Università di Trento

DIPARTIMENTO DI MATEMATICA
Doctoral Programme in Mathematics - Cycle XXXVII

Thesis for the Degree of Philosophiae Doctor

Dynamical C*-Algebras and Quadratic Perturbations for Fermions

Candidate:
Valentino Abram

Supervisor:
Prof. Romeo Brunetti

Referees:
Prof. Nicolò Drago
Prof. Kasia Rejzner

Academic Year 2023-2024

Abstract

We study interacting Fermionic fields on globally hyperbolic spacetimes in the context of dynamical C*-algebras. Our focus is on a particular class of interactions, that of kinetic perturbations; we provide their definition and we show that these induce, in the fermionic dynamical C*-algebra, symbols that obey the canonical anticommutation relation in a locally perturbed spacetime. We also study the adjoint action on the CAR algebra, a C*-subalgebra of the fermionic dynamical C*-algebra, induced by symbols associated with functionals that are compactly supported, self-adjoint, gauge-invariant, quadratic, and which induce normally hyperbolic operators (these are objects which can be interpreted as quadratic perturbations of the free Dirac Lagrangian); we show that this amounts to a Bogoliubov transformation of the CAR algebra. Said Bogoliubov transformation is shown to satisfy the Shale-Stinespring criterion for every gauge-invariant, quasifree, Hadamard state on the CAR algebra in a globally hyperbolic spacetime with compact Cauchy surfaces, and for the Minkowski vacuum; therefore the Bogoliubov transformation is implemented via unitary operators on the associated representation Fock space. In the case of the Minkowski vacuum, we also show that the phases of the unitary implementers can be chosen so that these satisfy the Bogoliubov causal factorization, provided that the representation of the perturbed CAR algebra satisfies twisted Haag duality.

Acknowledgements

First of all, I would like to thank my supervisor, prof. Romeo Brunetti, for giving me the opportunity to work on such a cutting-edge topic in mathematical physics, and for allowing me to pursue other projects on the side. His insights, guidance, and most importantly coda alla vaccinara and pastiera have been the fundamental fuel of this research project.
I also thank the mathematical physics group for our discussions; in particular, I would like to thank prof. Valter Moretti and prof. Enrico Pagani for being the kind and bright lecturers they are. Thanks are also due to prof. Klaus Fredenhagen for sharing his knowledge and for carefully listening to my progress on my research project during my stay at Hamburg.
A particular thanks goes to the referees, prof. Nicolò Drago and prof. Kasia Rejzner, for carefully reading my thesis and for their suggestions, which I believe significantly improved the final version of this work.
I am of course indebted to my doktorbrüder, Andrea and Antonio, and my dear colleague Anna for easing the burden of research during our breaks: thank you for sharing your viewpoint on mathematics-related issues, for listening to my complaints, especially during my time as a teaching assistant, and for always providing useful insights.
Working on some more applied projects with people having a background different than mine has been an invaluable experience; for this reason I also thank my friends Ankur and Gianmarco for getting me involved in their research activities.
Most importantly, my deepest gratitude goes towards Angelica, without whom I wouldn't have been able to see this doctorate to completion: thank you for keeping me sane and thank you for enduring this with me.

Contents

1 Introduction **1**

1.1 Outline of the thesis 2

2 Geometric and analytic preliminaries **4**

2.1 Basic notions in spin geometry on pseudo-riemannian manifolds 4

2.2 Topologies on spaces of sections of vector bundles 10

2.2.1 The Fréchet topology 10

2.2.2 The Whitney C^∞-topology 13

2.3 Fermionic functionals and Lagrangians 23

3 Different metrics and spinors **33**

3.1 Connecting different Lorentzian metrics 33

3.2 Generalized cylinders in spin geometry 35

3.2.1 The Dirac operator in generalized cylinders 41

3.2.2 Parallel transport in generalized cylinders 44

4 The dynamical C*-algebra for the Fermi fields **48**

4.1 Basic notions in enriched categories 48

4.2 The free algebra generated by formal S-matrices 63

4.3 The canonical anticommutation relations 72

4.4 A C*-norm on a $*$-subalgebra of the universal algebra $\mathfrak{A}$ 81

4.5 Kinetic perturbations and the canonical anticommutation relations 83

5 Automorphisms of the CAR algebra and their unitary implementation **90**

5.1 Transformations induced by quadratic functionals 90

5.2 The self-dual CAR algebra and its quasifree states 99

5.3 Quadratic functionals and the associated Bogoliubov transformations . . . 102

5.4 The phases and Bogoliubov's causal factorization 106

6 Conclusions and outlook **109**

Bibliography **110**

1 Introduction

Recently, D. Buchholz and K. Fredenhagen [14] developed a novel C*-algebraic framework to describe interacting quantum fields, that of *dynamical C^*-algebras*. Starting from a Lagrangian $\mathscr{L}$ for the scalar field on Minkowski spacetime $\mathbb{M}^d$, they consider a family of abstract symbols $\{S(F)\}_{F\in\mathcal{F}_{\mathrm{loc}}(M)}$, which the authors regard as local scattering matrices, labeled by local, polynomial functionals (i.e. interactions). The free group of these S-symbols is then quotiented by the ideal generated by two relations, both proven for S-matrices in the context of perturbative algebraic quantum field theory (see [36] and [20]):

(i) *Bogoliubov's causal factorization*, which encodes the causal structure of the background spacetime;

(ii) an integrated version of *Schwinger-Dyson's equation* (see [14, Appendix] and [12, Section 7]), which encodes the dynamical features of the chosen Lagrangian $\mathscr{L}$, i.e. the field equations.

Starting from this group, one constructs a complex $*$-algebra (where the involution is defined as to make the S-symbols unitary) which is then endowed with a C*-seminorm; the dynamical C*-algebra $\mathfrak{A}_{\mathscr{L}}$ is then its enveloping C*-algebra [34, Definition 10.1.10].
In [14], the authors construct a net of local algebras complying with Haag-Kastler axioms; moreover, they show that given the Lagrangian of the free-field, the associated dynamical C*-algebra contains as a subalgebra the Weyl C*-algebra. Finally, they show that given a Lagrangian $\mathscr{L}$ and considering the perturbed Lagrangian $\mathscr{L}+V$, it is possible to find, given a bounded, causally closed region $O \subseteq \mathbb{M}^d$, a region $\widehat{O}$ containing its closure such that $\mathfrak{A}_{\mathscr{L}+V}(O)$ is isomorphic to a subalgebra of $\mathfrak{A}_{\mathscr{L}}(\widehat{O})$.
In [15], the same authors enlarge the class of functionals they consider by allowing for *kinetic perturbations*, that is, compactly supported perturbations of the kinetic term of the free Lagrangian for the scalar field on Minkowski spacetime which mimic local perturbations due to gravity. They show that given a kinetic perturbation K, the relative S-symbols of the Weyl operators satisfy a modified version of the canonical commutation relations, which takes into account the metric deformation whose information is contained in K. In the same reference, the authors moreover show that the S-symbols associated to quadratic functionals (which include the kinetic perturbations) give rise to Bogoliubov transformations of the Weyl algebra; when considering the Minkowski vacuum state on the Weyl algebra, these are shown to satisfy Shale's criterion [41], and thus can be unitarily implemented on the representation Fock space. Finally, assuming that the perturbed Weyl operators on the bosonic Fock space satisfy Haag duality, the authors show that the phases of the unitary implementers of the Bogoliubov transformations can be chosen so that these operators obey Bogoliubov's causal factorization.
The framework of dynamical C*-algebras was subsequently extended to Fermionic fields in [12]. A straightforward extension is however impossible: indeed, in the Fermionic case, only a certain class of functionals (the gauge-invariant ones) admits a physical interpretation as the class of interactions, and thus as a family of labels of the S-symbols when

these are understood as local S-matrices. The issue is that if one were to consider only these functionals in the construction of a dynamical C*-algebra, one would be unable to incorporate the Schwinger-Dyson equation which requires linear (and thus non-gauge invariant) functionals. To solve this issue, the authors construct a functorial version of the so-called "η-trick" [20], called *covariant Grassmann multiplication*, which is used to alter the combinatorics of fermionic functionals to the bosonic case in a functorial way. This enables them to construct a functor which assign to each real, finite-dimensional Grassmann algebra G and abstract $\mathbb{Z}_2$-graded, complex $*$-algebra $\mathfrak{A}_G$; this algebra is constructed using S_G-symbols labeled by fermionic functionals tensored with the Grassmann variables and it incorporates the causal factorization (i) and the dynamics (ii). Due to the functoriality of the assignment, it is possible to extract a $\mathbb{Z}_2$-graded, complex $*$-algebra $\mathfrak{A}$; in [12, Section 5] the authors show that in the case of the Lagrangian of the free Dirac field on Minkowski spacetime $\mathbb{M}^4$, the algebra $\mathfrak{A}$ contains a C*-CAR-subalgebra, i.e. the canonical anticommutation relations.

1.1 Outline of the thesis

This last paper is the starting point of this thesis. Our goal was first of all to extend the construction provided by [12] to the free Dirac field on a globally hyperbolic, Lorentzian spin manifold; then among the functionals labeling the S-symbols our goal was to single out and study the behaviour of kinetic perturbations. On curved-spacetimes these are highly non-trivial as both the Dirac operator and the spinor bundles, whose space of sections is the configuration space for the Dirac field, depend on the metric. Finally, my goal was to study the behaviour of a certain class of quadratic perturbations and possibly prove that the S-symbols associated to them can be unitarily implemented on Fock space.

Chapter 2: geometric and analytic preliminaries

In Chapter 2 we review some of the necessary geometric and analytic notions, in order to also fix the notations used throughout the thesis. In Section 2.1 we briefly describe spin manifolds, spinor bundles and Dirac operators; in Section 2.2 we review two topologies one can endow the spaces of sections of spinor bundles with, the Fréchet topology (Section 2.2.1) and the Whitney C^∞-topology (Section 2.2.2). The first one is used to define fermionic functionals (see the brief review in Section 2.3), while the second one is used to show well-posedness of kinetic perturbations.

Chapter 3: different metrics and spinors

In Chapter 3 we begin by showing that if two globally hyperbolic metrics g_0 and g_1 are such that

(i) g_1 is a compact perturbation of g_0;

(ii) the gradient of the Cauchy temporal function of g_0 is timelike for g_1 and that the Cauchy slices $\{t_0 = k\}$ are spacelike hypersurfaces for g_1;

then these can be joined by a smooth one-parameter curve of Lorentzian metrics (see Section 3.1). In Section 3.2 we then use this fact together with some tools presented in [5] (which we describe in detail in Section 3.2.1 and 3.2.2) to define an invertible, linear map τ_0^1 which allows to relate sections of spinor bundles associated with the two different metrics.

Chapter 4: the dynamical C*-algebra for the Fermi field

In Chapter 4 we then generalize the contents of [12] to the globally hyperbolic case. In Section 4.1 we provide a brief review of enriched category theory, which is used to reformulate the definition of *covariant Grassmann multiplication* and results of [12, Section 3]. In Section 4.2 we consider the free Dirac lagrangian and define fermionic kinetic perturbations (see Definition 4.2.1), which we show to be well-defined fermionic functionals (Lemma 4.2.2). We then construct, using S-symbols, a functor and a natural transformation which we show to be a covariant Grassmann multiplication in the sense of Definition 4.1.6. These objects are the foundations of fermionic dynamical C*-algebras. In Section 4.3 we show that a universal algebra $\mathfrak{A}$, obtained from the covariant Grassmann multiplication thanks to Theorem 4.1.8, contains a subalgebra generated by self-adjoint, $\mathbb{R}$-linear symbols which obey the canonical anticommutation relations; we then use this fact to endow a $*$-subalgebra of $\mathfrak{A}$ with a C*-seminorm in Section 4.4. Finally, in Section 4.5 we show how kinetic perturbations act on the algebra of the canonical anticommutation relations; in particular we show that they give rise to "perturbed" symbols in $\mathfrak{A}$ (akin to the perturbed Weyl operators) obeying a modified version of the canonical anticommutation relations which takes into account the modified geometry of spacetime induced by the kinetic perturbation.

Chapter 5: automorphisms of the CAR algebra and their unitary implementation

In Chapter 5 we study the behaviour of the adjoint action of S-symbols associated to gauge-invariant, self-adjoint quadratic fermionic functionals, and in particular we show that they give rise to Bogoliubov transformations of the CAR algebra. In Section 5.3 we show that these Bogoliubov transformations satisfy the Shale-Stinespring criterion (see [42] and [8]) in two cases: for any given quasifree, Hadamard, gauge-invariant state in a globally hyperbolic spacetimes with compact Cauchy surfaces, and for the vacuum state on Minkowski spacetime. In Section 5.4 we also point out that using tools similar to those used in [15, Section 6] it is possible to prove that, in the case of the vacuum state on Minkowski spacetime, the phases of the unitary implementers of the Bogoliubov transformation can be chosen so that these unitary operators on Fock space satisfy the Bogoliubov causal factorization, *provided that twisted Haag duality for the perturbed CAR algebra holds.*

2 Geometric and analytic preliminaries

2.1 Basic notions in spin geometry on pseudo-riemannian manifolds

We begin the exposition by recalling some definitions and results in order to fix the notation used throughout the paper, borrowing some of the content from [1]. We refer the reader to [46], [29], [26] and [32] for more details.
We shall work on n-dimensional spacetimes, that is, couples (M, g) consisting of a connected, orientable, time-orientable smooth manifold[1] M and a non-degenerate, pseudo-riemannian metric g. We shall further suppose that two additional conditions hold:

(i) we assume (M, g) is *globally hyperbolic*: as pointed out in [6], this entails that given a normally hyperbolic operator, this admits retarded and advanced Green operators;

(ii) we also assume that the second Stiefel-Whitney class $w_2(TM)$ vanishes: this enables us to consider *spin structures* on TM. This will be explained in depth later.

On the manifold M modelling our spacetime, we shall consider *fiber bundles*, i.e., quadruples (B, M, π, F) where $\pi\colon B \to M$ is a smooth, surjective map and such that there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of the base manifold M (generally given by the domain of an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$) and an associated collection $\{\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F\}_{\alpha\in A}$ of diffeomorphisms, called *trivialization of the bundle*, such that

$$\pi_1 \circ \Phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)} \text{ for every } \alpha \in A \ . \tag{2.1}$$

Notice that given a point $p \in M$, the fiber $B_p \doteq \pi^{-1}(p)$ is diffeomorphic to F; this is the reason F is called *typical fiber*. We can consider the *transition functions* of the bundle, that is, the maps

$$\begin{aligned} \Phi_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times F &\to (U_\alpha \cap U_\beta) \times F \\ (p, f) \qquad &\mapsto \Phi_\alpha\left(\Phi_\beta^{-1}(p, f)\right) \ . \end{aligned}$$

By the condition (2.1), we have that $\Phi_{\alpha\beta}(p, f) = (p, g_{\alpha\beta}(p)(f))$ for some $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(F)$; the collection $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ is called *cocycle*. We also define the composition

$$\phi_\alpha \doteq (\varphi_\alpha \times \mathrm{id}_F) \circ \Phi_\alpha\colon \pi^{-1}(U_\alpha) \mapsto \varphi_\alpha(U_\alpha) \times F$$

In the context of quantum field theory, one usually deals with two kinds of fiber bundles:

(i) *vector bundles*, that is, fiber bundles whose typical fiber is a vector space and whose cocycle is such that $g_{\alpha\beta}(p) \in \mathrm{GL}(F)$ for every $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$;

(ii) *principal G-bundles*, that is, fiber bundles whose typical fiber is a Lie group G, whose total space P_G is a right G-manifold with right action $R_g\colon P_G \to P_G$ and whose trivialization $\{\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G\}$ consists of right G-equivariant maps. This entails that the cocycle $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(G)\}$ consists of left-translations [46, Lemma 27.7].

[1] We always assume that manifolds are Hausdorff and second-countable, and thus paracompact.

Given a principal G-bundle $\pi\colon P_G \to M$ and a representation $\rho\colon G \to \mathrm{GL}(V)$ of the group on a finite-dimensional vector space V, we can construct the *associated vector bundle* $\pi\colon P_G \times_\rho V \to M$ with typical fiber V, where

$$P_G \times_\rho V = P_G \times V/\sim, \qquad (p,v) \sim (R_g(p), \rho(g)^{-1}v)\ \forall g \in G$$

[46, Theorem 31.9] shows that there exist linear isomorphisms

$$\cdot^\flat\colon \Omega^q_\rho(P_G,V) \to \Gamma\left(\wedge^q T^*M \otimes P_G \times_\rho V\right) \qquad \cdot^\sharp\colon \Gamma\left(\wedge^q T^*M \otimes P_G \times_\rho V\right) \to \Omega^q_\rho(P_G,V) \tag{2.2}$$

between the spaces $\Omega^q_\rho(P_G,V)$ of V-valued, tensorial q-forms of type[2] ρ on P_G and the space of q-forms on M with values in $P_G \times_\rho V$.
Let $\mathfrak{g}$ be the Lie algebra of the Lie group G; Ehresmann connection 1-forms $\omega \in \Omega^1(P_G,\mathfrak{g})$ can be used to induce an *exterior covariant differentiation* $D\colon \Omega^q_\rho(P_G,V) \to \Omega^{q+1}_\rho(P_G,V)$,

$$D\varphi \doteq d\varphi + \omega \cdot_\rho \varphi \tag{2.3}$$

where

$$(\omega \cdot_\rho \varphi)_p(v_1,\dots,v_{q+1}) = \frac{1}{q!}\sum_{\sigma\in S_{q+1}} \mathrm{sgn}(\sigma)\rho_*(\omega_p(v_{\sigma(1)}))\varphi_p(v_{\sigma(2)},\dots,v_{\sigma(q+1)})$$

and $\rho_*\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the Lie algebra representation induced by the group representation ρ.
We can then use the exterior covariant derivative to endow the associated vector bundle $P_G \times_\rho V$ with a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(P_G \times_\rho V) \to \Gamma(P_G \times_\rho V)$,

$$\nabla_v s \doteq (D(s^\sharp))^\flat(v)\ .$$

This section of the associated vector bundle can be expressed locally as

$$(\nabla_v s)(p) = \left[\sigma(p), v(p)\left(s^\sharp \circ \sigma\right) + \rho_*\Big(\omega_{\sigma(p)}\Big(\sigma_{*p}(v_p)\Big)\Big)s^\sharp(\sigma(p))\right] \tag{2.4}$$

where $\sigma\colon U \to P_{GU} \doteq \pi^{-1}(U)$ is a *local* section of the principal G-bundle $\pi\colon P_G \to M$, usually deemed *local gauge choice.* The pullback $\sigma^*\omega \in \Omega^1(U,\mathfrak{g})$, which is such that

$$(\sigma^*\omega)(v)|_p = \omega_{\sigma(p)}\Big(\sigma_{*p}(v_p)\Big)\ ,$$

is called *local gauge potential.*
An important construction which can be carried out in the case of complex vector bundles is that of the *conjugate bundle.* Let us recall that given a complex vector space V, the

[2] These are V-valued q-forms ω on P_G such that $R_g^*\omega = \rho(g)^{-1}\omega$ for all $g \in G$ and such that, given vector fields $\{X_i\}_{1\le i\le q}$ on P_G, it holds that $\omega(X_1,\dots,X_q) = 0$ if $\pi_*(X_i) = 0$ for some $i \in \{1,\dots,q\}$.

conjugate vector space $\overline{V}$ is a vector space which is real-isomorphic to V and whose complex structure is the conjugate complex structure. The antilinear isomorphism between $\overline{V}$ and V shall be denoted with $C\colon \overline{V} \to V$, and it can be used to induce, starting from a linear map $L\colon V \to V$, a linear map $\overline{L}\colon \overline{V} \to \overline{V}$ by defining

$$\overline{L} \doteq C^{-1} \circ L \circ C \ .$$

Given a basis $\{\overline{e}_i\}_{i\in I}$ of $\overline{V}$, the components $\{\overline{L}^i_j\}_{i,j\in I}$ of $\overline{L}$ are the complex conjugates of the components of L in the basis $\{C\overline{e}_i\}_{i\in I}$, i.e.

$$\overline{L}^i_j = \overline{L^i_j} \ .$$

If V is endowed with a sesquilinear form, then one can show that $\overline{V} \simeq V^*$, where V^* denotes the dual space to V. These facts can be extended to the case of complex vector bundles in the following way: first of all, consider a vector bundle $E \xrightarrow{\pi} M$ with typical fiber V, whose cocycle is given by $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$; then one can construct another vector bundle with typical fiber $\overline{V}$ and whose cocycle is given by $\{\overline{g_{\alpha\beta}}\}_{\alpha,\beta\in A}$; this vector bundle is the conjugate vector bundle $\overline{E} \xrightarrow{\pi} M$. The conjugation map $C\colon \overline{V} \to V$ can be extended to a vector bundle anti-isomorphism $C\colon \overline{E} \to E$, and if E is endowed with a hermitian metric h, then

(i) h, which assigns to every point $p \in M$ a sesquilinear form h_p on E_p, can be understood as a map assigning to each $p \in M$ a bilinear map $h_p\colon \overline{E}_p \times E_p \to \mathbb{C}$;

(ii) we have an isomorphism $\overline{E} \simeq E^*$.

Let us now consider the tangent bundle $\pi\colon TM \to M$ on our spacetime; by the assumptions on (M, g) this is an oriented vector bundle endowed with a Lorentzian structure, and we can thus consider its oriented, pseudo-orthonormal frame bundle, that is, a principal $\mathrm{SO}^0_{r,1}$-bundle $\pi_P\colon P_{\mathrm{SO}^0_{r,1}}(M) \to M$ naturally associated to it[3]. As the second Stiefel-Whitney class $w_2(TM)$ of M vanishes, we can consider a *spin structure* on TM, that is, a principal $\mathrm{Spin}^0_{r,1}$-bundle $\pi_{P'}\colon P_{\mathrm{Spin}^0_{r,1}}(M) \to M$ coupled with a 2-sheeted covering

$$\xi\colon P_{\mathrm{Spin}^0_{r,1}} \to P_{\mathrm{SO}^0_{r,1}} \tag{2.5}$$

such that $\xi\left(R_g(p)\right) = R_{\xi_0(g)}\left(\xi(p)\right)$ for every $p \in P_{\mathrm{Spin}^0_{r,1}}$, $g \in \mathrm{Spin}^0_{r,1}$, with $\xi_0\colon \mathrm{Spin}^0_{r,1} \to \mathrm{SO}^0_{r,1}$ being the double covering map. This entails in particular that the cocycle $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathrm{SO}^0_{r,1})\}$ of the principal $\mathrm{SO}^0_{r,1}$-bundle is given by $\{\xi_0(s_{\alpha\beta})\}$, where $\{s_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathrm{Spin}^0_{r,1})\}$ is a cocycle for the principal $\mathrm{Spin}^0_{r,1}$-bundle.

Remark 2.1.1. We point out a few important dimensional considerations: if $r \neq 1$ then $\mathrm{Spin}_{r,1}$ is non-trivial, i.e. connected, over each connected component of $\mathrm{SO}_{r,1}$, while for

[3] Here, $\mathrm{SO}^0_{r,1}$ denotes the identity component of $\mathrm{SO}_{r,1}$. The same notation is used for the identity component of $\mathrm{Spin}_{r,1}$.

$r = 1$ we have $\mathrm{Spin}_{1,1} \simeq \mathrm{SO}_{1,1} \times \mathbb{Z}_2$ [29, §I, Proposition 2.10]. In particular, ξ_0 is the universal covering map of $\mathrm{SO}^0_{r,1}$ if $r \geq 3$, as in this case $\mathrm{Spin}^0_{r,1}$ is simply connected [29, §I, (2.35)]. If $r = 2$ we have $\pi_1(\mathrm{Spin}^0_{2,1}) \simeq \mathbb{Z}$, while if $r = 1$ we instead have $\pi_1(\mathrm{Spin}^0_{1,1}) \simeq C_1$ and $\mathrm{Spin}^0_{1,1} \simeq \mathrm{SO}^0_{1,1} \times \{0\}$. Notice that in this case we don't have a double covering map $\xi_0 \colon \mathrm{Spin}^0_{r,1} \to \mathrm{SO}^0_{r,1}$ between the identity components of the Spin(1, 1) and special orthogonal groups. This can be solved by considering instead the 2-sheeted covering $\xi_0 \colon \mathrm{SO}^0_{1,1} \times \mathbb{Z}_2 \to \mathrm{SO}^0_{1,1}$. ♠

By the Fundamental Theorem of Riemannian geometry, there exists a unique torsion-free, metric compatible connection $\omega \in \Omega^1(P_{\mathrm{SO}^0_{r,1}}, \mathfrak{so}^0_{r,1})$, the Levi-Civita connection; using the covering map (2.5) and the Lie algebra isomorphism $\xi_{0*} \colon \mathfrak{spin}^0_{r,1} \to \mathfrak{so}^0_{r,1}$, this connection can be pulled back to a connection 1-form $\omega^s \in \Omega^1(P_{\mathrm{Spin}^0_{r,1}}, \mathfrak{spin}^0_{r,1})$, usually called *spin connection.*
We can then consider the following: depending on the dimension $r + 1$ of our spacetime M, there exists a complex representation of the Spin group $\Delta^{\mathbb{C}}_{r,1} \colon \mathrm{Spin}^0_{r,1} \to \mathrm{GL}(V, \mathbb{C})$ which is irreducible in the odd dimensional case and reducible in the even dimensional one [29, §I, Proposition 5.15]. In both cases, the representation descends from a complex representation $\rho \colon \mathscr{C}\ell_{r,1} \to \mathrm{End}(V, \mathbb{C})$ of the Clifford algebra $\mathscr{C}\ell_{r,1}$. [29, §I, Theorem 5.7] shows that there exists a unique Clifford algebra representation if $r + 1$ is even, while there exist two inequivalent representations if $r + 1$ is odd; the induced Spin representation in the latter case doesn't depend on the choice of the chosen representation. These representations can be used to carry out the following construction:

(*i*) given a complex Spin representation $\Delta^{\mathbb{C}}_{r,1} \colon \mathrm{Spin}^0_{r,1} \to \mathrm{GL}(V, \mathbb{C})$, we can consider the *complex spinor bundle*

$$S(M) \doteq P_{\mathrm{Spin}^0_{r,1}} \times_{\Delta^{\mathbb{C}}_{r,1}} V \ ; \tag{2.6}$$

(*ii*) we can also consider the *Clifford algebra bundle*

$$\mathscr{C}\ell(M) \doteq P_{\mathrm{SO}^0_{r,1}} \times_{c\ell} \mathscr{C}\ell_{r,1} \tag{2.7}$$

where $c\ell \colon \mathrm{SO}^0_{r,1} \to \mathrm{Aut}(\mathscr{C}\ell_{r,1})$ denotes the unique extension to the Clifford algebra $\mathscr{C}\ell_{r,1}$ of the action of $\mathrm{SO}^0_{r,1}$ on $(\mathbb{R}^{r+1}, q_{r,1})$. It can be shown that the fiber of $\mathscr{C}\ell(M)$ at $p \in M$ is isomorphic to $\mathscr{C}\ell(T_pM, g_p)$.

These two bundles are related by means of the *Clifford module multiplication*

$$\mu \colon \mathscr{C}\ell(M) \otimes S(M) \to S(M)$$

which explicitly uses the fact that $\Delta^{\mathbb{C}}_{r,1}$ descends from an algebra representation of $\mathscr{C}\ell_{r,1}$. Thanks to the spin structure, this map can then be extended to yield a Clifford module multiplication

$$\cdot_\Gamma \colon \mathfrak{X}(M) \otimes \Gamma(S(M)) \to \Gamma(S(M)) \tag{2.8}$$

between vector fields and sections of the spinor bundle.
We shall indicate with

$$\nabla^s \colon \mathfrak{X}(M) \otimes \Gamma(S(M)) \to \Gamma(S(M)) \tag{2.9}$$

the covariant derivative induced as of (2.4) on the spinor bundle by the spin connection ω^s. This connection behaves well with respect to the Clifford multiplication, in the sense that it is a module derivation: given $s \in \Gamma(S(M))$ and $v, u \in \mathfrak{X}(M)$ we have

$$\nabla^s_u(v \cdot_\Gamma s) = (\nabla_u v) \cdot_\Gamma s + v \cdot_\Gamma \nabla^s_u s \ . \tag{2.10}$$

The Clifford multiplication can be combined with the covariant derivative ∇^s and with the musical isomorphism induced by the metric to yield a partial differential operator acting on sections of the spinor bundle, the *Dirac operator*

$$\begin{aligned} \mathbf{D} \colon \Gamma(S(M)) &\to \Gamma(S(M)) \\ s \quad &\mapsto \quad \mathbf{D}(s) \doteq i\left(\cdot_\Gamma(\nabla^s(s))\right) + m \end{aligned} \tag{2.11}$$

where $m \in [0, +\infty)$ is the mass of the Dirac field.
Given a local orthonormal frame $\{e_j\} \subset \Gamma(TM)$ and its dual $\{e^j\} \subset \Gamma(T^*M)$ and considering the local section $\sigma \colon U_\alpha \to P_{\mathrm{Spin}^0_{r,1}}(M)$, (2.11) can be written locally as

$$\mathbf{D}(s)(p) = \sum_j ie^j(p) \cdot_\Gamma \left[\sigma(p), e_j(p)\left(s^\sharp \circ \sigma\right) + \Delta^{\mathbb{C}}_{r,1*}\left(\omega^s_{\sigma(p)}(\sigma_{*p} e_j(p))\right) s^\sharp(\sigma(p)) + ms^\sharp(\sigma(p))\right] \ .$$

The local expression above can be written even more explicitly as

$$\mathbf{D}(s)(p) = ie^j(p) \cdot_\Gamma \left[\sigma(p), e_j(p)\left(s^\sharp \circ \sigma\right) + \frac{1}{2}\sum_{k<l} \Gamma^l_{kj} e^k(p) \cdot_V e_l(p) \cdot_V s^\sharp(\sigma(p)) + ms^\sharp(\sigma(p))\right] \tag{2.12}$$

where Γ^l_{jk} denote the (non-holonomic) Christoffel symbols of the Levi-Civita connection on M inducing the spin connection ω^s. We present the derivation of the explicit form of the Dirac operator in three different spacetimes.

Example 2.1.2. We provide three explicit examples of the Dirac operator computed using (2.12).

(i) Let us consider (flat) Minkowski spacetime $(M, g) = (\mathbb{R}^4, \eta)$ and the embedding of $\mathfrak{X}(\mathbb{R}^4)$ into $\Gamma(\mathscr{C}\ell(\mathbb{R}^4))$ given by $e_j \mapsto \gamma_j, \gamma_j(p) \equiv \gamma_j$ where $\{\gamma_j\}_{1\leq j\leq 4}$ are the usual Gamma matrices in the Dirac representation. By choosing the global pseudo-orthonormal frame $\{\partial_i\}_{1\leq i\leq 4}$ one can write, for a function $s \in \Gamma(S(\mathbb{R}^4)) \simeq C^\infty(\mathbb{R}^4, \mathbb{C}^4)$

$$\mathbf{D}(s) = i\gamma^j\partial_j s + m$$

which is the usual expression of the (massless) Dirac operator.

(ii) A more poignant example is that of the Dirac operator in Schwarzschild spacetime,

$$M = \mathbb{R} \times (r_g, +\infty) \times \mathbb{S}^1 \qquad g = -\left(1 - \frac{r_g}{r}\right) dt \otimes dt + \left(1 - \frac{r_g}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}$$

To carry out the computations, one considers a (in this case global) pseudo-orthonormal frame $\{e_j\}_{1\leq j\leq 4}$,

$$e_1 = \left(1 - \frac{r_g}{r}\right)^{-1/2} \partial_t \quad e_2 = \left(1 - \frac{r_g}{r}\right)^{1/2} \partial_r \quad e_3 = \frac{1}{r}\partial_\theta \quad e_4 = \frac{1}{r\sin(\theta)}\partial_\varphi$$

and the associated embedding $e_j \mapsto \gamma_j$ where, as before, $\{\gamma_j\}_{1\leq j\leq 4}$ are the Gamma matrices in the Dirac representation. In this non-holonomic, pseudo-orthornormal frame the Christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{im}\left(c_{mjk} + c_{mkj} - c_{jkm}\right)$$

where η is the flat metric and

$$[e_i, e_j] = c^m_{jk} e_m, \quad c_{jkp} = \eta_{pm} c^m_{jk}$$

One can also exploit the properties of Christoffel symbols and of the Clifford module multiplication to write

$$\frac{1}{4}\sum_{k,l} \Gamma_{mkj} e^k(p) \cdot_V e^j(p) = \frac{1}{8}\sum_{k,l} \Gamma_{mkj}[\gamma^k, \gamma^j]$$

A lengthy but straightforward computation then leads to

$$\begin{aligned}\mathbf{D}(s) = {} & i\gamma^1 \left(1 - \frac{r_g}{r}\right)^{-1/2} \partial_t s + i\gamma^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(\partial_r s - \frac{r_g}{4r^2}\left(1 - \frac{r_g}{r}\right)^{-1} s - \frac{1}{r} s\right) \\ & + i\gamma^3 \left(\frac{1}{r}\partial_\theta s - \frac{1}{2}\frac{\cot(\theta)}{r} s\right) + i\gamma^4 \frac{1}{r\sin(\theta)}\partial_\varphi s + m \ .\end{aligned}$$

(iii) Another interesting example is that of Friedman-Lemaitre-Robertson-Walker spacetime,

$$M = I \times J \times \mathbb{S}^2 \qquad g = S(t)^2 \left(-dt \otimes dt + dr \otimes dr + f(r)^2 g_{\mathbb{S}^2}\right) \ .$$

Proceeding in the same way as in the Schwarzschild case, a rather obnoxious computation leads to

$$\begin{aligned}\mathbf{D}(s) = {} & i\gamma^1 \left(\frac{1}{S(t)}\partial_t s - \frac{3}{2}\frac{\dot{S}(t)}{S(t)^2} s\right) + i\gamma^2 \left(\frac{1}{S(t)}\partial_r s - \frac{f'(r)}{S(t)f(r)} s\right) \\ & + i\gamma^3 \left(\frac{1}{S(t)f(r)}\partial_\theta s - \frac{\cot(\theta)}{S(t)f(r)} s\right) + i\gamma^4 \frac{1}{S(t)f(r)\sin(\theta)}\partial_\varphi s + m \ .\end{aligned}$$

✠

As one can see from the local expression presented above, the Dirac operator is a first-order partial differential operator; moreover, from the Weitzenböck formula [6, Example 4.3.3], $\mathbf{D}^2 = \Box^{\nabla^s} + \frac{R}{4} + m^2$ is a normally hyperbolic, second-order partial differential operator. Therefore, $\mathbf{D}$ admits unique advanced and retarded Green operators

$$S^{\pm} \colon \Gamma_c(S(M)) \to \Gamma(S(M)) \tag{2.13}$$

i.e.

$$\mathbf{D} \circ S^{\pm} = \mathrm{id}_{\Gamma(S(M))} \qquad S^{\pm} \circ \mathbf{D}|_{\Gamma_c(S(M))} = \mathrm{id}_{\Gamma_c(S(M))} \qquad \mathrm{Supp}(S^{\pm}(s)) \subseteq J^{\mp}(\mathrm{Supp}(s)) \ , \tag{2.14}$$

where $J^-(U)$ and $J^+(U)$ denote respectively the causal past and the causal future of $U \subset M$ according to the casual structure induced by the Lorentzian metric.
By [32, Proposition 12.1.65], we know that $S(M)$ admits a Hermitian metric such that the Clifford multiplication by $\alpha \in \mathfrak{X}(M)$ is an Hermitian automorphism of $S(M)$. We shall denote this Hermitian metric by $\langle \cdot, \cdot \rangle$.

2.2 Topologies on spaces of sections of vector bundles

2.2.1 The Fréchet topology

Throughout this Section, $\pi \colon E \to M$ is a smooth, complex vector bundle of rank r over the base manifold M of dimension n, and the space of its smooth sections is $\Gamma(E)$.

Definition 2.2.1. Let us consider a (maximal) atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ for M inducing local trivializations $\{\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r\}$ for E; then given two integers $1 \leq j \leq r$ and $m \in \mathbb{N}$, an index α and a compact subset $K \subset U_\alpha$, define the following family of seminorms on $\Gamma(E)$:

$$\|s\|_{j,\alpha,K,m} \doteq \sup_{\substack{k \in \mathbb{N}^n \\ |k| \leq m}} \sup_{x \in \varphi_\alpha(K)} \left| \partial^k (\phi_\alpha \circ s \circ \varphi_\alpha^{-1})^j (x) \right| \tag{2.15}$$

Notice that if $K \subseteq U_\alpha \cap U_\beta$, then we know that

$$\begin{aligned} \|s\|_{j,\alpha,K,m} &= \sup_{\substack{k \in \mathbb{N}^n \\ |k| \leq m}} \sup_{x \in \varphi_\alpha(K)} \left| \partial^k (\phi_\alpha \circ s \circ \varphi_\alpha^{-1})^j (x) \right| \\ &= \sup_{\substack{k \in \mathbb{N}^n \\ |k| \leq m}} \sup_{x \in \varphi_\alpha(K)} \left| \partial^k (\phi_\alpha \circ \phi_\beta^{-1})^j_k \circ (\phi_\beta \circ s \circ \varphi_\beta^{-1})^k \circ (\varphi_\beta \circ \varphi_\alpha^{-1})(x) \right| \end{aligned}$$

By using Faà di Bruno's formula [21], it is possible to expand the derivative of the composition into the sum of various terms, involving the partial derivatives of order up to m of the (local expression of the) cocycle of the trivialization, which we recall is a smooth map

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^r) \ ,$$

derivatives up to order m of the transition function

$$\varphi_{\beta\alpha} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \mapsto \varphi_\beta(U_\alpha \cap U_\beta)$$

and derivatives up to order m of $\phi_\beta \circ s \circ \varphi_\beta^{-1}$. Thus

$$\|s\|_{j,\alpha,K,m} \leq \underbrace{\sup_{\substack{k\in\mathbb{N}^n \\ |k|\leq m}} \sup_{x\in\varphi_\alpha(K)} \left\|\partial^k(\phi_\alpha \circ \phi_\beta^{-1})\right\| \sup_{\substack{k\in\mathbb{N}^n \\ |k|\leq m}} \sup_{x\in\varphi_\alpha(K)} \left\|\partial^k(\varphi_\beta \circ \varphi_\alpha^{-1})\right\|}_{k_{\alpha,\beta,K,m}} \|s\|_{k,\beta,K,m}$$

i.e. the various seminorms are equivalent on the intersection of the domains U_α and U_β. Notice that as the manifolds we consider are always second-countable, the atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ can be assumed countable; moreover, as the various U_α are locally compact and second countable, they admit an exhaustion by compact sets, i.e. a countable family $\{K_i^\alpha\}_{i\in\mathbb{N}}$ of compact subsets such that $K_i^\alpha \subseteq \mathrm{Int} K_{i+1}^\alpha$ and $\bigcup_{i\in\mathbb{N}} K_i^\alpha = U_\alpha$. These two conditions imply that by considering the family

$$\left\{\|\cdot\|_{j,\alpha,K_i^\alpha,m},\ 1 \leq j \leq r \quad \alpha \in A,\ K_i^\alpha \subseteq U_\alpha,\ m \in \mathbb{N}\right\}$$

we have a countable basis for the locally convex topology τ induced by the seminorms of Definition 2.2.1.

Let us denote with $\mathcal{E}(E)$ the locally convex topological vector space $(\Gamma(E), \tau)$, where τ is the topology described above; then,

Lemma 2.2.2. *$\mathcal{E}(E)$ is a complete, metrizable, locally convex topological vector space, and hence a Fréchet space.*

Proof. The fact that it is metrizable is an immediate consequence of the existence of the countable basis

$$\left\{\|\cdot\|_{j,\alpha,K_i^\alpha,m},\ 1 \leq j \leq r \quad \alpha \in A,\ K_i^\alpha \subseteq U_\alpha,\ m \in \mathbb{N}\right\}$$

of seminorms. For completeness, we argue as follows: consider a Cauchy sequence $\{s_n\} \subseteq \mathcal{E}(E)$, and given $(U_\alpha, \varphi_\alpha)$ consider as a compact subset $K = \{p\}$, any $1 \leq j \leq r$ and $m = 0$. We then have that for every $\varepsilon > 0$ t.e. $n_\varepsilon \in \mathbb{N}$ such that

$$\left|(\phi_\alpha \circ s_n)^j(p) - (\phi_\alpha \circ s_m)^j(p)\right| < \varphi \text{ if } n, m > n_\varepsilon$$

As $\mathbb{C}$ is complete, then $(\phi_\alpha \circ s_n)^j(p) \to (p, s^j(p))$ for some $s^j(p) \in \mathbb{C}$; proceeding in a similar fashion as before one can easily show that the r-uple $(s^1(p), \ldots, s^r(p))$ behaves well under a local trivialization change, thus yielding a map $s \colon M \to E$ which is such that $\pi \circ s = \mathrm{id}_M$. In order to show completeness, we just need to show that s is smooth; as smoothness is a local property, given any $p \in M$ we can fix any local chart $(U_\alpha, \varphi_\alpha)$ such that $p \in U_\alpha$, and then check the smoothness of $(\phi_\alpha \circ s \circ \varphi_\alpha^{-1})^j$ at p. We proceed by induction on the order m of derivatives: by hypothesis we know that given any compact $K \subseteq U_\alpha$ star-shaped with respect to $p = \varphi_\alpha^{-1}(x)$ we have

$$\sup_{\substack{k\in\mathbb{N}^n \\ |k|\leq m}} \sup_{y\in\varphi_\alpha(K)} \left|\partial^k(\phi_\alpha \circ s_n \circ \varphi_\alpha^{-1})^j(y) - \partial^k(\phi_\alpha \circ s_l \circ \varphi_\alpha^{-1})^j(y)\right| < \varepsilon \text{ if } n, l > n_\varepsilon$$

Thus $\partial^k(\phi_\alpha \circ s_n \circ \varphi_\alpha^{-1})^j$ converges to some continuous function v^k uniformly on $\varphi_\alpha(K)$. Then by applying Taylor's formula with integral remainder,

$$(\phi_\alpha \circ s_n \circ \varphi_\alpha^{-1})(y) = (\phi_\alpha \circ s_n \circ \varphi_\alpha^{-1})(x) + \sum_{k=1}^{n}(y-x)_k \int_0^1 \partial^k(\phi_\alpha \circ s \circ \varphi_\alpha^{-1})^j(tx + (1-t)y)\, dt$$

and by taking the limit, thanks to uniform convergence we arrive at

$$(\phi_\alpha \circ s \circ \varphi_\alpha^{-1})(y) = (\phi_\alpha \circ s \circ \varphi_\alpha^{-1})(x) + \sum_{k=1}^{n}(y-x)_k \int_0^1 v^k(x + t(y-x))\, dt$$

By considering $y = x + \lambda e_k$, dividing by λ and taking the limit we arrive at

$$\partial^k(\phi_\alpha \circ s \circ \varphi_\alpha^{-1})^j(x) = v^k(x)$$

Then by induction on m we have the desired smoothness. Thus $s_n \to s$ with $s \in \Gamma(E)$, i.e. $\mathcal{E}(E)$ is complete. □

Remark 2.2.3. It is possible to give an alternative description of the Fréchet topology on $\Gamma(E)$. Let $\nabla^E \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ and h be a connection and an hermitian metric on the vector bundle E, and let g be a Riemannian metric on M inducing the Levi-Civita connection ∇. Given a compact subset $K \subset M$ and an integer $m \in \mathbb{N}$ we define the following family of seminorms on $\Gamma(E)$:

$$\|s\|_{K,m,\nabla^E,h,g} \doteq \sup_{0\leq j\leq m} \sup_{x\in K} \left\|\nabla^{E^j} s(x)\right\|_{h,g} .$$

The topology induced by this family of seminorms is equivalent to the one previously described: indeed, given any local chart $(U_\alpha, \varphi_\alpha)$ inducing a local trivialization Φ_α we can locally write, given a section $s \in \Gamma(E)$ and a vector field $v \in \mathfrak{X}(M)$,

$$(\nabla^E_{\partial_i} s)^k(x) = \partial_i s^k(x) + \omega^k_{ij}(x) s^j(x) \qquad (\nabla_{\partial_i} v)^t(x) = \partial_i v^t(x) + \Gamma^t_{is}(x) v^s(x)$$

with ω^k_{ij} and Γ^t_{is} smooth. We now proceed by induction on m: given a compact set K, we can cover it with a finite number of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}_{1\leq\alpha\leq n}$. It is then possible to find a collection of compact sets $\{K_\alpha\}_{1\leq\alpha\leq n}$ such that $K_\alpha \subseteq U_\alpha$ and $K = \bigcup_\alpha K_\alpha$. Then using the local trivializations and the local charts associated to the chart domains we have, for $m = 0$,

$$\begin{aligned}\sup_{x\in K}\|s(x)\|_{h,g} &= \max_{1\leq\alpha\leq n} \sup_{x\in K_\alpha} \|s(x)\|_{h,g} = \max_{1\leq\alpha\leq n} \sup_{x\in K_\alpha} \sqrt{h_{k_\alpha l_\alpha}(x) s^{k_\alpha}(x) s^{l_\alpha}(x)} \\ &\leq c_{h,K} \max_{1\leq\alpha\leq n} \max_{1\leq k_\alpha\leq r} \underbrace{\sup_{x\in K_\alpha}\left|s^{k_\alpha}(x)\right|}_{\|s\|_{k_\alpha,\alpha,K_\alpha,0}} .\end{aligned}$$

We proceed in a similar fashion for $m = 1$: notice that due to the smoothness of h, g and the connection coefficient we have

$$\begin{aligned}\sup_{x\in K}\left\|\nabla^E s(x)\right\|_{h,g} &= \max_{1\leq\alpha\leq n}\sup_{x\in K_\alpha}\sqrt{(\partial_{i_\alpha}s^{k_\alpha}+\omega^{k_\alpha}_{i_\alpha j_\alpha}s^{j_\alpha}(x))(\partial_{l_\alpha}s^{v_\alpha}+\omega^{v_\alpha}_{l_\alpha j_\alpha}s^{j_\alpha}(x))g^{i_\alpha l_\alpha}h_{k_\alpha v_\alpha}}\\ &\leq c_{h,g,K,\nabla^E}\max_{1\leq\alpha\leq n}\max_{1\leq j_\alpha\leq r}\underbrace{\sup_{\substack{k\in\mathbb{N}^n\\|k|\leq 1}}\sup_{x\in K_\alpha}\left|\partial^k s^{j_\alpha}(x)\right|}_{\|s\|_{j_\alpha,\alpha,K_\alpha,1}}\ .\end{aligned}$$

Proceeding by induction one can show that similar results hold for higher order covariant differentiation, i.e.

$$\sup_{x\in K}\left\|\nabla^{E^j}s(x)\right\|_{h,g}\leq c_{h,g,K,\nabla^E,j}\max_{1\leq\alpha\leq n}\max_{1\leq j_\alpha\leq r}\underbrace{\sup_{\substack{k\in\mathbb{N}^n\\|k|\leq j}}\sup_{x\in K_\alpha}\left|\partial^k s^{j_\alpha}(x)\right|}_{\|s\|_{j_\alpha,\alpha,K_\alpha,j}}\ .$$

Thus the topology induced by the new collection of seminorms is finer than that induced by the seminorms in Definition 2.2.1. The converse implication also holds, and the proof is entirely analogous to the previous one; thus the two topologies are equivalent. This procedure also shows that the topology induced by $\{\|\cdot\|_{K,m,\nabla^E,h,g}\}$ is independent on the choice of ∇^E, h and g. ♠

Remark 2.2.4. If the base space M of the vector bundle is compact, then the Fréchet topology whose seminorms are defined in Definition 2.2.1 can be described in the following way: first of all, notice that if M is compact then we can restrict to a finite atlas $\{(U_\alpha,\varphi_\alpha)\}_{1\leq\alpha\leq n}$; secondly, proceeding as in Remark 2.2.3 we can cover K with a finite number of compact sets $\{K_\alpha\}_{1\leq\alpha\leq n}$ such that $K_\alpha\subseteq U_\alpha$. Then given a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n\to s$ in $\mathcal{E}(E)$, we have that

$$\sup_{x\in M}\sup_{\substack{k\in\mathbb{N}^n\\|k|\leq m}}\left|\partial^k\left(\phi_\alpha\circ(s_n-s)\right)^j(x)\right| = \max_{1\leq\alpha\leq n}\sup_{\substack{k\in\mathbb{N}^n\\|k|\leq m}}\sup_{x\in\varphi_\alpha(K_\alpha)}\left|\partial^k\left(\phi_\alpha\circ(s_n-s)\circ\varphi_\alpha^{-1}\right)^j(x)\right|<\varepsilon$$

if $n>n_\varepsilon$. Thus we see that if M is compact $s_n\to s$ in the Fréchet topology on $\Gamma(E)$ if s_n and all the partial derivatives of its components converge uniformly to s and the partial derivatives of its components. ♠

2.2.2 The Whitney C^∞-topology

We shall now briefly recall some facts and results about another class of topologies on the spaces of sections of vector bundles, namely the Whitney C^k-topologies. The key result which will be used later is Theorem 2.2.15; we refer the reader to [31] and [25] for more details.

Definition 2.2.5. Let M, N be smooth manifolds, and let us consider two C^k maps $f\colon U \to N$, $g\colon V \to N$ defined on U, V open neighbourhoods of a point $x \in M$. We say that f and g are *k-tangential* at x if $f(x) = g(x)$ and if given two local charts $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) centered at x and $f(x)$ respectively we have

$$D^j(\psi_\beta \circ f \circ \varphi_\alpha^{-1})\Big|_0 = D^j(\psi_\beta \circ g \circ \varphi_\alpha^{-1})\Big|_0 \quad \forall 1 \leq j \leq k \ .$$

This condition is easily seen to be independent of the chosen local charts.
Given the notion of k-tangentiality, we define the following equivalence relation: given $x \in M$, $y \in N$ and $f, g \in C^k(M, N)$ we say that $f \sim_k g$ if $f(x) = g(x) = y$ and f and g are k-tangential at x. The quotient $C^k(M, N)/\sim_k$ shall be denoted with $J^k(M, N)_{x,y}$; the equivalence classes are called *k-jets with source x and target y*; given a representative f of the equivalence class $z \in J^k(M, N)_{x,y}$, we write

$$z = j^k(f)[x] \ .$$

Let us define

$$J^k(M, N)_x = \bigsqcup_{y \in N} J^k(M, N)_{x,y} \qquad J^k(M, N) \doteq \bigsqcup_{x \in M} J^k(M, N)_x \ .$$

We will show that these spaces can be endowed with a (finite-dimensional) manifold structure; to do so, we first prove some ancillary results.

Lemma 2.2.6. *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open subsets, and let us consider the C^k maps $f_1, f_2\colon U \to V$ and $g_1, g_2\colon V \to \mathbb{R}^k$ such that $g_1 \circ f_1$ and $g_2 \circ f_2$ are well-defined, $f_1 \sim_k f_2$ at $x \in U$ and $g_1 \sim_k g_2$ at $f_1(x) \in V$. Then $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at x.*

Proof. By hypothesis we have $f_1(x) = f_2(x) = y$ and $g_1(y) = g_2(y)$; thus $g_1(f_1(x)) = g_2(f_2(x))$. Consider now the case $k = 1$: we have

$$D^1(g_1 \circ f_1)(x) = D^1(g_1)(y) \circ D^1(f_1)(x) = D^1(g_2)(y) \circ D^1(f_2)(x) = D^1(g_2 \circ f_2)(x) \ .$$

The same holds for $k > 1$: this can be easily seen by using Faà di Bruno's formula and the k-tangentiality of $g_1 \sim_k g_2$ and $f_1 \sim_k f_2$. □

Using local charts, the previous result applies without effort to C^k maps between smooth manifolds.

Corollary 2.2.7. *Let us consider $x \in M$, $y \in N$, $x' \in M'$, and two smooth maps $\varphi\colon U' \to U$ and $\psi\colon V \to V'$, with $x' \in U'$, $x = \varphi(x') \in U$, $y \in V$ and $y' = \psi(y) \in V'$. These two maps induce two maps*

$$\begin{aligned} \varphi^*\colon J^k(M, N)_x &\to J^k(M', N)_{x'} \\ j^k(f)[x] &\mapsto j^k(f \circ \varphi)[x'] \end{aligned} \qquad \begin{aligned} \psi_*\colon J^k(M, N)_{x,y} &\to J^k(M, N')_{x,y'} \\ j^k(f)[x] &\mapsto j^k(\psi \circ f)[x] \ . \end{aligned}$$

Moreover, given another manifold M'', a point $x'' \in M''$ and a smooth function $\varphi' \colon U'' \to U'$ such that $\varphi'(x'') = x'$, we have

$$\varphi'^* \circ \varphi^* = (\varphi \circ \varphi')^* \ .$$

Analogously, given N'', a point $y'' \in N''$ and $\psi' \colon V' \to V''$ such that $\psi'(y') = y''$, we have

$$\psi'_* \circ \psi_* = (\psi' \circ \psi)_* \ .$$

Proof. By Lemma 2.2.6, the maps φ^* and ψ_* are well-defined: indeed, given f and g two representatives of the k-jet $j^k(f)[x]$, then by the aforementioned Lemma we have that $f \circ \varphi \sim_k g \circ \varphi$ at x'; thus it maps the equivalence class $j^k(f)[x]$ to the equivalence class $j^k(f \circ \varphi)[x']$. The same applies to ψ_*.
Consider now

$$\varphi'^* \colon J^k(M', N)_{x'} \to J^k(M'', N)_{x''} \ .$$

We know that $\varphi^*(J^k(M,N)_x) \subseteq J^k(M',N)_{x'}$, and therefore the composition $\varphi'^* \circ \varphi^*$ is well-defined. Now, given an equivalence class $j^k(f)[x]$ we have

$$j^k(f)[x] \overset{\varphi^*}{\mapsto} j^k(f \circ \varphi)[x'] \overset{\varphi'^*}{\mapsto} j^k((f \circ \varphi) \circ \varphi')[x''] = j^k(f \circ (\varphi \circ \varphi'))[x''] = (\varphi \circ \varphi')^* j^k(f)[x] \ .$$

The proof proceeds similarly for $(\psi' \circ \psi)^* = \psi'^* \circ \psi^*$. □

Notice that if $\varphi = \mathrm{id}_M$ then $\varphi^* = \mathrm{id}_{J^k(M,N)_x}$. Thus in particular if φ is a diffeomorphism then φ^* is a bijection.
Consider now two open subsets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, and consider the set $J^k(U,V)$; on it we define the map

$$j^k(f)[x] \overset{\Phi_{U,V}}{\mapsto} (x, f(x), D^1 f(x), \dots, D^k f(x)) \in U \times V \times \mathrm{SL}^1(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \mathrm{SL}^k(\mathbb{R}^n, \mathbb{R}^m)$$

where $\mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m)$ denotes the set of j-linear, symmetric maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. This map is clearly well-defined: by the definition of k-tangentiality it doesn't depend on the choice of the representative of $j^k(f)[x]$, and it is injective: indeed, if given the representatives f of $j^k(f)[x]$ and g of $j^k(g)[x]$ we have

$$(x, f(x), D^1 f(x), \dots, D^k f(x)) = (x, g(x), D^1 g(x), \dots, D^k g(x))$$

then $f \sim_k g$ and thus $j^k(f)[x] = j^k(g)[x]$. Moreover, the maps is clearly surjective: indeed, given any object $(x_0, y, s_1, \cdots, s_k)$ in $U \times V \times \mathrm{SL}^1(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \mathrm{SL}^k(\mathbb{R}^n, \mathbb{R}^m)$ one can consider the C^k function $p \colon U \to V$,

$$p(x) = y + s_1(x - x_0) + \cdots + s_k(\underbrace{x - x_0, \cdots, x - x_0}_{k \text{ times}})$$

and its k-jet $j^k(p)[x_0]$. Notice that the spaces $\mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m)$ are finite-dimensional $\mathbb{R}$-vector spaces; therefore $U \times V \times \mathrm{SL}^1(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \mathrm{SL}^k(\mathbb{R}^n, \mathbb{R}^m)$ is an open subset of some euclidean space.
The maps $\{\Phi_{U,V}\}_{U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m}$ have the following property:

Lemma 2.2.8. *Consider $U, U' \subseteq \mathbb{R}^n$, $V, V' \subseteq \mathbb{R}^m$, and two diffeomorphisms $\varphi\colon U \to U'$ and $\psi\colon V \to V'$. Then the map*

$$\Phi_{U',V'} \circ \varphi^{-1^*} \circ \psi_* \circ \Phi_{U,V}^{-1}\colon U \times V \times \prod_{j=1}^{k} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m) \to U' \times V' \times \prod_{j=1}^{k} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m)$$

is smooth.

Proof. Consider a point $P = (x_0, y_0, s_1, \ldots, s_k) \in U \times V \times \mathrm{SL}^1(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \mathrm{SL}^k(\mathbb{R}^n, \mathbb{R}^m)$; by the previous observation we have that $\Phi_{U,V}^{-1}(P) = j^k(p)[x_0]$. Then

$$\varphi^{-1^*} \circ \psi_* j^k(p)[x_0] = j^k(\psi \circ p \circ \varphi^{-1})[\varphi(x_0)] \ .$$

Then

$$\Phi_{U',V'}(j^k(\psi \circ p \circ \varphi^{-1})[\varphi(x_0)]) = (\varphi(x_0), \psi(y_0), D^1\psi \circ p \circ \varphi^{-1}(\varphi(x_0)), \ldots, D^k\psi \circ p \circ \varphi^{-1}(\varphi(x_0)) \ .$$

As φ and ψ are smooth by hypotheses, it suffices to show that $P \mapsto D^j\psi \circ p \circ \varphi^{-1}(\varphi(x_0))$ is smooth for every $1 \leq j \leq k$. Consider $j = 1$: we have

$$D^1\psi \circ p \circ \varphi^{-1}(\varphi(x_0)) = D^1\psi(y_0) \circ s_1 \circ D^1\varphi^{-1}(\varphi(x_0)) \ .$$

This is clearly smooth in x_0, y_0 and s_1. Proceeding by induction on j one can see that the higher order derivatives are sum of compositions of derivatives of ψ and φ^{-1} and of s_j's up to the considered order; these objects are all smooth in P. $\square$

Corollary 2.2.9. *Given M and N smooth manifolds, given local charts $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ and $\psi\colon V \to \psi(V) \subseteq \mathbb{R}^m$, define the bijection*

$$\tau_{U,V} = \Phi_{\varphi(U),\psi(V)} \circ \psi_* \circ \varphi^{-1^*}\colon J^k(U, V) \to \varphi(U) \times \psi(V) \times \prod_{j=1}^{k} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m) \ .$$

Then the collection $\{(J^k(U, V), \tau_{U,V})\}_{U,V}$ is an atlas for $J^k(M, N)$, making it a finite-dimensional manifold.

Proof. Consider two local charts $\varphi'\colon U' \to \varphi(U')$ and $\psi'\colon V' \to \psi(V')$ such that $U \cap U' \neq \varnothing$, $V \cap V \neq \varnothing$. Then we can consider

$$\tau_{U,V}\colon J^k(U \cap U', V \cap V') \to \varphi(U \cap U') \times \psi(V \cap V') \times \prod_{j=1}^{k} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m)$$

$$\tau_{U',V'}\colon J^k(U \cap U', V \cap V') \to \varphi'(U \cap U') \times \psi'(V \cap V') \times \prod_{j=1}^{k} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m)$$

and the composition

$$\begin{aligned}\tau_{U,V}\circ\tau^{-1}_{U',V'} &= \Phi_{\varphi(U),\psi(V)}\circ\psi_*\circ\varphi^{-1*}\circ\varphi'^*\circ{\psi'_*}^{-1}\circ\Phi^{-1}_{\varphi'(U),\psi'(V)}\\ &= \Phi_{\varphi(U),\psi(V)}\circ\psi_*\circ{\psi'_*}^{-1}\circ\varphi^{-1*}\circ\varphi'^*\circ\circ\Phi^{-1}_{\varphi'(U),\psi'(V)}\\ &= \Phi_{\varphi(U),\psi(V)}\circ(\varphi'\circ\varphi^{-1})^*\circ(\psi\circ\psi'^{-1})_*\circ\Phi^{-1}_{\varphi'(U),\psi'(V)}\ .\end{aligned}$$

Notice that

$$\psi\circ\psi'^{-1}\colon\psi'(V\cap V')\to\psi(V\cap V')\qquad\varphi'\circ\varphi^{-1}\colon\varphi(U\cap U')\to\varphi'(U\cap U')$$

are diffeomorphisms; then by Lemma 2.2.8 we have that $\tau_{U,V}\circ\tau^{-1}_{U',V'}$ is smooth.
The topology on M is then the one whose basis is given by

$$\bigcup_{U,V}\left\{\tau^{-1}_{U,V}(O), O \text{ open in } \varphi(U)\times\psi(V)\times\prod_{j=1}^{k}\mathrm{SL}^j(\mathbb{R}^n,\mathbb{R}^m)\right\}\ .$$

As M and N are second countable, Hausdorff manifolds then so is $J^k(M,N)$. □

Remark 2.2.10. On $J^k(M,N)$ one can define a map $\pi\colon J^k(M,N)\to M\times N$: we know that each equivalence class z belongs to some $J^k(M,N)_{x,y}$; then

$$\pi(z)\doteq(x,y)\ .$$

This map is smooth: indeed, its given (U,φ) and (V,ψ) local charts for M and N its local expression can be written

$$(\varphi\times\psi)\circ\pi\circ\tau^{-1}_{U,V}=(\varphi\times\psi)\circ\pi\circ\varphi^*\circ\psi_*^{-1}\circ\Phi^{-1}_{\varphi(U),\psi(V)}\ .$$

When acting on the point $P=(x_0,y_0,s_1,\ldots,s_k)$ the expression on the right acts as

$$(\varphi\times\psi)\circ\pi\left(j^k(\psi^{-1}\circ p\circ\varphi)[\varphi^{-1}(x_0)]\right)=(x_0,y_0)\ .$$

This local expression shows that π is smooth and surjective as stated.
Notice that given a function $g\in C^k(M,N)$, we can define the map

$$\begin{aligned}j^k(g)\colon M&\to J^k(M,N)\\ x&\mapsto\quad j^k(g)[x]\ .\end{aligned}$$

Given local charts (U,φ) and (V,ψ) its local expression is then given by

$$\begin{aligned}\tau_{U,V}\circ j^k(g)\circ\varphi^{-1}(x_0)=\tau_{U,V}\left(j^k(g)[\varphi^{-1}(x_0)]\right) &= \Phi_{U,V}\circ\psi_*\circ\varphi^{-1*}\left(j^k(g)[\varphi^{-1}(x_0)]\right)\\ &= \Phi_{U,V}\left(j^k(\psi\circ g\circ\varphi^{-1})[x_0]\right)=\\ &= (x_0,\psi\circ g\circ\varphi^{-1}(x_0),D^1\psi\circ g\circ\varphi^{-1}(x_0),\ldots,D^k\psi\circ g\circ\varphi^{-1}(x_0))\ .\end{aligned}$$

This function is clearly C^k with respect to x_0. ♠

Definition 2.2.11. Let M, N be smooth manifolds; given any open subset $U \subseteq J^k(M,N)$, define

$$M^k(U) \doteq \left\{ f \in C^k(M,N) \text{ s.t. } j^k(f)[M] \subseteq U \right\} \ .$$

The collection $\{M^k(U)\}$, where U runs over all open subsets of $J^k(M,N)$, constitutes a basis for a topology on $C^k(M,N)$: indeed,

$$\bigcup_{U \subseteq J^k(M,N) \text{ open}} M^k(U) = C^k(M,N)$$

and $M^k(U) \cap M^k(V) = M^k(U \cap V)$, as is easily seen from Definition 2.2.11. Said topology is called *Whitney C^k topology.* Notice that $C^\infty(M,N) \subseteq C^k(M,N)$, and we can endow it with the subspace topology (let it be W^k) for every k. Then the Whitney C^∞ topology on $C^\infty(M,N)$ is defined as the topology whose basis is

$$W^\infty \doteq \bigcup_{k \in \mathbb{N}} W^k \ .$$

This is well-defined, as $W^k \subseteq W^l$ if $k \leq l$: indeed, consider the projection

$$\begin{aligned} \pi^l_k \colon J^l(M,N) &\to J^k(M,N) \\ j^l(f)[x] &\mapsto j^k(f)[x] \end{aligned}$$

which in local coordinates is given by

$$\begin{aligned} \tau^k_{U,V} \circ \pi^l_k \circ {\tau^l_{U,V}}^{-1} \colon \varphi(U) \times \psi(V) \times \prod_{j=1}^{l} \mathrm{SL}^j(\mathbb{R}^n, \mathbb{R}^m) &\to \varphi(U) \times \psi(V) \times \prod_{j=1}^{l} \mathrm{SL}^k(\mathbb{R}^n, \mathbb{R}^m) \\ (x_0, y_0, s_1, \ldots, s_l) &\mapsto (x_0, y_0, s_1, \ldots, s_k) \ . \end{aligned}$$

This map is evidently smooth; moreover, we have that given $U \subseteq J^k(M,N)$ open,

$$\begin{aligned} M^l({\pi^l_k}^{-1}(U)) =& \{ f \in C^\infty(M,N) \text{ s.t. } j^l(f)[M] \subseteq {\pi^l_k}^{-1}(U) \} \\ =& \{ f \in C^\infty(M,N) \text{ s.t. } \pi^l_k(j^l(f)[M]) \subseteq U \} = M^k(U) \ . \end{aligned}$$

Remark 2.2.12. Given a function $f \in C^\infty(M,N)$, a neighbourhood of f in the Whitney C^k topology W^k can be given in the following way: consider a continuous function $\delta \colon M \to \mathbb{R}^+$, and consider a metric $d \colon J^k(M,N) \times J^k(M,N) \to \mathbb{R}^+$, which exists because every finite-dimensional smooth manifold is metrizable. Now, define

$$B_\delta(f) \doteq \left\{ g \in C^\infty(M,N) \text{ s.t. } d(j^k(f)[x], j^k(g)[x]) \leq \delta(x) \ \forall x \in M \right\} \ .$$

This set is open. To show this, we proceed in the following way: define the mapping

$$\begin{aligned} \Delta \colon J^k(M,N) &\to \mathbb{R} \\ z &\mapsto \delta(\pi_1 \circ \pi(z)) - d(z, j^k(f)[\pi_1 \circ \pi(z)]) \ . \end{aligned}$$

This function is clearly continuous, being the composition and difference of continuous functions. Now, $\Delta^{-1}(\mathbb{R}^+)$ is clearly open in $J^k(M,N)$ by continuity of Δ; moreover, we have that

$$\begin{aligned} M^k(\Delta^{-1}(\mathbb{R}^+)) &= \left\{ g \in C^\infty(M,N) \text{ s.t. } \forall x \in M \ j^k(g)[x] \in \Delta^{-1}(\mathbb{R}^+) \right\} \\ &= \left\{ g \in C^\infty(M,N) \text{ s.t. } \forall x \in M \ \delta(x) - d(j^k(g)[x], j^k(f)[x]) \geq 0 \right\} = B_\delta(f) \ . \end{aligned}$$

The collection of $\{B_\delta(f)\}_\delta$ is actually a neighbourhood basis for f in the Whitney C^k topology: let O be an open neighbourhood of f, and let U be open in $J^k(M,N)$ and such that $f \in M^k(U) \subseteq O$. Define

$$m(x) \doteq \inf \left\{ d(j^k(f)[x], z), \ z \in (\pi_1 \circ \pi)^{-1}(x) \cap (J^k(M,N) \setminus U) \right\}$$

with $m(x) = \infty$ if $(\pi_1 \circ \pi)^{-1}(x) \subseteq U$, and notice that $m \colon M \to \mathbb{R}^+ \cup \{\infty\}$ is lower-semicontinuous: indeed, consider the interval $(a, +\infty)$ for $a > 0$, and consider $m^{-1}(a, +\infty)$; this is the set

$$\left\{ x \in X \mid \exists z \in (\pi_1 \circ \pi)^{-1}(x) \cap (J^k(M,N) \setminus U) \text{ s.t. } d(j^k(f)[x], z) > a \right\} \ .$$

Fix $x \in X$; then by the continuity of $d(\cdot,\cdot)$, we know that there exist $V, W \subseteq J^k(M,N)$ open neighbourhoods of $j^k(f)[x]$ and z respectively such that $d(z_v, z_w) > a$ for $z_u \in U$, $z_v \in V$. By considering $(\pi_1 \circ \pi)V \cap (\pi_1 \circ \pi)W$, which is nonempty and open as $\pi_1 \circ \pi$ is open and continuous, we then have that $m^{-1}(a, +\infty)$ contains an open neighbourhood of x, i.e. it is open.
As m is lower-semicontinuous, it admits a (necessarily positive) minimum on each compact subset K of M, and it is possible to find a continuous function $\delta_K \colon K \to \mathbb{R}^+$ such that $\delta_K(x) < m(x)$. By considering a partition of unity it is then possible to construct a globally defined object $\delta \colon M \to \mathbb{R}^+$ such that $\delta(x) < m(x)$ for all $x \in M$. Then if we consider $B_\delta(f)$ it holds that $B_\delta(f) \subseteq M^k(U)$: indeed,

$$B_\delta(f) = \left\{ g \in C^\infty(M,N) \text{ s.t. } \forall x \in M \ d(j^k(f)[x], j^k(g)[x]) < \delta(x) < m(x) \right\} \ .$$

But if $d(j^k(f)[x], j^k(g)[x]) < \delta(x) < m(x)$ for all $x \in M$ then it must hold that $j^k(g)[x] \in (\pi_1 \circ \pi)^{-1}(x) \cap U$, i.e. $j^k(g)[M] \subseteq U$.
Finally, it easily holds that given $B_\delta(f)$ and $B_\gamma(f)$, if we define the continuous function $\eta(x) \doteq \min\{\delta(x), \gamma(x)\}$ and consider $B_\eta(f)$ then $B_\eta(f) \subseteq B_\delta(f) \cap B_\gamma(f)$. ♠

Notice that if M were compact, then every $\delta \colon M \to \mathbb{R}^+$ would admit a global minimum, and we could bound it from below by $\frac{1}{n}$ for $n \in \mathbb{N}$ large enough; that is, $B_{\frac{1}{n}}(f) \subseteq B_\delta(f)$. Thus every $f \in C^\infty(M,N)$ admits a countable neighbourhood basis $\left\{ B_{\frac{1}{n}}(f) \right\}_{n \in \mathbb{N}}$, i.e. $(C^\infty(M,N), W^k)$ is first-countable if M is compact, and thus sequential: we can charachterize its topology via sequences. In particular, it follows that a sequence $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $(C^\infty(M,N), W^k)$ if and only if $\{f_n\}_{n \in \mathbb{N}}$ and all the

partial derivatives of order at most k converges uniformly to f. Thus in the case of M compact, the Fréchet topology described in Definition 2.2.1 and the Whitney C^∞-topology coincide (see Remark 2.2.4).
On the other hand, for non-compact M we have the following:

Lemma 2.2.13. *A sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C^\infty(M,N)$ converges to f in $(C^\infty(M,N), W^k)$ if and only if there exists a compact subset K of M such that $j^k(f_n) \to j^k(f)$ uniformly on K and all but a finite number of f_n's equal f outside of K.*

Proof. Let us first prove the direct implication by contradiction: suppose that $f_n \to f$ in $(C^\infty(M,N), W^k)$ but that there exists no $K \subset\subset M$ with the property described above. Let us consider a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of M such that $K_i \subsetneq \mathrm{Int}(K_{i+1})$ and $M = \bigcup_{i\in\mathbb{N}} K_i$. We know that there exist $x_0 \in M \setminus K_0$ and $n_0 \in \mathbb{N}$ such that $f_{n_0}(x_0) \neq f(x_0)$; as the sequence of compact sets is invading, $x_0 \in K_{l_0}$ for some l_0. In particular it holds that

$$d(j^k(f_{n_0})[x_0], j^k(f)[x_0]) > a_0 \ .$$

Define a function $\delta\colon K_{l_0} \to \mathbb{R}^+$ by $\delta(x) \equiv a_0$. We proceed by induction now: suppose that we have constructed the sequences $x_0, \ldots, x_k$, $n_0, \ldots, n_k$ and $l_0, \ldots, l_k$ such that

$$f_{n_k}(x_k) \neq f(x_k) \qquad x_k \in K_{l_k}$$

and that we have a continuous function $\delta\colon K_{l_k} \to \mathbb{R}^+$ with $\delta(x) = a_j$ if $x \in K_{l_j} \setminus K_{l_{j-1}}$. As by hypothesis there exists no $K \subset\subset M$ with the properties described in the Lemma, there exist $n_{k+1} > n_k$, $l_{k+1} > l_k + 1$ and $x_{k+1} \in K_{l_{k+1}} \setminus K_{l_k+1}$ such that

$$f_{n_{k+1}}(x_{k+1}) \neq f(x_{k+1}) \text{ and thus } d(j^k(f_{n_{k+1}})[x_{k+1}], j^k(f)[x_{k+1}]) > a_{k+1}$$

We then extend by continuity δ from K_{l_k} to $K_{l_{k+1}}$ by requiring that $\delta \equiv a_{k+1}$ on $K_{l_{k+1}} \setminus K_{l_k}$. Notice that as $l_{k+1} > l_k + 1$, this extension is possible. Then if we consider $B_\delta(f)$, there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ such that $f_{n_i} \notin B_\delta(f)$ for every i; but this is a contradiction, as $f_n \to f$ in $(C^\infty(M,N), W^k)$.
The other implication follows easily. $\square$

We are shall now state and prove the most important result of this Section; before that, we need an ancillary Lemma.

Lemma 2.2.14. *Let A, B, P be first-countable, Hausdorff topological spaces with P locally compact and paracompact; further suppose that there exist $\pi_a\colon A \to P$ and $\pi_b\colon B \to P$ continuous mappings. Consider the set*

$$A \times_P B \doteq \{(a,b) \in A \times B \ s.t.\ \pi_a(a) = \pi_b(b)\}$$

and endow it with the subspace topology. Let $K \subseteq A$, $J \subseteq B$ such that $\pi_a|_K$ and $\pi_b|_J$ are proper maps, and let U be an open neighbourhood of $K \times_P J$ in $A \times_P B$; then there exist open neighbourhoods U_A of K in A and U_B of J in B such that $U_A \times_P U_B \subseteq U$.

Proof. (i) First of all, notice that if $f\colon X \to Y$ is a continuous, proper map between Hausdorff topological spaces and Y is locally compact, then f is closed. Indeed, suppose that $Z \subseteq X$ is closed, and consider $y \in \overline{f(Z)}$. As Y is locally compact, there exits a compact neighbourhood V of y, and by construction there exists a sequence $\{y_i\}_{i\in\mathbb{N}}$ in $f(Z)$ which converges to y and which we can suppose is contained in V. By the hypotheses on f, we have that $S = Z \cap f^{-1}(V)$ is compact and non-empty; in particular there exist $\{x_i\}_{i\in\mathbb{N}} \subseteq S$ such that $f(x_i) = y_i$. As X is first-countable, we then have that S is sequentially compact, i.e. the sequence $\{x_i\}_{i\in\mathbb{N}}$ admits a subsequence $\{x_{i_k}\}_{k\in\mathbb{N}}$ converging to x in S. But then by continuity $f(x) = \lim_{k\to\infty} f(x_{i_k}) = \lim_{k\to\infty} y_{i_k} = y$, i.e. $y \in f(Z)$.

(ii) Notice that
$$A \times_P B = (\pi_a \times \pi_b)^{-1}(\Delta_p)$$
and thus it is closed in $A \times B$; thus its complement $C = A \times B \setminus A \times_P B$ is open, and so is $U \cup C$. Given any $p \in P$, consider
$$K_p \doteq K \cap \pi_a^{-1}(p) \qquad J_p \doteq J \cap \pi_b^{-1}(p)$$
which are compact as $\pi_a|_K$ and $\pi_b|_J$ are proper, and $K_p \times J_p = K_p \times_P J_p \subseteq U \cup C$. As K_p and J_p are compact, there exists $V_p \supseteq K_p$ and $W_p \supseteq J_p$ open and such that
$$K_p \times J_p \subseteq V_p \times W_p \subseteq U \cup C\ .$$
By (i), $\pi_a|_K$ and $\pi_b|_J$ are closed maps; therefore $\pi_a(K \setminus V_p)$ and $\pi_b(K \setminus W_p)$ are closed; thus
$$P_p \doteq P \setminus (\pi_a(K \setminus V_p) \cup \pi_b(K \setminus W_p))$$
is open, and the collection $\{P_p\}_{p\in P}$ constitutes an open cover of P. By the paracompactness of P, there exists a locally finite open refinement $\{P_\alpha\}_\alpha$ of said cover; in particular for a given α let $p(\alpha)$ be such that $P_\alpha \subseteq P_{p(\alpha)}$. We then define
$$V_\alpha \doteq V_{p(\alpha)} \cup \pi_a^{-1}(P \setminus P_\alpha) \qquad W_\alpha \doteq W_{p(\alpha)} \cup \pi_a^{-1}(P \setminus P_\alpha)$$
and $U_A = \cap_\alpha V_\alpha$, $U_B = \cap_\alpha W_\alpha$.

(iii) To conclude the proof, we need to show that $K \subseteq U_A$, $J \subseteq U_B$, that U_A and U_B are open and that $U_A \times_P U_B \subseteq U$. The first two inclusions are easily shown by showing that actually $K \subseteq V_\alpha$ and $J \subseteq W_\alpha$ for all α. Indeed, given $k \in K$ we can have that $k \in V_{p(\alpha)}$ or $k \notin V_{p(\alpha)}$; in the second case k lies in $K \setminus V_{p(\alpha)}$, and so $\pi_a(k) \notin P_{p(\alpha)}$. Therefore $\pi_a(k) \in P \setminus P_{p(\alpha)} \subseteq P \setminus P_\alpha$. The same applies to J.
To show that U_A and U_B are open, consider the following: since the cover $\{P_\alpha\}_\alpha$ is locally finite, given $x \in U_A$ there exist a finite collection $\{P_{\alpha_1}, \dots, P_{\alpha_n}\}$ such that $\pi_a(x) \in P_{\alpha_j}$. As these sets are open, there exists an open neighbourhood U_x of $\pi_a(x)$ contained in their intersection. Consider now the open set
$$U' \doteq \pi_a^{-1}(U_x) \cap V_{\alpha_1} \cap \dots \cap V_{\alpha_n}\ .$$

We claim that $U' \subseteq U_A$, that is, $U' \subseteq V_\alpha$ for all α. Clearly by its definition $U' \subseteq V_{\alpha_i}$ for all $1 \leq i \leq n$. On the other hand, for $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$ we have that $U_x \subseteq P \setminus P_\alpha$, and thus $\pi_a^{-1}(U_x) \subseteq V_\alpha$. The same holds for U_B.
Finally, we show that $U_A \times_P U_B \subseteq U$. Let $(x, y) \in U_A \times_P U_B$, and let $p = \pi_a(x) = \pi_b(y)$. As $\{P_\alpha\}_\alpha$ covers P, there exists P_α containing p, and as $x \in U_A$, then $x \in V_\alpha$. Now, as $p \in P_\alpha$, then necessarily $x \in V_{p(\alpha)}$. Analogously, $y \in W_{p(\alpha)}$. Then as $\pi_a(x) = \pi_b(y)$, the couple $(x, y) \in V_{p(\alpha)} \times W_{p(\alpha)}$ is necessarily in U.

□

Theorem 2.2.15. *Let M, N, P be smooth manifold with M compact; then the composition mapping*

$$\begin{aligned} \circ \colon C^\infty(M, N) \times C^\infty(N, P) &\to C^\infty(M, P) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

is continuous with respect to the Whitney C^k topology for every $k \in \mathbb{N} \cup \{\infty\}$.

Proof. In order to show that the result holds for $k = \infty$ it suffices to show it holds for finite k's. Thus let us consider the following: let $\pi \colon J^k(M, N) \to N$ and $\pi' \colon J^k(N, P) \to N$ be the (smooth) projections described in Remark 2.2.10. Consider the fiber product

$$D \doteq J^k(M, N) \times_N J^k(N, P)$$

and define the map

$$\begin{aligned} c\colon \qquad D \qquad &\to J^k(M, P) \\ (j^k(f_1)[x], j^k(f_2)[f_1(x)]) &\mapsto j^k(f_2 \circ f_1)[x] \ . \end{aligned}$$

By Lemma 2.2.6 this maps doesn't depend on the choice of the representatives f_1 and f_2, and moreover it is continuous. Let us now consider $f \in C^\infty(M, N)$ and $g \in C^\infty(N, P)$; notice that by definition of c,

$$j^k(g \circ f)[x] = c\left(j^k(f)[x], j^k(g)[f(x)]\right) \ \forall x \in M$$

i.e. $j^k(g \circ f)[M] = c(j^k(f)[M] \times_N j^k(g)[N])$. Moreover, as $j^k(f) \colon M \to J^k(M, N)$ is continuous and M is compact, then $\pi \colon j^k(f)[M] \to N$ is proper; the same holds for $\pi' \colon j^k(g)[N] \to N$, as $\pi' \circ j^k(g) \colon N \to N$ is the identity as can be inferred from Remark 2.2.10.
Consider now an open neighbourhood U of $j^k(g \circ f)[M]$ in $J^k(M, P)$; by the continuity of c, the compactness of M and the previous properties,

$$j^k(f)[M] \times_N j^k(g)[N] \subseteq c^{-1}(U)$$

with $\pi \colon j^k(f)[M] \to N$ and $\pi' \colon j^k(g)[N] \to N$ proper, and $c^{-1}(U)$ open. Therefore by Lemma 2.2.14 there exist V and W open neighbourhoods of $j^k(f)[M]$ in $J^k(M, N)$ and of $j^k(g)[N]$ in $J^k(N, P)$ respectively such that

$$V \times_N W \subseteq c^{-1}(U) \ .$$

Thus given any open neighbourhood $M^k(U)$ of $g \circ f$ in $(C^\infty(M, P), W^k)$, there exists an open neighbourhood $M^k(V) \times M^k(W)$ of (f, g) such that $\circ(M^k(V) \times M^k(W)) \subseteq M^k(U)$. $\square$

2.3 Fermionic functionals and Lagrangians

We shall now present a brief introduction to fermionic functionals, as presented in [12] and [37], in order to fix the notation. First of all, let us recall that the algebras associated to the Dirac field need to account for both the spinor and cospinor field, that is, sections of both the vector bundle $S_{r,1}(M)$ and its conjugate bundle $\overline{S_{r,1}(M)}$. In order to do so, one considers the Whitney sum of the two vector bundles $S_{r,1}(M) \oplus \overline{S_{r,1}(M)}$, which we shall denote with $S^{\oplus}_{r,1}(M)$. A section u of $S^{\oplus}_{r,1}(M)$ can be then understood as a couple (u_1, u_2) with $u_1 \in \mathcal{E}(S_{r,1}(M))$ and $u_2 \in \mathcal{E}(\overline{S_{r,1}(M)})$. The hermitian metric on $S_{r,1}(M)$, which induces a (bilinear) pairing

$$\mathcal{E}(S_{r,1}(M)) \times \mathcal{E}(\overline{S_{r,1}(M)}) \ni u, v \mapsto \langle v, u \rangle \in \mathcal{E}(M) \ ,$$

can be used to induce a symmetric and bilinear map (denoted with the same symbol)

$$\mathcal{E}(S^{\oplus}_{r,1}(M)) \times \mathcal{E}(S^{\oplus}_{r,1}(M)) \ni u, v \mapsto \langle v, u \rangle \doteq \langle v_2, u_1 \rangle + \langle u_2, v_1 \rangle \in \mathcal{E}(M) \ .$$

One can also define an involution on the space of sections $\mathcal{E}(S^{\oplus}_{r,1}(M))$ by using the conjugation maps $C\colon \overline{S_{r,1}(M)} \to S_{r,1}(M)$ and $C^{-1}\colon S_{r,1}(M) \to \overline{S_{r,1}(M)}$:

$$\mathcal{E}(S^{\oplus}_{r,1}(M)) \ni u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto u^* \doteq \begin{pmatrix} C^{-1}u_1 \\ Cu_2 \end{pmatrix} \in \mathcal{E}(S^{\oplus}_{r,1}(M)) \ .$$

Using the Dirac operator $\mathbf{D}\colon \mathcal{E}(S_{r,1}(M)) \to \mathcal{E}(S_{r,1}(M))$ and its adjoint $\mathbf{D}^* = C^{-1} \circ \mathbf{D} \circ C$ as well as the causal propagators $S\colon \mathcal{D}(S_{r,1}(M)) \to \mathcal{E}(S_{r,1}(M))$ and $S^*\colon \mathcal{D}(\overline{S_{r,1}(M)}) \to \mathcal{E}(\overline{S_{r,1}(M)})$ we construct the operators

$$\mathbf{D}^{\oplus} \doteq \mathbf{D} \oplus -\mathbf{D}^* \qquad S^{\oplus} \doteq S \oplus -S^* \ .$$

Notice that $S^{\oplus}$, which is the causal propagator for $\mathbf{D}^{\oplus}$, is formally self-adjoint: indeed, given $u, v \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ we have

$$\begin{aligned} \int_M \langle v, S^{\oplus} u \rangle \, d\mu_g &= \int_M \langle v_2, S u_1 \rangle - \langle S^* u_2, v_1 \rangle \, d\mu_g = \int_M - \langle S^* v_2, u_1 \rangle + \langle u_2, S v_1 \rangle \, d\mu_g \\ &= \int_M \langle S^{\oplus} v, u \rangle \, d\mu_g \ . \end{aligned}$$

Let $E \xrightarrow{\pi} M$ be a vector bundle with typical fiber V (in our case it will mostly be $S^{\oplus}_{r,1}(M)$) endowed with a symmetric bilinear metric $\langle \cdot, \cdot \rangle$; let us consider the space $\mathcal{E}(E)$ of sections of said vector bundle, equipped with the Fréchet topology described in Section 2.2.1. By [45, Theorem 51.5 and Corollary] we know that $\mathcal{E}(E)$ is nuclear, and therefore its projective and inductive tensor products coincide. We shall then consider the tensor products $\mathcal{E}(E)^{\widehat{\otimes} p}$; notice that by [10, Proposition III.8] we have that as topological vector spaces $\mathcal{E}(E)^{\widehat{\otimes} p} \simeq \mathcal{E}(E^{\boxtimes p})$, where $\boxtimes$ denotes the external tensor product of vector bundles. Now consider, given $\sigma \in S_p$, the linear operator $U_\sigma \colon \mathcal{E}(E)^{\otimes p} \to \mathcal{E}(E)^{\otimes p}$ such that

$$U_\sigma(u_1 \otimes \cdots \otimes u_p) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)} \ .$$

For each $\sigma \in S_p$, this map is continuous with respect to the projective topology: indeed, we know that said topology is given by the collection of seminorms $\{p_1 \otimes \cdots \otimes p_p\}$ with p_i a seminorm of $\mathcal{E}(E)$ and

$$(p_1 \otimes \cdots \otimes p_p)(u) = \inf \sum_n p_1(u_{1,n}) \cdots p_p(u_{p,n})$$

where the infimum is taken over all the finite linear combinations

$$\sum_n u_{1,n} \otimes \cdots \otimes u_{p,n} = u \ .$$

Now, consider, given u, any finite linear combination such that

$$u = \sum_n u_{1,n} \otimes \cdots \otimes u_{p,n} \ .$$

Then we get a finite linear representation of $U_\sigma u$, given by

$$U_\sigma u = \sum_n u_{\sigma(1),n} \otimes \cdots u_{\sigma(p),n} \ .$$

This entails in particular that given any seminorm $p_1 \otimes \cdots \otimes p_p$,

$$(p_1 \otimes \cdots \otimes p_p)(U_\sigma(u)) \leq \inf \sum_n p_1(u_{\sigma(1),n}) \cdots p_p(u_{\sigma(p),n})$$

where the infimum is taken over all finite representations of u. Now,

$$\sum_n p_1(u_{\sigma(1),n}) \cdots p_p(u_{\sigma(p),n}) = \sum_n p_{\sigma^{-1}(1)}(u_{1,n}) \cdots p_{\sigma^{-1}(p)}(u_{p,n})$$

and so

$$(p_1 \otimes \cdots \otimes p_p)(U_\sigma(u)) \leq (p_{\sigma^{-1}(1)} \otimes \cdots \otimes p_{\sigma^{-1}(p)})(u)$$

thus showing the continuity of U_σ. We then consider its continuous extension to $\mathcal{E}(E)^{\widehat{\otimes} p}$, which shall be denoted with the same symbol, and we then consider the map $A \colon \mathcal{E}(E)^{\widehat{\otimes} p} \to \mathcal{E}(E)^{\widehat{\otimes} p}$, defined by

$$A \doteq \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) U_\sigma \ .$$

Notice that A is continuous with respect to the projective topology on $\mathcal{E}(E)^{\otimes p}$, being a finite linear combination of continuous maps; it is also idempotent: indeed, we have that

$$A^2 = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) U_\sigma A \ .$$

Now, let us compute $U_\sigma A$:

$$U_\sigma A = \frac{1}{p!}\sum_{\tau\in S_p}\mathrm{sgn}(\tau)U_\sigma U_\tau = \frac{1}{p!}\sum_{\tau\in S_p}\mathrm{sgn}(\tau\sigma)\mathrm{sgn}(\sigma)U_{\tau\sigma} = \mathrm{sgn}(\sigma)\frac{1}{p!}\sum_{\tau\in S_p}\mathrm{sgn}(\tau\sigma)U_{\tau\sigma}$$

where the second equality follows from properties of permutations. Again, by properties of the permutations it follows that $S_p \ni \tau \mapsto \tau\sigma \in S_p$ is a bijection; thus

$$U_\sigma A = \mathrm{sgn}(\sigma)A$$

and we conclude that

$$A^2 = \frac{1}{p!}\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)^2 A = A\ .$$

Using A, we define the space of p-antisymmetric sections

$$\mathcal{E}^a(M^p, E^{\boxtimes p}) \doteq A\left(\mathcal{E}(E)^{\widehat{\otimes} p}\right)\ .$$

Notice that if we define $\wedge^p\mathcal{E}(E) \doteq A(\mathcal{E}(E)^{\otimes p})$, we then have

$$\mathcal{E}^a(M^p, E^{\boxtimes p}) = \overline{\wedge^p\mathcal{E}(E)}$$

where $\bar{\cdot}$ denotes the closure. Given $\{u_i\}_{1\le i\le p} \subset \mathcal{E}(E)$, we shall use the notation

$$u_1\wedge\cdots\wedge u_p = A(u_1\otimes\cdots\otimes u_p)$$

The configuration space for the Dirac field is then given by the (algebraic) direct sum

$$\mathcal{C}(E) \doteq \bigoplus\nolimits_{p\in\mathbb{N}}\mathcal{E}^a(M^p, E^{\boxtimes p})\ .$$

In the case $E = S^{\oplus}_{r,1}(M)$, the involution previously defined on $\mathcal{E}(E)$ can be extended to an involution $\cdot^*\colon \mathcal{C}(E)\to\mathcal{C}(E)$ in the following way: first of all, we define

$$(u_1\wedge\cdots\wedge u_p)^* \doteq u_p^*\wedge\cdots\wedge u_1^*$$

and we extend it by linearity to $\wedge^p\mathcal{E}(E)$. Noticing that continuity easily follows from an analogous reasoning as to that of U_σ, the involution can be extended to $\mathcal{E}^a(M^p, E^{\boxtimes p})$ and finally in an obvious way to $\mathcal{C}(E)$.

We are interested in *antisymmetric functionals* on the space of sections $\mathcal{E}(E)$, that is, elements of the strong dual of $\mathcal{C}(E)$:

$$\mathcal{F}(E) \doteq \mathcal{C}(E)' = \left(\bigoplus\nolimits_{p\in\mathbb{N}}\mathcal{E}^a(M^p, E^{\boxtimes p})\right)' = \prod_{p\in\mathbb{N}}\mathcal{E}^{a\prime}(M^p, E^{\boxtimes p}) \doteq \prod_{p\in\mathbb{N}}\mathcal{F}^p(E)\ .$$

Its elements can be regarded as sequences $\{F_p\}_{p\in\mathbb{N}}$ with $F_p\in\mathcal{F}^p(E)$. The duality pairing between $\mathcal{F}(E)$ and $\mathcal{C}(E)$ is given by

$$F(u) \doteq \sum_{p\in\mathbb{N}} F_p(u_p) \text{ for all } F\in\mathcal{F}(E),\ u = \{u_p\}_{p\in\mathbb{N}}\in\mathcal{C}(E)$$

where the sum is finite and thus always well-defined, $\mathcal{C}(E)$ being an algebraic direct sum.

Example 2.3.1. We propose here a few examples of elements of $\mathcal{F}^2(E)$ for the simple vector bundle $S_{3,1}(\mathbb{M}^4) = \mathbb{M}^4 \times \mathbb{C}^4$ over the 4-dimensional Minkowski spacetime $\mathbb{M}^4$:

(i) Given a compactly supported, smooth map $\mathbb{M}^4 \supset K \ni p \mapsto A(p) \in L(\mathbb{C}^4, \mathbb{C}^4)$ such that $A^\intercal = -A$, we define on $\mathcal{E}(S_{3,1}(\mathbb{M}^4))$

$$F(u, v) = \sum_{i,j=1}^{4} \int_{\mathbb{M}^4} u^i(p) A_{ij}(p) v^j(p)\, d^4p \; .$$

Clearly $F(v, u) = -F(u, v)$ and $F(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 F(u_1, v) + \lambda_2 F(u_2, v)$; by the universal property of the exterior product then there exists a unique linear map $\overline{F}\colon \wedge^2 \mathcal{E}(E) \to \mathbb{C}$ such that $\overline{F}(u \wedge v) = F(u, v)$. Now, given $u = \sum_n u^n \wedge v^n$, we have

$$\begin{aligned}
\left|\overline{F}(u)\right| &\le \sum_n \sum_{i,j=1}^{4} \int_{\mathbb{M}^4} \left|u_i^n(p) A_{ij}(p) v_j^n(p)\right| \, d^4p \\
&\le \sum_n \sum_{i,j=1}^{4} \sup_{p\in K} \left|u_i^n(p)\right| \sup_{p\in K} \left|v_j^n(p)\right| \sup_{p\in K} \left|A_{ij}(p)\right| |K| \\
&\le 8|K| \max_{1\le i<j\le 4} \sup_{p\in K} \left|A_{ij}(p)\right| \sum_n \max_{1\le i\le 4} \sup_{p\in K} \left|u_i^n(p)\right| \max_{1\le j\le 4} \sup_{p\in K} \left|v_j^n(p)\right| \\
&= 8|K| \max_{1\le i<j\le 4} \sup_{p\in K} \left|A_{ij}(p)\right| \sum_n p_{\bar{\imath},K,0}(u^n) p_{\bar{\jmath},K,0}(v^n)
\end{aligned} \; .$$

By considering the infimum over the finite representations of u we then have

$$\left|\overline{F}(u)\right| \le 8|K| \max_{1\le i<j\le 4} \sup_{p\in K} \left|A_{ij}(p)\right| (p_{\bar{\imath},K,0} \otimes p_{\bar{\jmath},K,0})(u)$$

and thus $\overline{F}$ is continuous on $\wedge^2\mathcal{E}(E)$ with respect to the (subspace) projective topology; its continuous extension therefore yields a functional $\overline{F} \in \mathcal{F}^2(E)$.

(ii) Given a skew-symmetric matrix $A \in M_4(\mathbb{C})$, a function $f \in \mathcal{D}(\mathbb{M}^4)$, and $k \in \{1, \dots, 4\}$ fixed, let us consider

$$F(u, v) = \sum_{i,j=1}^{4} \int_{\mathbb{M}^4} f(x) \partial_i u^k(p) A_{ij} \partial_j v^k(p)\, d^4p$$

As before, we have that $F(v, u) = -F(u, v)$ and $F(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 F(u_1, v) + \lambda_2 F(u_2, v)$, and thus there exists a linear map $\overline{F}\colon \wedge^2 \mathcal{E}(E) \to \mathbb{C}$ such that $\overline{F}(u\wedge v) = F(u, v)$. Again, considering $u = \sum_n u^n \wedge v^n$ we have

$$\begin{aligned}
\left|\overline{F}(u)\right| &\le \sum_n \sum_{i,j=1}^{4} \int_{\mathbb{M}^4} \left|f(x) \partial_i u_k^n(p) A_{ij} \partial_j u_k^n(p)\right| \, d^4p \\
&\le 8\left|\mathrm{Supp}(f)\right| \sup_{p\in\mathbb{M}^4} \left|f(p)\right| \max_{1\le i<j\le 4} \left|A_{ij}\right| \\
&\qquad \sum_n \max_{1\le i\le 4} \sup_{p\in\mathrm{Supp}(f)} \left|\partial_i u_k^n(p)\right| \max_{1\le j\le 4} \sup_{p\in\mathrm{Supp}(f)} \left|\partial_j v_k^n(p)\right|
\end{aligned}$$

Again, by taking the infimum over all the finite representations of u we get

$$\left|\overline{F}(u)\right| \leq 8\left|\mathrm{Supp}(f)\right| \sup_{p\in\mathbb{M}^4}\left|f(p)\right| \max_{1\leq i<j\leq 4}\left|A_{ij}\right| (p_{\bar{i},\mathrm{Supp}(f),1}\otimes p_{\bar{j},\mathrm{Supp}(f),1})(u)$$

and continuity of $\overline{F}$ follows; its continuous extension to $\mathcal{E}^a(\mathbb{M}^4\times\mathbb{M}^4, E^{\boxtimes 2})$ is then a well-defined object in $\mathcal{F}^2(E)$. ✠

We define the support of a fermionic functional as

$$\begin{aligned}\mathrm{Supp}(F) \doteq \Big\{ & x\in M \text{ s.t. for all } U\subseteq M \text{ open neighbourhood of } x\\ & \exists p\in\mathbb{N},\ \{u_i\}_{1\leq i\leq p}\subseteq\mathcal{E}(E) \text{ with } \mathrm{Supp}(u_i)\subseteq U \text{ and}\\ & F_p(u_1\wedge\cdots\wedge u_p)\neq 0\Big\}\,.\end{aligned}$$

We endow $\mathcal{F}(E)$ with the strong topology, that is, the topology given by the family of seminorms

$$p_B(F)\doteq\sup_{u\in B}\left|F(u)\right|$$

where $B\subseteq\mathcal{C}(E)$ is a bounded subset. From [39, p. 33] it follows that B is bounded in $\mathcal{C}(E)$ if and only if it is given by a set of the form

$$\prod_{\substack{p\in S\\ S\subset\mathbb{N}\text{ finite}}} B_p\times\{0\},\qquad B_p\subseteq\mathcal{E}^a(M^p,E^{\boxtimes p})\text{ bounded.}$$

Lemma 2.3.2. *$\mathcal{F}(E)$ endowed with the strong topology is a locally convex, nuclear topological vector space.*

Proof. We have already observed that $\mathcal{E}(E)$ equipped with the Fréchet topology is nuclear, and so in particular is $\mathcal{E}(M^p,E^{\boxtimes p})\simeq\mathcal{E}(M)^{\widehat{\otimes}p}$. Let us now consider the space $\mathcal{E}^a(M^p,E^{\boxtimes p})$; this is a closed subspace of a nuclear Fréchet space, and so it is nuclear and Fréchet [45, Proposition 50.1]. The algebraic direct sum $\mathcal{C}(E)$ is then a direct sum of nuclear, Fréchet topological vector spaces; it is therefore nuclear [45, Proposition 50.1] and Fréchet [39, p. 33]. By [45, Proposition 50.6] we then know that its strong dual is nuclear. □

$\mathcal{F}(E)$ can be endowed with an antisymmetric, pointwise product; we start by defining, given $F,G\in\mathcal{F}(E)$,

$$\begin{aligned}(F\wedge G)_p(u_1\wedge\cdots\wedge u_p)\doteq & \sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)\\ & \sum_{k=0}^{p}\frac{1}{k!(p-k)!}F_k(u_{\sigma(1)}\wedge\cdots\wedge u_{\sigma(k)})G_{p-k}(u_{\sigma(k+1)}\wedge\cdots\wedge u_{\sigma(p)})\end{aligned}$$

and then by extending by linearity to $\wedge^p\mathcal{E}(E)$. Notice that the fermionic functional $F\wedge G$ is then a sequence $\{(F\wedge G)_p\}_{p\in\mathbb{N}}$ of continuous functionals with respect to the projective topology: indeed,

$$\begin{aligned}\big|(F\wedge G)_p(u_1\wedge\cdots\wedge u_p)\big| \leq \sum_{\sigma\in S_p}\sum_{k=0}^{p}&\frac{1}{k!(p-k)!}\\ &\Big|F_k(u_{\sigma(1)}\wedge\cdots\wedge u_{\sigma(k)})\Big|\Big|G_{p-k}(u_{\sigma(k+1)}\wedge\cdots\wedge u_{\sigma(p)})\Big|\end{aligned}$$

and by exploiting the continuity of $\{F_k\}_{0\leq k\leq p}$ and $\{G_{p-k}\}_{0\leq k\leq p}$ we then have

$$\begin{aligned}\big|(F\wedge G)_p(u_1\wedge\cdots\wedge u_p)\big| \leq \sum_{\sigma\in S_p}\sum_{k=0}^{p}&\frac{C_{F,k}C_{G,p-k}}{k!(p-k)!}\\ &p_1^F(u_{\sigma(1)})\cdots p_k^F(u_{\sigma(k)})p_1^G(u_{\sigma(k+1)})\cdots p_{p-k}^G(u_{\sigma(p)})\ .\end{aligned}$$

By considering the maximum over the two sums, which is well-defined as both are finite, we then have that

$$\big|(F\wedge G)_p(u_1\wedge\cdots\wedge u_p)\big|\leq C\overline{p}_1(u_1)\cdots\overline{p}_p(u_p)$$

for certain seminorms $\{\overline{p}_i\}_{1\leq i\leq p}$ on $\mathcal{E}(E)$. Then given any $u\in\wedge^p\mathcal{E}(E)$, by representing it as a finite linear combination $u=\sum_n u_{1,n}\wedge\cdots\wedge u_{p,n}$ we have

$$\big|(F\wedge G)_p(u)\big|\leq C\sum_n\overline{p}_1(u_{1,n})\cdots\overline{p}_p(u_{p,n})$$

and by taking the infimum over all the representation on the right we have the desired seminorm on the projective tensor product. One can then extend $(F\wedge G)_p$ by continuity to $\mathcal{E}^a(M^p,E^{\boxtimes p})$. This product is associative and graded commutative.
Moreover, the product is also continuous with respect to the topology on $\mathcal{F}(E)$: consider a bounded set $B=\prod_{p\in S}B_p\times\{0\}$ in $C(E)$, and consider

$$p_B(F\wedge G)=\sup_{u\in B}\big|(F\wedge G)(u)\big|=\sup_{u\in B}\left|\sum_{p\in S}(F\wedge G)_p(u_p)\right|\leq|S|\sup_{u_{\overline{p}}\in B_{\overline{p}}}\big|(F\wedge G)_{\overline{p}}(u_{\overline{p}})\big|\ .$$

As $F\wedge G$ is continuous, the supremum over $B_{\overline{p}}$ is equal to the supremum over a dense subset of $B_{\overline{p}}$, which we can take to be the intersection of $B_{\overline{p}}$ with $\wedge^{\overline{p}}\mathcal{E}(E)$. Notice that $B_{\overline{p}}$ is bounded in $\mathcal{E}(E)^{\widehat{\otimes}\overline{p}}$, that is, there exist a seminorm $p_1\otimes\cdots\otimes p_{\overline{p}}$ and a constant $r>0$ such that

$$(p_1\otimes\cdots\otimes p_{\overline{p}})(u)<r\ \forall u\in B_{\overline{p}}\ .$$

By [45, Proposition 43.1] we then know that elements in $B_{\overline{p}}\cap\wedge^{\overline{p}}\mathcal{E}(E)$ can be written as

$$\sum_{k=1}^{n}\lambda_k u_1^k\wedge\cdots\wedge u_{\overline{p}}^k,\qquad \sum_{k=0}^{n}|\lambda_k|<r\ \text{and}\ p_i(u_i^k)\leq 1\ \forall i\in\{1,\ldots,\overline{p}\}\ . \tag{2.16}$$

Now, notice that given any $u_{\overline{p}} \in B_{\overline{p}} \cap \wedge^{\overline{p}}\mathcal{E}(E)$ we then have

$$\begin{aligned}\left|(F\wedge G)_{\overline{p}}(u_{\overline{p}})\right| \leq \sum_{k=0}^{n}|\lambda_k| \sum_{\sigma\in S_{\overline{p}}} \operatorname{sgn}(\sigma) \sum_{l=0}^{\overline{p}} \frac{1}{l!(\overline{p}-l)!} \\ \left|F_l(u^k_{\sigma(1)}\wedge\cdots\wedge u^k_{\sigma(l)})\right|\left|G_{\overline{p}-l}(u^k_{\sigma(l+1)}\wedge\cdots\wedge u^k_{\sigma(\overline{p})})\right|\end{aligned}$$

By (2.16), we can bound

$$\left|F_l(u^k_{\sigma(1)}\wedge\cdots\wedge u^k_{\sigma(l)})\right|\left|G_{\overline{p}-l}(u^k_{\sigma(l+1)}\wedge\cdots\wedge u^k_{\sigma(\overline{p})})\right| \leq \sup_{u_l\in B^\sigma_l}\left|F_l(u_l)\right| \sup_{u_{\overline{p}-l}\in B^\sigma_{\overline{p}-l}}\left|G_{\overline{p}-l}(u_{\overline{p}-l})\right|$$

with B^σ_l and $B^\sigma_{\overline{p}-l}$ bounded sets in $\mathcal{E}^a(M^l, E^{\boxtimes l})$ and $\mathcal{E}^a(M^{\overline{p}-l}, E^{\boxtimes \overline{p}-l})$ respectively, described by $(p_{\sigma(1)}\otimes\cdots p_{\sigma(l)})(u_l) \leq 1$ and $(p_{\sigma(l+1)}\otimes\cdots p_{\sigma(\overline{p}-l)})(u_{\overline{p}-l}) \leq 1$. By taking a suitable maximum over $l \in \{1,\dots,\overline{p}\}$, $\overline{p}-l \in \{1,\dots,\overline{p}\}$, and $\sigma \in S_{\overline{p}}$ (which is well-defined as all these sets are finite), we conclude that for all $u_{\overline{p}} \in B_{\overline{p}} \cap \wedge^{\overline{p}}\mathcal{E}(E)$

$$\left|(F\wedge G)_{\overline{p}}(u_{\overline{p}})\right| < r2^{\overline{p}} p_{B^{\tilde{\sigma}}_{\tilde{l}}}(F) p_{B^{\tilde{\sigma}}_{\overline{p}-\tilde{l}}}(G) \ .$$

Thus we have

$$p_B(F\wedge G) \leq |S|\, r2^{\overline{p}} p_{B^{\tilde{\sigma}}_{\tilde{l}}}(F) p_{B^{\tilde{\sigma}}_{\overline{p}-\tilde{l}}}(G)$$

and continuity of the antisymmetric product follows. $\mathcal{F}(E)$ can also be endowed with an involution $\cdot^*\colon \mathcal{F}(E) \to \mathcal{F}(E)$,

$$\{F_p\}_{p\in\mathbb{N}} \mapsto \left\{F^*_p\right\}_{p\in\mathbb{N}}, \qquad (F^*_p)(u_p) \doteq (-1)^p\overline{F_p(u^*_p)} \ . \tag{2.17}$$

The involution is continuous with respect to the topology on $\mathcal{F}(E)$: indeed, we have, given a bounded set $B = \prod_{p\in S} B_p \times \{0\}$

$$p_B(F) = \sup_{u\in B}\left|F^*(u)\right| = \sup_{u\in B}\left|\sum_{p\in S}(-1)^p\overline{F_p(u^*_p)}\right| \leq \sup_{u\in B}\sum_{p\in S}\left|F_p(u^*_p)\right|$$

Now, notice that as $\cdot^*\colon \mathcal{C}(E) \to \mathcal{C}(E)$ is continuous it maps bounded sets to bounded sets; thus we can consider B^* which would then yield

$$p_B(F) \leq \sup_{u\in B^*}\sum_{p\in S}\left|F_p(u_p)\right| \leq |S| \max_{p\in S}\left\{\sup_{p\in B^*_p}\left|F_p(u_p)\right|\right\} = |S|\, p_{B^*_{\tilde{p}}}(F)$$

as desired. These two operations make $\mathcal{F}(E)$ into a $\mathbb{Z}$-graded, topological $*$-algebra.
One can consider derivatives of fermionic functionals in the following way: given $F_p \in \mathcal{F}^p(E)$, the left derivative of F_p in the direction $h \in \mathcal{E}(E)$ is defined as 0 if $p = 0$, and if $p \geq 1$ it is defined on $\wedge^{p-1}\mathcal{E}(E)$ as

$$d_hF_p(u) \doteq F_p(h\wedge u) \ .$$

It is then extended by continuity to $\mathcal{E}^a(M^{p-1}, E^{\boxtimes p-1})$, thus yielding a linear and continuous map $d_h F_p\colon \mathcal{E}^a(M^{p-1}, E^{\boxtimes p-1}) \to \mathbb{C}$, i.e. $d_h F_p \in \mathcal{F}^{p-1}(E)$. One can then extend the map $d_h\colon \mathcal{F}^p(E) \to \mathcal{F}^{p-1}(E)$ to the whole algebra $\mathcal{F}(E)$ by considering

$$d_h\colon F = \{F_p\}_{p\in\mathbb{N}} \mapsto d_h F \doteq \{d_h F_p\}_{p\in\mathbb{N}} \ .$$

Now,

Lemma 2.3.3. *For any $h \in \mathcal{E}(E)$, d_h is a graded derivation.*

Proof. Due to linearity and continuity, it is sufficient to show the result holds on elements of the form

$$u_1 \wedge \cdots \wedge u_{p+q-1}, \qquad \{u_i\}_{1\le i\le p+q-1} \subset \mathcal{E}(E) \ .$$

Let us consider, given $F \in \mathcal{F}^p(E)$, $G \in \mathcal{F}^q(E)$, and $h \in \mathcal{E}(E)$,

$$d_h(F\wedge G)(u_1\wedge\cdots\wedge u_{p+q-1}) = F\wedge G(h\wedge u_1\wedge\cdots\wedge u_{p+q-1}) = \sum_{\sigma\in S_{p+q}} \frac{\operatorname{sgn}(\sigma)}{p!q!}$$
$$F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1}))G(\sigma(u_p)\wedge\cdots\wedge\sigma(u_{p+q-1}))$$

where $\sigma(h)$ acts on h by moving it to the spot indexed by $\sigma(1)$ and on u_i by moving it to the spot indexed by $\sigma(i+1)$. Now, we can decompose the set S_{p+q} into the two disjoint subsets

$$S_1 = \{\sigma \in S_{p+q} \text{ s.t. } \sigma(1) \in \{1,\dots,p\}\} \qquad S_2 = \{\sigma \in S_{p+q} \text{ s.t. } \sigma(1) \in \{p+1,\dots,p+q\}\}$$

and then consider

$$\begin{aligned} &d_h(F\wedge G)(u_1\wedge\cdots\wedge u_{p+q-1}) = \\ &\quad\sum_{\sigma\in S_1} \frac{\operatorname{sgn}(\sigma)}{p!q!} F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1}))G(\sigma(u_p)\wedge\cdots\wedge\sigma(u_{p+q-1})) \\ &\quad+\sum_{\sigma\in S_2} \frac{\operatorname{sgn}(\sigma)}{p!q!} F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1}))G(\sigma(u_p)\wedge\cdots\wedge\sigma(u_{p+q-1})) \ . \end{aligned}$$

Let us consider the first sum: as $\sigma(1) \in \{1,\dots,p\}$, thanks to the properties of the antisymmetric product we can write

$$\begin{aligned} F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1})) &= (-1)^{\sigma(1)-1}F(h\wedge\overline{\sigma}(u_1)\wedge\cdots\wedge\overline{\sigma}(u_{p-1})) \\ &= (-1)^{\sigma(1)-1}(d_hF)(\overline{\sigma}(u_1)\wedge\cdots\wedge\overline{\sigma}(u_{p-1})) \end{aligned}$$

where $\overline{\sigma} \in S_{p+q}$ is the permutation originating from the composition of the permutation τ that sends $\sigma(1)$ to 1 by pairwise exchanging elements in the wedge product and σ, i.e. $\overline{\sigma} = \tau\circ\sigma$. Notice then that $\operatorname{sgn}(\sigma)(-1)^{\sigma(1)-1} = \operatorname{sgn}(\overline{\sigma})$, that $\overline{\sigma}$ is, for each σ, a permutation of the last $p+q-1$ elements of $\{1,\dots,p+q\}$, and that as σ runs over all permutations in S_1 we get that $\overline{\sigma}$ runs over all permutations of $\{2,\dots,p+q\}$ p times. Thus

$$\begin{aligned}
&\sum_{\sigma\in S_1}\frac{\mathrm{sgn}(\sigma)}{p!q!}F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1}))G(\sigma(u_p)\wedge\cdots\wedge\sigma(u_{p+q-1}))\\
=&\sum_{\sigma\in S_1}\frac{\mathrm{sgn}(\overline{\sigma})}{p!q!}(d_hF)(\overline{\sigma}(u_1)\wedge\cdots\wedge\overline{\sigma}(u_{p-1}))G(\overline{\sigma}(u_p)\wedge\cdots\wedge\overline{\sigma}(u_{p+q-1}))\\
=&\sum_{\overline{\sigma}\in S_{p+q-1}}\frac{\mathrm{sgn}(\overline{\sigma})}{(p-1)!q!}(d_hF)(\overline{\sigma}(u_1)\wedge\cdots\wedge\overline{\sigma}(u_{p-1}))G(\overline{\sigma}(u_p)\wedge\cdots\wedge\overline{\sigma}(u_{p+q-1}))\\
=&((d_hF)\wedge G)(u_1\wedge\cdots\wedge u_{p+q-1})
\end{aligned}$$

As for the second sum, one can reason in an identical fashion; the only difference in this case is that by considering the permutation τ that sends $\sigma(1)$ to p, we get

$$\mathrm{sgn}(\sigma)(-1)^{\sigma(1)-p}=(-1)^p\mathrm{sgn}(\overline{\sigma})$$

where $\overline{\sigma}$ leaves 1 invariant. As we sum over all elements of S_2, we get all possible permutations $\overline{\sigma}$ of $\{1,\dots,p+q\}$ of the last $p+q-1$ terms q times; therefore

$$\begin{aligned}
&\sum_{\sigma\in S_2}\frac{\mathrm{sgn}(\sigma)}{p!q!}F(\sigma(h)\wedge\sigma(u_1)\wedge\cdots\wedge\sigma(u_{p-1}))G(\sigma(u_p)\wedge\cdots\wedge\sigma(u_{p+q-1}))\\
=&(-1)^p(F\wedge(d_hG))(u_1\wedge\cdots\wedge u_{p+q-1})\ ,
\end{aligned}$$

that is,

$$d_h(F\wedge G)=(d_hF)\wedge G+(-1)^pF\wedge(d_hG)$$

□

One can also consider higher order derivatives by iterating the left derivative: given $F_p\in\mathcal{F}^p(E)$, $k\leq p$ and $h_1,\dots,h_k\in\mathcal{E}(E)$, we define for $u\in\wedge^{p-k}\mathcal{E}(E)$

$$d^k_{h_k,\dots,h_1}F_p(u)=F_p(h_k\wedge\cdots\wedge h_1\wedge u)$$

and then proceed by continuity as before, obtaining for each $F\in\mathcal{F}(E)$ a jointly continuous map

$$d^kF\colon\mathcal{E}(E)^k\times\mathcal{C}(E)\to\mathbb{C}$$

which is easily seen to be multilinear and alternating in the first k entries, that is, equivalently, a continuous map

$$F^{(k)}\colon\mathcal{E}^a(M^k,E^{\boxtimes k})\times\mathcal{C}(E)\to\mathbb{C}$$

which is linear in the first entry. In particular, notice that these can be considered as an $\mathcal{F}(E)$-valued, compactly supported, antisymmetric distributional section of $E^{*\boxtimes k}\to M^k$, that is, an object of $\mathcal{E}^{a\prime}(M^k,E^{\boxtimes k})\widehat{\otimes}\mathcal{F}(E)$; notice that as both $\mathcal{E}^{a\prime}(M^k,E^{\boxtimes k})$ and $\mathcal{F}(E)$ are nuclear (by [45, Theorem 51.5 and Corollary] and Lemma 2.3.2 respectively), the

topological tensor product is uniquely defined.
We will be particularly interested in a peculiar subset of the graded algebra of fermionic functionals, namely *local* fermionic functionals $\mathcal{F}_{\mathrm{loc}}(E) \subseteq \mathcal{F}(E)$. These are those functionals $F \in \mathcal{F}(E)$ such that:

(i) for every $u \in \mathcal{C}(E)$ their wavefront set satisfies

$$\mathrm{WF}(F_u^{(n)}) \subseteq \Xi_n \doteq T^*\dot{M}^n \setminus \bigcup_{(p_1,\ldots,p_n)\in M^n} \left(\overline{V_{p_1}^+} \times \cdots \times \overline{V_{p_n}^+}\right) \cup \left(\overline{V_{p_1}^-} \times \cdots \times \overline{V_{p_n}^-}\right) ,$$

where $T^*\dot{M}^n$ denotes the cotangent bundle minus the image of the zero section, and $\overline{V}_{p_i}^{\pm}$ denotes the closure of the future/past causal cone at p_i;

(ii) the distributional support of $F_u^{(n)}$ lies in the thin diagonal $\Delta_n \hookrightarrow M^n$ of M^n;

(iii) $\mathrm{WF}(F_u^{(n)}) \perp T\Delta_n$ for every $u \in \mathcal{C}(E)$.

We refer the reader to [27] for an introduction to wavefront sets and their properties.
Having defined the local fermionic functionals, a *fermionic Lagrangian* is then a map $\mathscr{L} \colon \mathcal{D}(M) \to \mathcal{F}_{\mathrm{loc}}(E)$ such that

(i) $\mathscr{L}(f_1 + f_2 + f_3) = \mathscr{L}(f_1 + f_2) - \mathscr{L}(f_2) + \mathscr{L}(f_2 + f_3)$ for every $f_1, f_2, f_3 \in \mathcal{D}(M)$ such that $\mathrm{Supp}(f_1) \cap \mathrm{Supp}(f_3) = \varnothing$;

(ii) $\mathrm{Supp}(\mathscr{L}(f)) \subseteq \mathrm{Supp}(f)$ for every $f \in \mathcal{D}(M)$.

3 Different metrics and spinors

3.1 Connecting different Lorentzian metrics

Given a manifold M of dimension $r+1$, consider the vector bundle $\pi\colon \mathrm{Sym}^2(T^*M) \to M$ of rank 2 symmetric tensors, and let $\pi\colon \mathrm{Lor}(M) \to M$ be its subbundle of indefinite quadratic forms of signature $(r,1)$. Its smooth sections $\Gamma(\mathrm{Lor}(M))$ are the Lorentzian metrics on M.
We are interested in the case in which we are given two sections $g_0, g_1 \in \Gamma(\mathrm{Lor}(M))$ which satisfy three assumptions: first of all,

(i) (M, g_0) and (M, g_1) are globally hyperbolic;

(ii) $\mathrm{Supp}(g_1 - g_0)$ is a compact subset of M, i.e. there exists a compact set $K \subset M$ such that $g_1 \equiv g_0$ on $M \setminus K$.

We wish to determine whether there exists a curve $g_\lambda\colon [0,1] \to \Gamma(\mathrm{Lor}(M))$ joining the two metrics. To do so, we proceed as follows: first of all, by (i) and [7] we know that there exist isometries

$$\begin{aligned} (M, g_0) &\overset{\psi_0}{\simeq} (\mathbb{R} \times \Sigma_0, -\beta_0(t_0,p_0)dt_0 \otimes dt_0 + g_{t_0}) \\ (M, g_1) &\overset{\psi_1}{\simeq} (\mathbb{R} \times \Sigma_1, -\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}) \end{aligned} \tag{3.1}$$

with $\beta_i\colon \mathbb{R} \times \Sigma_i \to (0,+\infty)$ being the smooth lapse functions, h_{t_1} and g_{t_0} being smooth riemannian metrics on Σ_1 and Σ_0 respectively depending smoothly on t_1 and t_0. We can also choose ψ_0 and ψ_1 in the following way: given a spacelike Cauchy hypersurface $\Sigma_i \hookrightarrow S_i \subset M$ for g_i, we can choose ψ_i so that they map to $\mathbb{R} \times \Sigma_i$ and so that $S_i = \psi_i^{-1}(\{0\} \times \Sigma_i)$. Now, requirement (ii) means that if we consider a spacelike Cauchy surface $\Sigma \hookrightarrow S \subset M \setminus K$ for g_0, it will also be a spacelike Cauchy hypersurface for g_1; therefore the isometries in (3.1) can be chosen so that

$$\begin{aligned} (M, g_0) &\overset{\psi_0}{\simeq} (\mathbb{R} \times \Sigma, -\beta_0(t_0,p_0)dt_0 \otimes dt_0 + g_{t_0}) \\ (M, g_1) &\overset{\psi_1}{\simeq} (\mathbb{R} \times \Sigma, -\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}) \ . \end{aligned}$$

Moreover, if we consider the smooth pseudo-riemannian manifolds $(M \setminus K, g_0)$ and $(M \setminus K, g_1)$, these two are equal, i.e. we have an isometry between

$$(\mathbb{R} \times \Sigma \setminus \psi_0(K), -\beta_0(t_0,p_0)dt_0 \otimes dt_0 + g_{t_0})$$

and

$$(\mathbb{R} \times \Sigma \setminus \psi_1(K), -\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}) \ .$$

That is, there exists a diffeomorphism

$$\begin{aligned} \psi\colon \mathbb{R} \times \Sigma &\to \mathbb{R} \times \Sigma \\ (t_0, p_0) &\to (t_1, p_1) \end{aligned}$$

given by $\psi = \psi_1 \circ \psi_0^{-1}$, such that, denoting with K_0 the set $\psi_0(K)$,

$$\psi\colon \mathbb{R} \times \Sigma \setminus K_0 \to \mathbb{R} \times \Sigma \setminus \psi(K_0)$$

is an isometry, i.e.

$$\psi^* \left(-\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}\right) |_{\mathbb{R}\times\Sigma\setminus K_0} = \left(-\beta_0(t_0,p_0)dt_0 \otimes dt_0 + g_{t_0}\right) |_{\mathbb{R}\times\Sigma\setminus K_0} \ .$$

Given local coordinate neighbourhoods U and V on Σ, inducing local coordinates x^i and y^j respectively, the above requirement reads

$$\begin{cases} -(\beta_1 \circ \psi)\left(\dfrac{\partial \psi^{t_1}}{\partial t_0}\right)^2 + (h_{ij}\circ\psi)\dfrac{\partial \psi^{y^i}}{\partial t_0}\dfrac{\partial \psi^{y^j}}{\partial t_0} = -\beta_0 \\ -(\beta_1 \circ \psi)\dfrac{\partial \psi^{t_1}}{\partial t_0}\dfrac{\partial \psi^{t_1}}{\partial x^k} + (h_{ij}\circ\psi)\dfrac{\partial \psi^{y^i}}{\partial t_0}\dfrac{\partial \psi^{y^j}}{\partial x^k} = 0 \\ -(\beta_1 \circ \psi)\dfrac{\partial \psi^{t_1}}{\partial x^l}\dfrac{\partial \psi^{t_1}}{\partial x^k} + (h_{ij}\circ\psi)\dfrac{\partial \psi^{y^i}}{\partial x^l}\dfrac{\partial \psi^{y^j}}{\partial x^k} = g_{lk} \end{cases}$$

on $\mathbb{R} \times \Sigma \setminus K_0$. We can now state our third assumption:

(iii) we assume that the vector field ∂_{t_0} (which is, modulo a positive factor, the gradient of the Cauchy temporal function ∇t_0) is timelike for $\psi^* \left(-\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}\right)$, that ∂_{t_0} has the same time-orientation as $\psi_*^{-1}\left(\partial_{t_1}\right)$ and that the spacelike, Cauchy slices $\{t_0 = k\}$, $k \in \mathbb{R}$, are spacelike hypersurfaces for $\psi^* \left(-\beta_1(t_1,p_1)dt_1 \otimes dt_1 + h_{t_1}\right)$.

Assumption (iii) then implies that on K_0 we have

$$\begin{cases} -(\beta_1 \circ \psi)\left(\dfrac{\partial \psi^{t_1}}{\partial t_0}\right)^2 + (h_{ij}\circ\psi)\dfrac{\partial \psi^{y^i}}{\partial t_0}\dfrac{\partial \psi^{y^j}}{\partial t_0} < 0 \\ \dfrac{\partial \psi^{t_1}}{\partial t_0} > 0 \\ \left(-(\beta_1 \circ \psi)\dfrac{\partial \psi^{t_1}}{\partial x^k}\dfrac{\partial \psi^{t_1}}{\partial x^l} + (h_{ij}\circ\psi)\dfrac{\partial \psi^{y^i}}{\partial x^k}\dfrac{\partial \psi^{y^j}}{\partial x^l}\right)_{kl} \quad \text{is positive definite .} \end{cases}$$

Under these requirements and thanks to Sylvester's law of inertia, which tells us that the signature of the metric is preserved under coordinate transformations, it follows that $(\mathbb{R}\times\Sigma, \psi^*(-\beta_1 dt_1 \otimes dt_1 + h_{t_1}))$ is a pseudo-riemannian manifold of signature $(r,1)$; moreover, the same holds for convex combinations of $-\beta_0 dt_0 \otimes dt_0 + g_{t_0}$ and $\psi^*(-\beta_1 dt_1 \otimes dt_1 + h_{t_1})$. Therefore, we consider the manifold $\mathbb{R} \times \Sigma$ endowed with the family of metrics

$$[0,1] \ni \lambda \mapsto \mathscr{g}_\lambda \doteq (1-\lambda)\left(-\beta_0 dt_0 \otimes dt_0 + g_{t_0}\right) + \lambda\psi^*\left(-\beta_1 dt_1 \otimes dt_1 + h_{t_1}\right) \ .$$

Using the diffeomorphism $\psi_0\colon M \to \mathbb{R} \times \Sigma$, we then consider the pseudo-riemannian manifold $(M, g_\lambda \doteq \psi_0^*(\mathscr{g}_\lambda))$, and notice the following things:

(i) for $\lambda = 0$ we have that $\mathfrak{g}_0 = -\beta_0 dt_0 \otimes dt_0 + g_{t_0}$, and thus $\psi_0^* \left(-\beta_0 dt_0 \otimes dt_0 + g_{t_0}\right) = g_0$;

(ii) for $\lambda = 1$ we have that $\mathfrak{g}_1 = \psi^* \left(-\beta_1 dt_1 \otimes dt_1 + h_{t_1}\right)$. But as $\psi^* = (\psi_1 \circ \psi_0^{-1})^*$, we have that $\psi_0^* \psi^* = \psi_1^*$, and therefore $\psi_0^* \mathfrak{g}_1 = \psi_1^* \left(-\beta_1 dt_1 \otimes dt_1 + h_{t_1}\right) = g_1$;

(iii) the curve $\lambda \mapsto \mathfrak{g}_\lambda$ is smooth, and thus so is $\lambda \mapsto g_\lambda$; moreover, as $-\beta_0 dt_0 \otimes dt_0 + g_{t_0}$ and $\psi^* \left(-\beta_1 dt_1 \otimes dt_1 + h_{t_1}\right)$ coincide outside of K_0, we have that the curve is constantly equal to $g_0 \equiv g_1$ outside of K.

To summarize the results of this Section, we state the following Proposition:

Proposition 3.1.1. *Let g_0 and g_1 be two Lorentzian metrics on M satisfying the assumptions (i), (ii) and (iii). Then there exists a smooth curve $[0,1] \ni \lambda \mapsto g_\lambda$ such that $g_\lambda \equiv g_0$ outside of* $\mathrm{Supp}(g_0 - g_1)$.

We remark that the requirements on the two metrics g_0 and g_1 are essentially equal to those reported in [15].

3.2 Generalized cylinders in spin geometry

Following [5], we now present a construction which allows us to relate spinors associated to two different pseudo-riemannian metrics g_0 and g_1.
The starting point is to note that the following result holds:

Proposition 3.2.1 ([29, §I, Theorem 3.7])**.** *There exists an algebra isomorphism*

$$\mathscr{C}\ell^0_{r+1,s} \simeq \mathscr{C}\ell_{r,s}$$

where $\mathscr{C}\ell^0_{r+1,s}$ denotes the even subalgebra of the Clifford algebra $\mathscr{C}\ell_{r+1,s}$.

Proof. Let us consider a pseudo-orthornormal basis $\{e_i\}_{1 \le i \le r+s+1}$ of $\mathbb{R}^{r+1+s}$; we consider the set

$$\mathbb{R}^{r+s} = \langle e_i,\ i \neq r+1 \rangle$$

and construct an action of $\mathbb{R}^{r+s}$ on $\mathscr{C}\ell^0_{r+1,s}$ by defining, on the basis elements,

$$f(e_i) = e_{r+1} \cdot_{\mathscr{C}\ell} e_i$$

and then extending it by linearity to the whole $\mathbb{R}^{r+s}$. Notice that

$$\begin{aligned} f(v) \cdot_{\mathscr{C}\ell} f(v) &= v^i v^j f(e_i) \cdot_{\mathscr{C}\ell} f(e_j) = v^i v^j e_{r+1} \cdot_{\mathscr{C}\ell} e_i \cdot_{\mathscr{C}\ell} e_{r+1} \cdot_{\mathscr{C}\ell} e_j \\ &= -v^i v^j e_{r+1} \cdot_{\mathscr{C}\ell} e_{r+1} \cdot_{\mathscr{C}\ell} e_i \cdot_{\mathscr{C}\ell} e_j = v^i v^j e_i \cdot_{\mathscr{C}\ell} e_j \ . \end{aligned}$$

Thus by the universal property of Clifford algebras [29, §I, Proposition 1.1] there exists a unique algebra homomorphism $\tilde{f} \colon \mathscr{C}\ell_{r,s} \to \mathscr{C}\ell^0_{r+1,s}$ which restricts to f on $\mathbb{R}^{r+s} \subseteq \mathscr{C}\ell_{r,s}$.
Now, consider the action of $\tilde{f}$ on the basis

$$\{e_{i_1} \cdot_{\mathscr{C}\ell} \cdots \cdot_{\mathscr{C}\ell} e_{i_m},\ i_1, \ldots, i_m \in \{1, \ldots, r+s+1\} \setminus \{r+1\},\ m \in \mathbb{N}\}$$

of $\mathscr{C}\ell_{r,s}$; it can be shown that this is mapped to

$$\{e_{i_1} \cdot_{\mathscr{C}\ell} \cdots \cdot_{\mathscr{C}\ell} e_{i_m},\ i_1, \dots, i_m \in \{1, \dots, r+s+1\},\ m = 2k,\ k \in \mathbb{N}\}$$

that is, $\tilde{f}$ maps a basis of $\mathscr{C}\ell_{r,s}$ to a basis of $\mathscr{C}\ell^0_{r+1,s}$, i.e. it is an algebra isomorphism. $\square$

Notice that the algebra isomorphism given by Proposition 3.2.1 satisfies $\tilde{f}(\mathrm{Spin}_{r,s}) \subseteq \mathrm{Spin}_{r+1,s}$. Indeed, given $s \in \mathrm{Spin}_{r,s}$ and $v \in \mathbb{R}^{r+s+1}$ we have that $\tilde{\mathrm{Ad}}_{\tilde{f}(s)}(v) \in \mathbb{R}^{r+s+1}$, as

$$\begin{aligned}\tilde{\mathrm{Ad}}_{\tilde{f}(s)}(v) &= \tilde{f}(s) \cdot_{\mathscr{C}\ell} v \cdot_{\mathscr{C}\ell} \tilde{f}(s)^{-1} = s \cdot_{\mathscr{C}\ell} v \cdot_{\mathscr{C}\ell} s^{-1} = s \cdot_{\mathscr{C}\ell} (\mathbf{v} + v^{r+1} e_{r+1}) \cdot_{\mathscr{C}\ell} s^{-1} \\ &= s \cdot_{\mathscr{C}\ell} \mathbf{v} \cdot_{\mathscr{C}\ell} s^{-1} + v^{r+1} s \cdot_{\mathscr{C}\ell} e_{r+1} \cdot_{\mathscr{C}\ell} s^{-1} = \underbrace{s \cdot_{\mathscr{C}\ell} \mathbf{v} \cdot_{\mathscr{C}\ell} s^{-1}}_{\in \mathbb{R}^{r+s}} + v^{r+1} e_{r+1}\end{aligned}$$

where $\mathbf{v}$ denotes the projection of v onto $\mathbb{R}^{r+s} \subset \mathbb{R}^{r+s+1}$. The spinor norm[4] of $\tilde{f}(s)$ is equal to one:

$$N(\tilde{f}(s)) = e_{r+1} \cdot_{\mathscr{C}\ell} s \cdot_{\mathscr{C}\ell} s^t \cdot_{\mathscr{C}\ell} (-e_{r+1}) = N(s) = 1\ .$$

Therefore, given a spinor representation $\Delta^{\mathbb{C}}_{r+1,s} \colon \mathrm{Spin}^0_{r+1,s} \to \mathrm{GL}(V, \mathbb{C})$, we can define a spinor representation of $\mathrm{Spin}^0_{r,s}$,

$$\mathrm{Spin}^0_{r,s} \ni s \mapsto \tilde{f}(s) \mapsto \Delta^{\mathbb{C}}_{r+1,s}(\tilde{f}(s))$$

which can be understood as a restriction to $\mathrm{Spin}^0_{r,s} \subseteq \mathscr{C}\ell_{r,s} \simeq \mathscr{C}\ell^0_{r+1,1}$ of the Clifford multiplication given on $x \in \mathbb{R}^{r+s}$ by

$$V \ni v \mapsto (e_{r+1} \cdot_{\mathscr{C}\ell} x) \cdot_V v$$

where $\cdot_{\mathscr{C}\ell}$ denotes the Clifford algebra multiplication in $\mathscr{C}\ell_{r+1,s}$ and $\cdot_V$ denotes the Clifford module multiplication of $\mathscr{C}\ell_{r+1,s}$.
Consider now a manifold M whose second Stiefel-Whitney class $w_2(TM)$ is trivial, and consider a smooth one-parameter family $I \ni t \mapsto g_t$ of pseudo-riemannian metrics of signature $(r, 1)$. Using this family, we construct the pseudo-riemannian manifold

$$\mathcal{Z} = I \times M \qquad g_{\mathcal{Z}} = dt \otimes dt + g_t$$

whose metric $g_{\mathcal{Z}}$ has signature $(r+1, 1)$. Notice that the globally defined vector field $\partial_t \in \mathfrak{X}(\mathcal{Z})$ is such that $g_{\mathcal{Z}}(\partial_t, \partial_t) = 1$ and $g_{\mathcal{Z}}(\partial_z, v) = 0$ for every $v \in \mathfrak{X}(M) \subseteq \mathfrak{X}(\mathcal{Z})$; therefore the normal bundle N_M of M in $\mathcal{Z}$ is trivial. We shall denote the hypersurface $(\{t\} \times M, g_t) \simeq (M, g_t)$ with M_t. The following holds:

Proposition 3.2.2. *$\mathcal{Z}$ is a spin manifold iff M is such, i.e. spin structures on $\mathcal{Z}$ and spin strucutres on M are in 1-to-1 correspondence.*

[4]The spinor norm of an object in $\mathscr{C}\ell_{r,s}$ is defined to be $N(\phi) \doteq \phi \cdot_{\mathscr{C}\ell} \tilde{\alpha}(\phi^t)$, where $\tilde{\alpha}$ is the unique extension of $\alpha \colon \mathbb{R}^{r+s} \ni v \mapsto -v$ to $\mathscr{C}\ell_{r,s}$ and $\cdot^t \colon \mathscr{C}\ell_{r,s} \to \mathscr{C}\ell_{r,s}$ is the restriction of $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$ to $\mathscr{C}\ell_{r,s}$.

Proof. Let us show the two implications hold.

(i) Suppose $\mathscr{Z}$ is a spin manifold, i.e. it admits a spin structure $\theta\colon P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) \to P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})$ with

$$\theta(R_s(p)) = R_{\theta_0(s)}(\theta(p))$$

where $\theta_0\colon \mathrm{Spin}^0_{r+1,1} \to \mathrm{SO}^0_{r+1,1}$ denotes the double covering map. Given the inclusion map $i\colon M_t \hookrightarrow \mathscr{Z}$, one can consider the pullback bundles on M_t

$$i^*P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z}) = \left\{((t,x), f) \in M_t \times P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})\colon (t,x) = \pi'_{\mathscr{Z}}(f)\right\}$$

and

$$i^*P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) = \left\{((t,x), s) \in M_t \times P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})\colon (t,x) = \pi_{\mathscr{Z}}(s)\right\}$$

and notice that the map $\mathrm{id}\times\theta\colon i^*P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) \to i^*P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})$ is a spin structure: indeed,

$$\begin{aligned}(\mathrm{id}\times\theta)R_s((t,x),p) = (\mathrm{id}\times\theta)((t,x), R_sp) = ((t,x), \theta(R_s(p))) = ((t,x), R_{\theta_0(s)}(\theta(p))\\ = R_{\theta_0(s)}(\mathrm{id}\times\theta)((t,x),p)\end{aligned}$$

and clearly $\mathrm{id}\times\theta$ is a 2-fold covering.
Consider now the following: given a pseudo-orthonormal, time-oriented frame at $x \in M$ with respect to g_t, be it $\{e_1, \ldots, e_{r+1}\}$, we can define the map

$$f\colon \{e_i\}_{1\le i\le r+1} \mapsto \{\partial_t, e_i\}_{1\le i\le r+1}$$

which maps it to a pseudo-orthonormal, time-oriented frame at $(t,x) \in M_t$. One can use this map to embed the space of pseudo-orthonormal, time-oriented frames at $x \in M$ into the space of pseudo-orthonormal, time-oriented frames of $\mathscr{Z}$ at $(t,x) \in M_t$; thus we have a map which embeds each fiber $P_{\mathrm{SO}^0_{r,1}}(M)_x$ of the frame bundle into the fiber $i^*P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})_{(t,x)}$ and which extends to a principal bundle morphism $f\colon P_{\mathrm{SO}^0_{r,1}}(M) \to i^*P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})$. At this point, if we define

$$P_{\mathrm{Spin}^0_{r,1}}(M) \doteq (\mathrm{id}\times\theta)^{-1}(f(P_{\mathrm{SO}^0_{r,1}}(M)))$$

this is a principal $\mathrm{Spin}^0_{r,1}$-bundle (where $\mathrm{Spin}^0_{r,1}$ is embedded into $\mathrm{Spin}^0_{r+1,1}$ via the isomorphism given by Proposition 3.2.1). Then, the restriction of the spin structure on $i^*P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})$ gives a spin structure on $P_{\mathrm{SO}^0_{r,1}}(M)$: indeed, given $s \in \mathrm{Spin}^0_{r,1}$ and denoting with $\tilde{s} = \tilde{f}(s)$ (see the proof of Proposition 3.2.1 for the definition of $\tilde{f}$) we have, given $((t,x),p) \in P_{\mathrm{Spin}^0_{r,1}}(M)$,

$$(\mathrm{id}\times\theta)R_{\tilde{s}}((t,x),p) = ((t,x), \theta(R_{\tilde{s}}(p))) = \Big((t,x), R_{\theta_0(\tilde{s})}(f(q))\Big)\ .$$

As $\tilde{s} = \tilde{f}(s)$, it follows that $\theta_0(\tilde{s}) \in \mathrm{id} \otimes \mathrm{SO}^0_{r,1}$; thus in particular there exists $\xi_0(s)$, where $\xi_0 \colon \mathrm{Spin}^0_{r,1} \to \mathrm{SO}^0_{r,1}$ is the covering map, such that $\theta_0(\tilde{s}) = \mathrm{id} \otimes \xi_0(s)$; then we have

$$\Big((t,x), R_{\theta_0(\tilde{s})}(f(q))\Big) = \Big((t,x), f(R_{\xi_0(s)}(q))\Big) = f\left(x, R_{\xi_0(s)}(q))\right)$$

which is the proper behavior of a spin structure on M.

(ii) Suppose now that M is a spin manifold, i.e. it admits a spin structure $\xi \colon P_{\mathrm{Spin}^0_{r,1}}(M) \to P_{\mathrm{SO}^0_{r,1}}(M)$ such that

$$\xi(R_s(q)) = R_{\xi_0(s)}(\xi(q)) \ .$$

We have a map $\pi_2 \colon \mathscr{Z} \to M$ by which we can consider the pullback bundles

$$\pi_2^* P_{\mathrm{Spin}^0_{r,1}}(M) = \Big\{((t,x),q) \in \mathscr{Z} \times P_{\mathrm{Spin}^0_{r,1}}(M) \colon x = \pi(q)\Big\}$$

and

$$\pi_2^* P_{\mathrm{SO}^0_{r,1}}(M) = \Big\{((t,x),f) \in \mathscr{Z} \times P_{\mathrm{SO}^0_{r,1}}(M) \colon x = \pi(f)\Big\} \ .$$

They are a principal $\mathrm{Spin}^0_{r,1}$-bundle and a $\mathrm{SO}^0_{r,1}$-bundle on $\mathscr{Z}$ respectively. Now, by Proposition 3.2.1 we have an embedding

$$\tilde{f} \colon \mathrm{Spin}^0_{r,1} \hookrightarrow \mathrm{Spin}^0_{r+1,1}$$

and we can thus consider the bundle

$$\pi_2^* P_{\mathrm{Spin}^0_{r,1}}(M) \times_{\mathrm{Spin}^0_{r,1}} \mathrm{Spin}^0_{r+1,1}$$

which is a principal $\mathrm{Spin}^0_{r+1,1}$-bundle on $\mathscr{Z}$, as well as the bundle

$$\pi_2^* P_{\mathrm{SO}^0_{r,1}}(M) \times_{\mathrm{SO}^0_{r,1}} \mathrm{SO}^0_{r+1,1}$$

where in this case the embedding is given by $\mathrm{SO}^0_{r,1} \ni s \mapsto \mathrm{id} \otimes s$. Let us now consider the restriction of the map

$$(\mathrm{id} \times \xi \times \theta_0) \colon \pi_2^* P_{\mathrm{Spin}^0_{r,1}}(M) \times \mathrm{Spin}^0_{r+1,1} \to \pi_2^* P_{\mathrm{SO}^0_{r,1}}(M) \times \mathrm{SO}^0_{r+1,1}$$

to the quotient spaces above; this is well-defined thanks to the properties of the maps $\xi \colon P_{\mathrm{Spin}^0_{r,1}}(M) \to P_{\mathrm{SO}^0_{r,1}}(M)$ and $\tilde{f}$; moreover, it is clearly a double covering morphism between principal bundles, and it satisfies

$$(\mathrm{id} \times \xi \times \theta_0) R_s[((t,x),q,g)] = R_{\theta_0(s)}(\mathrm{id} \times \xi \times \theta_0)[((t,x),q,g)]$$

for every choice of $s \in \mathrm{Spin}^0_{r+1,1}$, i.e. it provides a spin structure on $\mathscr{Z}$.

□

Remark 3.2.3. This theorem can be proved via techinques from differential topology; namely, given the Künneth formula

$$H^k(I \times M, TI \boxtimes TM) \simeq \bigoplus_{p+q=k} H^p(I, TI) \otimes H^q(M, TM)$$

and the Poincaré lemma we know that

$$w_2(I \times M) = w_0(I) \otimes w_2(M) = w_2(M) \ .$$

From this formula one can conclude that as $w_2(M)$ vanishes, M being spin, we have that $w_2(I \times M)$ vanishes as well; the converse assertion also holds. ♠

Consider now the standard complex representation for $\mathrm{Spin}^0_{r,1}$. This is given by the restriction of a complex representation of $\mathscr{C}\ell_{r,1}$, which can be determined by considering [29, Theorem 4.3]:

$$\mathscr{C}\ell_{r,1} \otimes \mathbb{C} \simeq \mathrm{Mat}(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \text{ if } r+1 = 0 \mod 2$$
$$\mathscr{C}\ell_{r,1} \otimes \mathbb{C} \simeq \mathrm{Mat}(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \text{ if } r+1 = 1 \mod 2$$

An irreducible $\mathbb{C}$-module for $\mathscr{C}\ell_{r,1}$ descends from an irreducible module for $\mathscr{C}\ell_{r+1} \otimes \mathbb{C}$; thus from the above property one can understand that in the case when $r+1$ is even we have an irreducible complex module, be it $S_{r,1}$, while when $r+1$ is odd we have two irreducible modules, $S^+_{r,1}$ and $S^-_{r,1}$, which can be distinguished by the action of the complex volume element $\omega^{\mathbb{C}}$. Moreover,

(i) if $r+1$ is even, we have that if one restricts the module $S_{r,1}$ to the even subalgebra $\mathscr{C}\ell^0_{r,1} \subseteq \mathscr{C}\ell_{r,1}$, then the module is reducible, $S_{r,1} = S^0_{r,1} \oplus S^1_{r,1}$, and the two irreducible components can again be distinguished by the action of the complex volume element $\omega^{\mathbb{C}}$;

(ii) if $r+1$ is odd, then the two inequivalent irreducible modules $S^+_{r,1}$ and $S^-_{r,1}$ give rise to the same module when restricted to the even subalgebra $\mathscr{C}\ell^0_{r,1} \subseteq \mathscr{C}\ell_{r,1}$, which shall be denoted with $S_{r,1}$.

Therefore we have that the complex representation space of $\mathrm{Spin}^0_{r,1}$ has dimension $2^{\lfloor (r+1)/2 \rfloor}$, and moreover if $r+1$ is even said representation space is reducible, while if $r+1$ is odd it is irreducible. This, combined with Proposition 3.2.1, gives rise to an interesting effect: we can embed $\mathrm{Spin}^0_{r,1}$ into $\mathrm{Spin}^0_{r+1,1}$, and then according to the parity of $r+1$ we have different behaviors of the restriction. Namely, if $r+1$ is even then $r+2$ is odd, and thus we have two inequivalent $\mathscr{C}\ell_{r+1,1}$-modules which give rise to the same $\mathscr{C}\ell^0_{r+1,1}$-modules whose complex dimension is

$$2^{\lfloor (r+2)/2 \rfloor} = 2^{\lfloor (r+1)/2 \rfloor}$$

i.e. $S_{r+1,1} = S_{r,1}$. In this case the representation space of $\mathrm{Spin}^0_{r,1}$ induced by the $\mathscr{C}\ell_{r,1}$-module coincides with that induced by the $\mathscr{C}\ell^0_{r+1,1}$-module, although in the second case $\mathrm{Spin}^0_{r,1}$ acts on $S_{r,1}$ via the action

$$\mathbb{R}^{r+1} \otimes S_{r,1} \ni (v \otimes z) \mapsto (e_{r+1} \cdot_{\mathscr{C}\ell} v) \cdot_{S_{r,1}} z \ .$$

In the case when $r+1$ is odd, then $r+2$ is even, and thus we have an irreducible complex $\mathscr{C}\ell_{r+1,1}$-module $S_{r+1,1}$ of complex dimension

$$2^{\lfloor (r+2)/2 \rfloor} = 2^{\lfloor (r+1)/2 \rfloor + 1} \ .$$

Recall moreover that when restricted to $\mathscr{C}\ell^0_{r+1,1}$ this module is reducible, with $S_{r+1,1} = S^0_{r+1,1} \oplus S^1_{r+1,1}$ and

$$\dim(S^0_{r+1,1}) = \dim(S^1_{r+1,1}) = \dim(S^+_{r,1}) = \dim(S^-_{r,1}) = 2^{\lfloor (r+1)/2 \rfloor} \ .$$

Due to the action of $\omega^{\mathbb{C}} \in \mathrm{Spin}^0_{r,1}$ on $S^0_{r+1,1}$ via the embedding $\tilde{f}$, it then follows that

$$S^0_{r+1,1} \simeq S^+_{r,1} \qquad S^1_{r+1,1} \simeq S^-_{r,1}$$

as $\mathscr{C}\ell_{r,1}$-modules.
Given these representations $\Delta^{\mathbb{C}}$ of the Spin groups, one can proceed as in Section 2.1, constructing the associated vector bundles $S_{r,1}(M)$ and $S_{r+1,1}(\mathscr{Z})$. Consider then the pullback bundle along $i\colon M_t \hookrightarrow \mathscr{Z}$,

$$i^* S_{r+1,1}(\mathscr{Z}) = \left\{ ((t,x), v) \in M_t \times S_{r+1,1}(\mathscr{Z}) \colon \pi(v) = (t,x) \right\} \ .$$

This is a vector bundle with typical fiber $S_{r+1,1}$ and base M_t, whose cocycle is given by

$$\left\{ \Delta^{\mathbb{C}}_{r+1,1}(g_{\alpha\beta}) \colon \{t\} \times (U_\alpha \cap U_\beta) \to \mathrm{GL}(S_{r+1,1}, \mathbb{C}) \right\}_{\alpha,\beta}$$

where $g_{\alpha\beta}$ is a cocycle for $P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})$. It turns out that if $r+1$ is even then

$$i^* S_{r+1,1}(\mathscr{Z}) \simeq S_{r,1}(M)$$

while if $r+1$ is odd we have

$$S^0_{r+1,1}(\mathscr{Z}) \simeq S_{r,1}(M) \text{ and } S^1_{r+1,1}(\mathscr{Z}) \simeq S_{r,1}(M)$$

where the choice between the two subbundles depends on whether the chosen $\mathscr{C}\ell_{r,1}$-module is $S^+_{r,1}$ or $S^-_{r,1}$. Indeed, consider the following in the case $r+1$ even: if one fixes a principal $\mathrm{Spin}^0_{r+1,1}$-bundle on $\mathscr{Z}$ with cocycle given by $\{g_{\alpha\beta} \colon I \times (U_\alpha \cap U_\beta) \to \mathrm{Spin}^0_{r+1,1}\}_{\alpha,\beta}$, then the procedure depicted in the proof of Proposition 3.2.2 describes a way to construct a principal $\mathrm{Spin}^0_{r,1}$-bundle on M. Using this procedure, one can construct, as stated above, the bundle $S_{r,1}(M)$, with cocycle given by

$$\left\{ \Delta^{\mathbb{C}}_{r,1}(\tilde{g}_{\alpha\beta}) \colon U_\alpha \cap U_\beta \to \mathrm{GL}(S_{r,1}, \mathbb{C}) \right\}_{\alpha,\beta} \ .$$

In order to show that the bundles are isomorphic, it sufficies to show that the two cocycles are equivalent. But recall that $P_{\mathrm{Spin}^0_{r,1}}(M)$ is defined as

$$(\mathrm{id} \times \theta)^{-1}(\mathfrak{f}(P_{\mathrm{SO}^0_{r,1}}(M)) \subseteq i^* P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})$$

and thus the cocycle $\tilde{g}_{\alpha\beta}$ is just the restriction of $g_{\alpha\beta}$ to $P_{\mathrm{Spin}^0_{r,1}}(M)$, that is, $g_{\alpha\beta}(t,x) \in \theta_0^{-1}(\mathrm{id}\otimes\mathrm{SO}^0_{r,1})$ for every $x \in U_\alpha \cap U_\beta$. Thus we have that $\Delta^{\mathbb{C}}_{r+1,1}(g_{\alpha\beta})$ is a representation of $\theta_0^{-1}(\mathrm{id}\otimes\mathrm{SO}^0_{r,1}) \simeq \mathrm{Spin}^0_{r,1}$ onto a complex vector space of dimension $2^{\lfloor (r+2)/2 \rfloor} = 2^{\lfloor (r+1)/2 \rfloor}$. As for $r+1$ even we have a unique representation of $\mathrm{Spin}_{r,1}$ we have the desired equivalence of cocycles, and thus the vector bundles $i^* S_{r+1,1}(\mathscr{Z})$ and $S_{r,1}(M)$ are isomorphic. This property is of the utmost importance, as we can now interpret sections of $S_{r,1}(M_t)$ (i.e. spinors on M when the spinor bundle is constructed using the metric g_t) as sections of the pullback bundle $i^* S_{r+1,1}(\mathscr{Z}) = S_{r+1,1}(\mathscr{Z})\Big|_{M_t}$. Thus, by a well-known extension procedure one can consider them as sections of $S_{r+1,1}(\mathscr{Z})$ defined in a neighbourhood of M_t.

Remark 3.2.4. The construction in the proof of Proposition 3.2.2 also describes a way to relate the different Clifford multiplications on M_t and $\mathscr{Z}$. We show the results holding for the case of even $r+1$, the one in the case $r+1$ odd following suit. Thanks to Proposition 3.2.2, we can focus on the case in which we have a spin structure on $\mathscr{Z}$, $\theta\colon P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) \to P_{\mathrm{SO}^0_{r+1,1}}(\mathscr{Z})$. Then the Clifford multiplication

$$\mu_M : \mathscr{C}\ell(M_t) \otimes S_{r,1}(M_t) \to S_{r,1}(M_t)$$

is defined in the following way: we know that there exists a Clifford multiplication

$$\mu_{\mathscr{Z}} : \mathscr{C}\ell(\mathscr{Z}) \otimes S_{r+1,1}(\mathscr{Z}) \to S_{r+1,1}(\mathscr{Z})$$

which descends from the restrictions to quotients of the map, denoted with the same name,

$$\begin{aligned} \mu_{\mathscr{Z}} : P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) \times (\mathscr{C}\ell_{r+1,1} \otimes S_{r+1,1}) &\to P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z}) \times S_{r+1,1} \\ (p, \phi \otimes v) \qquad &\mapsto \qquad (p, \phi \cdot_{S_{r+1,1}} v)\ . \end{aligned}$$

Now, recall that $P_{\mathrm{Spin}^0_{r,1}}(M_t) \subseteq P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})$ and that $\mathscr{C}\ell_{r,1} \overset{\tilde{f}}{\simeq} \mathscr{C}\ell^0_{r+1,1}$, thanks to Proposition 3.2.1; therefore, as $S_{r+1,1} = S_{r,1}$ in the even case, we can consider the composition

$$\overbrace{(p, \phi \otimes v)}^{\in P_{\mathrm{Spin}^0_{r,1}}(M_t) \times (\mathscr{C}\ell_{r,1} \otimes S_{r,1})} \quad \mapsto \quad \underbrace{(p, \tilde{f}(\phi) \otimes v)}_{\in P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})|_{M_t} \times (\mathscr{C}\ell_{r+1,1} \otimes S_{r+1,1})} \quad \overset{\mu_{\mathscr{Z}}}{\mapsto} \quad \overbrace{\left(p, \tilde{f}(\phi) \cdot_{S_{r+1,1}} v\right)}^{\in P_{\mathrm{Spin}^0_{r,1}(M_t)} \times S_{r,1}}\ .$$

This map is well-defined on the quotients with respect to the group actions defining the associated vector bundles, and thus descends to a map which is the Clifford multiplication μ_M. ♠

3.2.1 The Dirac operator in generalized cylinders

We now wish to explore the relation between the connection on $\mathscr{Z}$ and the Dirac operator on $S_{r+1,1}(\mathscr{Z})$ and those on M_t and $S_{r,1}(M_t)$. We refer the reader to [6] and [35] for further

details.
Consider, given a fixed $t \in I$, the hypersurface $M_t \hookrightarrow \mathcal{Z}$; its normal bundle is trivial, being spanned by the unit vector field $\partial_t \in \mathfrak{X}(\mathcal{Z})$. We shall denote ∂_t with ν. The second fundamental form on M_t is the bilinear, symmetric map defined as

$$\mathrm{II}(x,y) \doteq g_{\mathcal{Z}}(\nabla^{\mathcal{Z}}_x \nu, y), \quad x, y \in \mathfrak{X}(M_t) \ .$$

Notice that

$$g(\nabla^{\mathcal{Z}}_x \nu, \nu) = \frac{1}{2}\nabla^{\mathcal{Z}}_x (g(\nu,\nu)) = 0$$

i.e. $\nabla^{\mathcal{Z}}_x \nu \in \mathfrak{X}(M_t)$; this property allows us to define a $C^\infty(M)$-linear map called the *shape operator* or *Weingarten map*

$$\begin{aligned} S \colon \mathfrak{X}(M_t) &\to \mathfrak{X}(M_t) \\ x \quad &\mapsto -\nabla^{\mathcal{Z}}_x \nu \ . \end{aligned}$$

Notice that using the shape operator it follows that

$$\mathrm{II}(x,y) = -\mathrm{I}(S(x), y)$$

where $\mathrm{I}(x,y)$ denotes the first fundamental form, i.e. the restriction of $g_{\mathcal{Z}}$ to M_t. Moreover, given any $x, y \in \mathfrak{X}(M_t)$,

$$\begin{aligned} \nabla^{\mathcal{Z}}_x y &= \nabla^{\mathcal{Z}}_x y - g_{\mathcal{Z}}(\nabla^{\mathcal{Z}}_x y, \nu)\nu + g_{\mathcal{Z}}(\nabla^{\mathcal{Z}}_x y, \nu)\nu \\ &= \underbrace{\nabla^{\mathcal{Z}}_x y + \mathrm{II}(x,y)\nu}_{\nabla^{M_t}_x y} + \mathrm{I}(S(x), y)\nu \ . \end{aligned}$$

Let us focus on the highlighted term: this is indeed the covariant derivative with respect to the Levi-Civita connection on M_t. As a matter of fact,

$$\nabla^{M_t}_x y - \nabla^{M_t}_y x = \nabla^{\mathcal{Z}}_x y - \nabla^{\mathcal{Z}}_y x + \left(\mathrm{II}(x,y) - \mathrm{II}(y,x)\right)\nu = [x,y]$$

as $\nabla^{\mathcal{Z}}$ is torsion-free and $\mathrm{II}(\cdot,\cdot)$ is symmetric; moreover, given any $x, y, z \in \mathfrak{X}(M_t)$

$$\begin{aligned} (\nabla^{M_t}_x g_t)(y,z) &= \nabla^{M_t}_x (g_t(y,z)) - g_t(\nabla^{M_t}_x y, z) - g_t(y, \nabla^{M_t}_x z) \\ &= \nabla^{\mathcal{Z}}_x (g_{\mathcal{Z}}(y,z)) - g_{\mathcal{Z}}(\nabla^{\mathcal{Z}}_x y + \mathrm{II}(x,y)\nu, z) - g_{\mathcal{Z}}(y, \nabla^{\mathcal{Z}}_x z + \mathrm{II}(x,z)\nu) \\ &= (\nabla^{\mathcal{Z}}_x g_{\mathcal{Z}})(y,z) = 0 \ . \end{aligned}$$

This entails the following: given a pseudo-orthonormal, local frame $\{e_i\}_{1\le i\le n}$ for (M_t, g_t), let us complete it to a local pseudo-orthornormal frame for $\mathcal{Z}$ by adding ν in position 0; then

$$\begin{aligned} {\Gamma^{\mathcal{Z}}_{ij}}^k &= (\nabla^{\mathcal{Z}}_{e_i} e_j)^k = (\nabla^{M_t}_{e_i} e_j)^k = {\Gamma^{M_t}_{ij}}^k \\ {\Gamma^{\mathcal{Z}}_{ij}}^\nu &= (\nabla^{\mathcal{Z}}_{e_i} e_j)^\nu = \mathrm{I}(S(e_i), e_j) \\ {\Gamma^{\mathcal{Z}}_{i\nu}}^j &= (\nabla^{\mathcal{Z}}_{e_i} \nu)^j = g_{\mathcal{Z}}\left(\nabla^{\mathcal{Z}}_{e_i}\nu, \frac{e_j}{\varepsilon_j}\right) = -\varepsilon_j g_{\mathcal{Z}}(\nu, \nabla^{M_t}_{e_i} e_j) = -\varepsilon_j {\Gamma^{\mathcal{Z}}_{ij}}^\nu = -\varepsilon_j \mathrm{I}(S(e_i), e_j) \end{aligned} \tag{3.2}$$

where $\varepsilon_j = g_{\mathcal{Z}}(e_j, e_j) = \pm 1$. Notice that in our case the expression $(S(x), y)$ can be written in a more elegant and explicit way if the vector fields $x, y \in \mathfrak{X}(M_t)$ are coordinate vector fields: indeed, using the Koszul formula we get

$$\begin{aligned}
\mathrm{I}(S(x), y) = -g_{\mathcal{Z}}(\nabla^{\mathcal{Z}}_x \nu, y) = -\frac{1}{2}\Big(& x(g_{\mathcal{Z}}(\nu, y)) + \nu(g_{\mathcal{Z}}(x, y)) - y(g_{\mathcal{Z}}(x, \nu)) \\
& - g_{\mathcal{Z}}([\nu, x], y) - g_{\mathcal{Z}}([x, y], \nu) - g_{\mathcal{Z}}([\nu, y], x)\Big) \\
= -\frac{1}{2}\nu\left(g_t(x, y)\right) \doteq -\frac{1}{2}\dot{g}_t(x, y) \ .
\end{aligned}$$

Equations (3.2) allow us to use (2.12) to write a local expression for $\nabla^{s,\mathcal{Z}}_{e_i} s$ when $s \in \Gamma(S_{r,1}(M_t))$: given a local section $\sigma \colon U \to P_{\mathrm{Spin}^0_{r+1,1}}(\mathcal{Z})$ and the previously defined local pseudo-orthornormal frame, for $p \in M_t$

$$\begin{aligned}
\nabla^{s,\mathcal{Z}}_{e_i} s(p) &= \left[\sigma(p), e_i(p)\left(s^\sharp \circ \sigma\right) + \frac{1}{2}\sum_{k<l} \Gamma^{\mathcal{Z}\,l}_{ik} \varepsilon_k e_k(p) \cdot_{\mathcal{Z}} e_l(p) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p)\right] \\
&= \Bigg[\sigma(p), e_i(p)\left(s^\sharp \circ \sigma\right) + \frac{1}{2}\sum_{l=1}^{r+1} \Gamma^{\mathcal{Z}\,l}_{i\nu} \nu(p) \cdot_{\mathcal{Z}} e_l(p) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p) \\
&\quad + \frac{1}{2}\sum_{1 \le k < l \le r+1} \Gamma^{\mathcal{Z}\,l}_{ik} \varepsilon_k e_k(p) \cdot_{\mathcal{Z}} e_l(p) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p)\Bigg] \\
&= \Bigg[\sigma(p), e_i(p)\left(s^\sharp \circ \sigma\right) - \frac{1}{2}\nu(p) \cdot_{\mathcal{Z}} S(e_i) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p) \\
&\quad - \nu(p) \cdot_{\mathcal{Z}} \nu(p) \cdot_{\mathcal{Z}} \frac{1}{2}\sum_{1 \le k < l \le r+1} \Gamma^{M_t\,l}_{ik} \varepsilon_k e_k(p) \cdot_{\mathcal{Z}} e_l(p) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p)\Bigg] \\
&= \Bigg[\sigma(p), e_i(p)\left(s^\sharp \circ \sigma\right) - \frac{1}{2}\nu(p) \cdot_{\mathcal{Z}} S(e_i) \cdot_{\mathcal{Z}} (s^\sharp \circ \sigma)(p) \\
&\quad + \frac{1}{2}\sum_{1 \le k < l \le r+1} \Gamma^{M_t\,l}_{ik} \varepsilon_k e_k(p) \cdot_{M_t} e_l(p) \cdot_{M_t} (s^\sharp \circ \sigma)(p)\Bigg] \\
&= \tilde{\nabla}^{s,M_t}_{e_i} s(p) - \frac{1}{2}\nu \cdot_{\mathcal{Z}} \cdot S(e_i) \cdot_{\mathcal{Z}} s(p)
\end{aligned}$$

where

$$\tilde{\nabla}^{s,M_t}_{e_i} \doteq \begin{cases} \nabla^{s,M_t}_{e_i}, & r+1 \text{ even} \\ \begin{pmatrix} \nabla^{s,M_t}_{e_i} & 0 \\ 0 & \nabla^{s,M_t}_{e_i} \end{pmatrix}, & r+1 \text{ odd} \ . \end{cases}$$

The Dirac operators $\mathbf{D}_{\mathcal{Z}}$ applied to a section $s \in \Gamma(S_{r,1}(M_t))$ satisfies then the following:

$$\begin{aligned}
\mathbf{D}_{\mathcal{Z}} s(p) &= i\nu \cdot_{\mathcal{Z}} \nabla^{s,\mathcal{Z}}_{\nu} s(p) + i\sum_{i=1}^{r+1} \varepsilon_i e_i \cdot_{\mathcal{Z}} \nabla^{s,\mathcal{Z}}_{e_i} s(p) \\
&= i\nu \cdot_{\mathcal{Z}} \nabla^{s,\mathcal{Z}}_{\nu} s(p) + i\sum_{i=1}^{r+1} \varepsilon_i e_i \cdot_{\mathcal{Z}} \left(\tilde{\nabla}^{s,M_t}_{e_i} s(p) - \frac{1}{2}\nu \cdot_{\mathcal{Z}} \cdot S(e_i) \cdot_{\mathcal{Z}} s(p)\right) \\
&= i\nu \cdot_{\mathcal{Z}} \nabla^{s,\mathcal{Z}}_{\nu} s(p) - \nu \cdot_{\mathcal{Z}} \left(i\sum_{i=1}^{r+1} e_i \cdot_{M_t} \tilde{\nabla}^{s,M_t}_{e_i} s(p)\right) \\
&\quad + \frac{i}{2}\nu \cdot_{\mathcal{Z}} \sum_{i=1}^{r+1} \varepsilon_i e_i \cdot_{\mathcal{Z}} S(e_i) \cdot_{\mathcal{Z}} s(p)\ .
\end{aligned}$$

Now, if $\{e_i\}_{1\le i\le r+1}$ is a pseudo-orthonormal basis given by the eigenvectors of S with associated eigenvalues (principal curvatures) $\{\kappa_i\}_{1\le i\le r+1}$, then

$$\sum_{i=1}^{r+1} \varepsilon_i e_i \cdot_{\mathcal{Z}} S(e_i) \cdot_{\mathcal{Z}} s(p) = \sum_{i=1}^{r+1} \varepsilon_i \kappa_i e_i \cdot_{\mathcal{Z}} e_i \cdot_{\mathcal{Z}} s(p) = -\sum_{i=1}^{r+1} \kappa_i \varepsilon_i^2 s(p) = (r+1) H s(p)$$

where H denotes the mean curvature of M_t in $\mathcal{Z}$; therefore we have

$$\mathbf{D}_{\mathcal{Z}} s(p) = i\nu \cdot_{\mathcal{Z}} \nabla^{s,\mathcal{Z}}_{\nu} s(p) - \nu \cdot_{\mathcal{Z}} \tilde{\mathbf{D}}_{M_t} - i\frac{r+1}{2} H\nu \cdot_{\mathcal{Z}} s(p)$$

where

$$\tilde{\mathbf{D}}_{M_t} \doteq \begin{cases} \mathbf{D}_{M_t} & r+1 \text{ even} \\ \begin{pmatrix} \mathbf{D}_{M_t} & 0 \\ 0 & -\mathbf{D}_{M_t} \end{pmatrix} & r+1 \text{ odd}\ . \end{cases}$$

3.2.2 Parallel transport in generalized cylinders

Consider a vector field $\mathbf{V} \in \mathfrak{X}(M_t)$ and a section $s \in \Gamma(S_{r,1}(M_t))$; as previously stated, we can view both of them as objects in $\mathfrak{X}(\mathcal{Z})$ and $\Gamma(S_{r+1,1}(\mathcal{Z}))$ respectively. The same can be done with the section $\mathbf{V} \cdot_{\Gamma_{M_t}} s$; due to the way the Clifford module multiplication

$$\cdot_{M_t} \colon \mathfrak{X}(M_t) \otimes S_{r,1}(M_t) \to S_{r,1}(M_t)$$

is defined (see Remark 3.2.4), we have that

$$\mathbf{V} \cdot_{M_t} s\Big|_{M_t} = \nu \cdot_{\mathcal{Z}} \mathbf{V} \cdot_{\mathcal{Z}} s\Big|_{M_t}\ .$$

At this point, consider, for any fixed $x \in M$ and $t_0, t_1 \in I$ the curve $\gamma_x \colon [t_0, t_1] \ni t \mapsto (t, x) \in \mathcal{Z}$, and the parallel transport in $\mathcal{Z}$ along γ_x, that is, we consider the Cauchy problem

$$\begin{cases} \nabla^{s,\mathcal{Z}}_{\dot{\gamma}} s = 0 \\ s(t_0, x) = v \end{cases} \tag{3.3}$$

for a given $v \in S_{r+1,1}(\mathscr{Z})_{(t_0,x)} \simeq S_{r,1}(M_{t_0})_x$. The sections $s_1, s_2 \in \Gamma(S_{r+1,1}(\mathscr{Z}))$ which are the unique solutions of (3.3) associated to the initial data $v_1, v_2 \in S_{r+1,1}(\mathscr{Z})_{(t_0,x)}$ satisfy the following important properties:

(i) $(s_1(t,x), s_2(t,x))_{S_{r+1,1}(\mathscr{Z})} = (s_1(t_0,x), s_2(t_0,x))_{S_{r+1,1}(\mathscr{Z})} = (v_1, v_2)_{S_{r+1,1}(\mathscr{Z})_{(t_0,x)}} = (v_1, v_2)_{S_{r,1}(M)_x}$: indeed, as the connection $\nabla^{s,\mathscr{Z}}$ on $S_{r+1,1}(\mathscr{Z})$ is hermitian we know that
$$\nu(s_1, s_2) = (\nabla^{s,\mathscr{Z}}_{\dot{\gamma}} s_1, s_2) + (s_1, \nabla^{s,\mathscr{Z}}_{\dot{\gamma}} s_2) = 0 \ ,$$
i.e.
$$(s_1(t,x), s_2(t,x)) = (s_1(t_0,x), s_2(t_0,x)) \ ;$$

(ii) parallel transport is well-behaved with respect to Clifford multiplication: indeed, consider the Cauchy problem
$$\begin{cases} \nabla^{s,\mathscr{Z}}_{\dot{\gamma}} s = 0 \\ s(t_0,x) = \mathbf{v} \cdot_{M_{t_0}} v = \nu|_{(t_0,x)} \cdot_{\mathscr{Z}} \cdot \mathbf{v} \cdot_{\mathscr{Z}} \cdot v \end{cases}$$
where $\mathbf{v} \in T_x M \subseteq T_{(t_0,x)}\mathscr{Z}$ and $v \in S_{r,1}(M_{t_0})_x \simeq S_{r+1,1}(\mathscr{Z})_{(t_0,x)}$. We claim that the solution s is given by
$$s = \nu \cdot_{\mathscr{Z}} \mathbf{V} \cdot_{\mathscr{Z}} \cdot s_v$$
where $\mathbf{V}$ is the parallel transport of $\mathbf{v} \in T_{(t_0,x)}\mathscr{Z}$ along γ_x and s_v is the parallel transport of $v \in S_{r+1,1}(\mathscr{Z})_{(t_0,x)}$ along γ_x. Indeed, as the connection $\nabla^{s,\mathscr{Z}}$ is compatible with the Clifford multiplication we have
$$\nabla^{s,\mathscr{Z}}_{\dot{\gamma}}(\nu \cdot_{\mathscr{Z}} \mathbf{V} \cdot_{\mathscr{Z}} \cdot s_v) = \left(\nabla^{\mathscr{Z}}_{\dot{\gamma}} \nu\right) \cdot_{\mathscr{Z}} \mathbf{V} \cdot_{\mathscr{Z}} s_v + \nu \cdot_{\mathscr{Z}} \left(\nabla^{\mathscr{Z}}_{\dot{\gamma}} \mathbf{V}\right) \cdot_{\mathscr{Z}} s_v + \nu \cdot_{\mathscr{Z}} \mathbf{V} \cdot_{\mathscr{Z}} \left(\nabla^{s,\mathscr{Z}}_{\dot{\gamma}} s_v\right) \ .$$
This is equal to 0 as $\nabla^{\mathscr{Z}}_{\nu} \nu = 0$: indeed, given $x \in \mathfrak{X}(M_t)$ a local coordinate vector field,
$$\begin{aligned} 0 = g_{\mathscr{Z}}(\nu, x) = \nu(g_{\mathscr{Z}}(\nu, x)) &= (\nabla^{\mathscr{Z}}_{\nu} g_{\mathscr{Z}})(\nu, x) + g_{\mathscr{Z}}(\nabla^{\mathscr{Z}}_{\nu} \nu, x) + g_{\mathscr{Z}}(\nu, \nabla^{\mathscr{Z}}_{\nu} x) \\ &= g_{\mathscr{Z}}(\nabla^{\mathscr{Z}}_{\nu} \nu, x) + \underbrace{g_{\mathscr{Z}}(\nu, \nabla^{\mathscr{Z}}_{x} \nu)}_{=0} = g_{\mathscr{Z}}(\nabla^{\mathscr{Z}}_{\nu} \nu, x) \ . \end{aligned}$$
At the same time, as $g_{\mathscr{Z}}(\nu, \nu) = 1$,
$$0 = \nu(g_{\mathscr{Z}}(\nu, \nu)) = (\nabla^{\mathscr{Z}}_{\nu} g_t)(\nu, \nu) + 2 g_{\mathscr{Z}}(\nabla^{\mathscr{Z}}_{\nu} \nu, \nu) = g_{\mathscr{Z}}(\nabla^{\mathscr{Z}}_{\nu} \nu, \nu) \ .$$
The two equalities above mean $\nabla^{\mathscr{Z}}_{\nu} \nu = 0$.

Combining these two results, the linearity of (3.3) as an ODE and the fact that parallel transport is invertible we have, if we evaluate the solution s of (3.3) at t_1, an isometric isomorphism
$$\tau^{t_1}_{t_0} \colon S_{r+1,1}(\mathscr{Z})_{(t_0,x)} \to S_{r+1,1}(\mathscr{Z})_{(t_1,x)} \iff \tau^{t_1}_{t_0} \colon S_{r,1}((M, g_{t_0}))_x \to S_{r,1}((M, g_{t_1}))_x$$

which is such that
$$\tau_{t_0}^{t_1}\left(\mathbf{v}\cdot_{M_{t_0}} v\right) = \left(\tau_{t_0}^{t_1}\mathbf{v}\right)\cdot_{M_{t_1}}\left(\tau_{t_0}^{t_1} v\right) \ .$$
Let us now consider a local expression of (3.3): in particular given a local section $\sigma\colon U \to P_{\mathrm{Spin}^0_{r+1,1}}(\mathscr{Z})$ we can write $s(p) = [\sigma(p), s^\sharp \circ \sigma(p)]$; then by defining $\varphi \doteq s^\sharp \circ \sigma(p)$, by considering a local pseudo-orthonormal frame on M (and extending it to a local pseudo-orthonormal frame on $\mathscr{Z}$) we then have, disregarding the local section of $P_{\mathrm{Spin}_{r+1,1}}(\mathscr{Z})$,

$$\begin{aligned}\nabla^{s,\mathscr{Z}}_{\dot\gamma} s(t,x) &= \partial_t\varphi(t,x) + \frac{1}{2}\sum_{k<l}\Gamma^{\mathscr{Z}\ l}_{\nu k}\varepsilon_k e_k(t,x)\cdot_{\mathscr{Z}} e_l(t,x)\cdot_{\mathscr{Z}}\varphi\\ &\overset{(3.2)}{=} \partial_t\varphi(t,x) + \frac{1}{4}\sum_{1\le k<l\le r+1}\left(\varepsilon_l \dot g_{tkl} + \varepsilon_l g_{\mathscr{Z}}([\nu,e_k],e_l)\right)\varepsilon_k e_k(t,x)\cdot_{\mathscr{Z}} e_l(t,x)\cdot_{\mathscr{Z}}\varphi\end{aligned}$$

Thus the local expression of (3.3) can be written as

$$\begin{cases}\dot\varphi(t,x) = -\dfrac{1}{4}\displaystyle\sum_{1\le k<l\le r+1}\left(\varepsilon_l \dot g_{tkl} + \varepsilon_l g_{\mathscr{Z}}([\nu,e_k],e_l)\right)\dot g_{tsk}\varepsilon_k e_k(t,x)\cdot_{\mathscr{Z}} e_l(t,x)\cdot_{\mathscr{Z}}\varphi\\ \varphi(t_0,x) = v\ .\end{cases} \tag{3.4}$$

Notice how the right-hand side of the differential equation depends smoothly on t and x via the smooth family $I \ni t \mapsto g_t$ and its derivative. Thus by the existence of solutions of equations depending on parameters (see [16, Theorem 10.2.3, 10.7.3] and [19, Theorem B.3]) we know that there exist open neighbourhoods I of t_0, U of x and V of v, and a smooth function $s\colon I\times U\times V \to \mathbb{C}^{2^{\lfloor (r+1)/2\rfloor}}$ which is a solution of (3.4). For each choice of $v \in V$, these local solutions behave well with respect to a coordinate change, and thus give a global object in $\Gamma(S_{r+1,1}(\mathscr{Z}))$. Consider now a section $\sigma \in \Gamma(S_{r,1}(M_{t_0}))$; this induces a smooth map
$$f_\sigma\colon M \ni x \mapsto (t_1, \sigma(x)) \in I \times S_{r,1}(M_{t_0})$$
whose local expression is given by $U \ni x \mapsto (t_1, x, \sigma(x)) \in I\times U\times \mathbb{C}^{2^{\lfloor (r+1)/2\rfloor}}$. We can then consider the composition

$$U \xrightarrow{f_\sigma} I\times U\times\mathbb{C}^{2^{\lfloor (r+1)/2\rfloor}} \xrightarrow{s} \mathbb{C}^{2^{\lfloor (r+1)/2\rfloor}} \ .$$

This locally defined map behaves well with respect to coordinate changes; notice that this map is akin to evaluating a section in $\Gamma(S_{r+1,1}(\mathscr{Z}))$ at $t = t_1$, and thus defines a smooth section in $\Gamma(S_{r,1}(M_{t_1}))$. Thus we have a map

$$\begin{aligned}\tau_{t_0}^{t_1}\colon \Gamma(S_{r,1}(M_{t_0})) &\to \Gamma(S_{r,1}(M_{t_1}))\\ \sigma &\mapsto s\circ f_\sigma\ .\end{aligned}$$

Remark 3.2.5. The previous constructions and results can be carried out in the same way for $\overline{S_{r,1}(M)}$, thus yielding a way to map sections of $S^\oplus_{r,1}(M, g_{t_0})$ to sections $S^\oplus_{r,1}(M, g_{t_1})$; said map will be again denoted with $\tau_{t_0}^{t_1}$. ♠

Taking into account Proposition 3.1.1, we summarize the results of this Section with the following:

Proposition 3.2.6. *Given two metrics g_0 and g_1 on M satisfying the assumptions (i), (ii) and (iii) in Section 3.1, there exists an invertible map*

$$\tau_0^1 \colon \Gamma(S_{r,1}(M, g_0)) \to \Gamma(S_{r,1}(M, g_1))$$

which allows us to relate spinors associated to the metrics.

4 The dynamical C*-algebra for the Fermi fields

4.1 Basic notions in enriched categories

We provide here a brief introduction to categories, functors and natural transformations enriched over a monoidal category; these notions are necessary in the case of fermionic dynamical C*-algebras because they are needed to prove Theorem 4.1.8 (see [12, Section 3]). We refer the reader to [28] for further details.

Definition 4.1.1. A symmetric monoidal category $(\mathfrak{V}, \otimes, e)$ is a triple consisting of a category $\mathfrak{V}$, a bifunctor $\otimes\colon \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ called the monoidal product and an object $e \in \mathfrak{V}$ called the unit object. Such a cateogory is also equipped with natural isomorphism expressing symmetry, associativity and the behaviour of the unit object:

$$V \otimes W \simeq W \otimes V \qquad V \otimes (W \otimes Z) \simeq (V \otimes W) \otimes Z \qquad e \otimes V \simeq V \simeq V \otimes e$$

for all $V, W, Z \in \mathfrak{V}$.

We are mainly interested in the case of $(\mathfrak{Vec}_{\mathbb{R}}, \otimes_{\mathbb{R}}, \mathbb{R})$, where $\mathfrak{Vec}_{\mathbb{R}}$ denotes the category of $\mathbb{R}$-vector spaces with linear maps as morphisms and $\otimes_{\mathbb{R}}$ denotes the tensor product of $\mathbb{R}$-vector spaces.

Definition 4.1.2. Given a (symmetric) monoidal category $(\mathfrak{V}, \otimes, e)$, a category enriched over $\mathfrak{V}$ (for short $\mathfrak{V}$-category) $\underline{\mathfrak{C}}$ consists of:

(i) a collection of objects, which we shall indicate with $\underline{\mathfrak{C}}$;

(ii) for each pair $x, y \in \underline{\mathfrak{C}}$ of objects, a hom-object $\underline{\mathfrak{C}}(x, y) \in \mathfrak{V}$;

(iii) for each object $x \in \underline{\mathfrak{C}}$ a morphism in $\mathfrak{V}$

$$\mathrm{id}_x\colon e \to \underline{\mathfrak{C}}(x, x)$$

(iv) for each triple $x, y, z \in \underline{\mathfrak{C}}$ a morphism in $\mathfrak{V}$

$$\circ\colon \underline{\mathfrak{C}}(y, z) \otimes \underline{\mathfrak{C}}(x, y) \to \underline{\mathfrak{C}}(x, z)$$

such that

$$\begin{tikzcd}
\underline{\mathfrak{C}}(z,w) \otimes \underline{\mathfrak{C}}(y,z) \otimes \underline{\mathfrak{C}}(x,y) \arrow[r, "\mathbb{1}\otimes\circ"] \arrow[d, "\circ\otimes\mathbb{1}"] & \underline{\mathfrak{C}}(z,w) \otimes \underline{\mathfrak{C}}(x,z) \arrow[d, "\circ"] \\
\underline{\mathfrak{C}}(y,w) \otimes \underline{\mathfrak{C}}(x,y) \arrow[r, "\circ"] & \underline{\mathfrak{C}}(x,w)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathfrak{C}}(x,y) \otimes e \arrow[r, "\mathbb{1}\otimes\mathrm{id}_x"] \arrow[dr, "\simeq"'] & \underline{\mathfrak{C}}(x,y) \otimes \underline{\mathfrak{C}}(x,x) \arrow[d, "\circ"] \\
& \underline{\mathfrak{C}}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathfrak{C}}(y,y) \otimes \underline{\mathfrak{C}}(x,y) \arrow[d, "\circ"'] & e \otimes \underline{\mathfrak{C}}(x,y) \arrow[l, "\mathrm{id}_y\otimes\mathbb{1}"'] \arrow[dl, "\simeq"] \\
\underline{\mathfrak{C}}(x,y) &
\end{tikzcd}$$

In our case, we are mainly interested in the following: first of all, we consider two categories, $\mathfrak{Grass}$ and $\mathfrak{Alg}^{\mathbb{Z}_2}$, whose objects are finite dimensional, real Grassmann algebras and $\mathbb{Z}_2$-graded unital, associative complex algebras, and whose morphisms are unital algebra homomorphisms of even degree and $\mathbb{Z}_2$-graded, unital algebra homomorphisms respectively.
Starting from these two categories, we consider the $\mathfrak{Vec}$-categories $\mathbb{R}\mathfrak{Grass}$ and $\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ defined in the following way:

(i) the objects of $\mathbb{R}\mathfrak{Grass}$ and $\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ are the objects of $\mathfrak{Grass}$ and $\mathfrak{Alg}^{\mathbb{Z}_2}$, respectively;

(ii) for each pair $G, H \in \mathbb{R}\mathfrak{Grass}$, the $\mathbb{R}$-vector space $\mathbb{R}\mathfrak{Grass}(G,H)$ is the $\mathbb{R}$-vector space generated by the morphisms in $\mathfrak{Grass}(G,H)$ using the linear structure of the Grassmann algebras, and analogously for $\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,B)$ with $A, B \in \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$;

(iii) given $G \in \mathbb{R}\mathfrak{Grass}$, the morphism id_G in $\mathfrak{Vec}_{\mathbb{R}}$ is given by

$$\mathrm{id}_G \colon \mathbb{R} \ni \lambda \mapsto \lambda \mathbb{1} \in \mathbb{R}\mathfrak{Grass}(G,G)$$

and analogously for $A \in \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ and id_A;

(iv) for each triple $G, H, F \in \mathbb{R}\mathfrak{Grass}$, the composition morphism

$$\circ \colon \mathbb{R}\mathfrak{Grass}(H,F) \otimes \mathbb{R}\mathfrak{Grass}(G,H) \to \mathbb{R}\mathfrak{Grass}(G,F)$$

is the linear extension of the usual composition of morphisms (notice that this exploits the linear structure of the Grassmann algebras), and the same goes for $A, B, C \in \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ and

$$\circ \colon \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(B,C) \otimes \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,B) \to \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,C) \ .$$

Definition 4.1.3. Given a $\mathfrak{V}$-category $\underline{\mathfrak{C}}$, the underlying category $\mathfrak{C}_0$ is a category whose objects are given by the same objects of $\underline{\mathfrak{C}}$, and whose home-sets are given by

$$\mathfrak{C}_0(x,y) = \mathfrak{V}(e, \underline{\mathfrak{C}}(x,y)) = \mathrm{Hom}(e, \underline{\mathfrak{C}}(x,y)) \ .$$

with composition of $f \in \mathfrak{C}_0(x,y)$ and $g \in \mathfrak{C}_0(y,z)$ given by

$$e \xrightarrow{\quad \simeq \quad} e \otimes e \xrightarrow{\quad g\otimes f \quad} \underline{\mathfrak{C}}(y,z) \otimes \underline{\mathfrak{C}}(x,y) \xrightarrow{\quad \circ \quad} \underline{\mathfrak{C}}(x,z)$$

Following the above example of the $\mathfrak{Vec}$-categories $\mathbb{R}\mathfrak{Grass}$ and $\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$, the underlying categories $\mathfrak{Grass}_0$ and $\mathfrak{Alg}_0^{\mathbb{Z}_2}$ have as objects the same objects as $\mathfrak{Grass}$ and $\mathfrak{Alg}^{\mathbb{Z}_2}$, while the hom-sets are

$$\mathfrak{Grass}_0(G,H) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}\mathfrak{Grass}(G,H)) \qquad \mathfrak{Alg}_0^{\mathbb{Z}_2}(A,B) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,B))$$

respectively. Notice that we can identify

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}\mathfrak{Grass}(G,H)) \simeq \mathbb{R}\mathfrak{Grass}(G,H) \qquad \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,B)) \simeq \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(A,B) \ .$$

Thus the hom-sets in $\mathfrak{Grass}_0$ and $\mathfrak{Alg}_0^{\mathbb{Z}_2}$ are respectively the $\mathbb{R}$-vector spaces generated by morphism in $\mathfrak{Grass}$ and $\mathfrak{Alg}^{\mathbb{Z}_2}$ exploiting the linear structure on $\mathfrak{Grass}$ and $\mathfrak{Alg}^{\mathbb{Z}_2}$.

Definition 4.1.4. A functor enriched over $\mathfrak{V}$ (for short $\mathfrak{V}$-functor) $\mathfrak{F}\colon \underline{\mathfrak{C}} \to \underline{\mathfrak{D}}$ between $\mathfrak{V}$-categories $\underline{\mathfrak{C}}$ and $\underline{\mathfrak{D}}$ is given by a map

$$\underline{\mathfrak{C}} \ni x \mapsto \mathfrak{F}(x) \in \underline{\mathfrak{D}}$$

together with morphisms $\mathfrak{F}_{x,y}\colon \underline{\mathfrak{C}}(x,y) \to \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y))$ in $\mathfrak{V}$ for each $x, y \in \underline{\mathfrak{C}}$ such that

$$\begin{tikzcd}
\underline{\mathfrak{C}}(y,z) \otimes \underline{\mathfrak{C}}(x,y) \arrow[r, "\circ"] \arrow[d, "\mathfrak{F}_{y,z}\otimes\mathfrak{F}_{x,y}"] & \underline{\mathfrak{C}}(x,z) \arrow[d, "\mathfrak{F}_{x,z}"] \\
\underline{\mathfrak{D}}(\mathfrak{F}(y),\mathfrak{F}(z)) \otimes \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \arrow[r, "\circ"'] & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(z))
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{\mathfrak{F}(x)}"'] & \underline{\mathfrak{C}}(x,x) \arrow[d, "\mathfrak{F}_{x,x}"] \\
 & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(x))
\end{tikzcd}$$

Given a $\mathfrak{V}$-functor $\mathfrak{F}\colon \underline{\mathfrak{C}} \to \underline{\mathfrak{D}}$ between the $\mathfrak{V}$-categories $\underline{\mathfrak{C}}$ and $\underline{\mathfrak{D}}$, one can define the underlying functor $\mathfrak{F}_0\colon \mathfrak{C}_0 \to \mathfrak{D}_0$ between the underlying categories in the following way: as a map, it behaves as

$$\mathrm{Obj}(\mathfrak{C}_0) = \mathrm{Obj}(\underline{\mathfrak{C}}) \ni x \mapsto \mathfrak{F}_0(x) = \mathfrak{F}(x) \in \mathrm{Obj}(\underline{\mathfrak{D}}) = \mathrm{Obj}(\mathfrak{D}_0)$$

while on a morphism $\chi \in \mathfrak{C}_0(x,y)$ we have that $\mathfrak{F}_0(\chi) \in \mathfrak{D}_0(\mathfrak{F}_0(x),\mathfrak{F}_0(y))$ is given by

$$\mathfrak{F}_{x,y} \circ \chi\colon e \to \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \ .$$

Definition 4.1.5. Given two $\mathfrak{V}$-functors $\mathfrak{F}$ and $\mathfrak{G}$ bewteen the $\mathfrak{V}$-categories $\underline{\mathfrak{C}}$ and $\underline{\mathfrak{D}}$, a natural transformation enriched over $\mathfrak{V}$ (for short $\mathfrak{V}$-natural transformation) $\alpha\colon \mathfrak{F} \Rightarrow \mathfrak{G}$ consists of a family $\{\alpha_x\colon e \to \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(x))\}_{x\in\underline{\mathfrak{C}}}$ of morphisms in $\mathfrak{V}$ such that

$$\begin{tikzcd}
\underline{\mathfrak{C}}(x,y) \arrow[r, "\mathfrak{F}_{x,y}"] \arrow[d, "\mathfrak{G}_{x,y}"'] & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \arrow[d, "(\alpha_y)_*"] \\
\underline{\mathfrak{D}}(\mathfrak{G}(x),\mathfrak{G}(y)) \arrow[r, "(\alpha_x)^*"'] & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(y))
\end{tikzcd}$$

where the maps $\{(\alpha_x)^*\colon \underline{\mathfrak{D}}(\mathfrak{G}(x),\mathfrak{G}(y)) \to \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(y))\}$ and $\{(\alpha_y)_*\colon \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \to \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(y))\}$ are defined via the following diagrams:

$$\begin{tikzcd}
\underline{\mathfrak{D}}(\mathfrak{G}(x),\mathfrak{G}(y)) \arrow[r, "\simeq"] \arrow[drr, "(\alpha_x)^*"'] & \underline{\mathfrak{D}}(\mathfrak{G}(x),\mathfrak{G}(y)) \otimes e \arrow[r, "\mathbb{1}\otimes\alpha_x"] & \underline{\mathfrak{D}}(\mathfrak{G}(x),\mathfrak{G}(y)) \otimes \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(x)) \arrow[d, "\circ"] \\
 & & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(y))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \arrow[r, "\simeq"] \arrow[drr, "(\alpha_y)_*"'] & e\otimes\underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \arrow[r, "\alpha_y\otimes\mathbb{1}"] & \underline{\mathfrak{D}}(\mathfrak{F}(y),\mathfrak{G}(y))\otimes\underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{F}(y)) \arrow[d, "\circ"] \\
& & \underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(y))
\end{tikzcd}$$

Also in the case of a $\mathfrak{V}-$natural transformation $\alpha\colon \mathfrak{F}\Rightarrow\mathfrak{G}$ it is possible to consider the underlying natural transformation $\alpha_0\colon \mathfrak{F}_0\Rightarrow\mathfrak{G}_0$. We recall that this is given by a collection of morphisms $\{(\alpha_0)_x\colon \mathfrak{F}_0(x)\to\mathfrak{G}_0(x)\}_{x\in\mathfrak{C}_0}$ in $\mathfrak{D}_0$ such that given a morphism $\chi\colon x\to y$ in $\mathfrak{C}_0$ we have

$$\begin{tikzcd}
\mathfrak{F}_0(x) \arrow[r, "\mathfrak{F}(\chi)"] \arrow[d, "(\alpha_0)_x"] & \mathfrak{F}_0(y) \arrow[d, "(\alpha_0)_y"] \\
\mathfrak{G}_0(x) \arrow[r, "\mathfrak{G}(\chi)"] & \mathfrak{G}_0(y)
\end{tikzcd}$$

In this case, we define

$$\mathrm{Hom}(e,\underline{\mathfrak{D}}(\mathfrak{F}(x),\mathfrak{G}(x)))\ni(\alpha_0)_x\doteq\alpha_x\ .$$

Using Definitions 4.1.3 and 4.1.5 it is then easy to see that the above diagram commutes. We are now in a position to give the following definition:

Definition 4.1.6. Let

$$C\colon \mathbb{R}\mathfrak{Grass}\to\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$$

be the $\mathfrak{Vec}$-functor which acts as

$$\mathbb{R}\mathfrak{Grass}\ni G\mapsto G_{\mathbb{C}}\doteq G\otimes\mathbb{C}$$

where $G_{\mathbb{C}}$ is understood as a $\mathbb{Z}_2$-graded algebra, the grading coming from the graded structure of G, and which is such that for each $G,H\in\mathbb{R}\mathfrak{Grass}$, the map $C_{G,H}\colon \mathbb{R}\mathfrak{Grass}(G,H)\to\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}},H_{\mathbb{C}})$ is given by the complexification of $\mathbb{R}$-linear maps.
A covariant Grassmann multiplication consists of a couple $(\mathfrak{G},\iota)$ given by a $\mathfrak{Vec}$-functor

$$\mathfrak{G}\colon \mathbb{R}\mathfrak{Grass}\to\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$$

and a $\mathfrak{Vec}$-natural transformation

$$\iota\colon C\Rightarrow\mathfrak{G}$$

such that for every $G\in\mathbb{R}\mathfrak{Grass}$, the morphism $\iota_G\in\mathrm{Hom}_{\mathbb{R}}(\mathbb{R},\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}},\mathfrak{G}(G)))\simeq\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}},\mathfrak{G}(G))$ (which we recall is the $\mathbb{R}$-vector space generated by the morphisms in $\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}},\mathfrak{G}(G))$) is such that for all homogeneous elements $\eta\in G$ and $a\in\mathfrak{G}(G)$ we have

$$\iota_G(\eta\otimes 1)a=(-1)^{\deg(\eta)\deg(a)}a\iota_G(\eta\otimes 1)\ .$$

Remark 4.1.7. We can consider the underlying functor $\mathfrak{G}_0 \colon \mathfrak{Grass}_0 \to \mathfrak{Alg}_0^{\mathbb{Z}_2}$ and the underlying natural transformation $\iota_0 \colon C_0 \Rightarrow \mathfrak{G}_0$. The first is given by a functor

$$\mathfrak{G}_0 \colon \mathrm{Obj}(\mathfrak{Grass}_0) = \mathrm{Obj}(\mathbb{R}\mathfrak{Grass}) \ni G \mapsto \mathfrak{G}_0(G) = \mathfrak{G}(G) \in \mathrm{Obj}(\mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}) = \mathrm{Obj}(\mathfrak{Alg}_0^{\mathbb{Z}_2})$$

such that given $G, H \in \mathfrak{Grass}_0$ and a morphism $\chi \in \mathfrak{Grass}_0(G,H) \simeq \mathbb{R}\mathfrak{Grass}(G,H)$, which can be thus written as

$$\chi = \sum_{i\in I} \lambda^i \chi_i, \qquad \{\lambda^i\}_{i\in I} \subseteq \mathbb{R},\ \{\chi_i\}_{i\in I} \subseteq \mathfrak{Grass}(G,H)\ ,$$

we have

$$\mathfrak{G}(\chi) = \mathfrak{G}_{G,H}\left(\sum_{i\in I} \lambda^i \chi_i\right) = \sum_{i\in I} \lambda^i \mathfrak{G}_{G,H}(\chi_i)$$

as we recall that $\mathfrak{G}_{G,H}$ is a morphism in $\mathfrak{Vec}_{\mathbb{R}}$ and thus $\mathbb{R}$-linear.
The underlying natural transformation ι_0 is given by the family of maps $\{(\iota_0)_G \colon C_0(G) \to \mathfrak{G}_0(G)\}_{G\in\mathfrak{Grass}_0}$ which are defined as

$$(\iota_0)_G = \iota_G\ .$$

It is clear that that for each $G \in \mathfrak{Grass}_0$ we have, on homogeneous elements $\eta \in G$ and $a \in \mathfrak{G}_0(G)$

$$\left((\iota_0)_G(\eta \otimes 1)\right) a = (-1)^{\deg(\eta)\deg(a)} a \left((\iota_0)_G(\eta\otimes 1)\right)\ .$$

Notice that this behaviour is exactly the one of [12, Definition 3.1]. ♠

Given Remark 4.1.7, we can then state the following Theorem, borrowing it from [12, Theorem 3.2]:

Theorem 4.1.8. *Consider a covariant Grassmann multiplication $(\mathfrak{G}, \iota)$ as of Definition 4.1.6; there exists a universal $\mathbb{Z}_2$-graded, unital complex algebra $\mathfrak{A}$ such that, given the functor*

$$\mathfrak{G}^{\mathfrak{A}} \colon \mathbb{R}\mathfrak{Grass}_0 \to \mathbb{R}\mathfrak{Alg}_0^{\mathbb{Z}_2}$$

that acts on objects as $\mathbb{R}\mathfrak{Grass}_0 \ni G \mapsto \mathfrak{G}^{\mathfrak{A}}(G) \doteq G \otimes \mathfrak{A} \in \mathbb{R}\mathfrak{Alg}_0^{\mathbb{Z}_2}$ and on morphisms as $\mathbb{R}\mathfrak{Grass}_0(G,H) \ni \chi \mapsto \mathfrak{G}^{\mathfrak{A}}(\chi) \in \mathbb{R}\mathfrak{Alg}_0^{\mathbb{Z}_2}(\mathfrak{G}^{\mathfrak{A}}(G), \mathfrak{G}^{\mathfrak{A}}(H))$,

$$\mathfrak{G}^{\mathfrak{A}}(\chi)(\eta \otimes a) \doteq (\chi(\eta)) \otimes a\ \forall \eta \in G,\ a \in \mathfrak{A}$$

there exists a natural transformation $\sigma \colon \mathfrak{G}_0 \Rightarrow \mathfrak{G}^{\mathfrak{A}}$, i.e. a collection of morphisms $\sigma_G \colon \mathfrak{G}_0(G) \to \mathfrak{G}^{\mathfrak{A}}(G)$, $\sigma_G \in \mathbb{R}\mathfrak{Alg}_0^{\mathbb{Z}_2}(\mathfrak{G}_0(G), \mathfrak{G}^{\mathfrak{A}}(G))$, such that

$$\begin{tikzcd}
\mathfrak{G}_0(G) \arrow[r, "\mathfrak{G}_0(\chi)"] \arrow[d, "\sigma_G"'] & \mathfrak{G}_0(H) \arrow[d, "\sigma_H"] \\
\mathfrak{G}^{\mathfrak{A}}(G) \arrow[r, "\mathfrak{G}^{\mathfrak{A}}(\chi)"] & \mathfrak{G}^{\mathfrak{A}}(H)
\end{tikzcd}$$

for each $\chi \in \mathbb{R}\mathfrak{Grass}_0(G,H)$.

Proof. The proof is constructive; in the first part, we explicitly construct the universal algebra $\mathfrak{A}$. As every object in $\mathfrak{Grass}$ is isomorphic to $\wedge\mathbb{R}^n$ for some $n \in \mathbb{N}$, we shall deal only with this case.

(i) Let us consider $\wedge\mathbb{R}^n$ whose generators are $\{\eta_I\}_{I\subseteq\{1,\dots,n\}}$ with $I = \{i_1 < \dots < i_{|I|}\}$ and $\eta_I = \eta_{i_1}\cdots\eta_{i_{|I|}}$, and $\wedge\mathbb{R}^m$ whose generators are $\{\theta_J\}_{J\subseteq\{1,\dots,m\}}$. First of all, notice that $\chi\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^m$ is a morphism in $\mathbb{R}\mathfrak{Grass}_0$ if and only if

$$\chi(\eta_I) = \sum_{J\subseteq\{1,\dots,m\}} c_{IJ}\theta_J, \qquad c_{IJ} \neq 0 \text{ if } |I| = |J| \mod 2,\ |J| \geq |I| \ .$$

One implication is trivial: indeed, if $\chi \in \mathbb{R}\mathfrak{Grass}_0(\wedge\mathbb{R}^n, \wedge\mathbb{R}^m)$ then it can be written as $\sum_{i=1}^n c_i\chi_i$, with $\chi_i\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^m$ being a morphism in $\mathfrak{Grass}(\wedge\mathbb{R}^n, \wedge\mathbb{R}^m)$. Thus

$$\chi(\eta_I) = \sum_{i=1}^n c_i\chi_i(\eta_I) = \sum_{i=1}^n c_i\chi_i(\eta_{i_1})\cdots\chi_i(\eta_{i_{|I|}}) \ .$$

As by assumption χ_i is of even degree $2d$ for every $1 \leq i \leq n$, then

$$\chi_i(\eta_{i_j}) = \sum_{\substack{J\subseteq\{1,\dots,m\}\\ |J|=1+2d}} c_{i_j}^J\theta_J \ .$$

Thus

$$\chi(\eta_I) = \sum_{i=1}^n c_i \sum_{\substack{J_1,\dots,J_{|I|}\subseteq\{1,\dots,m\}\\ |J_1|=\dots=|J_{|I|}|=1+d}} c_{i_1}^{J_1}\cdots c_{i_{|I|}}^{J_{|I|}} \underbrace{\theta_{J_1}\cdots\theta_{J_{|I|}}}_{\text{degree } |I|(1+2d)}$$

and the implication immediately follows. On the other hand, we do the following: we define the map $E_{JI}(\eta_{I'}) \doteq \delta_{I,I'}\theta_J$ with $|I| = |J| \mod 2$ and $|I| \leq |J|$; then we can write

$$\chi(\eta_{I'}) = \sum_{J\subseteq\{1,\dots,m\}} c_{I'J}E_{JI'}(\eta_{I'}) = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ J\subseteq\{1,\dots,m\}}} c_{IJ}E_{JI}(\eta_{I'}) \ .$$

We will now write the E_{JI}'s as linear combination of even degree homomorphisms in $\mathfrak{Grass}(\wedge\mathbb{R}^n, \wedge\mathbb{R}^m)$. To do so, we define, given any $I \subseteq \{1,\dots,n\}$, the projector in $\mathbb{R}^n$

$$P_I\eta_i \doteq \begin{cases} \eta_i, & i \in I \\ 0, & \text{otherwise} \end{cases}$$

and we extend it to $\wedge\mathbb{R}^n$. Notice that this gives rise to a zero-degree map. Using these homomorphisms, we define another homomorphism

$$E_I \doteq P_I \circ (\mathrm{id}_{\wedge\mathbb{R}^n} - P_{I\setminus\{i_1\}}) \circ \dots \circ (\mathrm{id}_{\wedge\mathbb{R}^n} - P_{I\setminus\{i_{|I|}\}}) \ .$$

It can be easily seen that this projects onto the one-dimensional subspace generated by η_I. As $|I| < |J|$ and as $|I| = |J| \mod 2$, we can partition J into $|I|$ non-empty subsets $J_1,\dots,J_{|I|}$ such that

$$J_k \ni j_k > j_l \in J_l \text{ if } k > l \text{ and } |J_k| = 1 \mod 2 \ .$$

Using this partition, we define the homomorphism $\chi^{JI}\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^m$ by its action on the generators

$$\chi^{JI}(\eta_i) \doteq \begin{cases} \theta_{J_i}, & i \in I \\ 0, & \text{otherwise} \end{cases}$$

and then extending it to $\wedge\mathbb{R}^n$. Notice that given $\eta = \sum_{I' \subseteq \{1,\dots,n\}} c^{I'} \eta_{I'}$,

$$\begin{aligned}(\chi^{JI} \circ E_I)(\eta) = \chi^{JI}(a^I \eta_I) = a^I \chi^{JI}(\eta_I) = a^I \theta_J = \sum_{I' \subseteq \{1,\dots,n\}} \delta_{I,I'} a^{I'} \theta_J \\ = \sum_{I' \subseteq \{1,\dots,n\}} a^{I'} E_{JI}(\eta_{I'}) = E_{JI}(\eta) \ .\end{aligned}$$

As χ^{JI} and E_I are degree zero homomorphism in $\mathfrak{Grass}(\wedge\mathbb{R}^n, \wedge\mathbb{R}^m)$, we have our assertion.

Given the covariant Grassmann multiplication whose $\mathfrak{Vec}$-functor is $\mathfrak{G}\colon \mathbb{R}\mathfrak{Grass} \to \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$, we define

$$\pi_K \doteq \mathfrak{G}_0(P_K) \qquad \rho_K \doteq \mathfrak{G}_0(E_K) \ .$$

Clearly it holds that

$$\pi_K \pi_J = \mathfrak{G}_0(P_K)\mathfrak{G}_0(P_J) = \mathfrak{G}_0(P_K P_J) = \mathfrak{G}_0(P_{K\cap J}) = \pi_{K\cap J}$$

and

$$\begin{aligned}\rho_K = \mathfrak{G}_0(E_K) = \mathfrak{G}_0\left(P_K \circ (\mathrm{id}_{\wedge\mathbb{R}^n} - P_{K\setminus\{k_1\}}) \circ \cdots \circ (\mathrm{id}_{\wedge\mathbb{R}^n} - P_{K\setminus\{k_{|K|}\}})\right) \\ = \pi_K \circ (\mathrm{id}_{\mathfrak{G}_0(\wedge\mathbb{R}^n)} - \pi_{K\setminus\{k_1\}}) \circ \cdots \circ (\mathrm{id}_{\mathfrak{G}_0(\wedge\mathbb{R}^n)} - \pi_{K\setminus\{k_{|K|}\}}) \ .\end{aligned}$$

(ii) The collection $\{\rho_K\}_{K\subseteq\{1,\dots,n\}}$ has the following properties:

$$\rho_K \rho_J = \mathfrak{G}_0(E_K)\mathfrak{G}_0(E_J) = \mathfrak{G}_0(E_K E_J) = \mathfrak{G}_0(\delta_{K,J} E_K) = \delta_{JK}\rho_K$$

$$\sum_{K\subseteq\{1,\dots,n\}} \rho_K = \sum_{K\subseteq\{1,\dots,n\}} \mathfrak{G}_0(E_K) = \mathfrak{G}_0\left(\sum_{K\subseteq\{1,\dots,n\}} E_K\right) = \mathfrak{G}_0(\mathrm{id}_{\wedge\mathbb{R}^n}) = \mathrm{id}_{\mathfrak{G}_0(\wedge\mathbb{R}^n)} \ .$$

Moreover,

$$\rho_K(ab) = \sum_{J\subseteq K} \rho_J(a)\rho_{K\setminus J}(b) \quad \text{for any } a, b \in \mathfrak{G}_0(\wedge\mathbb{R}^n) \ .$$

Indeed, consider, given any $\lambda \in \mathbb{R}^n$, the homomorphism $\chi_\lambda\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^n$ defined by $\chi_\lambda(\eta_i) \doteq \lambda_i \eta_i$. By its definition and the definition of E_K it follows that

$$E_K \chi_\lambda = \chi_\lambda E_K = \lambda_K E_K \implies \mathfrak{G}_0(\chi_\lambda)\rho_K = \rho_K \mathfrak{G}_0(\chi_\lambda) = \lambda_K \rho_K$$

which entails that

$$\begin{aligned}\lambda_K \rho_K(\rho_J(a)\rho_I(b)) &= \mathfrak{G}_0(\chi_\lambda)\rho_K(\rho_J(a)\rho_I(b))\\ &= \rho_K(\mathfrak{G}_0(\chi_\lambda)(\rho_J(a))\mathfrak{G}_0(\chi_\lambda)(\rho_I(b)))\\ &= \lambda_J\lambda_I\rho_K(\rho_J(a)\rho_I(b))\end{aligned}$$

As this holds for any $\lambda \in \mathbb{R}^n$, we have that $\rho_K(\rho_J(a)\rho_I(b)) = 0$ unless $J \cap I = \varnothing$, $K = J \cup I$. Then

$$\begin{aligned}\rho_K(ab) = \rho_K\left(\left(\sum_{I\subseteq\{1,\dots,n\}} \rho_I(a)\right)\left(\sum_{J\subseteq\{1,\dots,n\}} \rho_J(b)\right)\right) &= \sum_{I,J\subseteq\{1,\dots,n\}} \rho_K(\rho_I(A)\rho_J(b))\\ = \sum_{J\subseteq\{1,\dots,n\}} \rho_K(\rho_J(a)\rho_{K\setminus J}(b)) = \sum_{J\subseteq\{1,\dots,n\}} \rho_J(a)\rho_{K\setminus J}(b)\end{aligned}$$

(iii) Consider now the algebras $\mathfrak{G}_0(\wedge\mathbb{R}^n)$ and $\mathfrak{G}_0(\wedge\mathbb{R}^m)$; we define

$$\mathfrak{A}^n \doteq \rho_{\{1,\dots,n\}}\mathfrak{G}_0(\wedge\mathbb{R}^n) \qquad \mathfrak{A}^m \doteq \rho_{\{1,\dots,m\}}\mathfrak{G}_0(\wedge\mathbb{R}^m)$$

and we also define the product

$$\begin{aligned}\cdot\colon \mathfrak{A}^n \times \mathfrak{A}^m &\to \mathfrak{A}^{m+n}\\ (a,b) &\mapsto a\cdot b\end{aligned}$$

by defining it on homogeneous elements in $\mathfrak{A}^n$ in the following way:

$$a\cdot b \doteq (-1)^{m\deg(a)}\mathfrak{G}_0\left(\chi^{m+n}_{\{m+1,\dots,m+n\}}\right)(a)\mathfrak{G}_0\left(\chi^{m+n}_{\{1,\dots,m\}}\right)(b)$$

where given $J \subseteq \{1,\dots,n\}$, the map $\chi^n_J\colon \wedge\mathbb{R}^{|J|} \to \wedge\mathbb{R}^n$ is a homomorphism defined on the generators as

$$\chi(\theta_i) \doteq \eta_{j_i}\ .$$

The product is then extended by linearity in its first entry to the whole $\mathfrak{A}^n$. Notice that this product is well-defined: indeed,

$$\begin{aligned}\pi_{\{1,\dots,m+n\}\setminus\{k\}}(a\cdot b) &= \pm\pi_{\{1,\dots,m+n\}\setminus\{k\}}\left(\mathfrak{G}_0\left(\chi^{m+n}_{\{m+1,\dots,m+n\}}\right)(a)\mathfrak{G}_0\left(\chi^{m+n}_{\{1,\dots,m\}}\right)(b)\right)\\ &= \pm\mathfrak{G}_0\left(P_{\{1,\dots,m+n\}\setminus\{k\}}\circ\chi^{m+n}_{\{m+1,\dots,m+n\}}\right)(a)\mathfrak{G}_0\left(P_{\{1,\dots,m+n\}\setminus\{k\}}\circ\chi^{m+n}_{\{1,\dots,m\}}\right)(b)\\ &= \pm\mathfrak{G}_0\left(\chi^{m+n}_{\{m+1,\dots,m+n\}}\circ P_{\{1,\dots,n\}\setminus\{k-m\}}\right)(a)\mathfrak{G}_0\left(\chi^{m+n}_{\{1,\dots,m\}}\circ P_{\{1,\dots,m\}\setminus\{k\}}\right)(b)\end{aligned}$$

where

$$P_{\{1,\dots,n\}\setminus\{k-m\}} = \mathrm{id}_{\wedge\mathbb{R}^n} \text{ if } k \le m \qquad P_{\{1,\dots,m\}\setminus\{k\}} = \mathrm{id}_{\wedge\mathbb{R}^m} \text{ if } k > m\ .$$

Notice that as $a \in \mathfrak{A}^n = \rho_{\{1,\dots,n\}}\mathfrak{G}_0(\wedge\mathbb{R}^n)$ then if $k > m$

$$\mathfrak{G}_0\left(\chi^{m+n}_{\{m+1,\dots,m+n\}} \circ P_{\{1,\dots,n\}\setminus\{k-m\}}\right)(a) = 0 \ .$$

Analogously, if $k \leq m$ then

$$\mathfrak{G}_0\left(\chi^{m+n}_{\{1,\dots,m\}} \circ P_{\{1,\dots,m\}\setminus\{k\}}\right)(b) = 0 \ .$$

Thus

$$\begin{aligned}\rho_{\{1,\dots,m+n\}}(a \cdot b) &= \pi_{\{1,\dots,m+n\}} \circ (\mathrm{id}_{\mathfrak{G}_0(\wedge\mathbb{R}^n)} - \pi_{\{1,\dots,m+n\}\setminus\{1\}}) \circ \cdots \\ &\quad \circ (\mathrm{id}_{\mathfrak{G}_0(\wedge\mathbb{R}^n)} - \pi_{\{1,\dots,m+n\}\setminus\{m+n\}})(a \cdot b) \\ &= \pi_{\{1,\dots,m+n\}}(a \cdot b) = a \cdot b \ .\end{aligned}$$

This product is also associative: indeed, given $a \in \mathfrak{A}^n$, $b \in \mathfrak{A}^m$ and $c \in \mathfrak{A}^k$ all homogeneous we have

$$\begin{aligned}(a \cdot b) \cdot c &= (-1)^{m\deg(a)}\mathfrak{G}_0\left(\chi^{m+n}_{\{m+1,\dots,m+n\}}\right)(a)\mathfrak{G}_0\left(\chi^{m+n}_{\{1,\dots,m\}}\right)(b) \cdot c \\ &= (-1)^{m\deg(a)+k(\deg(a)+\deg(b))}\mathfrak{G}_0\left(\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{m+1,\dots,m+n\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{1,\dots,m\}}\right)(b)\mathfrak{G}_0\left(\chi^{m+n+k}_{\{1,\dots,k\}}\right)(c) \ .\end{aligned}$$

Notice that

$$\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{m+1,\dots,m+n\}} : \theta_i \mapsto \eta_{k+n+i}$$

i.e. $\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{m+1,\dots,m+n\}} = \chi^{m+n+k}_{\{k+m+1,\dots,k+n+m\}}$, and that

$$\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{1,\dots,m\}} : \theta_i \mapsto \eta_{k+i}$$

i.e. $\chi^{m+n+k}_{\{k+1,\dots,k+m+n\}} \circ \chi^{m+n}_{\{1,\dots,m\}} = \chi^{m+n+k}_{\{k+1,\dots,k+m\}} = \chi^{n+m+k}_{\{1,\dots,k+m\}} \circ \chi^{m+k}_{\{k+1,\dots,k+m\}}$; thus

$$\begin{aligned}(a \cdot b) \cdot c &= (-1)^{m\deg(a)+k(\deg(a)+\deg(b)}\mathfrak{G}_0\left(\chi^{m+n+k}_{\{k+m+1,\dots,k+n+m\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{n+m+k}_{\{1,\dots,k+m\}} \circ \chi^{m+k}_{\{k+1,\dots,k+m\}}\right)(b)\mathfrak{G}_0\left(\chi^{m+n+k}_{\{1,\dots,k+m\}} \circ \chi^{m+k}_{\{1,\dots,k\}}\right)(c) \\ &= (-1)^{(m+k)\deg(a)}\mathfrak{G}_0\left(\chi^{m+n+k}_{\{k+m+1,\dots,k+n+m\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{n+m+k}_{\{1,\dots,k+m\}}\right)\left((-1)^{k\deg(b)}\mathfrak{G}_0\left(\chi^{m+k}_{\{k+1,\dots,k+m\}}\right)(b)\mathfrak{G}_0\left(\chi^{m+k}_{\{1,\dots,k\}}\right)(c)\right) \\ &= (-1)^{(m+k)\deg(a)}\mathfrak{G}_0\left(\chi^{m+n+k}_{\{k+m+1,\dots,k+n+m\}}\right)(a)\mathfrak{G}_0\left(\chi^{n+m+k}_{\{1,\dots,k+m\}}\right)(b \cdot c) \\ &= a \cdot (b \cdot c) \ .\end{aligned}$$

(iv) We now wish to define a directed system using the $\{\mathfrak{A}^n\}_{n\in\mathbb{N}}$. To this end, we define, for $k \geq n$, the linear maps $\iota_{k,n} \colon \mathfrak{A}^n \to \mathfrak{A}^k$,

$$\iota_{k,n}(a) \doteq \begin{cases} a, & k = n \\ \iota_{\wedge\mathbb{R}^k}\left(\eta_1 \cdots \eta_{k-n} \otimes 1\right) \mathfrak{G}_0\left(\chi^k_{\{k-n+1,\dots,k\}}\right)(a), & k > n \ . \end{cases}$$

This is a well-defined directed system: indeed, $\iota_{k,n} \circ \iota_{n,m} = \iota_{k,m}$. This follows from the following computation:

$$\begin{aligned} \iota_{k,n}\left(\iota_{n,m}(a)\right) &= \iota_{\wedge\mathbb{R}^k}\left(\eta_1 \cdots \eta_{k-n} \otimes 1\right) \mathfrak{G}_0\left(\chi^k_{\{k-n+1,\dots,k\}}\right) \Bigg(\iota_{\wedge\mathbb{R}^n}\left(\eta_1 \cdots \eta_{n-m} \otimes 1\right) \\ &\quad \mathfrak{G}_0\left(\chi^n_{\{n-m+1,\dots,n\}}\right)(a) \Bigg) \\ &= \iota_{\wedge\mathbb{R}^k}\left(\eta_1 \cdots \eta_{k-n} \otimes 1\right) \iota_{\wedge\mathbb{R}^k} \circ \chi^k_{\{k-n+1,\dots,k\}}\left(\eta_1 \cdots \eta_{n-m} \otimes 1\right) \\ &\quad \mathfrak{G}_0\left(\chi^k_{\{k-n+1,\dots,k\}} \circ \chi^n_{\{n-m+1,\dots,n\}}\right)(a) \\ &= \iota_{\wedge\mathbb{R}^k}\left(\eta_1 \cdots \eta_{k-m} \otimes 1\right) \mathfrak{G}_0\left(\chi^k_{\{k-m+1,\dots,k\}}\right)(a) = \iota_{k,m}(a) \ . \end{aligned}$$

(v) We will now show that the maps $\{\iota_{k,n}\}_{k\geq n}$ behave well with respect to the product defined in (iii), that is,

$$\iota_{n,m}(a) \cdot \iota_{l,k}(b) = \iota_{n+l,k+m}(a \cdot b) \ .$$

Let us consider $a \in \mathfrak{A}^m$ and $b \in \mathfrak{A}^k$ homogeneous elements, and let us focus separately on the two sides of the equality. The right-hand side can be written as

$$\begin{aligned} &(-1)^{k\deg(a)} \iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1 \cdots \eta_{n+l-k-m} \otimes 1\right) \mathfrak{G}_0\left(\chi^{n+l}_{\{n+l-k-m+1,\dots,n+l\}} \circ \chi^{k+m}_{\{k+1,\dots,m\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{n+l-k-m+1,\dots,n+l\}} \circ \chi^{k+m}_{\{1,\dots,k\}}\right)(b) \\ &= (-1)^{k\deg(a)} \iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1 \cdots \eta_{n+l-k-m} \otimes 1\right) \mathfrak{G}_0\left(\chi^{n+l}_{\{n+l-m+1,\dots,n+l\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{n+l-k-m+1,\dots,n+l-m\}}\right)(b) \end{aligned}$$

while the left-hand side is given by

$$\begin{aligned} &(-1)^{l(\deg(a)+n-m)} \mathfrak{G}_0\left(\chi^{n+l}_{\{l+1,\dots,n+l\}}\right) \left(\iota_{\wedge\mathbb{R}^n}\left(\eta_1 \cdots \eta_{n-m} \otimes 1\right) \mathfrak{G}_0\left(\chi^n_{\{n-m+1,\dots,n\}}\right)(a) \right) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{1,\dots,l\}}\right) \left(\iota_{\wedge\mathbb{R}^l}\left(\eta_1 \cdots \eta_{l-k} \otimes 1\right) \mathfrak{G}_0\left(\chi^l_{\{l-k+1,\dots,l\}}\right)(b) \right) \\ &= (-1)^{l(\deg(a)+n-m)} \iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_{l+1} \cdots \eta_{l+n-m} \otimes 1\right) \mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-m+1,\dots,n+l\}}\right)(a) \\ &\quad \iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1 \cdots \eta_{l-k} \otimes 1\right) \mathfrak{G}_0\left(\chi^{n+l}_{\{l-k+1,\dots,l\}}\right)(b) \\ &= (-1)^{l(\deg(a)+n-m)+(l-k)(\deg(a)+n-m)} \iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1 \cdots \eta_{l-k}\eta_{l+1} \cdots \eta_{l+n-m} \otimes 1\right) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-m+1,\dots,n+l\}}\right)(a) \mathfrak{G}_0\left(\chi^{n+l}_{\{l-k+1,\dots,l\}}\right)(b) \end{aligned}$$

where we have used the naturality of $\iota\colon C \Rightarrow \mathfrak{G}$ and its graded centrality (cfr. Definition 4.1.6). Consider now the permutation $\sigma \in S_{n+l}$

$$\sigma(i) \doteq \begin{cases} l-k+n-m+(i-(l-k)), & l-k+1 \leq i \leq l \\ l-k+(i-l), & l+1 \leq i \leq l+n-m \\ i, & \text{otherwise} \end{cases}$$

whose sign is $\mathrm{sgn}(\sigma) = (-1)^{k(n-m)}$, and let $\chi_\sigma\colon \wedge\mathbb{R}^{n+l} \to \wedge\mathbb{R}^{n+l}$ be the associated homomorphism. Notice that

$$\begin{aligned} \mathfrak{G}_0(\chi_\sigma)\rho_{\{1,\dots,n+l\}} &= \mathfrak{G}_0\left(\chi_\sigma \circ E_{\{1,\dots,n+l\}}\right) = (-1)^{\mathrm{sgn}(\sigma)}\mathfrak{G}_0\left(E_{\{1,\dots,n+l\}}\right) \\ &= (-1)^{\mathrm{sgn}(\sigma)}\rho_{\{1,\dots,n+l\}} \ . \end{aligned}$$

Thus

$$\begin{aligned} &(-1)^{l(\deg(a)+n-m)+(l-k)(\deg(a)+n-m)}\iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1\cdots\eta_{l-k}\eta_{l+1}\cdots\eta_{l+n-m}\otimes 1\right) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-m+1,\dots,n+l\}}\right)(a)\mathfrak{G}_0\left(\chi^{n+l}_{\{l-k+1,\dots,l\}}\right)(b) \\ &= (-1)^{k\deg(a)}\mathfrak{G}_0(\chi_\sigma)\Bigg(\iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1\cdots\eta_{l-k}\eta_{l+1}\cdots\eta_{l+n-m}\otimes 1\right) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-m+1,\dots,n+l\}}\right)(a)\mathfrak{G}_0\left(\chi^{n+l}_{\{l-k+1,\dots,l\}}\right)(b)\Bigg) \\ &= (-1)^{k\deg(a)}\iota_{\wedge\mathbb{R}^{n+l}}\left(\eta_1\cdots\eta_{l+n-m-k}\otimes 1\right)\mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-m+1,\dots,n+l\}}\right)(a) \\ &\quad \mathfrak{G}_0\left(\chi^{n+l}_{\{l+n-k-m+1,\dots,l+n-m\}}\right)(b) \end{aligned}$$

and we therefore see that the two sides coincide.

(vi) We are now in a position to define the universal algebra $\mathfrak{A}$: using the directed system constructed in (iv), we define $\mathfrak{A}$ as the limit of the directed system $\{\mathfrak{A}^n, \iota_{n,m}\}$, i.e.

$$\mathfrak{A} \doteq \varinjlim \mathfrak{A} = \coprod_{n\in\mathbb{N}} \mathfrak{A}^n/\sim$$

where given $a_n \in \mathfrak{A}^n$ and $a_m \in \mathfrak{A}^m$ we say that $a_n \sim a_m$ if there exists $k \geq n$, $k \geq m$ such that $\iota_{k,n}(a_n) = \iota_{k,m}(a_m)$. The canonical homomorphism $\iota_n\colon \mathfrak{A}^n \to \mathfrak{A}$ are then given by $\iota_n\colon a_n \mapsto [a_n]$. The operations in $\mathfrak{A}$ are defined in the following way: given $[a], [b] \in \mathfrak{A}$, there exist $a_m \in \mathfrak{A}^m$ and $b_k \in \mathfrak{A}^k$ such that $[a] = \iota_m(a_m)$ and $[b] = \iota_k(b_k)$; then we define

$$\begin{aligned} [a]+[b] &= \iota_m(a_m)+\iota_k(b_k) \doteq \iota_{m+k}(\iota_{m+k,m}(a_m)+\iota_{m+k,k}(b_k)) \\ \lambda[a] &= \lambda\iota_m(a_m) \doteq \iota_m(\lambda a_m) \\ [a]\cdot[b] &= \iota_m(a_m)\cdot\iota_k(b_k) \doteq \iota_{m+k}(a_m\cdot b_k) \ . \end{aligned}$$

Owing to (v) the product is well-defined. The algebra $\mathfrak{A}$ can also be endowed with a $\mathbb{Z}_2$-grading in the following way: first of all, an element $[a] \in \mathfrak{A}$ is homogeneous if there exists $a_n \in \mathfrak{A}^n$ homogeneous such that $[a] = \iota_n(a_n)$. Then we define

$$\deg([a]) = \deg(\iota_n(a_n)) \doteq \deg(a_n) + n \bmod 2 \ .$$

This is again well-defined; therefore $\mathfrak{A}$ is a complex, unital, $\mathbb{Z}_2$-graded algebra.

We now turn to the construction of the natural transformation $\sigma \colon \mathfrak{G}_0 \Rightarrow \mathfrak{G}^{\mathfrak{A}}$; thus we need to construct the family $\{\sigma_{\wedge\mathbb{R}^n}\}_{n\in\mathbb{N}}$. We proceed in the following way:

(vii) consider $J \subseteq \{1, \ldots, n\}$ with $j_1 < \cdots < j_{|J|}$; then we define the homomorphism $\chi_n^J \colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^{|J|}$ by its action on the generators:

$$\chi_n^J(\eta_k) \doteq \begin{cases} \eta_i, & k = j_i \text{ for some } i \in \{1, \ldots, |J|\} \\ 0, & \text{otherwise .} \end{cases}$$

Notice that

$$\chi^n_{\{j_1,\ldots,j_{|J|}\}} \circ \chi_n^J = P_J \qquad \chi_n^J \circ \chi^n_{\{j_1,\ldots,j_{|J|}\}} = \mathrm{id}_{\wedge\mathbb{R}^{|J|}} \ .$$

Using these maps, we define

$$\sigma_{\wedge\mathbb{R}^n}(a) \doteq \sum_{I\subseteq\{1,\ldots,n\}} \eta_I \otimes \left(\iota_{|I|} \circ \mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I\right)(a) \ .$$

(viii) We need to show that the collection $\{\sigma_{\wedge\mathbb{R}^n}\}_{n\in\mathbb{N}}$ is actually a natural transformation. Thus let us consider $\chi \in \mathbb{R}\mathfrak{Grass}_0(\wedge\mathbb{R}^n, \wedge\mathbb{R}^m)$; by (i) we know that

$$\chi(\eta_I) = \sum_{J\subseteq\{1,\ldots,m\}} c_{IJ}\theta_J \qquad c_{IJ} \neq 0 \text{ if } |I| = |J| \bmod 2,\ |J| \geq |I| \ .$$

Therefore

$$\mathfrak{G}^{\mathfrak{A}}(\sigma_{\wedge\mathbb{R}^n}(a)) = \sum_{I\subseteq\{1,\ldots,n\}} \sum_{J\subseteq\{1,\ldots,m\}} c_{IJ}\theta_J \otimes \left(\iota_{|I|} \circ \mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I\right)(a) \ .$$

As $|J| \geq |I|$ we can write $\iota_{|I|} = \iota_{|J|} \circ \iota_{|J|,|I|}$; thus

$$\mathfrak{G}^{\mathfrak{A}}(\sigma_{\wedge\mathbb{R}^n}(a)) = \sum_{I\subseteq\{1,\ldots,n\}} \sum_{J\subseteq\{1,\ldots,m\}} c_{IJ}\theta_J \otimes \left(\iota_{|J|} \circ \iota_{|J|,|I|} \circ \mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I\right)(a) \ .$$

Notice that $\mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I = \mathfrak{G}_0\left(E_{\{1,\ldots,|I|\},I}\right)$; then

$$\iota_{|J|,|I|} \circ \mathfrak{G}_0\left(E_{\{1,\ldots,|I|\},I}\right) = \iota_{\wedge\mathbb{R}^{|J|}}(\eta_1 \cdots \eta_{|J|-|I|} \otimes 1)\mathfrak{G}_0\left(\chi^{|J|}_{\{|J|-|I|+1,\ldots,|I|\}} \circ E_{\{1,\ldots,|I|\},I}\right) \ .$$

As $|J| = |I| \mod 2$ (cfr. the definiton of E_{JI} in (i)) we then conclude that

$$\mathfrak{G}^{\mathfrak{A}}(\sigma_{\wedge\mathbb{R}^n}(a)) = \sum_{I\subseteq\{1,\dots,n\}} \sum_{J\subseteq\{1,\dots,m\}} c_{IJ}\theta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0 \left(E_{\{1,\dots,|J|\},I} \right) \right) (a) \ .$$

On the other hand we have

$$\sigma_{\wedge\mathbb{R}^m} \left(\mathfrak{G}_0(\chi)(a) \right) = \sum_{J\subseteq\{1,\dots,m\}} \theta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0 \left(\chi_m^J \right) \circ \rho_J \circ \mathfrak{G}_0(\chi) \right) (a) \ .$$

Now, as shown in (i) we can write

$$\chi = \sum_{\substack{I\subseteq\{1,\dots,n\} \\ K\subseteq\{1,\dots,m\}}} c_{IK} E_{KI}$$

and it follows that

$$\begin{aligned} \mathfrak{G}_0 \left(\chi_n^J \right) \circ \rho_J \circ \mathfrak{G}_0(\chi) &= \mathfrak{G}_0 \left(\chi_m^J \circ E_J \circ \sum_{\substack{I\subseteq\{1,\dots,n\} \\ K\subseteq\{1,\dots,m\}}} c_{IK} E_{KI} \right) \\ &= \mathfrak{G}_0 \left(\chi_m^J \circ \sum_{I\subseteq\{1,\dots,n\}} c_{IJ} E_{JI} \right) \ . \end{aligned}$$

Therefore

$$\sigma_{\wedge\mathbb{R}^m} \left(\mathfrak{G}_0(\chi)(a) \right) = \sum_{J\subseteq\{1,\dots,m\}} \sum_{I\subseteq\{1,\dots,n\}} c_{IJ}\theta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0 \left(\chi_m^J \right) \circ \mathfrak{G}_0 \left(E_{JI} \right) \right) (a) \ .$$

Now, recall that $E_{JI} = \chi^{JI} \circ E_I$ (cfr. (i)); thus $\mathfrak{G}_0(E_{JI}) = \mathfrak{G}_0(\chi^{JI}) \circ \rho_I$, and thus

$$\sigma_{\wedge\mathbb{R}^m} \left(\mathfrak{G}_0(\chi)(a) \right) = \sum_{J\subseteq\{1,\dots,m\}} \sum_{I\subseteq\{1,\dots,n\}} c_{IJ}\theta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0 \left(\chi_m^J \circ \chi^{JI} \right) \circ \rho_I \right) (a) \ .$$

Notice that

$$\mathfrak{G}_0 \left(\chi_m^J \circ \chi^{JI} \right) \circ \rho_I = \mathfrak{G}_0 \left(E_{\{1,\dots,|J|\},I} \right)$$

and thus

$$\sigma_{\wedge\mathbb{R}^m} \left(\mathfrak{G}_0(\chi)(a) \right) = \sum_{J\subseteq\{1,\dots,m\}} \sum_{I\subseteq\{1,\dots,n\}} c_{IJ}\theta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0 \left(E_{\{1,\dots,|J|\},I} \right) \right) (a)$$

and we see that the two coincide.

(ix) Finally, we show that $\sigma_{\wedge\mathbb{R}^n}$ is an algebra homomorphism. Linearity follows easily from its definition; we thus need to show that given $a, b \in \mathfrak{G}_0(\wedge\mathbb{R}^n)$ (which we suppose to be homogeneous),

$$\sigma_{\wedge\mathbb{R}^n}(a \cdot b) = \sigma_{\wedge\mathbb{R}^n}(a) \cdot \sigma_{\wedge\mathbb{R}^n}(b) \ .$$

Let us start by examining the object on the right-hand side: we have

$$\begin{aligned}
\sigma_{\wedge\mathbb{R}^n}(a) \cdot \sigma_{\wedge\mathbb{R}^n}(b) &= \left(\sum_{I \subseteq \{1,\dots,n\}} \eta_I \otimes \left(\text{ч}_{|I|} \circ \mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I \right)(a) \right) \\
&\quad \cdot \left(\sum_{J \subseteq \{1,\dots,n\}} \eta_J \otimes \left(\text{ч}_{|J|} \circ \mathfrak{G}_0\left(\chi_n^J\right) \circ \rho_J \right)(b) \right) \\
&= \sum_{I,J \subseteq \{1,\dots,n\}} (-1)^{|J|\deg(a)} \eta_J \eta_I \\
&\quad \otimes \text{ч}_{|I|+|J|} \left(\mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I(a) \cdot \mathfrak{G}_0\left(\chi_n^J\right) \circ \rho_J(b) \right) \ .
\end{aligned}$$

Notice that $\eta_I\eta_J \neq 0$ if and only if $I \cap J = \varnothing$; in this case we define $K = J \cup I$, with $k_1 < \dots < k_{|K|}$. Then the object on the right-hand side of the tensor product can then be written as

$$\begin{aligned}
&\text{ч}_{|K|} \left(\mathfrak{G}_0\left(\chi_n^I\right) \circ \rho_I(a) \cdot \mathfrak{G}_0\left(\chi_n^J\right) \circ \rho_J(b) \right) = (-1)^{|J|\deg(a)} \\
&\text{ч}_{|K|} \left(\left(\mathfrak{G}_0\left(\chi^{|K|}_{\{|J|+1,\dots,|K|\}} \circ \chi_n^I \right) \circ \rho_I \right)(a) \left(\mathfrak{G}_0\left(\chi^{|K|}_{\{1,\dots,|J|\}} \circ \chi_n^J \right) \circ \rho_J \right)(b) \right) \ .
\end{aligned}$$

By considering the permutation $\sigma \in S_n$ that maps $\{j_1, \dots, j_{|J|}, i_1, \dots, i_{|I|}\}$ with $j_1 < \dots < j_{|J|}$ and $i_1 < \dots < i_{|I|}$ to $\{k_1, \dots, k_{|K|}\}$ while leaving the rest of the indices untouched, we can consider the associated homomorphism $\chi_\sigma \colon \wedge \mathbb{R}^n \to \wedge\mathbb{R}^n$; we then have

$$\chi^{|K|}_{\{|J|+1,\dots,|K|\}} \circ \chi_n^I = \chi_n^K \circ \chi_\sigma \qquad \chi^{|K|}_{\{1,\dots,|J|\}} \circ \chi_n^J = \chi_n^K \circ \chi_\sigma \ .$$

Therefore

$$\begin{aligned}
\sigma_{\wedge\mathbb{R}^n}(a) \cdot \sigma_{\wedge\mathbb{R}^n}(b) &= \sum_{\substack{I,J \subseteq \{1,\dots,n\} \\ I \cap J = \varnothing}} \eta_J \eta_I \otimes \left(\text{ч}_{|K|} \circ \mathfrak{G}_0\left(\chi_n^K \circ \chi_\sigma \right) \right) (\rho_I(a)\rho_J(b)) \\
&= \sum_{\substack{I,J \subseteq \{1,\dots,n\} \\ I \cap J = \varnothing}} (-1)^{\operatorname{sgn}(\sigma)} \eta_K \otimes \left(\text{ч}_{|K|} \circ \mathfrak{G}_0\left(\chi_n^K \circ \chi_\sigma \right) \right) (\rho_I(a)\rho_J(b)) \\
&= \sum_{K \subseteq \{1,\dots,n\}} \sum_{I \subseteq K} \eta_K \otimes \left(\text{ч}_{|K|} \circ \mathfrak{G}_0\left(\chi_n^K \right) \right) (\rho_I(a)\rho_{K \setminus I}(b)) \ .
\end{aligned}$$

The object on the left is instead given by

$$\begin{aligned}\sigma_{\wedge\mathbb{R}^n}(a\cdot b) &= \sum_{K\subseteq\{1,\ldots,n\}} \eta_K \otimes \left(\natural_{|K|}\circ \mathfrak{G}_0\left(\chi_n^K\right)\circ\rho_K\right)(a\cdot b)\\ &\overset{\text{(ii)}}{=} \sum_{K\subseteq\{1,\ldots,n\}}\sum_{I\subseteq K} \eta_K \otimes \left(\natural_{|K|}\circ \mathfrak{G}_0\left(\chi_n^K\right)\right)(\rho_I(a)\rho_{K\setminus I}(b))\end{aligned}$$

and we see that the two coincide.

(x) To complete the proof, we need to show the universality of $\mathfrak{A}$. Let $\mathfrak{A}'$ be another unital, complex, $\mathbb{Z}_2$-graded algebra, and let $\sigma'\colon \mathfrak{G}_0 \Rightarrow \mathfrak{G}^{\mathfrak{A}'}$ be the associated natural transformation. We need to define a homomorphism $\tau\colon \mathfrak{A}\to\mathfrak{A}'$ such that

$$(\mathrm{id}_{\wedge\mathbb{R}^n}\otimes\tau)\circ\sigma_{\wedge\mathbb{R}^n} = \sigma'_{\wedge\mathbb{R}^n} \text{ for all } n\in\mathbb{N}\ .$$

Consider now $[a]\in\mathfrak{A}$; then there exists $a_n\in\mathfrak{A}^n = \rho_{\{1,\ldots,n\}}\mathfrak{G}_0(\wedge\mathbb{R}^n)$ such that $[a]=\iota_n(a_n)$. We then define τ by requiring that

$$\eta_{\{1,\ldots,n\}}\otimes\tau(a) = \sigma'_{\wedge\mathbb{R}^n}(a_n) \text{ for all } n\in\mathbb{N}\ .$$

Consider now any $b_m\in\mathfrak{G}_0(\wedge\mathbb{R}^m)$; we have

$$\begin{aligned}(\mathrm{id}_{\wedge\mathbb{R}^m}\otimes\tau)\circ\sigma_{\wedge\mathbb{R}^m}(b_m) &= (\mathrm{id}_{\wedge\mathbb{R}^m}\otimes\tau)\left(\sum_{J\subseteq\{1,\ldots,m\}}\eta_J\otimes\left(\natural_{|J|}\circ\mathfrak{G}_0\left(\chi_m^J\right)\circ\rho_J\right)(b_m)\right)\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\eta_J\otimes\tau\left(\natural_{|J|}\circ\mathfrak{G}_0\left(\chi_m^J\right)\circ\rho_J\right)(b_m)\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\chi_J^m(\eta_{|J|})\otimes\tau\left(\natural_{|J|}\circ\mathfrak{G}_0\left(\chi_m^J\right)\circ\rho_J\right)(b_m)\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\mathfrak{G}^{\mathfrak{A}'}\left(\chi_J^m\right)\left(\eta_{|J|}\otimes\tau\left(\natural_{|J|}\circ\mathfrak{G}_0\left(\chi_m^J\right)\circ\rho_J\right)(b_m)\right)\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\mathfrak{G}^{\mathfrak{A}'}\left(\chi_J^m\right)\sigma'_{\wedge\mathbb{R}^{|J|}}\left(\mathfrak{G}_0\left(\chi_m^J\right)\circ\rho_J\right)(b_m)\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\mathfrak{G}^{\mathfrak{A}'}\left(\chi_J^m\circ\chi_m^J\right)\sigma'_{\wedge\mathbb{R}^m}(\rho_J(b_m))\\ &= \sum_{J\subseteq\{1,\ldots,m\}}\sigma'_{\wedge\mathbb{R}^m}(\rho_J(b_m)) = \sigma'_{\wedge\mathbb{R}^m}(b_m)\ .\end{aligned}$$

□

4.2 The free algebra generated by formal S-matrices

First of all, let us fix a Lorentzian manifold (M, g_0) and let us define the Dirac Lagrangian

$$\begin{aligned}\mathscr{L}\colon \mathcal{D}(M) &\to \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M,g_0))\\ f &\mapsto \quad \mathscr{L}(f)\end{aligned}$$

with $\mathscr{L}(f)$ defined on $\wedge^2\mathcal{E}(S^{\oplus}_{r,1}(M,g_0))$

$$\mathscr{L}(f)[u\wedge v] \doteq \int_M \langle fu, \mathbf{D}^{\oplus}_0(fv)\rangle \; d\mu_{g_0} \; .$$

This object is evidently antisymmetric and bilinear. Moreover, notice that $\mathscr{L}(f)^* = \mathscr{L}(f)$ for every $f \in \mathcal{D}(M)$: indeed,

$$\begin{aligned}
\mathscr{L}(f)^*[u\wedge v] &= \overline{\mathscr{L}(f)[v^*\wedge u^*]} = \overline{\int_M \langle fv^*, \mathbf{D}^{\oplus}_0(fu^*)\rangle \; d\mu_g} = \\
&= \overline{\int_M \langle fC^{-1}v_1, \mathbf{D}_0(fCu_2)\rangle \; d\mu_g} - \overline{\int_M \langle \mathbf{D}^*_0(fC^{-1}u_1), fCv_2\rangle \; d\mu_g} \\
&= \overline{\int_M \langle fv_1, \mathbf{D}_0(fCu_2)\rangle \; d\mu_g} - \overline{\int_M \langle \mathbf{D}_0(fu_1), fCv_2\rangle \; d\mu_g} \\
&= \int_M \langle CC^{-1}\mathbf{D}_0C(fu_2), fv_1\rangle \; d\mu_g - \int_M \langle C(fv_2), \mathbf{D}_0(fu_1)\rangle \; d\mu_g \\
&= -\left(\int_M \langle -\mathbf{D}^*_0(fu_2), fv_1\rangle \; d\mu_g + \int_M \langle fv_2, \mathbf{D}_0(fu_1)\rangle \; d\mu_g\right) \\
&= -\int_M \langle fv, \mathbf{D}^{\oplus}_0(fu)\rangle \; d\mu_g = -\mathscr{L}(f)[v\wedge u] = \mathscr{L}(f)[u\wedge v] \; .
\end{aligned}$$

It is also *gauge-invariant*: that is, for every $t\in\mathbb{R}$ we have that $\mathscr{L}(f)\circ\chi_t = \mathscr{L}(f)$, where χ_t is the antisymmetric extension to $\mathcal{E}^a(M^2, S^{\oplus}_{r,1}(M)^{\boxtimes 2})$ of

$$\chi_t\colon \mathcal{E}(S^{\oplus}_{r,1}(M)) \ni \begin{pmatrix}u_1\\u_2\end{pmatrix} \mapsto \begin{pmatrix}e^{it}u_1\\e^{-it}u_2\end{pmatrix} \; .$$

We then proceed as in [12], that is, we construct a Grassmann covariant multiplication $(\mathfrak{A}, \iota)$ with $\mathfrak{A}\colon \mathbb{R}\mathfrak{Grass} \to \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ a $\mathfrak{Vec}$-functor and $\iota\colon C \Rightarrow \mathfrak{A}$ a $\mathfrak{Vec}$-natural tranformation. The construction of these objects explicitly uses fermionic functionals as depicted in Section 2.3; among these, we point out a particular class of functionals which will be widely used in Section 4.5:

Definition 4.2.1. Given a metric g_1 such that g_1 and g_0 satisfy the assumptions of Section 3.1, consider then a smooth curve $I \ni t \mapsto g_t$ connecting the two. We can then construct the map τ^1_0 as in Section 3.2, and use it to map spinors in $\mathcal{E}(S^{\oplus}_{r,1}(M,g_0))$ to spinors in $\mathcal{E}(S^{\oplus}_{r,1}(M,g_1))$. We then define an associated *kinetic perturbation* as the functional

$$K[u\wedge v] \doteq \int_M \langle \tau^1_0(fu), \mathbf{D}^{\oplus}_1\tau^1_0(fv)\rangle_1 \; d\mu_{g_1} - \int_M \langle fu, \mathbf{D}^{\oplus}_0(fv)\rangle_0 \; d\mu_{g_0}$$

where $f\in\mathcal{D}(M)$ is any test function such that $f\equiv 1$ on $\mathrm{Supp}(g_1 - g_0)$.

Lemma 4.2.2. *K is a well-defined object in $\mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$. Moreover, $K^* = K$ and it is also gauge-invariant.*

Proof. First of all, notice that K doesn't actually depend on the choice of f, as outside of $\mathrm{Supp}(g_1 - g_0)$ we have that $g_1 \equiv g_0$ and thus there $\mathbf{D}^{\oplus}_1 = \mathbf{D}^{\oplus}_0$ and $d\mu_{g_1} = d\mu_{g_0}$; moreover, due to the properties of the curve $I \ni t \mapsto g_t$ joining g_0 and g_1 we have that τ^1_0 is actually the identity outside of $\pi^{-1}(\mathrm{Supp}(g_1 - g_0))$. This also shows that K is compactly supported as desired. Clearly, K is also bilinear and antisymmetric; it is also continuous with respect to the Fréchet topology. Indeed, as outside of $\pi^{-1}(\mathrm{Supp}(g_1 - g_0))$ the parallel transport map τ^1_0 acts as the identity, we have that

$$\left\|\tau^1_0 u\right\|_{D,n} = \|u\|_{D,n} \quad \forall D \subset\subset M \setminus \mathrm{Supp}(g_1 - g_0) \ .$$

Thus we can focus our attention on the behaviour of the parallel transport on $S = \mathrm{Supp}(g_1 - g_0)$. Thus, let us consider the map

$$\tau^1_0 \colon \Gamma\left(S_{r,1}(M_{t_0})\Big|_S\right) \to \Gamma\left(S_{r,1}(M_{t_1})\Big|_S\right) \ .$$

Notice, as pointed out in Section 3.2, that given a section σ, $\tau^1_0\sigma = s \circ f_\sigma$. It is easily seen that if $\sigma_n \to \sigma$ in $\mathcal{E}(S, S_{r,1}(M_0))$, then $f_{\sigma_n} \to f_\sigma$ in $\mathcal{E}(S, I \times S_{r,1}(M_0))$. Then, by the continuity (cfr. Theorem 2.2.15) of the composition

$$\circ \colon C^\infty\left(S, I \times S_{r,1}(M_0)\Big|_S\right) \times C^\infty\left(I \times S_{r,1}(M_0)\Big|_S, S_{r,1}(M_1)\Big|_S\right) \to C^\infty\left(S, S_{r,1}(M_1)\Big|_S\right)$$

with respect to the Whitney C^∞ topology and by the fact that the Fréchet topology and the Whitney C^∞ topology coincide on $C^\infty\left(S, I \times S_{r,1}(M_0)\Big|_S\right)$ and $C^\infty\left(S, S_{r,1}(M_1)\Big|_S\right)$ thanks to the compactness of S (cfr. Section 2.2.2 after Remark 2.2.12), we infer that if $\sigma_n \to \sigma$ in $\mathcal{E}(S, S_{r,1}(M_0))$ then $\tau^1_0\sigma_n \to \tau^1_0\sigma$ in $\mathcal{E}(S, S_{r,1}(M_1))$. Thus the desired continuity of τ^1_0 follows, and so of K.
The claim about the involution and the gauge invariance follow from a straightforward computation. $\square$

Remark 4.2.3. In order to simplify the notation, in the following we will use the following notation:

$$(u, v) \doteq \int_M \langle u, v \rangle \, d\mu_g$$

where $u, v \in \Gamma(E)$, where E may be $S_{r,1}(M)$ or $S^{\oplus}_{r,1}(M)$. The context will make it clear on which bundle we are working on.
Moreover, whenever needed, we shall specify with $\langle \cdot, \cdot \rangle_{0,1}$ whether the hermitian metric is the one on $S^{\oplus}_{r,1}(M, g_0)$ or $S^{\oplus}_{r,1}(M, g_1)$, and with $(\cdot, \cdot)_{0,1}$ whether the integration is carried out using the measure $d\mu_{g_0}$ or $d\mu_{g_1}$. ♠

Remark 4.2.4. Let us denote with $\mathcal{K} \subseteq \mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ the subset of kinetic perturbations. Notice that this is not a vector subspace, as in general given $K_1, K_2 \in \mathcal{K}$ it holds that $K_1 + K_2 \notin \mathcal{K}$.
We examine two cases:

(i) suppose that K_1 and K_2 satisfy $\mathrm{Supp}(K_1) \cap \mathrm{Supp}(K_2) = \varnothing$. Denoting with g_1 and g_2 the associated metrics (which both satisfy the assumptions of Section 3.1), then it follows easily that $g_1 + g_2 - g_0$ also satisfies the assumptions of Section 3.1. One can thus consider the curve joining g_0 and $g_1 + g_2 - g_0$, and the associated kinetic perturbation K_+. Due to the disjoint supports of K_1 and K_2, the curve connecting g_0 and $g_1 + g_2 - g_0$ just restricts to the one joining g_0 and g_1 and g_0 and g_2 on $\mathrm{Supp}(K_1)$ and $\mathrm{Supp}(K_2)$ respectively; this, combined with the same support properties of g_1 and g_2, the freedom in the choice of the cutoff function f in Definition 4.2.1, the way that the Dirac operator is defined and the way integration on manifolds is carried out, implies that in this case
$$K_+ = K_1 + K_2$$
which entails in particular that $K_1 + K_2 \in \mathcal{K}$.

(ii) If $\mathrm{Supp}(K_1) \cap \mathrm{Supp}(K_2) \neq \varnothing$, then the situation is more delicate. First of all, notice that outside of $\mathrm{Supp}(K_1) \cap \mathrm{Supp}(K_2)$ the situation is entirely analogous to case (i); therefore the problem lies in the intersection of the supports. The first issue is that in general given two metrics g_1 and g_2 which satisfy the assumption in Section 3.1, the metric $g_1 + g_2 - g_0$ (which corresponds to the perturbation of the metric g_0 obtained by adding to it the individual perturbations $g_1 - g_0$ and $g_2 - g_0$) doesn't necessarily satisfy the requirements of Section 3.1. Secondly and most importantly, due to the definition of kinetic perturbations (and in particular owing to the differential equations (3.4) of the parallel transport) it holds that $K_1 + K_2 \notin \mathcal{K}$.

Therefore, when dealing with sums of functionals possibly containing kinetic perturbations one has to pay particular attention. ♠

Given a kinetic perturbation K, we have a corresponding globally hyperbolic metric g_K (this is just the metric denoted by g_1 in Section 3.1, which is a compact perturbation of g_0). With $J^{\pm}_K$ we shall denote the causal structures computed with respect to said metric, while with $J^{\pm}_0$ we denote the causal structures computed with respect to the unperturbed metric g_0.
We are now in a position to carry out the following construction:

Definition 4.2.5. Given a real Grassmann algebra $G \in \mathbb{R}\mathfrak{Grass}$, we construct the $\mathbb{Z}_2$-graded algebra $\mathfrak{A}_G$ by considering the free algebra over $\mathbb{C}$ generated by the symbols $\{S_G(F)\}_{f \in G \otimes \mathcal{F}_{\mathrm{loc}}(E)}$ under the following conditions:

(i) $\{S_G(F)\}_{F \in (G \otimes \mathcal{F}_{\mathrm{loc}}(E))^0} \subseteq \mathfrak{A}^0_G$;

(ii) the product is $\mathbb{Z}_2$-graded, i.e.
$$\mathfrak{A}^{\alpha}_G \mathfrak{A}^{\beta}_G \subseteq \mathfrak{A}^{\alpha+\beta}_G, \qquad \alpha, \beta \in \{0, 1\} \ ;$$

(iii) given $P, Q, N \in (G\otimes\mathcal{F}_{\mathrm{loc}}(E))^0$ with P and Q such that $\mathrm{Supp}(P)\cap J^-_N(\mathrm{Supp}(Q)) = \varnothing$, we have that

$$S_G(P + Q + N) = S_G(P + N)S_G(N)^{-1}S_G(Q + N)\ . \tag{4.1}$$

In order to impose the next requirements, we notice the following: any functional $F \in \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ gives rise to a family of maps $\{F_G\}_{G\in\mathfrak{Grass}}$ defined by

$$F_G\colon G \otimes \mathcal{C}(E) \ni \eta \otimes u \mapsto \eta \otimes F(u) \in G \otimes \mathbb{C} \tag{4.2}$$

which commutes with any morphism $\chi\colon G \to G'$. By the even-rules-principle, we know that knowledge of $F \in \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ is equivalent to the knowledge of $F_G \circ \exp$ on even elements in $(G \otimes \mathcal{C}(S^{\oplus}_{r,1}(M)))^0$, in particular on elements of the form

$$\sum_{i\in I} \eta_i \otimes u^i,\ \ \eta_i \in G^1, u^i \in \mathcal{E}(S^{\oplus}_{r,1}(M)) \subseteq \mathcal{C}(S^{\oplus}_{r,1}(M))^1\ .$$

Indeed,

$$\exp\left(\sum_{i\in I}\eta_i \otimes u^i\right) = \prod_{i\in I}\left(\mathrm{id}_{G\otimes\mathcal{C}(E)} + \eta_i \otimes u^i\right)$$

and thus

$$F_G \circ \exp\left(\sum_{i\in I}\eta_i \otimes u^i\right) = \sum_{n=0}^{|I|}\sum_{i_1<\cdots<i_n} \eta_{i_n}\cdots\eta_{i_1} \otimes F_n(u^{i_1} \wedge \cdots \wedge u^{i_n})\ .$$

We then define the shift of a functional $F \in \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ by an element $\mathbf{h} = \sum_{j\in J}\theta_j\otimes v^j$ by considering, on any even element of the above form,

$$(F_G \circ \exp)^{\mathbf{h}}\left(\sum_{i\in I}\eta_i \otimes u^i\right) = F_G \circ \exp\left(\sum_{i\in I}\eta_i \otimes u^i + \sum_{j\in J}\theta_j \otimes v^j\right)\ .$$

The object on the left can be written as

$$\sum_{n=0}^{|I|}\sum_{i_1<\cdots<i_n} \eta_{i_n}\cdots\eta_{i_1} \otimes (F^{\mathbf{h}}_G)_n(u^{i_1} \wedge \cdots \wedge u^{i_n})$$

where thanks to the right-hand side we can infer

$$(F^{\mathbf{h}}_G)_n(u^{1_i} \wedge \cdots \wedge u^{i_n}) = \sum_{k=0}^{|J|}\sum_{j_1<\cdots<j_k} \theta_{j_k}\cdots\theta_{j_1} \otimes F_{n+k}(u^{i_1} \wedge \cdots \wedge u^{i_n} \wedge v^{j_1} \wedge \cdots \wedge v^{j_k})\ .$$

The shift $F^{\mathbf{h}} \in G\otimes\mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ is then defined as the collection of functionals $\{(F^{\mathbf{h}}_G)_n\}_{n\in\mathbb{N}}$. Notice that if $\mathbf{h} = 0 \otimes 0$ we then have that

$$(F_G \circ \exp)^{\mathbf{h}}\left(\sum_{i\in I}\eta_i \otimes u^i\right) = F_G \circ \exp\left(\sum_{i\in I}\eta_i \otimes u^i\right)$$

and so $F^{\mathbf{h}} = \{\mathrm{id}_G \otimes F_n\}_{n\in\mathbb{N}} = \mathrm{id}_G \otimes F$.
We are interested in defining shifts of the Lagrangian when the shift $\mathbf{h}$ is constructed using compactly supported functions $u^i \in \mathcal{D}(S^{\oplus}_{r,1}(M))$; these will be denoted by $\mathscr{L}(f)^{\mathbf{h}}$ for every $f \in \mathcal{D}(M)$. Then we define the functionals

$$\delta\mathscr{L}_{\mathbf{h}} \doteq \mathscr{L}(f)^{\mathbf{h}} - \mathrm{id}_G \otimes \mathscr{L}(f),\ f \equiv 1 \text{ on } \mathrm{Supp}(\mathbf{h}) \ .$$

It can be easily shown that these objects don't depend on the choice of f, and that $\delta\mathscr{L}_{\mathbf{h}} \in (G \otimes \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M)))^0$. We are now in a position to state the other requirement:

(iv) given $F \in G \otimes \mathscr{F}_{\mathrm{loc}}(E)$ and $\mathbf{h} = \sum_{h\in J} \theta_j \otimes v^j$ with $\theta_j \in G^1$ and $v^j \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ we require that

$$S_G(\delta\mathscr{L}_{\mathbf{h}})S_G(F) = S_G(F)S_G(\delta\mathscr{L}_{\mathbf{h}}) = S_G(F^{\mathbf{h}} + \delta\mathscr{L}_{\mathbf{h}}) \ . \tag{4.3}$$

Remark 4.2.6. Notice that if $\mathbf{h} = 0 \otimes 0$, then $\delta\mathscr{L}_{\mathbf{h}} = 0 \otimes 0$, and (4.3) entails that

$$S_G(0 \otimes 0) = \mathrm{id}_{\mathfrak{A}_G} \ .$$

Now, clearly $0 \otimes 0 \in (G \otimes \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M)))^0$, and by the previous equality and by (4.1) we have that

$$S_G(P + Q + 0) = S_G(P)S_G(Q)$$

if $\mathrm{Supp}(P) \cap J_0^-(\mathrm{Supp}(Q)) = \varnothing$. Notice that if $\mathrm{Supp}(P)$ and $\mathrm{Supp}(Q)$ are spacelike separated then we can also write

$$S_G(P + Q + 0) = S_G(Q + P + 0) = S_G(Q)S_G(P)$$

and thus we have

$$S_G(P)S_G(Q) = S_G(Q)S_G(P) \ . \tag{4.4}$$

♠

Remark 4.2.7. Consider a fixed $\eta \in G^0$ and the constant functional $F_c \in \mathscr{F}^0_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$. Then $\mathrm{Supp}(\eta \otimes F_c) = \varnothing$, and so $\eta \otimes F_c$ and $P \in (G \otimes \mathscr{F}_{\mathrm{loc}(S^{\oplus}_{r,1}(M)})^0$ satisfy (4.4) for every choice of $P \in \left(G \otimes \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))\right)^0$, i.e.

$$S_G(\eta \otimes F_c)S_G(P) = S_G(P)S_G(\eta \otimes F_c)$$

and moreover

$$S_G(\eta \otimes F_{c_1})S_G(\eta \otimes F_{c_2}) = S_G(\eta \otimes F_{c_1} + \eta \otimes F_{c_2}) = S_G(\eta \otimes F_{c_1+c_2}) \ .$$

Thus for any fixed $\eta \in G^0$, $\{S_G(\eta \otimes F_c)\}_{c\in\mathbb{C}}$ is a central subgroup. ♠

The $\mathfrak{Vec}$-functor $\mathfrak{A}\colon \mathbb{R}\mathfrak{Grass} \to \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}$ thus assigns to each $G \in \mathbb{R}\mathfrak{Grass}$ the $\mathbb{Z}_2$-graded algebra $\mathfrak{A}_G$; its behaviour on morphisms is then defined in the following way:

(i) first of all, given any morphism $\chi\colon G \to H$ we can define a morphism $\chi_{\mathfrak{A}}\colon \mathfrak{A}_G \to \mathfrak{A}_H$ by defining it on the generators as

$$\chi_{\mathfrak{A}}(S_G(F)) \doteq S_H\left(\chi_{\mathcal{F}_{\mathrm{loc}}}(F)\right) \ ,$$

with $\chi_{\mathcal{F}_{\mathrm{loc}}}(\eta \otimes F) \doteq \chi(\eta) \otimes F$.

(ii) We can now define, for any given couple $G, H \in \mathbb{R}\mathfrak{Grass}$, the morphism

$$\mathfrak{A}_{G,H}\colon \mathbb{R}\mathfrak{Grass}(G,H) \to \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(\mathfrak{A}_G, \mathfrak{A}_H)$$

by defining it on morphisms in $\mathfrak{Grass}(G,H)$ and then extending it by $\mathbb{R}$-linearity to $\mathbb{R}\mathfrak{Grass}(G,H)$. In particular we define

$$\mathfrak{A}_{G,H}(\chi) \doteq \chi_{\mathfrak{A}} \ .$$

It is easily seen that the requirements of Definition 4.1.4 are satisfied.

In order to have a covariant Grassmann multiplication as of Definition 4.1.6, we then need to exhibit a $\mathfrak{Vec}$-natural transformation

$$\iota\colon C \Rightarrow \mathfrak{A}$$

that is, a family of morphisms $\{\iota_G\}_{G\in\mathbb{R}\mathfrak{Grass}} \subseteq \mathrm{Hom}(\mathbb{R}, \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}}, \mathfrak{A}_G)) \simeq \mathbb{R}\mathfrak{Alg}^{\mathbb{Z}_2}(G_{\mathbb{C}}, \mathfrak{A}_G)$ in $\mathfrak{Vec}_{\mathbb{R}}$ satisfying Definition 4.1.5 and 4.1.6.
We then proceed in the following way:

(i) we require that for any $\eta \in G$, the morphism $\iota_G\colon G_{\mathbb{C}} \to \mathfrak{A}_G$ satisfies

$$\iota_G(e^{i(\eta\otimes z)}) = S_G(\eta \otimes F_z) \ \forall z \in \mathbb{C} \ .$$

Observe that if $\eta \in G^1$, noticing that $F_1 = \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}$, it then follows that $e^{i(\eta\otimes 1)} = \mathrm{id}_G + \eta \otimes i$, and thus

$$S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) = \iota_G(\mathrm{id}_G + \eta \otimes i) = \iota_G(\mathrm{id}_G) + i\iota_G(\eta \otimes 1) = \mathrm{id}_{\mathfrak{A}_G} + i\iota_G(\eta \otimes 1) \ .$$

Therefore we conclude that the above requirement entails that for every $\eta \in G^1$

$$\iota_G(\eta \otimes 1) = i(\mathrm{id}_{\mathfrak{A}_G} - S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})) \ .$$

As we have defined ι_G on $G^1 \otimes 1$, we have defined it on the generators of $G_{\mathbb{C}}$, and thus on every element of $G_{\mathbb{C}}$: moreover, notice that $\iota_G(G_{\mathbb{C}})$ is a subalgebra of $\mathfrak{A}_G$.

(ii) ι_G also respects the grading: indeed, we know that given $\eta_1, \eta_2 \in G^1$

$$\begin{aligned}\mathfrak{A}^0_G \ni S_G(\eta_1\eta_2 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) =& \iota_G(e^{i(\eta_1\eta_2\otimes 1)}) = iS_G(\eta_1 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})S_G(\eta_2 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) \\ &- iS_G(\eta_1 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) - iS_G(\eta_2 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) - (i-1)\mathrm{id}_{\mathfrak{A}_G}\end{aligned}$$

and decomposing the right side into the even and odd parts we arrive at

$$0 = S_G(\eta_1 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^1 (S_G(\eta_2 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^0 - \mathrm{id}_{\mathfrak{A}_G}) + S_G(\eta_2 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^1 (S_G(\eta_1 \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^0 - \mathrm{id}_{\mathfrak{A}_G}) \ .$$

Thus, either

$$S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^1 = 0 \text{ or } S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^0 = \mathrm{id}_{\mathfrak{A}_G}$$

for every $\eta \in G^1$. If $S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^1 = 0$ were to hold, then one finds inconsistencies when considering the fact that $\eta_1 \wedge \eta_2 = -\eta_2 \wedge \eta_1$ and carrying out the computations; thus we conclude that $S_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})^0 = \mathrm{id}_{\mathfrak{A}_G}$, and therefore

$$\iota_G(\eta \otimes 1) \in \mathfrak{A}_G^1 \text{ if } \eta \in G^1$$

i.e. ι_G is a (linear combination of) degree 0 maps between $\mathbb{Z}_2$-graded algebras as required.

(iii) As far as the behaviour with respect to morphism is concerned, notice that for any $\eta \in G^1$,

$$\begin{aligned}(\chi_{\mathfrak{A}} \circ \iota_G)(\eta \otimes 1) &= \chi_{\mathfrak{A}} \left(i\mathrm{id}_{\mathfrak{A}_G} - iS_G(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}}) \right) = i\mathrm{id}_{\mathfrak{A}_{G'}} - iS_G(\chi_{\mathcal{F}_{\mathrm{loc}}}(\eta \otimes \mathrm{id}_{\mathcal{F}_{\mathrm{loc}}})) \\ &= (\iota_{G'} \circ \chi_{\mathbb{C}})(\eta \otimes 1) \ .\end{aligned}$$

As this holds for the generators, it holds for any object in G. But notice that this is precisely the behaviour required by Definition 4.1.5.

Thus we have a covariant Grassmann multiplication $(\mathfrak{A}, \iota)$ in the sense of Definition 4.1.6, and we can thus apply Theorem 4.1.8, which yields a universal unital, $\mathbb{Z}_2$-graded complex algebra $\mathfrak{A}$ as well as embeddings $\sigma_G \colon \mathfrak{A}_G \to G \otimes \mathfrak{A}$ which are such that

$$\sigma_{G'} \circ \chi_{\mathfrak{A}} = \chi \circ \sigma_G$$

for every morphism $\chi \colon G \to G'$. We want to endow this algebra with an involution; to do so, we first define it on $\mathfrak{A}_G$ by first defining it on the generators as

$$(S_G(F))^* \doteq S_G(F^*)^{-1}$$

where the involution on $G \otimes \mathcal{F}_{\mathrm{loc}}$ is defined on the tensor product of homogeneous elements by

$$(\eta \otimes F)^* \doteq (-1)^{\deg(\eta)\deg(F)} \eta^* \otimes F^* \ .$$

Notice that the involution on G is defined by $\eta_i^* = \eta_i$ on the generators, and $(\eta_1 \wedge \cdots \eta_n)^* = \eta_n^* \wedge \cdots \wedge \eta_1^*$. One can then induce an involution on $\mathfrak{A}$ via the maps $\sigma_G \colon \mathfrak{A}_G \to G \otimes \mathfrak{A}$ by requiring that

$$\sigma_G(S_G(F))^* = \sigma_G(S_G(F)^*) = \sigma_G(S_G(F^*))^{-1} \ .$$

Remark 4.2.8. Given $\eta_i\eta_j \otimes c \in G \otimes \mathbb{C}$ with η_i, η_j generators of G, we consider

$$\sigma_G(S_G(\eta_i\eta_j \otimes F_c)) \ .$$

We can write it as

$$\sigma_G(S_G(\eta_i\eta_j \otimes F_c)) = \sum_{I\subset\{1,\dots,n\}} \eta_I B^I(\eta_1, \dots, \eta_n, c) \ .$$

Given a morphism $\chi \colon \wedge \mathbb{R}^n \to G$, $\chi \colon e_i \mapsto \eta_i$, we consider

$$\sigma_G(S_G(\eta_i\eta_j \otimes F_c)) = \sigma_G\left(S_G\left(\chi_{\mathcal{F}_{\mathrm{loc}}}(e_ie_j \otimes F_c)\right)\right) = \chi\left(\sigma_{\wedge\mathbb{R}^n}\left(S_{\wedge\mathbb{R}^n}\left(e_ie_j \otimes F_c\right)\right)\right) \ .$$

Thus we can suppose that $G = \wedge\mathbb{R}^n$. Consider now, given $\lambda \in \mathbb{R}^n$, $\chi_\lambda \colon \wedge \mathbb{R}^n \to \wedge\mathbb{R}^n$, $\chi_\lambda(e_i) = \lambda_i e_i$. We have

$$\sum_{I\subset\{1,\dots,n\}} \lambda_I e_I B^I(e_1, \dots, e_n, c) = \sum_{I\subset\{1,\dots,n\}} \lambda_I e_I B^I(\lambda_1 e_1, \dots, \lambda_n e_n, c)$$

that is,

$$B^I(e_1, \dots, e_n, c) = B^I(\lambda_1 e_1, \dots, \lambda_n e_n, c) \ \forall \lambda \in \mathbb{R}^n \ .$$

This entails that B^I doesn't actually depend on $\{e_1, \dots, e_n\}$; moreover, by considering $\lambda = 0$ we get

$$\begin{aligned}\mathrm{id}_{\wedge\mathbb{R}^n} \otimes B^{\varnothing}(c) &= \chi_\lambda\left(\sigma_{\wedge\mathbb{R}^n}\left(S_{\wedge\mathbb{R}^n}\left(e_ie_j \otimes F_c\right)\right)\right) = \sigma_{\wedge\mathbb{R}^n}\left(S_{\wedge\mathbb{R}^n}\left(\chi_{\lambda\mathcal{F}_{\mathrm{loc}}}\left(e_ie_j \otimes F_c\right)\right)\right) \\ &= \mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}}\end{aligned}$$

i.e. $B^{\varnothing}(c) = \mathrm{id}_{\mathfrak{A}}$. Now, by considering that we can write

$$\chi_{\lambda\mathcal{F}_{\mathrm{loc}}}(e_ie_j \otimes F_c) = e_ie_j \otimes F_{\lambda_i\lambda_j c}$$

we conclude that

$$B^I(\lambda_i\lambda_j c) = \lambda_I B^I(c) \text{ for every } I \subset \{1, \dots, n\} \text{ and } \lambda \in \mathbb{R}^n \ .$$

This implies that $B^I(c)(1 - \lambda_I) = 0$ for every $\lambda \in \mathbb{R}^n$ such that $\lambda_\eta = \lambda_\theta = 1$; thus $B^I(c) = 0$ if $I \not\subset \{\eta, \theta\}$, i.e.

$$\sigma_{\wedge\mathbb{R}^n}(S_{\wedge\mathbb{R}^n}(e_ie_j \otimes F_c)) = \mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}} + e_i \otimes B^i(c) + e_j \otimes B^j(c) + e_ie_j \otimes B^{ij}(c) \ .$$

By fixing now $\lambda_i = 0$ and $\lambda_j = 1$ one can show that $B^i(c) = 0$ for all $c \in \mathbb{C}$, and the same happens inverting the role of i and j, i.e. $B^j(c) = 0$; therefore

$$\sigma_{\wedge\mathbb{R}^n}(S_{\wedge\mathbb{R}^n}(e_ie_j \otimes F_c)) = \mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}} + e_ie_j \otimes B^{ij}(c) \ .$$

As by definition we have

$$S_{\wedge\mathbb{R}^n}(e_ie_j \otimes F_c) = \iota_{\wedge\mathbb{R}^n}(e^{i(e_ie_j\otimes c)}) = \mathrm{id}_{\mathfrak{A}_{\wedge\mathbb{R}^n}} + \iota_{\wedge\mathbb{R}^n}(e_ie_j \otimes ic)$$

it also follows that

$$\mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}} + e_ie_j \otimes B^{ij}(c) = \mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}} + \sigma_{\wedge\mathbb{R}^n}(\iota_{\wedge\mathbb{R}^n}(e_ie_j \otimes ic))$$

i.e.

$$e_ie_j \otimes B^{ij}(c) = ic\sigma_{\wedge\mathbb{R}^n}(\iota_{\wedge\mathbb{R}^n}(e_ie_j \otimes 1)) \ ,$$

that is, $B^{ij}(c)$ is $\mathbb{C}$-linear, even in $\mathfrak{A}$ and it holds that

$$-ie_ie_j \otimes B^{ij}(1) = \sigma_{\wedge\mathbb{R}^n}(\iota_{\wedge\mathbb{R}^n}(e_ie_j \otimes 1)) \ .$$

From the fact that for every morphism $\chi\colon \wedge\mathbb{R}^n \to G$ we have $\sigma_G \circ \iota_G \circ \chi_{\mathbb{C}} = \chi \circ \sigma_{\wedge\mathbb{R}^n} \circ \iota_{\wedge\mathbb{R}^n}$ it follows that

$$B^{ij}(1) = B^{\chi(ij)}(1) \ \forall e_i, e_j \in \wedge\mathbb{R}^n \text{ and } \forall \chi\colon \wedge\mathbb{R}^n \to G$$

that is,

(i) by considering any permutation $\sigma \in S_n$ and the associated morphism χ_σ, we have that $B^{ij}(1)$ doesn't depend on the choice of i and j;

(ii) by considering the morphism $\chi\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^{n+2}$,

$$\chi(e_k) \doteq \begin{cases} e_k \wedge e_{n+1}, & k = i \\ -e_k \wedge e_{n+2}, & k = j \\ e_k, & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned} &e_ie_je_{n+1}e_{n+2} \otimes B^{ij}(1)B^{(n+1)(n+2)}(1) = (-ie_ie_j \otimes B^{ij}(1))(-ie_{n+1}e_{n+2} \otimes B^{(n+1)(n+2)}(1)) \\ &= \sigma_{\wedge\mathbb{R}^{n+2}}(\iota_{\wedge\mathbb{R}^{n+2}}(e_ie_je_{n+1}e_{n+2} \otimes 1)) = -ie_ie_je_{n+1}e_{n+2} \otimes B^{ij(n+1)(n+2)}(1) \\ &= -ie_ie_je_{n+1}e_{n+2} \otimes B^{\chi(ij)}(1) \end{aligned}$$

i.e., denoting with $B = B^{ij}(1) = B^{(n+1)(n+2)}(1)$,

$$B^2 = iB \ .$$

By the definition of the involution we also get $B^* = -B$; moreover, as $\{S_{\wedge\mathbb{R}^n}(e_ie_j \otimes F_c))\}_{c\in\mathbb{C}}$ is a central subgroup of $\mathfrak{A}_{\wedge\mathbb{R}^n}$ we have that $B \in Z(\mathfrak{A})$. A non-trivial solution of these equations is given by $B = i\mathrm{id}_{\mathfrak{A}}$, which then yields

$$\sigma_{\wedge\mathbb{R}^n}(S_{\wedge\mathbb{R}^n}(e_ie_j \otimes c)) = \mathrm{id}_{\wedge\mathbb{R}^n\otimes\mathfrak{A}} + ie_ie_j \otimes c \ .$$

There may be other solutions though; an analogous problem occurred in [14], where the central subgroup $\{S(F_c)\}_{c\in\mathbb{R}}$ of S-symbols associated to constant functionals was set to $\{e^{ic}\mathrm{id}\}_{c\in\mathbb{R}}$. Here we do the same: thus in the following we will use $B = i\mathrm{id}_{\mathfrak{A}}$ when needed. ♠

4.3 The canonical anticommutation relations

We now wish to find a way to extend certain functionals in $\mathscr{F}_{\text{loc}}(S^{\oplus}_{r,1}(M))$, acting on $\mathcal{C}(S^{\oplus}_{r,1}(M))$, to objects in $G \otimes \mathscr{F}_{\text{loc}}(S^{\oplus}_{r,1}(M))$ acting on $G \otimes \mathcal{C}(S^{\oplus}_{r,1}(M))$. Recall that $\mathcal{C}(S^{\oplus}_{r,1}(M))$ is $\mathbb{Z}$-graded, and therefore has a natural $\mathbb{Z}_2$-grading; then we can consider, given *homogeneous elements* $\eta, \theta \in G$ and $u, v \in \mathcal{C}(S^{\oplus}_{r,1}(M))$,

$$(\eta \otimes u)(\theta \otimes v) \doteq (-1)^{\deg(\theta)\deg(u)}(\eta\theta) \otimes (u \wedge v) \ .$$

Every functional F induces a map $F_G \colon G \otimes \mathcal{C}(S^{\oplus}_{r,1}(M)) \to G \otimes \mathbb{C}$ given by

$$F_G(\eta \otimes u) = \eta \otimes F(u) \ .$$

We thus define, given $\eta, \theta \in G$ and $u, v \in \mathcal{E}(S^{\oplus}_{r,1}(M))$,

$$(\theta v, u\eta)_G \doteq \theta(v, u)\eta$$

where we used the notation

$$\eta u \doteq (\eta \otimes \mathrm{id}_{\mathcal{C}})(\mathrm{id}_G \otimes u) = -(\mathrm{id}_G \otimes u)(\eta \otimes \mathrm{id}_{\mathcal{C}}) \doteq -u\eta \ .$$

Using this, we can extend the Lagrangian in the following way: given $h = \sum_{i\in I} u^i \eta_i = -\sum_{i \in I} \eta_i \otimes u^i$ with $\{\eta_i\}_{i\in I} \subset G^1$ and $\{u_i\}_{i\in I} \subset \mathcal{E}(S^{\oplus}_{r,1}(M))$ we consider, given $f \in \mathcal{D}(M)$, $\mathscr{L}(f)_G\left(\exp(h)\right)$ (because of the even-rules principle). Notice that as $\mathscr{L}(f) \in \mathscr{F}^2_{\text{loc}}(S^{\oplus}_{r,1}(M))$, using the extension (4.2) we have

$$\mathscr{L}(f)_G\left(\exp(h)\right) = \mathscr{L}(f)_G\left[\frac{hh}{2}\right]$$

and in particular it holds

$$\begin{aligned}
\mathscr{L}(f)_G\left[\frac{1}{2}\sum_{i,j}(u^i\eta_i)(u^j\eta_j)\right] &= \frac{1}{2}\sum_{i\neq j}\eta_j\eta_i \otimes \mathscr{L}(f)[u^i \wedge u^j] = \frac{1}{2}\sum_{i\neq j}\eta_j\left(fu^i, \mathbf{D}^{\oplus}_0(fu^j)\right)\eta_i \\
&= \frac{1}{2}\sum_{i\neq j}\eta_j\left(-(\mathbf{D}^*_0(fu^j_2), fu^i_1) + (fu^i_2, \mathbf{D}_0(fu^j_1))\right)\eta_i \\
&= \frac{1}{2}\sum_{i\neq j}(\mathbf{D}^*_0(fu^j_2)\eta_j, (fu^i_1)\eta_i)_G + (fu^i_2\eta_i, \mathbf{D}_0(fu^j_1\eta_j))_G \\
&= \left(fh_2, \mathbf{D}_0(fh_1)\right)_G \ .
\end{aligned}$$

Given this extension, one can also compute the objects $\delta\mathscr{L}_{\mathbf{h}}$ with $\mathbf{h} = \sum_{j\in J}\theta_j v^j$:

$$\begin{aligned}
\delta\mathscr{L}_{\mathbf{h}}[h] &= (\mathscr{L}(f)_G \circ \exp)^{\mathbf{h}}[h] - (\mathscr{L}(f) \circ \exp)[h] \\
&= (\mathscr{L}(f)_G \circ \exp)[h + \mathbf{h}] - (\mathscr{L}(f)_G \circ \exp)[h] \\
&= \left(f(h_2 + \mathbf{h}_2), \mathbf{D}_0(f(h_1 + \mathbf{h}_1))\right)_G - (fh_2, \mathbf{D}_0(fh_1))_G \\
&= (fh_2, \mathbf{D}_0(f\mathbf{h}_1))_G + (f\mathbf{h}_2, \mathbf{D}_0(f\mathbf{h}_1))_G + (f\mathbf{h}_2, \mathbf{D}_0(fh_1))_G \\
&= (h_2, \mathbf{D}_0\mathbf{h}_1)_G + (\mathbf{h}_2, \mathbf{D}_0\mathbf{h}_1)_G + (\mathbf{h}_2, \mathbf{D}_0h_1)_G \\
&= (h_2, \mathbf{D}_0\mathbf{h}_1)_G + (\mathbf{D}^*_0\mathbf{h}_2, h_1)_G + (\mathbf{h}_2, \mathbf{D}_0\mathbf{h}_1)_G \\
&= \mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0\mathbf{h})[h] + (\mathbf{h}_2, \mathbf{D}_0\mathbf{h}_1)_G
\end{aligned}$$

where we have defined the smeared double Dirac field

$$\mathfrak{D}_G(\mathbf{s})[h] \doteq (h_2, \mathbf{s}_1)_G + (\mathbf{s}_2, h_1)_G$$

and where $\tilde{\mathbf{D}}_0^{\oplus} = \mathbf{D}_0 \oplus \mathbf{D}_0^*$ (notice the sign change with respect to $\mathbf{D}_0^{\oplus} = \mathbf{D}_0 \oplus -\mathbf{D}_0^*$).

Remark 4.3.1. If we define

$$\begin{aligned} \psi(s) \colon \mathcal{E}(S_{r,1}(M)) &\to \quad \mathbb{C} \\ u \quad &\mapsto (s,u) \end{aligned}$$

and

$$\begin{aligned} \overline{\psi(s)} \colon \mathcal{E}(S_{r,1}(M)) &\to \quad \mathbb{C} \\ u \quad &\mapsto (u,s) \end{aligned}$$

and extend them in the same fashion as [12, Section 5] to objects in $G \otimes \mathcal{E}(S_{r,1}(M))$ and $G \otimes \mathcal{D}(S_{r,1}(M))$, then we can make the following observation.
Notice that if $u^i \in \mathcal{E}(S^{\oplus}_{r,1}(M))$ and $v^j \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ in h and $\mathbf{h}$ are given by

$$u^i = (u^i_1, u^i_2) = (u^i, C^{-1}u^i) \qquad v^j = (v^j_1, v^j_2) = (v^j, C^{-1}v^j)$$

for $u^i \in \mathcal{E}(S_{r,1}(M)),\ v^j \in \mathcal{D}(S_{r,1}(M))$ then

$$\begin{aligned} \mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\mathbf{h})[h] &= \sum_{i,j} \Big((u^i\eta_i, \mathbf{D}_0(\theta_j v^j)) + (\mathbf{D}_0(\theta^j v^j), u^i\eta_i) \Big) \\ &= \sum_{i,j} \Big(\psi(\mathbf{D}_0(\theta_j v^j))[u^i\eta_i] - \overline{\psi(\mathbf{D}_0(\theta_j v^j))}[u^i\eta_i] \Big) \\ &= \psi(\mathbf{D}_0\mathbf{h})[h] - \overline{\psi(\mathbf{D}_0\mathbf{h})}[h] \end{aligned}$$

with $\mathbf{D}_0\mathbf{h} = \sum_j \theta_j \mathbf{D}_0 v^j$. Moreover, notice that

$$\mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\mathbf{h})[h] = -(\mathbf{D}_0^{\oplus}\mathbf{h}, h)_G \ .$$

♣

Proposition 4.3.2. *Given $s = \sum_{j=1}^n \eta_j s^j$, $\eta_j \in G^1$ generators of the Grassmann algebra G of dimension n and $s^j \in \mathcal{D}(S^{\oplus}_{r,1}(M))$, we have the decomposition*

$$\sigma_G\left(S_G(\mathfrak{D}_G(s))\right) = \sum_{k=0}^n \frac{i^k}{k!} \sum_{i_1<\dots<i_k} \eta_{i_k}\cdots\eta_{i_1} \otimes B^k(s^{i_1},\dots,s^{i_k})$$

with $B^k \colon \mathcal{D}(S^{\oplus}_{r,1}(M)) \to \mathfrak{A}$ alternating, k-linear and $\mathbb{R}$-linear.

Proof. We proceed as in Remark 4.2.8; as stated there, it suffices to prove the result in the case $G = \wedge\mathbb{R}^n$; we will drop the subscript from now on.
We know that $\sigma(S(\mathfrak{D}(s))) \in \wedge\mathbb{R}^n \otimes \mathfrak{A}$; thus we can write

$$\sigma(S(\mathfrak{D}(s))) = \sum_{I\subset\{1,\dots,n\}} e_I \otimes B^I(s_1,\dots,s_n,e_1,\dots,e_n)$$

with $B^I(s_1,\dots,s_n,e_1,\dots,e_n) \in \mathfrak{A}$. Consider now $\lambda \in \mathbb{R}^n$, and the associated homomorphism $\chi_\lambda\colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^n$ which maps $e_i \mapsto \lambda_i e_i$. Applying that to $\sigma(S(\mathfrak{D}(s)))$ we have

$$\chi_\lambda(\sigma(S(\mathfrak{D}(s)))) = \sum_{I\subset\{1,\dots,n\}} \lambda_I e_I \otimes B^I(s_1,\dots,s_n,e_1,\dots,e_n)\ .$$

On the other hand, we know that $\chi_\lambda \circ \sigma = \sigma \circ \chi_{\lambda\mathfrak{A}}$, where σ now maps to the exterior algebra $\wedge\mathbb{R}^n$ transformed through χ_λ; thus we have

$$\sum_{I\subset\{1,\dots,n\}} \lambda_I e_I \otimes B^I(s_1,\dots,s_n,e_1,\dots,e_n) = \sum_{I\subset\{1,\dots,n\}} \lambda_I e_I \otimes B^I(s_1,\dots,s_n,\lambda_1 e_1,\dots,\lambda_n e_n)$$

i.e.

$$B^I(s_1,\dots,s_n,e_1,\dots,e_n) = B^I(s_1,\dots,s_n,\lambda_1 e_1,\dots,\lambda_n e_n) \text{ for every } I \subset \{1,\dots,n\},\ \lambda \in \mathbb{R}^n\ .$$

Thus B^I doesn't actually depend on $e_1,\dots,e_n$, i.e.

$$B^I(s_1,\dots,s_n,e_1,\dots,e_n) = B^I(s_1,\dots,s_n)\ .$$

Now, notice that $\chi_\lambda(s) = \sum_{i=1}^n (\lambda_i e_i) \otimes s^i = \sum_{i=1}^n e_i \otimes (\lambda_i s^i)$, i.e. we can interpret $\chi_\lambda(s)$ as an object in $\wedge\mathbb{R}^n \otimes \mathcal{C}(S^{\oplus}_{r,1}(M))$ where $\wedge\mathbb{R}^n$ is *not* transformed by χ_λ; in this case we have the equality

$$\sum_{I\subset\{1,\dots,n\}} \lambda_I e_I \otimes B^I(s_1,\dots,s_n) = \sum_{I\subset\{1,\dots,n\}} e_I \otimes B^I(\lambda_1 s_1,\dots,\lambda_n s_n)$$

i.e.

$$\lambda_I B^I(s_1,\dots,s_n) = B^I(\lambda_1 s_1,\dots,\lambda_n s_n)\ .$$

This implies two things:

(i) $B^I(s_1,\dots,s_n)$ depends only on those s_i's such that $i \in I$, i.e.

$$B^I(s_1,\dots,s_n) = B^I(s_{i_1},\dots,s_{i_{|I|}})\ ;$$

(ii) B^I is homogeneous of degree 1 in each of its entries with respect to $\mathbb{R}$.

Notice moreover that if $\lambda = 0 \in \mathbb{R}^n$, we then have $\chi_\lambda(s) = 0 \in G \otimes \mathcal{C}(S^{\oplus}_{r,1}(M))$, and thus

$$B^{\varnothing}(s_1,\dots,s_n) = \chi_0(\sigma(S(\mathfrak{D}(s)))) = \sigma\left(S(\chi_{0\mathcal{F}_{\mathrm{loc}}}(\mathfrak{D}(s)))\right) = \sigma(S(\mathfrak{D}(0))) = \sigma(S(0)) = \mathrm{id}_{G\otimes\mathfrak{A}}$$

i.e.

$$B^{\varnothing}(s_1,\dots,s_n) = \mathrm{id}_{\mathfrak{A}} \text{ for every choice of } s \ .$$

We now wish to show that B^I is linear and antisymmetric in its entries. To do so, consider a permutation $p \in S_n$, and the associated morphism $\chi_p \colon \wedge \mathbb{R}^n \to \wedge \mathbb{R}^n$; we then have

$$\begin{aligned}\sum_{I\subset\{1,\dots,n\}} e_{p(i_1)}\cdots e_{p(i_{|I|})} \otimes B^I(s_{i_1},\dots,s_{i_{|I|}}) &= \chi_p(\sigma(S(\mathfrak{D}(s)))) = \sigma\left(S(\chi_p(\mathfrak{D}(s)))\right) \\ &= \sum_{I\subset\{1,\dots,n\}} e_I \otimes B^I(s^{p^{-1}(i_1)},\dots,s^{p^{-1}(i_{|I|})}) \ .\end{aligned}$$

But as

$$\sum_{I\subset\{1,\dots,n\}} e_{p(i_1)}\cdots e_{p(i_{|I|})} \otimes B^I(s_{i_1},\dots,s_{i_{|I|}}) = \sum_{I\subset\{1,\dots,n\}} \mathrm{sgn}(p|_I) e_I \otimes B^I(s^{i_1},\dots,s^{i_{|I|}})$$

we then conclude that

$$\mathrm{sgn}(p|_I) B^I(s^{i_1},\dots,s^{i_{|I|}}) = B^I(s^{p^{-1}(i_1)},\dots,s^{p^{-1}(i_{|I|})})$$

i.e. B^I is antisymmetric in its entries as desired. Finally, let us show that B^I is $\mathbb{R}$-linear in each entry. To this end, let us consider the morphism $\chi_k \colon \wedge \mathbb{R}^n \to \wedge \mathbb{R}^n$ defined on the generators as

$$\chi_k(e_i) = \begin{cases} e_i & i \neq k+1 \\ e_k & i = k+1 \ . \end{cases}$$

We then have

$$\sum_{I\subset\{1,\dots,n\}} \chi_k(e_I) \otimes B^I(s^{i_1},\dots,s^{i_{|I|}}) = \sigma(S(\mathfrak{D}(\tilde{s})))$$

where $\tilde{s} = \sum_{i=1}^n e_i \tilde{s}^i$,

$$\tilde{s}^i = \begin{cases} s^i & i \neq k \\ s^k + s^{k+1} & i = k \\ 0 & i = k+1 \ . \end{cases}$$

Moreover,

$$\chi_k(e_I) = \begin{cases} e_I & k+1 \neq I \\ e_{i_1}\cdots e_k \cdots e_{i_{|I|}} & k+1 \in I,\ k \notin I \\ 0 & k, k+1 \in I \ . \end{cases}$$

These two entail that if $k+1 \in I, k \notin I$, denoting with $\tilde{I} = \{i_1,\dots,k,\dots,i_{|I|}\}$ we have

$$B^{\tilde{I}}(s^{i_1},\dots,s^k,\dots,s^{i_{|I|}}) + B^{\tilde{I}}(s^{i_1},\dots,s^{k+1},\dots,s^{i_{|I|}}) = B^{\tilde{I}}(s_{i_1},\dots,s_k+s_{k+1},\dots,s_{i_{|I|}}) \ .$$

As k was generic, we have verified that B^I is actually $\mathbb{R}$-linear in each entry.
Finally, consider the following: as a function on $|I|$ entries, the dependence of B^I on I is

due to the elements $s^{i_1},\dots,s^{i_{|I|}}$ acting as its arguments. B^I doesn't actually depend on the chosen entries of I. Thus we can define, for each $k \in \mathbb{N}$,

$$B^k(s^k,\dots,s^1) \doteq \frac{k!}{i^k} B^{\{1,\dots,k\}}(s^1,\dots,s^k)$$

and we can thus write

$$\sigma(S(\mathfrak{D}(s))) = \sum_{k=0}^{n} \frac{i^k}{k!} \sum_{i_1<\dots<i_k} e_{i_k}\cdots e_{i_1} \otimes B^k(s^{i_1},\dots,s^{i_k})$$

with B^k an alternating $\mathbb{R}$- and k-linear map on $\mathcal{D}(S^{\oplus}_{r,1}(M))$ □

Remark 4.3.3. Consider now $s = \theta u$, with $u \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ and $\theta \in G^1$ a generator; according to Proposition 4.3.2 we have

$$\sigma_G(S_G(\mathfrak{D}_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u) \ .$$

Notice that in this case the functional $\mathfrak{D}_G(s) \in (G \otimes \mathcal{F}_{\mathrm{loc}})^0$; therefore we necessarily have $B^1(u) \in \mathfrak{A}^1$. We now apply the involution on both sides: we then have

$$\mathrm{id}_{G\otimes\mathfrak{A}} - i(\theta \otimes B^1(u))^* = \mathrm{id}_{G\otimes\mathfrak{A}} - i(-1)(\theta^* \otimes B^1(u)^*) = \mathrm{id}_{G\otimes\mathfrak{A}} + i(\theta \otimes B^1(u)^*) \ .$$

On the other hand, due to the properties of the involution we have

$$\sigma_G(S_G(\mathfrak{D}_G(s)))^* = \sigma_G(S_G(\mathfrak{D}_G(s))^*) = \sigma_G(S_G(\mathfrak{D}_G(s)^*)^{-1}) = \sigma_G(S_G(\mathfrak{D}_G(s)^*))^{-1} \ .$$

We need to compute $\mathfrak{D}_G(s)^*$: notice that

$$\mathfrak{D}_G(\theta u)[v\eta] = (v_2\eta, \theta u_1)_G + (\theta u_2, v_1\eta)_G = \theta\underbrace{((u_2,v_1) - (v_2,u_1))}_{\mathfrak{D}(u)[v]}\eta$$

and that

$$\begin{aligned}\mathfrak{D}(u)^*[v] &= (-1)\overline{\mathfrak{D}(u)[v^*]} = (-1)\left(\overline{(u_2, Cv_2) - (C^{-1}v_1, u_1)}\right) = \\ &= (-1)\left((v_2, Cu_2) - (C^{-1}u_1, v_1)\right) = -\mathfrak{D}(-u^*)[v] \ .\end{aligned}$$

Therefore

$$\mathfrak{D}_G(s)^* = (\theta \otimes \mathfrak{D}(u))^* = (-1)(\theta^* \otimes \mathfrak{D}(u)^*) = \theta \otimes \mathfrak{D}(-u^*) = \mathfrak{D}_G(-\theta u^*) \ ;$$

if we consider the expansion for $\sigma_G(S_G(\mathfrak{D}_G(s)^*))$ we have

$$\sigma_G(S_G(\mathfrak{D}_G(s)^*)) = \mathrm{id}_{G\otimes\mathfrak{A}} - i\theta \otimes B^1(u^*)$$

whose inverse is given by

$$\mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u^*) \ .$$

Comparing the two expression we conclude that

$$B^1(u)^* = B^1(u^*) \text{ for ever } u \in \mathcal{D}(S^{\oplus}_{r,1}(M)) \ .$$

In particular, if $u = u_f \doteq \begin{pmatrix} f \\ C^{-1}f \end{pmatrix}$ for some $f \in \mathcal{D}(S_{r,1}(M))$ we then have $u_f^* = u_f$ and thus

$$B^1(u_f)^* = B^1(u_f)$$

i.e. for every $f \in \mathcal{D}(S_{r,1}(M))$ $B^1(u_f) \in \mathfrak{A}$ is a $\mathbb{R}$-linear, self-adjoint element. ♠

Remark 4.3.4. Using a procedure analogous to that of Proposition 4.3.2, it is possible to show that given an even functional $F \in \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))^0$, it holds that

$$\sigma_G(S_G(\mathrm{id}_G \otimes F)) = \mathrm{id}_G \otimes A_F$$

for some $A_F \in \mathfrak{A}^0$. Indeed, as previously pointed out we can stick to the case $G = \wedge\mathbb{R}^n$; it then follows that

$$\sigma_G(S_G(\mathrm{id}_G \otimes F)) = \sum_{I \subseteq \{1,\dots,n\}} e_I \otimes A_F^I \ , \qquad \{A_F^I\}_{I \subseteq \{1,\dots,n\}} \subseteq \mathfrak{A} \ .$$

Consider now any $\lambda \in \mathbb{R}^n$, and the morphism $\chi_\lambda \colon \wedge\mathbb{R}^n \to \wedge\mathbb{R}^n$ which acts on the generators as $e_i \mapsto \lambda_i e_i$. Clearly

$$\chi_\lambda(\mathrm{id} \otimes F) = \mathrm{id} \otimes F$$

and

$$\chi_\lambda\left(\sigma(S(\mathrm{id} \otimes F))\right) = \sum_{I \subseteq \{1,\dots,n\}} \lambda_I e_I \otimes A_F^I \ .$$

But due to the properties of the natural transformation σ and to the definition of the morphism $\chi_{\lambda\mathfrak{A}}$ it then follows that

$$\sum_{I \subseteq \{1,\dots,n\}} \lambda_I e_I \otimes A_F^I = \sum_{I \subseteq \{1,\dots,n\}} e_I \otimes A_F^I \iff \sum_{I \subseteq \{1,\dots,n\}} e_I \otimes (\lambda_I - 1) A_F^I = 0 \ \forall \lambda \in \mathbb{R}^n \ .$$

As $\{e_I\}_{I \subseteq \{1,\dots,n\}}$ is a basis of $\wedge\mathbb{R}^n$, it follows that

$$(\lambda_I - 1) A_F^I = 0 \ \forall I \subseteq \{1,\dots,n\}, \ \forall \lambda \in \mathbb{R}^n \ .$$

If $I \neq \varnothing$ then it must hold that $A_F^I = 0$; on the other hand, if $I = \varnothing$ then it may be that $A_F^I \neq 0$. Thus it follows that

$$\sigma(S(\mathrm{id} \otimes F)) = \mathrm{id} \otimes A_F^{\varnothing} \ .$$

As $\mathrm{id} \otimes F \in (\wedge\mathbb{R}^n \otimes \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M)))^0$, by definition $S(\mathrm{id} \otimes F) \in \mathfrak{A}^0_{\wedge\mathbb{R}^n}$, and due to the fact that $\sigma_{\wedge\mathbb{R}^n}$ is a $\mathbb{Z}_2$-graded algebra homomorphism it follows that $A_F^{\varnothing} \in \mathfrak{A}^0$. ♠

Theorem 4.3.5. *The family of self-adjoint, $\mathbb{R}$-linear objects $\{B^1(u_f)\}_{f\in\mathcal{D}(S_{r,1}(M))}\subset\mathfrak{A}$ satisfies the canonical anticommutation relations,*

$$\{B^1(u_f),B^1(u_g)\}=2\mathrm{Re}(f,iS_0g)\ .$$

Proof. Let us consider $s=\sum_i^n\eta_is^i$ and $t=\sum_i^n\theta_jt^j$ with $\{s^i\}_i,\{t^j\}_j\subset\mathcal{D}(S^{\oplus}_{r,1}(M))$ and $\{\eta_i\}_i,\{\theta_j\}_j\subset G^1$. It is possible to decompose

$$s=s'+\tilde{\mathbf{D}}^{\oplus}_0h$$

by considering a Cauchy surface $\Sigma\subset M$ such that $\Sigma\cap J^-(\mathrm{Supp}(t))=\varnothing$ and a smooth function $\chi\in\mathcal{E}(M)$ such that $\chi\equiv1$ on the past Cauchy development $D^-_0(\Sigma)$ and $\chi\equiv0$ on the future Cauchy development $D^+_0(\Sigma)$ of Σ minus an open neighbourhood of Σ, and defining

$$h\doteq\chi\tilde{S}_0^{-\oplus}s\ .$$

Then clearly $\mathrm{Supp}(h)\subseteq J^+_0(\mathrm{Supp}(s))\cap D^-(\Sigma)=J^+_0(\mathrm{Supp}(s))\cap J^-_0(\Sigma\cap J^+_0(\mathrm{Supp}(s)))$, and as $\Sigma\cap J^+_0(\mathrm{Supp}(s))$ is compactly supported because of the global hyperbolicity of (M,g_0) and of the properties of Cauchy surfaces we have that

$$J^+_0(\mathrm{Supp}(s))\cap J^-_0(\Sigma\cap J^+_0(\mathrm{Supp}(s)))\subset\subset M$$

because of global hyperbolicity; thus h is a compactly supported smooth section of $S^{\oplus}_{r,1}(M)$. Then if we define

$$s'\doteq s-\tilde{\mathbf{D}}^{\oplus}_0h$$

we have that $s'\in\mathcal{D}(S^{\oplus}_{r,1}(M))$ and that

$$s'=s-\tilde{\mathbf{D}}^{\oplus}_0h=\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}_0^{-\oplus}s-\tilde{\mathbf{D}}^{\oplus}_0h=\tilde{\mathbf{D}}^{\oplus}_0\left(\underbrace{(1-\chi)\tilde{S}_0^{-\oplus}s}_{\mathrm{Supp}\subset D^+(\Sigma)}\right)$$

i.e. $\mathrm{Supp}(s')\cap J^-_0(\mathrm{Supp}(t))=\varnothing$. Now, recall that

$$\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0h)=\delta\mathscr{L}_h-(h_2,\mathbf{D}_0h_1)_G$$

and thus

$$\mathfrak{D}_G(s)=\mathfrak{D}_G(s')+\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0h)=\mathfrak{D}_G(s')+\delta\mathscr{L}_h-(h_2,\mathbf{D}_0h_1)_G\ .$$

Then it follows that

$$\begin{aligned}S_G(\mathfrak{D}_G(s))&=S_G(\mathfrak{D}_G(s')+\delta\mathscr{L}_h-(h_2,\mathbf{D}_0h_1)_G)\\&\overset{(4.3)}{=}S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s')^{-h}-(h_2,\mathbf{D}_0h_1)_G)\ .\end{aligned}$$

Now, notice that

$$\mathfrak{D}_G(s')^{-h}[u\eta]=\mathfrak{D}_G(s')[u\eta-h]=\mathfrak{D}_G(s')[u\eta]-(h_2,s'_1)_G-(s'_2,h_1)_G$$

and thus

$$S_G(\mathfrak{D}_G(s)) = S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s') - \underbrace{(h_2, s'_1)_G - (s'_2, h_1)_G - (h_2, \mathbf{D}_0 h_1)_G}_{\text{constant functionals}=c}) \ .$$

Consider now

$$\begin{aligned} S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t)) &= S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s') - c)S_G(\mathfrak{D}_G(t)) \\ &\overset{(4.1)}{=} S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s') + \mathfrak{D}_G(t) - c) \\ &= S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s'+t) - c) \ . \end{aligned}$$

Notice that

$$\begin{aligned} \mathfrak{D}_G(s'+t) = \mathfrak{D}_G(s - \tilde{\mathbf{D}}_0^{\oplus} h + t) &= \mathfrak{D}_G(s+t) + \mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}(-h)) \\ &= \mathfrak{D}_G(s+t) + \delta\mathscr{L}_{-h} - (h_2, \mathbf{D}_0 h_1)_G \end{aligned}$$

and using again (4.3) we have

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t)) = S_G(\mathfrak{D}_G(s+t)^h + \delta\mathscr{L}_{-h}^h + \delta\mathscr{L}_h - (h_2, \mathbf{D}_0 h_1)_G - c) \ .$$

Consider now

$$\begin{aligned} \mathfrak{D}_G(s+t)^h - (h_2, \mathbf{D}_0 h_1)_G - c &= \mathfrak{D}_G(s+t) + (h_2, s_1 + t_1)_G + (s_2 + t_2, h_1) \\ &\quad - (h_2, s'_1)_G - (s'_2, h_1)_g - 2(h_2, \mathbf{D}_0 h_1)_G \\ &= \mathfrak{D}_G(s+t) + (h_2, t_1)_G + (t_2, h_1)_G \end{aligned}$$

and

$$\begin{aligned} \delta\mathscr{L}_{-h}^h[u\eta] = \delta\mathscr{L}_{-h}[u\eta + h] &= (\mathscr{L}(f)_G \circ \exp)^{-h}[u\eta + h] - (\mathscr{L}(f)_G \circ \exp)[u\eta + h] \\ &= (\mathscr{L}(f)_G \circ \exp)[u\eta] - (\mathscr{L}(f)_G \circ \exp)[u\eta + h] = -\delta\mathscr{L}_h \ . \end{aligned}$$

We therefore conclude that

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t)) = S_G(\mathfrak{D}_G(s+t) + (h_2, t_1)_G + (t_2, h_1)_G) \ .$$

Recall that $h = \chi \tilde{S}_0^{-\oplus} s$ and thus

$$h_2 = \chi S_0^{-*} s_2 \qquad h_1 = \chi S_0^- s_1$$

and as $\chi \equiv 1$ on $\mathrm{Supp}(t)$ we have

$$(h_2, t_1)_G + (t_2, h_1)_G = (S_0^{-*} s_2, t_1)_G + (t_2, S_0^- s_1)_G \doteq E(s,t)$$

and we have

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t)) = S_G(\mathfrak{D}_G(s+t))S_G(E(s,t)) \ .$$

From this we have

$$\begin{aligned}S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t))S_G(-E(s,t)) &= S_G(\mathfrak{D}_G(s+t)) = S_G(\mathfrak{D}_G(t+s))\\ &= S_G(\mathfrak{D}_G(t))S_G(\mathfrak{D}_G(s))S_G(-E(t,s))\end{aligned}$$

i.e.

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t)) = S_G(\mathfrak{D}_G(t))S_G(\mathfrak{D}_G(s))S_G(E(s,t) - E(t,s))$$

with

$$\begin{aligned}E(s,t) - E(t,s) &= (S_0^{-*}s_2, t_1)_G + (t_2, S_0^- s_1)_G - (S_0^{-*}t_2, s_1)_G - (s_2, S_0^- t_1)_G\\ &= (s_2, S_0^+ t_1)_G + (t_2, S_0^- s_1)_G - (t_2, S_0^+ s_1)_G - (s_2, S_0^- t_1)_G\\ &= -(s_2, S_0 t_1)_G + (t_2, S_0 s_1)_G \ .\end{aligned}$$

Let us now consider the particular case $s = \eta u_f$ and $t = \theta u_g$, with $\eta, \theta \in G^1$ generators; then by Remark 4.3.3 we have

$$\sigma_G(S_G(\mathfrak{D}_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta \otimes B^1(u_f) \qquad \sigma_G(S_G(\mathfrak{D}_G(t))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u_g)$$

and thus

$$\sigma_G(S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u_g) + i\eta \otimes B^1(u_f) + \eta\theta \otimes B^1(u_f)B^1(u_g)$$

and

$$\sigma_G(S_G(\mathfrak{D}_G(t))S_G(\mathfrak{D}_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u_g) + i\eta \otimes B^1(u_f) + \theta\eta \otimes B^1(u_g)B^1(u_f) \ .$$

Moreover, notice that in this case we have that

$$E(s,t) - E(t,s) = -(\eta C^{-1} f, \theta S_0 g)_G + (\theta C^{-1} g, \eta S_0 f)_G = \eta(f, S_0 g)\theta + \eta(g, S_0 f)\theta \ .$$

By considering Remark 4.2.8 we then have

$$\sigma_G(S_G(\eta\theta \otimes c)) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\theta \otimes c \ .$$

We thus have that

$$\begin{aligned}\mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u_g) + i\eta B^1(u_f) + \eta\theta \otimes B^1(u_f)B^1(u_g) &= \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta \otimes B^1(u_g) + i\eta B^1(u_f)\\ &\quad + \eta\theta \otimes \Big(-B^1(u_g)B^1(u_f) + i\left((f, S_0 g) + (g, S_0 f)\right)\Big)\end{aligned}$$

from which it follows that

$$\{B^1(u_f), B^1(u_g)\} = (f, iS_0 g) + (g, iS_0 f) = 2\mathrm{Re}(f, iS_0 g) \doteq 2\mathrm{Re}\,\langle f, g\rangle_{S_0} \ .$$

$\square$

4.4 A C*-norm on a ∗-subalgebra of the universal algebra $\mathfrak{A}$

We now wish to endow a ∗-subalgebra of $\mathfrak{A}$ with a C*-norm. We first consider the unital ∗-subgalgebra $\mathfrak{A}_e$ generated by

$$\mathfrak{A}_e \doteq \left\{A_F,\ F \in \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))^0 \text{ s.t. } F^* = F\right\} \subseteq \mathfrak{A}^0$$

where $A_F \in \mathfrak{A}^0$ is obtained as described in Remark 4.3.4. Notice that as

$$S_G(\mathrm{id}_G \otimes F)^* = S_G(\mathrm{id}_G \otimes F^*)^{-1} = S_G(\mathrm{id}_G \otimes F)^{-1}$$

we have

$$\mathrm{id}_G \otimes A_F^* = \sigma_G(S_G(\mathrm{id}_G \otimes F))^* = \sigma_G(S_G(\mathrm{id}_G \otimes F)^{-1}) = \mathrm{id}_G \otimes A_F^{-1}$$

i.e. A_F is unitary. Now, we consider the set $\{B^1(u_f)\}_{f\in\mathcal{D}(S_{r,1}(M))}$; notice that given any C*-seminorm on the ∗-algebra generated by it we have, if $\mathrm{Re}(f, iS_0 f) = 1$, $B^1(u_f)^2 = \mathrm{id}_{\mathfrak{A}}$; then if $\mathrm{Re}(f, iS_0 f) \neq 0$ it follows that

$$\left\|B^1(u_f)\right\|^2 = \left\|B^1(u_f)^* B^1(u_f)\right\| = \left\|B^1(u_f)^2\right\| \overset{\text{Th. 4.3.5}}{=} \mathrm{Re}\,(f, iS_0 f)\ .$$

Moreover, if $\mathrm{Re}(f, iS_0 f) = 0$ then $f = \mathbf{D}_0\psi$ for some $\psi \in \mathcal{D}(S_{r,1}(M))$: indeed, we know that $(\cdot, iS_0\cdot)\colon \mathcal{D}(S_{r,1}(M)) \times \mathcal{D}(S_{r,1}(M)) \to \mathbb{C}$ is positive semidefinite as shown in [17], and therefore $(f, iS_0 f) \geq 0$ for every $f \in \mathcal{D}(S_{r,1}(M)$; this in turn implies that

$$0 \leq |(g, iS_0 f)| \leq \sqrt{(g, iS_0 g)}\sqrt{(f, iS_0 f)} = \sqrt{(g, iS_0 g)}\sqrt{\mathrm{Re}(f, iS_0 f)} = 0\ \forall g \in \mathcal{D}(S_{r,1}(M))\ .$$

Thus $(g, iS_0 f) = 0$ for every $g \in \mathcal{D}(S_{r,1}(M))$, i.e. $S_0 f = 0$. But one can show that $\ker(S_0) = \mathbf{D}_0\mathcal{D}(S_{r,1}(M))$. Indeed, clearly $\mathbf{D}_0\mathcal{D}(S_{r,1}(M)) \subset \ker(S_0)$; on the other hand, suppose that $f \in \ker(S_0)$. Then $S_0 f = 0$ entails that $S_0^- f = S_0^+ f \doteq \psi$; then we know that

$$\mathrm{Supp}(\psi) \subseteq J^+(\mathrm{Supp}(f)) \cap J^-(\mathrm{Supp}(f))$$

where the last set is compact due to the global hyperbolicity of M. Thus $\psi \in \mathcal{D}(S_{r,1}(M))$, and we have $\mathbf{D}_0\psi = \mathbf{D}_0 S_0^- f = f$, i.e. $f \in \mathbf{D}_0\mathcal{D}(S_{r,1}(M))$. So in particular if we consider f to be a representative of

$$[f] \in \mathcal{D}(S_{r,1}(M))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M))$$

we have that $[f] = 0$, i.e. the positive semidefinite inner product $(\cdot, iS_0\cdot)$ is actually positive definite on it; thus if we restrict the family $\{B^1(u_f)\}_{f\in\mathcal{D}(S_{r,1}(M))}$ to the above quotient space we have that $\mathrm{Re}(f, iS_0 f) = 0$ entails that $B^1(u_f) = 0$, and the C*-seminorm defined by

$$\left\|B^1(u_f)\right\| \doteq \sqrt{\mathrm{Re}(f, iS_0)}$$

is actually a C*-norm.

Consider now the ∗-subalgebra $\mathcal{B}$ of $\mathfrak{A}$ generated by

$$\left\{B^1(u_f)\right\}_{f\in\mathcal{D}(S_{r,1}(M))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M))} \cup \mathfrak{A}_e\ .$$

We have the following result (cfr. [12, Theorem 6.2]):

Proposition 4.4.1. *Every* C* *seminorm on $\mathcal{B}$ is bounded from above.*

Proof. Notice that any element $A \in \mathcal{B}$ can be written as

$$A = \sum_{i=1}^{n} c_i B^1(u_{f_1}) \cdots B^1(u_{f_{j_i}}) A_{F_1} \cdots A_{F_{k_i}} \quad \{j_i\}_{1 \leq i \leq n}, \{k_i\}_{1 \leq i \leq n} \subseteq \mathbb{N} \ .$$

In order to ease the notation we write

$$c_i B^1(u_{f_1}) \cdots B^1(u_{f_{j_i}}) \doteq C_i^{j_i} \qquad A_{F_1} \cdots A_{F_{k_i}} \doteq U_i^{k_i}$$

and

$$A = \sum_{i=1}^{n} C_i^{j_i} U_i^{k_i} \ .$$

Notice that $C_i^{j_i}$ belongs, for every $i \in \mathbb{N}$, to the $*$-algebra generated by $\{B^1(u_f)\}$, while $U_i^{k_i}$ is unitary, being the product of unitary objects. Then consider any C*-seminorm $p \colon \mathcal{B} \to \mathbb{R}^+$: we have

$$p(A) = p\left(\sum_{i=1}^{n} C_i^{j_i} U_i^{k_i}\right) \leq \sum_{i=1}^{n} p\left(C_i^{j_i} U_i^{k_i}\right) \leq \sum_{i=1}^{n} p\left(C_i^{j_i}\right) p\left(U_i^{k_i}\right) \leq \sum_{i=1}^{n} p\left(C_i^{j_i}\right) \ .$$

Then by the previous observation we know that we can write

$$p(C_i^{j_i}) = \left\| C_i^{j_i} \right\|$$

where $\|\cdot\|$ is the C*-norm on the $*$-algebra generated by $\{B^1(u_f)\}$ obtained previously. $\square$

Definition 4.4.2. Let $A \in \mathcal{B}$; we define the map $\|\cdot\|_0 : \mathcal{B} \to \overline{\mathbb{R}}$ as

$$\|A\|_0 \doteq \sup_p p(A)$$

where the supremum is taken over all C*-seminorms $p \colon \mathcal{B} \to \mathbb{R}$.

In the general case, $\|\cdot\|_0$ is a function with values in the extended real numbers; however, in our case $\|A\|_0 < +\infty$ for every $A \in \mathcal{B}$ as shown in Proposition 4.4.1. In this case we call $\|\cdot\|_0$ the *Gelfand-Naimark seminorm*. This is indeed a C*-seminorm, as [34, Proposition 10.1.2, Theorem 9.5.14, Theorem 10.1.3] show.
Notice that the Gelfand-Naimark seminorm is well-defined, as the set of C*-seminorms is non-empty: indeed, consider the *trace state* $\omega_0 \colon \mathcal{B} \to \mathbb{C}$,

$$\omega_0(A) = \begin{cases} 1, & A = \mathrm{id}_{\mathfrak{A}} \\ 0, & \text{otherwise} \ . \end{cases}$$

This state can be used to induce a C*-seminorm on $\mathcal{B}$ by

$$p_0(A) \doteq \sqrt{\omega_0(A^* A)}$$

which is easily seen[5] to actually be a C*-seminorm.

Definition 4.4.3. Let us consider the reducing ideal of $\mathcal{B}$ (usually denoted with $\mathcal{B}_R$), that is, the $*$-stable two-sided ideal given by $\mathcal{B}_R = \{A \in \mathcal{B} \text{ such that } \|A\|_0 = 0\}$. Consider then the quotient $\mathcal{B}/\mathcal{B}_R$, which is again a $*$-algebra; on said algebra define the norm

$$\|[A]\| \doteq \|A\|_0 \ .$$

Then the completition of $(\mathcal{B}/\mathcal{B}_R, \|\cdot\|)$, which we shall simply denote with $\mathfrak{B}$, is a C*-algebra (cfr. [34, Theorem 10.1.11]).

Remark 4.4.4. One may wonder whether $\mathfrak{B}$ is trivial. That this is not the case follows from the fact that

$$p(B^1(u_f)) = \sqrt{\mathrm{Re}(f, iS_0 f)}$$

for every seminorm $p\colon \mathcal{B} \to \mathbb{R}^+$; thus given any $f \in \mathcal{D}(S_{r,1}(M))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M))$ we have that $[B^1(u_f)] \neq 0$. Moreover, in the seminorm p_0 we have

$$p_0(A_F) = \sqrt{\omega_0(A_F^* A_F)} = 1$$

for every $A_F \in \mathfrak{A}_e$; thus $\|A_F\|_0 \neq 0$, and the associated classes are thus different from zero in $\mathfrak{B}$. ♠

4.5 Kinetic perturbations and the canonical anticommutation relations

Given a kinetic perturbation K as of Definition 4.2.1, that is, an object in $\mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M, g_0))$, we can consider its extension to an even functional in $G \otimes \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ given by $\mathrm{id}_G \otimes K$. Then, we define the objects in $\mathfrak{A}_G$

$$S^K_G(F) \doteq S_G(\mathrm{id}_G \otimes K)^{-1} S_G(F + \mathrm{id}_G \otimes K) \ . \tag{4.5}$$

Notice that these objects satisfy an analogous version of the requirements in Definition 4.2.5, indeed

(i) clearly if $F \in (G \otimes \mathcal{F}_{\mathrm{loc}})^0$ then $F + \mathrm{id}_G \otimes K \in (G \otimes \mathcal{F}_{\mathrm{loc}})^0$; thus by (i) in Definition 4.2.5 we have that $S_G(\mathrm{id}_G \otimes K)^{-1} S_G(F + \mathrm{id}_G \otimes K) \in \mathfrak{A}^0_G$. This entails that $\{S^K_G(F)\}_{F \in (G \otimes \mathcal{F}_{\mathrm{loc}})^0} \subseteq \mathfrak{A}^0_G$;

(ii) given $S^K_G(F) \in \mathfrak{A}^\alpha_G$, $S^K_G(H) \in \mathfrak{A}^\beta_G$ we have that

$$S^K_G(F) = S_G(\mathrm{id}_G \otimes K)^{-1} S_G(F + \mathrm{id}_G \otimes K) \qquad S^K_G(H) = S_G(\mathrm{id}_G \otimes K)^{-1} S_G(H + \mathrm{id}_G \otimes K)$$

[5] Subadditivity follows from the Cauchy-Schwarz property of states, [9, Lemma 2.3.10]; as shown in [40], submultiplicativity actually follows from the C*-property which is easily deduced from the definition of ω_0.

and thus $S_G(F+\mathrm{id}_G\otimes K)\in\mathfrak{A}_G^\alpha$, $S_G(H+\mathrm{id}_G\otimes K)\in\mathfrak{A}_G^\beta$. This entails that

$$S_G^K(F)S_G^K(H)=S_G(\mathrm{id}_G\otimes K)^{-1}\underbrace{\underbrace{S_G(F+\mathrm{id}_G\otimes K)S_G(\mathrm{id}_G\otimes K)^{-1}}_{\in\mathfrak{A}_G^\alpha}S_G(H+\mathrm{id}_G\otimes K)}_{\in\mathfrak{A}_G^{\alpha+\beta}}$$

and thus $S_G^K(F)S_G^K(H)\in\mathfrak{A}_G^{\alpha+\beta}$.

(iii) Given a morphism $\chi\colon G\to G'$, we have that the morphism $\chi_{\mathfrak{A}}\colon\mathfrak{A}_G\to\mathfrak{A}_{G'}$ acts on $S_G^K(F)$ in the following way:

$$\begin{aligned}\chi_{\mathfrak{A}}(S_G^K(F))&=\chi_{\mathfrak{A}}\left(S_G(\mathrm{id}_G\otimes K)^{-1}\right)\chi_{\mathfrak{A}}\left(S_G(F+\mathrm{id}_G\otimes K)\right)\\&=S_{G'}(\mathrm{id}_{G'}\otimes K)^{-1}S_{G'}(\chi_{\mathscr{F}_{\mathrm{loc}}}(F)+\mathrm{id}_{G'}\otimes K)\\&=S_{G'}^K(\chi_{\mathscr{F}_{\mathrm{loc}}}(F))\ ;\end{aligned}$$

(iv) Given $P,Q,N\in(G\otimes\mathscr{F}_{\mathrm{loc}})^0$ with N such that $N+K$ is a well-defined kinetic perturbation, then

$$S_G^K(P+Q+N)=S_G(\mathrm{id}_G\otimes K)^{-1}S_G(P+Q+N+\mathrm{id}_G\otimes K)$$

and if $\mathrm{Supp}(P)\cap J^-_{N+K}(\mathrm{Supp}(Q))=\varnothing$ we then have

$$\begin{aligned}S_G^K(P+Q+N)&\overset{(4.1)}{=}S_G(\mathrm{id}_G\otimes K)^{-1}\\&\qquad S_G(P+N+\mathrm{id}_G\otimes K)S_G(N+\mathrm{id}_G\otimes K)^{-1}S_G(Q+N+\mathrm{id}_G\otimes K)\\&=\ S_G^K(P+N)S_G^K(N)^{-1}S_G^K(Q+N)\ .\end{aligned}$$

Notice that if $N=0$ then it follows that if $\mathrm{Supp}(P)\cap J^-_K(\mathrm{Supp}(Q))=\varnothing$ then

$$S_G^K(P+Q)=S_G^K(P)S_G^K(Q)$$

and thus if P and Q are spacelike separated with respect to the causal structure induced by K we have

$$S_G^K(P)S_G^K(Q)=S_G^K(Q)S_G^K(P)\ ;$$

(v) Let us define the functional

$$\delta\mathscr{L}_{\mathbf{h}}^K\doteq\delta\mathscr{L}_{\mathbf{h}}+(\mathrm{id}_G\otimes K)^{\mathbf{h}}-(\mathrm{id}_G\otimes K)$$

where $(\mathrm{id}_G \otimes K)^{\mathbf{h}}$ is defined according to the same procedure as before; then

$$\begin{aligned}
S_G^K(\delta\mathscr{L}_{\mathbf{h}}^K)S_G^K(F) &= S_G(\mathrm{id}_G \otimes K)^{-1}S_G(\delta\mathscr{L}_{\mathbf{h}} + (\mathrm{id}_G \otimes K)^{\mathbf{h}}) \\
&\quad S_G(\mathrm{id}_G \otimes K)^{-1}S_G(F + \mathrm{id}_G \otimes K) \\
&\overset{(4.3)}{=} S_G(\mathrm{id}_G \otimes K)^{-1}S_G(\delta\mathscr{L}_{\mathbf{h}})S_G(F + \mathrm{id}_G \otimes K) \\
&= S_G(\mathrm{id}_G \otimes K)^{-1} \\
&\quad S_G(F^{\mathbf{h}} + \delta\mathscr{L}_{\mathbf{h}} + (\mathrm{id}_G \otimes K)^{\mathbf{h}} - \mathrm{id}_G \otimes K + \mathrm{id}_G \otimes K) \\
&= S_G^K(F^{\mathbf{h}} + \delta\mathscr{L}_{\mathbf{h}}^K) \ .
\end{aligned}$$

The same goes for $S_G^K(F)S_G^K(\delta\mathscr{L}_{\mathbf{h}}^K)$.

Notice thus that the symbols $\{S_G^K(F)\}_{F\in G\otimes\mathscr{F}_{\mathrm{loc}}} \subseteq \mathfrak{A}_G$ generate a subalgebra $\mathfrak{A}_G^K$ in $\mathfrak{A}_G$ which obeys the same rules as the algebra $\mathfrak{A}_G$, only for an altered version of spacetime (the metric g_0 is replaced with g_K) and the Lagrangian considered in the dynamical relation is given by $\mathscr{L} + K$. We can also define another family of injective morphisms $\{\iota_G^K \colon G_{\mathbb{C}} \to \mathfrak{A}_G^K\}_{G\in\mathfrak{Grass}}$, by requiring that

$$\iota_G^K(e^{i\eta\otimes z}) = S_G^K(\eta \otimes F_z) \ \forall z \in \mathbb{C}$$

for every $G \in G$. By repeating the same steps carried out after Definition 4.2.5 one gets the correct behaviour with respect to morphisms.
Consider now $s = \sum_{i=1}^n \eta_i s^i$, with $\{\eta_i\} \subseteq G^1$ generators of the Grassmann algebra G of dimension n and $\{s_i\} \subseteq \mathcal{D}(S_{r,1}^{\oplus}(M, g_0))$; we wish to study

$$S_G^K(\mathfrak{D}_G(s)) \text{ and } S_G^K(\mathfrak{D}_G(s))S_G^K(\mathfrak{D}_G(t)) \ ,$$

with $t = \sum_{j=1}^n \theta_j t^j$. The first result is a decomposition similar to that of Proposition 4.3.2: indeed, as $S_G^K(\mathfrak{D}_G(s)) \in \mathfrak{A}_G$, we can map it to $G \otimes \mathfrak{A}$ using σ_G, and we obtain:

Proposition 4.5.1.

$$\sigma_G\left(S_G^K(\mathfrak{D}_G(s))\right) = \sum_{k=0}^n \frac{i^k}{k!} \sum_{i_k<\cdots<i_i} \eta_{i_k}\cdots\eta_{i_1} \otimes B_K^k(s^{i_1}, \ldots, s^{i_k})$$

with $B^k \colon \mathcal{D}(S_{r,1}^{\oplus}(M, g_0)) \to \mathfrak{A}$ alternating, k-linear and $\mathbb{R}$-linear.

Proof. The proof is the same as that of Proposition 4.3.2. □

Remark 4.5.2. Consider $s = \eta u$, with $\eta \in G^1$ generator and $u \in \mathcal{D}(S_{r,1}^{\oplus}(M, g_0))$; then by Proposition 4.5.1 we have

$$\sigma_G(S_G^K(\mathfrak{D}_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta \otimes B_K^1(u)$$

with $B_K^1(u) \in \mathfrak{A}^1$ as $\mathfrak{D}_G(u) \in (G \otimes \mathscr{F}_{\mathrm{loc}}(S_{r,1}^{\oplus}(M, g_0)))^0$. By applying the involution on both sides we have

$$\sigma_G(S_G^K(\mathfrak{D}_G(s)))^* = \mathrm{id}_{G\otimes\mathfrak{A}} - i(\eta \otimes B_K^1(u))^* = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta \otimes B_K^1(u)^* \ .$$

On the other hand, we know that

$$\begin{aligned}\sigma_G(S_G^K(\mathfrak{D}_G(s)))^* &= \sigma_G\left((S_G^K(\mathfrak{D}_G(s)))^*\right)\\ &= \sigma_G\left((S_G(\mathrm{id}_G\otimes K)^{-1}S_G(\mathfrak{D}_G(s)+\mathrm{id}_G\otimes K))^*\right)\\ &= \sigma_G\left(S_G(\mathfrak{D}_G(s)^*+(\mathrm{id}_G\otimes K)^*)^{-1}S_G((\mathrm{id}_G\otimes K)^*)\right)\\ &= \sigma_G\left(S_G(\mathfrak{D}_G(s)^*+\mathrm{id}_G\otimes K^*)^{-1}S_G(\mathrm{id}_G\otimes K^*)\right)\ .\end{aligned}$$

Recall that by Lemma 4.2.2 it holds that $K^*=K$, and thus

$$\sigma_G(S_G^K(\mathfrak{D}_G(s)))^* = \sigma_G\left(S_G(\mathfrak{D}_G(s)^*+\mathrm{id}_G\otimes K)^{-1}S_G(\mathrm{id}_G\otimes K)\right) = \sigma_G(S_G^K(\mathfrak{D}_G(s)^*))^{-1}\ .$$

We have already shown that $\mathfrak{D}_G(s)^*=\mathfrak{D}_G(-\eta u^*)$; therefore

$$\sigma_G(S_G^K(\mathfrak{D}_G(s)^*))^{-1} = \left(1-i\eta\otimes B_K^1(u^*)\right)^{-1} = 1+i\eta\otimes B_K^1(u^*)$$

and thus it follows that $B_K^1(u)^*=B_K^1(u^*)$, and again, if $u=u_f$ then

$$B_K^1(u_f)^* = B_K^1(u_f)\ .$$

♠

Theorem 4.5.3. *The family of self-adjoint, $\mathbb{R}$-linear objects $\{B_K^1(u_f)\}_{f\in\mathcal{D}(S_{r,1}(M,g_0))}\subset\mathfrak{A}$ satisfies the perturbed canonical anticommutation relations,*

$$\{B_K^1(u_f),B_K^1(u_g)\} = 2\mathrm{Re}(\tau_0^1 f, iS_K\tau_0^1 g)_1\ .$$

Proof. As in the proof of Theorem 4.3.5, let us start by considering $s=\sum_i^n\eta_i s^i$ and $t=\sum_j^n\theta_j t^j$, and studying

$$S_G^K(\mathfrak{D}_G(s))S_G^K(\mathfrak{D}_G(t))\ .$$

In the following, we denote with μ the Radon-Nikodym derivative of $d\mu_{g_1}$ with respect to $d\mu_{g_0}$, which in this case is a smooth, non-vanishing function given by

$$\mu = \sqrt{\frac{\det g_1}{\det g_0}}\ .$$

As in the previous case, given a Cauchy surface Σ such that $\Sigma\cap J_K^-(\mathrm{Supp}(t))=\varnothing$ and a smooth function χ such that $\chi\equiv 1$ on $D_K^-(\Sigma)$ and $\chi\equiv 0$ on $D_K^+(\Sigma)\setminus U_\Sigma$, we consider a decomposition

$$s = s' + \mu\tau_1^0\tilde{\mathbf{D}}_K^{\oplus}\tau_0^1\mathbf{h}$$

with $\mathbf{h}$ and s' given by $\mathbf{h}=\chi\tau_1^0\tilde{S}_K^{-\,\oplus}\tau_0^1(s\mu^{-1})$ and

$$\begin{aligned}s' &\doteq \mu(s\mu^{-1}) - \mu\tau_1^0\tilde{\mathbf{D}}_K^{\oplus}\tau_0^1\mathbf{h} = \mu\tau_1^0(\tilde{\mathbf{D}}_K^{\oplus}\tilde{S}_K^{-\,\oplus})\tau_0^1(s\mu^{-1}) - \mu\tau_1^0\tilde{\mathbf{D}}_K^{\oplus}\tau_0^1\left(\chi\tau_1^0\tilde{S}_K^{-\,\oplus}\tau_0^1(s\mu^{-1})\right)\\ &= \mu\tau_1^0\tilde{\mathbf{D}}_K^{\oplus}\left(1-\chi\right)\tilde{S}_K^{-\,\oplus}\tau_0^1(s\mu^{-1})\end{aligned}$$

The first equality shows that $s' \in G\otimes\mathcal{D}(S^{\oplus}_{r,1}(M))$, while the last one shows that $\mathrm{Supp}(s')\cap J^-_K(\mathrm{Supp}(t)) = \varnothing$. Notice that this equality follows from the fact that for each $x \in M$ the map

$$\tau_0^1 \colon S^{\oplus}_{r,1}(M, g_0)_x \to S^{\oplus}_{r,1}(M, g_1)_x$$

is linear, and thus $\tau_0^1 \colon \mathcal{E}(S^{\oplus}_{r,1}(M, g_0)) \to \mathcal{E}(S^{\oplus}_{r,1}(M, g_1))$ is tensorial. Therefore,

$$S^K_G(\mathfrak{D}_G(s)) = S^K_G(\mathfrak{D}_G(s' + \mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h})) = S^K_G(\mathfrak{D}_G(s') + \mathfrak{D}_G(\mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h}))\ .$$

We now need to understand $\mathfrak{D}_G(\mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h})$. To do so, recall that we consider $\delta\mathscr{L}^K_{\mathbf{h}}$ in the dynamical relation; therefore, let us examine that object. Recall that

$$\delta\mathscr{L}^K_{\mathbf{h}}[h] = \delta\mathscr{L}_{\mathbf{h}} + (\mathrm{id}_G \otimes K)^{\mathbf{h}} - (\mathrm{id}_G \otimes K)$$

and therefore we need to understand $(\mathrm{id}_G \otimes K)^{\mathbf{h}}$. This is given by

$$\begin{aligned}((\mathrm{id}_G \otimes K)\circ \exp)^{\mathbf{h}}[h] &= ((\mathrm{id}_G \otimes K) \circ \exp)[h + \mathbf{h}] \\ &= (\tau_0^1(f(h+\mathbf{h}))_2, \mathbf{D}_K\tau_0^1(f(h+\mathbf{h}))_1)_{1_G} - (f(h+\mathbf{h})_2, \mathbf{D}_0(f(h+\mathbf{h})_1))_{0_G}\end{aligned}$$

with $f \in \mathcal{D}(M)$ such that $f \equiv 1$ on $\mathrm{Supp}(g_K - g_0)$. We can in particular pick $f \equiv 1$ on $\mathrm{Supp}(g_K - g_0) \cup \mathrm{Supp}(\mathbf{h})$, which then yields

$$\begin{aligned}((\mathrm{id}_G \otimes K)\circ \exp)^{\mathbf{h}}[h] &= ((\mathrm{id}_G \otimes K) \circ \exp)[h] + \underbrace{(\tau_0^1 h_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G} + (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1 h_1)_{1_G}}_{\mathfrak{D}_G(\mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h})[h]} \\ &\quad + (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G} \underbrace{-(h_2, \mathbf{D}_0\mathbf{h}_1)_{0_G} - (\mathbf{h}_2, \mathbf{D}_0 h_1)_{0_G}}_{-\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0\mathbf{h})[h]} - (\mathbf{h}_2, \mathbf{D}_0\mathbf{h}_1)_{0_G} \\ &= \mathfrak{D}_G(\mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h})[h] + ((\mathrm{id}_G \otimes K) \circ \exp)[h] + (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G} - \delta\mathscr{L}_{\mathbf{h}}[h]\end{aligned}$$

i.e.

$$\delta\mathscr{L}^K_{\mathbf{h}} = \mathfrak{D}_G(\mu\tau_1^0\tilde{\mathbf{D}}^{\oplus}_K\tau_0^1\mathbf{h}) + (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G}\ .$$

Thus

$$\begin{aligned}S^K_G(\mathfrak{D}_G(s)) = S^K_G\left(\mathfrak{D}_G(s') + \delta\mathscr{L}^K_{\mathbf{h}} - (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G}\right) = S^K_G(\delta\mathscr{L}^K_{\mathbf{h}})S^K_G(\mathfrak{D}_G(s')^{-\mathbf{h}} \\ - (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G})\ .\end{aligned}$$

Now,

$$\mathfrak{D}_G(s')^{-\mathbf{h}}[h] = \mathfrak{D}_G(s')[h - \mathbf{h}] = \mathfrak{D}_G(s')[h] - (\mathbf{h}_2, s'_1)_{0_G} - (s'_2, \mathbf{h}_1)_{0_G}$$

and we conclude that

$$\begin{aligned}S^K_G(\mathfrak{D}_G(s)) = S^K_G(\delta\mathscr{L}^K_{\mathbf{h}})S^K_G\Big(&\mathfrak{D}_G(s') \\ &- \underbrace{(\mathbf{h}_2, s'_1)_{0_G} - (s'_2, \mathbf{h}_1)_{0_G} - (\tau_0^1\mathbf{h}_2, \mathbf{D}_K\tau_0^1\mathbf{h}_1)_{1_G}}_{\text{constant functional } c}\Big)\ .\end{aligned}$$

Thus

$$\begin{aligned}
S^K_G(\mathfrak{D}_G(s))S^K_G(\mathfrak{D}_G(t)) &= S^K_G(\delta\mathscr{L}^K_{\mathbf{h}})S^K_G(\mathfrak{D}_G(s'+t)-c)\\
&= S^K_G(\delta\mathscr{L}^K_{\mathbf{h}})S^K_G(\mathfrak{D}_G(s+t)-\mathfrak{D}_G(\tau^0_1\tilde{\mathbf{D}}^{\oplus}_K\tau^1_0 h)-c)\\
&= S^K_G(\delta\mathscr{L}^K_{\mathbf{h}})S^K_G(\mathfrak{D}_G(s+t)+\delta\mathscr{L}^K_{-\mathbf{h}}-(\tau^1_0\mathbf{h}_2,\mathbf{D}_K\tau^1_0\mathbf{h}_1)_{1G}-c)\\
&= S^K_G(\mathfrak{D}_G(s+t)^{\mathbf{h}}+\delta\mathscr{L}^{K\ \mathbf{h}}_{-\mathbf{h}}-(\tau^1_0\mathbf{h}_2,\mathbf{D}_K\tau^1_0\mathbf{h}_1)_{1G}-c+\delta\mathscr{L}^K_{\mathbf{h}})\ .
\end{aligned}$$

Now, notice that

$$\delta\mathscr{L}^{K\ \mathbf{h}}_{-\mathbf{h}} = \delta\mathscr{L}^{\mathbf{h}}_{-\mathbf{h}} + (K^{-\mathbf{h}})^{\mathbf{h}} - K^{\mathbf{h}} = -\delta\mathscr{L}_{\mathbf{h}} + K - K^{\mathbf{h}} = -\delta\mathscr{L}^K_{\mathbf{h}}$$

and that

$$\mathfrak{D}_G(s+t)^{\mathbf{h}} - c - (\tau^1_0\mathbf{h}_2,\mathbf{D}_K\tau^1_0\mathbf{h}_1)_1 = \mathfrak{D}_G(s+t) + (\mathbf{h}_2,t_1)_{0G} + (t_2,\mathbf{h}_1)_{0G}\ .$$

These imply that

$$S^K_G(\mathfrak{D}_G(s))S^K_G(\mathfrak{D}_G(t)) = S^K_G(\mathfrak{D}_G(s+t)+(\mathbf{h}_2,t_1)_{0G}+(t_2,\mathbf{h}_1)_{0G})\ .$$

Now, recall that $\mathbf{h} = \chi\tau^0_1\tilde{S}^{-\ \oplus}_K\tau^1_0(s\mu^{-1})$ with $\chi\equiv 1$ on $\mathrm{Supp}(t)$. Therefore

$$(\mathbf{h}_2,t_1)_{0G} + (t_2,\mathbf{h}_1)_{0G} = (S^{-\ *}_K\tau^1_0 s_2,\tau^1_0 t_1)_{1G} + (\tau^1_0 t_2, S^-_K\tau^1_0 s_1)_{1G} \doteq E_K(s,t)\ .$$

Then we can show that

$$S^K_G(\mathfrak{D}_G(s))S^k_G(\mathfrak{D}_G(t)) = S^K_G(\mathfrak{D}_G(t))S^K_G(\mathfrak{D}_G(s))S^K_G(E_K(s,t)-E_K(t,s))$$

with

$$\begin{aligned}
E_K(s,t)-E_K(t,s) &= (S^{-\ *}_K\tau^1_0 s_2,\tau^1_0 t_1)_{1G} + (\tau^1_0 t_2,S^-_K\tau^1_0 s_1)_{1G} - (S^{-\ *}_K\tau^1_0 t_2,\tau^1_0 s_1)_{1G}\\
&\quad - (\tau^1_0 s_2,S^-_K\tau^1_0 t_1)_{1G}\\
&= -(\tau^1_0 s_2,S_K\tau^1_0 t_1)_{1G} + (\tau^1_0 t_2,S_K\tau^1_0 s_1)_{1G}\ .
\end{aligned}$$

By considering $s=\eta u_f$ and $t=\theta u_g$ with $\eta,\theta\in G^1$ generators of the Grassmann algebra and $f,g\in\mathcal{D}(S_{r,1}(M,g_0))$, thanks to Remark 4.5.2 we have that

$$\sigma_G(S^K_G(\mathfrak{D}^K_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B^1_K(u_f) \qquad \sigma_G(S^K_G(\mathfrak{D}^K_G(t))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\theta\otimes B^1_K(u_g)$$

which imply that

$$\sigma_G(S^K_G(\mathfrak{D}^K_G(s))S^K_G(\mathfrak{D}^K_G(t))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B^1_K(u_f) + i\theta\otimes B^1_K(u_g) + \eta\theta\otimes B^1_K(u_f)B^1_K(u_g)$$

and

$$\sigma_G(S^K_G(\mathfrak{D}^K_G(t))S^K_G(\mathfrak{D}^K_G(s))) = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B^1_K(u_f) + i\theta\otimes B^1_K(u_g) + \theta\eta\otimes B^1_K(u_g)B^1_K(u_f)\ .$$

Moreover, by Remark4.2.8 we know that

$$\begin{aligned}\sigma_G\Big(S_G^K(E_K(s,t)-E_K(t,s))\Big) &= \mathrm{id}_{G\otimes\mathfrak{A}} \\ &\quad + i\eta\theta\otimes\Big((\tau_0^1 f, S_K\tau_0^1 g)_1 + (\tau_0^1 g, S_K\tau_0^1 f)_1\Big) \ .\end{aligned}$$

Therefore

$$\begin{aligned}&\mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B_K^1(u_f) + i\theta\otimes B_K^1(u_g) + \eta\theta\otimes B_K^1(u_f)B_K^1(u_g) \\ &\quad = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B_K^1(u_f) + i\theta\otimes B_K^1(u_g) + \theta\eta\otimes B_K^1(u_g)B_K^1(u_f) \\ &\quad + \eta\theta\otimes\Big(-B_K^1(u_g)B_K^1(u_f) + i\Big((\tau_0^1 f, S_K\tau_0^1 g)_1 + (\tau_0^1 g, S_K\tau_0^1 f)_1\Big)\Big)\end{aligned}$$

that is,

$$\begin{aligned}\{B_K^1(u_f), B_K^1(u_g)\} &= (\tau_0^1 f, iS_K\tau_0^1 g)_1 + (\tau_0^1 g, iS_K\tau_0^1 f)_1 = 2\mathrm{Re}(\tau_0^1 f, iS_K\tau_0^1 g)_1 \\ &\doteq 2\mathrm{Re}\,\langle f, g\rangle_{S_K} \ .\end{aligned}$$

□

5 Automorphisms of the CAR algebra and their unitary implementation

5.1 Transformations induced by quadratic functionals

We now wish to analyze, given a gauge-invariant functional $Q \in \mathscr{F}^2_{\text{loc}}(S^{\oplus}_{r,1}(M, g_0))$ such that $Q^* = Q$, the behavior of the adjoint action of objects $S_G(\text{id}_G \otimes Q) \in \mathfrak{A}_G$ on the symbols of the form $S_G(\mathfrak{D}_G(s))$, with $s = \sum_i^n \eta_i s^i$, $\{\eta_i\}_i$ generators of the n-dimensional Grassmann algebra G and $\{s^i\} \subset \mathcal{D}(S^{\oplus}_{r,1}(M, g_0))$.
The first result is the following one:

Lemma 5.1.1. *Given $F \in (G \otimes \mathscr{F}_{\text{loc}})^0$, then*

$$S_G(\mathfrak{D}_G(s))S_G(F) = S_G(\mathfrak{D}_G(s) + F^{\tilde{S}_0^{-\oplus}s}) \qquad S_G(F)S_G(\mathfrak{D}_G(s)) = S_G(\mathfrak{D}_G(s) + F^{\tilde{S}_0^{+\oplus}s}) \ .$$

Proof. As in the proofs of Theorems 4.3.5 and 4.5.3, consider a decomposition

$$s = s' + \tilde{\mathbf{D}}_0^{\oplus} h$$

with $\text{Supp}(s') \cap J_0^+(\text{Supp}(F)) = \varnothing$. Then we know that

$$\mathfrak{D}_G(s) = \mathfrak{D}_G(s' + \tilde{\mathbf{D}}_0^{\oplus} h) = \mathfrak{D}_G(s') + \delta\mathscr{L}_h - (h_2, \mathbf{D}_0 h_1)_G$$

and that

$$\begin{aligned} S_G(\mathfrak{D}_G(s)) &= S_G(\mathfrak{D}_G(s') + \delta\mathscr{L}_h - (h_2, \mathbf{D}_0 h_1)_G) = S_G(\delta\mathscr{L}_h) S_G(\mathfrak{D}_G(s')^{-h} - (h_2, \mathbf{D}_0 h_1)_G) \\ &= S_G(\delta\mathscr{L}_h) S_G(\mathfrak{D}_G(s') - \underbrace{(h_2, s_1')_G - (s_2', h_1)_G - (h_2, \mathbf{D}_0 h_1)_G}_{\text{constant functional}=c}) \ . \end{aligned}$$

Then

$$\begin{aligned} S_G(\mathfrak{D}_G(s))S_G(F) &= S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s') - c)S_G(F) = S_G(\delta\mathscr{L}_h)S_G(\mathfrak{D}_G(s') + F - c) \\ &= S_G(\mathfrak{D}_G(s')^h + F^h - c + \delta\mathscr{L}_h) = S_G(\mathfrak{D}_G(s')^h + F^h - c + \mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus} h) + (h_2, \mathbf{D}_0 h_1)_G) \\ &= S_G(\mathfrak{D}_G(s) + F^h + (h_2, s_1')_G + (s_2', h_1)_G + (h_2, \mathbf{D}_0 h_1)_G - c) = S_G(\mathfrak{D}_G(s) + F^h) \ . \end{aligned}$$

Now, recall that $h = \chi \tilde{S}_0^{-\oplus} s$, with $\chi \equiv 1$ on $\text{Supp}(F)$; therefore $F^h = F^{\tilde{S}_0^{-\oplus}s}$, i.e.

$$S_G(\mathfrak{D}_G(s))S_G(F) = S_G(\mathfrak{D}_G(s) + F^{\tilde{S}_0^{-\oplus}s}) \ .$$

The second equality follows from similar computations and by considering the decomposition

$$h \doteq \chi \tilde{S}_0^{+\oplus} s \qquad s' \doteq s - \tilde{\mathbf{D}}_0^{\oplus} h$$

with $\chi \in \mathcal{E}(M)$ such that $\chi \equiv 1$ on the future Cauchy development $D^+(\Theta)$ of a Cauchy surface Θ with $\Theta \cap J_0^+(\text{Supp}(F)) = \varnothing$ and $\chi \equiv 0$ on $D^-(\Theta) \setminus U$, with U an open neighbourhood of M. $\square$

Corollary 5.1.2. *If $s=\eta u$, then it follows that for any $F\in(G\otimes\mathcal{F}_{\mathrm{loc}})^0$*

$$S_G(\mathfrak{D}_G(s))S_G(F)S_G(\mathfrak{D}_G(s))^{-1}=S_G(F^{\tilde{S}^{\oplus}s})\ .$$

Proof. Notice that by the proof of Theorem 4.3.5 we have

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(t))=S_G(\mathfrak{D}_G(s+t))S_G(E(s,t))$$

with $E(s,t)=(S_0^{-*}s_2,t_1)_G+(t_2,S_0^{-}s_1)_G$. Now, notice that if $t=-\eta u$ then we have $s+t=0$ and thus $\mathfrak{D}_G(s+t)=\mathfrak{D}_G(0)=0\in G\otimes\mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M,g_0))$. At the same time, we have $E(s,t)=0$ due to it containing objects of the form $\eta\eta$ with $\eta\in G^1$; thus

$$S_G(\mathfrak{D}_G(s))S_G(\mathfrak{D}_G(-s))=S_G(\mathfrak{D}_G(-s))S_G(\mathfrak{D}_G(s))=\mathrm{id}_{\mathfrak{A}_G}\ .$$

This entails, together with Lemma 5.1.1,

$$\begin{aligned}S_G(\mathfrak{D}_G(s))S_G(F)S_G(\mathfrak{D}_G(s))^{-1}&=S_G(\mathfrak{D}_G(s)+F^{\tilde{S}_0^{-\oplus}s})S_G(\mathfrak{D}_G(-s))\\&=S_G(\mathfrak{D}_G(-s)+\underbrace{\mathfrak{D}_G(s)^{-\tilde{S}_0^{+\oplus}s}}_{=\mathfrak{D}_G(s)\text{ because of }\eta\eta\text{ terms}}+F^{\tilde{S}_0^{-\oplus}s-\tilde{S}_0^{+\oplus}s})\\&=S_G(\mathfrak{D}_G(-s)+\mathfrak{D}_G(s)+F^{\tilde{S}_0^{\oplus}s})=S_G(F^{\tilde{S}_0^{\oplus}s})\ .\end{aligned}$$

□

We wish now to compute, given $Q\in\mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M,g_0))$,

$$S_G(\mathrm{id}_G\otimes Q)^{-1}S_G(\mathfrak{D}_G(s))S_G(\mathrm{id}_G\otimes Q)\ .$$

By Lemma 5.1.1 we have

$$S_G(\mathrm{id}_G\otimes Q)^{-1}S_G(\mathfrak{D}_G(s))S_G(\mathrm{id}_G\otimes Q)=S_G(\mathrm{id}_G\otimes Q)^{-1}S_G(\mathfrak{D}_G(s)+(\mathrm{id}_G\otimes Q)^{\tilde{S}_0^{-\oplus}s})\ .$$

We thus need to compute $(\mathrm{id}_G\otimes Q)^{\tilde{S}_0^{-\oplus}s}$: this is given by, on elements $h=\sum_i\theta_i\otimes h^i$,

$$\begin{aligned}((\mathrm{id}_G\otimes Q)\circ\exp)^{\tilde{S}_0^{-\oplus}s}[h]&=((\mathrm{id}_G\otimes Q)\circ\exp)[h+\tilde{S}_0^{-\oplus}s]\\&=(\mathrm{id}_G\otimes Q)\left[\frac{1}{2}\left(hh+(\tilde{S}_0^{-\oplus}s)(\tilde{S}_0^{-\oplus}s)+(h(\tilde{S}_0^{-\oplus}s)+(\tilde{S}_0^{-\oplus}s)h)\right)\right]\\&=((\mathrm{id}_G\otimes Q)\circ\exp)[h]+\frac{1}{2}(\mathrm{id}_G\otimes Q)[(\tilde{S}_0^{-\oplus}s)(\tilde{S}_0^{-\oplus}s)]\\&\quad+\frac{1}{2}(\mathrm{id}_G\otimes Q)[(\tilde{S}_0^{-\oplus}s)h+h(\tilde{S}_0^{-\oplus}s)]\ .\end{aligned}$$

For the last term, notice that both h and $\tilde{S}_0^{-\oplus}s$ are even, and thus they commute: we thus have

$$\frac{1}{2}(\mathrm{id}_G\otimes Q)[h(\tilde{S}_0^{-\oplus}s)+(\tilde{S}_0^{-\oplus}s)h]=(\mathrm{id}_G\otimes Q)[h(\tilde{S}_0^{-\oplus}s)]\ .$$

Notice that if $s = \eta u$ we have
$$(\tilde{S}_0^{-\oplus} s)(\tilde{S}_0^{-\oplus} s) = 0$$
and thus in this case
$$((\mathrm{id}_G \otimes Q) \circ \exp)^{\tilde{S}_0^{-\oplus} s}[h] = ((\mathrm{id}_G \otimes Q) \circ \exp)[h] + (\mathrm{id}_G \otimes Q)[h(\tilde{S}_0^{-\oplus} s)] \ .$$
Recall that $h = \sum_i \theta_i \otimes h^i$; we thus have
$$(\mathrm{id}_G \otimes Q)[h(\tilde{S}_0^{-\oplus} s)] = -\sum_i \theta_i \eta \otimes Q[h^i \wedge \tilde{S}_0^{-\oplus} u] \ .$$

Now, notice that $Q \in \mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M, g_0))$, i.e. $Q \in \mathcal{E}^{a\prime}(M^2, S_{r,1}(M)^{\oplus} \boxtimes S_{r,1}(M)^{\oplus})$; thus by the Schwartz kernel Theorem (see for instance [44, Theorem 1.5.1]) we know that there exists a continuous linear operator $P_Q^{\oplus} \colon \mathcal{E}(S^{\oplus}_{r,1}(M)) \to \mathcal{E}(S^{\oplus}_{r,1}(M))$ such that
$$Q[h^i \wedge \tilde{S}_0^{-\oplus} u] = (h^i, P_Q^{\oplus} \tilde{S}_0^{-\oplus} u) \ .$$

Clearly $P_Q^{\oplus}$ is a partial differential operator with smooth, compactly supported coefficients; moreover, the following holds:

Lemma 5.1.3. *Given $Q \in \mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M, g_0))$ such that $Q = Q^*$ and Q is gauge-invariant, then*
$$P_Q^{\oplus} = P_Q \oplus -P_Q^* \qquad P_Q^* = C^{-1} \circ P_Q \circ C \ .$$

Proof. Notice that we can write the operator $P_Q^{\oplus} \colon \mathcal{E}(S^{\oplus}_{r,1}(M)) \to \mathcal{E}(S^{\oplus}_{r,1}(M))$ as
$$P_Q^{\oplus} = \begin{pmatrix} P_Q^{11} & P_Q^{12} \\ P_Q^{21} & P_Q^{22} \end{pmatrix}$$
with the operators $P_Q^{11} \colon \mathcal{E}(S_{r,1}(M)) \to \mathcal{E}(S_{r,1}(M))$, $P_Q^{12} \colon \mathcal{E}(\overline{S_{r,1}(M)}) \to \mathcal{E}(S_{r,1}(M))$, $P_Q^{21} \colon \mathcal{E}(S_{r,1}(M)) \to \mathcal{E}(\overline{S_{r,1}(M)})$ and $P_Q^{22} \colon \mathcal{E}(\overline{S_{r,1}(M)}) \to \mathcal{E}(\overline{S_{r,1}(M)})$.

(i) We know that Q is gauge-invariant; therefore for any $t \in \mathbb{R}$
$$\begin{aligned}(u, P_Q^{\oplus} v) = Q[u \wedge v] = Q[\chi_t u \wedge \chi_t v] = (\chi_t u, P_Q^{\oplus} \chi_t v) = (e^{-it} u_2, e^{it} P_Q^{11} v_1) \\ + (e^{-it} u_2, e^{-it} P_Q^{12} v_2) + (e^{it} P_Q^{21} v_1, e^{it} u_1) + (e^{-it} P_Q^{22} v_2, e^{it} u_1) \\ = (u_2, P_Q^{11} v_1) + (P_Q^{22} v_2, u_1) + e^{-i2t}(u_2, P_Q^{12} v_2) + e^{i2t}(P_Q^{21} v_1, u_1) \ .\end{aligned}$$

In order for gauge-invariance to hold we thus need $P_Q^{12} = P_Q^{21} = 0$.

(ii) We know that Q is skew-symmetric; therefore
$$\begin{aligned}(u_2, P_Q^{11} v_1) + (P_Q^{22} v_2, u_1) = Q[u \wedge v] = -Q[v \wedge u] = -(v_2, P_Q^{11} u_1) - (P_Q^{22} u_2, v_1) \\ = (-P_Q^{11*} v_2, u_1) + (u_2, -P_Q^{22*} v_1)\end{aligned}$$

which entails $P_Q^{22} = -P_Q^{11*}$ and $P_Q^{11} = -P_Q^{22*}$.

(iii) Finally, we know that $Q = Q^*$; therefore,

$$\begin{aligned}
(P_Q^{11*}u_2, v_1) + (v_2, P_Q^{22*}u_1) &= (u_2, P_Q^{11}v_1) + (P_Q^{22}v_2, u_1) = Q[u \wedge v] = Q^*[u \wedge v] = \\
&= \overline{(C^{-1}v_1, P_Q^{11}Cu_2)} + \overline{(P_Q^{22}C^{-1}u_1, Cv_2)} \\
&= (C^{-1}P_Q^{11}Cu_2, v_1) + (v_2, CP_Q^{22}C^{-1}u_1)
\end{aligned}$$

i.e. $P_Q^{11*} = C^{-1}P_Q^{11}C$ and $P_Q^{22*} = CP_Q^{22}C^{-1}$.

$\square$

Therefore,

$$\begin{aligned}
(\mathrm{id}_G \otimes Q)[h(\tilde{S}_0^{-\oplus}s)] &= -\sum_i \theta_i\eta \otimes (h^i, P_Q^{\oplus}\tilde{S}_0^{-\oplus}u) = (h_2, P_Q S_0^- s_1)_G + (P_Q^* S_0^{-*}s_2, h_1)_G \\
&= \mathfrak{D}_G(\tilde{P}_Q\tilde{S}_0^{-\oplus}s)[h]
\end{aligned}$$

with $\tilde{P}_Q^{\oplus} = P_Q \oplus P_Q^*$ (notice the sign change with respect to $P_Q^{\oplus}$). Therefore,

$$\mathfrak{D}_G(s)[h]+(\mathrm{id}_G\otimes Q)[h(\tilde{S}_0^{-\oplus}s)] = \mathfrak{D}_G(s)[h]+\mathfrak{D}_G(\tilde{P}_Q\tilde{S}_0^{-\oplus}s)[h] = \mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{-\oplus}s\right)[h]$$

and

$$\begin{aligned}
&S_G(\mathrm{id}_G \otimes Q)^{-1}S_G(\mathfrak{D}_G(s) + (\mathrm{id}_G \otimes Q)^{\tilde{S}_0^{-\oplus}s}) \\
&\quad = S_G(\mathrm{id}_G \otimes Q)^{-1}S_G\left(\mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{-\oplus}s\right) + \mathrm{id}_G \otimes Q\right) \\
&\quad = S_G^Q\left(\mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{-\oplus}s\right)\right) \ .
\end{aligned}$$

We can assume the operator $\mathbf{D}_0 + P_Q$ is Green hyperbolic[6]; then there exist retarded and advanced propagators $\tilde{S}_Q^{\pm\oplus} \colon \mathcal{D}(S_{r,1}^{\oplus}(M, g_0)) \to \mathcal{E}(S_{r,1}^{\oplus}(M, g_0))$ such that

$$\left(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus}\right)\tilde{S}_Q^{\pm\oplus} = \tilde{S}_Q^{\pm\oplus}\left(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus}\right)\Big|_{\mathcal{D}(S_{r,1}^{\oplus}(M))} = \mathrm{id}_{\mathcal{D}(S_{r,1}^{\oplus}(M))} \ .$$

Notice that given $u \in \mathcal{D}(S_{r,1}^{\oplus}(M))$,

$$\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{\pm\oplus}u = u - \underbrace{\tilde{P}_Q^{\oplus}\tilde{S}_Q^{\pm\oplus}u}_{\mathrm{Supp}\subset\subset M} \ .$$

Now, consider that

$$\mathfrak{D}_G(s) = \mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_Q^{+\oplus}s\right) = \mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{+\oplus}\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s\right)$$

[6] This is the case whenever Q is a kinetic perturbation for instance, or when Q represents the interaction with an external potential.

where the equality is due to the fact that propagators map compactly supported sections to spacelike compact sections, and the fact that the identity (2.14) for propagators can be extended to spacelike compact sections. Then, given $s = \eta u$,

$$\begin{aligned}\mathfrak{D}_G(s) + \mathrm{id}_G \otimes Q &= \mathfrak{D}_G\left((\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{-\oplus}\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s\right) + \mathrm{id}_G \otimes Q \\ &= \mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s) + \mathfrak{D}_G(\tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus}\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s) + \mathrm{id}_G \otimes Q \\ &= \mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s) + (\mathrm{id}_G \otimes Q)^{\tilde{S}_0^{+\oplus}\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s} \ .\end{aligned} \tag{5.1}$$

By Lemma 5.1.1 we then have

$$\begin{aligned}S_G^Q(\mathfrak{D}_G(s)) &= S_G(\mathrm{id}_G \otimes Q)^{-1}S_G(\mathfrak{D}_G(s) + \mathrm{id}_G \otimes Q) \\ &= S_G(\mathrm{id}_G \otimes Q)^{-1}S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s) + (\mathrm{id}_G \otimes Q)^{\tilde{S}_0^{+\oplus}\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s}) \\ &= S_G(\mathrm{id}_G \otimes Q)^{-1}S_G(\mathrm{id}_G \otimes Q)S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s)) \\ &= S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}s)) \ .\end{aligned} \tag{5.2}$$

With this we conclude that whenever $s = \eta u$,

$$S_G(\mathrm{id}_G \otimes Q)^{-1}S_G(\mathfrak{D}_G(s))S_G(\mathrm{id}_G \otimes Q) = S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})S_0^{-\oplus}s)) \ .$$

Notice that the operator $T_Q^{\oplus\,-1} \doteq \tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})S_0^{-\oplus}$ admits an inverse given by

$$T_Q^{\oplus} \doteq \tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{-\oplus}\left(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus}\right)\tilde{S}_0^{+\oplus}$$

and both of these are linear operators acting on $\mathcal{D}(S_{r,1}^{\oplus}(M))$. Notice that clearly $T_Q^{\oplus}$ has values in $\mathcal{D}(S_{r,1}^{\oplus}(M))$: indeed, we know that $\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{-\oplus}$ maps compactly supported functions to compactly supported functions as shown before, and

$$\left(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus}\right)\tilde{S}_0^{\oplus} = \mathrm{id}_{\mathcal{D}(S_{r,1}^{\oplus}(M))} + \underbrace{\tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus}}_{\text{maps to } \mathcal{D}(S_{r,1}^{\oplus}(M))}$$

We can thus rewrite

$$S_G(\mathrm{id}_G \otimes Q)S_G(\mathfrak{D}_G(s))S_G(\mathrm{id}_G \otimes Q)^{-1} = S_G(\mathfrak{D}_G(T_Q^{\oplus}s)) \ . \tag{5.3}$$

Proposition 5.1.4. *Given a quadratic perturbation $Q \in \mathscr{F}^2_{\mathrm{loc}}(S_{r,1}^{\oplus}(M, g_0))$, there exists an object $A_Q \in \mathfrak{A}^0$ such that*

$$A_Q B^1(u) A_Q^{-1} = B^1(T_Q^{\oplus}u) \ .$$

Proof. Let us apply $\sigma_G\colon \mathfrak{A}_G \to G\otimes\mathfrak{A}$ to (5.3) in the case $s=\eta u$: it yields

$$\sigma_G(S_G(\mathrm{id}_G\otimes Q))\sigma_G(S_G(\mathfrak{D}_G(s)))\sigma_G(S_G(\mathrm{id}_G\otimes Q))^{-1} = \sigma_G(S_G(\mathfrak{D}_G(T_Q^{\oplus}s)))\ .$$

By Proposition 4.3.2 we have

$$\sigma_G(S_G(\mathrm{id}_G\otimes Q))\left(\mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B^1(u)\right)\sigma_G(S_G(\mathrm{id}_G\otimes Q))^{-1} = \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta\otimes B^1(T_Q^{\oplus}u)$$

i.e.

$$\sigma_G(S_G(\mathrm{id}_G\otimes Q))\eta\otimes B^1(u)\sigma_G(S_G(\mathrm{id}_G\otimes Q))^{-1} = \eta\otimes B^1(T_Q^{\oplus}u)\ .$$

By Remark 4.3.4 we know that

$$\sigma_G(S_G(\mathrm{id}_G\otimes Q)) = \mathrm{id}_G\otimes A_Q$$

with $A_Q\in\mathfrak{A}^0$; it then clearly holds that

$$\sigma_G(S_G(\mathrm{id}_G\otimes Q))^{-1} = \mathrm{id}_G\otimes A_Q^{-1}\ .$$

Then

$$\eta\otimes B^1(T_Q^{\oplus}u) = \sigma_G(S_G(\mathrm{id}_G\otimes Q))\eta\otimes B^1(u)\sigma_G(S_G(\mathrm{id}_G\otimes Q))^{-1} = \eta\otimes(A_Q B^1(u)A_Q^{-1})\ .$$

From this we conclude that there exists $A_Q\in\mathfrak{A}^0$ such that

$$A_Q B^1(u) A_Q^{-1} = B^1(T_Q^{\oplus}u)\ .$$

$\square$

The operators $T_Q^{\oplus}$ induced by quadratic functionals $Q\in\mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M,g_0))$ have a particular behaviour:

Proposition 5.1.5. *Given $P,Q,N\in\mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ such that*

$$\mathrm{Supp}(P)\cap J_N^-(\mathrm{Supp}(Q)) = \varnothing$$

then

$$T^{\oplus}_{P+N}{T^{\oplus}_N}^{-1}T^{\oplus}_{N+Q} = T^{\oplus}_{N+P+Q}$$

(provided that the objects $P+N$, $N+Q$ and $N+P+Q$ are well-defined as functionals in $\mathcal{K}$).

Proof. We prove the result in the following way: first of all, let us define

$${T_P^N}^{\oplus} \doteq {T_N^{\oplus}}^{-1}T^{\oplus}_{N+P}\ .$$

We will show that under the hypothesis, the relation

$${T_P^N}^{\oplus}{T_Q^N}^{\oplus} = {T_{P+Q}^N}^{\oplus}$$

holds. Now, notice that on test functions it holds that

$$T_Q^{N\oplus}(\tilde{\mathbf{D}}_0^\oplus \tilde{S}_N^{+\oplus}) = \tilde{\mathbf{D}}_0^\oplus \tilde{S}_N^{+\oplus}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\overbrace{\tilde{S}_0^{-\oplus}\tilde{\mathbf{D}}_0^\oplus \tilde{S}_{N+Q}^{-\oplus}}^{\tilde{S}_{N+Q}^{-\oplus}}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)\underbrace{\tilde{S}_0^{+\oplus}\tilde{\mathbf{D}}_0^\oplus \tilde{S}_N^{+\oplus}}_{\tilde{S}_N^{+\oplus}}$$
$$= (\tilde{\mathbf{D}}_0^\oplus \tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_{N+Q}^{-\oplus}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)\tilde{S}_N^{+\oplus}\right] . \tag{5.4}$$

As $\mathrm{Supp}(P) \cap J_N^-(\mathrm{Supp}(Q)) = \varnothing$, there exist a Cauchy hypersurface Σ and an open neighbourhood $U \subset M$ of it such that

$$U \cap J_N^+(\mathrm{Supp}(P)) = U \cap J_N^-(\mathrm{Supp}(Q)) = \varnothing .$$

Let us consider a smooth function χ such that $\chi \equiv 1$ on $J_N^-(\Sigma) \setminus U$ and $\chi \equiv 0$ on $J_N^+(\Sigma)$. Now, let us consider a function $f \in \mathcal{D}(S_{r,1}^\oplus(M))$ such that $\mathrm{Supp}(f) \subset U$, and let us study $T_Q^{N\oplus} f$. We define

$$g = (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)\tilde{S}_N^{+\oplus} f = f + \tilde{P}_Q^\oplus \tilde{S}_N^{+\oplus} f .$$

Using χ and g, we define $h_Q \doteq \chi(\tilde{S}_{N+Q}^{-\oplus} g)$. Notice how h_Q is supported in $(J_N^-(\Sigma) \cap J_{N+Q}^+(\mathrm{Supp}(g))$ which is compact. Then we consider

$$g_Q \doteq g - (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)h_Q$$

which is a smooth, compactly supported function whose support lies in U. Thus we have a decomposition

$$g = g_Q + (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)h_Q$$

for $g_Q, h_Q \in \mathcal{D}(S_{r,1}^\oplus(M))$ with $\mathrm{Supp}(g_Q) \subset U$ and $\mathrm{Supp}(h_Q) \cap J_N^+(\mathrm{Supp}(P)) = \varnothing$. This entails that

$$(\tilde{S}_{N+Q}^{-\oplus} - \tilde{S}_N^{-\oplus})g_Q = \tilde{S}_{N+Q}^{-\oplus}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_N^{-\oplus} g_Q - \underbrace{\tilde{S}_{N+Q}^{-\oplus}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)\tilde{S}_N^{-\oplus} g_Q}_{\tilde{S}_N^{-\oplus} g_Q \text{ is past compact}}$$
$$= -\tilde{S}_{N+Q}^{-\oplus}\tilde{P}_Q^\oplus \tilde{S}_N^{-\oplus} g_Q = 0$$

where the last term vanishes because $J_N^+(\mathrm{Supp}(g_Q)) \cap \mathrm{Supp}(Q) = \varnothing$, as $\mathrm{Supp}(g_Q) \subset U$; therefore

$$(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_{N+Q}^{-\oplus} g_Q = (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_N^{-\oplus} g_Q = g_Q$$

which yields

$$\begin{aligned}
&(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_{N+Q}^{-\oplus}(\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)\tilde{S}_N^{+\oplus} f \\
&\quad = (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)\tilde{S}_{N+Q}^{-\oplus}(g_Q + (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus + \tilde{P}_Q^\oplus)h_Q) \\
&\quad = g_Q + (\tilde{\mathbf{D}}_0^\oplus + \tilde{P}_N^\oplus)h_Q .
\end{aligned}$$

This yields

$$T_Q^{N\oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})f = (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})(g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q) \ .$$

Now we consider

$$\begin{aligned} T_P^{N\oplus}T_Q^{N\oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})f &= T_P^{N\oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})(g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q) \\ &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_P^{\oplus})\tilde{S}_N^{+\oplus}\right](g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q) \\ &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}\left[g_Q + \tilde{P}_P^{\oplus}\tilde{S}_N^{+\oplus}g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q + \tilde{P}_P^{\oplus}h_Q\right] \ . \end{aligned}$$

By the way h_Q is constructed and by the properties of Σ, we have that $\mathrm{Supp}(P) \cap \mathrm{Supp}(h_Q) = \varnothing$, and thus $\tilde{P}_P^{\oplus}h_Q = 0$; moreover, as $\mathrm{Supp}(g_Q) \subset U$ and as $J_N^+(\mathrm{Supp}(P)) \cap U = \varnothing$, we also have that $\tilde{P}_P^{\oplus}\tilde{S}_N^{+\oplus}g_Q = 0$. Thus we have

$$\begin{aligned} T_P^{N\oplus}T_Q^{N\oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})f &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}\left[g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q\right] \\ &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q\right] \ . \end{aligned}$$

By definition, $g_Q = g - (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_Q^{\oplus})h_Q$, which can be written as

$$g_Q = (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_Q^{\oplus})\left(\tilde{S}_N^{+\oplus}f - h_Q\right) \ .$$

As $\mathrm{Supp}(f) \subset U$, we have that $\tilde{S}_N^{+\oplus}f$ is supported in $J_N^-(U)$; thus $\tilde{P}_P^{\oplus}\tilde{S}_N^{+\oplus}f = 0$. Therefore we can actually write

$$g_Q = (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_Q^{\oplus} + \tilde{P}_P^{\oplus})\left(\tilde{S}_N^{+\oplus}f - h_Q\right) \ .$$

Moreover, we also have

$$\begin{aligned} \left(\tilde{S}_{N+P+Q}^{-\ \oplus} - \tilde{S}_{N+P}^{-\ \oplus}\right)g_Q &= \tilde{S}_{N+P+Q}^{-\ \oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_P^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}g_Q \\ &\quad - \tilde{S}_{N+P+Q}^{-\ \oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_P^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}g_Q \\ &= -\tilde{S}_{N+P+Q}^{-\ \oplus}\tilde{P}_Q^{\oplus}\tilde{S}_{N+P}^{-\ \oplus}g_Q = 0 \end{aligned}$$

where the last equality follows from the fact that $\mathrm{Supp}(g_Q) \subset U$ and that $\mathrm{Supp}(Q) \cap J_{N+P}^+(U) = \varnothing$. These relations yield

$$\begin{aligned} T_P^{N\oplus}T_Q^{N\oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})f &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P}^{-\ \oplus}g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q\right] \\ &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P+Q}^{-\ \oplus}g_Q + (\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})h_Q\right] \\ &= (\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})\left[(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus})\tilde{S}_{N+P+Q}^{-\ \oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_N^{\oplus} + \tilde{P}_P^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_N^{+\oplus}f\right] \ . \end{aligned}$$

But notice that the last expression is equal to

$$T_{P+Q}^{N\ \oplus}(\tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_N^{+\oplus})f \ .$$

We therefore have our assertion if $\mathrm{Supp}(f) \subset U$, where U is a neighbourhood of the Cauchy surface Σ separating P and Q. But any function $f \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ can be written as $f = f_U + (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N)g_U$, with $\mathrm{Supp}(f_U) \subset U$ and $f_U, g_U \in \mathcal{D}(S^{\oplus}_{r,1}(M))$. Then this decomposition yields

$$T^{N\oplus}_P T^{N\oplus}_Q (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) f = T^{N\oplus}_P T^{N\oplus}_Q (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(f_U + (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N)g_U) = T^{N}_{P+Q}{}^{\oplus} (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) f_U \\ + T^{N\oplus}_P T^{N\oplus}_Q (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \ .$$

If we consider the last term, we have

$$\begin{aligned} & T^{N\oplus}_P T^{N\oplus}_Q (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \\ & = T^{N\oplus}_P (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) \left[(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) \tilde{S}^{-}_{N+Q}{}^{\oplus} (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N + \tilde{P}^{\oplus}_Q) \tilde{S}^{+\oplus}_N \right] (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \\ & = T^{N\oplus}_P (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \\ & = (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) \left[(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) \tilde{S}^{-}_{N+P}{}^{\oplus} (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N + \tilde{P}^{\oplus}_P) \tilde{S}^{+\oplus}_N \right] (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \\ & = (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U = T^{N}_{P+Q}{}^{\oplus} (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N)(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) g_U \end{aligned}$$

which yields the result

$$T^{N\oplus}_P T^{N\oplus}_Q (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) f = T^{N}_{P+Q}{}^{\oplus} (\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) f \ .$$

Now, notice that given any function $f \in \mathcal{D}(S^{\oplus}_{r,1}(M))$ we can define the function $g_f \in \mathcal{D}(S^{\oplus}_{r,1}(M))$,

$$g_f \doteq (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N) \tilde{S}^{+\oplus}_0 f$$

which is such that

$$(\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N) g_f = \tilde{\mathbf{D}}^{\oplus}_0 \circ \underbrace{\tilde{S}^{+\oplus}_N (\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N)}_{\text{id on future-compact functions}} \circ \tilde{S}^{+\oplus}_0 f = f \ .$$

It therefore holds that

$$T^{N\oplus}_P T^{N\oplus}_Q f = T^{N}_{P+Q}{}^{\oplus} f$$

for any $f \in \mathcal{D}(S^{\oplus}_{r,1}(M))$. But this yields

$$T^{\oplus}_N{}^{-1} T^{\oplus}_{N+P} T^{\oplus}_N{}^{-1} T^{\oplus}_{N+Q} = T^{\oplus}_N{}^{-1} T^{\oplus}_{N+P+Q} \iff T^{\oplus}_{N+P} T^{\oplus}_N{}^{-1} T^{\oplus}_{N+Q} = T^{\oplus}_{N+P+Q} \ .$$

$\square$

Remark 5.1.6. Notice that if $\mathrm{Supp}(P) \cap J^{-}_0(\mathrm{Supp}(Q)) = \varnothing$, then Proposition 5.1.5 simply yields

$$T^{\oplus}_P T^{\oplus}_Q = T^{\oplus}_{P+Q} \ .$$

♠

We now wish to understand the behaviour of $T_Q^{\oplus}$ more thoroughly. To do so, we need to expand on the construction of the complex Clifford algebra (see [4] and [30]) and on Bogoliubov transformations, and for the sake of convenience we will use the self-dual CAR algebra, as introduced by [3].

5.2 The self-dual CAR algebra and its quasifree states

First we restate that the inner product $\langle\cdot,\cdot\rangle_{S_0}$, which we recall is given by

$$\langle f,g\rangle_{S_0} = (f, iS_0 g)$$

is positive semidefinite on $\mathcal{D}(S_{r,1}(M))$ [17]. As $\langle\cdot,\cdot\rangle_{S_0}$ is positive semidefinite, we then have

$$0 \leq \left|\langle f,g\rangle_{S_0}\right| \leq \|f\|_{S_0}\|g\|_{S_0} \ .$$

Thus if $\|g\|_{S_0} = 0$ $\langle f,g\rangle_{S_0} = 0$ for every $f \in \mathcal{D}(S_{r,1}(M))$, i.e. $S_0 g = 0$. But this entails that $g \in \mathbf{D}_0\mathcal{D}(S_{r,1}(M))$ (cfr. 4.4). Thus $\mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0}$ is positive definite, symmetric, $\mathbb{R}$-bilinear on the quotient space

$$V \doteq \mathcal{D}(S_{r,1}(M))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M)) \qquad Q\colon \mathcal{D}(S_{r,1}(M)) \to V$$

which is a real vector space endowed with the scalar product $\mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0}$. We can then consider its completion

$$\mathcal{H} \doteq \overline{V}^{\mathrm{Re}\langle\cdot,\cdot\rangle_{S_0}}$$

which yields a real, infinite dimensional Hilbert space. The family of $\mathbb{R}$-linear operators $\{B^1(u_f)\}_{f\in\mathcal{D}(S_{r,1}(M,g_0))}$ can then be restricted to V to yield an $\mathbb{R}$-linear map defined on a dense subset of $\mathcal{H}$,

$$V \ni f \mapsto B^1(u_f) \in \mathfrak{A} \ .$$

Thus $\mathfrak{A}$ contains a family of self-adjoint, real-linear objects $\{B^1(u_f)\}_{f\in V}$ defined on a dense subset of the real Hilbert space $(\mathcal{H}, \mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0})$ and which satisfy

$$\left\{B^1(u_f), B^1(u_g)\right\} = 2\mathrm{Re}\,\langle f,g\rangle_{S_0} \ .$$

This can be used to define a norm on the $*$-subalgebra of $\mathfrak{A}$ generated by $\{B^1(u_f)\}_{f\in V}$; its completion is the so-called *Clifford algebra* $\mathfrak{C}(\mathcal{H}, \mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0})$ built on $(\mathcal{H}, \mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0})$.
As mentioned, the self-dual CAR algebra is better suited to study Bogoliubov transformations. Notice that the space

$$\left\{ u_f = \begin{pmatrix} f \\ C^{-1}f \end{pmatrix},\ f \in \mathcal{D}(S_{r,1}(M,g_0)) \right\}$$

is the real space Re_Γ of the complex vector space $\mathcal{D}(S_{r,1}^{\oplus}(M,g_0)) \simeq \mathcal{D}(S_{r,1}(M,g_0)) \oplus \mathcal{D}(\overline{S_{r,1}(M,g_0)})$ endowed with the conjugation

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \overset{\Gamma}{\mapsto} \begin{pmatrix} Cu_2 \\ C^{-1}u_1 \end{pmatrix}$$

We can endow $\mathcal{D}(S^{\oplus}_{r,1}(M,g_0))$ with the positive semidefinite inner product

$$\langle u,v\rangle_{\oplus} \doteq \frac{1}{2}\left((u_1, iS_0 v_1) + (u_2, -iS_0^* v_2)\right)$$

which is such that the restriction of $\langle\cdot,\cdot\rangle_{\oplus}$ to Re_{Γ} is given by $\mathrm{Re}\,\langle\cdot,\cdot\rangle_{S_0}$. By an argument analogous to the previous one, if we consider

$$V^{\oplus} = \mathcal{D}(S_{r,1}(M,g_0))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M,g_0)) \oplus \mathcal{D}(\overline{S_{r,1}(M,g_0)})/\mathbf{D}_0^*\mathcal{D}(\overline{S_{r,1}(M,g_0)})$$

with projection $Q\colon \mathcal{D}(S_{r,1}(M,g_0) \oplus \mathcal{D}(\overline{S_{r,1}(M,g_0)}) \to V^{\oplus}$, then $\langle\cdot,\cdot\rangle_{\oplus}$ is a positive definite hermitian inner product. Notice that Γ descends to a well-defined map on $V^{\oplus}$. On such quotient space, we can consider the symbols

$$\psi(u) \doteq \frac{1}{\sqrt{2}}\Big(B^1(\mathrm{Re}u) + iB^1(\mathrm{Im}u)\Big) \qquad \mathrm{Re}u = \frac{1}{2}\left(u + \Gamma u\right) \quad \mathrm{Im}u = \frac{1}{2i}\left(u - \Gamma u\right)$$

where both $\mathrm{Re}u$ and $\mathrm{Im}u$ belong to Re_{Γ}. Then define also

$$\psi(u)^* \doteq \frac{1}{\sqrt{2}}\Big(B^1(\mathrm{Re}u) - iB^1(\mathrm{Im}u)\Big) \ .$$

Notice that $\psi(u)^* = \psi(\Gamma u)$ and that

$$\{\psi(u)^*, \psi(v)\} = \langle u,v\rangle_{\oplus} \ .$$

Thus the symbols $\{\psi(u)\}_{u\in V^{\oplus}}$ generate a $*$-subalgebra of $\mathfrak{A}$ deemed *self-dual CAR algebra* and usually denoted with $\mathfrak{A}_{SDC}(\mathcal{H}^{\oplus},\Gamma)$, where $\mathcal{H}^{\oplus} = \overline{V}^{\langle\cdot,\cdot\rangle_{\oplus}}$.
Following [2], a state $\omega\colon \mathfrak{A}_{SDC}(\mathcal{H}^{\oplus},\Gamma) \to \mathbb{C}$ is said to be quasi-free if

$$\begin{aligned}
&\omega(\psi(u_1)\cdots\psi(u_{2n+1})) = 0\\
&\quad \omega(\psi(u_1)\cdots\psi(u_{2n})) = (-1)^{n(n-1)/2}\sum_{p\in\widehat{S_{2n}}}\mathrm{sgn}(p)\prod_{i=1}^{n}\omega(\psi(u_{p(i)})\psi(u_{p(i+n)}))
\end{aligned}$$

where $\widehat{S_{2n}}$ denotes the subset of S_{2n} consisting of those permutations such that $p(1) < \cdots < p(n)$ and $p(i) < p(i+n)$ for all $i \in \{1,\ldots,n\}$. As pointed out in [2, Lemma 3.2], there exists a bijection between quasi-free states on $\mathfrak{A}_{SDC}(\mathcal{H}^{\oplus},\Gamma)$ and the set

$$Q(\mathcal{H}^{\oplus}) = \Big\{A \in \mathfrak{B}(\mathcal{H}^{\oplus}) \text{ s.t.} 0 \le A = A^* \le \mathrm{id},\ A + \Gamma A\Gamma = \mathrm{id}\Big\} \ .$$

This bijection is given explicitly by

$$\omega_A(\psi(u)^*\psi(v)) = \langle u, Av\rangle_{\oplus} \ .$$

If $A^2 = A$, then A is called a basis projection and the state ω_A is a *Fock state*. These correspond to the pure states, as pointed out in [30]. A state ω is deemed *gauge invariant* if it holds that

$$\omega\circ\alpha_\lambda = \omega, \qquad \alpha_\lambda\colon \psi((u_1,u_2)) \mapsto \psi\left((e^{i\lambda}u_1, e^{-i\lambda}u_2)\right) \ .$$

A quasi-free state is gauge invariant if and only if $A = A_1 \oplus C^{-1}(\mathrm{id} - A_1)C$ for some operator $0 \leq A_1 = A_1^* \leq \mathrm{id}$ on the Hilbert space $\overline{\mathcal{D}(S_{r,1}(M, g_0))}^{\langle\cdot,\cdot\rangle_{S_0}}$. The GNS representation $(\pi_P, \mathcal{H}_P, \Omega_P)$ associated to a gauge invariant Fock state ω_P is then given by

$$\pi_P(\psi(u)) = a(Pu)^* + a(\Gamma(\mathrm{id} - P)u)$$

where $a(\cdot)^*$ and $a(\cdot)$ denote the creation and annihilation operators on the Fock space $\mathcal{F}(P\mathcal{H}^\oplus)$, which can be written as

$$\mathcal{F}\left(P_1\overline{\mathcal{D}(S_{r,1}(M, g_0))}^{\langle\cdot,\cdot\rangle_{S_0}} \oplus C^{-1}(\mathrm{id} - P_1)C\overline{\mathcal{D}(\overline{S_{r,1}(M, g_0)})}^{\langle\cdot,\cdot\rangle_{-S_0}}\right) .$$

The group of unitary operators on $\mathcal{H}^\oplus$ commuting with Γ, which shall be denoted with $O(\mathcal{H}^\oplus, \Gamma)$ preserves the defining relations of $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma)$; therefore if one defines a right-action of $O(\mathcal{H}^\oplus, \Gamma)$ on $\{\psi(u)\}_{u \in V^\oplus}$ via

$$\alpha_T \colon \psi(u) \mapsto \psi(Tu)$$

the maps α_T are $*$-endomorphisms of $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma)$ named *Bogoliubov transformations.* Notice that $O(\mathcal{H}^\oplus, \Gamma)$ acts on the right also on $Q(\mathcal{H}^\oplus)$, via

$$R_T \colon A \mapsto T^*AT \ .$$

It is easy to see that

$$\omega_A \circ \alpha_T = \omega_{R_T A} \ .$$

The Shale-Stinespring criterion [42] (see also [8] for the criterion in the context of self-dual CAR algebras) states that given a gauge invariant, Fock state ω_P, a Bogoliubov transformation α_T is unitarily implementable in the representation π_P if and only if $P - T^*PT \in \mathfrak{B}_2(\mathcal{H}^\oplus)$, i.e. if it is an Hilbert-Schmidt operator.
It is often convenient to adopt an equivalent formulation of the self-dual CAR algebra in terms of Cauchy data on a Cauchy surface $\Sigma \hookrightarrow M$. It is possible to show (see [17] and [24]) that there exists a unitary transformation

$$U_\Sigma \colon (V^\oplus, \langle\cdot,\cdot\rangle_\oplus) \to \left(C_0^\infty(\Sigma, S_{r,1}(M)|_\Sigma) \oplus C_0^\infty(\Sigma, \overline{S_{r,1}(M)}|_\Sigma), \langle\cdot,\cdot\rangle_\Sigma\right)$$

explicitly given by

$$V^\oplus \ni (u, v) \mapsto (S_0 u|_\Sigma, S_0^* v|_\Sigma)$$

and where

$$\langle u, v\rangle_\Sigma = \int_\Sigma (u_1, \nu_\Sigma \cdot_\Gamma v_1)\, d\mu_g^\Sigma - \int_\Sigma (u_2, \nu_\Sigma \cdot_{\Gamma^*} v_2)\, d\mu_g^\Sigma \ .$$

This unitary transformation induces a $*$-algebra isomorphism between the self-dual CAR algebra $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma)$ and the self-dual CAR algebra $\mathfrak{A}_{SDC}(\mathcal{H}_\Sigma, \Gamma_\Sigma)$ where $\mathcal{H}_\Sigma$ denotes the completion

$$\mathcal{H}_\Sigma = \overline{C_0^\infty(\Sigma, S_{r,1}(M)|_\Sigma) \oplus C_0^\infty(\Sigma, \overline{S_{r,1}(M)}|_\Sigma)}^{\langle\cdot,\cdot\rangle_\Sigma}$$

and $\Gamma_\Sigma = U_\Sigma \Gamma U_\Sigma^{-1}$. Moreover, notice that if $T \in O(\mathcal{H}^\oplus, \Gamma)$, then $T_\Sigma = U_\Sigma T U_\Sigma^{-1} \in O(\mathcal{H}_\Sigma, \Gamma_\Sigma)$; these objects have the same behaviour as that of their counterpart in the self-dual algebra $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma)$, including the Shale-Stinespring condition.

5.3 Quadratic functionals and the associated Bogoliubov transformations

In Proposition 5.1.4 we have shown that given an allowed quadratic functional $Q \in \mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M,g_0))$, the adjoint action of the even elements $A_Q \in \mathfrak{A}$ associated to it on the generators of the Clifford algebra $\{B^1(u_f)\}_{f\in V}$ is given by

$$A_Q B^1(u_f) A_Q^{-1} = B^1(T^{\oplus}_Q u_f) \ .$$

Now, notice that due to its particular form, $T^{\oplus}_Q \Gamma = \Gamma T^{\oplus}_Q$. Moreover, we have that $T^{\oplus}_Q \equiv \mathrm{id}$ on $\tilde{\mathbf{D}}^{\oplus}_0 \mathcal{D}(S^{\oplus}_{r,1}(M))$: indeed,

$$T^{\oplus}_Q \tilde{\mathbf{D}}^{\oplus}_0 f = \tilde{\mathbf{D}}^{\oplus}_0 \overbrace{\tilde{S}^{-\oplus}_Q \left(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}_Q\right)}^{\mathrm{id}} \underbrace{\tilde{S}^{+\oplus}_0 \tilde{\mathbf{D}}^{\oplus}_0}_{\mathrm{id}} f = \tilde{\mathbf{D}}^{\oplus}_0 f$$

and thus it descends to a well-defined, linear, invertible operator $T^{\oplus}_Q \colon V^{\oplus} \to V^{\oplus}$. Moreover, due to Proposition 5.1.4 and Theorem 4.3.5 we have

Corollary 5.3.1. *For each $Q \in \mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$, the operator $T^{\oplus}_Q \colon V^{\oplus} \to V^{\oplus}$ is isometric with respect to the inner product $\langle \cdot, \cdot \rangle_{\oplus}$.*

Proof. Given the family $\{B^1(u_f)\}_{f\in V}$ of $\mathbb{R}$-linear, self adjoint objects in $\mathfrak{A}$, we know by Proposition 5.1.4 that for each $Q \in \mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ there exists an object $A_Q \in \mathfrak{A}^0$ such that

$$A_Q B^1(u_f) A_Q^{-1} = B^1(T^{\oplus}_Q u_f) \ .$$

Now, consider $\{\psi^*(T^{\oplus}_Q u), \psi(T^{\oplus}_Q v)\}$: by Theorem 4.3.5 it follows that

$$\begin{aligned}
\langle T^{\oplus}_Q u, T^{\oplus}_Q v\rangle_{\oplus} &= \{\psi^*(T^{\oplus}_Q u), \psi(T^{\oplus}_Q v)\} \\
&= \frac{1}{2}\left\{B^1(T^{\oplus}_Q \mathrm{Re}u) + iB^1(T^{\oplus}_Q \mathrm{Im}u), B^1(T^{\oplus}_Q \mathrm{Re}u) - iB^1(T^{\oplus}_Q \mathrm{Im}u)\right\} \\
&= \frac{1}{2} A_Q \left\{B^1(\mathrm{Re}u) + iB^1(\mathrm{Im}u), B^1(\mathrm{Re}u) - iB^1(\mathrm{Im}u)\right\} A_Q^{-1} \\
&= A_Q \langle u, v\rangle_{\oplus} A_Q^{-1} = \langle u, v\rangle_{\oplus} \ .
\end{aligned}$$

□

Therefore, $T^{\oplus}_Q \colon V^{\oplus} \to V^{\oplus}$ can be extended by continuity to a unitary operator $T^{\oplus}_Q \colon \mathcal{H}^{\oplus} \to \mathcal{H}^{\oplus}$ which commutes with Γ, i.e. $T^{\oplus}_Q \in O(\mathcal{H}^{\oplus}, \Gamma)$; this entails that $T^{\oplus}_Q$ induces a Bogoliubov endomorphism of the self-dual CAR algebra $\mathfrak{A}_{SDC}(\mathcal{H}^{\oplus}, \Gamma)$. The subsequent question is whether given a gauge invariant, Hadamard Fock state ω_P on $\mathfrak{A}_{SDC}(\mathcal{H}^{\oplus}, \Gamma)$, this Bogoliubov transformation is unitarily implementable; as previously stated, this amounts to show that

$$P - {T^{\oplus}_Q}^* P {T_Q}^{\oplus} \in \mathfrak{B}_2(\mathcal{H}^{\oplus}) \ .$$

To show this, let us consider the bidistribution on $\mathcal{D}(S_{r,1}(M,g_0)\oplus\mathcal{D}(\overline{S_{r,1}(M)})$ defined by

$$K(u,v) \doteq \langle Qu, (P - T_Q^{\oplus *} P T_Q^{\oplus}) Qv\rangle_{\oplus} = \langle Qu, PQv\rangle_{\oplus} - \langle QT_Q^{\oplus}u, PQT_Q^{\oplus}v\rangle_{\oplus} \ .$$

Notice that we can write $T_Q^{\oplus}$ in the following way:

$$T_Q^{\oplus} = \tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{-\oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{+\oplus} = (\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{-\oplus})(\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus}) \ .$$

By recalling that $T_Q^{\oplus}$ is invertible and that

$$T_Q^{\oplus -1} = \tilde{\mathbf{D}}_0^{\oplus}\tilde{S}_Q^{+\oplus}(\tilde{\mathbf{D}}_0^{\oplus} + \tilde{P}_Q^{\oplus})\tilde{S}_0^{-\oplus} = (\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})(\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{-\oplus})$$

one can see that

$$(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})(\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus}) = (\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus})(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus}) = \mathrm{id} \ ,$$

$$(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{-\oplus})(\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{-\oplus}) = (\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{-\oplus})(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{-\oplus}) = \mathrm{id} \ .$$

We can exploit this property to write

$$K(u,v) = \tilde{K}\left((\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus})u, (\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus})v\right)$$

where

$$\begin{aligned}\tilde{K}(u,v) \doteq\ & \overbrace{\langle Q(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})u, PQ(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})v\rangle_{\oplus}}^{D_1(u,v)} \\ & - \underbrace{\langle Q(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{-\oplus})u, PQ(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{-\oplus})v\rangle_{\oplus}}_{D_2(u,v)} \ .\end{aligned}$$

Now, suppose that both u and v have their support outside that of $J_0^+(\mathrm{Supp}(Q))$; then it follows that

$$(\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})u = u \qquad (\mathrm{id} - \tilde{P}_Q^{\oplus}\tilde{S}_Q^{+\oplus})v = v$$

and thus $D_1(u,v) = \langle Qu, PQv\rangle_{\oplus} = \omega_P(\psi^*(Qu)\psi(Qv))$, that is, outside of the future of the support of the quadratic functional Q, $D_1(u,v)$ is just the value of the two-point function of a Hadamard state, i.e. it is of Hadamard form. The same can be shown to hold for $D_2(u,v)$ outside of the past of the support of the functional Q; by [22], $D_1(u,v)$ and $D_2(u,v)$ are Hadamard over all of $M\times M$. Therefore, the bidistribution $\tilde{K}(u,v)$ is regular, being the difference of two Hadamard bidistributions, and it is induced by a smooth section $\tilde{k}\in\Gamma(M\times M, S_{r,1}^{\oplus}(M,g_0)\boxtimes S_{r,1}^{\oplus}(M,g_0))$. From this it follows that $K(u,v)$ is a regular distribution as well. To show this one could argue in the following way: notice that the composition of $\tilde{K}$ and $\mathrm{id} + \tilde{P}_Q^{\oplus}\tilde{S}_0^{+\oplus}$ is well-defined; moreover, as $\tilde{K}$ is a regular bidistribution, its wavefront set is empty, $\mathrm{WF}(\tilde{K}) = \varnothing$. Therefore, by [27, Theorem 8.2.14] it follows that $\mathrm{WF}(K) = \varnothing$, and thus K is a regular distribution as well. Moreover, notice that the integral kernel of K, denoted by $k\in\Gamma(M\times M, S_{r,1}^{\oplus}(M,g_0)\boxtimes S_{r,1}^{\oplus}(M,g_0))$ is a

bisolution of the Dirac operator $\mathbf{D}_0^{\oplus}$, and that $K(u,v) = 0$ if both u and v have support outside of $J_0(\mathrm{Supp}(Q))$.
Using the equivalent formulation in terms of initial data on any Cauchy surface $\Sigma \hookrightarrow M$, it is possible to write, for $u, v \in C_0^\infty(S_{r,1}(M)|_\Sigma) \oplus C_0^\infty(\overline{S_{r,1}(M)}|_\Sigma)$,

$$\begin{aligned} K(U_\Sigma^{-1}u, U_\Sigma^{-1}v) &= \langle U_\Sigma^{-1}u, U_\Sigma^{-1}U_\Sigma(P - T_Q^{\oplus *}PT_Q^{\oplus})U_\Sigma^{-1}v\rangle_{\oplus} \\ &= \langle u, U_\Sigma(P - T_Q^{\oplus *}PT_Q^{\oplus})U_\Sigma^{-1}v\rangle_\Sigma \doteq K_\Sigma(u,v) \ . \end{aligned}$$

Then as K is induced by a smooth bisolution of the Dirac operator in $\Gamma(M\times M, S_{r,1}^{\oplus}(M,g_0)\boxtimes S_{r,1}^{\oplus}(M,g_0))$, it follows that K_Σ is a bidistribution induced by a smooth section in $\Gamma(\Sigma\times\Sigma, S_{r,1}^{\oplus}(M,g_0)|_\Sigma\boxtimes S_{r,1}^{\oplus}(M,g_0)|_\Sigma)$. This property is of the utmost importance as it simplifies the proof that $P - T_Q^{\oplus *}PT_Q^{\oplus}$ is a Hilbert-Schmidt operator: indeed, the above equaility implies that it suffices to show that the integral kernel $k_\Sigma \in \Gamma(\Sigma\times\Sigma, S_{r,1}^{\oplus}(M,g_0)|_\Sigma\boxtimes S_{r,1}^{\oplus}(M,g_0)|_\Sigma)$ is actually square-integrable. Notice that due to the fact that $K(u,v) = 0$ if both u and v have their support outside of $J_0(\mathrm{Supp}(Q))$, then it follows that, defining $\mathscr{C} = \Sigma \cap J_0(\mathrm{Supp}(Q))$ which is a compact subset of Σ, the contribution to the L^2-norm of k_Σ are due to the integrals over $\mathscr{C}\times\mathscr{C}$, $\mathscr{C}\times\Sigma\setminus\mathscr{C}$ and $\Sigma\setminus\mathscr{C}\times\mathscr{C}$. The first one clearly poses no problem to the L^2-integrability of k_Σ (which we recall is a smooth section); thus let us focus on the other two sets.
Clearly if Σ is compact then k_Σ is in $L^2(\Sigma, d\mu_g^\Sigma)$; unfortunately, to our knowledge showing that the L^2-norm on those set is finite in a generic globally hyperbolic spacetimes and given a generic quasifree, Hadamard state is quite challengeing.
An instance in which the L^2 integrability holds is that of Minkowski spacetime $(M,g) = (\mathbb{M}^{r+1}, \eta)$ with ω_P being the vacuum state; the proof closely follows that in [47].
Consider the L^2-norm of k_Σ; as stated before, the contributions to this norm come from the integrals over $\mathscr{C}\times\mathscr{C}$, $\mathscr{C}\times\Sigma\setminus\mathscr{C}$ and $\Sigma\setminus\mathscr{C}\times\mathscr{C}$. We further split the last two domains in the following way: we consider a relatively compact neighborhood U of $\partial\mathscr{C}$ in Σ, and then we consider

$$U_1 = (\Sigma\setminus\mathscr{C})\cap U \ .$$

The integral over $\mathscr{C}\times\Sigma\setminus\mathscr{C}$ can then be written as the sum of an integral over $\mathscr{C}\times U_1$ and an integral over $\mathscr{C}\times(\Sigma\setminus\mathscr{C})\setminus U_1$. As k_Σ is a smooth section in $\Gamma(\Sigma\times\Sigma, S_{r,1}^{\oplus}(\mathbb{M}^{r+1},\eta)|_\Sigma\boxtimes S_{r,1}^{\oplus}(\mathbb{M}^{r+1},\eta)|_\Sigma)$, the first integral is clearly finite; we can thus focus on the integrals over

$$\mathscr{C}\times(\Sigma\setminus\mathscr{C})\setminus U_1 \text{ and } \mathscr{C}\times(\Sigma\setminus\mathscr{C})\setminus U_1 \ .$$

Now, we know that the integral kernel of $\tilde{K}$ is a smooth function given by the difference of two bidistributions D_1 and D_2. If the Cauchy surface Σ on which we compute the L^2-norm is such that $\Sigma\cap J_0^+(\mathrm{Supp}(Q)) = \varnothing$, then the integral kernel of D_1 is given by the Wightman two-point function of the Dirac field. Recall that if the Dirac field is massive, then for spacelike-separated points away from the diagonal (like in the case $\mathscr{C}\times(\Sigma\setminus\mathscr{C})\setminus U_1$) the Wightman two-point function is smooth and decays rapidly[7] and thus its L^2-integral

[7]See for instance [11, Appendix A] for the Wightman two-point function of the massive scalar field; the massless one can be obtained by considering the limit as $m \to 0^+$. The one for the Dirac field is obtained by applying the Dirac operator to the scalar one.

on $\mathscr{C} \times (\Sigma \setminus \mathscr{C}) \setminus U_1$ is finite; the same integrability holds for the Wightman two-point function of the massless Dirac field, although this is not a rapidly decreasing function[8]. The same argument applies to the integral kernel of D_2 if $\Sigma \cap J_0^-(\mathrm{Supp}(Q)) = \varnothing$; moreover, by the Dirac current conservation, the L^2-integrability of the kernel of D_2 also holds on Σ such that $\Sigma \cap J_0^+(\mathrm{Supp}(Q)) = \varnothing$, the Wightman function being a bisolution of the Dirac equation.
Now, the integral kernel k_Σ is given by the restriction of the (smooth) kernel of K, which we recall is given by

$$\tilde{K} \circ (\mathrm{id} + \tilde{P}_Q^\oplus \tilde{S}_0^{+\oplus}) \otimes (\mathrm{id} + \tilde{P}_Q^\oplus \tilde{S}_0^{+\oplus}) \; .$$

The integrability of this object is the same as that of $\tilde{K}$, thanks to the decay properties of the integral kernel of the retarded Green operator $\tilde{S}_0^{+\oplus}$ away from the diagonal: indeed, this object is rapidly decreasing for spacelike separated points in the case of the massive Dirac field, while its behaviour is the same as that of the Wightman two-point function in the massless case. Therefore, if we consider the L^2-norm of k_Σ on $\mathscr{C} \times (\Sigma \setminus \mathscr{C}) \setminus U_1$, this is finite; the same clearly holds for $(\Sigma \setminus \mathscr{C}) \setminus U_1 \times \mathscr{C}$. From this discussion it follows that $k_\Sigma \in L^2(\Sigma \times \Sigma, S_{r,1}^\oplus(\mathbb{M}^{r+1}, \eta)|_\Sigma \boxtimes S_{r,1}^\oplus(\mathbb{M}^{r+1}, \eta)|_\Sigma)$. Thus the operator $P - T_Q^{\oplus *} P T_Q^\oplus$ is an Hilbert-Schmidt operator, and the Bogoliubov transformation of the massive self-dual CAR algebra $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma)$ induced by $T_Q^\oplus$ is unitarily implementable via U_Q in the Fock representation associated to the Minkowski vacuum. We summarize the results of this section in the following Theorem:

Theorem 5.3.2. *Let $Q \in \mathscr{F}^2_{\mathrm{loc}}(S_{r,1}^\oplus(M))$ be gauge-invariant with $Q = Q^*$; let us consider a gauge-invariant, quasifree Hadamard Fock state ω on the (self-dual) CAR algebra generated by the symbols $\{B^1(u_f)\}_{f \in \mathcal{D}(S_{r,1}(M))/\mathbf{D}_0\mathcal{D}(S_{r,1}(M))}$. Then*

(i) if the Cauchy hypersurfaces of M are compact or

(ii) if $(M, g) = (\mathbb{M}^{r+1}, \eta)$ and ω is the vacuum state on Minkowski spacetime

there exist a unitary operator U_Q on the representation Fock space implementing the Bogoliubov transformation induced by Q, that is,

$$U_Q \mathbf{B}^1(u_f) U_Q^{-1} = \mathbf{B}^1(T_Q^\oplus u_f) \; .$$

Remark 5.3.3. Theorem 5.3.2 can be stated in the following, more suggestive way: in the two cases depicted above, we have extended the Fock representation of the CAR algebra $\mathfrak{A}_{SDC}(\mathcal{H}^\oplus, \Gamma) \subseteq \mathfrak{B}$ to the C*-subalgebra of $\mathfrak{B}$ generated by the CAR algebra and the objects $\Big\{A_Q,\ Q \in \mathscr{F}^2_{\mathrm{loc}}(S_{r,1}^\oplus(M)),\ Q = Q^*,\ Q \text{ gauge-invariant}\Big\}$, which are associated to compactly supported, quadratic perturbations of the Lagrangian. ♠

[8]Notice that in the case of the massless scalar field, the Wightman two-point function is L^2-integrable for dimension $r + 1 \geq 4$.

5.4 The phases and Bogoliubov's causal factorization

Proposition 5.1.5 entails that the unitary operators U_{P+N}, U_N, U_{N+Q} and U_{N+P+Q} satisfy the Bogoliubov causal factorization modulo a phase factor. In the case of the scalar field and of the vacuum state on Minkowski spacetime, it was shown in [15, Section 6], using a procedure akin to group cohomology, that the phases of the unitary operators $\{U_Q\}_{Q\in\mathscr{F}^2_{\mathrm{loc}}(\mathcal{E}(\mathbb{M}^{r+1}))}$ can be chosen so that the phase factor in the Bogoliubov causal factorization is equal to 1; as the proof relies solely on the properties of the allowed kinetic perturbations and of the algebraic properties of the S-symbols, an analogous procedure applies to our case, i.e. for the operators $\{U_Q\}_{Q\in\mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))}$.
In order to extend said proof to our case, we need to point out a few key assumptions though; we start by proving the following Lemma.

Lemma 5.4.1. *Consider $N, Q \in \mathscr{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))$ gauge-invariant with $N = N^*$, $Q = Q^*$; further suppose that N and $N+Q$ are well-defined kinetic perturbations. Then if $u_f \in \mathcal{D}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))$ is such that* $\mathrm{Supp}(u_f)$ *and* $\mathrm{Supp}(Q)$ *are spacelike separated with respect to the causal structure induced by N, then*

$$S^N_G(\mathrm{id}_G\otimes Q)S^N_G(\mathfrak{D}_G(\eta u_f)) = S^N_G(\mathfrak{D}_G(\eta u_f))S^N_G(\mathrm{id}_G\otimes Q)\ .$$

Proof. This follows from (5.2): indeed, denoting with $s = \eta u_f$, we can write

$$S^N_G(\mathfrak{D}_G(s)) = S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\,\oplus}_N s))\ .$$

Then

$$\begin{aligned}
S^N_G(\mathrm{id}_G\otimes Q)S^N_G(\mathfrak{D}_G(s)) &= S^N_G(\mathrm{id}_G\otimes Q)S^N_G(\mathfrak{D}_G(s))S^N_G(\mathrm{id}_G\otimes Q)^{-1}S^N_G(\mathrm{id}_G\otimes Q)\\
&= S_G(\mathrm{id}_G\otimes N)^{-1}S_G(\mathrm{id}_G\otimes(N+Q))S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s))\\
&\quad S_G(\mathrm{id}_G\otimes(N+Q))^{-1}S_G(\mathrm{id}_G\otimes N)S^N_G(\mathrm{id}_G\otimes Q)\\
&\overset{(5.3)}{=} S_G(\mathfrak{D}_G(T^{\oplus\,-1}_N T^{\oplus}_{N+Q}\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s))S^N_G(\mathrm{id}_G\otimes Q)\\
&= S_G(\mathfrak{D}_G(T^{N\,\oplus}_Q\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s))S^N_G(\mathrm{id}_G\otimes Q)\ .
\end{aligned}$$

By (5.4) we then have

$$T^{N\,\oplus}_Q\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s = (\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N)\left[(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N)\tilde{S}^{-\,\oplus}_{N+Q}(\tilde{\mathbf{D}}^{\oplus}_0 + \tilde{P}^{\oplus}_N + \tilde{P}^{\oplus}_Q)\tilde{S}^{+\,\oplus}_N\right] s\ .$$

We now use the hypothesis on Q and u_f: namely, if $\mathrm{Supp}(Q)$ and $\mathrm{Supp}(u_f)$ are spacelike separated with respect to N we then have

$$\tilde{P}^{\oplus}_Q\tilde{S}^{+\,\oplus}_N s = 0 \qquad \tilde{S}^{-\,\oplus}_{N+Q} s = \tilde{S}^{-\,\oplus}_N s$$

which entail that

$$\mathfrak{D}_G(T^{N\,\oplus}_Q\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s) = \mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0\tilde{S}^{+\,\oplus}_N s)$$

and thus

$$S^N_G(\mathrm{id}_G\otimes Q)S^N_G(\mathfrak{D}_G(s)) = S^N_G(\mathfrak{D}_G(s))S^N_G(\mathrm{id}_G\otimes Q)$$

as stated. □

This property descends to the objects in the algebra: indeed, we know that

$$\sigma_G(S^N_G(\mathrm{id}_G \otimes Q)) = \sigma_G(S_G(\mathrm{id}_G \otimes N)^{-1} S_G(\mathrm{id}_G \otimes (N+Q))) = \mathrm{id}_G \otimes \underbrace{A_N^{-1} A_{N+Q}}_{A^N_Q \in \mathcal{B}}$$

and that

$$\begin{aligned}\mathrm{id}_{G\otimes\mathfrak{A}} + i\eta \otimes B^1_N(u_f) &= \sigma_G(S^N_G(\mathfrak{D}_G(s))) = \sigma_G(S_G(\mathfrak{D}_G(\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N s))) \\ &= \mathrm{id}_{G\otimes\mathfrak{A}} + i\eta \otimes B^1(\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N u_f) \ .\end{aligned}$$

Thus also the family $\{B^1_N(u_f)\}_{f\in\mathcal{D}(S_{r,1}(\mathbb{M}^{r+1},\eta)/\mathbf{D}_0\mathcal{D}(S_{r,1}(\mathbb{M}^{r+1},\eta))}$ is contained in $\mathcal{B}$. The relation in Lemma 5.4.1 then reads

$$A^N_Q B^1_N(u_f) = B^1_N(u_f) A^N_Q$$

whenever $\mathrm{Supp}(Q)$ and $\mathrm{Supp}(u_f)$ are spacelike separated with respect to N. This behaviour is present also in the Fock space representation previously studied: indeed, we have

$$U^N_Q \mathbf{B}^1_N(u_f) = \mathbf{B}^1_N(u_f) U^N_Q, \qquad U^N_Q \doteq U_N^{-1} U_{N+Q}$$

where U_Q and U_{N+Q} are the unitary operators implementing T_Q and T_{N+Q} respectively and $\mathbf{B}^1_N(u_f)$ is the perturbed field operator on Fock space associated to $B^1(\tilde{\mathbf{D}}^{\oplus}_0 \tilde{S}^{+\oplus}_N u_f)$. To proceed further, we can consider the following: starting from the vacuum representation of the algebra $\mathfrak{B}$ on Fock space, we can consider its $*$-subalgebra $\mathfrak{B}_N(O)$ generated by

$$\left\{\mathbf{B}^1_N(u_f)\right\}_{\substack{f\in\mathcal{D}(S_{r,1}(\mathbb{M}^{r+1},\eta))/\mathbf{D}_0\mathcal{D}(S_{r,1}(\mathbb{M}^{r+1},\eta)) \\ \mathrm{Supp}(f)\subseteq O}}$$

where $O \subseteq \mathbb{M}^{r+1}$ is any bounded, causally closed region. By construction it follows that given $O_1 \subseteq O_2$ then $\mathfrak{B}_N(O_1) \subseteq \mathfrak{B}_N(O_2)$; thus the collection $\{\mathfrak{B}_N(O)\}_{O\subseteq\mathbb{M}^{r+1}}$ satisfies isotonoy and thus provides, modulo weak closure, a net of von Neumann algebras. These algebras can be endowed with an action of $U(1)$ in the following way: we already have an action of $U(1)$ on $\mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))$ by

$$\alpha_t(F) \doteq F \circ \chi_t$$

where χ_t was described in Section 4.2. Then after extending this action in an obvious way to $G \otimes \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))$ we can induce an action on $\mathfrak{A}_G$ by

$$\alpha_t(S_G(F)) \doteq S_G(\alpha_t(F))$$

and consequently on $\mathfrak{A}$ by requiring that

$$\alpha_t\left(\sigma_G(S_G(F))\right) \doteq \sigma_G\left(\alpha_t(S_G(F))\right) \ .$$

Notice that for gauge-invariant functionals as previously described this yields

$$\alpha_t(A_F) = A_F$$

as expected, while for $B^1(u_f)$ it holds that $\alpha_t(B^1(u_f)) = B^1(\chi_t u_f)$.
Under the additional assumption that the net of local algebras $\{\mathfrak{B}_N(O)\}_{O\subseteq\mathbb{M}^{r+1}}$ satisfies twisted Haag duality (cfr. [18] and [43]) with respect to the gauge group $U(1)$ and its action α_t, we then have by Lemma 5.4.1 that

$$U_Q^N \in \mathfrak{B}_N(O)^t$$

where O denotes the causal closure of $\mathrm{Supp}(Q)$ and ^{t} denotes the twist induced by the gauge group $U(1)$ via a Klein transformation. At the same time, it follows that if P is such that $\mathrm{Supp}(P)$ is spacelike separated from $\mathrm{Supp}(Q)$ with respect to N, then $U_P^N \in \mathfrak{B}_N(O')^t$, where O' denotes the spacelike complement of O. To conclude, notice then that

$$\left\{U_F^N,\ F \in \mathscr{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1},\eta))^2,\ F^* = F = F\circ\chi_t,\ \mathrm{Supp}(F) \subseteq O\right\}$$

are gauge-invariant elements of $\mathfrak{B}$, and thus they belong to the actual observable subalgebras of $\mathfrak{B}_N(O)^t$ and $\mathfrak{B}_N(O')^t$; for these subalgebras actual Haag duality holds, i.e. U_Q^N and U_P^N commute whenever $\mathrm{Supp}(Q)$ and $\mathrm{Supp}(P)$ are spacelike separated with respect to N.
This property is fundamental because it then entails the following: as stated at the beginning of this Section, by Proposition 5.1.5 we have

$$U_{P+N}U_N^{-1}U_{N+Q} = \alpha(N,P,Q)U_{N+P+Q}$$

with $\alpha(N,P,Q)$ a phase factor. Then if the supports of P and Q are spacelike separated with respect to N we can also write

$$U_{Q+N}U_N^{-1}U_{N+P} = \alpha(N,Q,P)U_{N+P+Q}\ .$$

Under the assumption of twisted Haag duality, we concluded that U_P^N and U_Q^N commute; but this entails that

$$\alpha(N,Q,P)U_{P+Q}^N = U_Q^N U_P^N = U_P^N U_Q^N = \alpha(N,P,Q)U_{P+Q}^N$$

i.e. the phase factors $\alpha(N,P,Q)$ and $\alpha(N,Q,P)$ are equal whenever P and Q are spacelike separated with respect to N. This property is a key property in the proof carried out in [15, Section 6], which can then be extended effortlessly to our case, yielding the following result:

Theorem 5.4.2. *Let us consider the collection of gauge-invariant, self-adjoint functionals $Q \in \mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(\mathbb{M}^{r+1}))$. On the antisymmetric Fock space associated to the Minkowski vacuum the unitary operators U_Q given by Theorem 5.3.2 are such that*

$$U_{P+N}U_N^{-1}U_{N+Q} = U_{N+P+Q}$$

if $\mathrm{Supp}(P) \cap J_N^-(\mathrm{Supp}(Q)) = \varnothing$, *i.e. they comply with Bogoliubov's causal factorization.*

6 Conclusions and outlook

In this thesis, we have shown how the C*-algebraic framework for interacting quantum field theories proposed in [14] for scalar fields on d-Minkowski spacetime and extended to Fermi fields in [12] can be used to model Fermionic fields on globally hyperbolic backgrounds via a Grassmann covariant multiplication $(\mathfrak{G}, \iota)$; this construction uses abstract algebras $\mathfrak{A}_G$ generated by symbols $\{S_G(F)\}$ where G denotes a real, finite-dimensional Grassmann algebra and the label F denotes objects in $G \otimes \mathcal{F}_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$. Notice that the dependence on the Grassmann algebra is functorial: there exist a universal, $\mathbb{Z}_2$-graded, complex $*$-algebra $\mathfrak{A}$ and a natural transformation $\sigma_G \colon \mathfrak{G} \Rightarrow \mathfrak{G}^{\mathfrak{A}}$, i.e. a collection of $*$-homomorphisms $\sigma_G \colon \mathfrak{A}_G \to G \otimes \mathfrak{A}$ well-behaved with respect to morphisms.
We have shown that $\mathfrak{A}$ contains a subalgebra, generated by the non-identity $\mathfrak{A}$-components of $\{\sigma_G(S_G(\mathfrak{D}_G(\eta u_f)))\}_{f \in \mathcal{D}(S_{r,1}(M))}$, obeying the canonical anticommutation relations of a Dirac field on a curved background; moreover, by considering a special class of objects in $\mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$, that of kinetic perturbations, we have shown that $\mathfrak{A}$ contains a "perturbed" version of the CAR algebra, generated by the non-identity $\mathfrak{A}$-components of $\{\sigma_G(S^K_G(\mathfrak{D}_G(\eta u_f)))\}_{f \in \mathcal{D}(S_{r,1}(M))}$, which obeys the canonical anticommutation relations of a Dirac field in the perturbed curved background associated to K.
Furthermore, by considering a particular class of functionals, that is, gauge-invariant, self-adjoint functionals $Q \in \mathcal{F}^2_{\mathrm{loc}}(S^{\oplus}_{r,1}(M))$ giving rise to normally hyperbolic operators, we have shown that the adjoint action of the $\mathfrak{A}$-component of $S_G(\mathrm{id}_G \otimes Q)$ on the generators of the CAR algebra gives rise to a Bogoliubov transformation of the latter; given a quasi-free, Hadamard, gauge-invariant state on the CAR algebra it was then shown that said Bogoliubov transformation satisfies the Shale-Stinespring condition if the globally hyperbolic spacetime under consideration has compact Cauchy surfaces, or if we are considering the Minkowski vacuum on $r+1$-Minkowski spacetime $\mathbb{M}^{r+1}$. In the latter case, under the assumption that under a functional of the above form the "perturbed" CAR algebras satisfies twisted Haag duality, it is then possible to extend the proof in [15] to yield a unitary representation of the Bogoliubov transformations which satisfy the Bogoliubov causal factorization.
Taking into account the recent developments in the algebraic framework, this work has various potential outlooks: on one hand it is possible to study in a sound context fermionic anomalies (as carried out in [13] for the scalar field), while on the other it is possible to study coherent and squeezed states for fermions (see [38], [33] as well as [23] where multi-excited fermionic states are introduced and shown to behave as the Weyl coherent states).

References

[1] Valentino Abram and Romeo Brunetti. "Møller Maps for Dirac Fields in External Backgrounds". In: *Mathematical Physics, Analysis and Geometry* 27.3 (July 2024). DOI: 10.1007/s11040-024-09487-8.

[2] Huzihiro Araki. "On Quasifree States of CAR and Bogoliubov Automorphisms". In: *Publications of the Research Institute for Mathematical Sciences* 6.3 (1971), pp. 385–442. DOI: 10.2977/prims/1195193913.

[3] Huzihiro Araki. "On the Diagonalization of a Bilinear Hamiltonian by a Bogoliubov Transformation". In: *Publications of the Research Institute for Mathematical Sciences, Kyoto University. Ser. A* 4.2 (1968), pp. 387–412. DOI: 10.2977/prims/1195194882.

[4] E. Balslev, J. Manuceau, and A. Verbeure. "Representations of anticommutation relations and Bogolioubov transformations". In: *Communications in Mathematical Physics* 8.4 (Dec. 1968), pp. 315–326. DOI: 10.1007/BF01646271.

[5] Christian Bär, Paul Gauduchon, and Andrei Moroianu. "Generalized cylinders in semi-Riemannian and spin geometry". In: *Mathematische Zeitschrift* 249.3 (2005), pp. 545–580. DOI: 10.1007/s00209-004-0718-0.

[6] Christian Bär, Nicolas Ginoux, and Frank Pfäffle. *Wave Equations on Lorentzian Manifolds and Quantization*. European Mathematical Society, 2007. DOI: 10.4171/037.

[7] Antonio N. Bernal and Miguel Sánchez. "Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes". In: *Communications in Mathematical Physics* 257.1 (May 2005), pp. 43–50. DOI: 10.1007/s00220-005-1346-1.

[8] Carsten Binnenhei. "Implementation of Endomorphisms of the CAR Algebra". In: *Reviews in Mathematical Physics* 07.06 (Aug. 1995), pp. 833–869. DOI: 10.1142/S0129055X95000323.

[9] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1: C^*- and W^*-Algebras, Symmetry Groups, Decomposition of States*. Springer Berlin Heidelberg, 1987. DOI: 10.1007/978-3-662-02520-8.

[10] Christian Brouder et al. "Properties of field functionals and characterization of local functionals". In: *Journal of Mathematical Physics* 59.2 (Feb. 2018), p. 023508. DOI: 10.1063/1.4998323.

[11] R. Brunetti, M. Dütsch, and K. Fredenhagen. "Perturbative algebraic quantum field theory and the renormalization groups". In: *Adv. Theor. Math. Phys.* 13.5 (2009), pp. 1541–1599.

[12] Romeo Brunetti et al. "C^*-algebraic approach to interacting quantum field theory: inclusion of Fermi fields". In: *Letters in Mathematical Physics* 112.5 (2022), p. 101. DOI: 10.1007/s11005-022-01590-7.

[13] Romeo Brunetti et al. “The Unitary Master Ward Identity: Time Slice Axiom, Noether’s Theorem and Anomalies”. In: *Annales Henri Poincaré* 24.2 (Feb. 2023), pp. 469–539. DOI: 10.1007/s00023-022-01218-5.

[14] Detlev Buchholz and Klaus Fredenhagen. “A C*-algebraic Approach to Interacting Quantum Field Theories”. In: *Communications in Mathematical Physics* 377.2 (2020), pp. 947–969. DOI: 10.1007/s00220-020-03700-9.

[15] Detlev Buchholz and Klaus Fredenhagen. “Dynamical C*-algebras and Kinetic Perturbations”. In: *Annales Henri Poincaré* 22.3 (2021), pp. 1001–1033. DOI: 10.1007/s00023-020-01002-3.

[16] Jean Dieudonné. *Foundations of modern analysis*. Academic Press, 1969.

[17] Jonathan Dimock. “Dirac quantum fields on a manifold”. In: *Transactions of the American Mathematical Society* 269.1 (Jan. 1982), pp. 133–147. DOI: 10.1090/S0002-9947-1982-0637032-8.

[18] Sergio Doplicher, Rudolf Haag, and John E. Roberts. “Fields, observables and gauge transformations I”. In: *Communications in Mathematical Physics* 13.1 (Mar. 1969), pp. 1–23. DOI: 10.1007/BF01645267.

[19] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000. DOI: 10.1007/978-3-642-56936-4.

[20] Michael Dütsch. “From Classical Field Theory to Perturbative Quantum Field Theory”. In: Springer International Publishing, 2019. DOI: 10.1007/978-3-030-04738-2.

[21] F. Faà di Bruno. “Sullo sviluppo delle funzioni”. In: *Annali di scienze matematiche e fisiche* 6 (1855), pp. 479–480.

[22] Stephen A. Fulling, Mark Sweeny, and Robert M. Wald. “Singularity structure of the two-point function in quantum field theory in curved spacetime”. In: *Communications in Mathematical Physics* 63.3 (Oct. 1978), pp. 257–264. DOI: 10.1007/BF01196934.

[23] Stefano Galanda, Albert Much, and Rainer Verch. “Relative Entropy of Fermion Excitation States on the CAR Algebra”. In: *Mathematical Physics, Analysis and Geometry* 26.3 (Aug. 2023), p. 21. DOI: 10.1007/s11040-023-09464-7.

[24] Christian Gérard and Théo Stoskopf. “Hadamard states for quantized Dirac fields on Lorentzian manifolds of bounded geometry”. In: *Reviews in Mathematical Physics* 34.04 (2022), p. 2250008. DOI: 10.1142/S0129055X22500088.

[25] Martin Golubitsky and Victor Guillemin. *Stable Mappings and Their Singularities*. New York, NY: Springer US, 1973. DOI: 10.1007/978-1-4615-7904-5.

[26] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973. DOI: 10.1017/CBO9780511524646.

[27] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003. DOI: 10.1007/978-3-642-61497-2.

[28] Gregory M. Kelly. "Basic Concepts of Enriched Category Theory". In: *Reprints in Theory and Applications of Categories* 10 (2005), pp. 1–136.

[29] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1990.

[30] J. Manuceau, F. Rocca, and D. Testard. "On the product form of quasi-free states". In: *Communications in Mathematical Physics* 12.1 (Mar. 1969), pp. 43–57. DOI: 10.1007/BF01646434.

[31] John N. Mather. "Stability of C^∞ Mappings: II. Infinitesimal Stability Implies Stability". In: *Annals of Mathematics* 89.2 (1969), pp. 254–291. DOI: 10.2307/1970668.

[32] Liviu I Nicolaescu. *Lectures on the Geometry of Manifolds*. 3rd. World Scientific, 2020. DOI: 10.1142/11680.

[33] Yoshio Ohnuki and Taro Kashiwa. "Coherent States of Fermi Operators and the Path Integral". In: *Progress of Theoretical Physics* 60.2 (Aug. 1978), pp. 548–564. DOI: 10.1143/PTP.60.548.

[34] Theodore W. Palmer. *Banach Algebras and the General Theory of *-Algebras*. Vol. 2. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2001. DOI: 10.1017/CBO9780511574757.

[35] Peter Petersen. *Riemannian Geometry*. Springer International Publishing, 2016. DOI: 10.1007/978-3-319-26654-1.

[36] Kasia Rejzner. *Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians*. Springer International Publishing, 2016. DOI: 10.1007/978-3-319-25901-7.

[37] Katarzyna Rejzner. "Fermionic Fields in the Functional Approach to Classical Field Theory". In: *Reviews in Mathematical Physics* 23.09 (2011), pp. 1009–1033. DOI: 10.1142/S0129055X11004503.

[38] Paul L. Robinson. "The Berezin Calculus". In: *Publications of the Research Institute for Mathematical Sciences* 35.2 (1999), pp. 123–194. DOI: 10.2977/prims/1195143948.

[39] H. H. Schaefer and M. P. Wolff. *Topological Vector Spaces*. Springer New York, 1999. DOI: 10.1007/978-1-4612-1468-7.

[40] Z. Sebestyén. "Every C*-seminorm is automatically submultiplicative". In: *Periodica Mathematica Hungarica* 10.1 (Mar. 1979), pp. 1–8. ISSN: 1588-2829. DOI: 10.1007/BF02018366.

[41] David Shale. "Linear Symmetries of Free Boson Fields". In: *Transactions of the American Mathematical Society* 103.1 (1962), pp. 149–167. DOI: 10.1090/S0002-9947-1962-0137504-6.

[42] David Shale and W. Forrest Stinespring. "States of the Clifford Algebra". In: *Annals of Mathematics* 80.2 (1964), pp. 365–381. DOI: 10.2307/1970397.

[43] Stephen J. Summers. “Normal product states for fermions and twisted duality for CCR- and CAR-type algebras with application to the Yukawa$_2$ quantum field model”. In: *Communications in Mathematical Physics* 86.1 (Mar. 1982), pp. 111–141. DOI: 10.1007/BF01205664.

[44] Nikolai N. Tarkhanov. *Complexes of Differential Operators*. Springer Netherlands, 1995. DOI: 10.1007/978-94-011-0327-5.

[45] François Treves. *Topological vector spaces, distributions and kernels*. Academic Press, Inc., 1967.

[46] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer International Publishing, 2017. DOI: 10.1007/978-3-319-55084-8.

[47] Robert M. Wald. “Existence of the S-matrix in quantum field theory in curved space-time”. In: *Annals of Physics* 118.2 (1979), pp. 490–510. DOI: 10.1016/0003-4916(79)90135-0.