,|e_k(\eta)|^2
\end{aligned}$$

Then, by means of the estimate in Lemma 2.7 and Hölder's inequality, for each $\theta \in (0,1)$ it holds

$$\int_\tau^t |s(\mu;r)|^2\,\mathrm{d}r \le C\,|t-\tau|^\theta\,\mu^{\theta-1}$$

as well as

$$\int_0^\tau [s(\mu_k; t-\sigma) - s(\mu_k; \tau-\sigma)]^2\,\mathrm{d}\sigma \le C\,|t-\tau|^\theta\,\mu^{\theta-1}.$$

Now, for any $\theta \in (0,\vartheta)$ it follows by interpolation from the estimates (2.5) that

$$|e_k(\xi) - e_k(\eta)| \le M_\beta\,\mu_k^\beta\,c_k\,|\xi-\eta|^\beta, \qquad k \in \mathbb{N}$$

and we obtain

$$\mathbb{E}|W_S(t,\xi) - W_S(\tau,\eta)|^2 \le C \sum_{k=1}^\infty \lambda_k^2\,\mu_k^{\theta-1}\,c_k^2\,|t-\tau|^\theta + C \sum_{k=1}^\infty \lambda_k^2\,\mu_k^{\beta-1}\,c_k^2\,|\xi-\eta|^{2\beta}$$

therefore, under assumption (3.17) we get

$$\mathbb{E}|W_S(t,\xi) - W_S(\tau,\eta)|^2 \le C_\theta\,|t-\tau|^\theta + C_\beta\,|\xi-\eta|^{2\beta}$$

for arbitrary $\theta, \beta < 1$. We notice that $W_S(t,\xi)$ is a Gaussian random field, hence

$$\mathbb{E}|W_S(t,\xi) - W_S(\tau,\eta)|^{2m} \le C_\theta\,|t-\tau|^{m\theta} + C_\beta\,|\xi-\eta|^{2m\beta}$$

and Kolmogorov's continuity theorem implies that $(t,\xi) \mapsto W_S(t,\xi)$ is Hölder continuous in $[0,T] \times \mathcal{O}$. $\square$

The last result of this section is a useful estimate that we shall apply in order to get the asymptotic behaviour of the solution for problem (1.1).

Lemma 3.7 *For any $m \in \mathbb{N}$ it holds*

$$\mathbb{E} \int_{\mathcal{O}} |W_S(t,\xi)|^{2m} \, \mathrm{d}\xi < C, \qquad \forall\, t \geq 0$$

and a constant C that depends on m, $|\mathcal{O}|$ and the constant M from assumption (2.5) but is independent of t.

Proof. Let us compute

$$\begin{aligned} v(t,\xi) = \mathbb{E}|W_S(t,\xi)|^2 &= \mathbb{E} \left| \sum_{k=1}^{\infty} \lambda_k \int_0^t s(\mu_k; t-\sigma) \, \mathrm{d}\beta_k(\sigma) \, e_k(\xi) \right|^2 \\ &= \sum_{k=1}^{\infty} \lambda_k^2 \int_0^t |s(\mu_k; t-\sigma)|^2 \, \mathrm{d}\sigma \, |e_k(\xi)|^2 \leq M^2 \sum_{k=1}^{\infty} \lambda_k^2 \, \mu_k^{-\alpha} \, c_k^2 \end{aligned} \tag{3.18}$$

and the above quantity is finite thanks to (2.15); therefore

$$\mathbb{E} \int_{\mathcal{O}} |W_S(t,\xi)|^{2m} \, \mathrm{d}\xi \leq c_m \int_{\mathcal{O}} |v(t,\xi)|^{2m} \, \mathrm{d}\xi \leq C$$

where the constant C is independent of t thanks to (3.18). □

4 The nonlinear equation

In this section we aim to prove existence and uniqueness of the solution for the general problem (1.1) in case $g(t) = g(u(t))$ is a dissipative mapping. The section is divided in two parts. In the first part, we consider Lipschitz continuous, dissipative mappings. In this case, we can use directly the state space setting of equation (1.3) and prove that the solution exists unique and that the system is ergodic and strongly mixing.

In the second part of the section, we study the case of general dissipative nonlinearities. In this case, we does not have any dissipative property of the leading operator $\mathcal{A}$ on the domain of the nonlinear term g. However, we can make direct estimates on the solution, which allow us to proceed with the construction of the solution by means of a suitable approximation procedure.

4.1 Lipschitz dissipative case

We are now going to define a concept of solution for equation (1.3) which is suitable in our framework. An $\mathcal{H}$-valued process $\phi(t) = (u(t), u_t)$ is a solution of (1.3) if $\phi \in L^2_{\mathcal{F}}(0,T; L^2(\Omega; \mathcal{H}))$ and it solves the mild form of equation (1.3), i.e.,

$$\phi(t) = \mathcal{T}(t)\bar{\phi} + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\phi(s)) \, \mathrm{d}s + W_T(t), \qquad t \geq 0. \tag{4.1}$$

We assume in this section that $g : Y \to Y$ is a Lipschitz continuous mapping. This means, recalling Hypothesis 2.2 with $d = 0$, that for some $L > 0$, $\gamma \in \mathbb{R}$, it holds:

$$\|g(x)\|_Y \le L\,(1 + \|x\|_Y) \text{ and } \|g(x_1) - g(x_2)\|_Y \le L\,\|x_1 - x_2\|_Y.$$

Theorem 4.1 *For arbitrary $T > 0$, there exists a unique solution $\phi \in C_{\mathcal{F}}([0,T];\mathcal{H})$ for equation (4.1).*

Proof. Problem (4.1) is equivalent to the following equation in $C_{\mathcal{F}}([0,T];\mathcal{H})$:

$$\phi(t) = \mathcal{T}(t)\phi(0) + G(\phi)(t) + W_T(t)$$

where the mapping $G : C_{\mathcal{F}}([0,T];\mathcal{H}) \to C_{\mathcal{F}}([0,T];\mathcal{H})$ is defined by

$$G(\phi)(t) = \int_0^t \mathcal{T}(t-s)\mathcal{F}(\phi(s))\,\mathrm{d}s.$$

It is an easy computation to show that G is well defined on the space $C_{\mathcal{F}}([0,T];\mathcal{H})$; we prove that G is a contraction with respect to the norm $\|\!|\cdot|\!\|_\beta$; from this it shall follow the thesis. Since $\mathcal{T}(t)$ is a contraction semigroup we get

$$\begin{aligned}
&\|\!|G(\phi_1) - G(\phi_2)|\!\|_\beta^2 = \sup_{t\in[0,T]} \mathbb{E}e^{-2\beta t}|G(\phi_1)(t) - G(\phi_2)(t)|_{\mathcal{H}}^2 \\
&\le \sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t e^{-\beta s}e^{-\beta(t-s)}\|\mathcal{T}(t-s)[\mathcal{F}(\phi_1(s)) - \mathcal{F}(\phi_2(s))]\|_{\mathcal{H}}\,\mathrm{d}s\right]^2 \\
&\le C \sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t e^{-\beta s}e^{-\beta(t-s)}\|g(x_1(s)) - g(x_2(s))\|_Y\,\mathrm{d}s\right]^2 \\
&\le C \sup_{t\in[0,T]} \left(\int_0^t e^{-\beta(t-s)}\,\mathrm{d}s\right)\left(\int_0^t e^{-\beta(t-s)}\,\mathbb{E}e^{-2\beta s}\|x_1(s) - x_2(s)\|_Y^2\,\mathrm{d}s\right) \\
&\le C \sup_{t\in[0,T]} \left(\sup_{s\in[0,T]} \mathbb{E}e^{-2\beta s}\|x_1(s) - x_2(s)\|_Y^2\right)\left[\int_0^t e^{-\beta(t-s)}\,\mathrm{d}s\right]^2 \\
&\le \frac{C}{\beta^2}\,\|\!|\phi_1(s) - \phi_2(s)|\!\|_\beta^2.
\end{aligned}$$

We have proved that G is a contraction on the space $C_{\mathcal{F}}([0,T];\mathcal{H})$ endowed with the equivalent norm $\|\!|\phi|\!\|_\beta^2$ for β large enough. Therefore, by a standard argument it follows that problem (4.1) has a unique solution on $[0,T]$. □

4.2 General dissipative case

Our first step is to define a concept of solution for equation (1.1) which is suitable in our framework. Recall from Section 3.2 that $C_{\mathcal{F}}([0,T];\mathcal{H})$ is the space of mean square continuous, adapted processes with values in $\mathcal{H}$.

Definition 4.2 *An $\mathcal{H}$-valued process $\phi(t) \in C_{\mathcal{F}}([0,T];\mathcal{H})$ is a solution of (1.1) if the first component $u(t)$ satisfies*

$$u(t) \in C_{\mathcal{F}}([0,T];Y) \cap L^{2d+2}(\Omega \times (0,T) \times \mathcal{O})$$

and ϕ solves the mild form of equation (1.1), i.e.,

$$\phi(t) = S(t)\bar{\phi} + \int_0^t S(t-s)\mathcal{F}(\phi(s))\,\mathrm{d}s + W_T(t), \qquad t \geq 0.$$

The first aim of this section is to prove a priori estimates that allow to prove that the process $u(t)$ satisfies the bounds in Definition 4.2. By using the analogies between the resolvent family $S(t)$ and the semigroup approach, we introduce the process $v(t) = u(t) - W_S(t)$; it follows that $v(t)$ solves the following equation with random coefficients

$$\tfrac{\mathrm{d}}{\mathrm{d}t}v(t) = k_0 Av(t) - \int_0^\infty k(s)Av_t(s)\,\mathrm{d}s + g(v(t) + W_S(t)) \tag{4.2}$$

with initial condition $(\bar{u}, \bar{u}_0)$.

Let us introduce the Yosida approximations p_α of p. It holds that $p_\alpha(r) = p(J_\alpha(r)) = \frac{1}{\alpha}(-J_\alpha(r) + r)$, where $J_\alpha(r) = (I + \alpha p)^{-1}(r)$.

Set further $g_\alpha(u) = \lambda u - p_\alpha(u)$.

Remark 4.1 *We shall need the following estimate for the approximants g_α. First compute*

$$p_\alpha(r) \cdot r = p_\alpha(r) \cdot (J_\alpha(r) + \alpha p_\alpha(r)) = p(J_\alpha(r)) \cdot J_\alpha(r) + \alpha(p_\alpha(r))^2$$

then we get

$$g_\alpha(r) \cdot r = \lambda r^2 - p(J_\alpha(r)) \cdot J_\alpha(r) - \alpha(p_\alpha(r))^2.$$

We introduce the following integro-differential equation with Lipschitz nonlinearity:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}v_\alpha(t) = k_0\, Av_\alpha(t) - \int_0^\infty k(s)Av_{\alpha;t}(s)\,\mathrm{d}s + g_\alpha(W_S(t) + v_\alpha(t)). \tag{4.3}$$

We take the scalar product of both sides for $v_\alpha(t)$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_\alpha(t)\|^2 &+ k_0\,\|\nabla v_\alpha(t)\|^2 \\ &= -\langle v_\alpha(t), \int_0^\infty k(\sigma)\,Av_{\alpha;t}(\sigma)\,\mathrm{d}\sigma\rangle + \langle v_\alpha(t), g_\alpha(W_S(t) + v_\alpha(t))\rangle \end{aligned} \tag{4.4}$$

which implies

$$\begin{aligned}&\frac{1}{2}\|v_\alpha(t)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|v_\alpha(0)\|^2_{L^2(\mathcal{O})} + k_0 \int_0^t \|v_\alpha(s)\|^2_{H^1_0(\mathcal{O})}\,\mathrm{d}s \\ &= -\int_0^t \int_0^\infty \rho'(s)\langle \nabla v_\alpha(s), \nabla v_\alpha(s-\sigma)\rangle\,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t \langle v_\alpha(s), g_\alpha(W_S(s) + v_\alpha(s))\rangle\,\mathrm{d}s.\end{aligned} \tag{4.5}$$

Lemma 4.3 *It holds*

$$\|v_{\alpha;t}\|^2_{\mathcal{X}} - \|v_{\alpha;0}\|^2_{\mathcal{X}} = \rho(0) \int_0^t \|v_\alpha(s)\|^2_{H^1_0} \, \mathrm{d}s + \int_0^t \int_0^\infty \rho'(\sigma) \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \, \mathrm{d}s. \tag{4.6}$$

Proof. We begin from the identity

$$\int_0^t \int_0^\infty \rho(\sigma) \tfrac{\mathrm{d}}{\mathrm{d}s} \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \, \mathrm{d}s = -\int_0^t \int_0^\infty \rho(\sigma) \tfrac{\mathrm{d}}{\mathrm{d}\sigma} \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \, \mathrm{d}s$$

Now, the left hand is equal to

$$\begin{aligned} &\int_0^t \int_0^\infty \rho(\sigma) \tfrac{\mathrm{d}}{\mathrm{d}s} \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \, \mathrm{d}s \\ &\qquad = \int_0^\infty \int_0^t \rho(\sigma) \tfrac{\mathrm{d}}{\mathrm{d}s} \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}s \, \mathrm{d}\sigma \\ &\qquad = \int_0^\infty \rho(\sigma) \, \|v_\alpha(t-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma - \int_0^\infty \rho(\sigma) \, \|v_\alpha(-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \end{aligned}$$

while the right hand side is

$$\begin{aligned} &-\int_0^t \int_0^\infty \rho(\sigma) \tfrac{\mathrm{d}}{\mathrm{d}\sigma} \|v_\alpha(s-\sigma)\|^2_{H^1_0} \, \mathrm{d}\sigma \, \mathrm{d}s \\ &\qquad = -\int_0^t \rho(\sigma) \|v_\alpha(s-\sigma)\|^2_{H^1_0} \Big|_{\sigma=0}^{\sigma=\infty} \, \mathrm{d}s + \int_0^t \int_0^\infty \rho'(\sigma) \|v_\alpha(s-\sigma)\|^2 \, \mathrm{d}\sigma \, \mathrm{d}s \\ &\qquad = \rho(0) \int_0^t \|v_\alpha(s)\|^2_{H^1_0} \, \mathrm{d}s + \int_0^t \int_0^\infty \rho'(\sigma) \|v_\alpha(s-\sigma)\|^2 \, \mathrm{d}\sigma \, \mathrm{d}s \end{aligned}$$

which leads to the thesis. □

Lemma 4.4 *Let us denote $y_\alpha(t) = J_\alpha(v_\alpha(t) + W_S(t))$. Then we can find constants γ_i, positive and finite, such that*

$$\begin{aligned} \int_0^t \langle v_\alpha(s), g_\alpha(W_S(s) + v_\alpha(s)) \rangle \, \mathrm{d}s \le \gamma_1 - \gamma_2 \int_0^t \langle p(y_\alpha(s)), y_\alpha(s) \rangle \, \mathrm{d}s \\ + \gamma_3 \int_0^t \|v_\alpha(s)\|^2_{L^2} \, \mathrm{d}s \end{aligned} \tag{4.7}$$

holds almost surely.

Proof. For any $t \in [0,T]$ we can estimate, taking into account Remark 4.1:

$$\begin{aligned} &\langle g_\alpha(v_\alpha(t) + W_S(t)), v_\alpha(t) \rangle \\ &\qquad = \lambda \, \langle v_\alpha(t) + W_S(t), v_\alpha(t) \rangle - \langle p_\alpha(v_\alpha(t) + W_S(t)), v_\alpha(t) + W_S(t) \rangle \\ &\qquad\quad + \langle p_\alpha(v_\alpha(t) + W_S(t)), W_S(t) \rangle \\ &\qquad \le \frac{3}{2} \lambda \|v_\alpha(t)\|^2_{L^2} + \frac{1}{2} \lambda \|W_S(t)\|^2_{L^2} - \langle p(J_\alpha(v_\alpha(t) + W_S(t))), J_\alpha(v_\alpha(t) + W_S(t)) \rangle \\ &\qquad\quad + \langle p(J_\alpha(v_\alpha(t) + W_S(t))), W_S(t) \rangle \end{aligned}$$

we introduce the notation $y_\alpha(t) = J_\alpha(v_\alpha(t)+W_S(t))$ and we consider the difference $-\langle p(y_\alpha), y_\alpha\rangle + \langle p(y_\alpha), W_S(t)\rangle$

by using Young's inequality in second term: $ab \le \frac{1}{p}\,\epsilon\, a^p + \frac{1}{q}\epsilon^{1-q} b^q$ we get the bound

$$\le -\langle p(y_\alpha(t)), y_\alpha(t)\rangle + \frac{2d+1}{2d+2}\,\epsilon \int_{\mathcal{O}} |p(y_\alpha(t))|^{(2d+2)/(2d+1)}\,\mathrm{d}\xi + \frac{1}{2d+2}\,\epsilon^{-2d-1}\int_{\mathcal{O}} |W_S(t)|^{2d+2}\,\mathrm{d}\xi$$

and, using the bounds in Hypothesis 2.2: $-r\cdot p(r) \le c_2 - c_3\, r^{2d+2}$, $|p(r)| \le c_1(1+r^{2d+1})$, by a suitable choice of ϵ we can find positive constants c_5 and c_6 such that

$$\le -c_5\langle p(y_\alpha(t)), y_\alpha(t)\rangle + c_6\left(1 + \int_{\mathcal{O}} |W_S(t,\xi)|^{2d+2}\,\mathrm{d}\xi\right).$$

By the estimates in Lemma 3.7, the quantity

$$\int_{\mathcal{O}} |W_S(t,\xi)|^2 + |W_S(t,\xi)|^{2d+2}\,\mathrm{d}\xi$$

is almost surely finite for any $t \in [0,T]$ and the thesis follows.

□

We sum the estimates in (4.5) and (4.6) and we use (4.7) to get

$$\begin{aligned}\frac{1}{2}\|v_\alpha(t)\|^2_{L^2(\mathcal{O})} + \frac{1}{2}\|v_{\alpha;t}\|^2_{\mathcal{X}} + k_0\int_0^t \|v_\alpha(s)\|^2_{H^1_0(\mathcal{O})}\,\mathrm{d}s + \gamma_2\int_0^t \langle p(y_\alpha(s)), y_\alpha(s)\rangle_{L^2(\mathcal{O})}\,\mathrm{d}s \\ \le \frac{1}{2}\|v_\alpha(0)\|^2 + \frac{1}{2}\|v_{\alpha;0}\|^2_{\mathcal{X}} + \gamma_1 + \gamma_3\int_0^t \|v_\alpha(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s + \int_0^t\int_0^\infty \rho'(s)\langle\nabla v_\alpha(s), \nabla v_\alpha(s-\sigma)\rangle\,\mathrm{d}\sigma\,\mathrm{d}s \\ + \frac{1}{2}\rho(0)\int_0^t \|v_\alpha(s)\|^2_{H^1_0}\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_0^\infty \rho'(\sigma)\|v(s-\sigma)\|^2_{H^1_0}\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned}$$

and we write

$$\begin{aligned}\frac{1}{2}\|v_\alpha(t)\|^2_{L^2(\mathcal{O})} + \frac{1}{2}\|v_{\alpha;t}\|^2_{\mathcal{X}} + (k_0-\rho(0))\int_0^t \|v_\alpha(s)\|^2_{H^1_0(\mathcal{O})}\,\mathrm{d}s + \gamma_2\int_0^t \langle p(y_\alpha(t)), y_\alpha(t)\rangle_{L^2(\mathcal{O})}\,\mathrm{d}t \\ \le \frac{1}{2}\|v_\alpha(0)\|^2 + \frac{1}{2}\|v_{\alpha;0}\|^2_{\mathcal{X}} + \gamma_1 + \gamma_3\int_0^t \|v_\alpha(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s - \int_0^t\int_0^\infty k(s)\langle\nabla v_\alpha(s), \nabla v_\alpha(s-\sigma)\rangle\,\mathrm{d}\sigma\,\mathrm{d}s \\ - \frac{1}{2}\int_0^\infty k(\sigma)\int_0^t \|v_\alpha(s)\|^2_{H^1_0}\,\mathrm{d}s\,\mathrm{d}\sigma - \frac{1}{2}\int_0^t\int_0^\infty k(\sigma)\|v_\alpha(s-\sigma)\|^2_{H^1_0}\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned}$$

which finally leads, by the positivity of k, to

$$\begin{aligned}\frac{1}{2}\|v_\alpha(t)\|^2_{L^2(\mathcal{O})} + \frac{1}{2}\|v_{\alpha;t}\|^2_{\mathcal{X}} + (k_0-\rho(0))\int_0^t \|v_\alpha(s)\|^2_{H^1_0(\mathcal{O})}\,\mathrm{d}s + \gamma_2\int_0^t \langle p(y_\alpha(t)), y_\alpha(t)\rangle_{L^2(\mathcal{O})}\,\mathrm{d}t \\ \le \frac{1}{2}\|v_\alpha(0)\|^2 + \frac{1}{2}\|v_{\alpha;0}\|^2_{\mathcal{X}} + \gamma_1 + \gamma_3\int_0^t \|v_\alpha(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s. \quad (4.8)\end{aligned}$$

Let us recall that $y_\alpha = J_\alpha(v_\alpha(t) + W_S(t))$; then an application of Gronwall's lemma in (4.8) and Hypothesis 2.1(2) imply that

$$v_\alpha \text{ is bounded in } L^\infty(0,T;L^2(\mathcal{O})) \cap L^2(0,T;H^1_0(\mathcal{O})).$$

Further, by Hypothsis 2.2(2), $-p(r) \cdot r \le c_2 - c_3 r^{2d+2}$ and it follows again from (4.8) that

$$y_\alpha \text{ is bounded in } L^{2d+2}((0,T) \times \mathcal{O}).$$

Now, since y_α belongs to $L^{2d+2}((0,T) \times \mathcal{O})$ and $|p(y_\alpha)| \le c\,(1 + |y_\alpha|^{2d+1})$, it follows that

$$p(y_\alpha) \text{ is bounded in } L^{(2d+2)/(2d+1)}((0,T) \times \mathcal{O}).$$

Therefore, there exists a sequence $\alpha_k \downarrow 0$ and processes $v^\star$, $y^\star$ and η such that

$$\begin{aligned} v_{\alpha_k} &\rightharpoonup v^\star &&\text{in } L^2((0,T) \times \mathcal{O}) \cap L^2(0,T;H^1_0(\mathcal{O})) \\ y_{\alpha_k} &\rightharpoonup y^\star &&\text{in } L^2((0,T) \times \mathcal{O}) \\ p(y_{\alpha_k}) &\rightharpoonup \eta &&\text{in } L^{(2d+2)/(2d+1)}((0,T) \times \mathcal{O}). \end{aligned} \tag{4.9}$$

Moreover, we have

$$y_\alpha + \alpha\, p(y_\alpha) = v_\alpha + W_S$$

and, passing to the limit for $\alpha_k \to 0$ we have

$$y_{\alpha_k} - v_{\alpha_k} \rightharpoonup W_S \qquad \text{in } L^{(2d+2)/(2d+1)}((0,T) \times \mathcal{O})$$

which means, by the uniqueness of the limit, that

$$y^\star = v^\star + W_S.$$

We introduce the antiderivative $B(r) = \int_0^r p(x)\,\mathrm{d}x$ and define, for $x \in L^{2d+2}((0,T)\times \mathcal{O})$, the function

$$\phi(x) = \int_0^T \int_{\mathcal{O}} B(x(t,\xi))\,\mathrm{d}\xi\,\mathrm{d}t.$$

Then $\phi(x)$ is a lower semicontinuous convex function having sub-gradient $p(x) \in L^{(2d+2)/(2d+1)}((0,T) \times \mathcal{O})$. Using convexity we have

$$\begin{aligned} \phi(y_{\alpha_k}) - \phi(x) &\le \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k})(y_{\alpha_k} - x)\,\mathrm{d}\xi\,\mathrm{d}t \\ &= \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k}) y_{\alpha_k}\,\mathrm{d}\xi\,\mathrm{d}t - \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k}) x\,\mathrm{d}\xi\,\mathrm{d}t. \end{aligned}$$

Using the lower semicontinuity we pass to the limit as $\alpha_k \to 0$, getting

$$\phi(y^\star) - \phi(x) \le \liminf_{\alpha_k \to 0} \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k}) y_{\alpha_k}\,\mathrm{d}\xi\,\mathrm{d}t - \int_0^T \int_{\mathcal{O}} p(y^\star) x\,\mathrm{d}\xi\,\mathrm{d}t.$$

Remark 4.2 *We are left to show that*

$$\liminf_{\alpha_k \to 0} \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k}) y_{\alpha_k} \,\mathrm{d}\xi \,\mathrm{d}t \le \int_0^T \int_{\mathcal{O}} \eta y^\star \,\mathrm{d}\xi \,\mathrm{d}t$$

which will allow us to conclude $\eta = p(y^\star)$.

Proof. Recall that $\liminf_n (f_n + g_n) \le \limsup_n f_n + \liminf_n g_n$ and, by induction, $\liminf_n \sum_k f_n^k \le \sum_k \limsup_n f_n^k$. Now we use an orthonormal basis of $L^2((0,T) \times \mathcal{O})$ to write

$$\begin{aligned}
\liminf_{\alpha_k \to 0} \int_0^T \int_{\mathcal{O}} p(y_{\alpha_k}) y_{\alpha_k} \,\mathrm{d}\xi \,\mathrm{d}t &= \liminf_{\alpha_k \to 0} \sum_{j=1}^{\infty} \langle p(y_{\alpha_k}), e_j\rangle \, \langle y_{\alpha_k}, e_j\rangle \\
&\le \sum_{j=1}^{\infty} \lim_{\alpha_k \to 0} \langle p(y_{\alpha_k}), e_j\rangle \, \langle y_{\alpha_k}, e_j\rangle \\
&= \sum_{j=1}^{\infty} \langle \eta, e_j\rangle \, \langle y^\star, e_j\rangle = \int_0^T \int_{\mathcal{O}} \eta(t,\xi) y^\star(t,\xi) \,\mathrm{d}\xi \,\mathrm{d}t.
\end{aligned}$$

□

We return to the approximating equation (4.3) which, in mild form, reads

$$\psi_\alpha(t) = \mathcal{T}(t)\bar\psi + \int_0^t \mathcal{T}(t-s)\mathcal{F}_\alpha(W_T(s) + \psi_\alpha(s)) \,\mathrm{d}s, \qquad \psi_\alpha(t) = (v_\alpha(t), v_{\alpha;t}(\cdot\,))$$

and pass to the limit as $\alpha_k \to 0$. Let $h \in \mathcal{H}$; then we have

$$\langle \psi_{\alpha_k}(t), h\rangle = \langle \mathcal{T}(t)\bar\psi, h\rangle + \int_0^t \langle \begin{pmatrix} \lambda(v_{\alpha_k}(s) + W_S(s)) - p(y_{\alpha_k}(s)) \\ 0 \end{pmatrix}, \mathcal{T}(t-s)^\star h\rangle \,\mathrm{d}s$$

and the convergences stated in (4.9) imply that, as $\alpha_k \to 0$, it holds

$$\begin{aligned}
\langle \psi^\star(t), h\rangle &= \langle \mathcal{T}(t)\bar\psi, h\rangle + \int_0^t \langle \lambda(v^\star(s) + W_S(s)) - \eta, S(t-s)^\star h\rangle \,\mathrm{d}s \\
&= \langle S(t)\bar\psi, h\rangle + \int_0^t \langle \begin{pmatrix} g(v^\star(s) + W_S(s)) \\ 0 \end{pmatrix}, \mathcal{T}(t-s)^\star h\rangle \,\mathrm{d}s
\end{aligned}$$

and the thesis follows from the arbitrariety of h.

5 Asymptotic properties of the nonlinear equation

In the last result of the paper we discuss the existence of an invariant measure for the nonlinear system (4.1). For the sake of simplicity, we shall only consider the case of a strictly decreasing polynomial nonlinearity g (which, however, does not imply that the nonlinear perturbation $\mathcal{F}$ is strictly dissipative). Let us state the assumption below:

$$\text{for some } \gamma \le 0 \text{ it holds } \langle g(x_1) - g(x_2), x_1 - x_2\rangle_Y \le \gamma \, \|x_1 - x_2\|^2.$$

For any $\bar{\phi} \in \mathcal{H}$, we let $\phi(t; s, \bar{\phi})$ be the solution of (4.1) with initial condition $\bar{\phi}$ at time s; we denote

$$P_t f(\bar{\phi}) = \mathbb{E}[f(\phi(t+s; s, \bar{\phi}))], \qquad f \in B_b(\mathcal{H}),$$

the *transition semigroup* associated to the nonlinear equation (4.1).

Theorem 5.1 *Tthere exists a unique $\zeta \in L^2(\Omega; \mathcal{H})$ such that*

$$\lim_{s \to -\infty} \phi(0; s, \bar{\phi}) = \zeta \qquad in L^2(\Omega; \mathcal{H}) \text{ for any } \bar{\phi} \in \mathcal{H}.$$

The law μ of ζ is the unique invariant measure for the nonlinear system (4.1), it is ergodic and strongly mixing.

Proof. The way we shall proceed is quite standard, compare [11, Proposition 3.16]. Thus, we only outline the construction here.

First step: extending the equation. We introduce another X-valued cylindrical Wiener process $W_1(t)$, independent of $W(t)$, and we define the two-sided Wiener process

$$\bar{W}(t) = \begin{cases} W(t), & t \geq 0, \\ W_1(t), & t < 0 \end{cases}$$

endowed with the filtration $\bar{\mathcal{F}}_t = \sigma(\bar{W}(s),\ s \leq t)$ for $t \in \mathbb{R}$. We may introduce the following form of equation (4.1), having initial condition at time $s \in \mathbb{R}$:

$$\begin{aligned} \mathrm{d}\phi(t) &= [\mathcal{A}\phi(t) + \mathcal{F}(\phi(t))]\,\mathrm{d}t + \mathcal{Q}\,\mathrm{d}\bar{W}(t) \\ \phi(s) &= \bar{\phi} \in \mathcal{H}. \end{aligned} \tag{5.1}$$

By previous Theorem 4.1 there exists a unique solution $\phi(t; s, \bar{\phi})$ of problem (5.1).

Second step: the solution is bounded at all times. Let us consider the stochastic convolution process $W_T(t; s)$ that is the mild solution of the linear problem

$$\mathrm{d}\phi(t) = \mathcal{A}\phi(t)\,\mathrm{d}t + \mathcal{Q}\,\mathrm{d}\bar{W}(t), \qquad \phi(s) = 0;$$

we remark that $\psi(t; s, \bar{\phi}) = \phi(t; s, \bar{\phi}) - W_T(t; s)$ is the mild solution of the problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \mathcal{A}\psi(t) + \mathcal{F}(\psi(t) + W_T(t; s)), \qquad \psi(s) = \bar{\phi}.$$

It follows, by computing the subdifferential of $\|\psi\|_{\mathcal{H}}$ that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_{\mathcal{H}} &= \langle \mathcal{A}\psi(t) + \mathcal{F}(\psi(t) + W_T(t; s)), \tfrac{\psi(t)}{\|\psi(t)\|_{\mathcal{H}}} \rangle_{\mathcal{H}} \\ &= \langle \mathcal{A}\psi(t) + \mathcal{F}(\psi(t) + W_T(t; s)) - \mathcal{F}(W_T(t; s)), \tfrac{\psi(t)}{\|\psi(t)\|_{\mathcal{H}}} \rangle_{\mathcal{H}} \\ &\quad + \langle \mathcal{F}(W_T(t; s)), \tfrac{\psi(t)}{\|\psi(t)\|_{\mathcal{H}}} \rangle_{\mathcal{H}} \\ &\leq -\delta\|\psi(t)\|_{\mathcal{H}} + \|\mathcal{F}(W_T(t; s))\|_{\mathcal{H}} \end{aligned}$$

and recalling the initial condition $\psi(s) = \bar{\phi}$, this implies

$$\|\psi(t; s, \bar{\phi})\| \leq e^{-\delta(t-s)}\|\bar{\phi}\|_{\mathcal{H}} + \int_s^t e^{-\delta(t-\sigma)}\|\mathcal{F}(W_T(\sigma; s)\|_{\mathcal{H}}\,\mathrm{d}\sigma.$$

We get, by using the definition of $\mathcal{F}$ and the bound on the growth of g in Hypothesis 2.2(i), that

$$\int_s^t e^{-\delta(t-\sigma)} \|\mathcal{F}(W_T(t;s)\|_{\mathcal{H}} \,\mathrm{d}\sigma \le C \int_s^t e^{-\delta(t-\sigma)} \left(1 + \| \, |W_S(\sigma;s)|^{2d+1} \, \|_{L^2(\mathcal{O})}\right) \mathrm{d}\sigma$$

and Lemma 3.7 implies that

$$\mathbb{E} \int_s^t e^{-\delta(t-\sigma)} \|\mathcal{F}(W_T(t;s)\|_{\mathcal{H}} \,\mathrm{d}\sigma \le C' \int_s^t e^{-\delta(t-\sigma)} \,\mathrm{d}\sigma$$

for a constant C'; therefore

$$\mathbb{E}\|\psi(t;s,\bar\phi)\| \le e^{-\delta(t-s)} \|\bar\phi\|_{\mathcal{H}} + C'_\delta$$

and, recalling the definition of $\psi(t) = \phi(t) - W_A(t)$ and by Proposition 2.10, we finally get for some $C > 0$

$$\mathbb{E}\|\phi(t;s,\bar\phi)\| \le C(1 + \|\bar\phi\|_{\mathcal{H}}). \tag{5.2}$$

Fourth step: convergence as the initial time goes to $-\infty$. We fix an initial condition $\bar\phi$ and initial times $s < \sigma$. We set $Z(t) = \phi(t;s,\bar\phi) - \phi(t;\sigma,\bar\phi)$ for $t \ge \sigma$. It follows that $Z(t)$ satisfies a deterministic equation with random coefficients

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Z(t) = \mathcal{A}Z(t) + F(\phi(t;s,x)) - F(\phi(t;\sigma,x)), \qquad Z(\sigma) = \phi(\sigma;s,\bar\phi) - \bar\phi.$$

As in previous step, we take the differential of $\|Z(t)\|_{\mathcal{H}}$ and we get

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \|Z(t)\|_{\mathcal{H}} = \langle \mathcal{A}Z(t) + F(\phi(t;s,x)) - F(\phi(t;\sigma,x)), \tfrac{Z(t)}{\|Z(t)\|} \rangle_{\mathcal{H}} \le -\delta \, \|Z(t)\|_{\mathcal{H}}$$

which implies

$$\mathbb{E}\|Z(t)\|_{\mathcal{H}} \le e^{-c(t-\sigma)} (\|\bar\phi\|_{\mathcal{H}} + \mathbb{E}\|\phi(\sigma;s,\bar\phi)\|_{\mathcal{H}})$$

which, by (5.2), implies

$$\mathbb{E}\|Z(t)\| \le C \, e^{-\delta(t-\sigma)} (1 + \|\bar\phi\|_{\mathcal{H}}).$$

Therefore, there exists a random variable ζ, the same for all $\bar\phi \in \mathcal{H}$, such that

$$\lim_{s \to -\infty} \mathbb{E}\|\phi(t;s,\bar\phi) - \zeta\|_{\mathcal{H}} = 0$$

and the law $\mu = \mathcal{L}(\zeta)$ is the unique invariant measure for the transition semigroup P_t, $t \ge 0$. ☐

We stress, again, the following point. Notice that previous theorem assures that if the initial condition $\phi(0) = \bar\phi = \binom{x}{y}$ satisfies $\mathcal{L}(X) = \mu$, then the process $\phi(t)$ is stationary. If we consider the projection on the first component, then it also becomes a stationary process. But this projection is the solution $u(t)$ of the nonlinear Volterra equation (1.1) with initial data $u(0) = x$ and $u_0 = y$.

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