

UNIVERSITY OF TRENTO

Department of Physics



PhD in Physics

**Performance of the electrostatic control force system for LISA
Test Masses: experimental results from LISA Pathfinder and new
on-ground measurements**

Supervisor:
Prof. William Joseph Weber

Candidate:
Vittorio Chiavegato

*A thesis submitted in fulfillment of the requirements
for the degree of Doctor of Philosophy in Physics*

Doctoral Cycle XXXVII

Abstract

The LISA mission will detect low frequency gravitational waves by interferometrically measuring the distance between free-falling Test Masses (TM) in space. These are six, 2 kg gold-platinum cubes, contained in three spacecrafts which fly in a triangular formation, 2.5 million km from one another. The TMs are in nearly perfect free-fall and are followed by the spacecrafts, there to shield them and keep them aligned with the interferometers. Spurious forces introduced by the spacecraft on the TMs are a major source of noise for the measurement and must be limited to the absolute minimum.

Several important sources of noise are linked to the system of forces used to control the TM dynamics. The whole system is usually referred to as electrostatic force actuation. The forces are generated by applying voltages to electrodes surrounding the TMs: noise in these voltages end up being one of the main disturbances for LISA. This thesis tackles actuation noise from multiple directions.

First, it contains the analysis of experiments performed on LISA Pathfinder (LPF), LISA precursor mission. Dedicated measurement campaigns allowed an in depth characterization of actuation noise, in particular voltage gain fluctuations and additive voltage noises. Good agreement with the current actuation noise model was found: through this, we estimate that actuation explains between 20 and 40% of LPF noise, and project the noise in the LISA configuration.

A detailed investigation of possible non-linearities in actuation voltages is reported. They were first measured in the laboratory on a LPF Front-End-Electronics (FEE) copy. Then, their impact on LPF and LISA was evaluated through simulations and development of analytic models, revealing that while they should not be a dominant noise source in LPF, they can become significant under certain conditions.

The second part of the thesis focuses on measuring actuation-related forces with the 1TM torsion pendulum, which was re-activated after a shut-down of seven years and updated in many aspects. It was then used for two measurement campaigns. The first was a demonstration that the instrument has the capability of measuring torque noise from actuation gain fluctuations, a test that could be previously performed only in space. The campaign successfully recovered fluctuation from some home-made actuation electronics, and it is planned to use the pendulum to test the new LISA FEE.

Finally, the 1TM facility was used for a second test campaign, to characterize the noisy charge fluctuations of the TM, induced by illumination with UV light. This is used in flight to control the TM charge, and can be a source of noise if too much power is used. The campaign completes a much wider characterization done with the 4TM torsion pendulum and demonstrates the accuracy of our models for this source of noise.

Contents

Acronyms	4
Parameters	5
Chapters content guide for the reader	6
1 Introduction to gravitational waves and their measurement	8
1.1 Measuring spacetime curvature	8
1.1.1 The challenges of GW measurement	11
1.2 Gravitational waves measurement	12
1.2.1 Strain from binary systems	14
1.3 Main noise sources in GW detectors	18
1.3.1 Strain sensitivity	18
1.3.2 Stray accelerations of the end mirrors	19
1.3.3 Laser frequency and readout noise	20
1.3.4 Putting all together	20
2 LISA and the Gravitational Reference System	25
2.1 The LISA Mission - Science Objectives	27
2.2 Goal sensitivity and design	29
2.2.1 Readout noise: 2.5 million km arms	29
2.2.2 Frequency noise: Time-Delay-Interferometry	30
2.3 Test Mass acceleration noise and the Gravitational Reference System	34
2.3.1 Gravitational Reference System	38
2.3.2 Putting it all together again	41
2.4 LISA Pathfinder	43
2.4.1 LPF results	46
3 Electrostatic force actuation in LISA	49
3.1 Why do we need force actuation?	49
3.1.1 Noise from ϕ actuation	51
3.2 Electrostatic force actuation and sensing	52

3.3	Model of GRS electrostatic forces	56
3.3.1	Test Mass force and voltage	58
3.3.2	Zeroth order electrostatic force model (centered TM)	59
3.3.3	Effect of stray voltages	60
3.3.4	Actuation voltages choice for simple TM control	63
3.3.5	Maximum forces and torques	67
3.3.6	Actuation and sensing circuit implementation in LPF	68
3.4	Actuation related noise sources	70
3.4.1	Gain fluctuations and white noise	71
3.4.2	In-band voltage fluctuations	75
3.4.3	TM charge fluctuations	77
4	LISA Pathfinder electrostatic force	
	actuation performance analysis	79
4.1	Gain fluctuations in LPF	79
4.1.1	Gain fluctuations noise model	81
4.1.2	LPF enhanced actuation runs	84
4.1.3	Estimation of electrode noises using Bayesian inference	87
4.1.4	Results and comparison with ground measurements	88
4.2	In-band voltage fluctuations in LPF	93
4.2.1	LPF charged TM runs, modeling and parameter estimation results . .	94
4.3	Actuation noise projections to LPF and LISA	98
	Conclusions	101
5	Actuation waveforms nonlinearities	102
5.1	Rounding error effect on LISA Pathfinder	103
5.2	Residual non linearities simulations	107
5.2.1	Simulations with injected non-linearities	109
5.3	Lab measurement: EM Light nonlinearities	111
5.3.1	Measurement in absence of digitization error	112
5.3.2	Measurement in presence of digitization error	126
5.4	Conversion of nonlinearities into acceleration noise	129
5.4.1	Simulations from laboratory measurements	129
5.4.2	Non-linearities spectrum	131
5.4.3	Analytic model for drift in commanded actuation	135
5.4.4	Effect of noisy commanded actuation	140
	Conclusions	146
6	Testing LISA actuation on-ground: the 1TM torsion pendulum	147
6.1	Small force measurements with torsion pendulums	147
6.1.1	1TM torsion pendulum	151

6.1.2	Pendulum torque estimation and noise floor	159
6.2	Enhanced actuation experiments	164
6.2.1	Actuation torque noise measurements	167
6.2.2	Gain fluctuations estimation from torque measurements	169
6.2.3	Direct measurement of home-made electronics voltage fluctuations . .	174
	Conclusions	179
7	Continuous UV discharge: random	
	charging noise characterization	181
7.1	UV discharge mechanism and model	181
7.1.1	Linearized apparent yield model	184
7.1.2	Fast discharge vs continuous discharge	184
7.1.3	Continuous discharge related noise sources	187
7.2	Measurement of discharge noise with torsion pendulum	189
7.2.1	TM charge measurement and control	189
7.2.2	Apparent yield measurement	194
7.2.3	UV induced charge noise measurement campaign	196
7.2.4	Contributions from phase fluctuations	201
	Conclusions	205
8	Important results summary	207
9	Appendix	209
9.1	Appendix A - PSD of delayed signals	209
9.2	Appendix B - Shot noise at the photodiode to phase noise	210
9.3	Appendix C - Bayesian parameter estimation of electrode noises	211
9.3.1	Parameter estimation from four actuation experiments at a single frequency	215
9.3.2	Parameter estimation from four actuation experiments with analytic functions	225
9.3.3	Details for low-frequency voltage noise parameter estimation	229
9.4	Appendix D - Analytic models for conversion of voltage non-linearities to acceleration noise from a noisy command	230
9.5	Appendix E - Autocollimator non-linearities	234
9.6	Appendix F - Suggested measurement for pendulum readout noise characterization	236
9.7	Appendix G - UV power measurement	239
9.8	Appendix H - Demodulated quadrature of charge measurement runs	241
	Acknowledgments	243
	Bibliography	244

Acronyms

ASD	Amplitude Spectral Density
CMS	Charge Management System
DFACS	Drag Free and Attitude Control System
DNL	Differential Non Linearity
DOF	Degree Of Freedom
EH	Electrode Housing
FEE	Front-End-Electronics
FT	Fourier Transform
GPRM	Grabbing, Positioning and Release Mechanism
GR	General Relativity
GRS	Gravitational Reference System
GW	Gravitational Wave
INL	Integral Non Linearity
LISA	Laser Interferometer Space Antenna
LPF	LISA Pathfinder
LSB	Least Significant Bit
MCF	Minimally Correlated Frequencies
MOSA	Moving Optical Sub-Assembly
OB	Optical Bench
PSD	Power Spectral Density
SC	Spacecraft
SNR	Signal to Noise Ratio
TDI	Time Delay Interferometry
TM	Test Mass
VC	Vacuum Chamber

Parameters

Parameter	Value	Description
M	1.9282 kg	LISA TM mass
C_{tot}	34.2 pF	TM to EH total capacitance
$C_{\text{inj,tot}}$	4.2 pF	Sum of injection electrodes to TM capacitance
C_x	1.17 pF	x -electrode capacitance to TM
C_y	0.94 pF	y -electrode capacitance to TM
C_z	0.69 pF	z -electrode capacitance to TM
$\frac{\partial C_x}{\partial x}$	291 pFm ⁻¹	x -electrode capacitance to TM, x derivative
$\frac{\partial C_{x,h}}{\partial x}$	-70 pFm ⁻¹	x -electrode capacitance to EH, x derivative
$\frac{\partial C_x}{\partial \phi}$	3.25 pFrad ⁻¹	x -electrode capacitance to TM, ϕ derivative
$\frac{\partial C_{x,h}}{\partial \phi}$	-0.78 pFrad ⁻¹	x -electrode capacitance to EH, ϕ derivative
$\frac{\partial^2 C_x}{\partial x^2}$	145 nFm ⁻²	x -electrode capacitance to TM, x second derivative
$\frac{\partial^2 C_x}{\partial \phi^2}$	26 pFrad ⁻²	x -electrode capacitance to TM, ϕ second derivative
$\frac{\partial^2 C_x}{\partial \phi \partial x}$	3.2 nFrad ⁻¹ m ⁻¹	x -electrode capacitance to TM, $\phi - x$ cross derivative

Table 2: Some important parameters for LISA electrostatic force actuation.

Chapters content guide for the reader

Here I give a brief, informal description and some motivation for the content of each chapter, so that the reader might understand immediately what could be of their interest.

Chapter 1. This chapter introduces the fundamental quantities that must be considered when designing a generic gravitational wave detector based on laser interferometry. I try to explain quantitatively to which level the relevant sources of noise must be suppressed, based on which kind of gravitational waves we want to observe. This chapter adopts a somewhat didactic tone and does not contain any original calculation; therefore it may be skipped without any issue by a reader familiar with gravitational wave detection. It may be useful instead for a person approaching gravitational waves detectors for the first time.

Chapter 2. Here the LISA mission is introduced and its fundamental design choices motivated. Particular focus is placed on the Gravitational Reference System and its test with the LISA Pathfinder mission. Also this chapter is somehow didactic and may be skipped by a reader familiar with LISA. The LPF section might instead be worth a look for someone not knowing the mission very well. It might instead be useful for someone approaching LISA and asking "Why was LISA designed like this?". I wrote down most of these first chapters to prepare some informal lectures that I was asked to give by younger students: other than personal studies, much comes from what I learned from my research group, especially my supervisor, from discussions and some of his lectures.

Chapter 3. Here the subject of electrostatic force actuation in LISA and LISA Pathfinder is addressed in detail. The conceptually important parts, like why we need actuation and how it affects the performance of the observatory, can be found at the beginning. Section 3.3 is instead much more technical and has a pretty heavy notation so the reader may decide to skip it, as all the important formulas are usually recalled when needed anyway. Section 3.4 might be worth a look instead, since it presents various noise sources which are studied in this thesis, highlighted by some green squares.

Chapter 4. This chapter discusses the analysis of some tests carried out during LPF flight, aimed at characterizing the electrostatic force actuation system. While I did not carry out the tests myself during LPF operations, this chapter reports my original contributions to the analysis and interpretation of the experiments, and was also published in [13]. The first campaign under investigation is a set of 4 noise runs in presence of enhanced electrostatic

force actuation, to characterize its associated noise. The results from this were very important for the understanding of LPF low-frequency noise, which remains a mystery and one of the biggest threats to LISA.

The second campaign is a re-analysis of LPF noise runs with a very charged TM published in [10]: those results are improved and linked more solidly to the actuation system.

Chapter 5. This chapter investigates if the LPF excess low frequency noise could be linked to nonlinearities in the actuation forces. One of these nonlinearities was already known to be linked to digitization errors in the actuation waveforms, and we looked for further inaccuracies. The problem was tackled through both simulations and measurements in the laboratory, which revealed some unexpectedly large nonlinearities in the voltages. Finally, we try to simplify the problem and develop a formalism to evaluate more easily how nonlinearities get converted to acceleration noise, without always have to recur to simulations. This allowed a deeper understanding of the quantities truly involved in the phenomenon and gave ideas on how to keep it under control. And no, unfortunately it does not seem to explain LPF excess noise.

Chapter 6. The 1TM torsion pendulum is introduced. I start with a motivation on why torsion pendulums are suited for the purpose of measuring small forces, and proceed with a description of the facility. I have included some details also to act as a reference for people that will work on it in the future. At the sensitivity level of the instrument many small things can cause noise, and spotting them is long and tedious: one attempts a change, then has to wait at least one night to see if something happened, the time needed to have a decent noise resolution in our band of interest, 0.1-10 mHz. Therefore reporting every change that had some beneficial effect can spare a lot of time. A reader not particularly interested in the facility structure may skip this part.

An "exploratory" measurement campaign to detect actuation noise with the torsion pendulum is then described. It constitutes a demonstration of the measurement, which allowed to identify the important aspects to be taken in account when we will actually try to measure the noise of the future LISA Front-End-Electronics.

Chapter 7. This chapter addresses the problem of noise in the TM charge, induced by the use of UV light from LEDs in LISA. This is necessary to eliminate the charge accumulated on the TM by cosmic rays, which would otherwise interact with the low-frequency fluctuations of the actuation voltages and result in TM acceleration noise. A small introduction to the topic is included at the chapter beginning. This shot noise was already observed with the 4TM torsion pendulum, but the sensitivity was at its limit and it was barely possible to test at the level required for LISA. The experimental campaign presented in this chapter complements that work, and exploits the enhanced 1TM sensitivity to better resolve the noise and deeply test our models for the TM discharging mechanism.

Chapter 1

Introduction to gravitational waves and their measurement

1.1 Measuring spacetime curvature

The paradigm of Einstein's theory of General Relativity (GR), on a completely qualitative level, can be read as

$$\text{Mass distribution} \xrightarrow{\text{fixes}} \text{Geometry of spacetime} \xrightarrow{\text{fixes}} \text{Geodesics (or particles trajectories)} \quad (1.1)$$

The presence of mass "bends" spacetime: in such bent space, the minimum distance between two points is not a straight Euclidean line, therefore, a particle that is not subject to any force will also follow a bent trajectory, called *geodesic*. We can observe the degree of bending of the trajectory to quantify the degree of bending of the underlying geometry. In fact, from an experimentalist point of view, chain (1.1) should be really read in reverse:

$$\text{Observe particles trajectories (or trace geodesics)} \xrightarrow{\text{infer}} \text{Geometry of spacetime} \xrightarrow{\text{infer}} \text{Mass distribution} \quad (1.2)$$

We observe the Earth traveling in ellipses, then we infer that there must be a mass in one of the ellipse foci, where we happen to find the Sun. We observe the velocity of stars in a galaxy, then infer that there must be some dark matter in that galaxy, otherwise the rotation velocities do not make sense. But we also observe that photons from a distant galaxy are more red-shifted the further the galaxy is, then we infer that spacetime, our Universe, must be expanding.

Free-falling test particles, or geodesics, are our way of probing the geometry of spacetime, of measuring curvature. To be more precise, we always need at least two test particles to detect a non-zero curvature (one could be us, carrying our own reference frame). Then, the symptom of curvature is the appearance of a relative acceleration between them. In Figures 1.1, 1.2 and 1.3 some visual examples to illustrate this concept are reported. It should then not be surprising that to detect gravitational effects we just need two free-falling particles and a way to measure and monitor their distance: easy, isn't it?

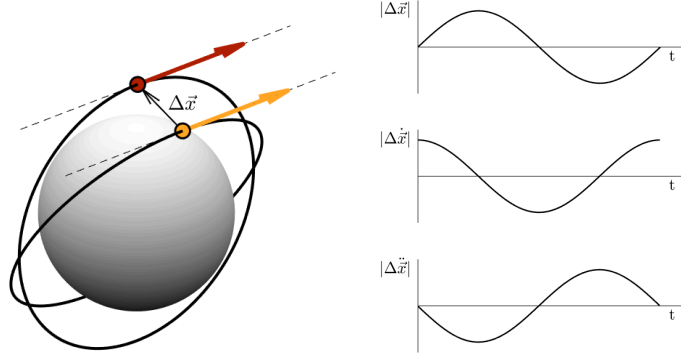


Figure 1.1: Free-falling particles that start on parallel geodesics do not remain parallel if they live on a sphere. Each of them believes to be going straight and could never realize to be living in a curved space. They can discover that by looking at their differential acceleration.

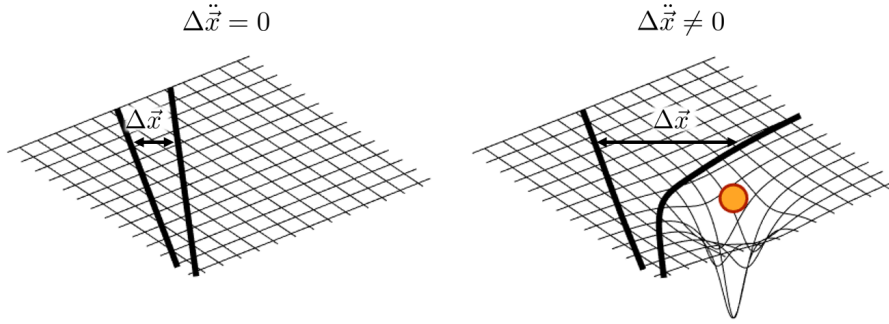


Figure 1.2: Two free particles experience a differential acceleration if spacetime is curved by the mass of a nearby star.

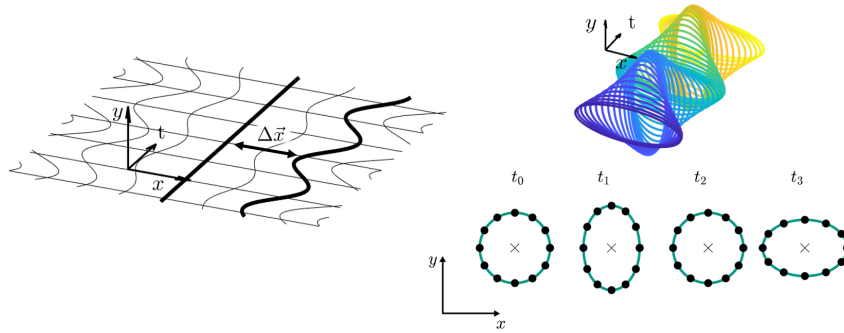


Figure 1.3: (Left) A free-falling observer sees a free falling particle following a sinusoidal geodesic if a plane GW is traveling along z . (Right) Effect of a $+$ polarized GW on a circle of free-falling particles.

Gravitational waves, an invaluable messenger

The geometry of spacetime is dynamical, not a fixed, immutable structure: if the distribution of masses changes in time, also the curvature does. It is in this bendable and stretchable fabric that lives the phenomenon of the Gravitational Wave (GW): we could call it a "curvature wave", literally a ripple that travels through spacetime at the velocity of light, bending the trajectories of free-falling particles. Their existence was predicted since the first days of General Relativity, as a solution to Einstein's equation. Emission of gravitational waves takes place in presence of accelerating, asymmetric masses and was indirectly observed experimentally for the first time by monitoring an electromagnetically visible, binary system of pulsars [67]: the orbital decay due to energy loss from gravitational radiation was in perfect agreement with the predictions of General Relativity.

However, only 10 years ago we started detecting GWs directly. We built instruments which are able to detect absurdly small changes in the geodesic motion of a set of free-falling bodies, and use their measurement to monitor the curvature of spacetime, looking for ripples in it. We are talking about the LIGO/Virgo Gravitational Wave detectors, which detected the first GW in 2015 [2] and have collected hundreds of them since then [3]. They could also detect an extremely important event, called GW170817 [1], which is the first binary neutron star merger with an electromagnetic counterpart. The field of gravitational waves and multi-messenger astrophysics received a huge acceleration from these discoveries.

These instruments do not only tell if there was a gravitational wave or not, they are able to measure *its shape*. The reason why this is so amazingly important is that by observing the spacetime ripple shape, we can understand the dynamics and the nature of the masses that produced it, and sometimes also where they are located in the Sky. A gravitational wave detector then becomes a telescope to see distant sources with unprecedented advantages:

- Pretty handy when the sources in question do not emit any light, black-holes for example.
- GWs are not easily scattered by a dust cloud as electromagnetic waves; on the contrary, their are almost not scattered at all by matter. So they reach us completely unperturbed, even if they come from the most dense places of the Universe like central cores of galaxies or ultra-dense, messy neutron clouds left after the collision of two neutron stars. They allow us to see precisely what happened in the most inaccessible and far places of the Universe.
- In these just mentioned environments, matter is extremely relativistic. GWs then also become a probe of the limits of GR.
- Like the Cosmic Microwave Background exists for electromagnetic waves, a gravitational wave background could exist as well, and its measurement would be of extreme cosmological relevance.

Obviously, accessing to such a spring of knowledge comes at a price.

1.1.1 The challenges of GW measurement

The measurement of GWs is extremely challenging. We can have a first, qualitative look at the difficulties by inspecting the measurement concept introduced in Sec. 1.1 once again:

Detect $\underbrace{\text{relative accelerations}}_{1.}$ of $\underbrace{\text{free-falling masses}}_{2.}$ by $\underbrace{\text{monitoring their distance}}_{3.}$.

1. **Relative accelerations:** The typical relative accelerations caused by GW are absurdly small. In particular, if we use two test-masses which are separated by a distance L , the change in relative distance δL due to a GW is of order

$$\frac{\delta L}{L} \sim 10^{-20} \quad (1.3)$$

This means that if the TMs are separated by 1 km, we have to measure a change in distance 1/100 of a nucleus size. If they are separated by 1 million kilometers, instead, they will move hundreds of pico-meters, the size of one atom. If we imagine that the change in length is sinusoidal with frequency ω , the corresponding acceleration is $|a| = \omega^2 \delta L$, which is less intuitive to interpret, but gives us an idea of the true "curvature observable" that we are interested in. Some relevant examples are summarized in Table 1.1

TMs relative accelerations from GW

	$L = 1 \text{ km } (\delta L \sim 10 \text{ am})$	$L = 1 \times 10^6 \text{ km } (\delta L \sim 10 \text{ pm})$
$f = 1 \text{ kHz}$	10^{-9} ms^{-2}	10^{-4} ms^{-2}
$f = 1 \text{ mHz}$	10^{-21} ms^{-2}	10^{-16} ms^{-2}

Table 1.1: Examples of relative accelerations between free-falling bodies for $\delta L/L = 10^{-20}$, as function on frequency ω of GW and distance L between test-masses.

2. **Free-falling test masses:** It turns out that finding a truly free-falling test mass is also a challenge. In fact, any force F on the mass will cause an acceleration

$$a = F/M. \quad (1.4)$$

The presence of noisy, spurious accelerations buries the acceleration due to GW in noise, making it impossible to detect. In Table 1.1 we can already see some reference values: when measuring a low frequency GW with a 1 million km detector, for example, we have to measure accelerations of fms^{-2} , and spurious forces must be accordingly small, so fN for a 1 kg mass. We will see in the next sections how difficult it can be to realize this "level of free-fall".

3. **Monitoring their distance:** Even if you were able to eliminate acceleration disturbances to the levels required at point 2. , you still need an apparatus which can measure variations of $1/100$ of a nucleus width on a distance of 1 km! The characteristic length measured by a laser, for example, is its wavelength, and the ones that we can use better are around $\lambda=1\mu\text{m}$. The effect we want to measure is just around 12 orders of magnitude lower than that...

1.2 Gravitational waves measurement

The quantity that we use to describe gravitational waves is called *strain*, indicated with letter h . It expresses how much spacetime deviates from flatness in presence of a gravity wave. As a first approach, this deviation is assumed to be very small, which means saying that the *metric tensor*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad |h_{\mu\nu}| \ll 1. \quad (1.5)$$

and $\eta_{\mu\nu}$ the Minkowski metric. In this regime, called *weak field limit*, Einstein's equations can be linearized, to then discover that spacetime ($h_{\mu\nu}$) obeys a wave-equation, as shown in basically any GR textbook [54, 23]. To obtain this result in a simple form, a special coordinate system is usually chosen, called the Transverse-Traceless Gauge (TT-Gauge), where one possible solution to the wave-equation is a monochromatic wave traveling along the z axis, of form

$$\bar{h}_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos[\omega(t - z/c)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t, z) & h_\times(t, z) & 0 \\ 0 & h_\times(t, z) & -h_+(t, z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.6)$$

where we can spot a few parameters: ω is the GW frequency and it travels at the speed of light c . There are no deformations along t and z , only along x and y , meaning the GW is *transverse*. There are other two independent parameters, h_+ and $h_\times$, called $+$ and $\times$ polarizations, which determine the directions in which spacetime contracts/stretches. The nature of this stretching is quadrupolar, meaning that spacetime is stretched in one direction and contracted along the perpendicular one, as sketched for a circle of free-falling particles in Figure 1.3.

Effect of gravitational waves on photons

Not only free-falling, massive particles are affected by GWs, but also photons. In particular, if a photon travels through space while a GW is passing, its travel time will be different from that of another photon traveling in flat space, free of GWs. Detectors that successfully detected GWs up to now are based on the observation of changes in photon travel time.

A very simplified setup that could perform this kind of measurement could look like the one in Figure 1.4. A free-falling laser emits some light which is sent to a distant, also free-falling

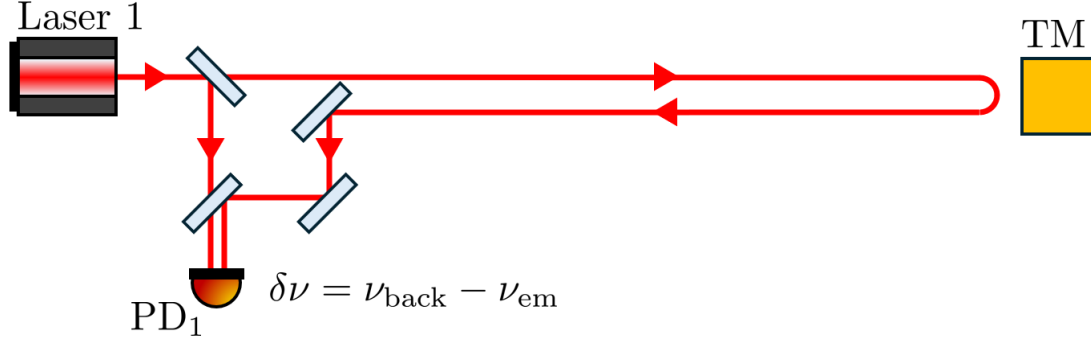


Figure 1.4: Very simplified setup for the measurement. A free-falling laser sends light to a distant, also free-falling, TM, then interferes the outgoing beam and the incoming one on a photodiode. If the two beams have different frequencies, they will generate a beat-note $\delta\nu$, containing the GW effect.

mirror. The beam reflected and comes back to the emission point after a round trip time $2T$: here, it is interfered with the outgoing laser. A change in light travel time will induce a time dependent phase-shift between the outgoing and incoming beams, which can be alternatively interpreted as a frequency shift. This frequency shift $\delta\nu_{GW}$ contains the information about the GW, and is derived in many textbooks, like [54]. For a $+$ polarized GW of amplitude $h_+(t)$, coming from a direction perpendicular to the detector axis in Figure 1.4, it reads

$$\frac{\delta\nu_{GW}(t)}{\nu_{em}} = -\frac{1}{2} [h_+(t) - h_+(t - 2T)] . \quad (1.7)$$

where ν_{em} is the frequency of the emitted laser. The observed frequency shift is proportional to the difference between the strain at the photon emission time $h_+(t - 2T)$ and that at detection time $h_+(t)$. Notice that the strain really does not depend on $h_+(t - T)$, the strain at the temporal location of the reflecting mirror. If we consider a "slow GW" - that has a period much larger than the photon round trip travel time $\omega \ll \pi/T$, or, alternatively, $h(t)$ changes on time scales much longer then the travel time T - we can write

$$\frac{\delta\nu_{GW}(t)}{\nu_{em}} = -T \frac{[h_+(t) - h_+(t - 2T)]}{2T} \approx -T \dot{h}_+(t) = -\frac{L}{c} \dot{h}_+(t) . \quad (1.8)$$

This underlines the fact that a larger L results in an enhancement of the photon frequency shift.

Alternative interpretation

We must imagine that if we did not know anything about GR (i.e. we were in a local inertial frame) and observed a frequency shift in the reflected laser beam that is received back at the initial emitting laser, we would conclude that the distant TM must be moving at velocity v

and we are actually seeing a Doppler shift, following the formula

$$\frac{\delta\nu}{\nu} = -\frac{v}{c} \quad \left(\rightarrow \frac{\delta\dot{\nu}}{\nu} = -\frac{g}{c} \right) \quad (1.9)$$

By using this identity in Eq. (1.8), we get that $v(t) \approx L\dot{h}_+(t)$, which is intuitively correct, since if $\dot{h} > 0$ it means that spaced is being elongated, and the masses are getting away from each other.

Now as discussed in Section 1.1, it is not really in the relative velocity of two TMs that one spots the effect of curvature, but in the *relative acceleration*. From this consideration and going back to the accurate formula (1.7), we understand that the real quantity of interest is

$$\Delta_{2T}\dot{h}(t) \equiv \frac{\delta\dot{\nu}_{GW}(t)}{\nu_{em}} = -\frac{1}{2} \left[\dot{h}_+(t) - \dot{h}_+(t-2T) \right]. \quad (1.10)$$

1.2.1 Strain from binary systems

Now that we have a way of measuring the GW strain h , we want to understand the size of this effect. To do so, we must talk about the sources of GW.

By working in the linearized regime of small metric deformations, Einstein's equation can be solved in the presence of a localized distribution of mass, $\rho(t, \mathbf{x})$ being its density. In the hypothesis that these masses are not moving at relativistic velocities, the 3 spatial components of the metric perturbation at a point $(t, \mathbf{x})$, at a great distance r from the source can be written through the *quadrupole formula* [23]

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{c^4 r} \ddot{I}_{ij}(t - r/c), \quad I_{ij} = \int \rho(t, \mathbf{x}) x^i x^j d^3x \quad (1.11)$$

where I_{ij} is the mass quadrupole moment. We will not do calculations with this formula, it is here to show that to have GW we need a system where the second derivative of the mass quadrupole moment is not zero. This system must then exhibit a certain degree of mass asymmetry, that must also change in time: obviously, more mass and a faster dynamics, implies an higher GW amplitude.

Binary systems

Compact binary systems are the prime example of a mass distribution that meet all the properties required above for the emission of GW. They consist of two bodies orbiting each other at a certain frequency: while they orbit, GWs are emitted and energy is radiated away. This causes the radius of the orbits to shrink, while the orbital frequency increases for angular momentum conservation. The increased velocity, in turn, increases the GW emission, speeding up the orbital shrinking, until they collide and finally relax to a spherically symmetric object. These well known phases of inspiral, merger and ringdown are the main features of the typical GW shape, represented in Figure 1.5.

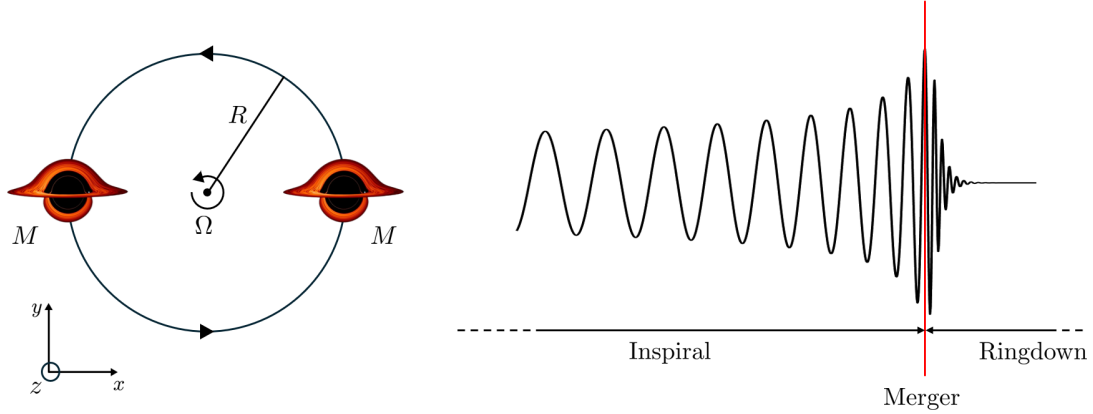


Figure 1.5: (Left) Two compact objects with the same mass in circular orbit used as a simplified example. Black-holes images from: NASA’s Goddard Space Flight Center/Jeremy Schnittman and Brian P. Powell. (Right) Example of the typical shape of a GW from a binary merger.

Binary systems which can emit a significant amount of GW must be very compact, so that they can reach very high orbital velocities (almost the speed of light) before colliding. Therefore, the protagonists of these events are surely black-holes and neutron stars. However, also other system like white dwarfs binaries can emit significant GWs.

While the majority of the inspiral can be very well modeled through (1.11) and does not depend so much on the nature of the involved objects, the dynamics of the merger are vastly different if the objects are black-holes, neutron-stars or something else. In this regime, bodies with the mass of multiple Suns can spin at nearly the velocity of light: all approximations done so far fail, and numerical relativity simulations are required, but still it is were we can extract the most information about the source and test the limits of GR. Measuring the wave at the merger moment is then one of our priorities.

To get an idea of the scales of a signal like the one in Figure 1.5, we consider the simple case of two objects of equal mass and a circular orbit, represented in the same Figure 1.5. In order to design a detector we need to know at least three things: the **frequency** of the signal, its **amplitude** and its **duration**. Rather than deriving their formulas which can be easily found on textbooks, we try to focus on their motivations and on understanding their orders of magnitude.

- **Frequency:** if the objects are far apart, we can use Kepler’s law to find the orbital frequency [23, p. 305]

$$\Omega = \sqrt{\frac{GM}{4R^3}}. \quad (1.12)$$

The GW frequency will then be

$$f_{gw} = 2\frac{\Omega}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{4R^3}}, \quad (1.13)$$

because the quadrupole moment is symmetric to interchange of the two masses (this result also follows by direct calculation from (1.11)). By inverting eq. (1.13) for R , f_{gw} can be used to determine at which point of the inspiral we are. It is then useful to describe the other GW features through f_{gw} .

As motivated before, f_{gw} will have some kind of maximum, the merger frequency. Using formula (1.13) will not give the exact result since it does not hold in this regime, but it will capture the correct scaling. Some examples. A binary system of neutron stars with $M=1.4 M_\odot$ each and a radius of ≈ 10 km will have

$$f_{\text{gw,merg}}^{\text{BNS}} \approx 2 \text{ kHz} \left(\frac{1.4 M_\odot}{M} \right) \quad (1.14)$$

If the two objects are black-holes, the collision will take place when the two event horizons touch, which means $R = R_s = 2GM/c^2$, the Schwarzschild radius. Then

$$R_s \approx 30 \text{ km} \left(\frac{M}{10 M_\odot} \right), \quad f_{\text{gw,merg}}^{\text{BBH}} = \frac{1}{4\pi\sqrt{2}} \frac{c^3}{GM} \approx 1 \text{ kHz} \left(\frac{10 M_\odot}{M} \right) \quad (1.15)$$

- **Amplitude:** it can be calculated from (1.11) to find that for both polarizations [23, p. 307]

$$h = \frac{2G^2}{c^4} \frac{M^2}{rR} \sim \frac{2(2\pi)^{2/3} G^{5/3}}{c^4} \frac{M^{5/3} f_{gw}^{2/3}}{r} \quad (1.16)$$

For an order of magnitude estimate, we will consider two cases. The first is for sources that are typically interest of Earth-based detectors, of several solar masses $M_\odot$ and merger frequencies in the kHz range

$$h \approx 10^{-22} \left(\frac{M}{M_\odot} \right)^{\frac{5}{3}} \left(\frac{f_{gw}}{1 \text{ kHz}} \right)^{\frac{2}{3}} \left(\frac{100 \text{ Mpc}}{r} \right) \quad (1.17)$$

the most famous of this category being neutron stars and black-holes up to hundred solar masses. The distance of 100 Mpc is the typical scale of observation of Earth-based detectors. The other example instead is more related to space-based detectors, regarding masses of order $10^{5-7} M_\odot$ and merger frequencies of mHz

$$h \approx 5 \times 10^{-19} \left(\frac{M}{10^6 M_\odot} \right)^{\frac{5}{3}} \left(\frac{f_{gw}}{1 \text{ mHz}} \right)^{\frac{2}{3}} \left(\frac{26 \text{ Gpc}}{r} \right). \quad (1.18)$$

The protagonists here can only be super-massive black-holes, the only compact objects which can have such masses. Given the $M^{5/3}$ dependence, these sources produce much larger strains and can be seen from great distances, which are usually better described in terms of red-shift z : in this case 26 Gpc corresponds to $z=3$.

- **Duration:** in the previous heuristic description, the inspiral speeds up with time. So, the binary will spend more time at large radii and low orbital frequencies, with respect

to small radii and high orbital frequencies. By properly integrating the power radiated away by the binary [23, p. 314], it is possible to calculate the so called *time to merger*

$$\tau = \left(\frac{1}{f} \frac{df}{dt} \right)^{-1} \propto c^5 M^{-5/3} G^{-5/3} f_{\text{gw}}^{-8/3}. \quad (1.19)$$

This quantity gives an order of magnitude of the time that a binary spends at a certain frequency. I believe it is easier to impress the numbers that come out of this formula by representing them in a sketch, like the one in Figure 1.6, which really summarizes all the qualitative and quantitative observations we did in this Section. These are the kind of signals we are looking for, at least for binary systems.

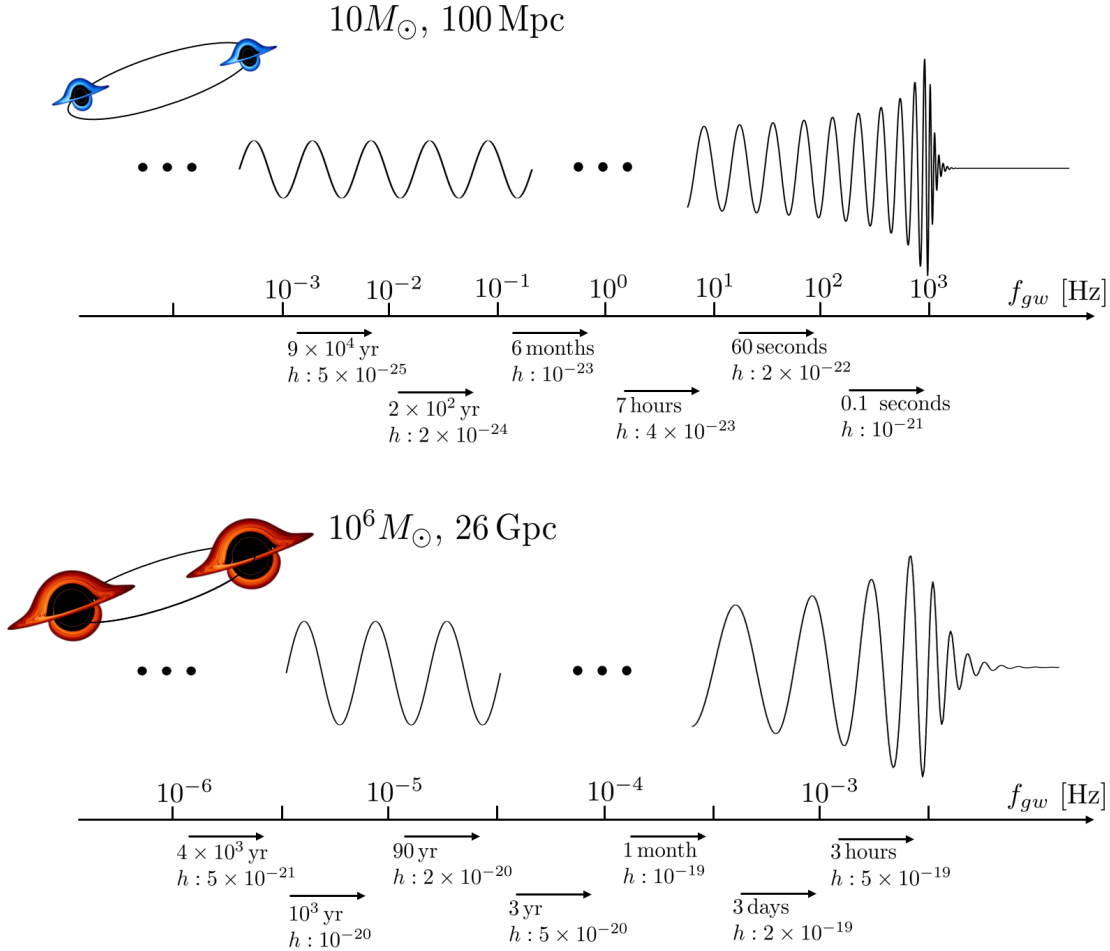


Figure 1.6: A qualitative drawing of the shape of a GW from a binary system, with orders of magnitude estimates of: frequency of the wave, amplitude and time spent at that frequency. The two proposed system are relevant examples for Earth-based detectors ($10M_{\odot}$, 100 Mpc), and space-based ones ($10^6 M_{\odot}$, 26 Gpc).

1.3 Main noise sources in GW detectors

Only now that we have a rough idea of the signals we expect, we can start discussing the consequences of not having easy access to truly free-falling test masses and to attometer length measurements over many kilometers: these two were the bullets discussed in Section 1.1.1. In particular, we will introduce noise due to stray accelerations of the inertial mirrors,, due to laser frequency fluctuations, and "readout noise" in the laser frequency measurement. Calculating and understanding all these terms, even with a very simplified approach, should allow to get an insight on why we need LISA and how it is designed.

1.3.1 Strain sensitivity

Before diving in the specific noise sources, let us introduce a generic noise $n(t)$ in what we identified as our signal, (1.10):

$$\Delta_{2T}\dot{h}(t) = -\frac{1}{2} \left[\dot{h}_+(t) - \dot{h}_+(t - 2T) \right] + n(t). \quad (1.20)$$

To quantitatively deal with this problem, where our goal is detecting a small signal into a noisy background, we move to the frequency domain and adopt the formalism of Wiener's optimal filter (also called matched filter). If we indicate with $S_n(\omega)$ the one-sided Power Spectral Density (PSD) of noise $n(t)$, then our capability of detecting a small signal $s(t)$ in the noisy background is quantified by the Signal to Noise Ratio (SNR):

$$\text{SNR}(\omega) = \frac{|\tilde{s}(\omega)|^2}{S_n(\omega)}, \quad (1.21)$$

where ω is the angular frequency and $\tilde{s}$ indicates the Fourier Transform (FT) of $s(t)$. We can calculate this quantity for our GW signal

$$FT \left[\frac{\dot{h}_+(t - 2T) - \dot{h}_+(t)}{2} \right] = \frac{i\omega}{2} \left(\tilde{h}_+ e^{i\omega 2T}(\omega) - \tilde{h}_+(\omega) \right) = -\omega e^{i\omega T} \sin(\omega T) \tilde{h}_+(\omega) \quad (1.22)$$

Then the SNR of (1.21) becomes

$$\text{SNR}(\omega) = |\tilde{h}_+(\omega)|^2 / \left(\frac{S_n(\omega)}{\omega^2 \sin^2 \omega T} \right). \quad (1.23)$$

This clearly highlights the quantity that fixes our sensitivity to GW, which we call

$$S_{\text{strain}}(\omega) = \frac{S_n(\omega)}{\omega^2 \sin^2 \omega T}. \quad (1.24)$$

This is a quantity that *depends only on the detector*. We choose the travel time T , we choose where and how to build it, therefore having to deal with certain noise sources in S_n . This quantity is the one we should be looking at during the design of a detector.

What strain sensitivity do we need?

Can we give a number for (1.24)? We need to know what to aim for when designing a detector.

We do it by hand-waving again, and we will give a very rough order of magnitude. In Figure 1.6 orders of magnitude estimates were given exactly for this purpose: let's take for example the two super-massive, $10^6 M_\odot$ black-holes at 26 Gpc. They spend 3 hours orbiting at ≈ 1 mHz before merger, with a typical $h \sim 10^{-18}$. Therefore, we approximate the GW signal as a sinusoidal wave at $\omega_0 = 2\pi \times 1$ mHz frequency, $h = 10^{-18}$ amplitude and of duration $T = 3$ h:

$$h(t) = h \sin(\omega_0 t) \Pi(t/T) \quad (1.25)$$

where $\Pi(t/T)$ is the box function, zero everywhere and 1 for a window of length T . For such a simple signal, Wiener's theory provides a simple formula for the error σ_h that we commit in estimating the amplitude h after recovering the signal, given it is affected by a certain noise.

$$r^2 = \left(\frac{\sigma_h}{h}\right)^2 = \left(\frac{1}{2\pi} \int_0^\infty \text{SNR}(\omega) d\omega\right)^{-1} \approx \frac{1}{h} \sqrt{\frac{S_{\text{strain}}(\omega_0)}{T}} \quad (1.26)$$

How well do we want to measure this signal? This is up to us (astrophysicists tell us precisely how well we should resolve these signals to understand the sources and test GR). Let's say we want a relative uncertainty of $r=1/100$, as a reference to start. Then by inverting (1.26)

$$(10^6 M_\odot) \quad S_{\text{strain}}^{1/2}(1 \text{ mHz}) = rh\sqrt{T} \approx 5 \times 10^{-19} \text{ Hz}^{-1/2} \quad (1.27)$$

We could also extend this reasoning to sources in the mass regime of order $M_\odot$ and to their consequently higher merger frequencies and shorter observation times. Always from Figure 1.6, we extract the data $\omega_0 = 2\pi \times 100$ Hz, $h = 10^{-21}$, $T = 0.1$ s. Then

$$(10 M_\odot) \quad S_{\text{strain}}^{1/2}(100 \text{ Hz}) = rh\sqrt{T} \approx 3 \times 10^{-24} \text{ Hz}^{-1/2} \quad (1.28)$$

We will use these reference values to constrain the sources of noise. So, in the next section we will try to give an estimate of $S_n(\omega)$ in (1.24), examining the main noise sources constituting $n(t)$ and how they are tackled in Earth and space-based detectors.

1.3.2 Stray accelerations of the end mirrors

Being in free-fall means being in complete absence of forces, which is a condition not easily realized. The test masses interact electromagnetically with their surroundings (possibly with the apparatus we use to measure their position) and they are hit by other particles moving around¹. As a result of these interactions, the distant TM in Figure 1.4 accelerates to a

¹If we are on Earth, they are also subject to gravity caused by masses moving around on Earth, like sea waves, or atmosphere masses, or cars. These cause true geodesic deviations, but they hide the much smaller ones that we are interested in, GWs!

velocity $v(t) \ll c$ ($v(t) > 0$ means it is getting away), then a photon bouncing on that mirror (at time $t - T$) will be classically Doppler shifted:

$$\frac{\nu_{back}}{\nu_{em}} = 1 - \frac{2v(t - T)}{c} \quad \rightarrow \quad \frac{\delta\nu_g(t)}{\nu_{em}} = -\frac{2v(t - T)}{c}, \quad (1.29)$$

the factor two coming from the fact that it "bounces" back elastically. In terms of *variations of the frequency shift* $\delta\dot{\nu}$, from (1.29) we have

$$\frac{\delta\dot{\nu}_g(t)}{\nu_{em}} = -\frac{2\dot{v}(t - T)}{c} = -\frac{2g_2(t - T)}{c}. \quad (1.30)$$

To complicate the things even further, we must now include the fact that also the emitter can be subjected to unwanted accelerations $g_1(t)$, which enter the previous equations at emission and detection

$$\frac{\delta\dot{\nu}_g(t)}{\nu_{em}} = \frac{g_1(t - 2T) - 2g_2(t - T) + g_1(t)}{c}. \quad (1.31)$$

1.3.3 Laser frequency and readout noise

We now address the issue of laser frequency noise. In fact, true lasers are not able to produce a perfectly stable monochromatic wave, but will exhibit a certain spectral width. As a result, the frequency of the light will vary with time. Always referring to Figure 1.4 if we imagine to emit a photon at time $t - 2T$ with frequency $\nu_{em} = \nu_1 + \delta\nu_f(t - 2T)$ and that there are no GW, nor acceleration noise, after its round trip at time t we will compare its frequency with our local laser, which will have in general a different frequency $\nu_1 + \delta\nu_f(t)$. To complete the picture, we must consider the noise introduced by the frequency measurement. As described in Figure 1.4, the incoming beam is interfered with the local laser and sent to a photodiode, which will add for example some shot noise at the time of the measurement $\delta\nu_{r1}(t)$. Putting everything into formulas

$$\begin{aligned} \frac{\dot{\nu}_{back} - \dot{\nu}_{em}}{\nu_{em}} &= \frac{[\dot{\nu}_1 + \delta\dot{\nu}_f(t)] - [\dot{\nu}_1 + \delta\dot{\nu}_f(t - 2T)] + \delta\dot{\nu}_{r1}(t)}{\nu_1} \\ &= \frac{\delta\dot{\nu}_f(t) - \delta\dot{\nu}_f(t - 2T) + \delta\dot{\nu}_{r1}(t)}{\nu_1}. \end{aligned} \quad (1.32)$$

We have the right to approximate the denominator to just ν_1 since corrections would result in quadratic terms in $\delta\nu$.

1.3.4 Putting all together

We can now summarize all the noise terms we found in our single arm detector by rewriting the complete $n(t)$ from (1.20):

$$\begin{aligned} n(t) &= \frac{1}{c} [g_1(t - 2T) - 2g_2(t - T) + g_1(t)] \\ &\quad + \frac{1}{\nu_1} [\delta\dot{\nu}_f(t) - \delta\dot{\nu}_f(t - 2T) + \delta\dot{\nu}_{r1}(t)] \end{aligned} \quad (1.33)$$

for which we want to calculate the spectrum $S_n(\omega)$. While we can safely assume that noises added in different spacecrafts are uncorrelated (for example g_1 and g_2), we must be careful in calculating the PSD of delayed signals like $g_1(t - 2T) + g_1(t)$, since they will be surely correlated and the spectrum will not be just a sum of the two. The calculation is carried out in Appendix 9.1 and here we use directly the result to obtain:

$$S_n(\omega) = \frac{1}{c^2} [4 \cos^2(\omega T) S_{g_1} + 4 S_{g_2}] + \frac{1}{\nu_1^2} [4 \sin^2(\omega T) S_{\delta \nu_f} + S_{\delta \nu_{r1}}] \quad (1.34)$$

This is the complete expression we were looking for, but we simplify it a little bit in order to make it easier to read and understand. First of all, we assume that there is no particular difference between the two spacecrafts, so $S_{g_1} = S_{g_2} = S_g$. Then we rewrite $S_{\delta \nu_f}$ in terms of relative laser frequency fluctuations, namely $S_{\delta \nu_f}/\nu_1^2 = \omega^2 S_{\delta \nu_f/\nu}$. Finally, to use a notation which is more diffused in the literature for the readout noise $S_{\delta \nu_{r1}}$, we imagine it is related to the measurement of a displacement, as if the noise was due to a motion of the observer. To do this, we use the following identity

$$\frac{|\dot{\nu}|}{\nu} = \frac{|\dot{x}|}{c} = \frac{|\ddot{x}|}{c} \quad (1.35)$$

then we can write

$$\frac{1}{\nu_1^2} S_{\delta \nu_{r1}}(\omega) = \frac{\omega^4}{c^2} S_{IFO}(\omega), \quad (1.36)$$

where $S_{IFO}(\omega)$ has units of m^2/Hz and stands for "interferometric". With all these transformations (1.34) reduces to

$$S_n(\omega) = \frac{4}{c^2} [\cos^2(\omega T) + 1] S_g + 4 \omega^2 \sin^2(\omega T) S_{\delta \nu_f/\nu} + \frac{\omega^4}{c^2} S_{IFO} \quad (1.37)$$

and we can finally calculate the strain sensitivity defined in (1.24):

$$S_{\text{strain}}(\omega) = \frac{1}{c^2 \omega^2 \sin^2 \omega T} [4(\cos^2(\omega T) + 1) S_g + \omega^4 S_{IFO}] + 4 S_{\delta \nu/\nu} \quad (1.38)$$

Now, before commenting this result, I think it is important to recall in which frame this derivation is valid: this is the sensitivity of a **one arm detector** ideally aligned to a **+** **polarized** gravitational wave, propagating along a direction **perpendicular** to the detector arm. Therefore, it does not represent exactly the sensitivity curve that you usually find on papers about LISA for example, which instead takes into account all the three arms and is the result of an average over all possible GW directions and polarization. Nonetheless, (1.38) still allows to capture the essential features of the detector sensitivity:

- **Stray accelerations noise:** $S_g(\omega)$ dominates in the low frequency limit, since it is multiplied by ω^{-4} , while it is suppressed at high frequencies. Thus, our sensitivity to low frequency GW is fixed by how well we can limit stray forces on the end mirrors. The work we do at the Experimental Gravitation Laboratory in Trento and much of what is contained in this thesis is dedicated to the study of force noise sources.

- **Interferometer noise:** $S_{IFO}(\omega)$ dominates at high frequencies and is mainly related to shot noise at the photodiode. This is an unavoidable noise source that arises from the stochastic nature of light and it is well known to improve as $1/\sqrt{P}$, where P is the light power on the photodiode. How this translates to our measurement of laser phase/frequency is calculated in Appendix 9.2 and reported here

$$S_{IFO}(\omega) \approx \left(\frac{\lambda}{2\pi} \right)^2 \frac{hc}{\lambda} \frac{1}{P_{back}}, \quad (1.39)$$

where P_{back} is the received power of the laser beam that comes back after bouncing on the far test mass and λ its wavelength. This is a white noise, let's see how it affects strain sensitivity (1.38):

$$S_{\text{strain}}^{IFO}(\omega) = \frac{\omega^2 S_{IFO}}{c^2 \sin^2 \omega T} \rightarrow \begin{cases} \omega T \ll 1 & \rightarrow S_{\text{strain}}^{IFO}(\omega) \approx \frac{1}{c^2 T^2} S_{IFO} \\ \omega T \gg 1 & \rightarrow S_{\text{strain}}^{IFO}(\omega) \geq \frac{\omega^2}{c^2} S_{IFO} \end{cases} \quad (1.40)$$

We extract two messages from this: first, the low frequency noise floor created by interferometer photon statistics is white and can be improved with larger photon travel times T . Second, for $\omega > 1/T$ we start losing sensitivity $\propto \omega^2$, therefore, if we are interested in a certain GW frequency f_{target} , **it is not convenient to have a detector much larger than $L = c/2\pi f_{\text{target}}$!** Notice also we are completely insensitive to GWs which have frequency equal to $1/2T$ and all its multiples. To put this in a more intuitive way, to detect a low frequency GW we must leave enough time to the photon to travel through it, or the effect will not be large enough: however, doing this, higher frequency GWs will shift the photon frequency back and forth multiple times during its travel, ending up cancelling the effect.

Summarizing, the choice of L fixes to which GW spectrum band we are sensitive, analogously to choosing the length of an antenna to detect microwaves rather than radio waves.

- **Laser frequency noise:** $S_{\delta\nu/\nu}(\omega)$ is instead a major issue since it enters directly in our strain PSD. As an example the required level of laser frequency fluctuations for LISA is $30 \text{ Hz}/\sqrt{\text{Hz}}$ down to 2 mHz [35], and this is already several orders of magnitude better than commercially available lasers (requires an ultra-stable cavity for stabilization). This number correspond to relative frequency fluctuations of $10^{-13} \text{ Hz}^{-1/2}$, which is still 7 orders of magnitude higher than what we need, i.e (1.27). This is the reason why our current one arm detector will never work, it is simply the wrong instrument to perform the measurement. To overcome this, our detector must have at least two arms: if the same laser is split and sent through both arms, when the two beams come back after bouncing on the far test mass, they can be recombined in order to cancel out frequency noise. We will see more in the next sections.

Through this chapter, the important formulas about gravitational wave detection were presented, always putting a focus on the orders of magnitude of the effects under consideration. Therefore, the chapter would not be complete without an order of magnitude estimation

of the quantities in (1.38).

We calculated the strain sensitivity required to measure different kinds of GWs in (1.27) and (1.28): we can use these estimates to infer what acceleration noise S_g and IFO noise S_{IFO} are required for the measurements. To do this, we first simplify (1.38) by working at low frequencies $\omega T = \omega L/c \ll 1$ and supposing that we found out a way of getting completely rid of $S_{\delta\nu/\nu}$, as explained at bullet 3. Then we have

$$S_{\text{strain}}(\omega) = \frac{1}{L^2} \left[\frac{8S_g(\omega)}{\omega^4} + S_{IFO}(\omega) \right]. \quad (1.41)$$

Again, here it is even more evident that the sensitivity improves with L , but we must always keep in mind that if our target is a GW of frequency f_{target} we must have (see discussion on IFO noise, bullet 2 in this section)

$$L \sim \frac{c}{2\pi f_{\text{target}}} \quad \begin{cases} f_{\text{target}} \sim 100 \text{ Hz} & \rightarrow L \sim 500 \text{ km} \\ f_{\text{target}} \sim 1 \text{ mHz} & \rightarrow L \sim 50 \times 10^6 \text{ km} \end{cases} \quad (1.42)$$

Supposing we are able to build detectors of such length, then what S_g and S_{IFO} would be required to meet the strain sensitivity requirement? Inverting (1.41), we calculate separately $S_g = L^2 \omega^4 S_{\text{strain}}/8$ and $S_{IFO} = L^2 S_{\text{strain}}$. Results are summarized in Table 1.2.

Toy detector requirements

h_{target}	L [km]	f_{target} [Hz]	$S_{\text{strain}}^{1/2}(f_{\text{target}})$ [Hz ^{-1/2}]	$S_g^{1/2}(f_{\text{target}})$ [ms ⁻² /Hz ^{1/2}]	$S_{IFO}^{1/2}(f_{\text{target}})$ [m/Hz ^{1/2}]
10^{-21}	500	100	3×10^{-24}	1×10^{-13}	1×10^{-18}
10^{-18}	50×10^6	10^{-3}	5×10^{-19}	2×10^{-13}	2.5×10^{-8}

Table 1.2: Detector spurious acceleration and IFO requirements to detect 3 periods of a sinusoidal wave of amplitude h_{target} with a relative error of 0.1%. This can be considered a more precise version of Table 1.1, but it is still just an order of magnitude estimate, to have a first idea of real detectors.

Using these values, we can now visualize the sensitivity curves for these two "toy" detectors in Figure 1.7. S_g and S_{IFO} are supposed to be white and equal to the values in Table 1.2, which is almost always **not true!** So these plots must be taken with a grain of salt, since they are meant to just give a first idea of these fundamental limiting sources and their orders of magnitude in a plot of sensitivity. We notice that with our choice of L , we achieve almost max sensitivity at the frequency of interest, but if we want to enlarge the detection band, it would be convenient to sacrifice a bit of sensitivity and use a lower L (compare the fuchsia curve to the black solid curve).

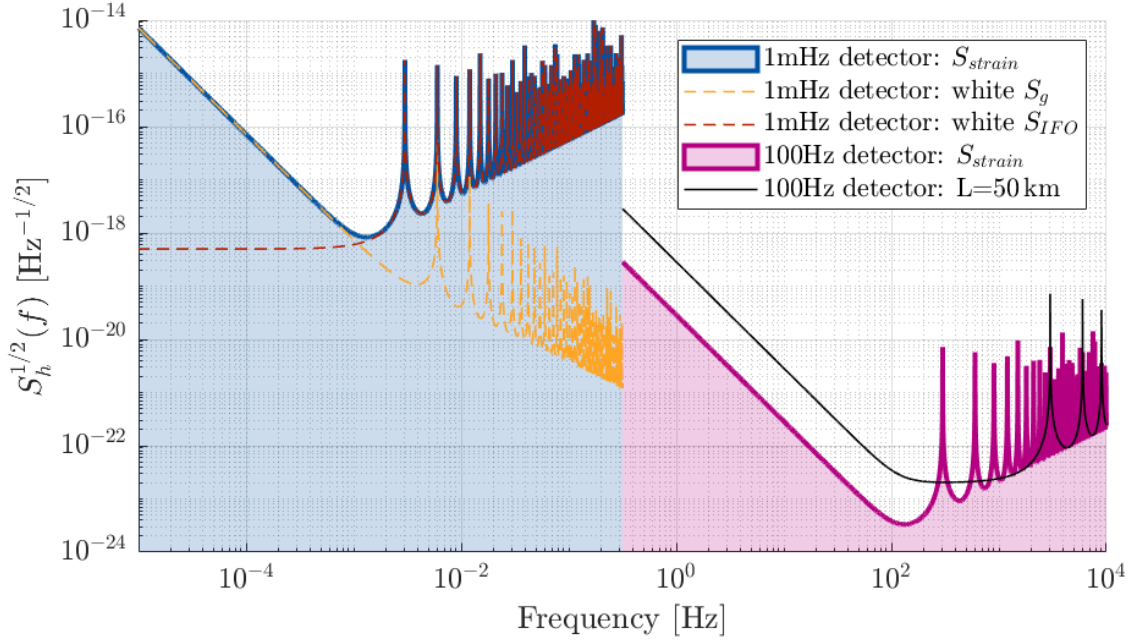


Figure 1.7: Strain sensitivity curves of the two "toy" detectors, assuming white S_g and S_{IFO} (this is usually not the case!). Blue and fuchsia use (1.41) with values in Table 1.2. Dotted lines show S_g and S_{IFO} separately for the low frequency GW detector. Black solid line uses the same PSD values of Table 1.2, but a length of $L=50$ km, to show the effect on the sensitivity curve.

We reached this chapter's goal of understanding which are the levels of free-fall and interferometric precision required when we are interested in measuring a source of a certain frequency and amplitude: these requirements are expressed in terms of the two quantities $S_g(f)$ and $S_{IFO}(f)$ that can be solely derived considering which scientific sources we want to observe. We specified very little about how to actually build this detector!

With these numbers in mind, one can then be creative on how to realize an apparatus which can actually achieve those numbers for $S_g(f)$ and $S_{IFO}(f)$ (and don't forget $S_{\delta\nu/\nu}$!). In the next chapter we will see how LISA tackles these aspects by having a look at the fundamental blocks of its design.

Chapter 2

LISA and the Gravitational Reference System

In the previous chapter we saw the fundamental quantities that must be addressed in building a GW detector: acceleration noise, interferometric noise and frequency noise. In this chapter we see more details of the actual detector design.

LIGO/Virgo and Kagra are Earth-based detectors with typical detection frequencies between 10 to 1000 Hz, therefore addressing in particular coalescences of Neutron Stars and Black-Holes up to masses of $100 M_{\odot}$. In Figure 2.1 the sensitivity curve of Advanced LIGO is reported, which is highly comparable to the one of Figure 1.7, designed ad-hoc for the same sources. The instruments are Michelson interferometers of 3-4 km length, in which the end-mirrors are the free-falling test masses whose relative motion is measured interferometrically. A sketch can be found in the same Figure 2.1. The mirrors are also designed to create resonant cavities with an amplification factor of order hundreds. A single photon bounces multiple times between the mirrors, so the GW wave signal is enhanced. We can not go into the details of every noise source determining the sensitivity curve, we limit ourselves to discuss the ones we already examined in the previous chapter.

- Laser frequency noise is not an issue: in an equal arm Michelson interferometer, the same laser beam is split in the two arms. When it is recombined, both beams share the same frequency noise, which cancels out.
- Shot noise: dominates at high frequency and is addressed with high laser power (which brings various other issues like coatings thermal noise) and light squeezing [61].
- TM acceleration noise: mirrors are very large (40 kg), in vacuum, suspended in pendulums with multiple stages, so that they are in "free-fall" along the horizontal axis. First of all, the detector has again a fundamental limit fixed by quantum fluctuations of radiation pressure noise. This gets worse when the laser power is increased, and can also be addressed with light squeezing. On top of this, there is a series of other acceleration noise sources affecting the instrument low frequency sensitivity. One for example

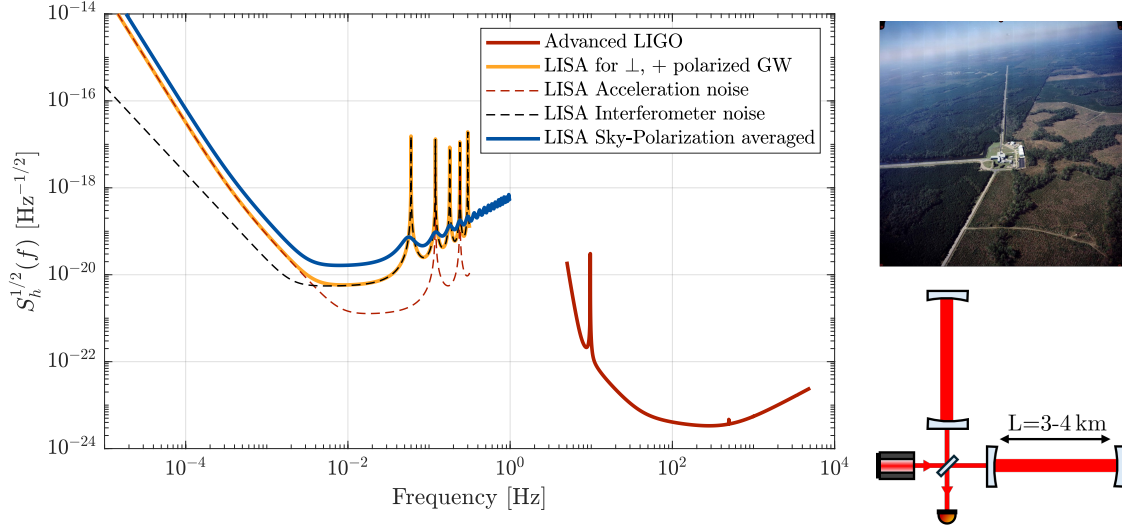


Figure 2.1: [Left] LISA and Advanced LIGO sensitivity curves, covering the low-frequency and high-frequency bands of the GW spectrum. (Blue) LISA sensitivity is a sky-polarization averaged estimate taken from [18]. (Orange) LISA sensitivity calculate with (1.38) and (2.2) and (2.1): the effect of averaging is a loss in sensitivity and a smoothing of the peaks. (Red) Advanced LIGO curve from [19]. [Right] LIGO Livingston aerial picture from <https://www.ligo.caltech.edu/> and a sketch of a Michelson interferometer with cavities.

is seismic noise transmitted through the pendulum suspension point to the mass. In a simple pendulum, the mass follows 1:1 the motion of the suspension point at frequencies lower than the pendulum resonance, while at higher frequencies, the motion of the suspension point is not transmitted directly to the mass, but it is suppressed as $1/f^2$ in amplitude. Adding multiple stages to the pendulum, this suppression factor is increased to $1/f^8$ [46]. On the other hand, this suggests that this noise explodes at low frequencies. On top of this, there is another, unavoidable source of acceleration noise which can never be eliminated called *Newtonian noise*. This is simply the gravitational disturbance from the motion of mass in the oceans, atmosphere, and earth crust, in addition to human activity. We have no control on these phenomena and unfortunately there is no way of screening the mirrors from gravity (they are actually designed to detect that). There are not many big masses on Earth which oscillate at frequencies higher than 10 Hz, that is why this noise is not an issue for high frequencies. But at low-frequencies, there is a wall of Newtonian noise. These instruments therefore do not look for GW at these frequencies, and the main issues come from maintaining the mirror cavities locked, compensating the low frequency "drifts" of the masses.

These arguments literally force us to go to space when trying to measure low frequency GWs. On top of that, we saw that the optimal length for measurement of mHz GWs is several

million km, which is not a length scale that can be realized on Earth. This said, we can introduce the LISA space mission.

2.1 The LISA Mission - Science Objectives

The Laser Interferometer Space Antenna (LISA) will be a GW detector in space, consisting of three spacecrafts (SC) orbiting the Sun along the Earth's orbit. They exchange laser beams, performing an interometric measurement of the kind presented in the previous Chapter 1 (Figure 1.4): the spacecrafts form an almost equilateral triangle with a side of 2.5 million kilometer, and in we provided a motivation for such a length scale, that means to maximize our sensitivity to 1-10 mHz GWs. In Figure 2.2 a sketch of LISA in orbit is included while in Figure 2.1 the target sensitivity curve is included near the Advanced LIGO one, to highlight their complementarity in covering the GW frequency spectrum. The launch is scheduled for 2035 and the mission will last nominally 4 years, extendable up to 10 years. The sensitivity curve is always a compromise between what we want to observe and what we think we can achieve from a technological point of view. Anyway, the window of the GW spectrum that LISA will open is overcrowded with sources and will fuel an incredible amount of science. An all round overview of the science that can be done with LISA can be found in the Definition Study Report, or Redbook [31]. Here, we include one plot from this source, Figure 2.2, to guide us in listing only some of the most interesting science cases.

- Massive black holes mergers ($10^5\text{--}7 M_\odot$) are one of LISA main targets. We find these objects at the core of galaxies, but their birth and evolution is mysterious. It is difficult to justify their existence with multiple mergers during their lifetime, therefore being able to see these objects at the early stages of the universe could lead to ground-breaking discoveries. Some representative signals are reported in Figure 2.2 as continuous yellow-purple lines, the color representing their SNR.
- Figure 2.2 is populated by a forest of gray dots, which represent the almost monochromatic signals coming from white-dwarfs binaries in our galaxy. After enough integration time, tens of thousands will be distinguished, while other can not: these end up constituting a background noise around 1 mHz (bump in blue). The red hexagons are binaries that we know electromagnetically, therefore we know what signal they should induce on LISA: we can use them to calibrate the instrument. From this, we understand that the observatory can be used to survey a huge number of binaries in the Milky Way, which are not accessible with light being too weak. This will give a huge amount of information on the galaxy evolution and populations.
- As red-orange lines just above the sensitivity floor, there are signals from Extreme Mass Ratio Inspirals (EMRI): these are the most difficult to detect, but would really contain a huge amount of science. These signals are GW from small black holes ($\sim 10 M_\odot$)

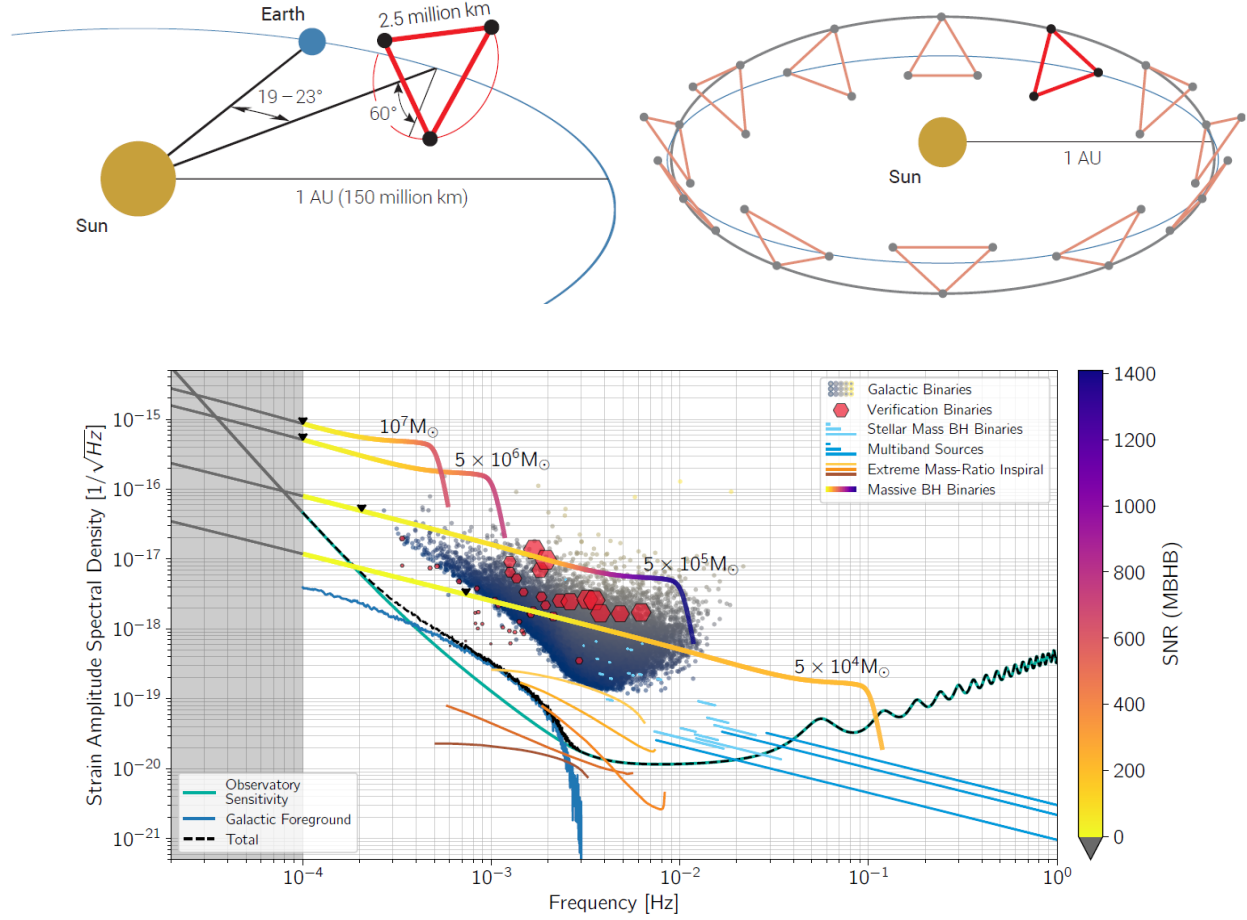


Figure 2.2: (Top) LISA constellation orbit, taken from [5]. (Bottom) LISA sensitivity curve compared with the power spectral density of various GW signals, discussed in the text. It is intuitive to understand that signals that lie above the sensitivity curve are well detectable, but the numbers on the plot are difficult to interpret: to understand how GW wave signals are turned into a PSD and plotted here, refer to [50]. Plot from [31].

orbiting around a super-massive one. They are literally test-particles which trace the geodesics of an extremely curved space, constituting one of the strongest tests of General Relativity.

- Also binary black holes of stellar origin spend part of their lifetime in the LISA band: some of these may be at a point of their evolution such that they can be first observed with LISA and end up merging in the LIGO band, opening the possibility of a multi-band observation. These are represented as light-blue straight lines in Figure 2.2. The first detected GW would have been one of those [56]. After the collection of many recent events with ground detectors, this is not expected to happen so frequently, still, LISA could give the alarm for the ground detectors months/years in advance of a merger.

- LISA could detect a GW background from the early Universe. This would come from a much earlier time than the CMB, which light could be released only after the Universe density decreased enough. This would have an unprecedented cosmological relevance, but the possibility of measuring this is under discussion, since it is not easy to understand how to distinguish it from instrumental noise.

2.2 Goal sensitivity and design

From an instrument point of view, what are the requirements to achieve LISA sensitivity? In Section 1.3.4 we saw that achieving a certain S_{strain} means requiring a certain level of TM acceleration noise S_g and interferometric noise S_{IFO} . For the LISA mission requirements adopted by ESA, these are set to [35, 31]

$$S_{\text{IFO}}(f) = \left[13.5 \text{ pm}/\sqrt{\text{Hz}}\right]^2 \left[1 + \left(\frac{2 \text{ mHz}}{f}\right)^4\right] \quad (2.1)$$

$$S_g(f) = \left(2.7 \text{ fms}^{-2}/\sqrt{\text{Hz}}\right)^2 \left[1 + \left(\frac{0.4 \text{ mHz}}{f}\right)^2\right] \left[1 + \left(\frac{f}{8 \text{ mHz}}\right)^4\right] \quad (2.2)$$

We see these are even more stringent numbers with respect to those derived by rules of thumb in Table 1.2: there we considered a longer detector, and focused on a 1 mHz supermassive-black hole binary. Extending the bandwidth and targeting even less bright sources obviously requires more stringent constraints on noise. In this section we will comment the design of LISA to have some clues on how these numbers are achieved. For acceleration noise, we will try to give a little more motivations, as this is central to the main research of this thesis.

2.2.1 Readout noise: 2.5 million km arms

The first problem to face is the fact that it is impossible to shine a laser to a mirror 2.5 million km apart and reflect it back to us with sufficient received power to allow precision interferometry. A more detailed (but not realistic) sketch of the laser locking system is depicted in Figure 2.3 and will help us understand better how different noise terms enter the measurement. A telescope of diameter $D \approx 30 \text{ cm}$ is used to send $P_e = 2 \text{ W}$ power laser (1064 nm) to the distant spacecraft. The gaussian nature of the laser causes the beam to spread in a cone with full-angle $\theta \approx 4\lambda/\pi D \approx 4.5 \mu\text{rad}$, which means that after 2.5 million km the diameter of the wavefront is $\theta L \approx 11.25 \text{ km}$. Some of this light ends up in an identical telescope on the distant spacecraft, which collects approximately 500 pW of power. In order to send back this signal, also spacecraft 2 must carry a laser: this is locked to the incoming 500 pW beam and sent back to SC1 with 2 W power again. The locking

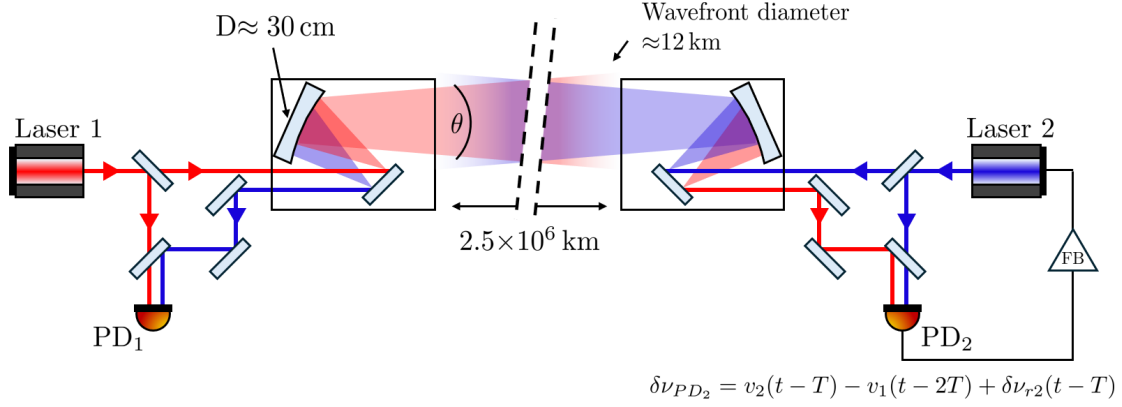


Figure 2.3: Sketch of LISA one-arm interferometric link. Light from the main laser is sent through a telescope to the distant SC, which collects the light, frequency locks its own laser to it and sends a beam back. More detailed description in the text.

process introduces some further readout noise in the final measurement; to understand this we follow the light path in Figure 2.3. The laser represented in red is emitted at time $t - 2T$ by SC1, with frequency $\nu_1(t - 2T)$. This arrives at SC2 at $t - T$, where it is interfered with laser 2 (frequency ν_2) on a photodiode which measures a beatnote plus readout noise $\delta\nu_{PD2} = \nu_2(t - T) - \nu_1(t - 2T) + \delta\nu_{r2}(t - T)$. This signal is then used in a feedback loop to adjust the frequency of laser 2 in order to make $\delta\nu_{PD2} = 0$ resulting in

$$\nu_2(t - T) = \nu_1(t - 2T) - \delta\nu_{r2}(t - T). \quad (2.3)$$

This is then sent back to SC1. Notice how the laser 2 frequency noise does not enter, thanks to the feedback loop, which only introduces some readout noise. This tells us that we need to correct a little bit expression (1.32) from the previous chapter, by also adding this readout noise

$$\frac{\dot{\nu}_{back} - \dot{\nu}_{em}}{\nu_{em}} = \frac{\delta\dot{\nu}_f(t) - \delta\dot{\nu}_f(t - 2T) + \delta\dot{\nu}_{r1}(t) - \delta\dot{\nu}_{r2}(t - T)}{\nu_1}. \quad (2.4)$$

With this modification, strain sensitivity (1.38) is modified only slightly: since the two readout noises are uncorrelated they sum incoherently and we get just a factor 2 in the spectrum

$$S_{\text{strain}}(\omega) = \frac{2}{c^2 \omega^2 \sin^2 \omega T} [2(\cos^2(\omega T) + 1)S_g + \omega^4 S_{IFO}] + 4S_{\delta\nu/\nu} \quad (2.5)$$

2.2.2 Frequency noise: Time-Delay-Interferometry

As anticipated earlier, it is in practice impossible to measure GW with a one arm interferometer because of laser frequency noise. Let us see how adding one arm at an angle α , as

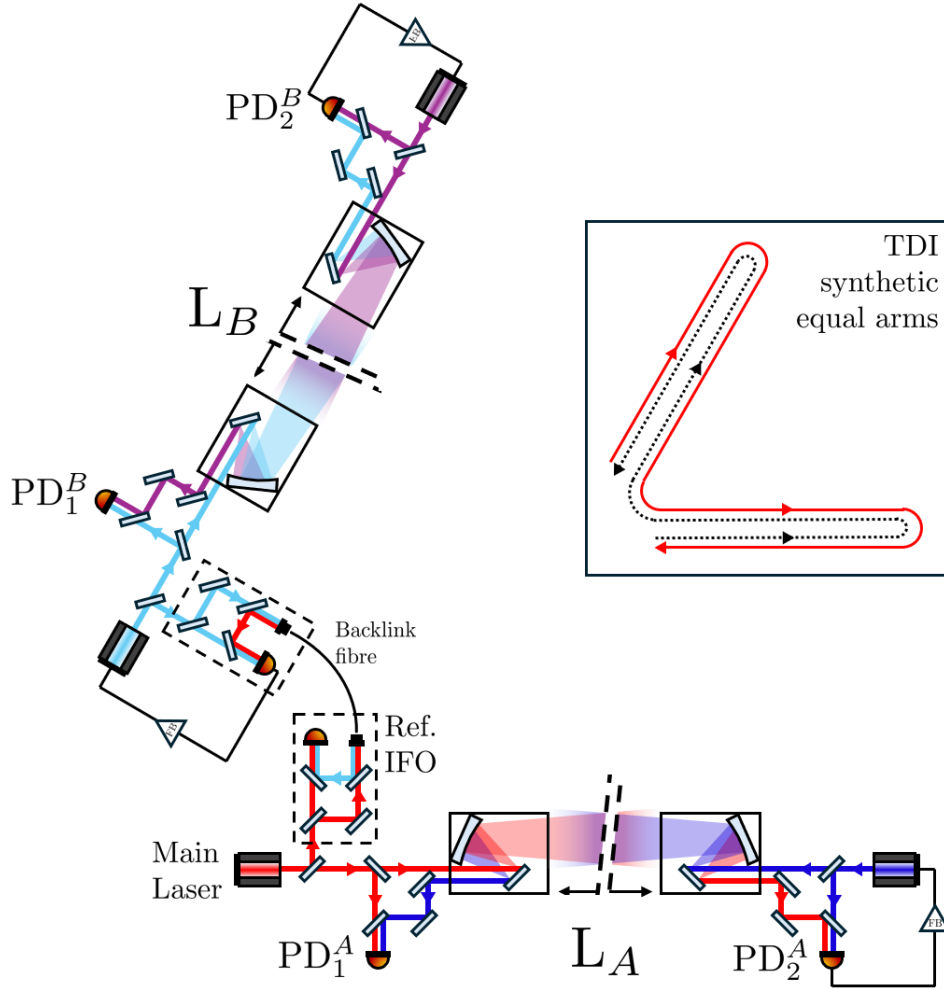


Figure 2.4: Sketch of an interferometer with two arms of different length and 60° angle. There are 4 lasers, which end up all being frequency offset locked to the main laser, as described in the text. Each arm end has two interferometers, a local one to interfere lasers of the same SC, and another one for interfering with the distant spacecraft one.

depicted in Figure 2.4 can help tackling this problem. First of all, on the main spacecraft there are two lasers: the red beam from the main one is brought to a *reference interferometer*, which is used to offset lock the other laser (and is useful for a set of other things which we can not discuss, that is why there is reference IFO for each laser). So, the lasers being locked share the same frequency noise, while all the readout noises and force disturbances are uncorrelated between the two arms and they will sum incoherently. The full calculation is long and here we just want to grasp the essence of laser frequency noise canceling. Therefore, we write down the signals at the two photodiodes PD_1^A and PD_1^B just considering the GW

signal and the laser frequency noise.

$$\frac{\delta\dot{\nu}^A(t)}{\nu_1} = -\frac{1}{2} [\dot{h}_+(t) - \dot{h}_+(t - 2T_A)] + \frac{1}{\nu_1} [\delta\dot{\nu}_f(t) - \delta\dot{\nu}_f(t - 2T_A)] \quad (2.6)$$

$$\frac{\delta\dot{\nu}^B(t)}{\nu_1} = -\frac{1}{2} [\dot{h}_+(t) - \dot{h}_+(t - 2T_B)] \cos 2\alpha + \frac{1}{\nu_1} [\delta\dot{\nu}_f(t) - \delta\dot{\nu}_f(t - 2T_B)] . \quad (2.7)$$

where the signal at photodiode B is multiplied by $(1 + \cos 2\alpha)$ because of the arm orientation in the $+$ polarization plane. If we combine the two signals as

$$\begin{aligned} \frac{\delta\dot{\nu}^A(t)}{\nu_1} - \frac{\delta\dot{\nu}^B(t)}{\nu_1} = & -\frac{1}{2} [\dot{h}_+(t) - \dot{h}_+(t - 2T_A)] + \frac{1}{2} [\dot{h}_+(t) - \dot{h}_+(t - 2T_A)] \cos 2\alpha \\ & + \frac{1}{\nu_1} [\delta\dot{\nu}_f(t - 2T_B) - \delta\dot{\nu}_f(t - 2T_A)] , \end{aligned} \quad (2.8)$$

We see that laser frequency noise does not cancel completely, but we combine the two signals delayed by a time $\Delta T = T_B - T_A$. If instead the arms have equal length $T_A = T_B = T$

$$\frac{\delta\dot{\nu}^A(t)}{\nu_1} - \frac{\delta\dot{\nu}^B(t)}{\nu_1} = -\frac{1}{2} [\dot{h}_+(t) - \dot{h}_+(t - 2T)] (1 - \cos 2\alpha) , \quad (2.9)$$

the laser frequency noise completely disappears. This actually the situation we mentioned in Earth based detectors, since the arm lengths are almost identical. They do not even have to register $\delta\dot{\nu}^A(t)/\nu_1$ and $\delta\dot{\nu}^B(t)/\nu_1$ and subtract them in post processing, since they can directly interfere the reflected beams in a beam splitter. The signal is multiplied by $(1 - \cos 2\alpha) = 3/2$ for $\alpha=60^\circ$, while the other sources of noise sum incoherently being related to different arms: this means multiplying the noise PSD by 2. The gain in SNR then is $(3/2)^2/2 = 9/8$: using two equal arms at 60° does not improve the SNR very much then, but is nonetheless essential for canceling the laser freq noise. We would get a factor 2 instead for $\alpha = 90^\circ$ (as in Earth based detectors).

Interferometry with unequal arms - LISA orbits

LISA is not an equal arms detector. The triangular constellation in Figure 2.2 "breaths", it is impossible to maintain a perfect equilateral triangle over many years without station keeping manoeuvres (which we do not want to do, otherwise the spacecrafts are not in free-fall). Its angles change of $\pm 1^\circ$, the interferometer armlengths vary through the year: they can differ up to $20\text{-}30 \times 10^3$ km, with relative velocities up to 10 ms^{-1} (equivalent to a Doppler shift of 10 MHz). These signals do not compete with the GW measurements, since they are at much lower frequencies (timescale $\sim$ year), but they pose serious challenges since they completely change the measurement range through the year, plus a re-orientation of the interferometric links is needed for the angular change. See the Science Orbit section of [31] for more information on this. The effect of having unequal arms will obviously be only a partial cancellation of the laser frequency noise. The calculation are cumbersome and

only the final result for the strain sensitivity is reported here, in the approximation of very low-frequencies as Eq.(1.41):

$$S_{\text{strain}}(\omega) \approx \frac{1}{L^2(\cos 2\alpha - 1)^2} \left[\frac{8S_g(\omega)}{\omega^4} + 2S_{IFO}(\omega) + 4\Delta L^2 S_{\delta\nu/\nu} \right]. \quad (2.10)$$

If we then consider that for LISA laser $S_{\delta\nu/\nu} \approx 10^{-13} \text{Hz}^{-/2}$ and $\Delta L \approx 30 \times 10^3 \text{km}$, we get $S_{\text{strain}} \approx 10^{-17}$, which is still 3 orders of magnitude from the 10^{-20} of Figure 2.2.

So is it all lost if we have unequal arms? Not quite. There exist more complicated combinations of the signals at the photodiodes which have again the result of cancelling laser frequency noise. These are called Time Delay Interferometry (TDI) combinations and we report just one, called TDI_X :

$$\delta\dot{\nu}_X(t) = [\delta\dot{\nu}^A(t) + \delta\dot{\nu}^B(t - 2T_A)] - [\delta\dot{\nu}^B(t) + \delta\dot{\nu}^A(t - 2T_B)] \quad (2.11)$$

The reader can check that this cancels frequency noise. The physical reason can be visualized in Figure 2.4 top right: the sum in Eq. (2.11) is like composing the frequency shifts accumulated along the red and the black dotted paths, which are equal in length $2L_A + 2L_B$.

The TDI_X variable in Eq. (2.11) is quite a different signal with respect to what we worked with up to now. This new combination actually suppresses the GW signal, but equally suppresses also the other noise sources like acceleration and readout noise, while canceling laser frequency noise. In other words, we do not lose SNR by performing the TDI operation and our sensitivity is still given by an expression like (2.10). One last questions before concluding this section. To which extent are we able to perform this subtraction? First of all, we need to know the times T_A and T_B for (2.11), with which precision? From (2.10) we can see that to meet the IFO noise requirement

$$2S_{\delta\nu/\nu}^{1/2} \Delta L < 10 \text{ pm}/\sqrt{\text{Hz}} \quad \rightarrow \quad \Delta L < 5 \text{ m} \quad \text{or} \quad T < 10 \text{ ns} \quad (2.12)$$

In LISA, data are sampled at a few Hz, but we see we still need nanosecond interpolation precision, which also means clock synchronization between the spacecrafts. The length measurements and clock synchronization are achieved by modulating further signals on the laser beam, but we do not have space to talk about this. On top of this, a seven orders of magnitude cancellation in post-processing is seriously challenging, requiring a measurement with high dynamic range, accuracy and precision. A lot of stringent requirements follow from this, many falling on the *phasemeter*, the system in charge of reading all the quadrant photodiodes of LISA and transform their signals in an interferometric phase measurement.

Further insights

- The just discussed TDI is called first-generation. There is a second one, more complicated, which accounts for the non-negligible fact that the arm length also change in time, that is $\dot{L} \neq 0$ [38, 57] .
- Experimental tests of noise cancellation in LISA conditions and in particular of the phasemeter performance are ongoing with dedicated and interesting facilities, including the Hexagonal Optical Bench at the AEI [55].

2.3 Test Mass acceleration noise and the Gravitational Reference System

The last noise source that we discuss is the most important for us, acceleration noise. In all the figures up to now, there were just some lasers shining light towards the distant spacecrafts, but we know that it is fundamental that these lasers are free-falling. Let's say we nail down the laser to a heavy spacecraft of 2500kg (approximate weight of LISA SC at launch [31]). So we go far enough from Earth, what kind of source in the vacuum of the solar system can hit a 2500 kg mass enough to disturb its geodesic motion? If we just look at the solar flux at 1 AU, the so called *solar constant* which is measured to be $G_{\text{SC}} = 1361 \text{ Wm}^{-2}$ [42], then we can calculate the acceleration of a body of mass M and exposed area A (let us say it is 20 m^2 for LISA) due to radiation pressure as

$$F_{\text{rp}} = \frac{2AG_{\text{SC}}}{c} \approx 180 \mu\text{N} \quad \rightarrow \quad a_{\text{rp}} = 2 \frac{A}{M} \frac{G_{\text{SC}}}{c} \approx 70 \text{ nm/s}^2 \quad (2.13)$$

where the factor 2 comes from the fact that incident light is mostly reflected by the spacecraft. This is a DC acceleration that would not bother our measurement: the problem arises when the solar flux starts to fluctuate in the frequency band of our interest. In [43] it is stated that the flux can show a variability of $\sim 0.01\%$ over time scales of minutes to hours. Which means acceleration fluctuations of our spacecraft of

$$\delta a_{\text{rp}} \approx 7 \text{ pm/s}^2, \quad (2.14)$$

over timescales of order minutes-hours. This simple calculation alone tells us that this solution is not enough to reach the $\text{fms}^{-2}/\sqrt{\text{Hz}}$ level that we need, and we are neglecting other phenomena, like micrometeorites impacts on the SC, for example. In short, it is not enough to just put an heavy spacecraft in space to achieve the required level of free-fall, even far from Earth.

A better strategy would be to have a way of measuring the non-gravitational (local) accelerations of the spacecraft, and then subtract them from the interspacecraft measurements. This can be achieved by using an accelerometer. Let us take one arm of our GW detector again,

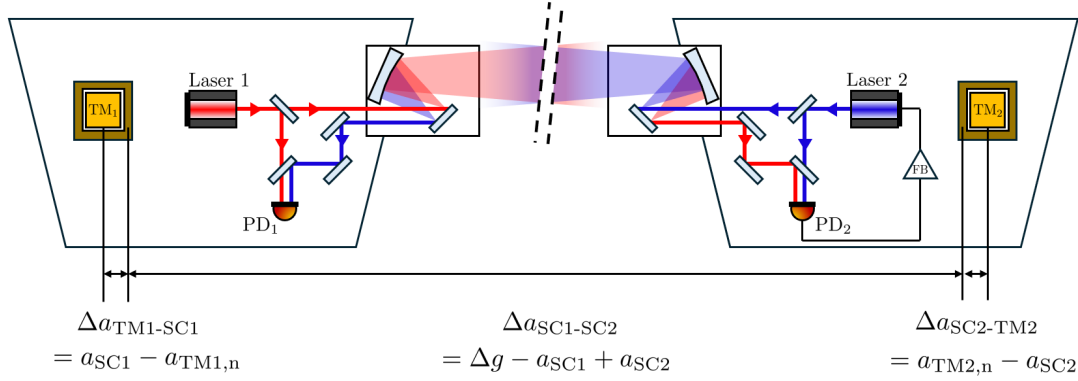


Figure 2.5: One arm of gravitational wave detector, with lasers mounted on spacecrafts and equipped with accelerometers. Measurements of local stray accelerations and interspacecraft acceleration (containing the GW signal) are reported at the bottom.

place the laser fixed to a spacecraft and equip it with an accelerometer, Figure 2.5. The two satellites measure their relative acceleration Δa_{SC-SC} interferometrically. This contains both accelerations due to gravity a_{GW} (what we want), and the spurious ones due to external disturbances, a_{SC1} and a_{SC2} , with respect to a local inertial frame. However, the accelerometer inside, literally another test-mass, is not directly subject to external disturbances. It is subject to *internal* disturbances $a_{TM,n}$, i.e. the ones that the spacecraft introduce on it¹. The device's purpose is exactly that of measuring the TM-SC differential acceleration, which will then contain the sum $a_{TM-SC} = a_{TM,n} - a_{SC}$ (it does not contain the gravity signal, since the tidal forces can be made very small due to the small distance, for example matching the TM and SC centers of mass). By **adding together the three measurements** in Figure 2.5 in post-processing

$$\Delta a_{SC-SC} - a_{SC2-TM2} + a_{TM1-SC1} = a_{GW} + a_{TM2,n} - a_{TM1,n} \quad (2.15)$$

we completely cancel out the noise of the SC motions, and we obtain just the acceleration due to GW, limited by the internal noise sources on the TM. This does not mean that our troubles are over yet. Are we able to realize an accelerometer that does not introduce noise on the TM to the $\text{fms}^{-2}/\sqrt{\text{Hz}}$ level?

The need for a Drag-Free system

What kind of system should we use then?

The accelerometer must be able to do three things: read the TM position with respect to the SC, apply control forces to the TM and ensure that it does not introduce force noise on the

¹Examples: a noisy electric field on the SC that interacts with the TM charge. Residual gas molecules hitting the TM in vacuum. Thermal deformations, thermal noise.. We could list infinite.

TM to the required level. There is a history of devices used for other low-orbit space missions, related to Earth gravitational field measurements like CHAMP, GRACE and GOCE [59], where the sensors demonstrated to be fit for the job. All of those employ variations of the same device structure: a dense, metallic, parallelepiped test mass, electrostatically "levitated" inside a shielding cage. It is surrounded by electrodes which simultaneously allow capacitive sensing of the TM position and the application of electrostatic forces, we will see what it looks like for LISA in a few sections.

Let us start by examining a GRACE-like accelerometer. The GRACE-FO Mission [4] objective is to monitor Earth's gravitational field to then reconstruct Earth's mass distribution for climate monitoring. The principle of detection is always the same, two satellites with an interferometric link orbit Earth at an altitude of 450 km, separated by 220 km distance: non uniformities of Earth's gravitational field cause relative accelerations of the spacecrafts, which are measured and the gravity field reconstructed. The low-Earth orbit introduces external disturbances on the satellites that are removed as described above through the use of an accelerometer. The design of this device is far from meeting LISA acceleration requirement, since it can achieve around $100 \text{ pms}^{-2}/\sqrt{\text{Hz}}$ with a rising $1/f$ tail at 5 mHz (Table 2 of [44]), but it also meant to apply much larger forces in the harsh low-orbit environment, causing spacecraft acceleration of several $\mu\text{m/s}^2$ [66]. The way the accelerometer works on GRACE is the following: the mass is kept at the housing center (fixed to the spacecraft) through electrostatic forces. If the satellite accelerates with respect to the TM, a force is applied *to the TM* in order to keep it centered. Then, the acceleration is read in the feedback forces that were applied.

If we were to use this system on LISA, what kind of forces should be applied to the TM? We said that the SC is subject to solar radiation pressure, which causes an acceleration of $\sim 70 \text{ nm/s}^2$; the TM is not subject to solar radiation directly, therefore, the SC would have to push it with a force of 70 nN/kg to drag it along. Now, without going into the details of the electrostatic force actuator, it is obvious that it will be affected by noise: to fulfill the acceleration requirement of $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$, the actuator should be able to apply 70 nN/kg with a noise of $3 \text{ fNkg}^{-1}/\sqrt{\text{Hz}}$ at 1 mHz, which means a relative stability of $40 \text{ ppb}/\sqrt{\text{Hz}}$. This is an extremely difficult requirement to meet, especially at low frequencies, where electronics are affect by various sources of red noise ($1/f$, $1/f^2$). For example, we will see in the next chapters that LISA very precise force actuators have noise of order $5 \text{ ppm}/\sqrt{\text{Hz}}$ at 1 mHz and $50 \text{ ppm}/\sqrt{\text{Hz}}$ at 0.1 mHz. This means that **we are not allowed to drag the TM along the measurement axes to make it follow the SC**.

To circumvent this problem, a Drag-Free system is implemented, which is here compared to the other case.

No Drag-Free	:	measure $a_{\text{TM-SC}}$	$\rightarrow$	apply force to the TM to keep it centered in the SC
Drag-Free	:	measure $a_{\text{TM-SC}}$	$\rightarrow$	apply force to the SC to keep it centered on the TM

In this configuration the *entire* 2500 kg spacecraft is controlled through μN thrusters so that it "follows" the TM along the measurement axes. The system is clearly more complicated

like this: the accelerometer is not just an independent system, a box that can be built and put in place, here the entire spacecraft controls depend on it. In turn, it allows the use of much more relaxed actuators (in terms of noise) and also solves a problem which we did not mention: the solar pressure would push the spacecraft out of orbit without a drag-free control. An on-board computer has the duty of calculating all the necessary forces to be applied to SC and TM to keep the drag-free environment: this system is called Drag Free and Attitude Control System (DFACS).

To summarize the concept, the drag-free system of LISA is usually visualized as really putting cubic test masses in free-fall and use them as a geodesic reference, which the spacecraft tries to follow with its thrusters, while performing the interferometric measurement in three pieces as expressed by Eq. (2.15) (TM_1-SC_1 , SC_1-SC_2 , SC_2-TM_2).

Notes

- To maintain the triangular interferometric link, the system can not be drag-free for all degrees of freedom. For example, a rotation of the TM must be compensated by applying torques to the TM, it would not be possible to rotate the entire spacecraft. So, drag-free only happens along the measurement axes and we will see what forces and torques we have to compensate along the other degrees of freedom (DOF) in Section 3.1.

Coupling to spacecraft motion

So does this mean that in drag-free mode the spacecraft motion can be as noisy as we want, since we subtract it away? No. First of all, noisier spacecraft control adds additional challenges in the composition of the TM-to-TM distance with Eq. (2.15). Secondly, we said that the spacecraft exerts forces on the TM, wanted and unwanted. Between these there are also TM position dependent forces, for example the electrostatic attraction of the plates surrounding the TM (which is non linear $\propto 1/d^2$), or the gravitational attraction of the various SC parts on the TM. Whatever the nature of these forces, we can expand them at first order around, let's say, the nominal zero position of the TM inside the SC, $F(x) \approx F_0 + \partial_x F|_0 x$ to write the TM acceleration

$$a_{TM,n} = \frac{F_0}{M} + \frac{1}{M} \frac{\partial F}{\partial x} \Big|_0 x + a_n = a_0 - \omega_{xx}^2 x + a_n \quad (2.16)$$

where a_n is any other disturbance, x is the displacement from zero position, and ω^2 is called the *stiffness* (usually $\omega^2 < 0$, so a force that tries to push the TM away). So if we move to power spectral densities

$$S_{g,SC \text{ coupling}}^{1/2} = \omega_{xx}^2 S_x^{1/2}, \quad (2.17)$$

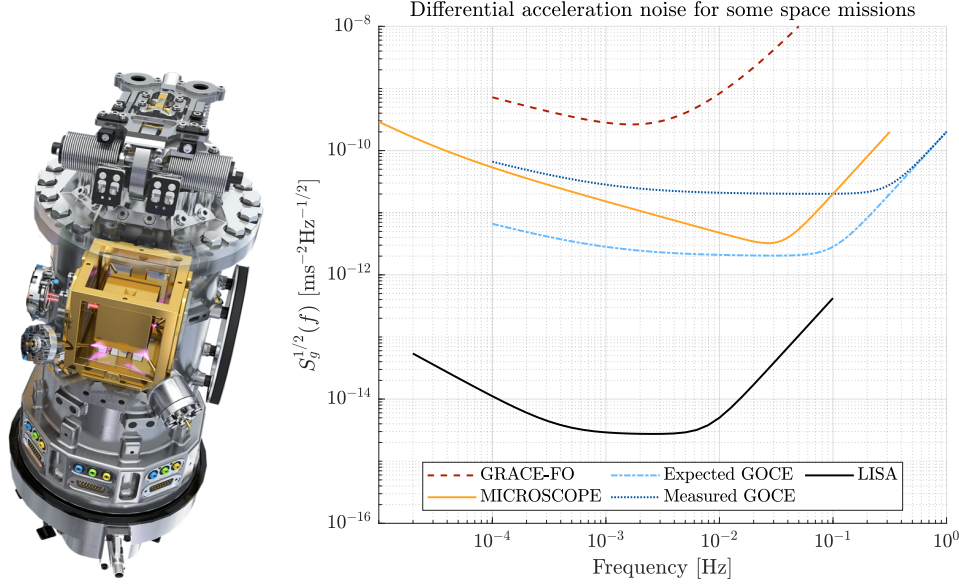


Figure 2.6: (Left) 3D rendering of the LISA GRS, described in the text, from ESA ATG/Medialab. (Right) Acceleration noise ASD for various accelerometers in space missions. Red dashed: design noise curve of GRACE-FO accelerometer [44] plus laser ranging noise [39] (the measurement is dominated by Earth gravity signal and accelerometer noise can not be measured directly). Yellow solid: MICROSCOPE differential acceleration in the reference accelerometer, deduced from [60]. Blue dotted and cyan dash-dotted: GOCE expected accelerometer noise and measured one [39] plus readout noise deduced from [59]. Black solid: LISA GRS acceleration noise requirement. Figure re-adapted and updated from Prof. William Joseph Weber lectures.

we see that the SC relative motion with respect to the TM causes a proportionally noisy force! Both ω^2 and SC noisy motion must then be limited to appropriate levels. Some numbers will be discussed in Section 3.1.

2.3.1 Gravitational Reference System

Let's get to the LISA accelerometer. It is called Gravitational Reference System (GRS) and it can be seen in Figure 2.6. It is designed to meet the following requirements [31]:

- Its role is exactly that of providing an end mirror which meets the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ free-fall requirement, and allow $10 \text{ pm}/\sqrt{\text{Hz}}$ interferometric sensitivity. As explained in [31], this is a system level requirement, meaning that the whole SC and control system is involved in reaching these requirements.
- More specifically to the GRS instead, it must provide TM position sensing at the $1 \text{ nm}/\sqrt{\text{Hz}}$ level in displacement and $100 \text{ nrad}/\sqrt{\text{Hz}}$ in rotation; then it also must be

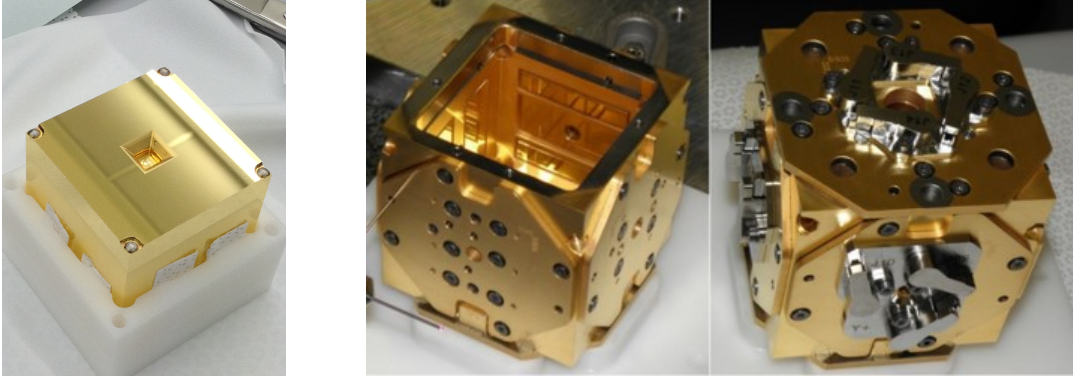


Figure 2.7: (Left) LISA Pathfinder Test Mass. (Right) LISA Pathfinder Electrode Housing, ESA Medialab.

able to apply nN forces and pNm torques. All of this is used to control and align the TM along the DOFs which are not drag-free.

To give a better idea of the performance that this instrument must reach, in Figure 2.6 the acceleration noise levels obtained (or designed) for other missions is reported and compared with the LISA requirement. The gap between this and any other mission is absurdly large, around five orders of magnitude: it is only after studying all of this history that I personally realized how crazy it could be to propose such a system in the past. For a clueless student that was already allowed to work on the finished product, like me, it may be difficult to immediately grasp how astonishing this system is. Notice also that some of the results on Figure 2.6 are relatively recent, obtained during the development of the GRS for LISA and the test mission LISA Pathfinder, so it was even more of a leap into the void.

Let us go through the GRS elements, first in a qualitative way. A more quantitative discussion about the system will be carried out in the next chapter, where we talk about electrostatic actuation and the associated noise sources, a major driver of the design.

1. **Test Mass (TM):** Figure 2.7, this is a 1.92 kg cube, gold coated, gold-platinum bulk alloy, 46 mm side. The choice of alloy was driven by achieving max density, low magnetic moment and susceptibility, to minimize the coupling with spurious magnetic fields. For comparison, GRACE test mass is 70 g. Gold coating allows to minimize surface imperfections that could cause what are called stray patch potentials, discussed in the next chapter.
2. **Electrode housing (EH):** Figure 2.7, a molybdenum Faraday cage, gold coated, with inserts for gold coated electrodes, used to provide 6 DOF position sensing and as force actuators. The electrodes are spaced by insulating materials which are completely "hidden" from the TM. Insulators can accumulate lots of charge and are difficult to discharge: large voltages then couple with other stray electrostatic fields to generate noisy forces on the TM. Even without insulators, centimeter-size gold surface patches

can exhibit stray biases up to 10-100 mV [6], which is a driver of the EH design. In particular, the force generated by these electrostatic interactions scale as d^{-2} , the distance between the surfaces. So the TM-EH gap along the measurement axis was maximized to 4mm, at the cost of sacrificing position sensitivity of the capacitive bridges, which allow a roughly $1.8 \text{ nm}/\sqrt{\text{Hz}}$ noise floor. **For this reason a third interferometer per TM is needed** to reach the $10 \text{ pm}/\sqrt{\text{Hz}}$ requirement. The gold, polished coating of the TM also allows to use it as an end mirror for the interferometer.

3. **Front-End-Electronics (FEE):** hosting the capacitive bridges for position sensing and in charge of generating the actuation voltages to be applied to the electrodes and control the TMs. Many of the experiments in this thesis end up falling in the performance evaluation of this sub-system.
4. **Charge Management System (CMS):** Cosmic rays deposit charges on the TM and this can be a source of acceleration noise, since it can couple with fluctuating electric fields present in the EH (for example due to electronics noise). Therefore, the TM must be discharged. Other missions in Figure 2.6 used a fine gold wire for this purpose: however, in MICROSCOPE the thermal noise associated with the wire seemed to be the limiting source of noise at the level of several $\text{pms}^{-2}/\sqrt{\text{Hz}}$ [60]. Even if it has been argued that this may have not been the case [49], theoretical predictions are still well above the LISA requirement, so it could not be used. The TM is never in direct contact with the surroundings. A UV discharge system is used instead: electrons are moved from TM to EH and vice-versa through photoelectric effect to keep the TM discharged. Gold coating also ensure that soft-UV light can exceed the work function and allow photoelectrons to be extracted. More information about this system can be found in Chapter 7.
5. **Caging Mechanism and Grabbing, Positioning and Release Mechanism:** Large TM-EH gaps and a heavy TM have other consequences. The TM must be locked during launch, otherwise it would destroy the vibrating SC. Eight fingers are used for this, which grab the TM with 1.2kN in specially coated inserts that avoid welding due to vibrations. In flight, two other fingers grab the TM with 1-70 N force, before releasing the other eight strong ones. The two fingers, called the GPRM, have to release the TM with $< 15 \mu\text{m/s}$ velocity: this is again because of the large gaps, which do not allow to apply forces high enough to stop the TM from hitting the EH wall for higher velocities. This requires careful engineering of the fingers tips retraction.
6. **Vacuum Chamber (VC):** it contains TM, EH and caging mechanism (and all the cables and fibers from the other subsystems). When in flight, the VC is vented to space through a tube called the Vent Duct. The VC duty is to protect the TM from the rest of the SC outgassing environment: in fact, brownian noise from residual gas molecules is one of the limiting noise sources at 1mHz, enhanced by the constrained geometry

of the TM-EH [27]. The residual pressure requirement is of $2\text{ }\mu\text{Pa}$: to achieve this at the start of the science measurement, it is important to do a bake out. Just venting to space, the system pressure decreases exponentially, and one year to get into LISA orbit would not be enough to reach the requirement. The baseline would be to do a bake out on ground, seal the VC and unseal it in space.

This should give a better idea of the complexity of this device. It is not simply a black box that can be built, placed on the SC and press the power button to read its data. It is deeply interconnected with all the other sub-systems on the satellite.

2.3.2 Putting it all together again

Now that also the GRS component was discussed, we can wrap up the design of LISA SC (at a basic level) that we discussed up to now. In Figure 2.8 our toy detector construction is reported and compared with an actual possible 3D rendering of the payload. Each of the arms end has one GRS, one telescope, and one Optical Bench (OB) which in turn hosts three interferometers: the Interspacecraft Interferometer (ISI), the Test Mass interferometer (TMI) - which we had to introduce in the previous section to compensate for the loss in capacitive sensitivity due to the large sensor gaps - and the Reference Interferometer (RFI). All of these elements constitute one Moving Optical Sub-Assembly (MOSA), again rendered in Figure 2.8. This whole system must be able to move throughout the mission to compensate for the $\pm 1^\circ$ annual angular breathing of the constellation, which is a huge complication.

The complete configuration of LISA is shown in Figure 2.9: there are three interferometric links, from which three unequal arms Michelson Interferometers can be synthesized. Three links also admit some redundancy: in the unfortunate case of 1-arm loss, the other two are sufficient to perform the TDI laser frequency cancellation. Three links means six MOSAs, six test masses, six optical benches, six telescopes. The question "was it possible to use just one TM per SC?" is not banal: such a system would require a way to steer the laser beams to be always aligned with the TMs of the constellation, and would not admit any redundancy. This is an important reason for preferring the two TM per SC solution.

Finally, each SC is equipped with cold-gas μN thrusters, that allow the drag-free control. This must happen only along the interferometric link axes. Question: how is it possible to follow the geodesic motion of both test masses at the same time? Section 3.1 is dedicated to that.

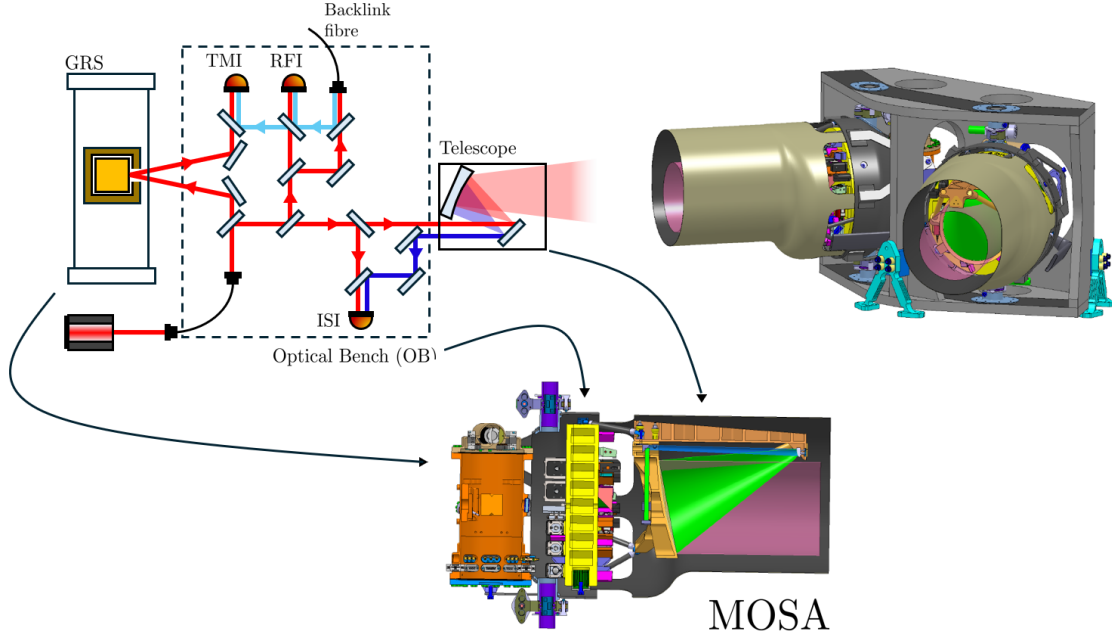


Figure 2.8: (Top left) Toy arm-end built up to now in this chapter, it is not representative of the real design, but includes the important elements. (Bottom) A 3D rendering of a MOSA for LISA, described in the text and taken from [5]. (Top right) Two MOSAs in a structure which would lie in the same spacecraft, from [5].

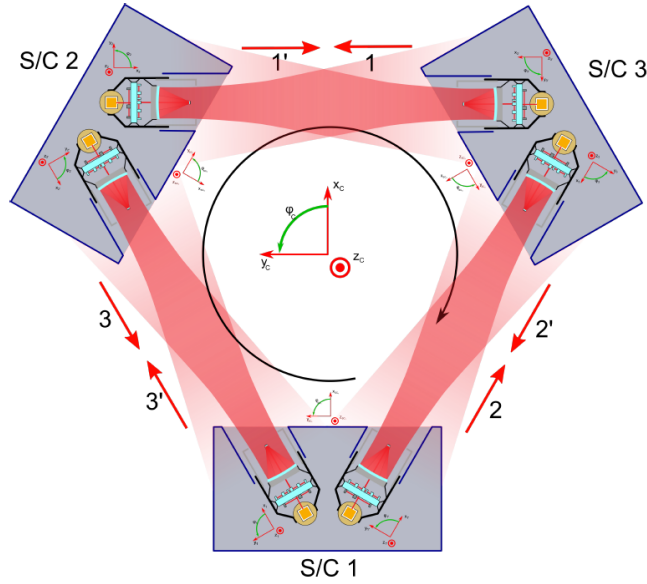


Figure 2.9: The three LISA interferometric links, each involving two MOSAs, from [52, 33], thanks to Lorenzo Sala.

2.4 LISA Pathfinder

In Figure 2.6 the required performance for LISA was compared to that of other missions. It should then be clear that it would have not been a good idea to fly the full, three spacecraft, LISA mission without being completely sure that the GRS-SC system could actually meet the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ free-fall level. This means that an experimental test was needed as a first step. The first problem is that on ground we have no instrument which is able to test acceleration noise at that level. Torsion pendulums are known to be one of the most precise facilities to probe small forces, and we will talk extensively about them in Chapter 6: some were explicitly designed and built to study the LISA GRS, like the ones in Trento [26, 25, 40, 30]. Still their sensitivity is ~ 1 -2 orders of magnitude away from being able to test at GRS required level. A further problem is that $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ is really not a requirement on just the GRS, but on the whole system, since it involves the drag-free control, the thrusters and so on. And this for sure we can not test on ground.

For these reasons, under the European Space Agency, another mission was studied, built and launched on December 3rd 2015, called LISA Pathfinder (LPF), with the precise objective of demonstrating that the acceleration requirement could be met.

The way to quantify the TM free-fall quality would be to measure it against another free-falling reference; from the previous discussion it emerged that it is not sufficient to put a single GRS on a drag-free satellite with thrusters and measure their relative acceleration, because the quality of the SC free-fall is not good enough (the TM-SC distance measurement would be dominated by the SC noisy motion). Therefore, the solution adopted is shown in Figure 2.10. Known as the **LISA Technology Package**, two GRS were put on board, separated by an optical bench. This is usually pictured as shrinking one of the LISA arms to 38 cm and measuring the differential acceleration between the two free-falling test masses. In this configuration, to be precise, only one of the two test masses can be truly drag-free with absolutely no forces applied along the measurement axes (see Section 3.1): the second test mass is forced to follow the first by application of electrostatic forces g_{2c} (expressed as force per unit mass). So, the interferometrically measured relative position is used in a feedback loop to control TM₂, while the position of TM₁ with respect to the SC is used to control the μN thrusters. This mechanism is nicely pictured in Figure 2.11. In this picture it should be easier to understand what quantities enter in the measurement of differential acceleration of the two TMs. In the inertial reference frame represented in Figure 2.11, we write down some definitions for the quantities in play in Table 2.1, so we can understand better what is measured and what not. In the same reference frame we can write the equation of motion for the two TMs

$$\begin{aligned}\ddot{x}_1 &= g_1 - \omega_1^2(x_1 - x_{SC}) \\ \ddot{x}_2 &= g_2 - \omega_2^2(x_2 - x_{SC}) + g_{2c}\end{aligned}\tag{2.18}$$

where we can recognize: the coupling to the SC motion through the stiffnesses ω_1^2 and ω_2^2 ; the applied forces g_{2c} only to TM₂; g_1 and g_2 any other source of spurious accelerations,

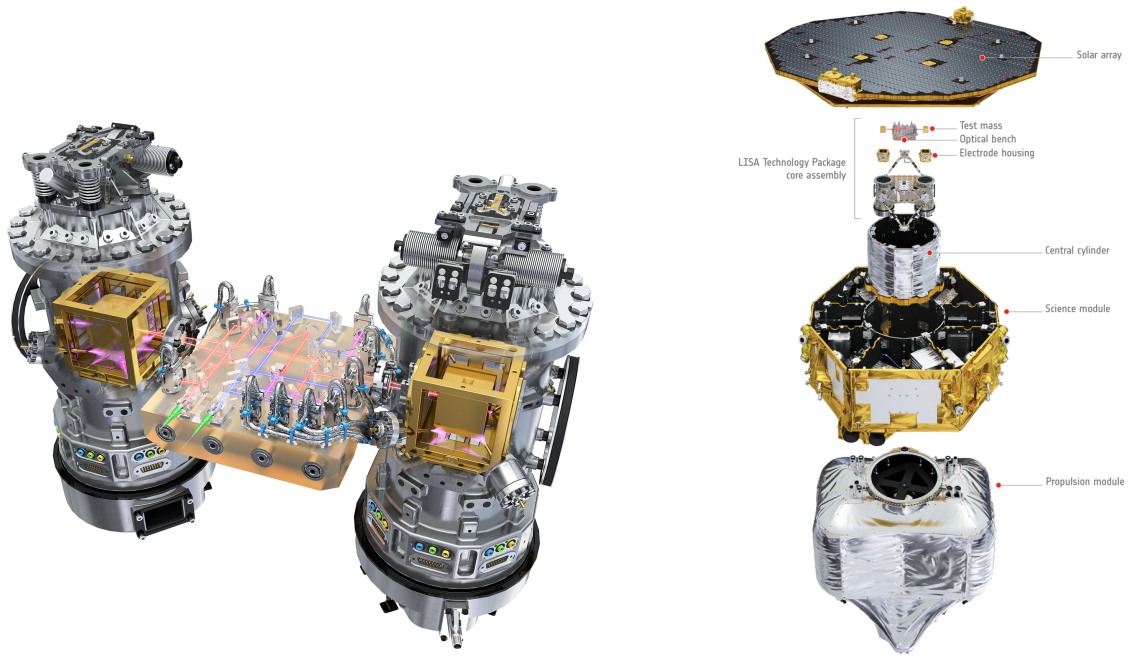


Figure 2.10: (Left) 3D cad of LISA Technology Package flown on LPF. (Right) Exploded view of the LPF spacecraft, it may be impressive to think how the whole spacecraft is bound to follow the motion of a 5cm cube. Both images from ESA/ATG Medialab.

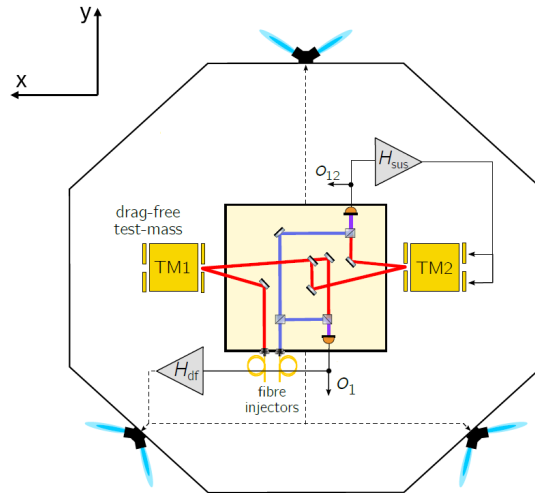


Figure 2.11: Nice sketch of LPF taken from [24], just reference frame is corrected. The closed feedback loops to control TM and SC motion are briefly described in the text.

Quantity	Description	IFO	CAP
x_1	TM ₁ position	not measurable	not measurable
x_2	TM ₂ position	not measurable	not measurable
x_{SC}	SC position	not measurable	not measurable
$x_1 - x_{SC}$	TM ₁ -SC relative position	measured $x_{1,OMS}$	measured $x_{1,GRS}$
$x_2 - x_{SC}$	TM ₂ -SC relative position	not measured	measured $x_{2,GRS}$
$x_2 - x_1$	TM ₁ -TM ₂ relative position	measured Δx_{OMS}	not measured

Table 2.1: Some of the quantities measured in LPF. In the "IFO" column it is indicated if the quantity is measured interferometrically and its notation. The analogous "CAP" column tells if it is measured through capacitive bridges.

which is what we are interested in studying. We want to cast these two equations in an expression in which only the IFO measured quantities of Table 2.1 appear. By subtracting the two equations and defining $\Delta\omega^2 = \omega_2^2 - \omega_1^2$, we obtain

$$\ddot{x}_2 - \ddot{x}_1 = g_2 - g_1 - \Delta\omega^2(x_1 - x_{SC}) - \omega_2^2(x_2 - x_1) + g_{2c}. \quad (2.19)$$

For all the quantities here we have a measurement in Table 2.1, therefore we can write that to obtain our quantity of interest $\Delta g \equiv g_2 - g_1$, we have to combine the measurements as follows

$$\Delta g = \Delta\ddot{x}_{OMS} - g_{2c} + \Delta\omega^2 x_{1,OMS} + \omega_2^2 \Delta x_{OMS}. \quad (2.20)$$

And this is indeed the main LPF observable, the one that must meet the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ requirement. But what about the other quantities?

- $\Delta\omega^2, \omega_2^2$ are stiffnesses mainly linked to gravitational gradients due to mass unbalances in the SC, and electrostatic force gradients due to interaction of the TM with the surrounding electrodes. Both parameters could be *measured* in flight through dedicated calibration experiments [8].
- g_{2c} is a very important quantity for this thesis, it is the linear force applied to TM₂ along the measurement axis. The data we have about this quantity is a telemetry that tells which force was *nominally* being applied by the electrodes every 0.1 s. But what is the *actual* force that we are applying? We will see that knowing this force is very important. As a first simple example, we report that the system had to be calibrated also for this, therefore Eq. (2.20) is usually found with g_{2c} substituted by λg_{2c} , $\lambda \approx 1.08$ being a calibration coefficient [8].

The requirement that was set for LPF was more relaxed than the LISA one and stopped at 1 mHz.

$$S_{\Delta g}^{1/2}(f) < 30 \frac{\text{fms}^{-2}}{\sqrt{\text{Hz}}} \sqrt{1 + \left(\frac{f}{3 \text{ mHz}}\right)^4} \quad (2.21)$$

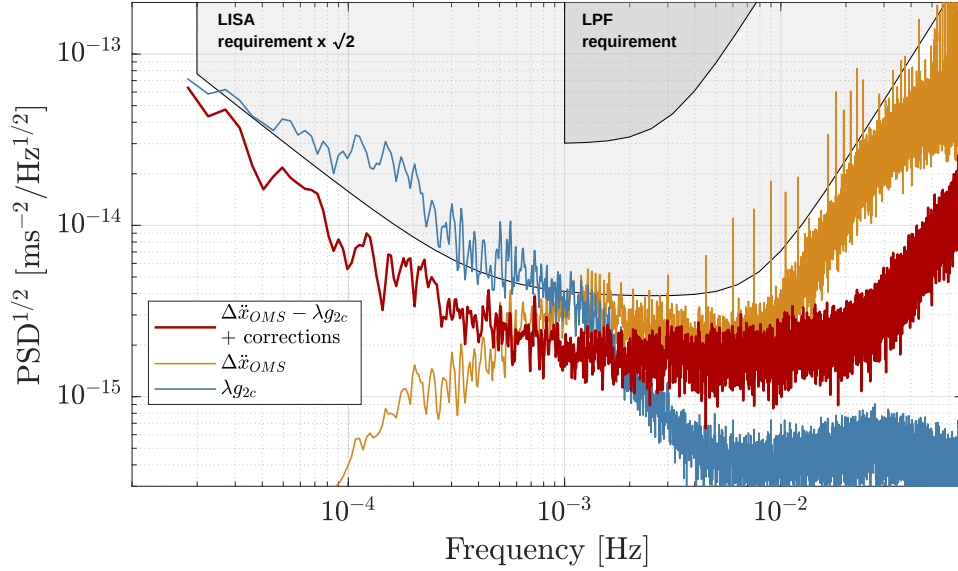


Figure 2.12: Acceleration noise ASD. Light grey area: LISA requirement (2.2) multiplied by $\sqrt{2}$ for the two TM system of LPF. Here the current $2.7 \text{ ms}^{-2}/\sqrt{\text{Hz}}$ requirement is being used, while plot in [7] uses $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$. Dark grey area: LPF requirement. Red curve: LPF acceleration noise measured in February 2017, including corrections for glitches, inertial forces, and actuation errors. Notice this is not the same data as [7], here we are using the latest version of L4 data, also in [12, 52]. Orange curve: $\Delta\ddot{x}_{OMS}$, IFO differential acceleration. Dark cyan curve: λg_{2c} , applied forces to TM₂. **Note:** all these spectra have been calculated excluding some time-series "slices" which had two huge glitches; this was done to show the typical shape of $\Delta\ddot{x}_{OMS}$ and λg_{2c} , avoiding their spectra being completely distorted by the glitches.

2.4.1 LPF results

It is not a secret that LISA Pathfinder was an incredible success. The measured level of differential acceleration noise in February 2017 is reported in red in Figure 2.12. The mission realized a level of free-fall which already exceeded the one required for LISA, demonstrating the readiness of our technology and giving the green light for the LISA mission.

However, the presented curve is not that easy to obtain. In Figure 2.12 I also included the Amplitude Spectral Densities (ASD) of $\Delta\ddot{x}_{OMS}$ and the raw time-series for the commanded forces λg_{2c} (see caption). Eq.(2.20) tells us how to calculate residual noise Δg using these two quantities². But from Figure 2.12 it already can be seen by eye that there is no way the orange and blue curves will sum to the red one. So where is the trick? Here we list a set of discovered, important effects that were removed from the data. For a more detailed discussion of these, see [52, 7, 16].

²The equation also needs the stiffness terms, but they turned out to be almost completely negligible [52].

- **Interferometer pick-up.** For frequencies between 20-200mHz a noisy "bump" affects the interferometer differential readout $\Delta\ddot{x}_{\text{OMS}}$: this was a pick up from the SC noisy motion, which could be first suppressed in flight by better TM beams alignment and also completely removed in post-processing [16].
- **Inertial forces.** The reference system of the SC is not an inertial one, especially for its noisy rotations. While following TM_1 along the translation DOF, both TMs angular DOFs were bound to be aligned with the spacecraft through the application of electrostatic forces. The SC in turn, used star tracker cameras to remain aligned with far stars: noise in these trackers resulted in noisy SC rotation. In this rotating reference frame, the TM were subject to inertial forces, like the centrifugal one, which had to be compensated by the actuation system, therefore they show up as "bumps" at low frequencies. Being able to calculate perfectly these forces, they could be removed from the data.
- **Glitches.** The stationarity and gaussianity of the LPF noise during the whole arc of the mission is astonishing [12], however, the data is populated by transient, non-gaussian events called "glitches" [17]. These are real force events which act on the TMs preferably in one direction, with an approximate rate of 1 per day. This force rises exponentially and also stops exponentially on timescales that can range from minutes to hours. The physical reason for these events has not yet been identified, but there was an extensive study of them [17, 24]. The main physical candidates under study at the moment are outgassing bursts. The functional form of these events is well modeled, so it can be fitted to the data and the glitches removed. If not done, they critically impact the noise spectrum of Figure 2.12, so they were one of LPF main issues and work is being done to understand how to deal with them in LISA.
- **Actuation forces inaccuracies.** Let us have a look at $\Delta\ddot{x}_{\text{OMS}}$ spectrum in Figure 2.12. At low frequencies it goes to zero, because the drag-free system tries to keep the TMs still with respect to each other, so there is effectively no motion over long timescales. This means that the low frequency Δg spectrum is *entirely* determined by the forces we are applying to keep the TMs still. As already anticipated, there could be a difference between the force that we command (a digital number) and the *actual* force that we end up applying. In this matter, it was discovered that the actuation voltages were affected by systematic errors of order $10\mu\text{V}$ between successive force commands. This was just an error related to how voltages were digitized, therefore it could be calculated knowing the digitization algorithm [47]. In this case, it is not completely correct to say that the error was "subtracted from the data", it is better to say that the real λg_{2c} where computed and used in the estimation of Δg . We will address this problem more in depth in Chapter 5.

LPF residual noise breakdown

Always referring to Figure 2.12, what do we understand of the red Δg noise?

- The region above ~ 20 mHz is very well understood, dominated by the readout noise of the interferometric measurement [15]. This is therefore not really noisy motion of the TMs.
- The region between ~ 1 mHz and ~ 20 mHz, where the noise seems almost white, is also very well understood. It is Brownian noise due to collisions with residual gas molecules in the GRS, therefore it is real force noise on the TMs. The level of this noise continuously decreased during the mission, as more and more molecules were vented to space through the Vent Duct, with pressure decreasing exponentially as expected from standard vacuum systems [12, 52].
- For frequencies lower than ~ 1 mHz a $1/f^2$ tail (in PSD) is observed. Many noise sources contribute to this low frequency limit, and they are analyzed and discussed in [12], which also uses results obtained in this thesis and published in [13]. Still, **approximately 50% of the observed noise is currently unexplained.**

Between the known low frequency noise sources, the most relevant one is associated to fluctuations of actuation forces. The estimation of this noise is subject of Chapter 4.

Chapter 3

Electrostatic force actuation in LISA

This Chapter is dedicated to the description of LISA and LISA Pathfinder electrostatic forces. First the reason we need them is explained and also their order of magnitude. Then we will see how they are generated electrostatically by electrodes surrounding the TMs. Finally we will go through the associated noise sources. A goal of this chapter for me was also that of developing a robust explanation and notation for the treatment of stray voltages in the GRS, which to my knowledge is not found in any document, except for a condensed presentation in some papers [6, 13] . The effect is not straightforward and can be confusing, so I hope that this chapter will be of help for people that have to deal with these phenomena.

3.1 Why do we need force actuation?

Focusing on one LISA spacecraft, we know it contains two TMs and both must be in free-fall along the long-interferometer axes. Following both test masses with one spacecraft may not be so simple: in Figure 3.1 a simplified sketch of the situation is proposed, which we examine by steps.

1. The various parts of the SC exert gravitational pulls on the TMs. Ideally, the SC is designed so that all these pulls balance each other, and the net force per unit mass left on the TMs is zero. Obviously, this balancing can never be perfect, therefore there will necessarily be some DC accelerations $\vec{g}_1$ and $\vec{g}_2$ of the TMs due to SC residual "self-gravity". If we did nothing, the TMs would hit the walls of the EH.
2. Since we are not allowed to apply forces along the measurement axes x_1 and x_2 (otherwise we would introduce force noise, Sec. 2.3), we will now show that it is possible to compensate the differential acceleration $\Delta\vec{g} = \vec{g}_2 - \vec{g}_1$ of the two TMs by using forces only along the perpendicular axes y_1 and y_2 , always referring to Figure 3.1. We call them $\vec{F}_1 = F_1\hat{y}_1$ and $\vec{F}_2 = F_2\hat{y}_2$, then the equations of motion for the two TMs are

$$\begin{cases} \vec{a}_1 = \vec{g}_1 + \frac{\vec{F}_1}{M} \\ \vec{a}_2 = \vec{g}_2 + \frac{\vec{F}_2}{M} \end{cases} \quad (3.1)$$

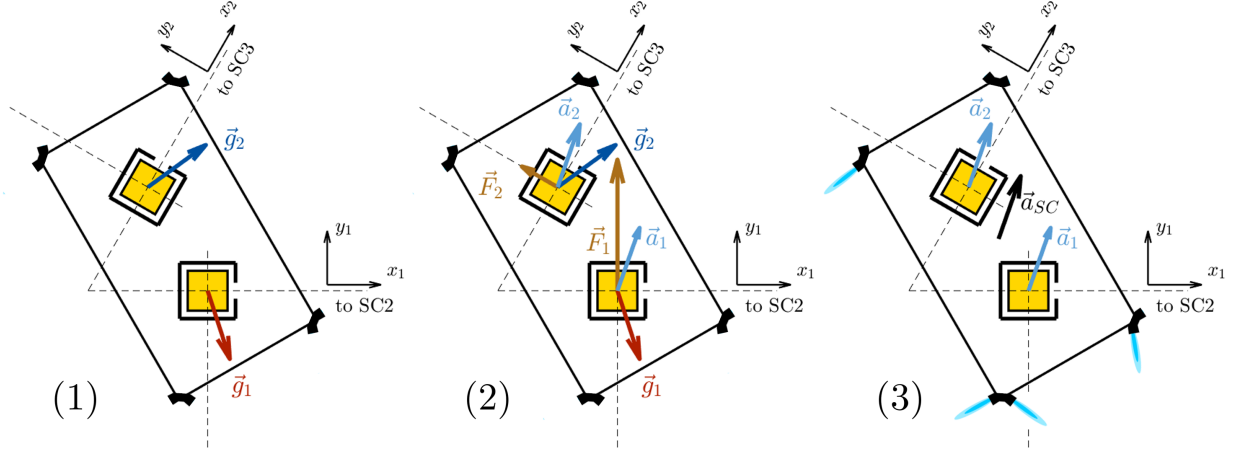


Figure 3.1: Simplified sketch of the LISA SC, containing two TMs, illustrating the strategy used to not perturb the free-fall along two axes. Described step-by-step in the text.

If we now require that $\vec{a}_1 = \vec{a}_2$, both TMs will accelerate in the same direction, so that the spacecraft can be forced to follow them both with the thrusters. Using this condition and the two previous equations

$$\frac{\vec{F}_1 - \vec{F}_2}{M} = \Delta\vec{g}_{\text{DC}} \quad (3.2)$$

Now, $\Delta\vec{g}_{\text{DC}}$ could be any vector in the xy -plane, but since $\hat{y}_1$ and $\hat{y}_2$ **are not parallel**, we can always project it on them and find unique coefficients. Choosing x_1, y_1 as reference frame (see Figure 3.1), so that $\hat{y}_2 = \cos\phi\hat{x}_1 + \sin\phi\hat{y}_1$, we can solve the equation and write down these coefficients

$$\begin{cases} F_2 = -M(\Delta\vec{g}_{\text{DC}} \cdot \hat{x}_1) / \cos\phi \\ F_1 = M[(\Delta\vec{g}_{\text{DC}} \cdot \hat{y}_1) - (\Delta\vec{g}_{\text{DC}} \cdot \hat{x}_1) \tan\phi] \end{cases} \quad (3.3)$$

3. Choosing these coefficients, both TMs have the same acceleration, equal to

$$\vec{a}_{\text{CM}} = \vec{g}_1 + [(\Delta\vec{g}_{\text{DC}} \cdot \hat{y}_1) - (\Delta\vec{g}_{\text{DC}} \cdot \hat{x}_1) \tan\phi] \hat{y}_1 \quad (3.4)$$

which can be followed by using the SC thrusters.

In LPF the two TMs vectors $\hat{y}_1$ and $\hat{y}_2$ are parallel, since the TMs are aligned! If $\Delta\vec{g}$ has a component along $\hat{x}$, there is no way of compensating that with a perpendicular force. That is why one test mass could not be drag-free.

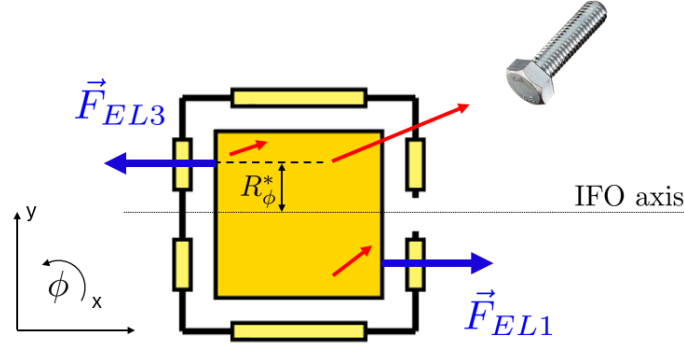


Figure 3.2: Sketch of how torques induced by gravitational unbalances of the SC on the TM are compensated through the application of equal and opposite forces. The screw just represents an unbalanced mass, which gravitational pull tries to rotate the TM.

3.1.1 Noise from ϕ actuation

Rotations can not be drag-free, because the TM must remain aligned with the long-arm interferometer, therefore we are forced to use electrostatic actuation. The most critical degree of freedom is the ϕ rotation around the TM z axis, because the electrodes providing the forces are disposed along the interferometer axis, as in Figure 3.2. The picture represents a point mass (represented by a screw) that exerts a gravitational pull Mg_{DC} on the TM. Since it is a cube, there will also be a gravitational torque $I\gamma_{\phi,\text{DC}}$, which scales as $1/d^5$. So, we follow the TM linear acceleration with the thrusters, and to compensate the torque we must use the electrodes: in this example, we are using just two of them¹. The equations of motion of the mass along x and ϕ are

$$\begin{cases} M\ddot{x} = Mg_{\text{DC}} + F_{EL1} - F_{EL3} \\ I\ddot{\phi} = I\gamma_{\phi,\text{DC}} + R_{\phi}^*(F_{EL1} + F_{EL3}) \end{cases} \quad (3.5)$$

where R_{ϕ}^* is the effective arm-length of the electrode on the TM. By setting $F_{EL1} = F_{EL3} = -I\gamma_{\phi,\text{DC}}/R_{\phi}^*$, we achieve both the objectives of preventing the TM from rotating and not applying a net electrostatic force along x . However, if we consider the fact that these forces could be affected by noise δF , then (3.5) becomes

$$\begin{cases} M\ddot{x} = Mg_{\text{DC}} + \delta F_{EL1} - \delta F_{EL3} \\ I\ddot{\phi} = R_{\phi}^*(\delta F_{EL1} + \delta F_{EL3}) \end{cases} \quad (3.6)$$

So if there is uncorrelated noise in our actuators, this will directly enter in the measurement. We must therefore have actuators which are able to satisfy the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ requirement on

¹Warning: it is not done like this in LISA, nor LPF. Here we are doing a simplification to just understand the noise source, the correct scheme will be given in the next sections.

their own, and we will see that this ends up being the most relevant noise source at low frequency in the budget of LISA and LPF. Notice that what we discussed up to know is completely independent of the actuators nature, they could as well have been pistons... In the following sections of this chapter we will go in the details of electrostatic actuators instead, and we will see what other noise sources they introduce.

How much force do we need?

Having an actuator with $6 \text{ fN}/\sqrt{\text{Hz}}$ noise is easier if we have to apply fN forces, it becomes troublesome if we need to apply μN . So what forces should we expect to have to apply? In this Section we mentioned just gravitational pulls/torque due to mass unbalances of the SC. These are not easy numbers to estimate, needed simulations in the past and experimental data from LPF. Here we give the levels required for LISA.

The most important requirement is on the ϕ DC torque

$$\text{LISA } \phi \text{ DC torque requirement} \quad |\gamma_\phi| < 1 \frac{\text{nrad}}{\text{s}^2} \quad (3.7)$$

which means forces of $\sim I_{zz}\gamma_\phi/R_\phi^* \approx 60 \text{ pN}$ ($R_\phi^* \approx 11 \text{ mm}$). Then, actuators with a few $\text{ppm}/\sqrt{\text{Hz}}$ at 1 mHz should be able to meet the acceleration requirement.

The levels for other degrees of freedom instead are more relaxed, since these forces will not enter *directly* in the IFO axis (they will in other subtle ways)²

$$|\Delta g_x|, |\Delta g_y|, |\Delta g_z|, < 500 \frac{\text{pm}}{\text{s}^2}, \quad |\gamma_\eta|, |\gamma_\theta| < 10 \frac{\text{nrad}}{\text{s}^2} \quad (3.8)$$

Notice these must be "system level" requirements, not just GRS, because the whole SC will contribute to the gravity field at the TM position. For more information on these numbers, see [65].

Are we really able to obtain these numbers in flight? Yes. Many are based on the actual performance of LPF, which was successfully balanced within its (more relaxed) requirements, initially estimated to be in the range $\pm 650 \text{ pm/s}^2$ for x and $\pm 8 \text{ nrad/s}^2$ for ϕ : the mission actually obtained orders of magnitude better performance. LPF obtained the unbalancing levels in Table 3.1. We will see that it was thanks to this level of balancing that LPF could satisfy the LISA acceleration noise requirement.

3.2 Electrostatic force actuation and sensing

The system must be able to provide TM control along 6 DOFs. This is achieved through the disposition of the electrodes around each TM is represented in Figure 3.3. There are

²The x, y, z axes here do not refer the interferometer axes of Figure 3.1, but to SC axes: the y axis passes through the TMs centers of mass, the x is perpendicular and in the constellation plane.

TM ₁				TM ₂			
Force	pm/s ²	Torque	nrad/s ²	Force	pm/s ²	Torque	nrad/s ²
g_c	0	γ_{θ_c}	0.50	g_c	5	γ_{θ_c}	-0.30
g_{y_c}	243	γ_{η_c}	-2.59	g_{y_c}	-243	γ_{η_c}	1.02
g_{z_c}	3	γ_{ϕ_c}	-0.97	g_{z_c}	3	γ_{ϕ_c}	0.15

Table 3.1: LPF commanded forces during February 2017 noise run. They can be assumed to be compensating just gravitational unbalances of the SC. From [65].

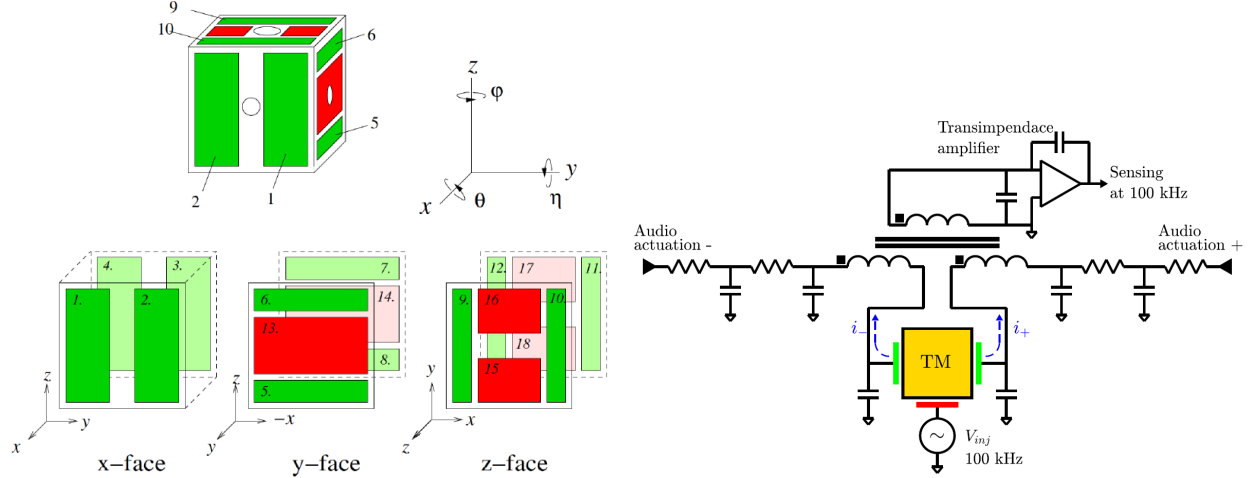


Figure 3.3: (Left) Spatial disposition of injection electrodes (red) and actuation-sensing electrodes (green). (Right) Capacitive bridge sketch, described in the text.

two kinds of electrodes: injection electrodes, in red, and sensing and actuation electrodes, in green. These last ones allow simultaneous sensing and actuation. While the emphasis of this chapter, and the thesis in general, is on the electrostatic actuation, the implementation of this is inseparable from the function of electrostatic sensing, in that they share a single circuitry. As such we begin with a brief description of capacitive sensing, described more in depth in [9]. A concise overall description of the actuation system can be found instead in [13].

Sensing

On the right of Figure 3.3 the sensing bridge is sketched for a single couple of electrodes along x , on which we focus. The TM is biased with a 100 kHz voltage through the injection electrodes: as a result, V_{TM} oscillates with a peak amplitude of ~ 0.6 V. This causes two currents i_+ and i_- to flow in the two green capacitors, which we call C_+ and C_- .

$$i_+ = j\omega C_+ V_{TM}, \quad i_- = j\omega C_- V_{TM} \quad (3.9)$$

These two currents enter a resonant bridge with two coils with the same number of turns. If the two currents are equal, no current will be induced in the secondary coil, otherwise

$$\Delta i = i_+ - i_- = j\omega V_{TM}(C_+ - C_-) \approx j\omega V_{TM} \frac{\partial C}{\partial x} 2x, \quad (3.10)$$

where we approximated that for a small displacement of the TM $C_+ \approx C_0 + \frac{\partial C}{\partial x}x$ and $C_- \approx C_0 - \frac{\partial C}{\partial x}x$. The 100 kHz current Δi is then converted to a voltage through a transimpedance amplifier and demodulated at 100 kHz. Thus we have a signal which is proportional to x .

Note

In missions that employ a discharge wire, the TM can be biased through that.

Force actuation

Always referring to Figure 3.3, actuation voltages are injected "after" the primary coils: the voltages are audio frequency sinusoids between 60-270 Hz (for LPF) and are filtered by two passive filters ($R = 3.3\text{ k}\Omega$, $C = 10\text{ nF}$) to eliminate any spurious component at 100 kHz and not interfere with sensing. Their generation will be discussed later in this thesis. Since the inductance of 5 mH is basically a short circuit for low frequency signals, the actuation voltages are directly applied to the electrodes.

The physical idea is that a voltage ΔV applied to the plates of a capacitor C , generates an *attractive* force between the plates

$$F = \frac{1}{2} \frac{\partial C}{\partial x} \Delta V^2. \quad (3.11)$$

In our case, one of the plates is the capacitor, the other is the TM itself. Figure 3.4 shows how electrodes are arranged around one axis, in this case x , but it is equivalent to any other axis. Since "we can only pull", the approximate actuation strategy is the following: to generate a positive force but no rotation, we activate $V_1 = V_2$. To generate a positive rotation but no force, we activate $V_1 = V_3$. Analogous for negative rotations and translations.

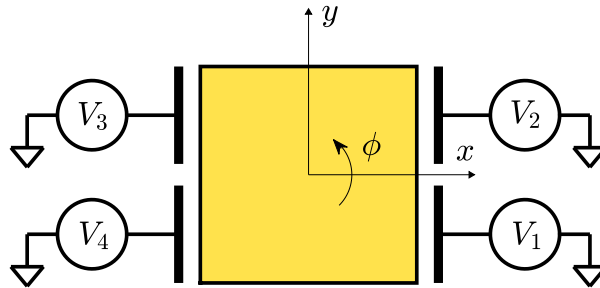


Figure 3.4: Sketch of the TM and electrodes providing x/ϕ actuation. The numbering of the voltage generators will be used throughout the whole thesis.

This seems straightforward, but it is not. First of all, the capacitance does not change linearly with the distance between the plates, so for example $C_1 = C_0 + |\partial_x C| x + \frac{1}{2} |\partial_x^2 C| x^2 + \dots$. Secondly, in our case $\Delta V = V_{EL} - V_{TM}$, but since the TM is electrically floating, its voltage will be determined by the voltages of all the surrounding conductors, namely 17 other electrodes ($V_{TM} = V_{TM}(V_{EL_i})$).

$$F = \frac{1}{2} \left(\left| \frac{\partial C}{\partial x} \right| + \left| \frac{\partial^2 C}{\partial x^2} \right| x + \left| \frac{\partial^2 C}{\partial x \partial \phi} \right| \phi + \dots \right) [V_{EL} - V_{TM}(V_{EL_i})]^2. \quad (3.12)$$

Like this, calculating which force to apply is pretty complicated, since it depends on position of the TM and all the other voltages. Furthermore, this would be a source of cross talks between commands from other degrees of freedom (entering through V_{TM}). In LISA and LPF this is greatly simplified by a smart actuation voltages scheme, which implements these three features

1. **A constant stiffness algorithm:** ensures that the electrostatic gradients are constant, or equivalently, that the electrostatic stiffness does not change when the applied forces are changed.
2. **TM voltage is always zero at the actuation frequencies:** this makes sure that V_{TM} does not depend on other electrode voltages, at least at the zeroth order.
3. **Orthogonal audio frequency actuation voltages:** this avoids again some cross-talk between DOFs, and most importantly, audio frequency voltages are *fundamental* to avoid interaction with stray DC electrostatic fields, which we will soon discuss.

In the next Section we will dissect the TM-EH electrostatic model and explain these features quantitatively, but it is pretty heavy. So, here I anticipate immediately the solution to these problems so that the reader may skip the next section, still grasping how actuation is done on LISA and LPF.

The choice of voltages that achieves the three bullets is

$$\begin{aligned} V_1(t) &= +V_{1x} \sin \omega_x t + V_{1\phi} \sin \omega_\phi t & (+V_{DC,1}) \\ V_2(t) &= -V_{1x} \sin \omega_x t + V_{2\phi} \cos \omega_\phi t & (+V_{DC,2}) \\ V_3(t) &= +V_{2x} \cos \omega_x t - V_{1\phi} \sin \omega_\phi t & (+V_{DC,3}) \\ V_4(t) &= -V_{2x} \cos \omega_x t - V_{2\phi} \cos \omega_\phi t & (+V_{DC,4}) \end{aligned} \quad (3.13)$$

With this choice, DC forces and torques become at zero order

$$\begin{aligned} \langle F_x \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{1x}^2 - V_{2x}^2) \\ \langle N_\phi \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{1\phi}^2 - V_{2\phi}^2). \end{aligned} \quad (3.14)$$

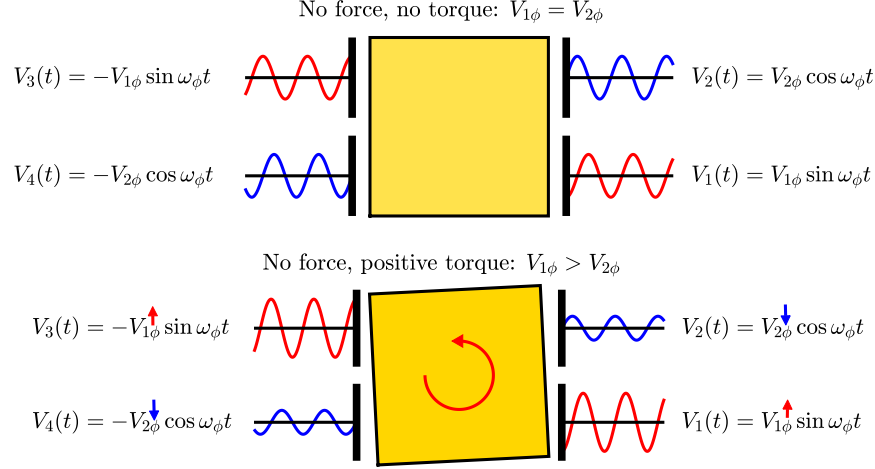


Figure 3.5: Example of ϕ actuation in LISA. Oscillating voltages, that all sum to zero, orthogonal with each other, are always on. An unbalance is created between couples of them to generate a torque.

In LISA, $V_{1x} = V_{2x} = 0$ always, since there is no x actuation (but there is y and z). We can not set any arbitrary value for $V_{1x/2x}$, and $V_{1\phi/2\phi}$: they must be chosen according to the *constant stiffness algorithm*, depicted in Figure 3.5. The voltages are always ON, fixed at a certain level, and an unbalance is created between them in order to generate a torque or a force. Let's suppose we want to apply a *commanded force* Mg_c and a *commanded torque* $I_{zz}\gamma_{\phi_c}$, then we choose

$$V_{1\phi/2\phi} = \sqrt{\frac{I_{zz}(\gamma_{\phi_0} \pm \gamma_{\phi_c})}{\left|\frac{\partial C_x^*}{\partial \phi}\right|}}, \quad V_{1x/2x} = \sqrt{\frac{M(g_0 \pm g_c)}{\left|\frac{\partial C_x^*}{\partial x}\right|}} \quad (3.15)$$

where γ_{ϕ_0} and g_0 are known as the *authorities*: while γ_{ϕ_c} and g_c can change in time to control the TM dynamics, the authorities are fixed numbers that determine the voltages level when no commands are given and the maximum amount of force that can be applied.

Finally, the possibility of applying small DC voltages $V_{DC,i}$ is required to compensate for the effects of stray electrostatic fields in the GRS.

3.3 Model of GRS electrostatic forces

Warning: This Section is a bit technical and heavy in the notation. It is not fundamental to understand what follows: the reader may go really quickly through this part or directly skip to Section 3.4 if not interested in the details.

The model that has been used for the analytical description of the TM-EH electrostatic

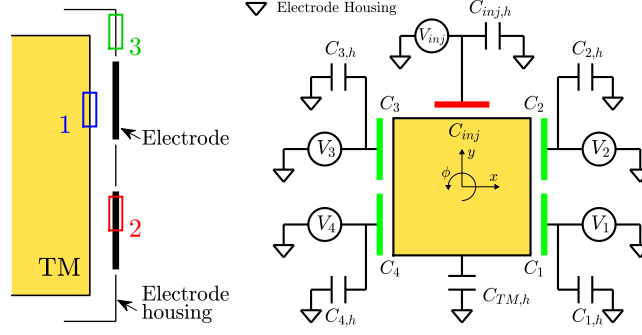


Figure 3.6: (Left) Sketch of the TM-EH interface. The small rectangles highlight "patches" of the conductors that are considered at a uniform potential. (Right) Sketch of the x/ϕ electrodes and TM capacitances.

interaction reads [21, 6]

$$F_q = \frac{1}{2} \sum_{\alpha < \beta} \frac{\partial C_{\alpha\beta}}{\partial q} (V_\alpha - V_\beta)^2 \quad (3.16)$$

where $C_{\alpha\beta}$ is the capacitance between conducting surface domains α and β . q is a generic displacement of the TM: setting $q = x$, we calculate F_x , the force along x ; for $q = \phi$ we calculate instead $F_\phi \equiv N_\phi$, the ϕ torque on the TM. This formula is effectively a generalization of the force between the plates of a single capacitor. The big number of elements unfortunately requires a terrible amount of notation, which should get better along the section.

A simplified sketch of the interface between TM and EH is reported on the left side of Figure 3.6. In an ideal world, each conductor should exhibit the same potential everywhere; unfortunately, due to imperfections and contamination of the surfaces inside the electrode housing, there will be small voltage differences between points of a same conductor [62]. Therefore, we imagine that each surface is composed of many small patches which have uniform voltage. In Figure 3.6 I highlighted with colours three examples of patches; we call V_{TM_j} the voltage of a TM patch (1), $V_{E_{in}}$ that of a patch n on electrode i (2) and V_{EH_k} the voltage of a patch on the guard ring (3). From (3.16), we can write

$$\begin{aligned} F_q = & \frac{1}{2} \sum_{j,k} \frac{\partial C_{TM_j EH_k}}{\partial q} [V_{TM_j} - V_{EH_k}]^2 && \text{TM-Guard ring} \\ & + \frac{1}{2} \sum_i \sum_{n,k} \frac{\partial C_{E_{in} EH_k}}{\partial q} [V_{E_{in}} - V_{EH_k}]^2 && \text{E}_i\text{-Guard ring} \\ & + \frac{1}{2} \sum_i \sum_{n,j} \frac{\partial C_{TM_j E_{in}}}{\partial q} [V_{TM_j} - V_{E_{in}}]^2 && \text{TM-E}_i \\ & + \frac{1}{2} \sum_{i,l} \sum_{n,m} \frac{\partial C_{E_{in} E_{lm}}}{\partial q} [V_{E_{in}} - V_{E_{lm}}]^2 && \text{E}_i\text{-E}_l \\ & + \text{TM-TM} + \sum_i \text{E}_i\text{-E}_i + \text{Guard ring-Guard ring} . \end{aligned} \quad (3.17)$$

The last three terms represent the interaction between patches belonging to the same conductor and they are not included explicitly: we will show that they can be neglected in our analysis. Furthermore, we will also neglect the sum involving $C_{E_{in} E_{lm}}$: these are very small with respect to the other ones in the system, so are their derivatives in q , since pairs of electrodes are shielded by the ground guard rings, by which they are surrounded. We can imagine that patches belonging to the same conductor will exhibit very similar voltages, therefore we can write

$$\begin{aligned} \text{TM patch } j: \quad & V_{TM_j} = V_{TM} + \delta V_{TM_j} \\ \text{Domain } n \text{ on electrode } i: \quad & V_{E_{in}} = V_i + \delta V_{E_{in}} \\ \text{EH patch } k: \quad & V_{EH_k} = \delta V_{EH_k} \end{aligned} \quad (3.18)$$

Where all the potentials are expressed relative to a nominal $V_{EH} = 0$ V and V_i are the voltages we can choose to apply to each electrode. Rewriting (3.17) with definitions (3.18), we discard all the δV^2 terms: the last line in (3.17) contains all terms of the form $\sum \partial_q C (\delta V_j - \delta V_i)^2$ and they are therefore also discarded. We note that this means neglecting the patch-patch interactions, addressed in [62], and studying only the interaction of patches with other large uniform voltages, like the actuation ones. After these considerations, we are left with terms

$$\begin{aligned} F_q = & \frac{1}{2} \sum_{j,k} \frac{\partial C_{TM_j EH_k}}{\partial q} [V_{TM}^2 + 2V_{TM}(\delta V_{TM_j} - \delta V_{EH_k})] \\ & + \frac{1}{2} \sum_i \sum_{n,k} \frac{\partial C_{E_{in} EH_k}}{\partial q} [V_i^2 + 2V_i(\delta V_{E_{in}} - \delta V_{EH_k})] \\ & + \frac{1}{2} \sum_i \sum_{n,j} \frac{\partial C_{TM_j E_{in}}}{\partial q} [(V_i - V_{TM})^2 + 2(V_{TM} - V_i)(\delta V_{TM_j} - \delta V_{E_{in}})] . \end{aligned} \quad (3.19)$$

Now, the following sums appeared

$$\begin{aligned} \sum_{j,k} C_{TM_j EH_k} &\equiv C_{TM,h} && \text{Total capacitance TM-Guard ring} \\ \sum_{n,k} C_{E_{in} EH_k} &\equiv C_{i,h} && \text{Capacitance Electrode } i\text{-Guard ring} \\ \sum_{n,j} C_{TM_j E_{in}} &\equiv C_i && \text{Capacitance Electrode } i\text{-TM} . \end{aligned} \quad (3.20)$$

3.3.1 Test Mass force and voltage

With the new definitions (3.20), we start by examining the result of (3.19) without stray voltages δV to discuss the terms related to applied electrode voltages and TM potential. The expression for the force is

$$F_q = \frac{1}{2} \sum_i^N \frac{\partial C_i}{\partial q} (V_i - V_{TM})^2 + \frac{1}{2} \sum_i^N \frac{\partial C_{i,h}}{\partial q} V_i^2 + \frac{1}{2} \frac{\partial C_{TM,h}}{\partial q} V_{TM}^2 \quad (3.21)$$

These are the most important terms in the force. The first sum runs over all $N = 18$ electrodes and considers only TM-electrodes capacitances. The second sum includes the capacitors between electrodes and EH. The last term is TM to EH. All these important capacitances are sketched in Figure 3.6 (right). Expanding the first term and defining

$$\frac{\partial C_i^*}{\partial q} := \frac{\partial C_i}{\partial q} + \frac{\partial C_{i,h}}{\partial q}, \quad C_{\text{tot}} := \sum C_i + C_{TM,h}, \quad (3.22)$$

- respectively the total capacitance of one electrode to TM and its surrounding guard ring, and the total TM capacitance - the final form of the force is

$$\text{Force} \quad F_q = \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_i^2 - V_{TM} \sum_i^N \frac{\partial C_i}{\partial q} V_i + \frac{1}{2} \frac{\partial C_{\text{tot}}}{\partial q} V_{TM}^2 \quad (3.23)$$

If one is not interested in stray electrostatic fields, this expression is more than enough and allows to calculate the force on the TM for any applied voltage. What is missing is an expression for V_{TM} , which is easily calculated by starting from its charge Q_{TM} , which is the sum of the charges accumulated on the capacitors formed with electrodes and with EH patches:

$$Q_{TM} = \sum_i \sum_{j,n} C_{TM_j E_{in}} (V_{TM_j} - V_{E_{in}}) + \sum_{j,k} C_{TM_j EH_k} (V_{TM_j} - V_{EH_k}), \quad (3.24)$$

then using definitions (3.18) and (3.20)

$$\begin{aligned} Q_{TM} &= \sum_i C_i (V_{TM} - V_i) + V_{TM} C_{TM,h} \\ &+ \sum_i \sum_{j,n} C_{TM_j E_{in}} (\delta V_{TM_j} - \delta V_{E_{in}}) + \sum_{j,k} C_{TM_j EH_k} (\delta V_{TM_j} - \delta V_{EH_k}). \end{aligned} \quad (3.25)$$

If we again imagine to neglect the stray biases δV , we can rearrange the expression with (3.22) to obtain

$$\text{TM voltage} \quad V_{TM} = \frac{Q_{TM} + \sum C_i V_i}{C_{\text{tot}}} \quad (3.26)$$

Again, (3.26) and (3.23) are more than enough to describe the whole system if one is not interested in stray electrostatic fields.

3.3.2 Zeroth order electrostatic force model (centered TM)

As we anticipated, the TM voltage depends on V_i , so looking at (3.23) seems like the force along one DOF depends on all the other DOFs commanded V_i . There is a total of 18 electrodes, 6 are injection electrodes which all apply a 100 kHz sinusoid for sensing purposes,

which is kept with constant amplitude at all times. With the other 12 actuation and sensing electrodes we can choose V_i such that

$$\sum_i^{12} C_i V_i \approx C_x^{(0)} \sum_{x \text{ elec.}} V_i + C_y^{(0)} \sum_{y \text{ elec.}} V_i + C_z^{(0)} \sum_{z \text{ elec.}} V_i = 0 \quad (3.27)$$

where we have used the fact that if the TM is centred, all capacitances of electrodes along a certain axis are equal at zeroth order. This is an approximation and when dealing with stray electrostatic fields, it is important to use the full (3.26): anyway, we will almost always apply V_i with a symmetry such that (3.27) is respected, ensuring that V_{TM} has just linear corrections in q (or equivalently, ensuring that V_{TM} is zero for a centered TM).

In this approximation

$$V_{TM} = \frac{Q_{TM} + C_{inj,tot} V_{inj} \sin \omega_0 t}{C_{tot}} \quad (3.28)$$

so the TM voltage will be just determined by its charge at DC, and will have a component at 100 kHz that will never interact with our audio frequency actuation waveforms. Therefore we can safely state that at low frequency

$$V_{TM}^{DC} = \frac{Q_{TM}}{C_{tot}} \quad \text{at } 0^{th} \text{ order.} \quad (3.29)$$

Finally, the charge on the TM can be made small through the discharge system (order mV) compared to the other voltages used to apply forces (order V), simplifying the force even more

Actuation force on TM for $V_{TM} \approx 0$

$$F_q = \frac{1}{2} \sum_{i=1}^{12} \frac{\partial C_i^*}{\partial q} V_i^2 \quad (3.30)$$

as long as (3.27) holds. This is the expression that should be used to calculate what voltages to get a certain force/torque on the TM. As long as V_{TM} is small, all the corrections to this formula are first order in TM displacement q .

3.3.3 Effect of stray voltages

Here we start addressing all the δV terms that were neglected in formulas (3.19) and (3.25). Starting by the former, stray voltages appear as the following terms:

$$\begin{aligned} F_q^{\text{stray}} = & V_{TM} \left[\sum_{j,k} \frac{\partial C_{TM_j EH_k}}{\partial q} (\delta V_{TM_j} - \delta V_{EH_k}) + \sum_i \sum_{j,n} \frac{\partial C_{TM_j E_{in}}}{\partial q} (\delta V_{TM_j} - \delta V_{E_{in}}) \right] \\ & + \sum_i V_i \left[\sum_{n,k} \frac{\partial C_{E_{in} EH_k}}{\partial q} (\delta V_{E_{in}} - \delta V_{EH_k}) + \sum_{j,n} \frac{\partial C_{TM_j E_{in}}}{\partial q} (\delta V_{E_{in}} - \delta V_{TM_j}) \right] \quad (3.31) \end{aligned}$$

All the δV enter the expression as particular sums: this immediately suggests that looking at the force, we will never be able to measure the individual δV of the patches, but some sort of average of them. In particular, the first line is the sum of all the interactions of TM stray patches with electrodes and guard ring patches, while in the second line square brackets we have the interaction of patches on a single electrode with the surrounding EH and TM. We define two new quantities meant just to simplify eq. (3.31), in particular

$$(C\delta V)_{TM}^s = \sum_{j,k} C_{TM_j EH_k} (\delta V_{TM_j} - \delta V_{EH_k}) + \sum_i \sum_{j,n} C_{TM_j E_{in}} (\delta V_{TM_j} - \delta V_{E_{in}}) \quad (3.32)$$

$$(C\delta V)_i^s = \sum_{n,k} C_{E_{in} EH_k} (\delta V_{E_{in}} - \delta V_{EH_k}) + \sum_{j,n} C_{TM_j E_{in}} (\delta V_{E_{in}} - \delta V_{TM_j}) \quad (3.33)$$

with the superscript "s" standing for "stray". By definition, these should be represent the charges accumulated on the TM and electrodes due to the "stray capacitors" of the patches. With this notation, Eq. (3.31) becomes

$$F_q^{\text{stray}} = V_{TM} \frac{\partial (C\delta V)_{TM}^s}{\partial q} + \sum_i V_i \frac{\partial (C\delta V)_i^s}{\partial q} \quad (3.34)$$

These stray biases are so "dangerous" because the last term is multiplied by the voltage that we apply to one electrode V_i : this can not be approximated as a small term, since it can happen that we have to apply large forces. Using the just introduced definitions, we can write the full expressions for the force in presence of stray biases

$$F_q = \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_i^2 - V_{TM} \sum_i^N \frac{\partial C_i}{\partial q} V_i + \frac{1}{2} \frac{\partial C_{\text{tot}}}{\partial q} V_{TM}^2 + V_{TM} \frac{\partial (C\delta V)_{TM}^s}{\partial q} + \sum_i V_i \frac{\partial (C\delta V)_i^s}{\partial q} \quad (3.35)$$

and for the TM voltage

$$V_{TM} = \frac{Q_{TM} + \sum C_i V_i + (C\delta V)_{TM}^s}{C_{\text{tot}}} \quad (3.36)$$

The need for AC actuation

Here we give some motivation for the use of **AC voltages** (at audio frequency to not interfere with the 100 kHz injection voltage). We look at the terms of equation (3.35) which involve the actuation voltages V_i :

$$F_q = \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_i^2 - V_{TM} \sum_i^N \frac{\partial C_i}{\partial q} V_i + \sum_i V_i \frac{\partial (C\delta V)_i^s}{\partial q} \quad (3.37)$$

1. We see from the second term that if we use DC V_i , we obtain a force which depends on V_{TM} . It is true that we can keep this term small with the UV discharge, but it is strongly preferable to have an actuation system which works independently of that. With AC V_i instead, the term $V_{TM}V_i$ oscillates at audio frequency and $\langle V_{TM}V_i \rangle = 0$.
2. If we used DC voltages again, also the first of term of (3.37) would be problematic. If we imagine that our actuators are affected by low frequency noise $V_i \rightarrow V_i + \delta V_i(t)$, then the first sum becomes

$$\frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_i^2 + \sum_i^N \frac{\partial C_i^*}{\partial q} V_i \delta V_i(t). \quad (3.38)$$

and $V_i \delta V_i(t)$ gives rise to a noisy force directly in the LISA band for x/ϕ electrodes. Very strict requirements should be imposed on $\delta V_i(t)$, which are difficult to achieve for electronics at 1 mHz. Using AC actuation ($V_i = V_{i0} \sin \omega_i t$) massively relaxes this problem. Eq. (3.38) becomes

$$\frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_{i0}^2 \sin^2 \omega_i t + \sum_i^N \frac{\partial C_i^*}{\partial q} V_{i0} \sin \omega_i t \delta V_i(t). \quad (3.39)$$

and if we look at the force we are causing at low frequency by taking its average, we have

$$\text{DC} \quad \frac{1}{4} \sum_i^N \frac{\partial C_i^*}{\partial q} V_{i0}^2 + \sum_i^N \frac{\partial C_i^*}{\partial q} \langle V_{i0} \sin \omega_i t \delta V_i(t) \rangle. \quad (3.40)$$

So, we can apply a desired force to the TM through the first term, while even if the electronics have low frequency noise, the product $\langle \delta V_i(t) V_{i0} \sin \omega_i t \rangle = 0$ since it oscillates at ω_i . We say that the noise was up-converted at the carrier frequency ω_i , moving it completely out of the LISA band. We will see that then the problem falls mainly on the stability of the actuation amplitude V_{i0} .

3. The last term of (3.37) gives completely analogous problems. The contingent distribution of patches introduces shear forces that can couple to any degree of freedom: to put it in formula, we can write for example at first order

$$\sum_i^N V_i \frac{\partial (C \delta V)_i^s}{\partial q} = \sum_i^N V_i \frac{\partial (C \delta V)_i^s}{\partial q} \Big|_0 + \sum_{i,p} V_i \frac{\partial^2 (C \delta V)_i^s}{\partial q \partial p} \Big|_0 p. \quad (3.41)$$

Again, if we used DC V_i , the first term would make our applied force dependent on the values of the DC stray biases, which we do not know, plus, analogously to (3.38) low frequency fluctuations of V_i would again be converted to force noise. The difference here is that while for (3.38) the relevant contributions come from the x/ϕ electrodes, stray patch voltages can result in shear forces, so all the 12 electrodes can be a disturbance through (3.41). This forces us to use AC actuation along *all* degrees of freedom.

Finally, the last term of (3.41) is second order and suggests that if we use DC V_i , cross-talks can appear with other DOFs. While this may be less relevant for LISA, we observe these phenomena frequently with torsion pendulums: we will see some examples in the thesis.

DC bias compensation strategy and orders of magnitude

Notice up to now we did not give any estimates of the stray patch voltages. These are not easy to estimate at all: parameters like $(C\delta V)^s$ strongly depend on the surfaces properties. In some particular cases, however, is to compensate the effects of the DC stray biases by applying some "small" DC voltages on our electrodes. To give an idea, if we perform the substitution

$$V_i \rightarrow V_i + V_{DC,i} \quad (3.42)$$

in Eq. (3.35), neglecting V_{TM}^2 and discarding the terms that involve products of small voltages, we have

$$\begin{aligned} F_q = & \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial q} V_i^2 - V_{TM} \sum_i^N \frac{\partial C_i}{\partial q} V_i \\ & + V_{TM} \left(\frac{\partial (C\delta V)_{TM}^s}{\partial q} - \sum_i^N \frac{\partial C_i}{\partial q} V_{DC,i} \right) + \sum_i^N V_i \frac{\partial (C\delta V)_i^s}{\partial q} + \sum_i^N V_i \frac{\partial C_i^*}{\partial q} V_{DC,i} \end{aligned} \quad (3.43)$$

The second line shows all the DC biases and the effect of DC voltages application: by proper choice of the $V_{DC,i}$, we can cancel certain parts of the terms involving DC biases. Compensating DC stray voltages with this method can vary from case to case, therefore we do not give a general rule here and we will see some applications later in the thesis, for example in Sec. 3.4.2. We just say that the $V_{DC,i}$ needed to compensate *the effects* of these stray biases in GRS like systems [22, 6] and also on LPF [10] have been found of order 10-100 mV. Therefore, it is somehow tempting to think that the stray biases themselves are of this order, but this picture is inaccurate to some degree and can lead to confusion.

3.3.4 Actuation voltages choice for simple TM control

Up to now we derived the most general expression for the TM force, discussed how to keep $V_{TM} \approx 0$ V at zeroth order and why it is better to use AC voltages because of stray patches. Now we will discuss the choice of actuation voltages in LISA and LPF to control in a simple way and simultaneously 1 translation and 1 rotation DOFs with four electrodes, like the ones in Figure 3.4. We consider a small V_{TM} and the fact that we are using just 4 electrodes

so that we can approximate the expression for the force and torques (3.23) as

$$F_x = \frac{1}{2} \sum_i^4 \frac{\partial C_i^*}{\partial x} V_i^2 - V_{TM} \sum_i^4 \frac{\partial C_i}{\partial x} V_i \quad (3.44)$$

$$F_\phi \equiv N_\phi = \frac{1}{2} \sum_i^4 \frac{\partial C_i^*}{\partial \phi} V_i^2 - V_{TM} \sum_i^4 \frac{\partial C_i}{\partial \phi} V_i \quad (3.45)$$

We report here the models of the four relevant electrodes capacitances³

$$\begin{aligned} C_1 = & C_x + \left| \frac{\partial C_x}{\partial x} \right| x + \left| \frac{\partial C_x}{\partial \phi} \right| \phi + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial x^2} \right| x^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \phi^2} \right| \phi^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \eta^2} \right| \eta^2 \dots \\ & \dots + \left| \frac{\partial^2 C_x}{\partial x \partial \phi} \right| x \phi + \left| \frac{\partial^2 C_x}{\partial y \partial \phi} \right| y \phi - \left| \frac{\partial^2 C_x}{\partial z \partial \eta} \right| z \eta \end{aligned} \quad (3.46)$$

$$\begin{aligned} C_2 = & C_x + \left| \frac{\partial C_x}{\partial x} \right| x - \left| \frac{\partial C_x}{\partial \phi} \right| \phi + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial x^2} \right| x^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \phi^2} \right| \phi^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \eta^2} \right| \eta^2 \dots \\ & \dots - \left| \frac{\partial^2 C_x}{\partial x \partial \phi} \right| x \phi + \left| \frac{\partial^2 C_x}{\partial y \partial \phi} \right| y \phi - \left| \frac{\partial^2 C_x}{\partial z \partial \eta} \right| z \eta \end{aligned} \quad (3.47)$$

$$\begin{aligned} C_3 = & C_x - \left| \frac{\partial C_x}{\partial x} \right| x + \left| \frac{\partial C_x}{\partial \phi} \right| \phi + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial x^2} \right| x^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \phi^2} \right| \phi^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \eta^2} \right| \eta^2 \dots \\ & \dots - \left| \frac{\partial^2 C_x}{\partial x \partial \phi} \right| x \phi - \left| \frac{\partial^2 C_x}{\partial y \partial \phi} \right| y \phi + \left| \frac{\partial^2 C_x}{\partial z \partial \eta} \right| z \eta \end{aligned} \quad (3.48)$$

$$\begin{aligned} C_4 = & C_x - \left| \frac{\partial C_x}{\partial x} \right| x - \left| \frac{\partial C_x}{\partial \phi} \right| \phi + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial x^2} \right| x^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \phi^2} \right| \phi^2 + \frac{1}{2} \left| \frac{\partial^2 C_x}{\partial \eta^2} \right| \eta^2 \dots \\ & \dots + \left| \frac{\partial^2 C_x}{\partial x \partial \phi} \right| x \phi - \left| \frac{\partial^2 C_x}{\partial y \partial \phi} \right| y \phi + \left| \frac{\partial^2 C_x}{\partial z \partial \eta} \right| z \eta \end{aligned} \quad (3.49)$$

The models for other electrodes can be found in [34]. The same expressions hold for C^* . Relevant values in this expression are reported in Table 2 at the beginning of the thesis. Let's start by examining just the leading order, that means we discard all the terms $\propto x$ or ϕ , that means also $V_{TM} \approx 0$ at leading order if we choose $\sum V_i = 0$. The force then becomes

$$\begin{aligned} F_x = & \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| (V_1^2 + V_2^2 - V_3^2 - V_4^2) \\ N_\phi = & \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_1^2 - V_2^2 + V_3^2 - V_4^2) \end{aligned} \quad (3.50)$$

As anticipated, we can apply a force without applying a torque by choosing for example $V_1 = -V_2$ (the minus sign is to respect $\sum V_i = 0$), but how can we apply a desired torque

³Note: this expansion is performed around a point where all the capacitances are supposed to be equal. In general this is not the point where your sensor reads position zero.

while applying a desired force, so simultaneously? The method used is to divide

$$\begin{aligned}
 V_1(t) &= +V_{1x}\chi_{1x}(t) + V_{1\phi}\chi_{1\phi}(t) \\
 V_2(t) &= -V_{1x}\chi_{1x}(t) + V_{2\phi}\chi_{2\phi}(t) \\
 V_3(t) &= +V_{2x}\chi_{2x}(t) - V_{1\phi}\chi_{1\phi}(t) \\
 V_4(t) &= -V_{2x}\chi_{2x}(t) - V_{2\phi}\chi_{2\phi}(t)
 \end{aligned} \tag{3.51}$$

where $\chi_{1x}(t)$ is an oscillating function with time averaged value of zero, for example a sine or cosine. The signs ensure again $\sum V_i = 0$. Now, if we perform the substitution

$$\begin{aligned}
 F_x = \left| \frac{\partial C_x^*}{\partial x} \right| & (V_{1x}^2\chi_{1x}^2 - V_{2x}^2\chi_{2x}^2 \\
 & + V_{1x}V_{1\phi}\chi_{1x}\chi_{1\phi} - V_{1x}V_{2\phi}\chi_{1x}\chi_{2\phi} + V_{2x}V_{1\phi}\chi_{2x}\chi_{1\phi} - V_{2x}V_{2\phi}\chi_{2x}\chi_{2\phi})
 \end{aligned} \tag{3.52}$$

We are interested in the force that we can produce at low frequency, so if we consider $\langle F \rangle$, we see that the terms $\langle \chi_{2x}\chi_{2\phi} \rangle$ are a source of cross talk between commands we give at different degrees of freedom. Therefore all the **waveforms are chosen to be orthogonal**, that is all the terms $\langle \chi_{2x}\chi_{2\phi} \rangle = 0$. This orthogonality condition also simplifies the calculations when including all the stiffness terms that we have neglected for now. A simple way to realize such condition is by choosing as waveforms sine and cosines at different frequencies for different DOFs, namely

$$\begin{aligned}
 V_1(t) &= +V_{1x} \sin \omega_x t + V_{1\phi} \sin \omega_\phi t \\
 V_2(t) &= -V_{1x} \sin \omega_x t + V_{2\phi} \cos \omega_\phi t \\
 V_3(t) &= +V_{2x} \cos \omega_x t - V_{1\phi} \sin \omega_\phi t \\
 V_4(t) &= -V_{2x} \cos \omega_x t - V_{2\phi} \cos \omega_\phi t
 \end{aligned} \tag{3.53}$$

that is the formula we already anticipated in (3.13). With this choice, DC forces and torques become

$$\begin{aligned}
 \langle F_x \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{1x}^2 - V_{2x}^2) \\
 \langle N_\phi \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{1\phi}^2 - V_{2\phi}^2)
 \end{aligned} \tag{3.54}$$

and we *finally* managed to get a way of commanding forces and torques simultaneously, without the risk cross-talks between the two. In Table 3.2 the LPF actuation frequencies are reported.

Constant stiffness algorithm

We do not do the calculations for the terms $\propto x$ and ϕ that we left behind, but assuming we use the proposed (3.53) and the capacitance models for C_i , $i = 1, 2, 3, 4$, (and remembering

DOF	$f = \omega/2\pi$ [Hz]	DOF	$f = \omega/2\pi$ [Hz]
x	60	ϕ	270
y	90	θ	240
z	120	η	180

Table 3.2: LPF actuation frequencies.

also to calculate V_{TM} to first order, not just zero) the result is

$$\begin{aligned}
 \langle F_x \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{1x}^2 - V_{2x}^2) \\
 &\quad + \frac{1}{2} \left[\left| \frac{\partial^2 C_x^*}{\partial x^2} \right| (V_{1x}^2 + V_{2x}^2 + V_{1\phi}^2 + V_{2\phi}^2) - \frac{4}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial x} \right|^2 (V_{1\phi}^2 + V_{2\phi}^2) \right] x \\
 \langle N_\phi \rangle &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{1\phi}^2 - V_{2\phi}^2) \\
 &\quad + \frac{1}{2} \left[\left| \frac{\partial^2 C_x^*}{\partial \phi^2} \right| (V_{1x}^2 + V_{2x}^2 + V_{1\phi}^2 + V_{2\phi}^2) - \frac{4}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial \phi} \right|^2 (V_{1x}^2 + V_{2x}^2) \right] \phi \\
 &\quad + \frac{1}{2} \left| \frac{\partial^2 C_x^*}{\partial \phi \partial y} \right| (V_{1x}^2 - V_{2x}^2) y
 \end{aligned} \tag{3.55}$$

These are not even all the terms, since we are just doing the calculations with electrodes 1 to 4. There are contributions to the stiffnesses from y-faces to N_ϕ for example, but we do not need to do all the calculation here, they can be all found in [34].

The message here is that the force we apply still appears complicated, because if we change V_{1x} we end up changing the stiffness of both degrees of freedom. The applied force still depends on the TM position. That is where this *constant stiffness algorithm* becomes useful. We require that

$$V_{1x}^2 + V_{2x}^2 = V_{\text{MAX}_x}^2 \quad V_{1\phi}^2 + V_{2\phi}^2 = V_{\text{MAX}_\phi}^2 \tag{3.56}$$

where V_{MAX} are constants. By holding the sum of the actuation amplitudes constant, the stiffnesses in (3.55) remain constant.

This means that our actuation strategy is the following: keep the actuation voltages always ON, then create an unbalance between them, in order to generate a desired force/torque as was represented already in Figure 3.5.

3.3.5 Maximum forces and torques

An immediate consequence of eq. (3.56) is that the maximum unbalance we can create between the voltages is (for example for ϕ)

$$\text{max unbalance:} \quad V_{1\phi} = 0, \quad V_{2\phi} = V_{\text{MAX}_\phi}, \quad (3.57)$$

or vice versa. V_{MAX} therefore is linked to the maximum torque we can apply, and this is ultimately fixed by the forces we have to compensate on the TM. To relate more directly the voltage amplitudes to the applied forces/torques we introduce some definitions

$$V_{1\phi/2\phi} = \sqrt{\frac{I_{zz}(\gamma_{\phi_0} \pm \gamma_{\phi_c})}{\left|\frac{\partial C_x^*}{\partial \phi}\right|}}, \quad V_{1x/2x} = \sqrt{\frac{M(g_0 \pm g_c)}{\left|\frac{\partial C_x^*}{\partial x}\right|}} \quad (3.58)$$

where I_{zz} is the moment of inertia of the TM. Now the forces can be recast in the following form

$$F_x^{\text{ACT}} = Mg_c - M\omega_{xx}^2 x \quad (3.59)$$

$$N_\phi^{\text{ACT}} = I_{zz}\gamma_{\phi_c} - I_{zz}\omega_{\phi\phi}^2 \phi \quad (3.60)$$

with resulting stiffnesses

$$\begin{aligned} \omega_{xx}^2 &= - \left[\left| \frac{\partial^2 C^*}{\partial x^2} \right| - \frac{4}{C_{\text{tot}}} \left| \frac{\partial C}{\partial x} \right|^2 \right] \frac{I_{zz}}{M \left| \frac{\partial C_x^*}{\partial \phi} \right|} \gamma_{\phi_0} - \frac{\left| \frac{\partial^2 C^*}{\partial x^2} \right|}{\left| \frac{\partial C_x^*}{\partial x} \right|} g_0 \\ \omega_{\phi\phi}^2 &= - \left[\left| \frac{\partial^2 C^*}{\partial \phi^2} \right| - \frac{4}{C_{\text{tot}}} \left| \frac{\partial C}{\partial \phi} \right|^2 \right] \frac{M}{I_{zz} \left| \frac{\partial C_x^*}{\partial x} \right|} g_0 - \frac{\left| \frac{\partial^2 C^*}{\partial \phi^2} \right|}{\left| \frac{\partial C_x^*}{\partial \phi} \right|} \gamma_{\phi_0} \end{aligned} \quad (3.61)$$

We underline again that these are only the stiffness contributions from x/ϕ electrodes, $\omega_{\phi\phi}$ misses a term from y DOF that can be found in [34]. We remind that $g_0 = 0$ for LISA. We can now see from (3.60) that γ_{ϕ_c} represents the torque we are commanding thorough the unbalance of $V_{1\phi}$ and $V_{2\phi}$, called *commanded torque*. γ_{ϕ_0} is called *actuation authority* and has two important roles:

1. It is the maximum torque we can apply ($|\gamma_{\phi_c}| < \gamma_{\phi_0}$). We must therefore choose it wisely: we will show that it must be kept as low as possible to minimize noise, but it must always leave enough range to control the TM.
2. It fixes actuation contribution to the stiffness. While along x we expect the gravitational stiffness to be dominant, ϕ stiffness will be dominated by actuation.

The same reasoning holds for the commanded force g_c and force authority g_0 .

Important note

We just explained how 4 electrodes can be used to provide actuation for 2 DOFs. We chose to use x/ϕ but it could be any other axis. However, we must not forget these two things:

- **LISA** will not have x actuation. Therefore $g_0 = 0$, $g_c = 0$, $V_{1x/2x} = 0$. It will have actuation along other translation DOFs.
- **LPF** does not have x actuation of TM_1 , but has for TM_2 .

Estimate of maximum electrode gaps

In the constant stiffness scheme that we just discussed, we understood that we must choose an authority for forces and one for torques, this fixes the maximum level of forces/torques that we can apply. In section 3.1.1 we stated that the levels of gravitational unbalances that we want to address are of order 500 pm/s^2 and $1\text{-}10 \text{ nrad/s}^2$, a bit more relaxed for LPF, up to a few nm/s^2 . As a reference, let us require that actuation must be able to generate 2 nN . The electronics gives 10 V with 2^{16} bits for a bit size of $V_{\text{LSB}} \approx 153 \mu\text{V}$. In the constant stiffness scheme we must "allocate some voltage" both for torques and forces: for this hand waving discussion let us say we put 5 V each. What is the maximum electrode-TM gap that we can have then? (recalling that we want large gaps to minimize spurious electrostatic interactions)

From (3.55) the maximum force that we can apply is

$$F_{\text{MAX}} = Mg_0 = \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| V_{\text{MAX}_x}^2, \quad (3.62)$$

and approximating the electrode capacitance as that of an infinite capacitor $C_x = A\epsilon_0/d$ ($A \approx 3 \text{ cm}^2$), we can invert the formula to obtain

$$d = \sqrt{\frac{1}{2} \frac{A\epsilon_0}{F_{\text{MAX}}} V_{\text{MAX}_x}^2} \approx 4 \text{ mm} \quad (3.63)$$

This being an hand waving argument, means that the fact that we got exactly 4 mm was pretty lucky, but it still gives the correct scales of the gaps that must be used. We do not do the calculations, but with these gaps, we can generate up to 45 nrad/s^2 angular accelerations, so more than enough for our required levels. Therefore, the limiting factor to the gaps is the force, having to move a 2 kg heavy cube.

3.3.6 Actuation and sensing circuit implementation in LPF

Now that all the elements required for actuation and sensing were discussed, we can give a look at the actual circuit design used to meet such requests in LPF. Starting from the

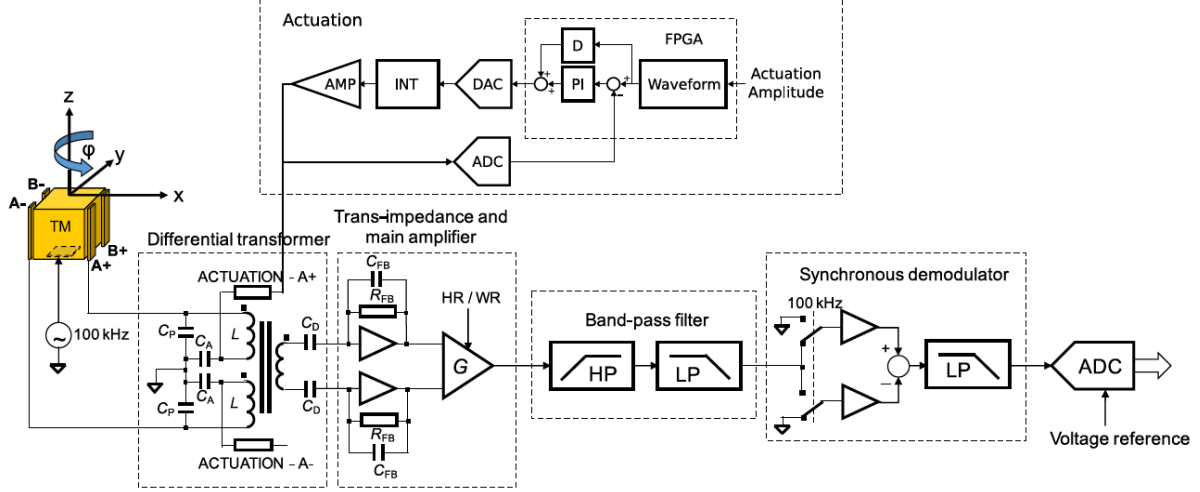


Figure 3.7: Capacitive sensing and actuation circuit for a couple of x/ϕ electrodes. Figure taken from [52] where it was adapted from [9] and [13].

sensing at the bottom, we can recognize the differential transformer and the transimpedance amplifier discussed at the beginning of this chapter, followed by a "synchronous demodulator" which extracts the amplitude of the 100 kHz voltage at the output of the amplifier. This system demonstrate on LPF to be able to measure the capacitance difference at a level between 0.7 and 1.8 aF/ $\sqrt{\text{Hz}}$, corresponding to a test mass displacement measurement noise of 1.2-2.4 nm/ $\sqrt{\text{Hz}}$ and rotation of 83-170 nrad/ $\sqrt{\text{Hz}}$ [9], enough to meet the required level for LISA.

On top of Figure 3.7 we see instead the actuation circuit for just one electrode. We discussed that it must be able to generate a waveform like $V_1(t) = +V_{1x} \sin \omega_x t + V_{1\phi} \sin \omega_\phi t + V_{DC,1}$, which we use as example. To do this, every 0.1 s it receives commands $V_{1x}, V_{1\phi}$ and $V_{DC,1}$ from the DFACS and uses lookup tables to build the two sine functions, re-scale them and sum them with $V_{DC,1}$: this process is better described in Chapter 5. After the digital waveform is complete (it has 1200 samples), it must be converted to an analog signal: this is done trough a Σ - Δ loop working at 96 kHz implemented with a DAC, an integrator and a digital PID controller. The feed-back signal for the PID is provided through an ADC, which ends up being the fundamental component fixing the output noise of the actuation circuit. We will see that in addition to causing uncorrelated fluctuations at each electrode, since the ADC voltage reference is shared between the 4 x/ϕ circuits, it could also cause correlated fluctuations on all of them.

The synthesized voltage are applied to the transformer primary windings two RC stages to not interfere with the 100 kHz sensing, and end up having a nominal bit resolution of 153 μV , corresponding to an effective force per unit mass quantization of order 100 fm/s². This is then further reduce at low frequency by the use of another Σ - Δ loop implemented in the DFACS, better discussed in Chapter 5.

3.4 Actuation related noise sources

Up to now we discussed how forces/torques are generated to control the TM dynamics. In particular, it was explained that the voltages must be always ON to maintain a constant stiffness, and they can be unbalanced to generate forces.

This section addresses the consequences of having voltages affected by noise in the surroundings of the TM. For this, we will only care about the four x/ϕ electrodes, which are the ones with maximum coupling with F_x , our main concern in LISA and LPF. Furthermore, dealing with noise we can consider the TM centered and neglect all the stiffnesses as second order effects, which means

$$F_x = \frac{1}{2} \sum_i^4 \frac{\partial C_i^*}{\partial x} \bigg|_0 V_i^2 - V_{TM} \left(\sum_i^4 \frac{\partial C_i}{\partial x} \bigg|_0 V_i - \frac{\partial (C\delta V)_{TM}^s}{\partial x} \bigg|_0 \right) + \sum_i^N V_i \frac{\partial (C\delta V)_i^s}{\partial x} \bigg|_0 \quad (3.64)$$

and

$$V_{TM} = \frac{Q_{TM} + (C\delta V)_{TM}^s}{C_{\text{tot}}}. \quad (3.65)$$

This last equation tells us that in this approximation the TM voltage does not depend at all on the commanded actuation voltages. Notice also that the last sum in the force runs on all electrodes, not only on the 4 x/ϕ electrodes.

Many of the noise sources that we will discuss are linked to the performance of the actuation electronics, which are in charge of the generation of the voltages V_i . We already explained that these are audio-frequency sine and cosines, with the possibility of applying DC voltages to correct possible stray biases. For a generic electrode nominally applying the oscillating voltage $V_{ic}(t)$ ("c" stands for "command"), we model the noise coming from the FEE as

$$V_i(t) = [1 + \alpha_i(t)] V_{ic}(t) + n_i(t) \quad (+V_{\text{DC},i}) \quad (3.66)$$

where

- $\alpha_i(t)$ are called **gain or amplitude fluctuations**. The generated waveforms amplitude stability is not perfect and varies in time. These are relevant only when some voltage is being applied. We also distinguish between correlated and uncorrelated fluctuations, so

$$\alpha_i(t) = \alpha_c(t) + \alpha_{uc,i}(t). \quad (3.67)$$

The four circuits providing actuation voltages for the x/ϕ electrodes of one test mass are placed on the same board and share a common voltage reference for the ADCs that fix the overall noise performance; fluctuations in this reference was expected to cause correlated noise. Uncorrelated fluctuations were instead observed on ground [32] and are still traced back to the ADC performance.

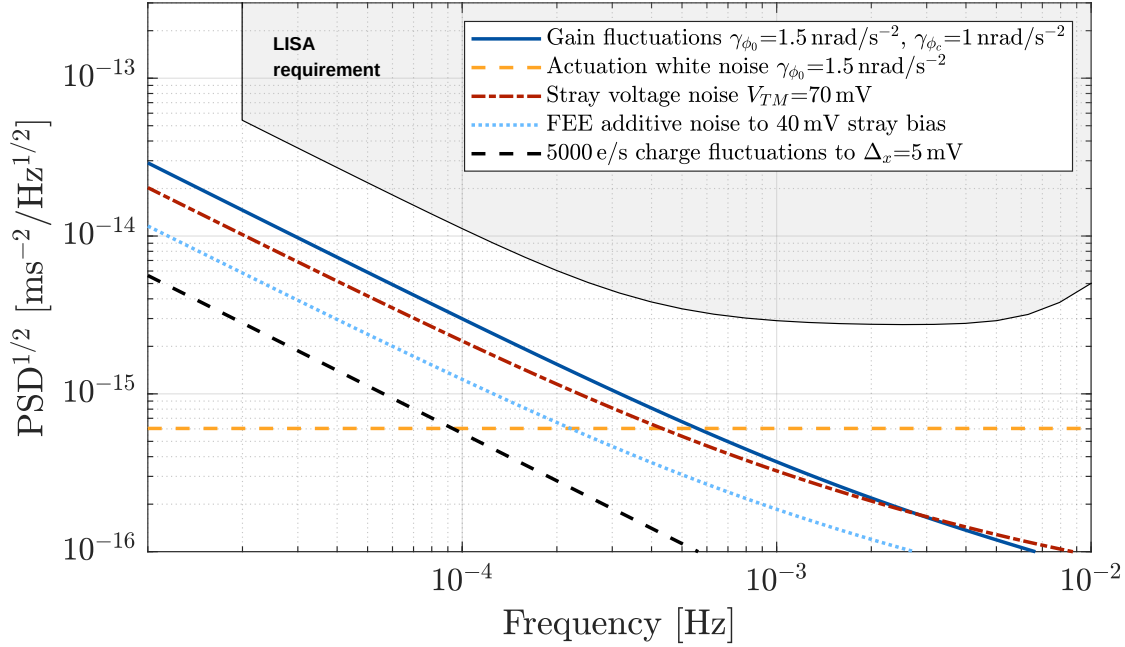


Figure 3.8: Actuation noise sources discussed in this chapter. For the description of each noise contribution, see the main text.

- $n_i(t)$ is instead **additive noise**. This is present even if no actuation is commanded and is basically electronic noise in the waveform generator circuits.

This model was the result of various on-ground investigations [32] and it is tested deeply in this thesis on the LPF data. The α_i and n_i fluctuations enter in various parts of (3.64), resulting in different sources of noise. The main ones will be discussed in the following and their estimates for LISA are all plotted in Figure 3.8: these contributions are among the biggest ones in the low frequency noise budget. For the complete list of associated noise source, see [35].

3.4.1 Gain fluctuations and white noise

The first sum of (3.64) represents the forces we voluntarily apply to the TM. Let us first examine just one electrode, for example electrode 1, insert our voltage noise model (3.66) and keep only the first orders in noise:

$$F_{x,1}(t) = \underbrace{\frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| V_{1c}^2(t)}_{F_{x,1c}(t)} + \underbrace{\frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| 2V_{1c}(t)\alpha_1(t)}_{\delta F_{x,1\alpha}(t)} + \underbrace{\frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| 2V_{1c}(t)n_1(t)}_{\delta F_{x,1n}(t)} \quad (3.68)$$

1. $F_{x,c1}(t)$ is the commanded force to that electrode.

2. $\delta F_{x,1\alpha}(t)$ represents how the force is affected by gain fluctuations. The noisy force is proportional to the force itself

$$\delta F_{x,1\alpha}(t) = 2\alpha_1(t)F_{x,c1}(t) \quad \rightarrow \quad S_{F_{x,1\alpha}}(\omega) = 4|F_{x,1c}|^2 S_{\alpha_1}(\omega) \quad (3.69)$$

In other words, **the more force we use, the more noise we get**. We argued many times that the main forces that must be counteracted are gravitational, so in the end if want to keep this noise low, we must have a very good balancing of the spacecraft. The balancing requirements were already presented in Sec. 3.1.1. The requirements on $\alpha(t)$ for LISA and the shape of their spectra are discussed below.

3. $\delta F_{x,1n}(t)$ involves a multiplication of and audio frequency signal $V_{1c}(t)$ (a sine function) with a broad-band, electronic noise $n_i(t)$. These kind of noises are usually white down to a certain "corner-frequency", and show a $1/f$ or $1/f^2$ behavior below that. Under the very reasonable assumption that $n_1(t)$ is white at the audio actuation frequency, then the result of the multiplication is a down-conversion of the noise $n_1(t)$. The PSD of down-converted noise from a signal of type $2n(t) \sin \omega_0 t$ is white and equal to $2S_n(\omega_0)$, which for our electrode means

$$S_{F_{x,1n}}(\omega) = \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right|^2 [|V_{1x}|^2 S_{n_1}(\omega_x) + |V_{1\phi}|^2 S_{n_1}(\omega_\phi)] \quad (3.70)$$

where we used the (3.53) scheme for actuation voltages. This noise source scales as $\sqrt{F}$.

Gain fluctuations in LISA

In LISA we have only ϕ actuation, which means

$$\begin{array}{ll} \text{LISA } \phi & V_1(t) = +V_{1\phi} \sin \omega_\phi t \\ \text{actuation} & V_2(t) = +V_{2\phi} \cos \omega_\phi t \\ & V_3(t) = -V_{1\phi} \sin \omega_\phi t \\ & V_4(t) = -V_{2\phi} \cos \omega_\phi t \end{array} \quad (3.71)$$

By inserting these in the first term of (3.64) and taking only the low frequency terms proportional to $\alpha_i = \alpha_c + \alpha_{uc,i}$ as defined in (3.67)

$$\begin{aligned} F_{x,\alpha} &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| [V_{1\phi}^2(\alpha_1 - \alpha_3) + V_{2\phi}^2(\alpha_2 - \alpha_4)] \\ &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| [V_{1\phi}^2(\alpha_{uc,1} - \alpha_{uc,3}) + V_{2\phi}^2(\alpha_{uc,2} - \alpha_{uc,4})] . \end{aligned} \quad (3.72)$$

Correlated fluctuations cancel out with this voltage configuration, they do not completely cancel out in LPF, as we will see in Chap. 4. Assuming that $\alpha_{uc,i}$ all have the same noise

spectrum, and linking $V_{1\phi/2\phi}$ to γ_{ϕ_c} and γ_{ϕ_0} through (3.58), we can calculate how much acceleration noise we get for a given commanded torque and authority

$$S_g^{\text{ACT,gain}}(\omega) = R^{*2} [\gamma_{\phi_0}^2 + \gamma_{\phi_c}^2] S_{\alpha_{uc}}(\omega) \quad (3.73)$$

where $R^* = I_{zz} \left| \frac{\partial C_x^*}{\partial x} \right| / M \left| \frac{\partial C_x^*}{\partial \phi} \right| \approx 33 \text{ mm}$ is an effective armlength converting angular into linear acceleration. The current requirements on γ_{ϕ_c} was already said to be 1 nrad/s^{-2} , so $\gamma_{\phi_0} = 1.5 \text{ nrad/s}^{-2}$ is allocated to have some control margin. Based on ground tests and results of LPF, which are presented in Chapter 4, the current requirement on $S_{\alpha_{uc}}$ is

$$S_{\alpha_{uc}}(f) < \left(50 \text{ ppm}/\sqrt{\text{Hz}} \right)^2 \left(\frac{0.1 \text{ mHz}}{f} \right)^2 + \left(4 \text{ ppm}/\sqrt{\text{Hz}} \right)^2 \left(\frac{1 \text{ mHz}}{f} \right)^2 \quad (3.74)$$

Acceleration noise corresponding to this requirements is plotted in Figure 3.8 as a solid blue line and constitutes the most important (and dangerous) noise source at low frequency, since it depends on the spacecraft balancing quality.

Note

Notice that from the first line of (3.72), it appears that the true important quantities to constrain would be $\alpha_1 - \alpha_3$ and $\alpha_2 - \alpha_4$, independently of our model of correlated/uncorrelated fluctuations. In my Master's thesis a simple and fast measurement was developed and tested, exactly addressing these quantities. The intention is use the technique for the on-ground testing of the future LISA FEE [29], discussed in Chapter 4. It could be a possibility to move the requirement from a model dependent α_{uc} to an independent $\alpha_1 - \alpha_3$.

In this thesis

- The first part of Chapter 4 is entirely dedicated to the quantification of the level of these gain fluctuations during the flight of LISA Pathfinder. The efficacy of the model is tested on 4 dedicated experiments runs. These fluctuations end up being the most important noise source that we know in the low frequency acceleration noise tail measured on LPF. The estimates are then used to predict noise for LISA.
- Being such an important source of low frequency noise, it is fundamental to test the LISA FEE electronics before flight. Chapter 6 is entirely dedicated to the description of a torsion pendulum test that can measure torque noise coming from gain fluctuations. In particular, it is exactly sensitive to the quantity (3.72). This is a test that does not rely on any electrostatic model, because it measures directly the LISA observable: acceleration of a test mass. Such a test was previously possible only in space with LPF.

Actuation white noise in LISA

Again, the terms proportional to n_i from the first sum of (3.64) when inserting 3.71 are

$$F_x = \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| [2V_{1\phi}(n_1 + n_3) \sin \omega_\phi t + 2V_{2\phi}(n_2 + n_4) \cos \omega_\phi t] . \quad (3.75)$$

This results in a down conversion of the noise: if we consider all the circuits equally noisy and uncorrelated

$$S_g^{\text{ACT,white}}(\omega) = \frac{2}{M} \left| \frac{\partial C_x^*}{\partial x} \right| R^* \gamma_{\phi_0} S_n(\omega_\phi) \quad (3.76)$$

The current requirement is

$$S_n(\omega_\phi) < \left[5 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 \quad (3.77)$$

This noise is reported in Figure 3.8 as an orange dashed line.

Additional actuation force inaccuracies/non-linearities

The generation of actuation waveforms in the FEE electronics is described in Chapter 5. During the LPF mission it was discovered that the rounding algorithm used to synthesize the digital waveforms was introducing systematic errors that depended on the commanded voltage amplitudes. In such a case, the model for actuation voltages can be written as

$$V_i(t) = [1 + \alpha_i(t) + \delta V_{nl}(V_{ic})] V_{ic}(t) + n_i(t) + \delta V_{nl,\text{DC}}(V_{ic}) . \quad (3.78)$$

These error are systematic in the sense that for a fixed choice of V_{ic} , they do not change in time, and the same error is committed every time that command is chosen.

The LPF rounding error was understood and removed from the data. However, there could be non-linearities of different nature, for example coming from the differential non linearity of the ADC/DAC components used in the FEE electronics circuits. How these inaccuracies impact the acceleration noise spectrum is not easy to evaluate, since they depend on $V_{ic}(t)$ and not just t .

In this thesis

The subject of Chapter 5 is an experimental search for other non-linearities in the actuation circuits. It then investigates how non-linearities impact acceleration noise with analytic and simulation approaches. The final result is that this can be an important source of noise and simple strategies are suggested to keep it under control. The results of this investigation done in collaboration with Lorenzo Sala are reported in a technical note [28].

3.4.2 In-band voltage fluctuations

Now that the first term of (3.64) has been fully discussed, we move to the remaining terms. The actuation voltages do not appear as V^2 anymore, therefore they do not have a DC component, just an audio frequency one which is well out of the LISA band. Discarding all the oscillating terms, Eq. (3.64) reduces to

$$F_x = -V_{TM} \left[\left| \frac{\partial C_x}{\partial x} \right| (n_1 + n_2 - n_3 - n_4) - \frac{\partial(C\delta V)_{TM}^s}{\partial x} \right]_0 + \left| \frac{\partial C_x}{\partial x} \right| (V_{DC,1} + V_{DC,2} - V_{DC,3} - V_{DC,4}) + \sum_i^N n_i \frac{\partial(C\delta V)_i^s}{\partial x} \Big|_0 \quad (3.79)$$

where this time we did not forget to also write down the possible $V_{DC,i}$ that can be applied to the electrodes. From this expression we see that if the additive noise in the actuation system $n_i(t)$ fluctuates at low frequencies, as many electronics noise do, it can be converted to a force through the TM voltage (or charge), or by coupling to stray biases on the corresponding electrode (last term). From our poor knowledge of electrostatic patch potentials $(C\delta V)^s$, it could even be they are oscillating too and therefore also couple to V_{TM} . For this reason this expression is usually rewritten as [6, 10]

$$F_x = -V_{TM} \left| \frac{\partial C_x}{\partial x} \right| \Delta_x + \sum_i^N n_i \left| \frac{\partial C_x}{\partial x} \right| \delta V_{x,i}, \quad (3.80)$$

where we defined

$$\Delta_x = \left[(n_1 + n_2 - n_3 - n_4) - \frac{1}{\left| \frac{\partial C_x}{\partial x} \right|} \frac{\partial(C\delta V)_{TM}^s}{\partial x} \right]_0 + (V_{DC,1} + V_{DC,2} - V_{DC,3} - V_{DC,4}) \quad (3.81)$$

and

$$\delta V_{x,i} = \frac{1}{\left| \frac{\partial C_x}{\partial x} \right|} \frac{\partial (C\delta V)_i^s}{\partial x} \bigg|_0. \quad (3.82)$$

Both definitions are meant to normalize the sum of stray voltages to the effect of one x electrode, which can then be represented by a voltage instead of the quantity $(C\delta V)^s$. This is the approach that can be found in many papers regarding this subject.

Low frequency voltage fluctuations coupling to TM charge

We mentioned that the TM is charged up by cosmic rays and it will never be discharged by contact with the surrounding surfaces. The charging rate was observed to be $+25 \text{ e/s}$ on average [11], which means $+10 \text{ mV/day}$. Surprisingly, it was also observed that the charging would saturate at $V_{TM} \sim 1 \text{ V}$. From (3.80) it can be seen that Δ_x fluctuations get multiplied by V_{TM} resulting in force noise.

$$S_g^{\delta\Delta_x}(f) = \frac{V_{TM}^2}{M^2} \left| \frac{\partial C_x}{\partial x} \right|^2 S_{\Delta_x}(f) \quad (3.83)$$

Therefore there are requirements both on the maximum value that V_{TM} must have (set to $\approx 70 \text{ mV}$) and to noise in Δ_x

$$S_{\Delta_x}(f) = \left[4.5 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 + \left[75 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 \left(\frac{0.1 \text{ mHz}}{f} \right) + \left[190 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 \left(\frac{0.1 \text{ mHz}}{f} \right)^2 \quad (3.84)$$

based on what was observed on LPF [10]. The associated noise is reported in Figure 3.8 as a red, dash-dotted line: it can be seen that for a 1 V TM the curve should be multiplied by ≈ 15 , surpassing the LISA requirement. Therefore a UV discharge system is necessary.

In this thesis

The analysis leading to the estimation of this noise in LISA Pathfinder was consolidated in this thesis studying some dedicated experiments. With respect to [10], the data used here are cleaned from some spurious actuation forces and a correlated analysis of differential acceleration and torque is used. Further analyses are brought in support of Δ_x noise being dominated by $n_i(t)$, the additive noises related to actuation circuits. All of this is subject of Section 4.2.

Low frequency voltage fluctuations coupling to single electrode stray bias

Even if we kept the TM neutral, we are still left with the last term of Eq. (3.80), in which noise n_i at each electrode mixes with its corresponding DC stray bias $\delta V_{x,i}$. The sum runs on all electrodes and also here the stray biases were normalized to an x electrode. The x/ϕ electrodes are expected to exhibit stray biases with the highest $\partial(C\delta V)_i^s/\partial x$ at zero order: the other electrodes can only contribute with shear forces instead and are expected to give

lower contributions. The x/ϕ electrodes are therefore the ones contributing most to the sum. We have ways of measuring and compensating combinations of these stray biases with small DC voltages applied to the electrodes, but not to compensate each of them singularly. They are therefore a way in which low frequency fluctuations of n_i affect the TM free-fall level.

$$S_g^n(f) = \frac{4}{M^2} \left| \frac{\partial C_x}{\partial x} \right|^2 \delta V_{x,i}^2 S_n(f) \quad (3.85)$$

Supposing the noise in Δ_x of previous section comes just from the 4 n_i , then each of them must have performed on LPF as

$$S_n(f) = \frac{1}{4} \left[4.5 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 + \left[75 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 \left(\frac{0.1 \text{ mHz}}{f} \right) + \left[190 \mu\text{V}/\sqrt{\text{Hz}} \right]^2 \left(\frac{0.1 \text{ mHz}}{f} \right)^2 \quad (3.86)$$

More difficult it to give an estimate of a typical $\delta V_{x,i}$, which as reference is set to $\approx 40 \text{ mV}$, based on the measured values of various multiple-electrode combinations of DC biases in the laboratory [6, 22] and in flight [10]. The noise with these parameters is a dotted cyan line in Figure 3.8.

3.4.3 TM charge fluctuations

Up to now we just talked about Δ_x fluctuations, but its definition involving stray biases in (3.82) means that it also has a DC value $\Delta_{x,\text{DC}}$, reported here:

$$\Delta_{x,\text{DC}} = \left[- \frac{1}{\left| \frac{\partial C_x}{\partial x} \right|} \frac{\partial (C \delta V)_{TM}^s}{\partial x} \right]_0 + (V_{\text{DC},1} + V_{\text{DC},2} - V_{\text{DC},3} - V_{\text{DC},4}) \quad (3.87)$$

When setting all the $V_{\text{DC},i} = 0$, the remaining stray term can again exhibit order 10-100 mV. This gives space for another kind of noise from (3.80)

$$\delta F_x = -\delta V_{TM} \left| \frac{\partial C_x}{\partial x} \right| \Delta_{x,\text{DC}} \approx -\frac{\delta Q_{TM}}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial x} \right| \Delta_{x,\text{DC}} \quad (3.88)$$

Charge burst

Before discussing the nature of TM charge fluctuations, equation (3.88) also suggests how to measure $\Delta_{x,\text{DC}}$, that is measuring

$$\frac{\partial F_x}{\partial Q_{TM}} = -\frac{1}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial x} \right| \Delta_{x,\text{DC}} \quad (3.89)$$

Operationally, this is done by a "charge burst", a rapidly induced change ΔQ_{TM} in the TM charge through UV light. The corresponding change in DC x force ΔF_x is then measured. From the ratio of this two quantities we can estimate $\Delta_{x,\text{DC}}$ with (3.89). The measured value can than be compensated to values $< 5 \text{ mV}$ (demonstrated in [6, 10]) by applying $V_{\text{DC},i}$ following the symmetry suggested by Eq. (3.87).

Fluctuations in the TM charge can be due to two mechanisms:

- **The environmental charging.** It was mentioned multiple times that cosmic rays charge the TM with $\lambda_{\text{net}} \approx +25 \text{ es}^{-1}$, but this is just the net result of a much higher traffic of positive and negative charges entering and leaving the TM. In particular the total amount of charges per unit time observed, regardless of their sign, was $\lambda_{\text{eff}} \sim 1300$ charges/s [10]. This was actually measured by maximizing exactly this source of noise: $\Delta_{x,\text{DC}}$ was set to 1V to enhance the force due to δQ_{TM} in (3.88). If the impact of each charge e is considered uncorrelated from the others, than the current i_{env} flowing to and from the TM is a series of spikes, or delta functions, which is a Poisson process. This was already mention in the discussion of shot noise and calculated in App. 9.2, the one-sided PSD of such current is given by

$$S_{i_{\text{env}}}(f) = 2e^2 \lambda_{\text{eff}}. \quad (3.90)$$

The charge is just the integral of the current, therefore

$$S_{Q_{\text{env}}}(f) = \frac{2e^2 \lambda_{\text{eff}}}{(2\pi f)^2} \quad (3.91)$$

This $1/f^2$ shape was observed in [10] and λ_{eff} estimated.

- **The continuous UV discharge.** In LPF, the TM was discharged every two weeks with a "fast discharge mode", using strong light to quickly reach the desired potential during a charge measurement. In LISA, the plan is to not interrupt any measurement, but use a series of continuous UV pulses to always keep the TM discharged. This mechanism will be described in Chapter 7. Still, leaving the light on during science mode introduces further charge traffic and noise on TM charge

$$S_{Q_{\text{UV}}}(f) = \frac{2e^2 \lambda_{\text{UV}}}{(2\pi f)^2}. \quad (3.92)$$

Less light means less noise, but also a much slower discharge: this must be able to deal with strong charging events like Solar Energetic Particles (SEP). $\lambda_{UV} < 3000 \text{ es}^{-1}$ was taken as a compromise. We will see in Chapter 7 that in our model this means a discharge time of ≈ 1 day.

Noise in LISA due to TM charge fluctuations with $\lambda_{\text{eff}} = 5000 \text{ e/s}$ and $\Delta_x = 5 \text{ mV}$ is reported as a dashed black line in Figure 3.8.

In this thesis

Chapter 7 is entirely dedicated to a measurement campaign of TM charge noise during continuous UV discharge, using UV LEDs envisioned for LISA. This was attempted with the 4TM torsion pendulum in [20], but the sensitivity was barely sufficient. The 1TM torsion pendulum allowed a very precise study of our discharge noise models and a verification that the UV discharge systems can meet the requirements.

Chapter 4

LISA Pathfinder electrostatic force actuation performance analysis

This chapter is dedicated to the analysis of the noise performance of the electrostatic forces used to control the TMs in LPF. It was said multiple times that the success of LISA Pathfinder was also thanks to the exceptional gravitational balancing (see Sec. 3.1.1 and some reference values in Table 3.1) that allowed lowering actuation authorities and therefore noise associated with actuators gain fluctuations. Still, the obtained noise is not completely explained: an accurate estimation of actuation noise is required to allow for further investigations, like the work in [12]. Understanding the electrostatic actuation performance is also important to set requirements on LISA and project how the noise will contribute to its sensitivity. That is what we do in this chapter.

Section 4.1 presents the analysis of four LPF runs with increased authorities, also showing how its noise would have looked like if LPF was not so perfectly balanced. A model of Δg as function of gain fluctuations at each electrode is then presented in detail and fitted to the experimental data with a Bayesian Monte Carlo approach, using not only differential acceleration of the TMs, but also the angular one. The results are used to assess the system performance, compare with ground measurements and project to other LPF runs and LISA. Section 4.2 is dedicated to the study of an experiment with highly charged TMs, to enhance their interactions with x stray electrostatic fields, as described in 3.4.2. This noise source is estimated, its origin discussed and projected to other runs and LISA. All the results presented were published in [13].

4.1 Gain fluctuations in LPF

Gain fluctuations were described in detail for LISA in Sec. 3.4.1. The way they affect LPF is the same, but the description becomes more complicated, since we have two TMs and also linear forces need to be applied to TM2. The configuration of TMs and x/ϕ electrodes is reported in Figure 4.1. Each applied voltage is labeled as V_{ij} with $i = 1, 2$ indicating the

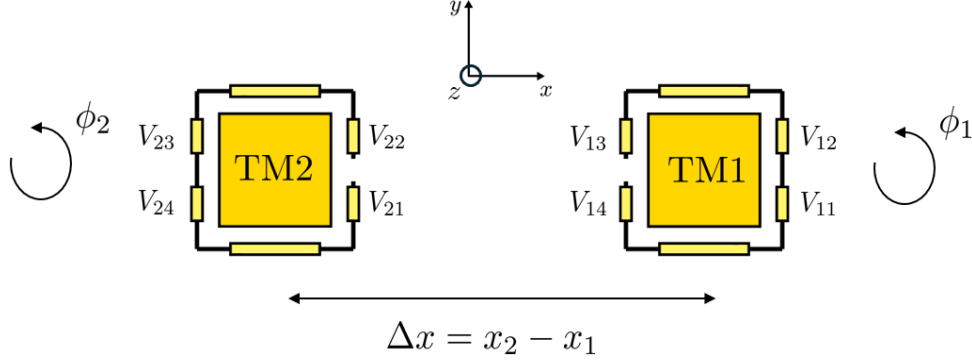


Figure 4.1: Configuration of the electrodes in LPF and reference frame.

TM, $j = 1, 2, 3, 4$ indicating the electrode. Also authorities will be labeled g_{i0} , γ_{i0} to indicate the TM, as well as commands g_{ic} , $\gamma_{\phi_{ic}}$. Let us start by examining just one electrode for TM₂, it applies, for $V_{TM} = 0$ - Eq. (3.30) and (3.53)-

$$\begin{aligned} F_{2,x}^{\text{ACT}} &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| V_{21}^2 = \frac{1}{4} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{2,1x}^2 + V_{2,1\phi}^2) \\ N_{2,\phi}^{\text{ACT}} &= \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| V_{21}^2 = \frac{1}{4} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{2,1x}^2 + V_{2,1\phi}^2) \end{aligned} \quad (4.1)$$

now, gain fluctuations of this electrode $\hat{\alpha}_{21}(t)$ result in force fluctuations $2\hat{\alpha}F$, and converting applied voltages to commanded forces and torques through (3.58), we obtain

$$\begin{aligned} \delta g_2^{\text{ACT}}(t) &= \frac{\delta F_{2,x}}{M} = \frac{1}{2} (g_{20} + g_{2c} + R_\phi^* \gamma_{\phi_{20}} + R_\phi^* \gamma_{\phi_{2c}}) \hat{\alpha}_{21}(t) = a_{12} \hat{\alpha}_{21}(t) \\ \delta \gamma_{\phi_2}^{\text{ACT}}(t) &= \frac{\delta N_{2,\phi}}{I_{zz}} = \frac{1}{2R_\phi^*} (g_{20} + g_{2c} + R_\phi^* \gamma_{\phi_{20}} + R_\phi^* \gamma_{\phi_{2c}}) \hat{\alpha}_{21}(t) = b_{12} \hat{\alpha}_{21}(t) \end{aligned} \quad (4.2)$$

with $R_\phi^* = I_{zz} \left| \frac{\partial C_x^*}{\partial x} \right| / M \left| \frac{\partial C_x^*}{\partial \phi} \right|$.

Again, we see that higher authorities and commands mean higher acceleration noise, but also equivalently higher angular acceleration noise: in particular, the noise in the two channels is exactly the same and this clearly suggests that a cross correlated analysis of both acceleration and torque could give valuable information. Each electrode fluctuations $\hat{\alpha}_{ij}$ will contribute with a different coefficient a_{ij} to linear and b_{ij} to angular accelerations. The differential angular acceleration was not mentioned in the previous chapters, so here we report how it is calculated from the optically measured quantities in LPF and the applied torques, including also Δg for comparison:

$$\begin{aligned} \Delta g &= \Delta \ddot{x}_{\text{OMS}} - g_{2c} + g_{1c} + \Delta \omega^2 x_{1,\text{OMS}} + \omega_2^2 \Delta x_{\text{OMS}} \\ \Delta \gamma_\phi &= \ddot{\phi}_{2,\text{OMS}} - \ddot{\phi}_{1,\text{OMS}} - \gamma_{\phi_{2c}} + \gamma_{\phi_{1c}} + \omega_2^2 \phi_{2,\text{OMS}} - \omega_1^2 \phi_{1,\text{OMS}}. \end{aligned} \quad (4.3)$$

The main difference between the two is that the TM tilts are measured separately and then combined while the differential linear acceleration is a single measurement. Then, g_{1c} is

usually 0, while the commanded torques are not. This said, the full noise model including all electrodes is presented in the next section.

4.1.1 Gain fluctuations noise model

We can start by writing down how each V_{ij} enters in the LPF observables

$$\begin{aligned}\Delta g^{\text{ACT}} &= \frac{F_{2,x} - F_{1,x}}{M} \\ &= \frac{1}{2M} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{21}^2 + V_{22}^2 - V_{23}^2 - V_{24}^2) - \frac{1}{2M} \left| \frac{\partial C_x^*}{\partial x} \right| (V_{11}^2 + V_{12}^2 - V_{13}^2 - V_{14}^2) \\ \Delta \gamma_\phi^{\text{ACT}} &= \frac{N_{2,\phi} - N_{1,\phi}}{I_{zz}} \\ &= \frac{1}{2I_{zz}} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{21}^2 - V_{22}^2 + V_{23}^2 - V_{24}^2) - \frac{1}{2I_{zz}} \left| \frac{\partial C_x^*}{\partial \phi} \right| (V_{11}^2 - V_{12}^2 + V_{13}^2 - V_{14}^2)\end{aligned}\quad (4.4)$$

then by moving to g_{i0} , g_{ic} , $\gamma_{\phi i0}$, $\gamma_{\phi ic}$ analogously to what we did in the previous paragraph for a single electrode, we can obtain an expression for the noise in Δg and $\Delta \gamma_\phi$

$$\begin{aligned}\delta \Delta g^{\text{ACT}}(t) &= \sum_{ij} a_{ij} \hat{\alpha}_{ij}(t) \\ \delta \Delta \gamma_\phi^{\text{ACT}}(t) &= \sum_{ij} b_{ij} \hat{\alpha}_{ij}(t).\end{aligned}\quad (4.5)$$

The expressions for all a_{ij} coefficients are reported below

$$\begin{aligned}a_{i1} &= \frac{(-1)^i}{2} (g_{i0} + g_{ic} + R_\phi^* \gamma_{\phi i0} + R_\phi^* \gamma_{\phi ic}) \\ a_{i2} &= \frac{(-1)^i}{2} (g_{i0} + g_{ic} + R_\phi^* \gamma_{\phi i0} - R_\phi^* \gamma_{\phi ic}) \\ a_{i3} &= \frac{(-1)^i}{2} (-g_{i0} - g_{ic} + R_\phi^* \gamma_{\phi i0} - R_\phi^* \gamma_{\phi ic}) \\ a_{i4} &= \frac{(-1)^i}{2} (-g_{i0} + g_{ic} - R_\phi^* \gamma_{\phi i0} + R_\phi^* \gamma_{\phi ic}) \\ b_{i1} &= \frac{a_{i1}}{R_\phi^*} & b_{i4} &= \frac{a_{i4}}{R_\phi^*} \\ b_{i2} &= -\frac{a_{i2}}{R_\phi^*} & b_{i3} &= -\frac{a_{i3}}{R_\phi^*}\end{aligned}\quad (4.6)$$

The model in this form is pretty general and just assumes fluctuations in the amplitude of the sinusoidal waveforms, there are no assumptions on the correlations between them. In general, they could all be correlated. The analysis that follows involves the spectra and cross spectra of these fluctuations: if we were to consider them all, we would have 8 spectra (1 for each electrode) and 28 cross spectra, for a total of 36 unknown spectra to be determined.

The experiments that were performed on LPF to enhance gain fluctuations noise are just 4 and could enhance only a limited amount of combinations of $\hat{\alpha}$, therefore in this work a minimal model is proposed including only the correlations which we think to be physically motivated. The result is a drastic reduction of parameters.

Assumptions on gain fluctuations

We hypothesize that fluctuations at each electrode are of the form

$$\hat{\alpha}_{ij}(t) = \alpha(t) + \alpha_i(t) + \alpha_{ij}(t) \quad (4.7)$$

where

- α are gain fluctuations common to all the 8 x/ϕ electrodes, which could be caused by a temperature change in the GRS FEE box which generates all these voltages.
- α_i , with $i = 1, 2$ are common fluctuations of all 4 electrodes of the same TM. ADCs in the 4 waveform generators share the same voltage reference. An instability in this will affect all 4 electrodes.
- α_{ij} are finally uncorrelated fluctuations at each electrode. These were not expected in the first stages of development and were observed with ground tests. They end up being the dominant contribution.

For a total of 11 generators, α , α_i and α_{ij} are all assumed to be uncorrelated between each other. This is not the only way in which this analysis could have been performed but as stated above, it relies on simple, physically motivated assumptions.

Inserting (4.7) in (4.5), we obtain

$$\begin{aligned} \delta\Delta g^{\text{ACT}}(t) &= a\alpha + \sum_i a_i\alpha_i(t) + \sum_{ij} a_{ij}\alpha_{ij}(t) \\ \delta\Delta\gamma_\phi^{\text{ACT}}(t) &= b\alpha + \sum_i b_i\alpha_i(t) + \sum_{ij} b_{ij}\alpha_{ij}(t). \end{aligned} \quad (4.8)$$

in which we introduced new coefficients

$$\begin{aligned} a &= \sum_{ij} a_{ij} = 2(g_{2c} - g_{1c}), & a_i &= \sum_j a_{ij} = (-1)^i 2g_{ic} \\ b &= \sum_{ij} b_{ij} = 2(\gamma_{\phi_{2c}} - \gamma_{\phi_{1c}}), & b_i &= \sum_j b_{ij} = (-1)^i 2\gamma_{\phi_{ic}} \end{aligned} \quad (4.9)$$

The 8 electrode correlated fluctuations enter the noise multiplied by $g_{2c} - g_{1c}$: this number must always be equal to $-\Delta g_{DC}$, the differential acceleration between the two TM, fixed mainly by gravitational unbalances. This has two consequences:

- Coefficients a and b can never be changed in an experiment by just acting on the actuation voltages. One should somehow change Δg_{DC} with a movement of masses, something that could not be done in LPF. This means that noise due to $\alpha(t)$ fluctuations can not be enhanced, so it can not be distinguished from a background noise of other nature. It is included here in the discussion of the model, because physically motivated and potentially impacting the noise, but it is not included in the analysis that follows, since un-resolvable.
- Coefficients a_1 and a_2 depend on the commanded force applied on a single TM. However, they must always respect $g_{2c} = g_{1c} - \Delta g_{DC}$. Therefore, if we want to enhance the effect of α_i fluctuations, we must apply a commanded force to both test masses, we can not do them separately. This can be done by just setting a value for g_{1c} and then let the control feedback loop apply g_{2c} so that TM₂ always follows TM₁. This means the SC also accelerates of g_{1c} . Notice however that changing g_{1c} does not change the b_i coefficients at all. This means that the effect of correlated fluctuations would show up only in Δg and not $\Delta \gamma_\phi$.

These considerations are even more stringent for the torque, since the DC gravitational torques must be compensated *on each test mass*, not just their differential angular acceleration: so it is not $\gamma_{\phi_{2c}} - \gamma_{\phi_{1c}}$ that must remain constant, but $\gamma_{\phi_{2c}}$ and $\gamma_{\phi_{1c}}$ separately. This means that we can only increase the torque authorities.

Cross Spectral Density model

Up to now the model was discussed in time-domain, with a final expression given by (4.8). Our objective is to recover estimates for the spectra of S_α , S_{α_i} and $S_{\alpha_{ij}}$, therefore we move to the frequency domain by taking the spectra and cross spectra of the expressions in (4.8). Since all the fluctuations are considered uncorrelated, the result is

$$\begin{aligned}
 S_{\Delta g}^{ACT}(f) &= AS_\alpha(f) + \sum_{i=1,2} A_i S_{\alpha_i}(f) + \sum_{ij} A_{ij} S_{\alpha_{ij}}(f) \\
 S_{\Delta \gamma_\phi}^{ACT}(f) &= BS_\alpha(f) + \sum_{i=1,2} B_i S_{\alpha_i}(f) + \sum_{ij} B_{ij} S_{\alpha_{ij}}(f) \\
 S_{\Delta g, \Delta \gamma_\phi}^{ACT}(f) &= CS_\alpha(f) + \sum_{i=1,2} C_i S_{\alpha_i}(f) + \sum_{ij} C_{ij} S_{\alpha_{ij}}(f)
 \end{aligned} \tag{4.10}$$

with

$$\begin{aligned}
 A &= a^2 & A_i &= a_i^2 & A_{ij} &= a_{ij}^2 \\
 B &= b^2 & B_i &= b_i^2 & B_{ij} &= b_{ij}^2 \\
 C &= ab, & C_i &= a_i b_i, & C_{ij} &= a_{ij} b_{ij}.
 \end{aligned} \tag{4.11}$$

As suggested previously, an experiment that wants to measure any S_α can rely on increasing the A and B coefficients by playing with authorities and commands. This strategy was adopted for a campaign in LPF, presented in the next Section.

Note on Cross Spectrum signs

The first two lines of (4.10) must be necessarily positive, since they are PSDs, therefore all A and B coefficients are also positive. The cross spectrum instead can be negative: in particular, all the electrodes that cause a Δg of the same sign of $\Delta\gamma_\phi$ will result in positive cross spectrum, namely electrodes 11, 14, 21 and 24 in Figure 4.1, which we call **bottom electrodes**. The respective coefficients C_{ij} will then be positive. Instead, the **top electrodes** 12, 13, 22, 23 will contribute negatively to the cross spectrum, with their C_{ij} coefficients negative. This suggests that including the cross spectrum in the analysis can help distinguishing if the noise is dominated by "top" or "bottom" electrodes.

4.1.2 LPF enhanced actuation runs

To study the noise from actuation forces and its agreement with the model of gain fluctuations presented here, four noise runs were carried out in May 2016, each one of approximately 2 days. Each of them had different authorities and commanded forces, which are reported in Table 4.3. The corresponding A and B coefficients are instead represented in Figure 4.2. The ASD of linear and angular accelerations measured during these runs is reported instead in Figure 4.4. It can be seen that for each experiment the ASD is represented in two ways: a continuous line represents the estimation with the standard Welch method [68]; the estimation by round discrete points instead is called here "MCF" standing for "Minimally Correlated Frequencies", a method described in [64, 63] and in [7] Supplemental Material. This provides points which are equally spaced in $\log f$, exhibit minimal correlation, with precision improving at higher frequencies. This is a useful tool for reducing the amount of frequencies entering the analysis while still maintaining the spectrum shape and correct statistics. It is particularly suited for "smooth" spectra (with no sharp peaks). These two methods are described more in details in Appendix 9.3. The cross spectra are not reported here because difficult to represent on a single plot, given that a logarithmic plot can not be used if we want to maintain their signs: also, the cross spectra increase of orders of magnitudes between different runs, so they do not fit in a single linear scale plot. Anyway, they can be found in Appendix 9.3 and a related quantity is shown in Figure 4.7.

Here we give a few comments about each experiment.

1. **UURLA.** (dark blue in Figure 4.4) This run uses the lowest possible authorities and commands, the same used for the majority of LPF noise runs, Figure 2.12 ([7]) included.
2. **Nominal.** (cyan) The authorities used for this run were the ones for which LPF was designed, from which the name "nominal" comes. If LPF gravitational balancing had not been so accurate, this would have been the low frequency noise floor and the LISA acceleration requirement could not have been met.

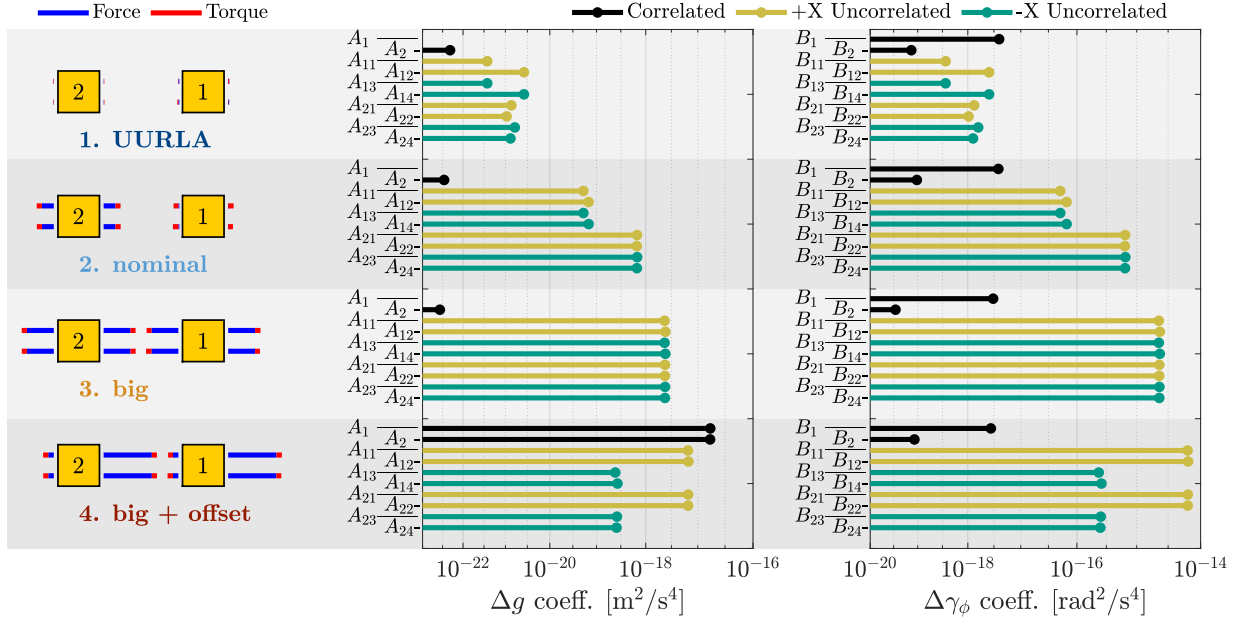


Figure 4.2: (Left) Sketch of applied forces, in blue, and torques, in red, in the four enhanced actuation experiments. They always respect the constant stiffness algorithm. (Right) Coefficients A_i , B_i and A_{ij} , B_{ij} , representing the coupling of correlated and uncorrelated gain fluctuations to linear and angular accelerations. They are calculated for the four experiments based on Table 4.3. Correlated coefficients are in black, while the A_{ij} and B_{ij} corresponding to positive electrodes are in dark yellow, while in green, the negative ones. From [13].

Experiment q	Name	Time (hours)	g_{1c} (pm/s ²)	g_{10} (pm/s ²)	$\gamma_{\phi 1c}$ (prad/s ²)	$\gamma_{\phi 10}$ (prad/s ²)	g_{2c} (pm/s ²)	g_{20} (pm/s ²)	$\gamma_{\phi 2c}$ (prad/s ²)	$\gamma_{\phi 20}$ (prad/s ²)
1	UURLA	61	0	0	-980	2170	-3	26	130	1450
2	Nominal	46	0	0	-960	15040	-3	1140	150	15040
3	Big	46	0	2590	-860	15040	-3	2590	90	15040
4	Big off	46	2060	2590	-820	15040	2060	2590	140	15040

Table 4.3: Average values of the authorities and commands during the four actuation runs. From [13].

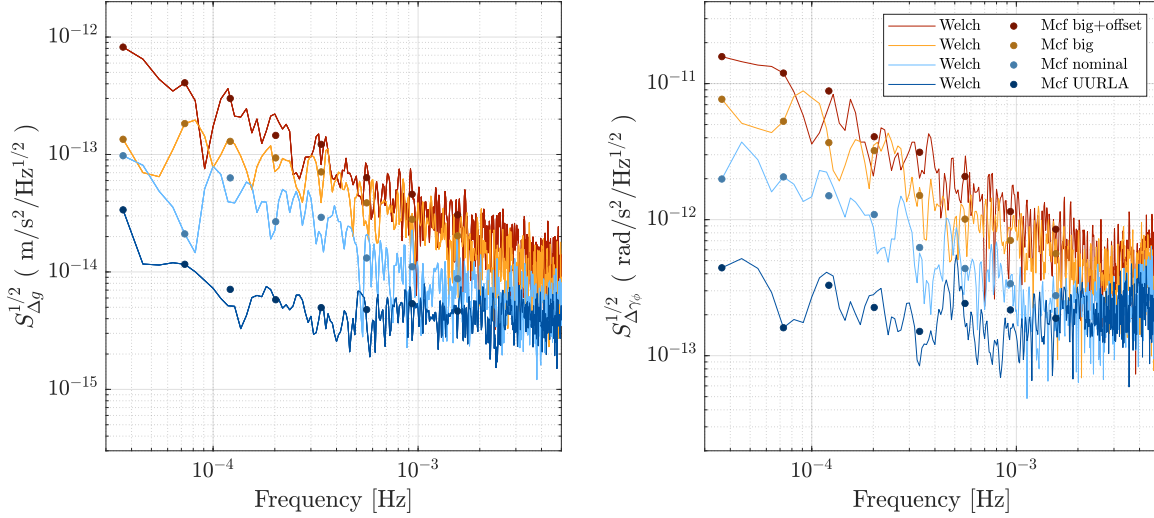


Figure 4.4: Experimental data for noise ASD of Δg and $\Delta \gamma_\phi$ for the four different actuation runs. They are presented both as continuous lines with the standard Welch method, and as single points with error-bars at minimally correlated frequencies (MCF), described in the text. Data are shown without uncertainties to not generate confusion with what follows. All the correct statistics of the PSD estimates are taken in account in the analysis.

3. **Big.** (orange) This one envisioned a further increase in authorities. Notice that both experiments 2 and 3 lead to a significant increase in noise with respect to UURLA, first of all demonstrating that actuation noise is well resolvable and that it seems to scale with applied force, as expected from gain fluctuations. Secondly, in our finer distinction between correlated and uncorrelated noises, in these experiments only the second ones were enhanced (Figure 4.2). This noise increase must then be linked just to those.
4. **Big + Offset.** (red) Here an out-of-loop g_{1c} is applied to TM_1 . As described in the previous Section, this should lead to an enhancement of noise from correlated fluctuations in Δg but not $\Delta \gamma_\phi$ (see also A_i vs B_i for Big and Big+Offset in Figure 4.2). Such a difference is not evident from Figure 4.4, already suggesting that correlated fluctuations are not playing a dominant role.
In addition, this experiment enhances the difference between + and - electrode uncorrelated contributions¹.

¹Considering also that adding the cross correlation to the analysis should remove some degeneracy between "top" and "bottom" electrodes, the only degeneracy which was not addressed in this campaign is the one between V_{1j} and V_{2j} , corresponding electrodes of different TMs.

4.1.3 Estimation of electrode noises using Bayesian inference

We use the just presented experimental data to estimate gain fluctuations with noise model (4.10). We do it with a Bayesian Markov Chain Monte Carlo (MCMC) algorithm, briefly summarized here. Starting from some **prior** assumptions on the model parameters, some possible values are proposed. The algorithm then evaluates the probability of observing the experimental data in case the parameters of the assumed model have the proposed values: this is done through a **likelihood** function, and a criterion is established to choose if the values are accepted or not. Then new values are proposed. Iterating this procedure, the parameters perform a random walk (guided by the likelihood) and explore their probability distribution, called **posterior**. The algorithm outputs are chains of values for each parameter of the model, which can be used to evaluate any other quantity predicted by the model, exhibiting the correct correlations.

The details of the MCMC used here can be a bit convoluted and require several pages (let's say, where much of my work resided). We do not want to interrupt the presentation of the important, physical results, therefore the in-depth description of the models, assumptions and discussion of priors and posteriors is moved to Appendix 9.3. Here we report only some important facts.

- **Background parameters.** The UURLA run does not contain only gain fluctuations noise, but also some background of other nature. Some parameters must be added to the model to account for this, and will be marginalized away.
- **Priors.** The parameters are assigned a prior, which represents our knowledge on them before comparison with the experimental data. We initially tried to be as uninformative as possible in these choices, putting what is called a log-uniform prior on every S_α generator: this means performing the search in the parameter order of magnitude, rather than in the parameter itself. It is important to use these priors for "scale parameters" as Power Spectral Densities. However, given the limited amount of data available and the degeneracy between some A , B , C coefficients in the model, these led to some non-idealities in the results. Still perfectly valid results, but with the feeling that something better could be done. So, the analysis has greatly benefited from the addition of a physically motivated hypothesis: gain fluctuations at different electrodes come from nominally identical circuits, so we expect them to be different, but not of several orders of magnitude. This is achieved by putting an uninformative log-uniform prior on the electrodes *average* noise, but just a uniform prior on the *difference* between couples of electrodes noises.
- **Likelihood.** The likelihood function used must be the correct one that describes $S_{\Delta g}$, $S_{\Delta \gamma_\phi}$ and their cross correlation, also accounting for the number of stretches used to calculate these experimental quantities. This tends to a Gaussian for a high number of stretches, but in this analysis sometimes we even have just 2 stretches, to reach the lowest possible frequencies. The correct likelihood can be found in App. 9.3.

- **Unresolved lower limits.** Some parameters end up having poorly resolved lower limits, while still exhibiting significant and robust upper limits. A criterion is established in App. 9.3 to identify these parameters and they can be found in this work as dash-dotted error bars.
- **Two kinds of fit.** The experimental data are analyzed first with a model for each separate MCF, in order not to assume any spectral shape of the noise. The results of these fits are usually reported as single points with error bars (with caps). Then some analytic functions are empirically chosen (that is: with not particular physical motivation but just looking at the spectral shape) to describe the whole frequency spectrum and the data at all frequencies is used to perform the fit. Analytic functions are useful to easily plot the projections in figures of other experiments or frequency bands. Results can be seen as "smooth" bands or colored areas.
- **Goodness of fit.** Such an algorithm will always output some posteriors for the parameters, even if the assumed model has nothing to do with the experimental data. Assessing how good the model is at describing the data can be quite challenging, for the likelihood is not a simple Gaussian; anyway, we try with something similar to a χ^2 test, called posterior predictive check. This suggests a slight over-fitting in the frequency-by-frequency analysis, since 87% of the experimental data seem to fall in our 68% predicted intervals. It indicates instead a very good fit for the analytic functions case, with 70% of the experimental data falling in our 68% predicted intervals.

4.1.4 Results and comparison with ground measurements

Results of the fits at single frequencies and to analytic functions are reported in Figure 4.5, compared with the spectra of the 4 runs calculated with the Welch method. The central values of the estimates are the medians, while the errorbars represent the 68% interval. As just mentioned, we did not just stop at the statement "the fit nicely reproduces the data", but performed a quantitative test that confirms it.

Together with the fit results, in black we plot also the posteriors associated to the sole actuation noise contributing to the UURLA noise run (it does not make sense to plot it for the other runs because they are completely dominated by it). In this contribution we also added the white noise coming from down-conversion at the actuation frequency. This contribution can not be resolved from the data, since degenerate with Brownian noise at these frequencies, but our knowledge of it is solid and understood, so we can safely add it afterwards. Actuation noise then seems to explain a large fraction of the observed Δg noise, but not all of it, something that is valid for all the other LPF noise runs, with an example in Sec. 4.3. For $\Delta\gamma_\phi$ instead it seem to explain all the noise for $f < 0.1$ mHz with the single frequency fit, while still a relevant fraction with the analytic functions.

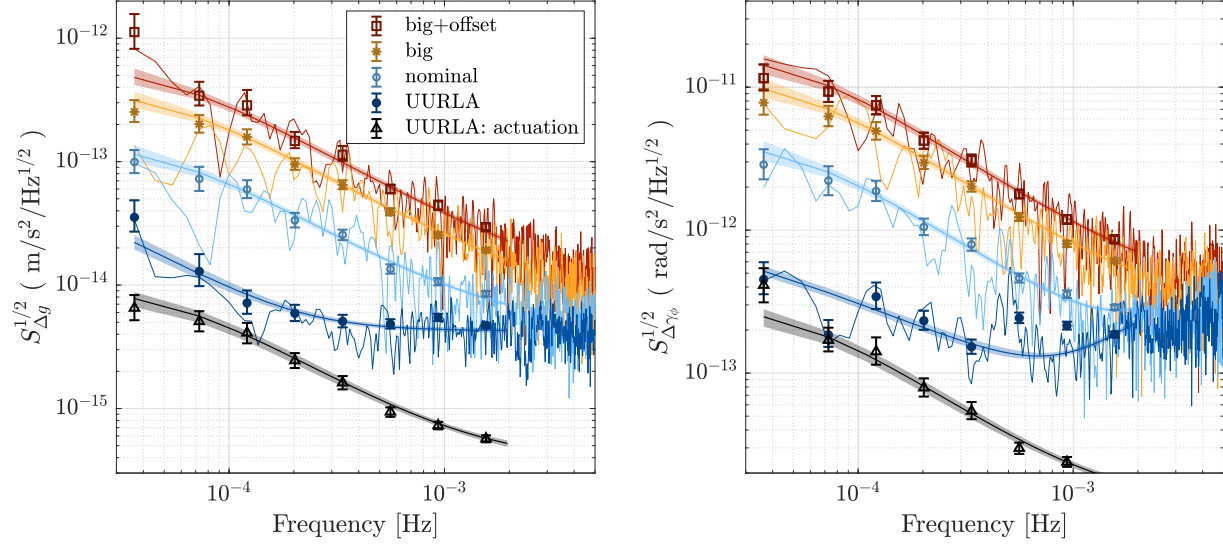


Figure 4.5: (Continuous lines) Spectra estimated with Welch method. (Points with error bars) Result of ASD posterior estimates from frequency-by-frequency fit. (Colored bands) Results of analytic smooth functions posterior estimates. (Black) Contribution of actuation noise to UURLA run. Adapted from [13].

Uncorrelated and correlated fluctuations estimates

From the MCMC chains we can easily reconstruct the posterior of uncorrelated gain fluctuations of each electrode: we could show all of them and we certainly find that some are noisier than others². However the distributions show substantial overlap between each other, especially between corresponding electrodes of different TMs. We are instead much more sensitive to particular combinations of the electrode noises which were enhanced during the 4 experiments. First off all the contributions from the groups of actuators providing +x and -x forces. We introduce their average noise as (refer to Figure 4.1)

$$\begin{aligned} S_{\alpha_{UC+}} &\equiv \frac{1}{4}(S_{\alpha_{11}} + S_{\alpha_{12}} + S_{\alpha_{21}} + S_{\alpha_{22}}) \\ S_{\alpha_{UC-}} &\equiv \frac{1}{4}(S_{\alpha_{13}} + S_{\alpha_{14}} + S_{\alpha_{23}} + S_{\alpha_{24}}) \end{aligned} \quad (4.12)$$

that we can calculate from the MCMC chains of the quantities involved. The left plot in Figure 4.6 shows their reconstructed posterior as violin plots, plus their 68% intervals. Both the frequency dependent fit and the single one reveal that positive electrodes are very well resolved and consistently noisier than the negative ones, which are then less constrained but with relevant upper limits. Now, since big-offset experiment enhanced only positive actuators, we could ask if the fact that we resolved positive actuators to be noisier could

²The noisiest couple seems to be the 12 and 22 electrodes, on average a factor 2-3 noisier than negative electrodes, and a factor ~ 1.5 noisier than 11 and 21.

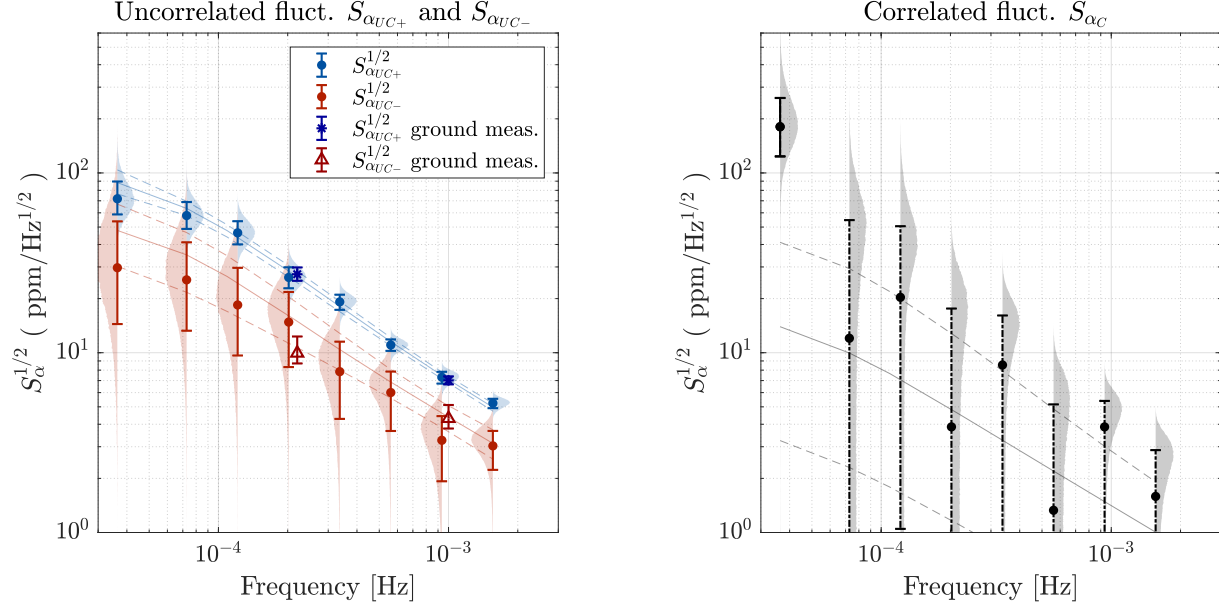


Figure 4.6: (Left) Average uncorrelated fluctuations $S_{\alpha_{UC+}}$ for +x actuators and -x $S_{\alpha_{UC-}}$. (Right) Estimates of board correlated fluctuations S_{α_C} . Shaded areas represent the posterior distributions for the quantities obtained with the frequency-by-frequency fit, and error bars are their 68% confidence interval. Dashed error bars indicate parameters with not significant lower limits. Results of analytic functions fit are solid lines, with dashed lines representing the 68% interval. Adapted from [13].

be somehow correlated with that. First of all, it should just make us more sensitive to the positive actuators, therefore "shrink" their posterior, but not move its central position substantially. Secondly, we managed to verify that **these estimates are consistent with ground measurements** done on the LPF electronics before flight and reported in [45]. These tests measured gain fluctuations in the form of differences in the actuation gain between identical electrodes on the two GRS, for example $\alpha_{11} - \alpha_{21}$ (such measurements were performed with a lock-in amplifier technique similar, but not identical, to that developed in [29] and used in Ch. 5 and 6). Access was obtained to the original measurements and the PSD for $S_{\alpha_{UC+}}$ and $S_{\alpha_{UC-}}$ calculated. Some points are reported in Figure 4.6 and agree with the recovered fluctuations, also confirming that the observed difference is not just an artifact and increasing our confidence in the overall result. What was not observed on ground is the flattening of the noise under 0.1 mHz, probably due to the lower temperature stability of laboratory measurements with respect to LPF environment (that I also observed while performing measurements for my Master's thesis [29]).

To the right of Figure 4.6, estimates of correlated fluctuations are reported. There was absolutely no resolution of the difference between S_{α_1} and S_{α_2} , so the quantity reported here

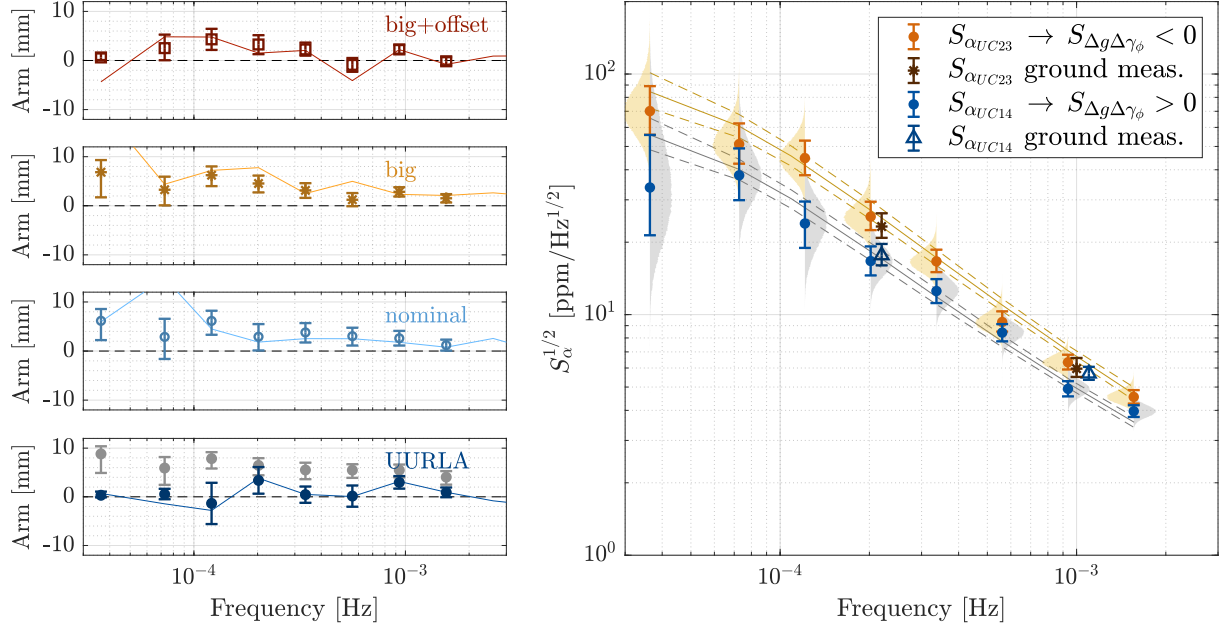


Figure 4.7: (Left) Effective arm during the 4 experiments. Solid line is the experimental data, error-bars are the fit reconstruction at single frequencies. (Right) Posteriors of uncorrelated fluctuations of top and bottom electrodes. Adapted from [13].

is

$$S_{\alpha_C} = \frac{1}{2}(S_{\alpha_1} + S_{\alpha_2}). \quad (4.13)$$

Except for the lowest frequency point, the posteriors do not exhibit a significant lower limit, largely determined by the prior choice. These again are indicated with dash-dotted error bars³. Anyway, they have solid upper limits which depend very weakly on the priors, which limit the contribution of these fluctuations to $\sim 50 \text{ ppm}/\sqrt{\text{Hz}}$ at 0.1 mHz and $\sim 6 \text{ ppm}/\sqrt{\text{Hz}}$ at 1 mHz. At this level they could never be relevant for LPF noise, since they affect it with coupling coefficient (given by (4.9)) proportional to g_{2c} , which was consistently below $\sim 20 \text{ pm/s}^2$ through the whole mission.

Cross correlation results

At the end of Sec. 4.1.1 it was underlined that including the cross correlation in the analysis could help removing degeneracy between "top" and "bottom" electrodes. Top electrodes 12, 13, 22, 23 will contribute negatively to the cross correlation, the bottom ones positively. To better visualize this effect, we imagine that the noisy Δg is the result of a single noisy force acting on the TM. If this noisy force is not applied to the TM center, but at a distance

³Also some fits removing these fluctuations from the model were performed: we were still able to recover the results for uncorrelated fluctuations.

r_ϕ (along y in Figure 4.1), it will also produce a correlated torque and a non-zero cross correlation. In this picture we introduce the effective arm-length

$$r_\phi = -\frac{I}{M} \frac{S_{\Delta g \Delta \gamma_{\phi c}}}{S_{\Delta g}} \quad (4.14)$$

with a sign convention that makes so that the application point is consistent with the positive y axis direction. We can do an example. Using model (4.10), if we imagine only the uncorrelated fluctuations of one electrode are active, then

$$S_{\Delta g}^{ACT} = A_{ij} S_{\alpha_{ij}}, \quad S_{\Delta g \Delta \gamma_\phi}^{ACT} = C_{ij} S_{\alpha_{ij}} \quad \rightarrow \quad r_\phi = -\frac{I}{M} \frac{C_{ij}}{A_{ij}} \approx \begin{cases} +11 \text{ mm} & \text{top el.} \\ -11 \text{ mm} & \text{bottom el.} \end{cases} \quad (4.15)$$

In the case all electrodes are active, but some are noisier then others, we will observe some average of them, and a value lower then 11 mm. Left Figure 4.7 reports a comparison between the experimental values and the ones reconstructed by the MCF MCMC. A consistently positive armlength is observed in the enhanced actuation runs, highlighting that the top electrodes are consistently more noisy at all frequencies. This is consistent with the plot on the right of Figure 4.7, which shows the average noise of top/bottom electrodes

$$\begin{aligned} S_{\alpha_{UC14}} &\equiv \frac{1}{4}(S_{\alpha_{11}} + S_{\alpha_{14}} + S_{\alpha_{21}} + S_{\alpha_{24}}) \\ S_{\alpha_{UC23}} &\equiv \frac{1}{4}(S_{\alpha_{12}} + S_{\alpha_{13}} + S_{\alpha_{22}} + S_{\alpha_{23}}) \end{aligned} \quad (4.16)$$

also in agreement with points obtained from ground measurements.

Going back to the r_ϕ estimates, for the UURLA run an armlength much more compatible with zero is observed: the reported gray points represent instead the quantity

$$r_\phi^{ACT} = -\frac{I}{M} \frac{S_{\Delta g \Delta \gamma_{\phi c}}^{ACT}}{S_{\Delta g}^{ACT}} \quad (4.17)$$

calculated through (4.10), and this represents the armlength that would be observed if the noise was only due to gain fluctuations. So the background must be hiding its contribution. Notice that while (4.14) is a general definition and can be calculated on any experiment, (4.17) is just related to actuation and can be only computed from the MCMC chains of the analysis proposed here. This said, in [12] r_ϕ is computed for all LPF noise runs, finding an average $r_\phi = (1.13 + 0.15) \text{ mm}$. We found out that this value is approximately reproduced if we do the following approximation

$$r_\phi = -\frac{I}{M} \frac{S_{\Delta g \Delta \gamma_{\phi c}}}{S_{\Delta g}} \approx -\frac{I}{M} \frac{S_{\Delta g \Delta \gamma_{\phi c}}^{ACT}}{S_{\Delta g}^{ACT}} \quad (4.18)$$

that means using the estimates obtained in this analysis to approximate the cross-correlation of other LPF runs. This is compatible with the hypothesis that all the cross-talk between

Δg and $\Delta \gamma_\phi$ noises observed through the mission was dominated by noise unbalances in actuation electrodes. This has very important consequences on our interpretation of the unexplained excess noise in LPF at low frequencies, which appears as a disturbance which must have no armlength.

4.2 In-band voltage fluctuations in LPF

In Sec. 3.4.2 we discussed how having a charged TM in a noisy electrostatic field can translate in acceleration noise. We recover expression (3.80) derived there and consider just the coupling to V_{TM} :

$$F_x = -V_{TM} \left| \frac{\partial C_x}{\partial x} \right| \Delta_x \quad (4.19)$$

where in our electrostatic model we defined

$$\begin{aligned} \Delta_x = & (n_1 + n_2 - n_3 - n_4) - \frac{1}{\left| \frac{\partial C_x}{\partial x} \right|} \frac{\partial (C\delta V)_{TM}^s}{\partial x} \bigg|_0 \\ & + (V_{DC,1} + V_{DC,2} - V_{DC,3} - V_{DC,4}). \end{aligned} \quad (4.20)$$

The way employed to study fluctuations $\delta \Delta_x(t)$ consisted in letting both TMs charge up to $V_{TM} \approx 1$ V and observe the resulting noise. A first analysis can be found in [10], which reports a noise level of $34 \mu\text{V}/\sqrt{\text{Hz}}$ at 1 mHz and $190 \mu\text{V}/\sqrt{\text{Hz}}$ at 0.1 mHz.

There are two reasons for which this experiment should be re-investigated:

- From (4.20) we see that also the commanded constant voltages $V_{DC,i}$ enter Δ_x . In principle, these should be just constant (and chosen by us) and not contribute at all to $\delta \Delta_x(t)$. However, the presence of a digitization error in waveform generation introduced a dependence of the kind $V_{DC,i} = V_{DC,i}(g_0, g_c(t), \gamma_{\phi_0}, \gamma_{\phi_c}(t))$ [47]. Therefore, the changing commands due to Drag-Free spacecraft control resulted in low frequency changes $V_{DC,i}$, affecting directly $\delta \Delta_x(t)$. We informally call this "DC Luigation" and its origin is discussed more in depth in Chap. 5. This phenomenon only critically affected these charged runs, not the other noise runs. Anyway, since deterministic, it was calculated and subtracted from the data regarding this experiment, with a slight but significant decrease in the observed noise. Considering this solved, we will drop all the $V_{DC,i}$ terms in the following to lighten the notation.

It is worth re-performing the analysis with these new data to better evaluate the truly stochastic noise contribution.

- In [10] only Δg is analyzed. Adding the differential angular acceleration measurement and cross correlation adds significant information to the analysis, since it can reveal the presence of correlated sources of noise. In fact, an expression completely analogous to (4.19) can be derived for torque

$$N_\phi = -V_{TM} \left| \frac{\partial C_x}{\partial \phi} \right| \Delta_\phi \quad (4.21)$$

where Δ_ϕ is a "rotational voltage unbalance"

$$\begin{aligned} \Delta_\phi = & (n_1 - n_2 + n_3 - n_4) - \left. \frac{1}{\left| \frac{\partial C_x}{\partial \phi} \right|} \frac{\partial (C\delta V)_{TM}^s}{\partial \phi} \right|_0 \\ & + (V_{DC,1} - V_{DC,2} + V_{DC,3} - V_{DC,4}). \end{aligned} \quad (4.22)$$

Notice that the term associated with stray patch voltages $(C\delta V)_{TM}^s$ could be completely different from that in (4.20) and exhibit different fluctuations. Also this data was corrected for the "DC luignation". The contribution of charge patches to Δ_ϕ is expected to be higher than Δ_x , because the latter gets the most important contributions from charge patches on EH x faces: patches on other faces can only contribute with shear forces, therefore less probable to constitute a dominant contribution over the x-faces ones. For Δ_ϕ instead both x and y EH faces contribute to direct stray electrostatic interactions, and in addition, y faces have even lower gaps (2.9mm) therefore increasing the interaction strength. A predominance of noise from charge patches would then probably be seen by detecting a noise in Δ_ϕ higher than Δ_x (we are not talking about orders of magnitude, but probably a factor 2-3).

In the following we describe the experimental runs, the models, the results of the Bayesian estimate and draw conclusions on the nature of stray electrostatic fields fluctuations.

4.2.1 LPF charged TM runs, modeling and parameter estimation results

The noise runs in this case are just two, performed between May 1 and May 4 2016, first with almost neutral $V_{TM_1} = -16$ mV and $V_{TM_2} = -24$ mV, to act as a background measurement for 39 hours, then with $V_{TM_1} = -1.066$ mV and $V_{TM_2} = -1.058$ V for 59 hours. The measured differential linear and angular accelerations are shown in Figure 4.8, this time accompanied by a plot of the cross correlation; it is easier to plot this time, since it basically does not change between the runs (differently from gain fluctuations experiments). An increase in noise is instead resolved over the backgrounds in the case of charged TMs, and we impute this to noise in Δ_x and Δ_ϕ . Notice that this is not a 2-orders of magnitude increase as it was for gain fluctuations experiments (Figure 4.4): combined with the fact that we have just two 80×10^3 s stretches for the PSD estimation at the lowest frequencies, will result in posteriors that are much less resolved compared to the previous analysis. The contribution to Δg noise from electrostatic fields comes from both TMs, so we introduce

$$\begin{aligned} \Delta g = & -\frac{1}{M} \left| \frac{\partial C_x}{\partial \phi} \right| (V_{TM_2} \Delta_{x_2} - V_{TM_1} \Delta_{x_1}) \approx -\frac{V_{TM}}{M} \left| \frac{\partial C_x}{\partial x} \right| \Delta(\Delta_x) \\ \Delta \gamma_\phi \approx & -\frac{V_{TM}}{M} \left| \frac{\partial C_x}{\partial \phi} \right| \Delta(\Delta_\phi) \end{aligned} \quad (4.23)$$

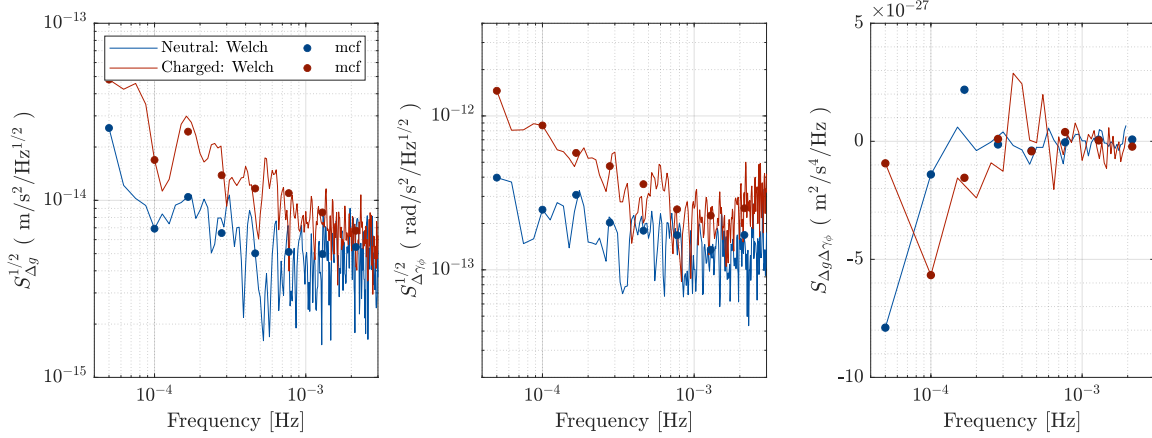


Figure 4.8: (Left) Differential linear acceleration ASD for the charged and neutral TMs experiments. (Center) Differential angular acceleration ASD. (Right) Cross Spectral Density.

approximating $V_{TM} \approx V_{TM_1} \approx V_{TM_2}$. We then can describe both the experimental runs PSDs using the following model

$$\begin{aligned}
 S_{\Delta g} &= S_{\Delta g}^{bg} + \left(\frac{\frac{\partial C_x}{\partial x} V_{TM}}{M} \right)^2 S_{\Delta(\Delta_x)} \\
 S_{\Delta \gamma_\phi} &= S_{\Delta \gamma_\phi}^{bg} + \left(\frac{\frac{\partial C_x}{\partial \phi} V_{TM}}{I} \right)^2 S_{\Delta(\Delta_\phi)} \\
 S_{\Delta g, \Delta \gamma_\phi} &= S_{\Delta g, \Delta \gamma_\phi}^{bg} + \frac{\frac{\partial C_x}{\partial x} \frac{\partial C_\phi}{\partial \phi} V_{TM}^2}{IM} S_{\Delta(\Delta_x), \Delta(\Delta_\phi)}
 \end{aligned} \tag{4.24}$$

where we introduced again some background parameters: while in the gain fluctuation enhancing runs, the UURLA runs had comparable contributions from actuation noise and background of some sort, here the neutral run is completely dominated by the background and almost negligible contribution from electrostatic fields. This means the the S^{bg} parameters basically just fit neutral run alone.

We perform the usual two kinds of fit: at single frequencies, and with analytic functions to describe the spectra, chosen to be a white noise, plus a $1/f$ and a $1/f^2$ component.

Model (4.24) has 2 parameters to describe Δg and at a single frequency we have 2 data points; the same holds for $\Delta \gamma_\phi$ and the cross correlation, for a total of 6 parameters. Therefore it seems that we could run 3 completely independent fits to the three quantities. This is the first analysis proposed in [13], actually the cross correlation is not even fitted since we are interested in $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$. The analysis is perfectly valid and finds comparable fluctuations in $\Delta(\Delta_\phi)$ and $\Delta(\Delta_x)$, but the data set is so restricted for each fit that the estimates at single frequencies have unresolved lower limits.

In this thesis a slightly different approach is proposed that uses also the cross correlation (third line of (4.24));

- as just discussed it does not add a ton of information, but still, the condition

$$S_{\Delta(\Delta_x), \Delta(\Delta_\phi)} \leq \sqrt{S_{\Delta(\Delta_x)} S_{\Delta(\Delta_\phi)}} \quad (4.25)$$

must always hold, so it constrains a little bit more the cases of very large $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$ happening at the same time.

- Running a fit with all the 6 parameters together allows to use the usual log-uniform prior on the average value of $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$ and a uniform one on their difference, a strategy already adopted for the gain fluctuations which improved the regularity of the results. A very conservative thermal noise limit of the actuation circuits was used as a lower limit for the parameters.

Details about the fit parameters can be found in Appendix 9.3.

Results of this fit are reported in Figure 4.9. While S_{Δ_g} and $S_{\Delta_{\gamma_\phi}}$ are resolved, the cross correlation is widely compatible with zero and poorly constrained: the large error bars are due to the very low number of stretches used and seem huge because of the linear scale. They are basically constrained just by (4.25). The reconstructed levels of $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$ are reported instead in left Figure 4.11⁴: the two quantities are compatible within uncertainties at almost all frequencies. The 68% interval for $S_{\Delta(\Delta_x)}$ smooth fit at 0.1 mHz is roughly $[135, 165] \mu\text{V}/\sqrt{\text{Hz}}$ while that reported in the earlier work [10] is $[225, 320] \mu\text{V}/\sqrt{\text{Hz}}$ (the paper reports single test mass noise result, so it was multiplied by $\sqrt{2}$). This reduction is due to the DC round-off error subtraction described previously.

The noise observed seems to be purely $1/f$, with only upper limits to $1/f^2$ that could kick-in at frequencies lower than 0.1 mHz.

Model for actuation noise dominated stray fields

We mentioned that finding $S_{\Delta(\Delta_x)} < S_{\Delta(\Delta_\phi)}$ would be an indication of a noise dominated by stray patch potentials fluctuations, but what we find instead is $S_{\Delta(\Delta_x)} \approx S_{\Delta(\Delta_\phi)}$ and absolutely no increase in cross correlation for the enhanced run. This is more compatible with a noise dominated by actuation low-frequency fluctuations $n_i(t)$, uncorrelated at each electrode. In this hypothesis, we can neglect the contribution of the patches and rewrite

$$\begin{aligned} \Delta(\Delta_x) &= (n_{21} - n_{24} - n_{21} + n_{24}) + (n_{22} - n_{23} - n_{12} + n_{13}) \equiv \Delta(\Delta_{14}) + \Delta(\Delta_{23}) \\ \Delta(\Delta_\phi) &= (n_{21} - n_{24} - n_{21} + n_{24}) - (n_{22} - n_{23} - n_{12} + n_{13}) \equiv \Delta(\Delta_{14}) - \Delta(\Delta_{23}) \end{aligned} \quad (4.26)$$

⁴The results for analytic functions are almost identical to [13], while at single frequencies they are much better constrained thanks to the newly introduced prior strategy.

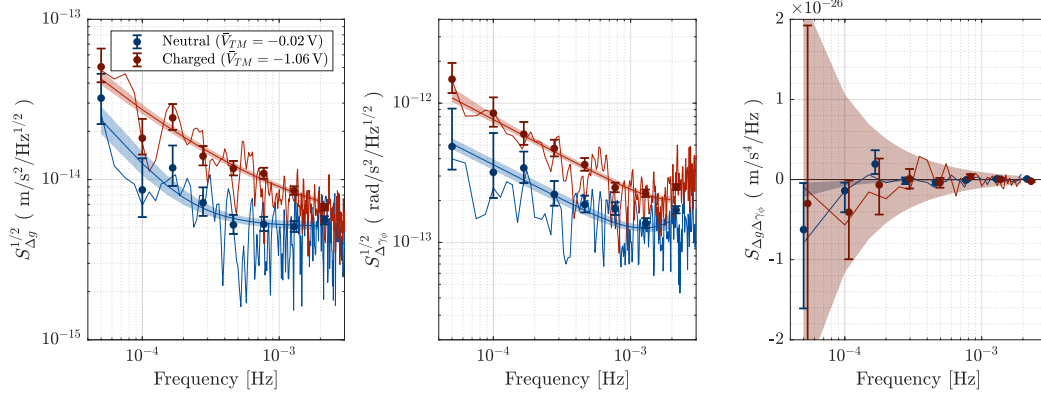


Figure 4.9: Fits to general model (4.24).

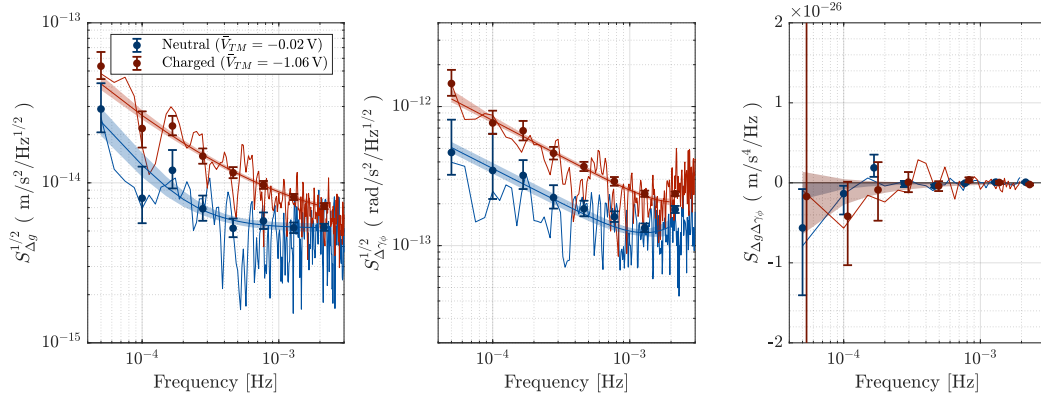


Figure 4.10: Fits in the hypothesis of noise only from actuation (4.27).

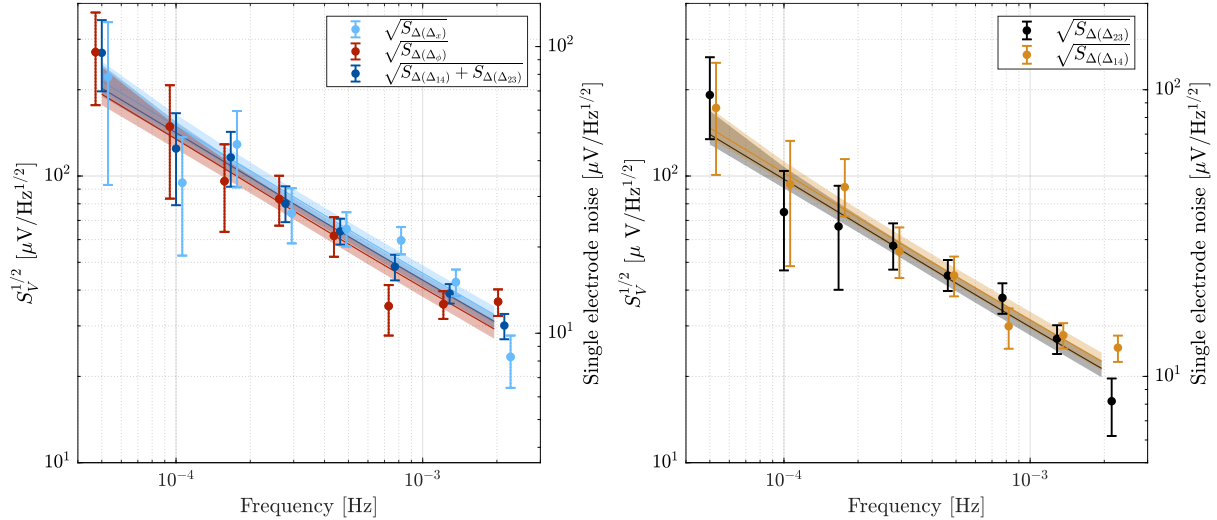


Figure 4.11: (Left) In-band voltage fluctuations ASD extracted from the fit to enhanced TM charge runs. Cyan represent the fluctuations extracted from Δ_g , red the ones from Δ_{γ_ϕ} , while dark blue estimates include also the assumption that the noise is only due to actuation noise. (Right) Same quantity, but restricted to the subgroups of the "top" and "bottom" electrodes. The right axes of the plots report the quantity normalized to a single electrode.

and calculating the spectra with this expression supposing all the generators uncorrelated, we obtain

$$\begin{aligned} S_{\Delta(\Delta_x)} &= S_{\Delta(\Delta_{14})} + S_{\Delta(\Delta_{23})} \\ S_{\Delta(\Delta_\phi)} &= S_{\Delta(\Delta_{14})} + S_{\Delta(\Delta_{23})} \\ S_{\Delta(\Delta_x), \Delta(\Delta_\phi)} &= S_{\Delta(\Delta_{14})} - S_{\Delta(\Delta_{23})} \end{aligned} \quad (4.27)$$

To put it in more physical terms, we will observe the same noise in Δg and $\Delta \gamma_\phi$ but the "top" and "bottom" electrodes will contribute with different signs to the cross correlation. I we find a non-zero cross spectrum, this will indicate the presence of some more noisy electrodes in one of the two groups. With this model we reduce the number of parameters from 6 to 5, but we also put a strong bound on the values that the cross correlation can assume.

This can be immediately seen from the results in Figure 4.10, with much more constrained results on the cross correlation, though still compatible with zero. The estimates of the quantities in (4.27) are in Figure 4.11, which do not reveal any disagreement with the previous, independent estimates of $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$. Also $S_{\Delta(\Delta_{14})}$ and $S_{\Delta(\Delta_{24})}$ (right Figure) are found equivalent at our level of resolution, supporting the hypothesis of equally noisy and uncorrelated fluctuations in the two groups of electrodes.

Supposing then that each electrode is contributing equally, on the right axes of Figure 4.11 the plotted quantities are normalized to a single electrode, showing a level of approximately $50 \mu\text{V}/\sqrt{\text{Hz}}$ at 0.1 mHz and $15 \mu\text{V}/\sqrt{\text{Hz}}$ at 1 mHz, which is perfectly in line (if not lower) with measurements done on-ground, pre-flight and post-flight on a LPF FEE prototype, something that I did during my Master's thesis [29].

4.3 Actuation noise projections to LPF and LISA

All the posteriors obtained from the previous analyses can be used to realize projections to other runs of LPF, a particularly interesting case being the February 2017 noise run. This is considered a benchmark for LPF and when we investigate sources of acceleration noise we often compare them with that ~ 14 days run. So we report our projections of actuation noise in Figure 4.12.

Actuation gain noise is calculated through (4.10) using A, B, C coefficients averaged for the February run, almost equal to that in May, being always in the UURLA configuration. The contribution from low-frequency fluctuations are calculated with (4.24) multiplied by an average TM voltage: this is calculated from two charge measurements performed before and after the run. V_{TM} is assumed to drift linearly between these two measured values and its square averaged, so $\bar{V}_{TM} = \sqrt{T^{-1} \int_0^T V_{TM}^2(t) dt}$.

We bring two messages away from this:

- At 0.1 mHz we explain 20-40% of the noise power in Δg with our model of gain fluctuations. At the time of writing this is still the major *known* source of noise in LPF, and

it seems to account for the majority of the differential angular acceleration noise too. The remaining fraction of noise power is still unexplained. Could it be still linked to actuation? Yes of course, it is possible that there are some effects which go beyond our linear gain noise model. These would have to scale as $\propto \sqrt{F}$ to become dominant at low applied forces over gain fluctuations, and non-linearities in the voltages could play such a role. In the next Chapter we analyze some possible sources of non-linearities and their impact on LPF and LISA noise.

- Low frequency fluctuations coupling to TM charge give a negligible contribution when keeping the TM voltage at a few tens of mV. Still, notice that if we could not use the UV discharge system, the TM would saturate at 1 V increasing the yellow contribution of a factor 25 and becoming the dominant source of acceleration noise. Furthermore, this is another reason for using AC actuation voltages: if they were DC, they would be of order ~ 0.7 to provide actuation at the 1 nrad/s² level required for LISA. Then the single electrode low frequency noise would couple to these DC voltages, and 4 of them would cause a noise 35 times larger than the yellow curve of Figure 4.12.

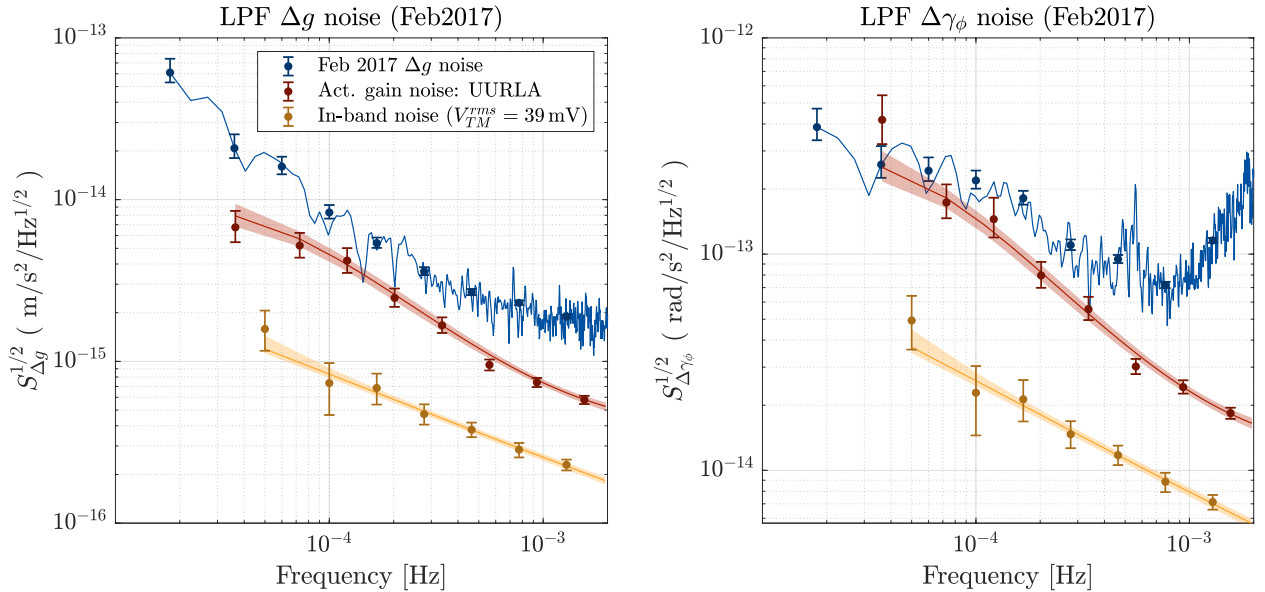


Figure 4.12: (Blue) LPF February 2017 noise ASD. The continuous line is the Welch estimate, while points are the MCF estimates: the error bars on these points are estimated through (9.19) in App. 9.3 (they are not the result of a fit as the red and yellow ones). The curve reported here differs from [13] and [7] because it uses the latest releases of Δg L4. (Red) Contribution from gain fluctuations. (Yellow) Contribution from in-band low-frequency voltage noise coupling to $V_{TM} \approx 39$ mV. For these two the error bars are instead the 68% posterior confidence interval from the fit. Adapted from [13].

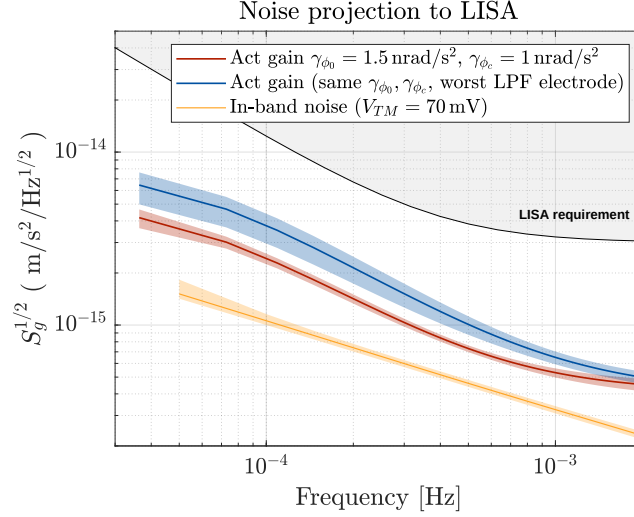


Figure 4.13: LISA acceleration noise from actuation assuming the same performance as the LPF FEE. From [13].

Projections to LISA

Given the performance of LPF FEE, what would be the actuation noise level in LISA? We realize a projection for that in Figure 4.13, only for the case of analytic functions. We use eq. (3.73) and (3.76), reported here

$$\begin{aligned} S_g^{\text{ACT,gain}}(\omega) &= R^{*2} [\gamma_{\phi_0}^2 + \gamma_{\phi_c}^2] S_{\alpha_{uc}}(\omega) \\ S_g^{\text{ACT,white}}(\omega) &= \frac{2}{M} \left| \frac{\partial C_x^*}{\partial x} \right| R^* \gamma_{\phi_0} S_n(\omega_\phi) \end{aligned} \quad (4.28)$$

derived in the case of 4 electrodes, absent x actuation and dominant uncorrelated fluctuations. The required level is $\gamma_{\phi_0} = 1.5 \text{ nrad/s}^2$, which gives $\approx 50\%$ margin to apply a maximum of $\gamma_{\phi_c} = 1 \text{ nrad/s}^2$. Then we use the posterior of $S_{\alpha_{uc}}$ from the MCMC, the average noise of all electrodes, while white noise is added afterwards. Also included, the case in which we use instead the posterior of $S_{\alpha_{uc12}}$, the noisiest electrode measured in LPF. Also included is the contribution from low frequency fluctuations and $V_{TM} = 70 \text{ mV}$, the target maximum TM charge in LISA. Again, it is a less important contribution, but if we left the TM undischarged, it would reach a voltage of $\approx 1 \text{ V}$ and the yellow curve should be multiplied by a factor ≈ 15 , eating the entire acceleration noise budget.

Conclusions

In this Chapter the noise performance of LPF electrostatic force actuation system was analyzed. First of all, even in the slightly more challenging LPF environment (which needs x actuation, differently from LISA) it proved to be suitable for the realization of the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$, in combination with an astonishing level of SC gravitational balancing, with residual DC pulls on the TMs of order pm/s^2 and nrad/s^2 . Before flight, this was budgeted to be of order $650 \pm 300 \text{ pm/s}^2$.

The dominant actuation noise source was expected to be related to gain fluctuations of the voltage waveform, so a measurement campaign was conducted during LPF flight to resolve these instabilities by enhancing them on purpose. Not only they were clearly resolved both in Δg and $\Delta \gamma_\phi$, but also allowed a deep testing of our actuation noise model through a Bayesian joint analysis of the two quantities. We could reconstruct the noise contribution from various subsets of electrodes, finding good agreement with pre-flight ground measurements. The noise resulted completely dominated by uncorrelated gain fluctuations, with positive electrodes being on average noisier than the negative ones, with a level of $7 \text{ ppm}/\sqrt{\text{Hz}}$ at 1 mHz and $40 \text{ ppm}/\sqrt{\text{Hz}}$ at 0.1 mHz . Correlated fluctuations between electrodes were found instead to be negligible with respect to uncorrelated ones, not exceeding the $6 \text{ ppm}/\sqrt{\text{Hz}}$ level at 1 mHz and $60 \text{ ppm}/\sqrt{\text{Hz}}$ at 0.1 mHz .

Projections realized through the consolidated model seem to account only for 20-40% of the LPF low frequency noise during the most quiet run in February 2017 (Figure 4.12), the remaining noise still being unknown. The excess noise is present throughout the whole mission and seems to exhibit zero correlations between Δg and $\Delta \gamma_\phi$ (the observed small correlations in the low frequency noise seem to be compatible with that from actuation). The possibility that the excess could be linked to actuation can not be excluded: it would have to fall out from our model and not scale as $\propto F$ but with a lower power law. An example could be non-linearities in the actuation waveforms amplitudes which would scale as $\propto \sqrt{F}$, and are investigated in Chapter 5.

Low frequency fluctuations of additive voltage noise in the actuation circuits were instead studied through a second campaign involving a charged TM. This had been already analyzed in the past [10], but used data affected by actuation rounding errors and did not include an analysis of the torque, as we do instead here. The recovered estimates of $\sim 50 \mu\text{V}/\sqrt{\text{Hz}}$ at 0.1 mHz are compatible with the hypothesis of noise just coming from the actuation circuit and not from stray charge patches. With the TM voltage kept under control, this was not a relevant source of noise for LPF (Figure 4.12), contributing at most 1% of the observed noise power.

Chapter 5

Actuation waveforms nonlinearities

The sources of noise presented in the previous chapter were described with a linear model and seem to agree reasonably well with observations. However, we can not exclude that the LPF noise excess could be still linked to the electrostatic force actuation system. In this Chapter we address the problem of non-linearities in actuation voltages. We will show that a relevant non-linearity could show up as an error in the waveform amplitude, like

$$V(t) = [V_0 + \delta V(V_0)] \sin \omega t, \quad (5.1)$$

Such deviation would then affect the resulting time-averaged force as

$$F(t) = \frac{1}{2} \frac{\partial C^*}{\partial x} V^2(t) = \frac{1}{4} \frac{\partial C^*}{\partial x} V_0^2 + \frac{1}{2} \frac{\partial C^*}{\partial x} V_0 \delta V(V_0) \quad (5.2)$$

which only becomes a source of problems if V_0 changes in time, otherwise it would just be a DC pull. This is always the case in LISA and LPF, because forces and torques are always used in a feedback loop to control the TM position. The resulting force noise would then be

$$\delta F(t) = \frac{1}{2} \frac{\partial C^*}{\partial x} V_0(t) \delta V(V_0(t)) \approx \frac{1}{2} \frac{\partial C^*}{\partial x} \bar{V}_0 \delta V(V_0(t)), \quad (5.3)$$

where we approximated $V(t) \approx \bar{V}_0$. Unfortunately, the name already suggests that these non-linearities $\delta V(V_0)$ can not be described by a simple analytic function and it is absolutely not straightforward to predict what will be the noise starting from $V_0(t)$.

In this chapter:

- We start by discussing a well known non-linearity in LPF of order 10 μV and how it was subtracted from the data. We will answer the questions "What if we did not subtract this non-linearity completely? Could the LPF excess noise be what is left?". Through simulations, we find that 1-2 μV error in the subtraction could explain part of the excess noise.
- We will then also address the question "Are there other non-linearities?" with a measurement campaign on a LPF FEE prototype, the EM Light. We find unexpected,

large non-linearities even of order $100\text{ }\mu\text{V}$ over large ranges of commanded voltages. Neighboring voltages instead seem to differ on average by $< 1\text{ }\mu\text{V}$, which end up being the relevant feature.

We also check experimentally that the non-linearities of bullet 1 were actually corrected to sub- μV level.

- Given the experimental result, it was important to assess how given some non-linearities and the time series of force and torque commands, these would affect acceleration noise. This is done through simulations and analytically, finding that non-linearities could potentially impact LISA but can be kept under control with simple operations. It is then shown that they could have constituted an interesting fraction of LPF excess noise but almost surely not constitute the most relevant source.

Regarding the first and third bullets, the work was carried out in close collaboration with Lorenzo Sala, whom I thank immensely. I think the continuous discussions, parallel working and cross checking really generated a huge amount of interesting results that would have never been possible for a single person in such a short time, and bringing a lot of satisfaction. A true confirmation of the superior power of doing science teaming up with others (especially if they are good as Lorenzo) with respect to working alone.

The contents of this chapter are also contained in a technical note [28].

5.1 Rounding error effect on LISA Pathfinder

A very simplified sketch of the chain generating the actuation waveforms for the x/ϕ electrodes is reported in Figure 5.1.

1. From the left, everything starts in the DFACS, which calculates the necessary forces and torques in order to control the TM dynamics at the current time sample, that is every 0.1 s . These 10 Hz time series are sent to Earth as telemetries.
2. Now, it needs to convert the forces to amplitudes of the voltage waveforms, in order to "ask" the FEE to apply them. In normal operations, a $\Sigma\text{-}\Delta$ loop is also used in the conversion, for which we give a simple explanation. The FEE has a final Least Significant Bit (LSB) of $\approx 153\text{ }\mu\text{V}$, therefore, it can only apply a set of discrete values of forces. Let us say the required force/torque is not an integer multiple of the Least Significant "Force" Bit, but falls between bit n and $n + 1$, for example $n + 0.8$. The $\Sigma\text{-}\Delta$ will then make the output oscillate (every 0.1 s) between bit n and $n + 1$, spending 80% of the time in $n + 1$ and 20% in n . At low frequency then, this will result in a DC force with amplitude $n + 0.8$. The $\Sigma\text{-}\Delta$ is said to effectively reduce the LSB. Notice these are only digital calculations in the DFACS, no voltages are generated yet. After the conversion, we have now V_{1x} , $V_{1\phi}$ and V_{DC} . This time series is at 10 Hz , but is sent to Earth as telemetry down-sampled at 1 Hz .

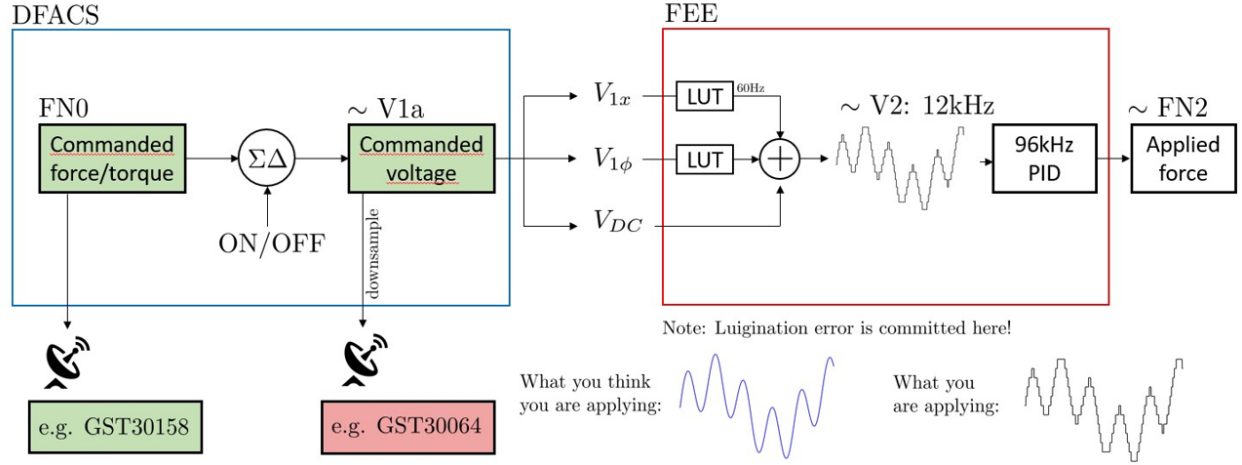


Figure 5.1: Waveform generation scheme, described extensively in the text.

3. The values of V_{1x} , $V_{1\phi}$ and V_{DC} are sent to the FEE FPGA, which now has to build the appropriate waveform to be applied during the current 0.1 s sampling time. Reference sines and cosines are stored in a 16-bit lookup table as a series of 1200 digital, discrete codes (so the updating frequency from one bit to the next is 12 kHz. Example: 60 Hz x actuation has 200 samples per period, 50 per quarter wave): they are re-scaled by V_{1x} , $V_{1\phi}$, and summed together and with V_{DC} , but still result in a waveform encoded on 16-bits. We recall that sines for the x degrees of freedom are at 60 Hz, while for ϕ 270 Hz, so their sum looks like the sketched one in Figure 5.1. Finally, there are other re-scaling operations to feed these still digital signal to the actual electronics turning them into analog voltages.
4. A 96 kHz PID then is at work to exactly convert each of the 1200 digital codes to an analog voltage, finally applied to the electrodes after some low-pass filters.

Let us now move to the rounding error, which is committed at bullet 3. , during the re-scaling. While the sine "peak" bit is exactly re-scaled to the desired amplitude, the other codes of the waveform are not re-scaled to integer multiples and must be rounded. When increasing the peak amplitude of 1 bit, all the others bits of the waveform might increase by 1 or 0, depending on their value (a small drawing can be found in Figure 5.2 left). Though completely deterministic, it is not obvious to state which bits will do what, and this results in non-linear changes in the effective amplitude of consecutive waveforms: so, instead of having a linear relation between the *commanded* voltage amplitude and *applied* one

$$V(t) = V_{app} \sin \omega t = V_{com} \sin \omega t \quad (5.4)$$

we end up with something non-linear of the form

$$V(t) = V_{app} \sin \omega t = [V_{com} + \delta V^{dig}(V_{com})] \sin \omega t \quad (5.5)$$

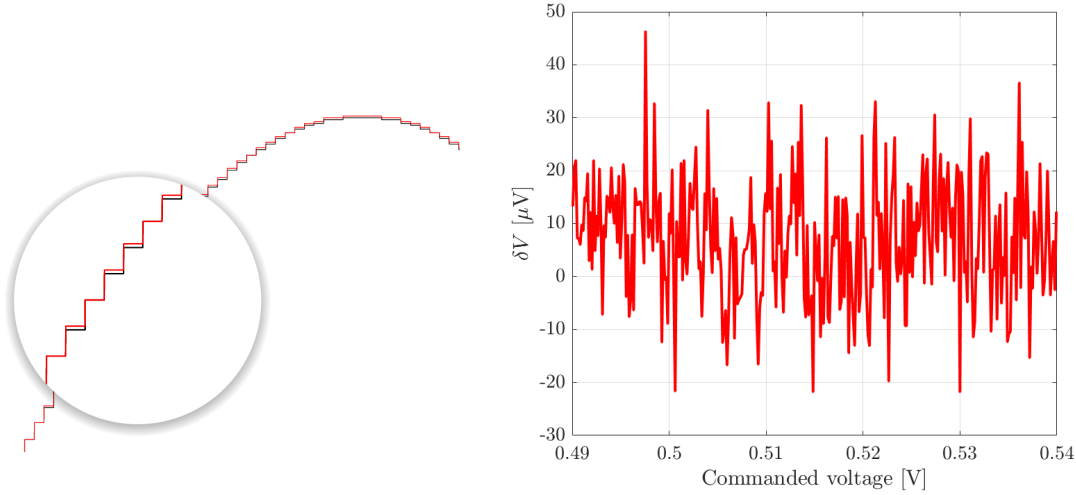


Figure 5.2: (Left) Slice of digital sine waveforms, differing by 1 LSB peak amplitude. The picture is a bit small but it must give the idea of what happens for a sufficient number of bits. Changing of 1 LSB the "peak" code of the digital sine, the others can increase of 1 or 0 (see zoomed circle). (Right) Error committed in the final, demodulated waveform amplitude, with respect to that of an ideal sine. This is represented for a range of neighboring commanded amplitudes; it looks like white noise, and indeed even though the single roundings are deterministic, their collective effect result in completely uncorrelated errors.

where $\delta V^{dig}(V_{com})$ is the so called "digitization error" or "Luigation"¹ and really looks uncorrelated between consecutive commands and of 10 μV amplitude. An example is included in right Figure 5.2.

Effect on LPF noise

Was this effect important for LISA Pathfinder? Yes it was. Figure 5.3 shows how acceleration noise used to look like before the understanding and subtraction of this error. It really classified as the dominant low frequency noise source!

I know this effect is a bit convoluted, so I take the time to repeat it once more. LPF Δg is a sum of the interferometric readout, plus the forces being applied to the TMs. At low frequency, the noise is *completely* determined by the forces that were being applied (we are basically reading Δg in the feedback forces necessary to keep the TMs at a fixed distance, see Figure 2.12 if there is confusion on this concept). It is then important to know very well what force we were applying, otherwise we fool ourselves. The red curve in Figure 5.3

¹The name refers to Luigi Ferraioli at ETH, who implemented a proper correction algorithm based on the roundoff scheme employed by the LPF FEE. The effect was first noticed and documented experimentally by Bill Weber and Davor Mance in the UTN laboratory with the LPF "ELM-light FEE" prototype.

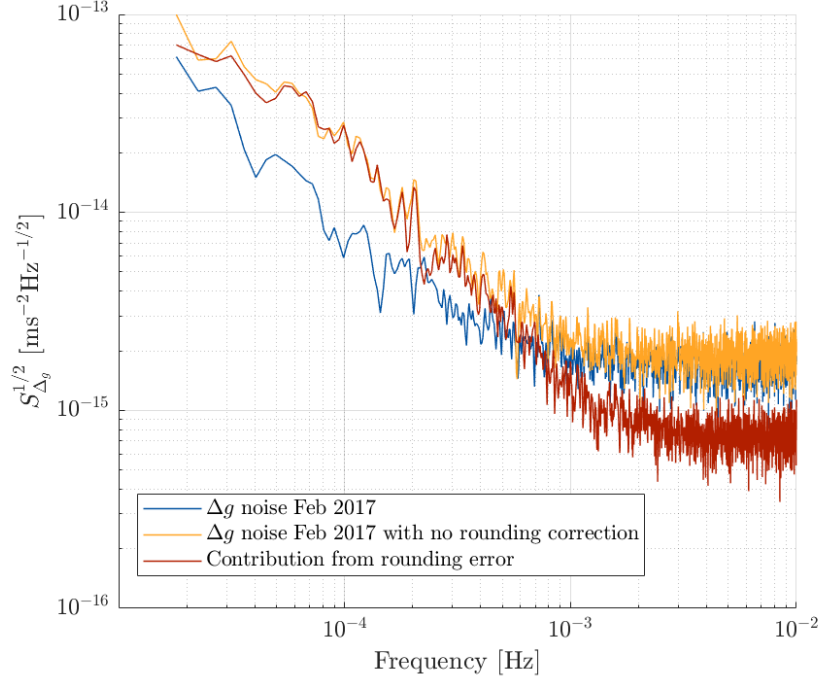


Figure 5.3: Acceleration noise in LPF February 2017 run, with and without the rounding correction.

is calculated using the telemetry forces provided by the DFACS (Figure 5.1), which is the *commanded force*. But we just explained that the *applied force* differs from it, because of the presence of digitization error. So, to calculate the blue curve of Figure 5.3 one has to take the commanded force telemetry, calculate the digitization error at each instant, then re-calculate the corrected forces and use them to build the Δg observable.

Now, some questions naturally arise in front of such noise improvement:

- Did we remove this effect completely? Is the remaining observed noise just a "bad" subtraction?
- Are there other kinds of bit-to-bit non-linearities in LPF/LISA actuation circuit?

Even in the data-sheet of a simple DAC (Digital-to-Analog-Converter) or ADC it is possible to find some information about two quantities:

1. **Differential non-linearity (DNL):** difference between bit N and bit N+1. In our case, we use a slightly different definition, referring to the difference in their systematic errors $\delta V_{N+1} - \delta V_N$.
2. **Integral non-linearity (INL):** the integral of the previous quantity, in our case just represented by the absolute systematic error of bit N, δV_N .

Could it be that we have this kind of things in LPF and LISA? How big should they be to actually impact the noise significantly? We try to answer this question by simulations in the next section, after a slightly better formalization of the effect of a generic non-linearity.

5.2 Residual non linearities simulations

The configuration of actuation voltages of LPF and LISA, applied to the 4 electrodes providing x/ϕ actuation, are reported here:

$$\text{LPF} \quad \begin{cases} V_{1c}(t) = V_{1x} \sin \omega_x t + V_{1\phi} \sin \omega_\phi t + V_{DC1} \\ V_{2c}(t) = -V_{1x} \sin \omega_x t + V_{2\phi} \cos \omega_\phi t + V_{DC2} \\ V_{3c}(t) = V_{2x} \cos \omega_x t - V_{1\phi} \sin \omega_\phi t + V_{DC3} \\ V_{4c}(t) = -V_{2x} \cos \omega_x t - V_{2\phi} \cos \omega_\phi t + V_{DC4} \end{cases} \quad (5.6)$$

$$\text{LISA} \quad \begin{cases} V_{1c}(t) = V_{1\phi} \sin \omega_\phi t + V_{DC1} \\ V_{2c}(t) = V_{2\phi} \cos \omega_\phi t + V_{DC2} \\ V_{3c}(t) = -V_{1\phi} \sin \omega_\phi t + V_{DC3} \\ V_{4c}(t) = -V_{2\phi} \cos \omega_\phi t + V_{DC4} \end{cases} \quad (5.7)$$

Let us take just the first lines of eq. (5.6) and illustrate how a systematic non-linearity would appear on the waveform amplitudes. The waveform is a set of discrete codes, sampled at 12 kHz. We can imagine an error signal that is committed at every point of the waveform, and that depends on the commanded voltages

$$V_{1c}(t) = V_{1x} \sin \omega_x t + V_{1\phi} \sin \omega_\phi t + V_{DC1} + \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC1}, t). \quad (5.8)$$

Now, this error signal can be very complicated, since the rounding changes almost randomly from code to code (at 12 kHz) and can therefore have a wide spectrum. Also, notice that this error is systematic, so it is periodic and has the same periodicity of $V_{1c}(t)$. However, the only way in which this signal can be relevant, is by "beating" against a big carrier present in $V_{1c}(t)$, for example $V_{1x} \sin \omega_x t$. With this we mean that since $F \propto V^2$, if we look at $\langle F \rangle$

averaged over an actuation period T

$$\begin{aligned}
 \langle F \rangle \propto & \frac{V_{1x}^2}{2} + \frac{V_{1\phi}^2}{2} + V_{DC_1}^2 \\
 & + V_{1x} \frac{2}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \sin \omega_x t \, dt \\
 & + V_{1\phi} \frac{2}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \sin \omega_\phi t \, dt \\
 & + V_{DC_1} \frac{2}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \, dt \\
 & + \frac{1}{T} \int_0^T \delta V_{1c}^2(V_{1x}, V_{1\phi}, V_{DC_1}, t) \, dt,
 \end{aligned} \tag{5.9}$$

the last line is completely negligible, while in the other ones the small, non-linear effect gets multiplied by potentially large voltages V_{1x} , $V_{1\phi}$, eventually becoming relevant. If we introduce the definitions

$$\begin{aligned}
 \delta V_{1x}(V_{1x}, V_{1\phi}, V_{DC_1}) &= \frac{2}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \sin \omega_x t \, dt \\
 \delta V_{1\phi}(V_{1x}, V_{1\phi}, V_{DC_1}) &= \frac{2}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \sin \omega_\phi t \, dt \\
 \delta V_{DC_1}(V_{1x}, V_{1\phi}, V_{DC_1}) &= \frac{1}{T} \int_0^T \delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t) \, dt,
 \end{aligned} \tag{5.10}$$

we can rewrite the expression for the actuation voltage (5.8) such that these non-linearities appear as errors in *effective amplitudes of the waveforms*

$$\begin{aligned}
 V_{1c}(t) = & [V_{1x} + \delta V_{1x}(V_{1x}, V_{1\phi}, V_{DC_1})] \sin \omega_x t \\
 & + [V_{1\phi} + \delta V_{1\phi}(V_{1x}, V_{1\phi}, V_{DC_1})] \sin \omega_\phi t \\
 & + V_{DC_1} + \delta V_{DC_1}(V_{1x}, V_{1\phi}, V_{DC_1}) \\
 & + \delta v(V_{1x}, V_{1\phi}, V_{DC_1}, t),
 \end{aligned} \tag{5.11}$$

where δv is everything that was left of the error signal. The square of this expression results exactly in (5.9)². Note that that every amplitude depends on the other commands, meaning for example that even for a constant V_{1x} , if $V_{1\phi}$ changes, different codes and different roundings are used in the generation of the total waveform, therefore also the error δV_{1x} changes.

Then we can divide the error in two contributions, the rounding, deterministic "Luigation" error (indicated with "dig"), and the other non-linear effects (indicated with "nl"):

$$\delta V_{1x}(V_{1x}, V_{1\phi}, V_{DC_1}) = \delta V_{1x}^{dig}(V_{1x}, V_{1\phi}, V_{DC_1}) + \delta V_{1x}^{nl1}(V_{1x}, V_{1\phi}, V_{DC_1}). \tag{5.12}$$

²An alternative way of seeing this is that we took the error signal $\delta V_{1c}(V_{1x}, V_{1\phi}, V_{DC_1}, t)$ and extracted its ω_x and ω_ϕ Fourier components, which definition are exactly those in (5.10). Then δv is just the error signal without the frequencies ω_ϕ , ω_x and DC.

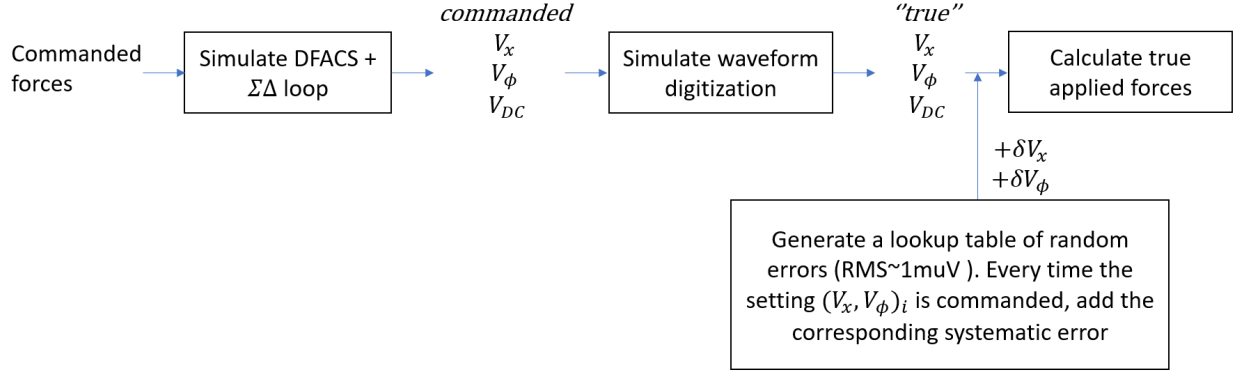


Figure 5.4: Scheme of the correction routine used in LPF [47], called `computeAppliedForcesAndTorques`. It starts from the commanded forces telemetry and simulates the DFACS operations in flight. Then it considers the effect of the digitization error to calculate the voltage waveform amplitudes that were truly applied to the electrodes. We intervene before the last stage, where we inject further systematic errors in the actuation waveform amplitudes to see their effect on the spectra.

What we are interested in is measuring the term δV_{1x}^{nl1} and try to understand to which accuracy we are able to subtract the term δV_{1x}^{dig} . Notice that V_{1x}^{nl1} has 1 in the superscript, meaning that is related to electrode 1; since these non-linear errors might depend on the actuation circuit components, they will be surely different for different actuation circuits.

The orders of magnitude of these signals are

$$V_{1x} \sim 1 \text{ V}, \quad \delta V_{1x}^{dig} \sim 10 \text{ } \mu\text{V}, \quad \delta V_{1x}^{nl} \sim ? \text{ } \mu\text{V}; \quad (5.13)$$

the first is determined by the forces that we have to apply, the second we know how to calculate, the estimate of the third is something we show now.

5.2.1 Simulations with injected non-linearities

To see which size of non-linearity is needed to impact LPF noise, we injected some further non-linearities in the correction routine, as explained in Figure 5.4. So we calculate on one side F_{leg} , standing for "legacy", which are the official forces used in the latest LPF data, corrected for the digitization error; on the other side we calculate $F_{n\mu V}$, with the injection of randomly generated, Gaussian, uncorrelated systematic errors with dispersion $n \text{ } \mu\text{V}$ RMS **on every electrode**, both for x and ϕ actuation. Then we calculate the contribution of the non-linearities alone, by

$$\Delta g_{NL} = \frac{F_{\text{leg}} - F_{n\mu V}}{M}. \quad (5.14)$$

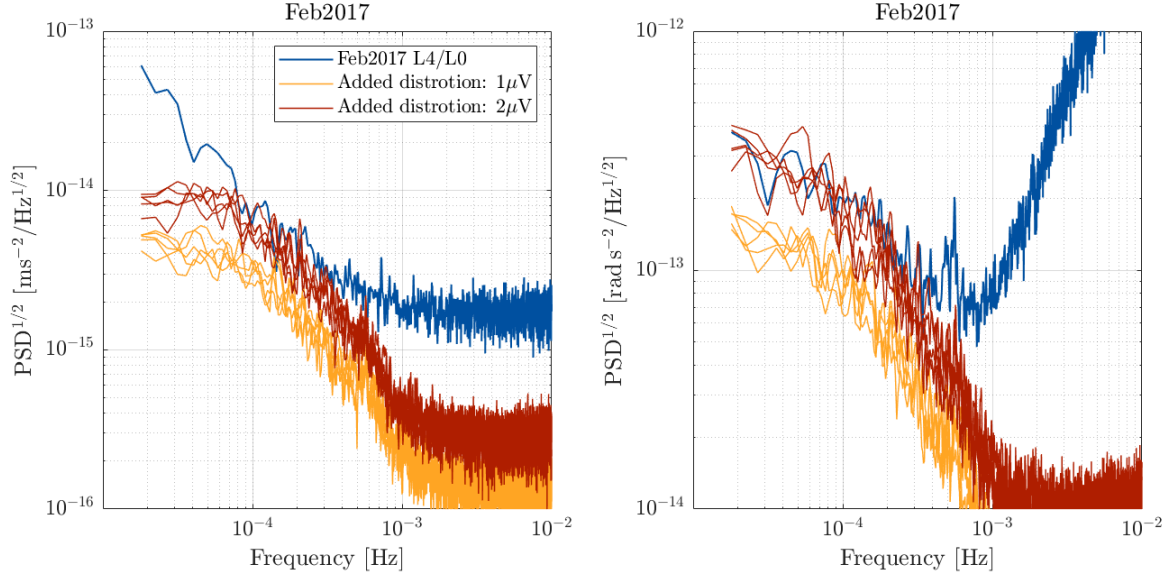


Figure 5.5: Simulated contributions to Δg (left) and $\Delta\gamma_\phi$ (right) from residual non-linearities in actuation voltage amplitudes, repeated five times each. They are compared with the blue "benchmark" noise observed in February 2017 noise run.

This quantity is plotted in Figure 5.5 for the LPF February 2017 run, both for Δg and the differential angular acceleration, which uses a definition analogous to (5.14). Five repetitions are included to show how much the result varies for different generations of systematic errors. We can see that the size of non-linearities necessary to critically impact LPF noise is between 1-2 μV , probably 2 μV being already too much, since we already explained much of the low frequency torque noise with just gain fluctuations. Still, the shape of the noise from the added distortion does not show the $1/f^2$ behavior of Δg , but seems to whiten. At this point we were not really asking ourselves why, but in Sec. 5.4 we will show that we understood this spectrum very well, and we even have analytic expressions for it. A similar version of these results is also reported in [12].

Impact on other experiments

Besides from the just reported simulations for the February 2017 noise runs, there are many other runs in LPF in which we could perform these simulations, and not only noise runs. The question was "Could we see an enhancement of the force from these $\sim 1 \mu\text{V}$ non-linearities in other experiments?". In particular we worked on two important experiments: the "calibration tone" and the " Σ - Δ OFF" runs. Especially the second is pretty convoluted to explain and Lorenzo Sala led the analyses of these, with me acting more as a continuous cross check with my own codes and discussing the results. So those analyses are not included in this thesis, which is already heavy enough. I just give a brief description of the two, which

are also mentioned in [12, p. 39]

- **Calibration tone:** An out of loop, sinusoidally oscillating, 10 mHz, 100 fN force g_{2c} was injected on TM₂. This should not be seen in Δg , since the control loop applies the necessary forces to keep the two TMs relative distance fixed. If the force is not properly calibrated, a 10 mHz peak is left in Δg spectrum, so it can be used to extract the correct calibration coefficient λ , introduced in (2.20). Initial failures in the complete elimination of this peak were an indication of the presence of the digitization error: λ seemed non-stationary and further harmonics appeared in the spectrum. The correction of the digitization error drastically improved this condition, removing all harmonics and also the main peak. So, does adding our 1-2 μV make the peak appear again? No. We found that these non-linearities are not enough to show a relevant peak in that experiment.
- **Sigma-Delta OFF:** a noise run in which the DFACS $\Sigma\text{-}\Delta$ loop was switched off, pretty straight forward. So there is no effective reduction of the 153 μV bit size. A large increase in low frequency noise was detected, interpreted as an enhancement of the digitization error, because it seemed to be completely removed by the `computeAppliedForcesAndTorques` routine. We discovered that this was not true, in fact the majority of the effect removed by the routine was just the difference between the commanded force and its version rounded to the already known 153 μV bit size. The Luigation effect instead, was not particularly enhanced at low frequencies during this run, but just at the level of the other noise runs. It was enhanced at higher frequencies, but remained completely hidden in the brownian noise.

5.3 Lab measurement: EM Light nonlinearities

The previous result is pretty astonishing, since it seems that non-linearities of order $\sim 1 \mu\text{V}$ out of full amplitudes of $\sim 1 \text{ V}$ could critically impact the noise. Was LPF FEE accurate to this level? Did we subtract the $\sim 10 \mu\text{V}$ Luigation effect to the sub- μV level? The only way to answer this was to perform measurements directly on a prototype of LPF FEE. Fortunately, we have one in the laboratory called EM Light, which is identical to LPF but not all the components are space-qualified. It is currently used to control one of the torsion pendulum facilities, actively used during these years, so I had to borrow it for approximately 1 month and try to perform as many measurements as possible, to not interrupt everyone activities for too long.

Given the amplitude of $\sim 1 \text{ V}$, an error of $1 \mu\text{V}$ is not easily detected. For example, the input ADC of a lock-in amplifier in the lab has 16 bits, and if we were forced to use a 1 V full-scale, the minimum resolution would be $1 \text{ V}/2^{16} \approx 15 \mu\text{V}$. This is the motivation for introducing a measurement procedure that cancels the big V_{1x} and allows to measure just

the errors.

We perform two main kinds of measurements:

1. One that automatically cancels the digitization error, in order to spot only sources from other possible non-linearities δV_{1x}^{nl} .
2. The other in presence of the digitization error δV_{1x}^{dig} , to check how well we are able to remove it in post-processing.

5.3.1 Measurement in absence of digitization error

The strategy to measure δV_{1x}^{nl} consists in just summing the output of two electrodes, with DC and ϕ actuation off. With these settings the first two lines of eq. (5.6) then become

$$\begin{aligned} V_{1c}(t) &= \left[V_{1x} + \delta V_{1x}^{dig}(V_{1x}, 0, 0) + \delta V_{1x}^{nl1}(V_{1x}, 0, 0) \right] \sin \omega_x t \\ V_{2c}(t) &= \left[-V_{1x} + \delta V_{1x}^{dig}(-V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0) \right] \sin \omega_x t, \end{aligned} \quad (5.15)$$

and from what we know about how the rounding is done in LPF electronics, the error committed for a positive sinusoid is exactly the same as the one committed for a negative one, meaning that

$$\delta V_{1x}^{dig}(-V_{1x}, 0, 0) = -\delta V_{1x}^{dig}(V_{1x}, 0, 0). \quad (5.16)$$

Therefore, summing the two electrode outputs of eq. (5.15)

$$V_{1c}(t) + V_{2c}(t) = \left[\delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0) \right] \sin \omega_x t. \quad (5.17)$$

This is still a signal at $\omega_x = 2\pi 60$ Hz, therefore we can feed this to a lock-in amplifier using as a reference for example $V_{3c}(t) = V_{2x} \cos \omega_x t$, and the wanted amplitude will be on the output quadrature Y. This is then sampled by an ADC at 10Hz. The apparatus is sketched in Figure 5.6. **The measurement** is performed by exploring a set of different commanded V_{1x} , usually by ramping from a certain V_{min} to a V_{max} and spending a certain measurement time on each command.

What should we expect to find on the X lock-in output? Nothing, according to these calculations; however, in eq. (5.11) we discarded the term δv , which actually gets demodulated by the lock-in. We should therefore find

$$X = \frac{2}{T} \int_0^T [\delta V_{1c}(V_{1x}, 0, 0, t) + \delta V_{2c}(-V_{1x}, 0, 0, t)] \cos \omega_x t \, dt. \quad (5.18)$$

Nonetheless, for the systematic errors discussed so far, this quantity should be zero anyway. Figure 5.7 contains an example that should help understanding why this is the case. As was mentioned several times, the waveforms are sequences of discrete voltages. If we now imagine the set of voltages used for the first quarter of the sinusoid, it is easy to understand that those same voltages will be used also for the second quarter of the sinusoid, given its symmetry.

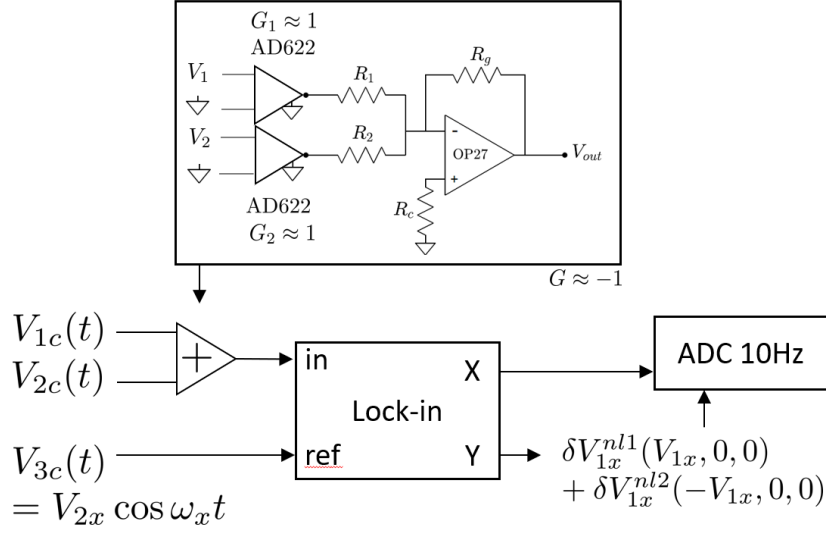


Figure 5.6: Simplified scheme of the measurement apparatus.

This means that any error committed in the first quarter, will be exactly mirrored in the second quarter (we could say the function is even with respect to $\pi/2$). By multiplying by $\cos \omega_x t$ (which is odd with respect to $\pi/2$) the function assumes opposite signs in the two quarters, making the integral vanish. The same holds for the third and fourth quarters. The same does not hold when multiplying by $\sin \omega_x t$. This was the concept of the measurement. In reality it is slightly more complicated due to white noise in actuation circuits and unbalances in electrode outputs. The first problem led to prefer a differential, modulated measurement, the second problem can be easily corrected. Motivations and solutions are explained in the following paragraphs.

Noise in actuation circuit - modulation

A first test run is reported in Figure 5.8. The measurement is described extensively in the caption, and here we report just the main conclusions from it.

- The measurement is plagued by a 50 μHz , large oscillation, already observed in the past on the same electronics [29]. This makes it completely impossible to do an absolute measurement of δV^{nl} by just averaging its DC value. The oscillation is not a perfect sinusoid and it is almost impossible to remove from the data.
- Actuation circuits have a background white noise of a few $\mu\text{V}/\sqrt{\text{Hz}}$.

These facts forced us to perform a modulated, differential measurement at 0.1 Hz. More in detail, we chose a reference V_{1x}^{ref} , and keep changing the command between V_{1x}^{ref} and the V_{1x} we want to measure. An example of what is obtained is reported in Figure 5.9 (top); a sinusoid is then least square fitted to the obtained square wave. To renormalize the peak-to-peak amplitude of the sinusoid to the peak-to-peak of the square wave, one must finally

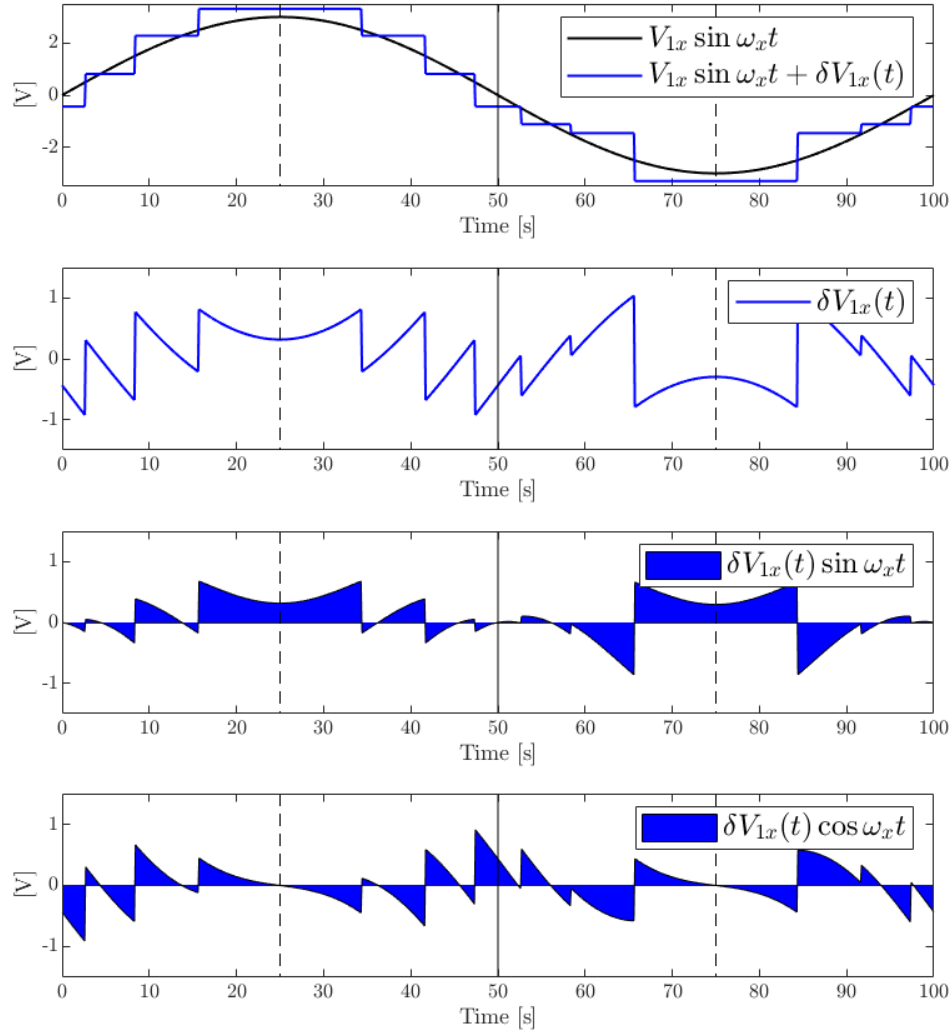


Figure 5.7: Visual example to understand the behaviour of systematic non-linear errors. From the top: [1] A perfect sinusoid and a dramatic example of rounding + non-linearity. Notice how voltages corresponding to the same "code" have the same error. [2] The actual error with respect to perfect sinusoid. Notice that the errors are not the same in the positive and negative parts of the waveform. [3,4] The same error multiplied by sin and cos. The areas are highlighted to help the reader understand that integral (5.18) in the fourth plot vanishes by symmetry in the first half of the waveform, and separately in the second, while eq. (5.10) in the third plot does not vanish.

multiply by $\pi/4$. An example is reported again Figure 5.9 (bottom). The expected error on the amplitude of the reconstructed square wave can be estimated from Wiener theory

$$\sigma = 2 * \frac{\pi}{4} * \sqrt{\frac{S_0}{T_{meas}}}, \quad (5.19)$$

where S_0 is the background white noise, T_{meas} the measurement time dedicated to a certain V_{1x} , $\pi/4$ for normalization and the factor 2 is needed to obtain the peak-to-peak amplitude. We can invert the formula and express it in terms of some relevant parameters

$$T_{meas} = 250 \text{ s} \left(\frac{\sqrt{S_0}}{5 \mu\text{V}/\sqrt{\text{Hz}}} \right)^2 \left(\frac{0.5 \mu\text{V}}{\sigma} \right)^2. \quad (5.20)$$

Considering that we want to characterize an effect of order $\sim 1 \mu\text{V}$ for many different V_{1x} commands, we are forced to a tradeoff. Usually 300-500 s where dedicated to a single command, in order to investigate a few hundreds of commands in 1-2 days.

Correction for channels unbalances and constant phase-shifts

When summing the voltage of two electrodes, we must consider that they do not have perfectly unitary gain (we call their gain β_1 and β_2); this also holds for the input channels of the summing amplifier (G_1 and G_2). We rewrite the gains as

$$G_{1/2} = 1 \pm \frac{\Delta G}{2}, \quad \beta_{1/2} = 1 \pm \frac{\Delta \beta}{2}. \quad (5.21)$$

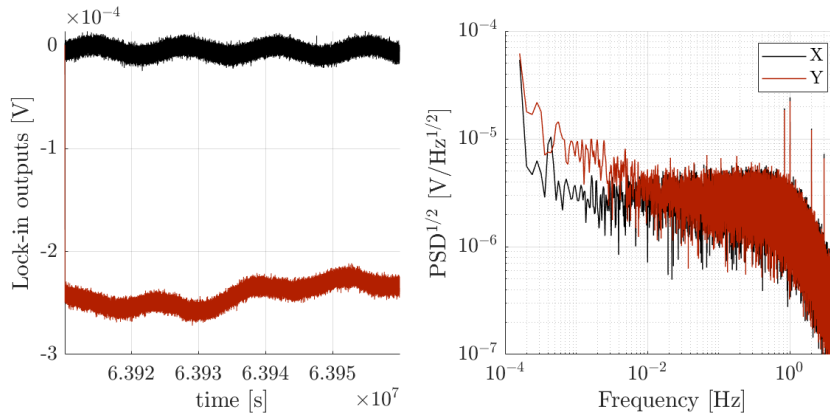


Figure 5.8: First trial of a DC measurement of $\delta V_{1x}^{nl1} + \delta V_{1x}^{nl2}$. (Left) Outputs of lock-in during a measurement in which V_{1x} was being changed of one LSB ($153 \mu\text{V}$) approximately every 150s, starting from $V_{1x}=0.49 \text{ V}$ to $V_{1x}=0.52 \text{ V}$. While X is almost zero, Y changes due to the presence of the non-linearities (and unbalances). A very low-frequency oscillation at $50 \mu\text{Hz}$ can be noticed in the measurement. This signal is present even in the absence of any commanded voltage. (Right) Amplitude spectral densities of the signals on the left. While X seems to be almost completely white, at low frequency Y is not, since we are changing the non-linearities in time by changing V_{1x} .

Furthermore, the two signals being summed could have some unwanted phase shifts ϕ_1 and ϕ_2 with respect to the reference being used. Taking account of these effects, the sum of the two signals of eq. (5.15), keeping just the first order terms, becomes

$$\begin{aligned} G_1\beta_1V_{1c}(t) + G_2\beta_2V_{2c}(t) &= \\ &= [(\Delta G + \Delta\beta)V_{1x} + \delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0)] \sin \omega_x t + (\phi_1 - \phi_2)V_{1x} \cos \omega_x t. \end{aligned} \quad (5.22)$$

Therefore, only the Y channel is affected by gain unbalances, while only the X channel is affected by phase shifts (at first order). It is possible to discriminate ΔG and $\Delta\beta$ by switching the input channels of the summing amplifier, which results in

$$\begin{aligned} G_2\beta_1V_{1c}(t) + G_1\beta_2V_{2c}(t) &= \\ &= [(-\Delta G + \Delta\beta)V_{1x} + \delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0)] \sin \omega_x t + (\phi_1 - \phi_2)V_{1x} \cos \omega_x t. \end{aligned} \quad (5.23)$$

These measurements are straightforward and fast, since it is sufficient to quickly ramp the commanded V_{1x} and the Y signal will be proportional to it with a slope $m_{12} = \Delta G + \Delta\beta$, which can be extracted with a linear fit; the input channels are then exchanged and the new

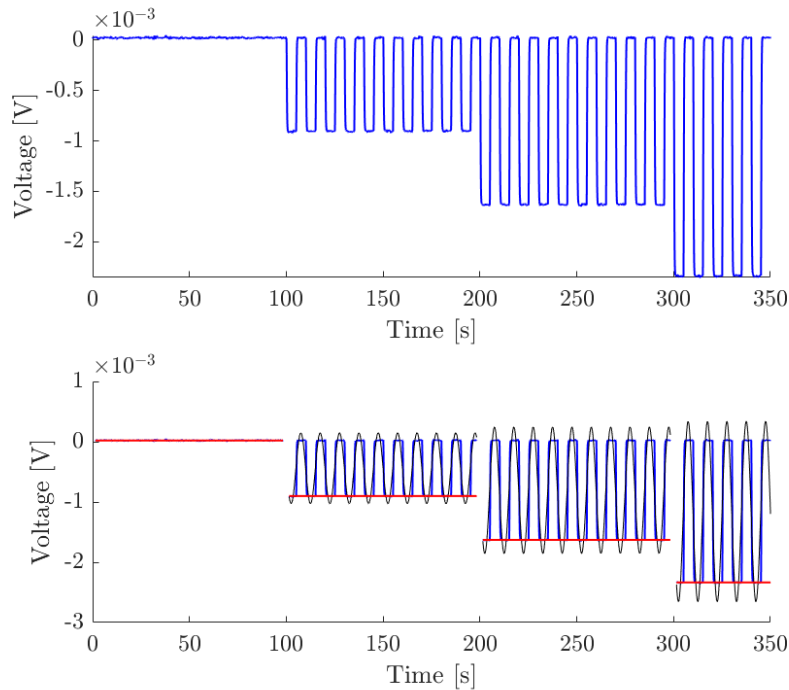


Figure 5.9: (Top) Example of lock-in output during a modulated measurement. (Bottom) Example of least square fit to a sinusoid (black) and reconstructed square wave amplitude (red).

	EL1,EL2 (V_{1x})	EL3,EL4 (V_{2x})	EL5,EL6 (V_{1y})	EL7,EL8 (V_{2y})
$\Delta\beta$ [ppm]	581	-950	-5000	-4900
$\phi_1 - \phi_2$ [μ rad]	-15	213	-169	-214

Table 5.1: Unbalances of different some relevant couples of electrodes.

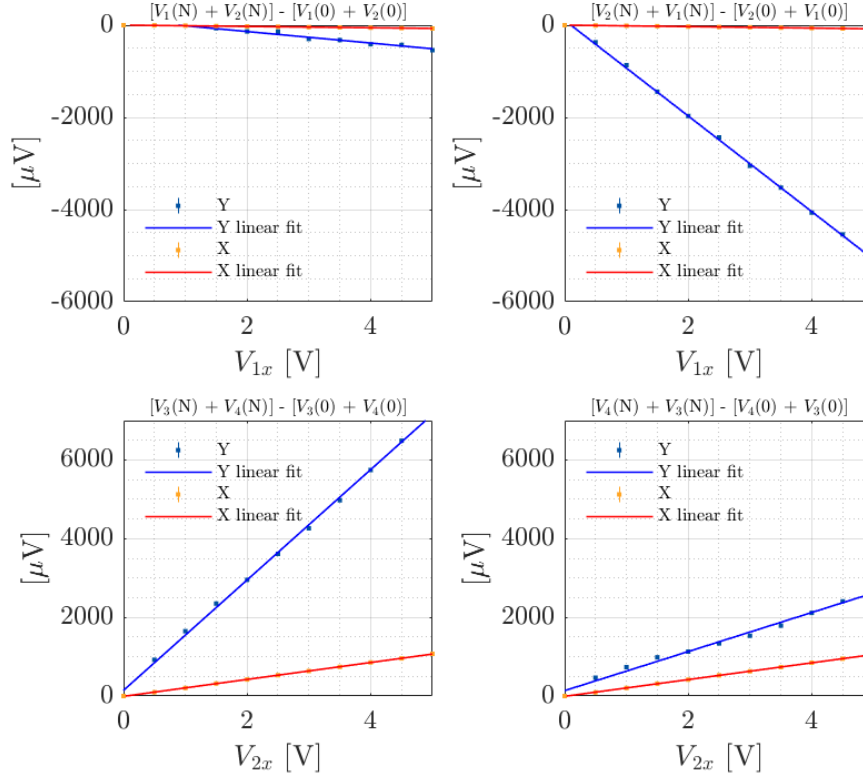


Figure 5.10: Measurement runs performed as explained in the text to measure unbalances and phase-shifts. In the top plots V_{1x} is ramped from 0 to 5 V, exchanging the amplifier input channels; the same is done for V_{2x} in the bottom plots.

slope will be $m_{21} = -\Delta G + \Delta\beta$. We can then calculate ΔG and $\Delta\beta$ as

$$\Delta G = \frac{m_{12} - m_{21}}{2}, \quad \Delta\beta = \frac{m_{12} + m_{21}}{2}. \quad (5.24)$$

For the X channels instead, the two measurement should give the same result. The unbalance was measured for the couples of electrodes providing V_{1x} , V_{2x} , V_{1y} and V_{2y} (not $V_{1\phi}$ and $V_{2\phi}$) which are reported in Figure 5.10. A consistent value of $\Delta G = 460$ ppm was found from all runs, the values for $\Delta\beta$ and $\phi_1 - \phi_2$ are reported in Table 5.1. From this point on, we will show measurements already corrected for both these effects, therefore containing only the non-linearities δV^{nl} .

Results

The results of two measurements targeting a wide range of commands for V_{1x} (EL1,EL2) V_{2x} (EL3,EL4) are reported in Figure 5.11. We recall that the measured non-linearities should be seen in the Y demodulation, while the X component should ideally be zero. Some clearly resolved structures are present in the Y channels, absent in the X one, which seem almost periodic in the commanded voltage and of order $\sim 100 \mu\text{V}$. The two couples of electrodes seem to differ in the magnitude of the nonlinearities, but not in the periodicity. This was a puzzling, because we were looking for non-linearities of order μV and found over $100 \mu\text{V}$.

The subject of the next Section will be to show that these $100 \mu\text{V}$ oscillations do not matter that much for us, because in LISA Pathfinder runs, much smaller ranges of commands were used (order hundreds over two weeks). What really matters is the difference between neighbouring bits, and how fast you change between these bits (1 change every 1.5 hours in Feb2017 LPF noise run). For this reason, more detailed measurements were performed in the range of commands used during LPF Feb2017 runs. To gather some statistics, this was done for multiple couples of electrodes, as time permitted. So in Figures 5.12 and 5.13 are reported investigations command-by-command of the same voltages of Figure 5.11, in a much more restricted set of commands. We do not show it here to not be too redundant, but the results perfectly overlap with the ones of Figure 5.11. These Figures show the direct result of the measurement in the top panel (what we called the INL, Integral Nonlinearity, for example $\delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0)$), while the bottom panel shows the difference between consecutive errors, which we called the DNL, Differential Nonlinearity. An overall view of the results is given after all the Figures.

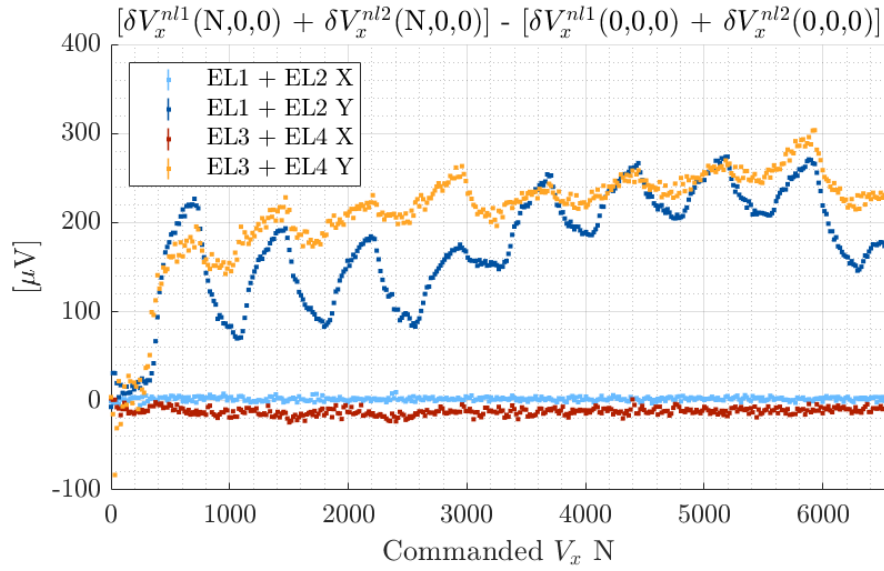


Figure 5.11: Nonlinearities for V_{1x} (EL1,EL2) V_{2x} (EL3,EL4). The x-axis is not expressed in terms of a voltage, but of $N = V_{1x}/V_{LSB}$ to always keep in mind that only discrete voltages can be applied, multiples of $V_{LSB} = 153 \mu\text{V}$. Title shows the quantity on the y-axis: as explained in the text, the measurement is differential between a reference command and the N^{th} .

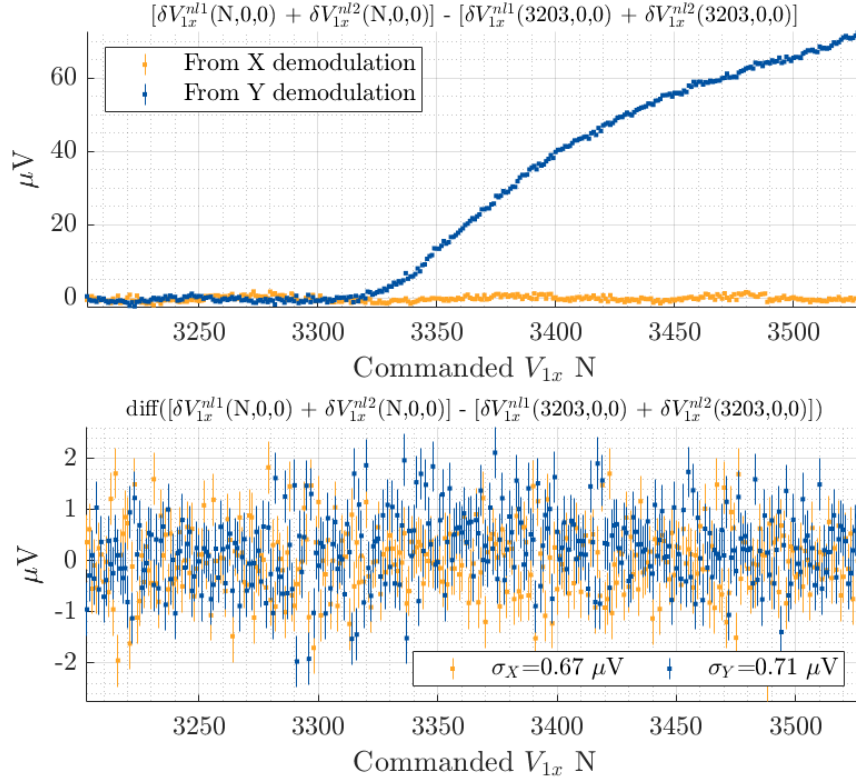


Figure 5.12: (Top) Measurement of the sum of integral nonlinearities of V_{1x} on electrodes 1 and 2, corrected for DC gain unbalance and phase shift as explained in the text. Each command is integrated for 413 s, for a total of 38 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $4.6 \mu V / \sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $0.35 \mu V$, following Eq. 5.19. (Bottom) DNL calculated as the difference between consecutive commands of the top panel. Uncertainties are just multiplied by $\sqrt{2}$.

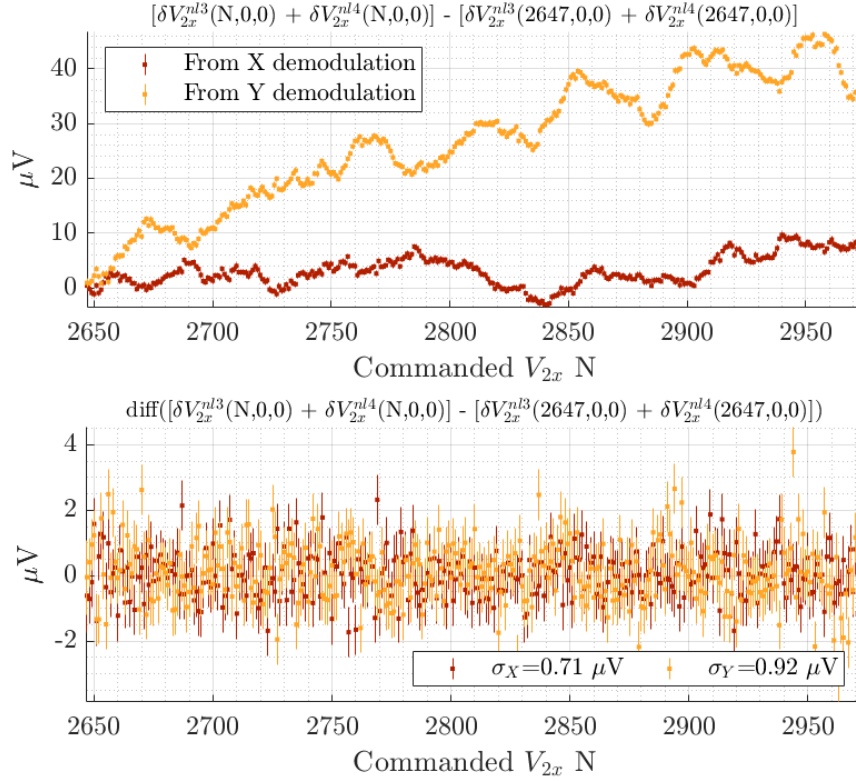


Figure 5.13: (Top) Measurement of the sum of integral nonlinearities of V_{2x} on electrodes 3 and 4, corrected for DC gain unbalance and phase shift as explained in the text. Each command is integrated for 413 s, for a total of 38 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $7 \mu\text{V}/\sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $0.5 \mu\text{V}$, following Eq. 5.19. (Bottom) DNL calculated as the difference between consecutive commands of the top panel. Uncertainties are just multiplied by $\sqrt{2}$.

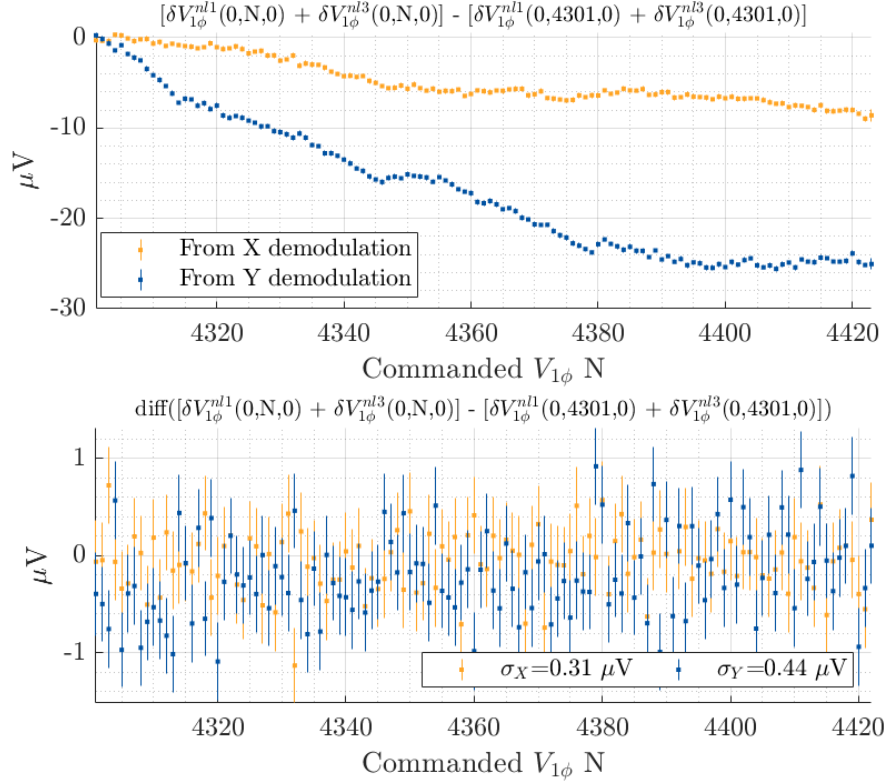


Figure 5.14: (Top) Measurement of the sum of integral nonlinearities of $V_{1\phi}$ on electrodes 1 and 3, **NOT** corrected for DC gain unbalance and phase shift, I did not take the time to measure it. Each command is integrated for 1100s, for a total of 35 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $6 \mu V / \sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $0.45 \mu V$, following Eq. 5.19. (Bottom) DNL calculated as the difference between consecutive commands of the top panel. Uncertainties are just multiplied by $\sqrt{2}$.

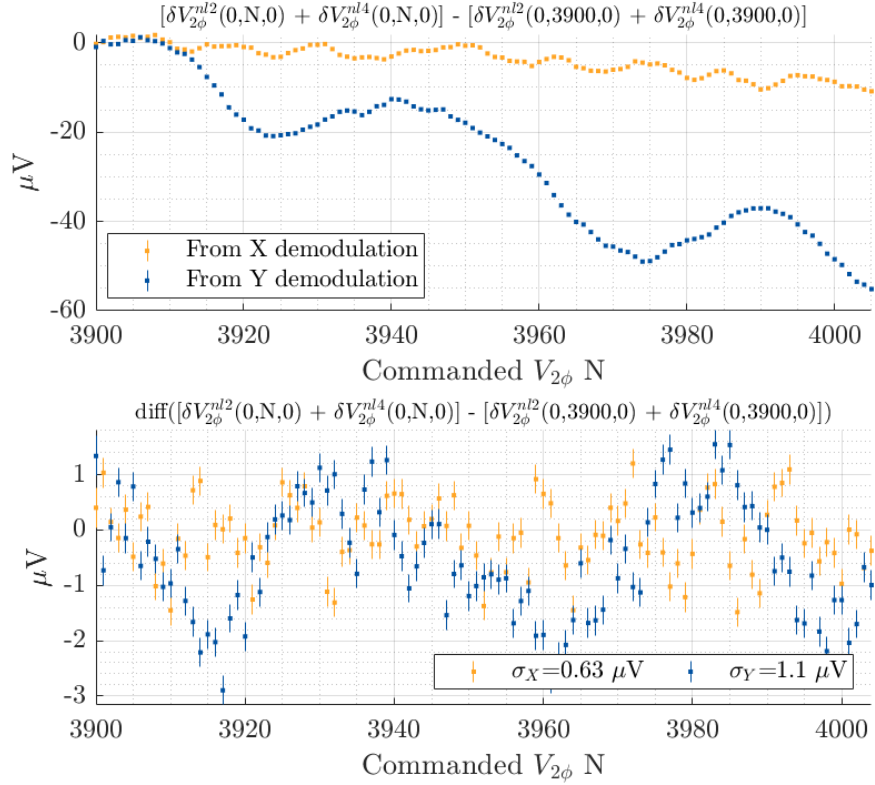


Figure 5.15: (Top) Measurement of the sum of integral nonlinearities of $V_{2\phi}$ on electrodes 2 and 4, **NOT** corrected for DC gain unbalance and phase shift, I did not take the time to measure it. Each command is integrated for 2100s, for a total of 62 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $5.5 \mu V / \sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $0.2 \mu V$, following Eq. 5.19. (Bottom) DNL calculated as the difference between consecutive commands of the top panel. Uncertainties are just multiplied by $\sqrt{2}$.

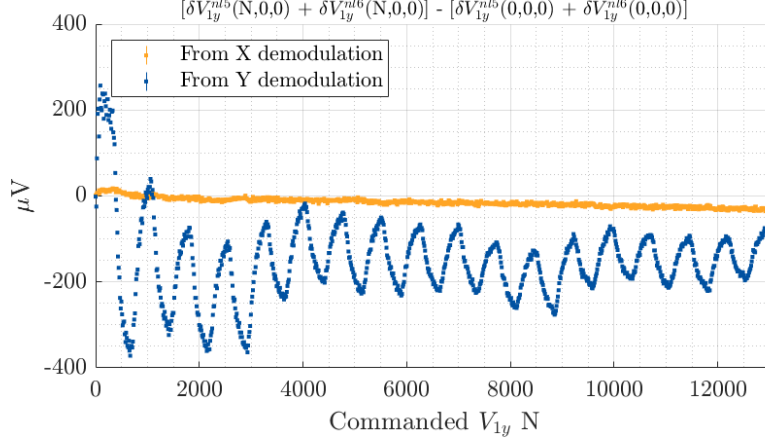


Figure 5.16: Measurement of the sum of integral nonlinearities of V_{1y} on electrodes 6 and 7, corrected for DC gain unbalance and phase shift. Command every 2 mV are integrated for 23 s, for a total of 6 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $6 \mu\text{V}/\sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $2 \mu\text{V}$, following Eq. 5.19.

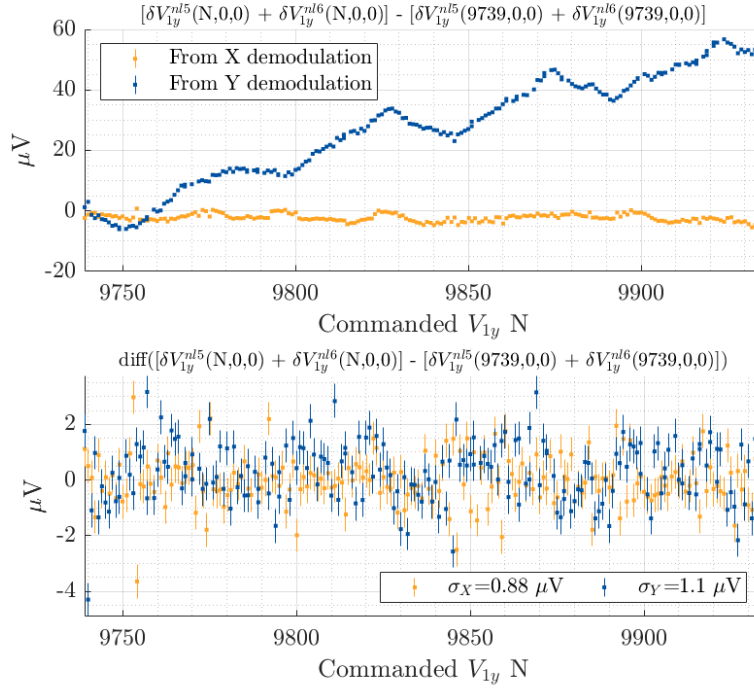


Figure 5.17: (Top) Measurement of the sum of integral nonlinearities of V_{1y} on electrodes 6 and 7, corrected for DC gain unbalance and phase shift as explained in the text. Each command is integrated for 290 s, for a total of 15 hours of measurement. The level of white noise at 0.1 Hz for this measurement is $5 \mu\text{V}/\sqrt{\text{Hz}}$, which means an uncertainty on the measurement of $0.45 \mu\text{V}$, following Eq. 5.19. (Bottom) DNL calculated as the difference between consecutive commands of the top panel. Uncertainties are just multiplied by $\sqrt{2}$.

I try to summarize what I think are the important points to notice from these measurements.

- The integral nonlinearities clearly exhibit some structures, which are worse when exploring large number of commands: being more precise, over variations of hundreds of commands, we observe changes in the nonlinearities of order hundreds of μV , which also seem to have some sort of periodicity of order 600-700 bits. While the nonlinearities differ between actuation circuits, the periodicity seems to be exactly the same. **Note:** some measurements were performed on subsets of commands, with different setup settings, like lock-in full scale, commands sequence, integration time, to see if the result is reproducible. These large oscillation are found unchanged in every test.
- On the other hand, on short ranges of commands, the difference in nonlinearities becomes less dramatic. This behavior is better spotted looking at the differential nonlinearities, and a good overall figure of merit is their standard deviation over the whole range, reported in every Figure as σ_Y . The value of σ_Y is always found lower or equal to $1\ \mu\text{V}$ (keep in mind, this is the sum of two electrodes, if we want that of only one we should divide by $\sqrt{2}$). Considering that we could obtain uncertainties of order $0.5\ \mu\text{V}$ in few days of measurement, it may seem that we find σ_Y systematically higher than the measurement uncertainty: a good example can be seen in Figure 5.15, a very long measurement of a hundred of bits in which we tried to have small uncertainty with respect to the nonlinearities scattering.

However, there are a few facts that may suggest that we are underestimating the uncertainty on the measurement of each command (discussed in the following bullet), so to be completely conservative, it is suggested to interpret σ_Y as **upper limits** to the real DNL dispersion.

- In the first bullet it was mentioned that measurements on small subsets of commands were performed to check the reproducibility of the results. While this was more than enough to confirm the presence of large INL, it does not give enough information on the command-to-command DNL. So we lack a test in which we check if the measurement of DNL is reproducible within uncertainty. A source of doubt is for sure the measurement in the X channels, which ideally should be zero and in some runs is clearly not, for example Figure 5.13, showing oscillation almost of the same order as the Y channel. A first possible explanation can be seen in Eq. 5.23, where X is shown to be proportional to $\phi_1 - \phi_2$, the phases of the actuation waveforms being summed. If $\phi_1 - \phi_2$ slowly oscillate through the measurement (for example for temperature variations), it could cause signal variations in X, while not affecting Y at all. However, there are a few strange facts that in my opinion exclude this hypothesis. First, the fluctuations in $\phi_1 - \phi_2$ should be completely independent from the ones found in Y. So, why do the fluctuations of X and Y look so much alike in every measurement (see Figure 5.11)? Even in the DNL, σ_X is always *slightly* lower than σ_Y : if they were independent, some-

times we should find $\sigma_X > \sigma_Y$. Second, we can for example link an oscillation in $\phi_1 - \phi_2$ to a change in the delays τ_1, τ_2 introduced by the actuation low pass-filters. The relation is simple:

$$\phi_1 - \phi_2 = \omega_{ACT}(\tau_1 - \tau_2). \quad (5.25)$$

τ_1, τ_2 can change differently because of different temperature coefficients of the components. But if this is the case, the effect should be less important for V_{1x} than $V_{1\phi}$ of almost a factor 5, since $\omega_x = 60$ Hz and $\omega_\phi = 270$ Hz. This seems not to be the case. Then what could it be? It seems also strange to have so largely underestimated the measurement noise at 0.1 Hz: this seems to be stationary through the measurement (indicated in each Figures caption), and, even if it was higher, it should result in a larger, uncorrelated scattering of the X and Y measurements. What we instead observe for example in X of Figure 5.13, is that consecutive commands are largely correlated. One possibility is that there is a dependence of the actuation waveform phase on the command, namely $\phi = \phi(V_{1x})$: this was a problem cited in the past by people working on the FEE, but still it is difficult to understand how it could be linked to nonlinearities. A symptom to check this could be if the X values are reproducible in different measurements.

Not enough attention was dedicated to the X aspect during the campaign, in the rush, so it is **strongly suggested** to perform again at least one of the measurements, for example the one of Figure 5.13, which seems to be the most dramatic and check: 1. if the DNL are reproducible, 2. the X values are systematic or not.

5.3.2 Measurement in presence of digitization error

Here we are trying to answer one of the question we posed in the introduction: "Did we remove the digitization error completely from LPF data?". To address this, we performed a similar measurement to the previous section, but adding a small $V_{2\phi} = 0.5$ mV. Always staring from eq. (5.6) we have

$$\begin{aligned} V_{1c}(t) &= \left[V_{1x} + \delta V_{1x}^{dig}(V_{1x}, 0, 0) + \delta V_{1x}^{nl1}(V_{1x}, 0, 0) \right] \sin \omega_x t \\ V_{2c}(t) &= \left[-V_{1x} + \delta V_{1x}^{dig}(-V_{1x}, V_{2\phi}, 0) + \delta V_{1x}^{nl2}(-V_{1x}, V_{2\phi}, 0) \right] \sin \omega_x t + [V_{2\phi} + \dots] \sin \omega_\phi t. \end{aligned} \quad (5.26)$$

The sum of the two is then

$$\begin{aligned} V_{1c}(t) + V_{2c}(t) &= \\ &= \left[\delta V_{1x}^{dig}(V_{1x}, 0, 0) + \delta V_{1x}^{dig}(-V_{1x}, V_{2\phi}, 0) + \delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, V_{2\phi}, 0) \right] \sin \omega_x t \\ &\quad + [V_{2\phi} + \dots] \sin \omega_\phi, \end{aligned} \quad (5.27)$$

which again has the effect of cancelling the big carrier V_{1x} , but leaves the digitization error. We use the same setup in Figure 5.6. When we feed this signal to a lock-in, only the component

at ω_x is demodulated, not the ω_ϕ ; still, it is important to keep $V_{2\phi}$ small to fit in the lock-in dynamic range, otherwise we would be forced to increase the full-scale and lose the resolution needed for the measurement.

Results

The strategy to collect the data is exactly the same as the previous section, including the modulation and the DC gain corrections. A measurement done for the same commands V_{1x} , electrode 1 and 2, as in Figure 5.12 is reported in Figure 5.18. The difference between the two figures is clear and it can also be seen that the digitization error really seems uncorrelated between consecutive commands and of order 10 μV . Since we have the algorithm to calculate the digitization error $\delta V_{1x}^{dig}(V_{1x}, 0, 0) + \delta V_{1x}^{dig}(-V_{1x}, V_{2\phi}, 0)$, also that is reported in Figure 5.18: it is already possible to see that there is a good match in the first hundreds of commands. We underline that there is no fitting procedure, this is exactly the prediction of the algorithm compared to experimental data.

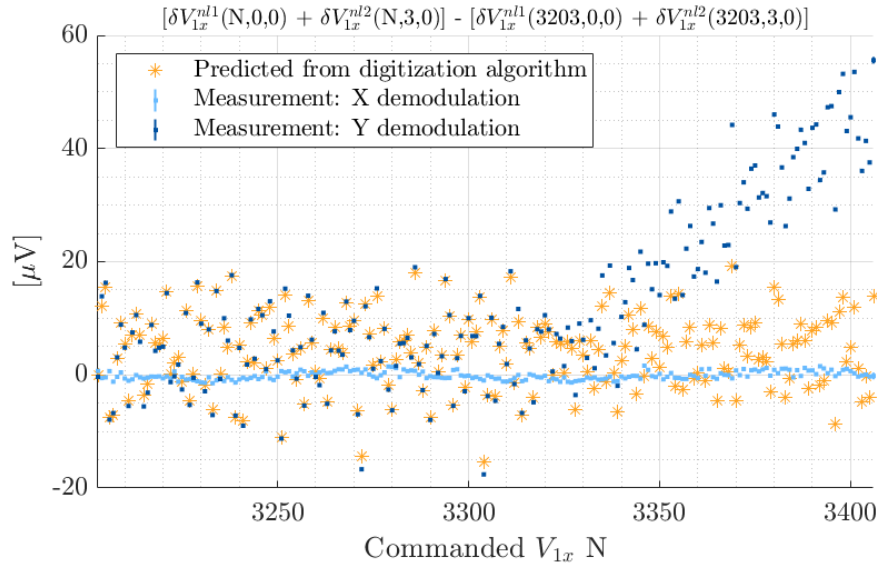


Figure 5.18: Measurement of non-linearities for EL1 and EL2, analogous to Figure 5.12, but with contribution from digitization error. The experimental points (in blue) are compared with the error predicted from the digitization algorithm. Absolutely no fitting procedure was performed to obtain the shown degree of agreement.

The result of subtracting the predicted error from the measured one is reported in Figure 5.19, where it is compared to the measurement *without* digitization error of the previous section. To show how good the match is, the difference between the two measurements is taken in the bottom plot, showing a sub- μV subtraction, compatible with the dispersion obtained for the measurement without digitization error. **This is an amazing result** and demonstrates: 1. the correctness of the digitization error algorithm, 2. the correctness of the measurement procedure of the nonlinearities (it is like a demonstration that we can measure something we know! So the nonlinearities of Figure 5.11 are real, not an artifact of the measurement).

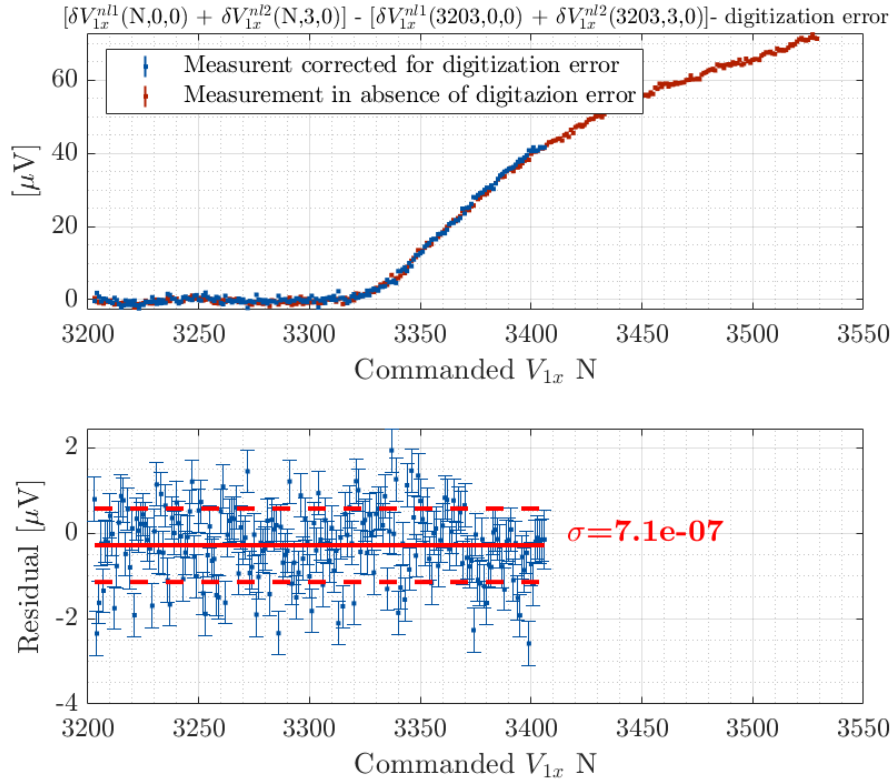


Figure 5.19: (Top panel) [Red] Measured INL for V_{1x} in absence of digitization error described in Figure 5.12. [Blue] Measured INL in presence of digitization error, corrected in dead-reckoning through the algorithm. (Bottom Panel) Difference between the two curves in the top panel. **Warning:** this is not the DNL as in all the previous figures!

Some final comments about this measurement. We successfully subtracted the digitization error, but why is what remains exactly equal to the nonlinearities measured in absence of digitization error? In principle they should be different, since the presence of $V_{2\phi}$ makes you use different DAC codes: this is expressed in Eq. 5.27, which says that we are measuring $\delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, V_{2\phi}, 0)$, not $\delta V_{1x}^{nl1}(V_{1x}, 0, 0) + \delta V_{1x}^{nl2}(-V_{1x}, 0, 0)$ as in Figure 5.12.

The answer could be that since $V_{2\phi}$ is so small (3 LSB), and nonlinearities of neighbouring commands are similar, it does not really modify the average effect on the waveform effective amplitude. A difference should be seen for larger $V_{2\phi}$. In this direction, there was a suggestion by ETHZ to further test this behaviour by completely changing the ADC codes used by adding a V_{DC} to one or both the electrodes. In fact, the first suspect to be the source of nonlinearities is the ADC present in the feedback loop of actuation: if commanding V_{DC} completely changes the nonlinearities, then it should be a further indication that the source could be the ADC.

5.4 Conversion of nonlinearities into acceleration noise

From now on we will give the digitization error less attention for two reasons: firstly, we experimentally showed that it was eliminated to the sub- μV level for LPF, secondly, the new FEE waveform generation algorithm for LISA will be designed so that this error is not present. We therefore focus mainly on possible other sources of non-linearities, assuming as their reference levels the ones measured in the laboratory.

5.4.1 Simulations from laboratory measurements

The initial simulations proposed in Figure 5.5 were performed to estimate the impact of non-linearities on acceleration noise: the added distortions were assumed uncorrelated between consecutive commands, like the quantization effect. However, the previous section shows clearly that this is not true for the other kinds of non-linearities, which instead show a very high degree of correlation. What looks more uncorrelated instead seems to be the DNL, the difference of non-linearities between consecutive commands. So we repeated the simulations (according to the procedure already explained in Figure 5.4), but instead of generating a random lookup table of uncorrelated errors, we just used the measured values obtained in laboratory resulting in Figure 5.20. The contribution is surely not negligible, but it is not even the one that can explain the observed excess noise. Notice that despite using the laboratory measurements, which exhibited 10-100 μV non-linearities, the noise is not that dramatic. Again, we can guess that this is because only a few commands are explored over several hours, and these typically differ by less than $\sim 1 \mu\text{V}$. If more bits were explored in less time, then we could expect a much larger effect.

Note 1. There is an important difference between the simulations of Figure 5.20 and 5.5 though. It was stated many times that the non-linearities depend on $\delta V(V_{1x}, V_{1\phi})$ (using EL1 as example), so in principle one needs an error for every possible couple of $[V_{1x}, V_{1\phi}]$. This could be done for the initial, randomly generated simulations. From the lab instead, we have much less data, namely just a column vector $[V_{1x}, 0]$ from Figure 5.12 and a row vector $[0, V_{1\phi}]$ from Figure 5.14. We have no way to go around this, measuring the whole

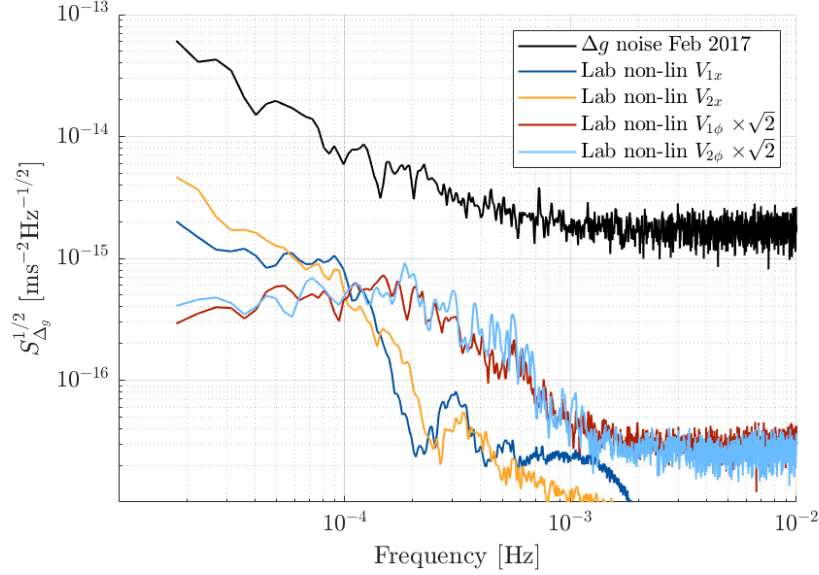


Figure 5.20: Simulated contribution of non-linearities to acceleration noise in Feb2017 run, using the laboratory measurements. See note in the text. The ϕ contributions are multiplied by $\sqrt{2}$ because there are two TMs.

table would take several weeks for just one electrode. Therefore, we are forced to simulate the errors on the amplitudes used during February 2017 not exactly as it should be done, that is

$$V_{1c}(t) = [V_{1x} + \delta V_{1x}(V_{1x}, V_{1\phi}, V_{DC1})] \sin \omega_x t + [V_{1\phi} + \delta V_{1\phi}(V_{1x}, V_{1\phi}, V_{DC1})] \sin \omega_\phi t, \quad (5.28)$$

but using the approximation

$$V_{1c}(t) \approx [V_{1x} + \delta V_{1x}(V_{1x}, 0, 0)] \sin \omega_x t + [V_{1\phi} + \delta V_{1\phi}(0, V_{1\phi}, 0)] \sin \omega_\phi t. \quad (5.29)$$

We could certainly find other ways of doing this injection, but they are all equivalently inaccurate. We still think they give valuable information: if we look at the contribution to the force, it reads (see eq. (5.3))

$$\begin{aligned} \delta F &= \frac{1}{2} \frac{\partial C^*}{\partial x} [V_{1x}(t) \delta V_{1x}(V_{1x}, V_{1\phi}) + V_{1\phi}(t) \delta V_{1\phi}(V_{1x}, V_{1\phi})] \\ &\approx \frac{1}{2} \frac{\partial C^*}{\partial x} [\bar{V}_{1x} \delta V_{1x}(V_{1x}, V_{1\phi}) + \bar{V}_{1\phi} \delta V_{1\phi}(V_{1x}, V_{1\phi})] \end{aligned} \quad (5.30)$$

where the approximation $V_{1x}(t) \approx \bar{V}_{1x}$ is really safe for LPF, since the voltage have slow oscillations around a much larger average voltage. This means that *the shape of the noise is entirely determined by how the non-linearities are changing in time*. Another bit of information to add is that there are no situations in which we rapidly jump from a couple $[V_{1x}, V_{1\phi}]$ to one which differs of several bits: we can say that even if $[V_{1x}, V_{1\phi}]$ changes every 0.1 s, it

stays within a small subset of neighboring couples (of order 10 couples, to give a number) over hours of measurement. When we are forced to do the further approximation

$$\delta F \approx \frac{1}{2} \frac{\partial C^*}{\partial x} [\bar{V}_{1x} \delta V_{1x}(V_{1x}, 0) + \bar{V}_{1\phi} \delta V_{1\phi}(0, V_{1\phi})] \quad (5.31)$$

we are basically assuming that $\delta V_{1x}(V_{1x}, 0)$ roughly represents the magnitude of $\delta V_{1x}(V_{1x}, V_{1\phi})$ and how it changes in time. Since $\delta V_{1x}(V_{1x}, 0)$ and $\delta V_{1x}(0, V_{1\phi})$ resulted pretty similar in amplitudes and correlations, we think that $\delta V_{1x}(V_{1x}, V_{1\phi})$ changing between neighboring commands couples can be quite well reproduced by how $\delta V_{1x}(V_{1x}, 0)$ changes between V_{1x} neighboring commands. The same holds for $V_{1x}(0, V_{1\phi})$ and we do not expect to entirely get the contribution wrong, if not for some small factors.

This was confirmed by generating a 2D table $\delta V_{1x}(V_{1x}, V_{1\phi})$, with 2D red correlations similar to the ones observed in the "single column" measurements $\delta V_{1x}(V_{1x}, 0)$, and found completely analogous results.

Note 2. The just discussed problem of not having a complete table for $\delta V_{1x}(V_{1x}, V_{1\phi})$ is an issue just for LISA Pathfinder projections. LISA has no x actuation, so the only nonlinearities we care about are $\delta V_{1\phi}(V_{1\phi})$, and no table is needed for this. In principle they also depend on the commanded V_{DC} , but there is no intention of changing its value during any measurement, so it was promptly discarded from this discussion. If the simulations of Figure 5.20 may be affected by some inaccuracies for LPF, each of the curves would be perfectly valid for LISA, assuming the same LPF voltage time series are used.

We need to simplify the problem

Up to now we just proceeded through measurements and simulations. This gave us an idea of the $\sim 1 \mu\text{V}$ size of non-linearities and the approximate level of noise that could have affected LPF. But we never tried to investigate why the spectra obtained in Figure 5.20 look like that. Why should there be a difference between the contribution from ϕ actuation and x ? Where does the shape come from? Do uncorrelated non-linearities like the luignation have a different impact on acceleration noise with respect to correlated non-linearities, like the ones that we measured in the lab?

Without even approximate answers to these questions, the only way we can realize projections to LISA is always through simulations. It is difficult to give even a rough estimate of the noise, and so to understand the guidelines that must be followed to keep it under control. We think we understood quite a lot in this direction. At the moment there are too many variables in the game, so we shall try to simplify as much as we can.

5.4.2 Non-linearities spectrum

The difference between the digitization error and the non-linearities measured in the lab is that the former looks uncorrelated between neighboring commands, the latter does not. Can

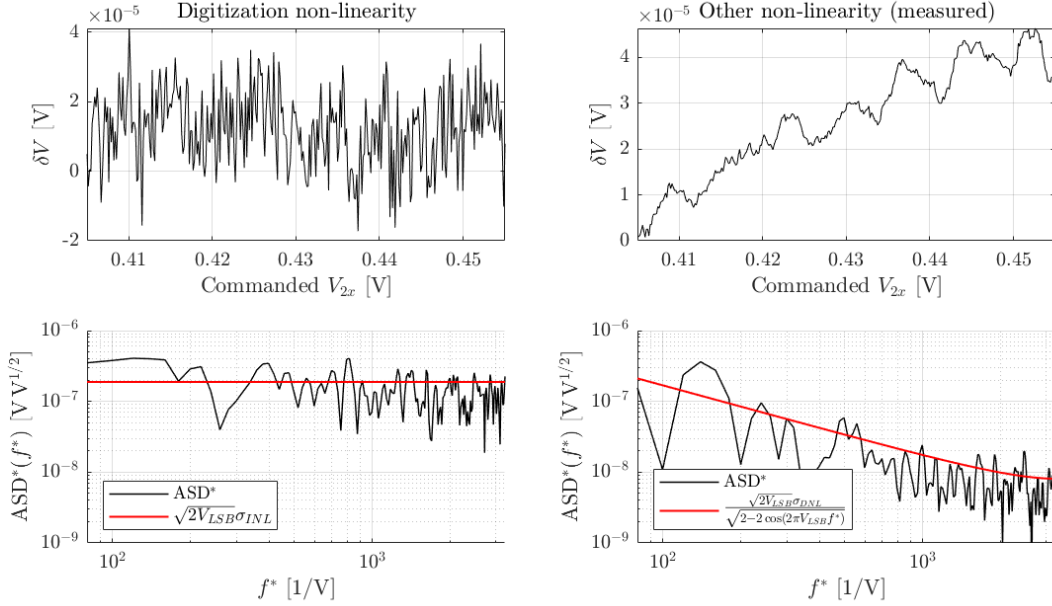


Figure 5.21: (Top) On the Left, example of digitization non linearities for a series of commanded amplitudes. On the right, non-linearities measured in the lab for the same interval of commands. (Bottom) ASD* of the series on top, compared with their analytic approximations (in red) described in the text.

we quantify this?

Analogously to how we calculate the PSD of a time series $V(t_i)$, sampled every T seconds, we can also do it with the series of non-linearities expressed as $\delta V(V_i)$, sampled every V_{LSB} . The x-axis of such a plot can not be a frequency in [Hz], but will be the inverse of a voltage, that we call f^* ; the y-axis is even worse since it cannot be $[V/\sqrt{\text{Hz}}]$, but it will be a "1/V density", so $[V\sqrt{V}]$, something that we usually call PSD* (or ASD* for the square root). The Nyquist frequency will be $f_{Nyq}^* = 1/2V_{LSB}$. Figure 5.21 reports the two main example. On the left-top plot, the digitization error for a set of few hundreds commands, and its ASD* as a black line in the left-bot plot. We can see that it is a flat spectrum, which agrees with our intuition that the commands are uncorrelated.

On the right of Figure 5.21 we have instead one "digitization error-free" measurement from the laboratory, for the same commanded V_{2x} . Its spectrum is not flat at all, as expected. However, even if not reported here, the DNL calculated from this measurement exhibits a quite white spectrum³. As a result, since the shown δV is nothing more then an integral of the DNL, it will have a $1/f^{*2}$ dependence.

So the first simplification that we introduce is that we consider just two main kinds of non-linearities:

³It is not exactly white because of the strange periodicities in Figure 5.11, that for very low f^* show up as peaks. But the rest of the spectrum, especially for higher f^* , is quite well approximated as white.

1. **White INL:** Errors that have white integral non-linearity, like the Luigation effect. We can describe this noise with just one number, that is its dispersion σ_{INL} (10 μ V for luigation). Considering our sample with limited sampling frequency, the relation between PSD and σ is very simple, and for out PSD* reads

$$S_{\delta V}^*(f^*) = 2V_{LSB}\sigma_{INL}^2. \quad (5.32)$$

2. **White DNL:** Errors that have white differential non-linearity, like those measured in the lab. Also here we can use a single number σ_{DNL} to describe the noise

$$S_{\delta V}^*(f^*) = \frac{2V_{LSB}\sigma_{DNL}^2}{2 - 2\cos(2\pi V_{LSB}f^*)} \xrightarrow{2\pi f^* \ll V_{LSB}^{-1}} \frac{2V_{LSB}\sigma_{DNL}^2}{(2\pi V_{LSB}f^*)^2}. \quad (5.33)$$

The cumbersome cosine function at the denominator emerges because of the finite sampling frequency, but at low f^* it is just a $1/f^{*2}$, as expected from the integral of a white process.

Both these formulas can be seen in Figure 5.21, with σ_{INL} and σ_{DNL} calculated directly from the data in the top plots. The agreement might not be perfect, probably due to the fact that the underlying data show some degree of non-whiteness, but it clearly reproduces the data trends and it is more than enough in our effort of simplifying the problem.

Note on PSD*

The standard interpretation of a PSD is that it describes a noisy process in time: if we repeat the measurement, we will always measure a different time-series, but the PSD we calculate will remain the same. The PSD* is a little different: once we measure the non-linearities of a device, those are *fixed*, so if we measure them again, we find the same result! If we measure another copy of the device, instead, we will find different ones. The PSD* than must be interpreted as a quantity that describes the statistical properties of non-linearities of a class of nominally identical devices, which exhibit differences due to manufacturing.

Can we predict acceleration noise from these spectra?

Now, it was said many times that force noise can be written as

$$\delta F = \frac{1}{2} \frac{\partial C^*}{\partial x} V(t) \delta V(V(t)) = \frac{1}{2} \frac{\partial C^*}{\partial x} \bar{V} \delta V(V(t)) \quad (5.34)$$

therefore the noise spectrum only depends on the spectrum of $\delta V(V(t))$, so we write

$$S_{\Delta g}(f) = \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \bar{V}^2 S_{\delta V(V(t))}(f) \quad (5.35)$$

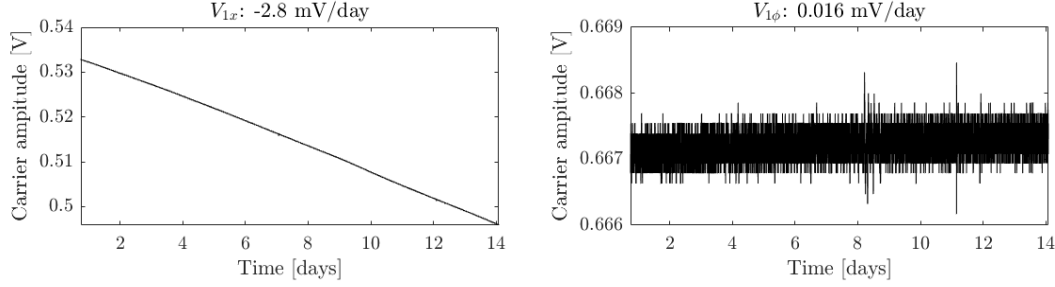


Figure 5.22: Examples of voltages applied during February 2017 LPF noise run. (Left) Voltage applied for x actuation, drifting almost linearly. (Right) Voltage for ϕ actuation.

Now, let us say we know that our actuation circuits are characterized by a certain PSD*. Is there any way to go from

$$S_{\delta V}^*(f^*) \xrightarrow{V(t)} S_{\delta V(V(t))}(f) \quad ? \quad (5.36)$$

This obviously requires some knowledge of $V(t)$ and I do not think there is a general solution for an arbitrary $V(t)$ and one at some point has to do simulations. I hope someone can tell me that I am wrong on this matter. However, there was the hope that some simple cases of $V(t)$ could be solved analytically.

Let us have a look at the typical voltage time series observed in LPF. Figure 5.22 shows the applied voltages for x and ϕ actuation during February 2017 noise runs, that show the two limiting behaviors exhibited by almost all the LPF noise runs.

- V_{1x} shows what looks like a linear drift. Why does it do so? Because the differential gravitational pull on the TMs slowly changes in time due to fuel tanks deployment, which is a very well understood phenomenon. As a consequence, the force g_{2c} necessary to keep the TMs distance constant changes accordingly, and so the applied voltages. The February 2017 LPF drift is $dg_c/dt \sim 0.3 \text{ pm s}^{-2}/\text{day}$.

This effect will **not** be present in LISA, because we have no x actuation.

- $V_{1\phi}$ exhibits instead a negligible drift compared to the noisy change in its commands. Also here, the TMs angular DOFs are kept aligned with the spacecraft, which rotation is noisy, due to noise in the star-tracker camera. So a noisy γ_{ϕ_c} must be applied to the TMs, which finally means a noisy $V_{1\phi}$.

Note: also γ_{ϕ_c} can drift! But not because of the tank deployment, which effect is much lower due to the TM symmetry. The mechanism driving a γ_{ϕ_c} drift is the TM charging, in fact we have encountered multiple times that the TM voltage couples to stray electrostatic fields as

$$N_\phi = -V_{TM} \left| \frac{\partial C}{\partial \phi} \right| \Delta_\phi. \quad (5.37)$$

Before December 2016 the DC value of Δ_ϕ was $\sim 100 \text{ mV}$, and V_{TM} increased linearly of $\sim 10 \text{ mV/day}$ due to cosmic ray charging. As a result, a $d\gamma_{\phi_c}/dt \sim 3.5 \text{ prad/s}^2/\text{day}$

	$d\gamma_{\phi_c}/dt$	dV_{ϕ}/dt	dg_c/dt	dV_x/dt
Apr. 2016	$4.5 \text{ prad s}^{-2}/\text{d}$	1.0 mV/d	$0.15 \text{ pm s}^{-2}/\text{d}$	1.4 mV/d
Feb. 2017	$0.08 \text{ prad s}^{-2}/\text{d}$	0.1 mV/d	$0.33 \text{ pm s}^{-2}/\text{d}$	3.3 mV/d

Table 5.2: Drift magnitude of commanded forces, torques, and relative voltages on LPF, before and after DC voltage compensation (respectively, Apr. 2016 and Feb. 2017). It is noticeable that Δ_{ϕ} compensation suppressed the rotational drift $d\gamma_{\phi_c}/dt$, whereas it did not suppress the linear one, dg_c/dt , as on LPF it was dominated by gas tank depletion.

was observed, or equivalently $dV_{1\phi}/dt \sim 1 \text{ mV/day}$.

After December 2016 Δ_{ϕ} was compensated with the application of proper DC voltages, suppressing the drift of about a factor 100. That is why Figure 5.22 does not show any drift.

This effect **will be** also present in LISA.

The important numbers relative to LPF drifts are summarized in Table 5.2. This should hint why we chose to study two very specific limit cases.

1. **A noiseless, linearly drifting voltage.** For these cases we have truly analytical solutions.
2. **A Gaussian, stationary voltage noise.** I always had the hope that knowing $S_{V(t)}$ (and not $V(t)$) could allow the establishment of the (5.36) link. So, not with the objective of finding the exact $\delta V(t)$ time series, but only assess its statistical properties, starting from those in $S_{V(t)}(f)$ and $S_{\delta V}^*(f^*)$. I did not manage to do find this mysterious relation, and I could only provide some approximate (rule of thumb) analytic models for the case in which $S_{V(t)}(f)$ is white noise, which is also a relevant case for LISA. For other cases, we still take advantage of simulations.

5.4.3 Analytic model for drift in commanded actuation

The idea of investigating a linear drift was entirely of Lorenzo Sala, and also a good part of the calculations, so credits to him and thanks for insisting to proceed also along this line. I think it is important to include it in this thesis to have a complete treatment of the non-linearities problem in LISA. It is also an occasion to explain this in a more linear way, since [28] is pretty technical.

So let us assume that

$$V(t) = mt + q. \quad (5.38)$$

This means that, as time passes, we will explore each voltage command, one by one. Given a series of non-linearities $\delta V(V_i)$ like the black ones in Figure 5.23 (top left), we will cross them, one by one, and obtain a black time-series $\delta V(V(t))$ with exactly the same shape, in

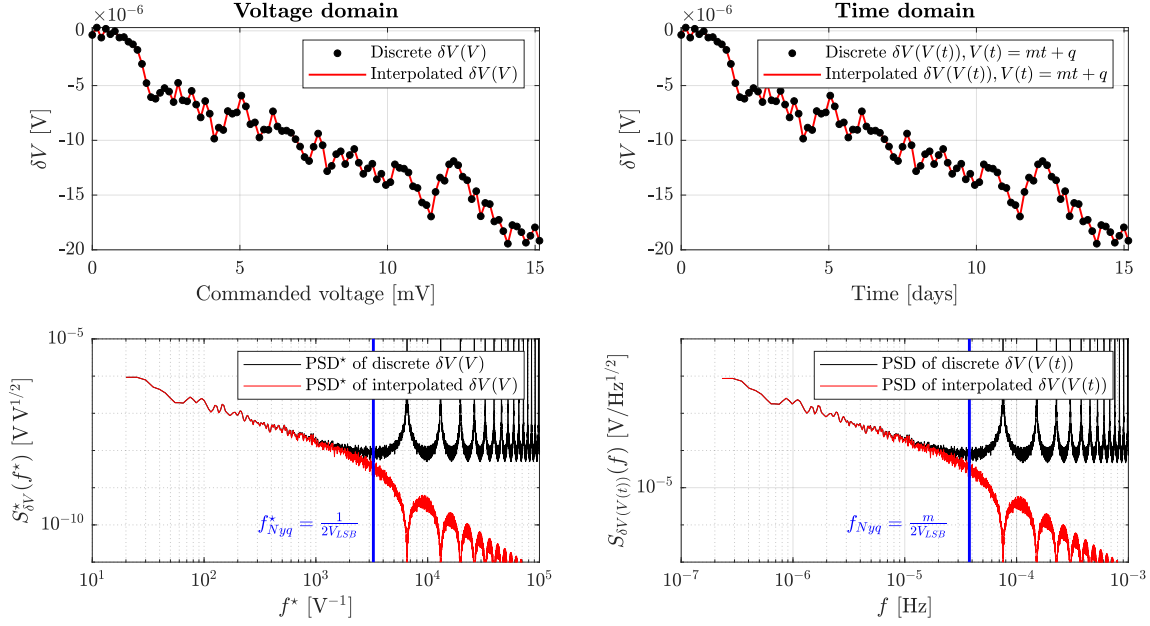


Figure 5.23: (Top left) An example of a few, simulated, white DNL nonlinearities, as black points. In red, their linearly interpolated version, as described in the text. (Bottom left) PSD* of the signals in the plot above. (Top right) The non-linearities of the top left plot, converted in a time-series by a linear drift $V(t) = mt$ with $m = 1$ mV/day. (Bottom right) PSD of the signals in the plot above. The curve shape is obviously identical to the bottom left one, but its amplitude and frequency are fixed by the value of m .

Figure 5.23 (top right). From this, we understand that also $S_{\delta V}^*(f^*)$ and $S_{\delta V(V(t))}(f)$ must also have the same shape! These are also shown in the bottom plots of Figure 5.23, always in black. We will now try to derive a formula to link $S_{\delta V}^*(f^*)$ and $S_{\delta V(V(t))}(f)$. There is one problem though, $\delta V(V)$ is discrete, sampled every V_{LSB} : if we convert it to a time series by just using (5.38), the samples will be spaced by $dt = V_{LSB}/m$. If the typical $m \sim 1$ mV/day, then $dt=3.7$ hours. But $V(t)$ is a 10 Hz time series, and so must be $\delta V(V(t))$. What happens between one non-linearity and the following one? For this, we recall that *in time domain* there is a Σ - Δ loop at work: this is effectively oscillating between the bit n and $n + 1$ to obtain a voltage that is a weighted average of them. In other words, it is performing a linear interpolation. The same happens for the non-linearities: we spend a fraction of the time affected by one non-linearity, and the remaining fraction affected by the other, so, the effect of the Σ - Δ is a linear interpolation between neighboring non-linearities. In Figure 5.23 we can see discrete series of non-linearities linear interpolation as a continuous red line. Considering a continuous signal for both $\delta V(V)$ and $\delta V(V(t))$ instead of a discrete one has two advantages: first, it removes the problem of not matching sampling frequencies; second,

it allows us to do calculations with the continuous Fourier transform⁴.

What is the spectrum of the newly interpolated data? It is represented in red in the bottom plots of Figure 5.23. We can see that red and black spectra are perfectly equal at low frequencies, but around the Nyquist frequency the linearly interpolated one exhibits some kind of filter. A linear interpolation is a combination of neighboring samples. Whenever we combine two or more delayed versions of a same time series, we obtain some degree of filtering of the data (digital filter are actually done like this). When we move from the discrete $\delta V(V)$ to its continuous, linearly interpolated version, the PSD^* is filtered, and changes as follows

$$\begin{array}{ccc} \text{Discrete} & & \text{Interpolated} \\ S_{\delta V}^*(f^*) & \longrightarrow & S_{\delta V, \text{int}}^*(f^*) = S_{\delta V}^*(f^*) \text{sinc}^4(\pi V_{\text{LSB}} f^*) \end{array} \quad (5.39)$$

The demonstration is not included here, it may be found in [28]. Notice that if the left side was calculated properly, it is always defined on the whole f^* axis, and shows periodicity $2f_{\text{Nyq}}^* = 1/V_{\text{LSB}}$.

Now we can finally establish the connection between $S_{\delta V, \text{int}}^*(f^*)$ and $S_{\delta V}(f)$. To do this, we start from scratch, calculating the Fourier transform of $n(t) = \delta V(V(t))$

$$\begin{aligned} \tilde{n}(\omega) &= \int n(t) e^{-i\omega t} dt = \int \delta V(V(t)) e^{-i\omega t} dt = \int \delta V(mt + q) e^{-i\omega t} dt = \\ &= \frac{1}{m} \int f(V) e^{-i\omega(V-q)/m} dV = \frac{1}{m} e^{i\omega q/m} \tilde{\delta V}(\omega/m) \end{aligned} \quad (5.40)$$

We recall the definitions of time-domain and voltage-domain PSDs through Wiener-Khinchin theorem:

$$\begin{aligned} S_{\delta V(V(t))}(\omega) &\equiv \lim_{\Delta T \rightarrow \infty} \frac{1}{\Delta T} \langle \tilde{n}(\omega) \tilde{n}^*(\omega) \rangle \\ S_{\delta V, \text{int}}^*(\omega/m) &\equiv \lim_{\Delta V \rightarrow \infty} \frac{1}{\Delta V} \langle \tilde{\delta V}(\omega/m) \tilde{\delta V}^*(\omega/m) \rangle, \end{aligned} \quad (5.41)$$

so that, using Eq. (5.40),

$$\begin{aligned} S_{\delta V(V(t))}(\omega) &\equiv \lim_{\Delta T \rightarrow \infty} \frac{1}{\Delta T} \langle \tilde{n}(\omega) \tilde{n}^*(\omega) \rangle = \lim_{\Delta T \rightarrow \infty} \frac{m}{m^2 \Delta T} \langle \tilde{\delta V}(\omega/m) \tilde{\delta V}^*(\omega/m) \rangle = \\ &= \frac{1}{m} S_{\delta V, \text{int}}^*(\omega/m) \end{aligned} \quad (5.42)$$

So, we found an exact analytical relation between the PSD^* of the non-linearities and the time-domain PSD of the signal, converted by a linear voltage drift m , in presence of a Σ - Δ loop

⁴Note that the linear interpolation argument is not a simplification introduced by us, it is really happening because of the Σ - Δ ! The real approximation is assuming $10 \text{ Hz} \sim \infty$, but this has absolutely no influence on the low frequency behavior.

$$f = mf^*, \quad S_{\delta V(V(t))}(f) = \frac{1}{m} S_{\delta V, int}^*(f/m) \quad (5.43)$$

From this, we can immediately grasp that the higher m , the higher the frequencies that will be impacted by the non-linearities, but their spectrum will be suppressed by a factor m .

White DNL and INL impact on acceleration noise

We can now collect all the information gathered up to now to write a complete form for acceleration noise due to the simplified models of non-linearities that we introduced, white INL and DNL. We use Eq. (5.35), (5.43) and (5.39)

$$\begin{aligned} \text{General:} \quad S_{\Delta g}(f) &= \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \bar{V}^2 S_{\delta V(V(t))}(f) \\ &= \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \frac{\bar{V}^2}{m} S_{\delta V, int}^*(f/m) \\ &= \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \frac{\bar{V}^2}{m} S_{\delta V}^*(f/m) \text{sinc}^4(\pi V_{\text{LSB}} f/m) \end{aligned} \quad (5.44)$$

This is the general expression of the noise for 1 electrode, affected by non-linearities with a $S_{\delta V}^*(f^*)$ of arbitrary form.

If we focus on our two simple cases of white INL and DNL, we wrote their $S_{\delta V}^*(f^*)$ in Eq. (5.32) and (5.33), for which we obtain

$$\text{White INL:} \quad S_{\Delta g}(f) = \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \frac{\bar{V}^2}{m} 2V_{\text{LSB}} \sigma_{\text{INL}}^2 \text{sinc}^4(\pi V_{\text{LSB}} f/m) \quad (5.45)$$

$$\text{White DNL:} \quad S_{\Delta g}(f) = \frac{1}{4M^2} \left| \frac{\partial C^*}{\partial x} \right|^2 \frac{\bar{V}^2}{m} 2V_{\text{LSB}} \sigma_{\text{DNL}}^2 \frac{\text{sinc}^4(\pi V_{\text{LSB}} f/m)}{2 - 2 \cos(2\pi V_{\text{LSB}} f/m)} \quad (5.46)$$

To demonstrate that this actually works, we report again Figure 5.24, just V_{2x} , which exhibits a drift dominated time series, with $m = 3.3 \text{ mV/day}$, and compare the result of the simulation described in Section 5.4.1, with the analytic estimates obtained in this section.

Impact on LISA

Now that we demonstrated that the analytic formulas very well represent the effect of non-linearities on noise due to a voltage drift, we evaluate their impact on LISA. We have only ϕ actuation, and we said tha a drift in $V_{1\phi}$ must be linked to one in γ_{ϕ_c} . In particular we can calculate it in the constant stiffness algorithm as

$$V_{1\phi/2\phi} = \sqrt{\frac{I_{zz}(\gamma_{\phi_0} \pm \gamma_{\phi_c})}{\left| \frac{\partial C^*}{\partial \phi} \right|}} \rightarrow m \equiv \frac{dV_{\phi}}{dt} = \frac{1}{2} \sqrt{\frac{I_{zz}}{\left| \frac{\partial C^*}{\partial \phi} \right| (\gamma_{\phi_0} + \gamma_{\phi_c})}} \frac{d\gamma_{\phi_c}}{dt}. \quad (5.47)$$

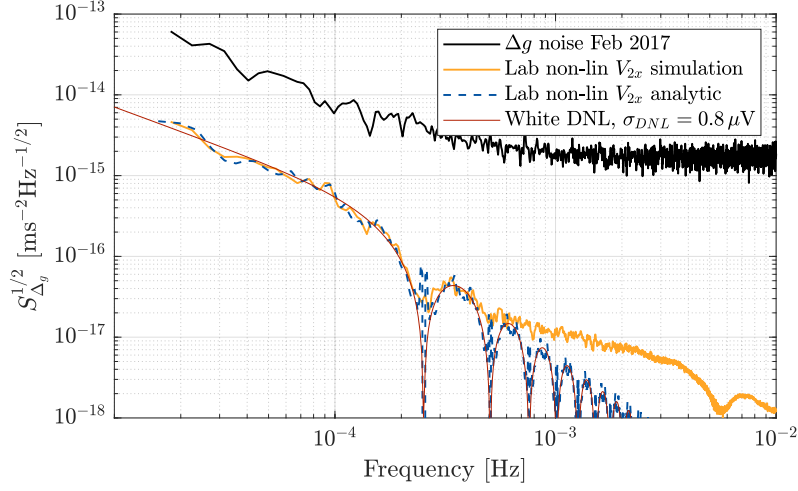


Figure 5.24: Δg noise from V_{2x} non-linearities measured in the lab. Orange curve is completely simulated, assigning the corresponding non-linearity when a certain command is used (so we need to know $V(t)$). The blue dashed line instead uses (5.44), with $m = 3.3 \text{ mV/day}$ and the $S_{\delta V}^*(f^*)$ calculated on the lab data (we did not need $V(t)$, only m). The red thin line is instead a completely analytic model from (5.46), assuming a white DNL of $0.8 \mu\text{V}$.

We also said in (5.37) that a source of $d\gamma_{\phi_c}/dt$ can be a dV_{TM}/dt due to cosmic ray charging. Assuming a worst case scenario charging rate of 50 e/s [11] and a $\Delta_\phi \sim 100 \text{ mV}$

$$\frac{d\gamma_{\phi_c}}{dt} = -\frac{1}{I_{zz}} \frac{dV_{TM}}{dt} \left| \frac{\partial C}{\partial \phi} \right| \Delta_\phi = -\frac{1}{I_{zz} C_{\text{tot}}} \frac{dQ_{TM}}{dt} \left| \frac{\partial C}{\partial \phi} \right| \Delta_\phi \sim 8 \text{ prads}^{-2}/\text{day} \quad (5.48)$$

In Figure 5.25 the noise curves projected for such drifts are compared to the LISA acceleration requirement in both the cases of white DNL and INL. Also the case $d\gamma_{\phi_c}/dt = 2 \text{ prad/s}^2$ is included to show how the noise changes: the higher the drift, the higher the frequency with which we convert the non-linearities into acceleration noise, so the higher the frequency band impacted. We see that both cases result in noise which does not violate the requirement and it impacts just the $< 0.1 \text{ mHz}$ band; still, it would rank as the second most impactful low-frequency actuation noise source after gain fluctuations. For this reason we thought it was important to include it in the LISA Performance and Error Budget [35], with an allocation of $d\gamma_{\phi_c}/dt = 2 \text{ prad/s}^2$, chosen so that the effect is pushed "just out" of the LISA lower band. We underline one last time that we think γ_{ϕ_c} drift sources can be kept under control. They should be linked to TM charging, interacting with stray DC biases: we can compensate the DC bias, but we can also keep the TM charge low with continuous illumination. Drifts in γ_{ϕ_c} due to rotations of the MOSA for the constellation angular breathing should be of much lower entity, happening on much larger timescales.

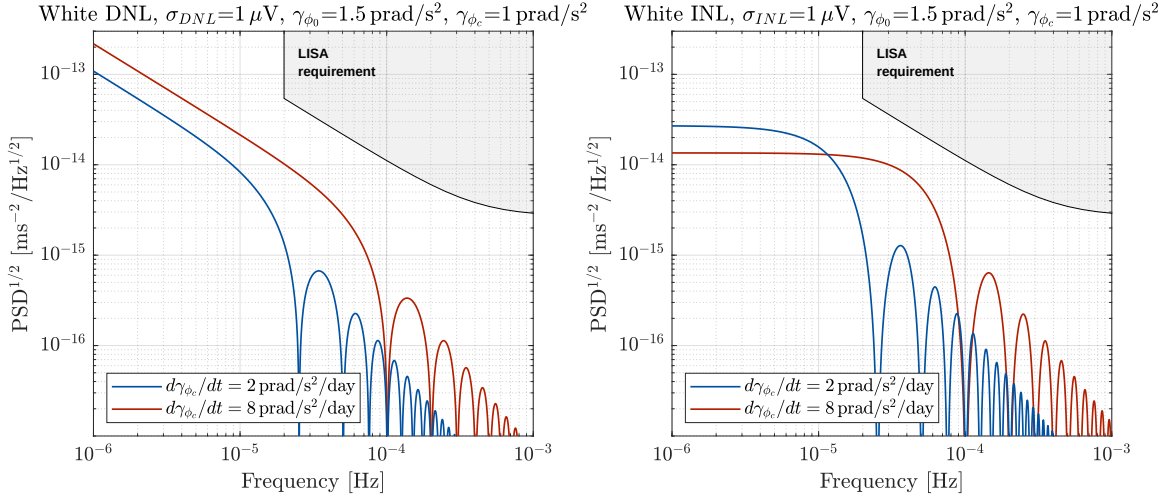


Figure 5.25: Acceleration noise in LISA due to conversion of voltage nonlinearities from a γ_{ϕ_c} drift. (Left) Effect of white DNL. (Right) White INL.

5.4.4 Effect of noisy commanded actuation

Now that we discussed the effect of drifting commands, we move to the case of γ_{ϕ_c} being a Gaussian, stationary voltage noise. How are non-linearities converted to acceleration noise in this case?

Let us start by comparing the amount of torque that was applied to the TMs of LPF with what will be applied to LISA, in Figure 5.26. We can see that the torque applied to each LPF TM ($\gamma_{\phi_{c2}}$ in the figure) was absurdly higher than the actual differential rotations of the two TMs $\Delta\gamma_{\phi_c}$: this was due to the noisy rotation of the spacecraft trying to noisily align with the star tracker, and dragging both the TMs with it. The digitization non-linearities converted by these noisy commands were the ones giving the dominant contribution to acceleration noise. In LISA, the commanded torque $\gamma_{\phi_c}(t)$ for each TM must comply with the requirement

$$S_{\gamma_{\phi_c}}^{1/2} < 0.47 \text{ rad s}^{-2}/\sqrt{\text{Hz}} \sqrt{1 + \left(\frac{60 \mu\text{Hz}}{f}\right)^2}. \quad (5.49)$$

much lower than LPF and set in order to limit crosstalk from applied ϕ torque into x acceleration, but it is relevant to this current question of acceleration noise from actuation non-linearities.

While in the previous case, it was possible to derive analytical results for the noise by assuming a deterministic, linear function for $\gamma_{\phi_c}(t)$ here we are dealing with stochastic processes, and the derivation of an analytical result is not straightforward. For example, can we expect that since the LISA $\gamma_{\phi_c}(t)$ noise requirement is lower than the one in LPF, then we will have less noise from non-linearities? Difficult to say!

We started with simulations for this case. We simulate many cases, to derive the dependencies of force noise on the relevant parameters: with those at hand, we could try to match

with some tentative analytic models. The simulations follow the following scheme:

1. Start from a certain $\gamma_{\phi_c}(t)$, generating the time series accordingly with the spectrum of the LISA requirement (5.49). In the LPF case, this could be the real telemetry.
2. Calculate the applied voltage through $V_{1\phi/2\phi} = \sqrt{I_{zz}(\gamma_{\phi_0} \pm \gamma_{\phi_c}) / \left| \frac{\partial C^*}{\partial \phi} \right|}$. Check how many codes (multiples of V_{LSB}) are being used.
3. Generate nonlinearities for each of the voltages used.
4. When non-integer multiples of V_{LSB} are required, we use again the linear interpolation between neighboring non-linearities, discussed in the previous sections, to approximate the action of the DFACS Sigma-Delta loop.
5. Repeat for each $V(t)$, building a time series $\delta V(t)$.
6. Calculate the force on each electrode by using

$$\delta F = \frac{1}{2} \frac{\partial C^*}{\partial x} V(t) \delta V(t) \quad (5.50)$$

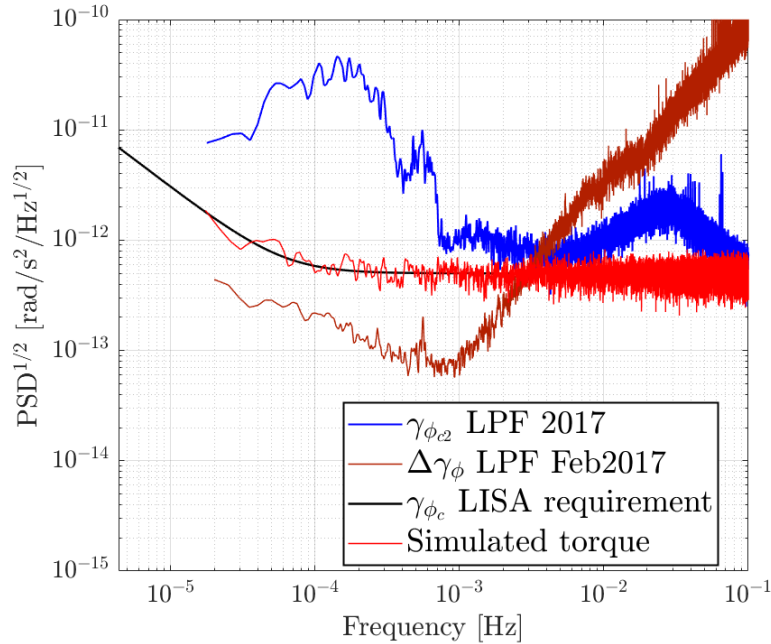


Figure 5.26: Comparison between the ASD of torque commands in LPF and LISA. (Blue) Torque applied on TM2 in LPF February 2017, the torque on TM1 is basically the same. (Dark red) Differential torque $\Delta\gamma_{\phi_c} = \gamma_{\phi_{c2}} - \gamma_{\phi_{c1}}$ in LPF February 2017. (Black) Maximum noise in commanded torque required for LISA. (Red) ASD of time-series simulated according to the requirement.

7. If few γ_{ϕ_c} commands are used, then a large variability in the result is observed among repetitions of the simulation. This is due to the fact that we can be "lucky" and randomly happen to extract small non-linearities for the few commands used, or "unlucky" on the other side. For this reason the simulations are repeated, all the results collected and usually represented as 68% interval.

Impact of a white DNL on LISA and LPF

The simulation results for LISA in presence of white DNL are reported in Figure 5.27, just assuming requirement (5.49) and for a bunch of values of σ_{DNL} for each electrode. We can immediately see that the contribution is negligible even in absurd conditions $\sigma_{DNL} = 10 \mu\text{V}$. In terms of relevance of the noise, results are not much different for white INL. LPF instead is another story; Figure 5.28 shows the simulated contribution of a white DNL with $\sigma_{DNL} = 0.7 \mu\text{V}$ on each electrode and for both ϕ and x DOFs (we already know that the x contribution could be described very well analytically). The contribution from ϕ actuation is instead the one that needed simulation and resulted on average a bit higher than the first simulations with laboratory measurement in Figure 5.20, probably because V_ϕ measurements in lab exhibited σ_{DNL} significantly lower than $0.7 \mu\text{V}$ (see Figure 5.14 and 5.15). Since we do not know what kind of non-linearities could have affected LPF FEE, this is an indication that in some pretty unlucky scenarios they could have been an important contribution at 0.1 mHz . However, it does not seem that the noise shape matches with the $S_{\Delta g}$ excess, so it is even less likely that this mechanism could be the explanation of LPF mystery noise.

The chapter could pretty much end here, since we evaluated the impacts of two limiting behaviors on LISA and LPF noise: a purely drifting command and a stationary noisy one. If both are present, one obtains some kind of average effect, since a larger noise on top of a drifting command tends to delete its effect. This is a well known phenomenon called "dithering": digital white noise is added on top of a clean audio signal which is being distorted by some non-linear effects. The noise averages over multiple non-linearities and as a result the signal is not distorted anymore.

What bothered me about these results is that both Figures 5.27 and 5.28 show simulated noise that has a shape *very* similar to their respective commands PSD, $S_{\gamma_{\phi_c}}$ in Figure 5.26. This made me really think that there could be an analytic approximate model to bypass the simulations. As anticipated, I could not find this relation except for the case of a white noisy command $S_{\gamma_{\phi_c}}$. Here I will briefly present the results, while the derivation is moved to the Appendix.

Analytic models for a white noise in commanded torque

At some point during the investigation, we were asked to provide at least some parametric fits to roughly represent how acceleration noise scales for LISA with $\sigma_{DNL/INL}$ and with $S_{\gamma_{\phi_c}}$, in a range of values that could be reasonable. For this purpose, Lorenzo Sala set up a

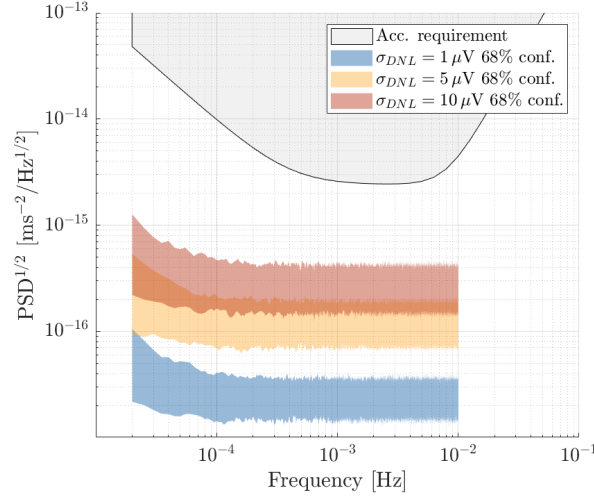


Figure 5.27: Simulation of acceleration noise in LISA, from actuation nonlinearities converted by a noisy torque command. The simulations exhibit a certain variability of the results, so they were repeated 100 times and the 68% at each frequency is shown.

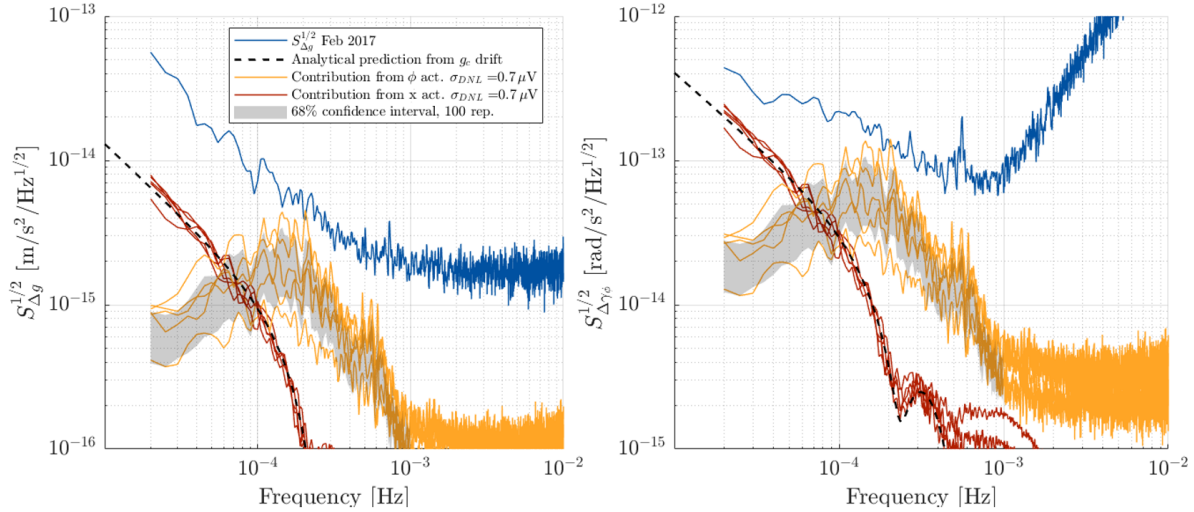


Figure 5.28: Simulation of acceleration noise in LISA, from actuation nonlinearities converted by a noisy torque and force commands. The analytical model correctly reproduces the x , drift converted part. The ϕ simulations exhibit a certain variability of the results, so they were repeated 100 times and the 68% at each frequency is shown.

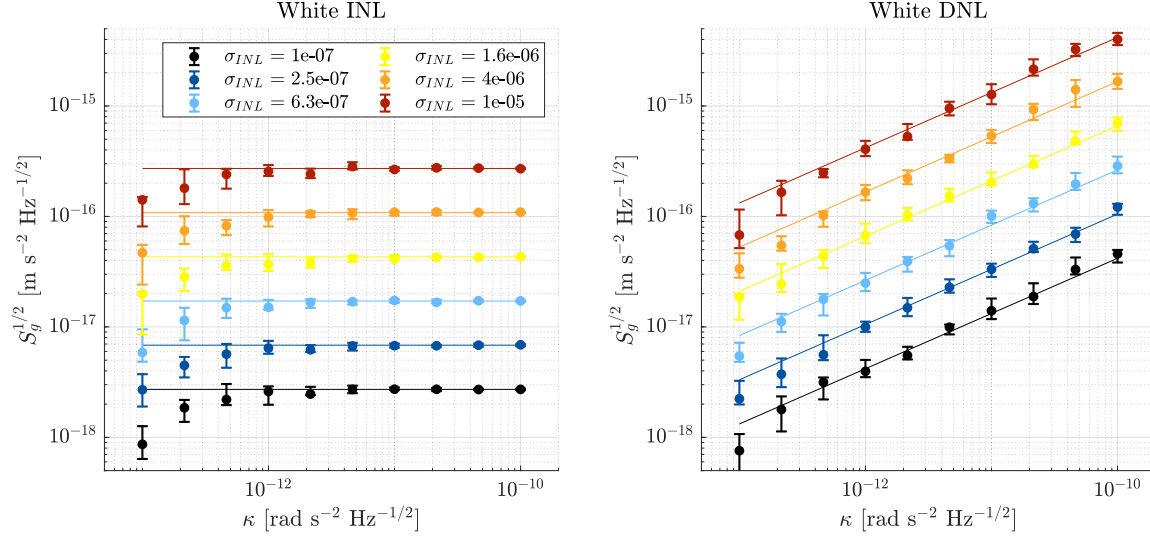


Figure 5.29: Average acceleration noise PSD $S_{\Delta g}$ from conversion of nonlinearities by a noisy command $S_{\gamma_{\phi_c}}$ defined in (5.51). Results of the simulations are shown as discrete points with error bars for many values of κ , white DNL (right) and white INL (left). Continuous lines are predictions from the analytic model for white noisy command.

long series of simulations like the ones presented in Figure 5.27, for many different values of $\sigma_{DNL/INL}$ and of the parameter κ , which just represents the white noise level of the LISA requirement (5.49):

$$S_{\gamma_{\phi_c}}^{1/2} = \kappa \sqrt{1 + \left(\frac{60 \mu\text{Hz}}{f} \right)^2}. \quad (5.51)$$

As we already saw in Figure 5.27, the resulting acceleration noise spectrum is almost white, so he just took the average value over the whole spectrum (with uncertainty) and included all the data as discrete points with error bars in Figure 5.29. It is possible to see first of all that the noise is linear in both σ_{INL} and σ_{DNL} . The behavior in κ instead may seem strange.

- For the white INL case, it seems to saturate to a plateau for high κ , so increasing the noisy command over a certain threshold does not increase this source of noise anymore!
- For the white DNL case instead, the noise seems to increase linearly in κ (in power).

The analytic models were derived through a series of physical intuitions and not from a rigorous mathematical treatment, which is still lacking. The reader is warned that the presented formulas may be incorrect for multiplicative factors, but we believe that the dependences on the relevant parameters are correct, as they agree almost perfectly with simulations.

Both are derived under two assumptions plus a small simplification:

1. The spectrum of commanded torque $S_{\gamma_{\phi_c}}$ is white and equal to κ^2 .

2. The command noise is high enough such that many voltage commands are used. The condition expressing this is:

$$\kappa \gg \frac{\sqrt{2/f_s} V_{\text{LSB}}}{\sqrt{\frac{I}{4 \frac{\partial C^*}{\partial \phi} \gamma_{\phi_0}}}} = 0.3 \text{ prads}^{-2} / \sqrt{\text{Hz}}, \quad (5.52)$$

where f_s is the sampling frequency and the standard parameters $V_{\text{LSB}} = 153 \mu\text{V}$, $\gamma_{\phi_0} = 1.5 \text{ nrad s}^{-2}$ were used.

3. $\langle \gamma_{\phi_c} \rangle$ is assumed to be 0, but the results can be easily generalized to non-zero case. This just allows to consider equal the noise from all electrodes, simplifying the already ugly expressions.

For both white DNL and white INL cases, we have:

$$\langle S_g^{1/2} \rangle = \frac{\sqrt{4}}{2M} \frac{\partial C^*}{\partial x} \sqrt{\frac{I \gamma_{\phi_0}}{\frac{\partial C^*}{\partial \phi}}} \sigma_{\delta V} \sqrt{\frac{2}{f_s}}, \quad (5.53)$$

where $\sigma_{\delta V}$ differs in the cases of white DNL and INL.

- For a white INL

$$\sigma_{\delta V, \text{INL}} = \sqrt{\frac{2}{3}} \sigma_{\text{INL}}, \quad (5.54)$$

meaning that the resulting acceleration noise level is white and independent of the level of torque noise! This model can be seen in Figure 5.29 (left) as solid lines. The approximation becomes better as κ increases, better fulfilling condition (5.52).

- For a white DNL, the expression becomes more complicated:

$$\sigma_{\delta V, \text{DNL}} = \frac{\sigma_{\text{DNL}}}{\sqrt{2}} \sqrt{\frac{1}{V_{\text{LSB}}} \sqrt{\frac{I}{4 \frac{\partial C^*}{\partial \phi} \gamma_{\phi_0}}} \frac{\kappa}{\sqrt{2/f_s}}}. \quad (5.55)$$

Using this result in (5.53), still result in a white noise, but its level shows now a dependence on κ , plotted in Figure 5.29 (right) as solid lines. The dependences V_{LSB} and γ_{ϕ_0} were also checked with simulations like the ones in Figure 5.29 and found highly compatible results.

The agreement of these models with the simulated data is very good exactly after the threshold (5.52): below this limit, the models act as upper limits for the noise, so they can still safely be used to realize projections of this noise for LISA. The other advantage of these formulas is that now we know how the noise scales with all the relevant parameters of the system. The feeling is that these formulas must admit some kind of generalization to an arbitrary spectrum, there was simply not enough time for me to demonstrate if there is or not, and would be cool to do it in the future. The derivation of these formulas is moved to Appendix 9.4.

Conclusions

In this chapter we investigated the impact of nonlinearities in the actuation waveforms amplitudes on LISA and LPF acceleration noise.

First, a well known digitization error, already removed from LPF, was described: possible inaccuracies in the subtraction were studied through simulations, finding that 1-2 μV errors could explain a large fraction of the LPF excess noise at low frequencies.

We looked for nonlinearities of such magnitude by conducting a measurement campaign on a LPF FEE prototype, the EM Light, finding large nonlinearities even of order 100 μV , showing correlations between successive commands. The differential non-linearities were instead found to be kind of uncorrelated, with standard deviations of $< 1 \mu\text{V}$. These ended up being real quantity impacting LPF noise, so in the end they could explain only a small fraction of the low frequency excess noise. A direct measurement also confirmed a sub- μV subtraction of the digitization error.

With the knowledge gathered through the measurements and simulations, simplified models were developed to calculate the impact of nonlinearities on LISA acceleration noise. We understood that the digitization error can be described as nonlinearities with *white INL* spectrum, while the additional nonlinearities measured in the lab are better described by a *white DNL*.

We discovered that these can cause acceleration noise if the commanded forces/torques drift in time; these drifts affected LPF and some will also be present on LISA, mainly linked to cosmic ray charging of the TM. It is possible to keep them under control through DC bias compensation: it is suggested to keep them under $2 \text{prad/s}^2/\text{day}$ for this source of noise to be completely irrelevant.

Finally, also a noisy commanded torque can convert nonlinearities to acceleration noise. This is expected to be completely irrelevant for LISA if the commanded torque noise requirement is respected. Approximate analytic models also for this case were derived.

Nonlinearities being added as a relevant source noise in the LISA noise budget means that the FEE will probably have to be tested to check that they are not exceeding the level of $\sigma_{DNL} \approx 1 \mu\text{V}$. It is clearly not possible to test all the commands of all the electrodes on LISA, so the nature of the tests are under discussion.

Chapter 6

Testing LISA actuation on-ground: the 1TM torsion pendulum

In these Chapter we abandon LISA Pathfinder and fall back to the ground, discussing what can we do on Earth to test acceleration noise of LISA TMs. In particular, the 1TM torsion pendulum facility is introduced and its current status reported. Then, we discuss an experimental campaign aimed at measuring actuation noise on ground with a strategy similar to LPF, presented in Chapter 4. The measurement constitutes a demonstration of a technique that can be used to test the performance of the future LISA Front End Electronics.

6.1 Small force measurements with torsion pendulums

It should be clear up to now that our interest is measuring the acceleration noise of a TM contained in an EH with large gaps, and our target sensitivity would be $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$. In the context of performing a macroscopic force measurement with the maximum possible sensitivity, torsion pendulums have proved to be the best instruments at our disposal. Why is it so?

In general, to measure a low-frequency force F , the best probe would be again a free-falling object, subject only to F itself. Its equation of motion for its x translation is simply

$$F(t) = m\ddot{x}(t) \xrightarrow{\text{Fourier}} \tilde{F} = -m\omega^2\tilde{x}. \quad (6.1)$$

By measuring $x(t)$, we can then reconstruct $F(t)$ with this simple equation. The first difficulty that we meet is that we can not have a perfect position measurement, but it will be affected by noise $x_n(t)$. This will act as a source of confusion, adding to the true position

$$\tilde{F}_{\text{meas}} = -m\omega^2(\tilde{x} + \tilde{x}_n). \quad (6.2)$$

If the true motion $\tilde{x}$ is not significantly higher than the noisy one, then it will be difficult to detect it. In other words, the spectrum of the measured force will be

$$S_{F_{\text{meas}}} = S_F + m^2\omega^4 S_{x_n} \quad (6.3)$$

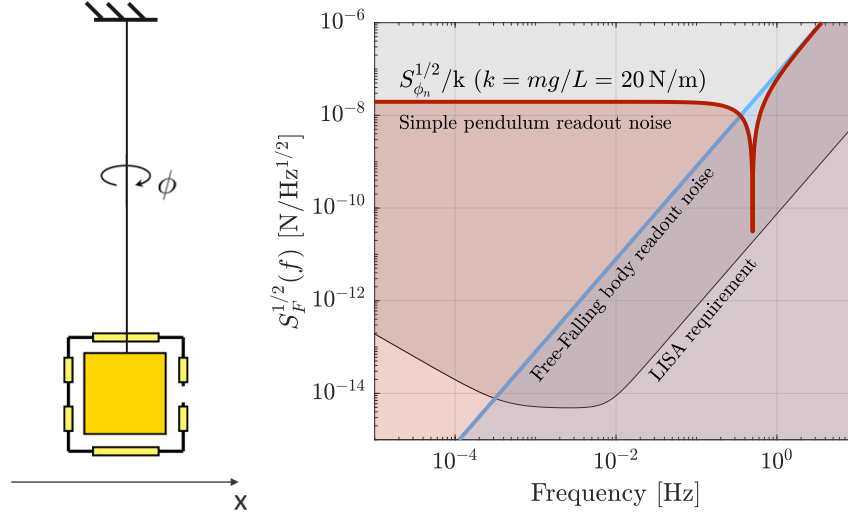


Figure 6.1: (Left) Sketch of a LISA TM suspended in a pendulum/torsion pendulum, inside an EH. (Right) Force noise ASD, comparing the required level of testing for LISA (gray area) with what can be achieved with a readout noise of $1 \text{ nm}/\sqrt{\text{Hz}}$ and a free-falling mass (cyan) or a simple pendulum of 1 m length on the Earth surface (red), looking at its x position.

supposing that readout noise is uncorrelated from other forces. It then becomes the only limiting factor to our measurement. Still, such a system would be perfect to study low frequency forces, since S_{x_n} is suppressed by a factor ω^4 , as depicted by the cyan line in Figure 6.1. Unfortunately, we already mentioned that finding a free falling body is a serious challenge, even in space: let alone putting a LISA like TM into an EH and simulate its free-fall on-ground. First of all, we are not in free-fall down here, the floor is pushing ourselves upwards with hundreds of Newton: could we think of putting the GRS onto a falling elevator and measure the TM motion? Absolutely not, since we are interested in the mHz frequency, so the elevator should be falling for hours. What we can hope for is a system that can simultaneously stop the fall of the mass along the vertical direction, to keep it in place in the EH, but leave the motion of the TM as free as possible along the $x - y$ directions. The first thing that comes to mind is hanging the TM with a thread inside the EH, as pictured in Figure 6.1, a pendulum.

This system is quite different from the one described by Eq. (6.1), its equation of motion being

$$m\ddot{x} = -\frac{mg}{L}(x - x_0) - \beta\dot{x} + F_{\text{ext}} + F \quad (6.4)$$

Here¹

- F is again the force we want to measure, linked to TM-EH interaction.

¹Notice that this expression is actually valid at first order for *any* particle in a well potential. One could think if there are other systems, different from a pendulum, which could achieve the same dynamics.

- $-\frac{mg}{L}(x - x_0)$ is the restoring term, in this case driven by gravity.
- x_0 is the equilibrium position of the pendulum, fixed by the position of the suspension point, which can change in time.
- $\beta\dot{x}$ is a viscous damping term.
- F_{ext} represents any disturbance that is not related to a TM-EH interaction or the pendulum dynamics.

Turning this again in power spectral densities, we have

$$S_{F_{\text{meas}}} = S_F + S_{F_{\text{ext}}} + \left(\frac{mg}{L}\right)^2 S_{x_0} + \left| -m\omega^2 + i\omega\beta + \frac{mg}{L} \right|^2 S_{x_n} \quad (6.5)$$

The first reason for which such a system can not work is that the restoring force, encoded in mg/L , is literally huge with respect to the forces that we want to measure. It is said to be too *stiff*. This means that at low frequency, the readout noise is $(mg/L)^2 g S_{x_n}$ and is not suppressed anymore as ω^4 in power, as compared with a free particle in Figure 6.1. This suggests that if we want to identify another candidate system, we have to take one in which the restoring force is as small as possible (after all $mg/L, \beta \rightarrow 0$ means that (6.4) $\rightarrow$ (6.1)).

The second reason is the transmission of suspension point x_0 motion to the TM, so all the seismic noise is directly converted to a force and it is known to worsen at low frequencies. To beat this, one would have to use another system that either is able to strongly suppress this transmission (with multiple pendulum stages for example), or that has a much more stable equilibrium position.

The third reason is again linked to the fact that gravity from surrounding masses is large with respect to the signals that we want to measure. Even a human of 80 kg at 10 m distance from a 100 g mass can cause a force of 5 pN. It is not easy to estimate what would be the contribution of this gravitational background, but it was certainly observed that this is a measurable source of noise during working days for example [30].

The system that emerges as a solution to these problems is one not driven by gravity, with a *very small restoring force* (since it fixes our low frequency sensitivity) and as much decoupled as possible from seismic noise and other gravitational disturbances. It is already under our eyes in Figure 6.1, we just have to look at the right degree of freedom: ϕ , making it a proper *torsion pendulum*. The equation of motion has the same form of (6.4),

$$I\ddot{\phi} = -\Gamma(\phi - \phi_0) - \beta\dot{\phi} + N_{\text{ext}} + N \quad (6.6)$$

which allows to write the torque power spectrum a

$$S_{N_{\text{meas}}} = S_N + S_{N_{\text{ext}}} + \Gamma^2 S_{\phi_0} + \frac{S_{\phi_n}}{|H(\omega)|^2} \quad H(\omega) = \frac{1}{-I\omega^2 + i\omega\beta + \Gamma} \quad (6.7)$$

but now Γ , the restoring constant of the fiber, is entirely fixed by the fiber properties. Its choice becomes crucial to the sensitivity of the apparatus in two ways:

1. The value of Γ depends on the material chosen, the length L and the radius r of the fiber as

$$\Gamma = \frac{\pi r^4}{2L} \left(E + \frac{mg}{\pi r^2} \right) \approx \frac{\pi r^4}{2L} E, \quad (6.8)$$

where E is the elastic modulus of the material. The typical E scale for metals is 100 GPa and the term of (6.8) $\propto m$ is nowhere near to these values for the typical fibers that we use, therefore can be safely neglected. We then understand that we must aim at making the fiber as thin as possible to lower Γ . Still we can not lower it more than

$$r > \sqrt{\frac{mg}{Y\pi}}, \quad (6.9)$$

with Y the yield strength of the material. As a result, it turns out that we are not allowed to hang an entire 2 kg LISA test mass, because this would force us to use fibers of hundreds of μm , which then do not allow to lower Γ enough. For this reason we use a hollow TM, as described in the next Section.

2. If the fiber has a temperature different from zero, thermal noise is inevitably linked to some dissipation, by the fluctuation-dissipation theorem. This behavior is well described in the frequency domain by introducing a spring constant with an imaginary part $\Gamma(1+i\delta)$, where δ is observed to be constant over a large frequency band [53]. This quantity can be equivalently expressed in terms of the quality factor Q as $\delta = 1/Q$. With a viscous damping term β due to residual gas around the torsion pendulum and fiber internal damping, a thermal noise floor arises

$$S_{N_{th}} = 4k_b T \left(\beta + \frac{\Gamma}{\omega Q} \right). \quad (6.10)$$

While β can be addressed by putting the apparatus in a high vacuum chamber, the internal friction is truly a limiting factor and must drive the choice of the fiber material with a very high quality factor. Therefore, if we were able to really isolate the pendulum from any disturbance, its ultimate noise floor in a torque measurement would be

$$S_{N_{\text{floor}}} = S_{N_{th}} + \frac{S_{\phi_n}}{|H(\omega)|^2} \quad (6.11)$$

If a low Γ will address the problems of readout noise, looking at the ϕ DOF also greatly reduces the coupling to seismic noise, since rotations of the suspension point are much more suppressed than linear motions. Finally, the use of a pendulum with a symmetric inertial member greatly reduces its interaction with the gravitational pull of surrounding objects: for example, the torque induced on a cubic test mass by the gravitational pull of a nearby mass will scale as r^{-5} .

These were all the ingredients needed to understand why a torsion pendulum is the right instrument to measure small, low frequency forces. However, we must underline that with the

simple apparatus proposed in Figure 6.1, we had to sacrifice the ability of measuring linear forces acting along x , to instead measure torques. Therefore, this 1TM torsion pendulum can only be used to measure forces with a non-zero lever arm, a huge limitation when we are interested in all the linear forces acting on the LISA TM. For that, the correct instrument has to be one which can convert all the linear forces acting on the TM into a torsion around a fiber. In this regard, we will briefly mention the 4TM torsion pendulum in the next section. Before moving to the actual description of the facility, we introduce some useful notation, that expresses the pendulum important quantities in terms of easily measurable ones. The transfer function, including the imaginary part of Γ and neglecting the viscous damping β is

$$H(\omega) = \frac{1}{-I\omega^2 + \Gamma(1 + i\delta)} = \frac{1}{\Gamma \left(1 - \frac{\omega^2}{\omega_0^2} + \frac{i}{Q}\right)} \quad (6.12)$$

where

$$\omega_0 = \sqrt{\frac{\Gamma}{I}} = \frac{2\pi}{T_0} \quad (6.13)$$

is the resonant frequency of the pendulum, linked to its period T_0 . The quality factor Q is related to the damping of the oscillation of the pendulum and roughly expresses the number of oscillations done by the pendulum until its energy is $1/e$ of the initial one.

6.1.1 1TM torsion pendulum

The facility was developed in Trento starting in 2000 and a series of people built, improved and used the instrument through the years, documented in various PhD theses [48, 58, 51, 41]. The core features of the facility were already in place when I started my PhD, so I had the opportunity to use a fully functional instrument to perform measurements with an already astonishing sensitivity. I happened to work on many parts of the facility to substitute and update them, as they were suffered some failures; now the pendulum is renewed in many parts. Some pictures and a sketch of the 1TM torsion pendulum can be seen in Figure 6.2.

- The central core is obviously the **TM and EH**, the subject under test; a picture is shown in Figure 6.2. The TM is a hollow, gold coated aluminum cube, and weights approximately 70 g; this allows the usage of a thin fiber. It makes the pendulum rather insensitive to disturbances that can couple to the bulk properties of the TM, for example magnetic fields, that however are usually originated outside of the GRS. A hollow TM maximizes instead the effect of surface interactions between EH and TM, which are the ones that we are mainly interested in. Isolated by a quartz ring, the TM is connected to the torsion fiber through a shaft, which passes through the z electrode housing hole and also hosts a gold coated-glass mirror, used to read the torsion of the fiber through an optical lever. There are also some "fingers" or "stoppers", to prevent the TM to rotate too much and hit the walls of the EH. The electrical isolation of the TM is important to reproduce flight conditions; it can be charged up or discharged

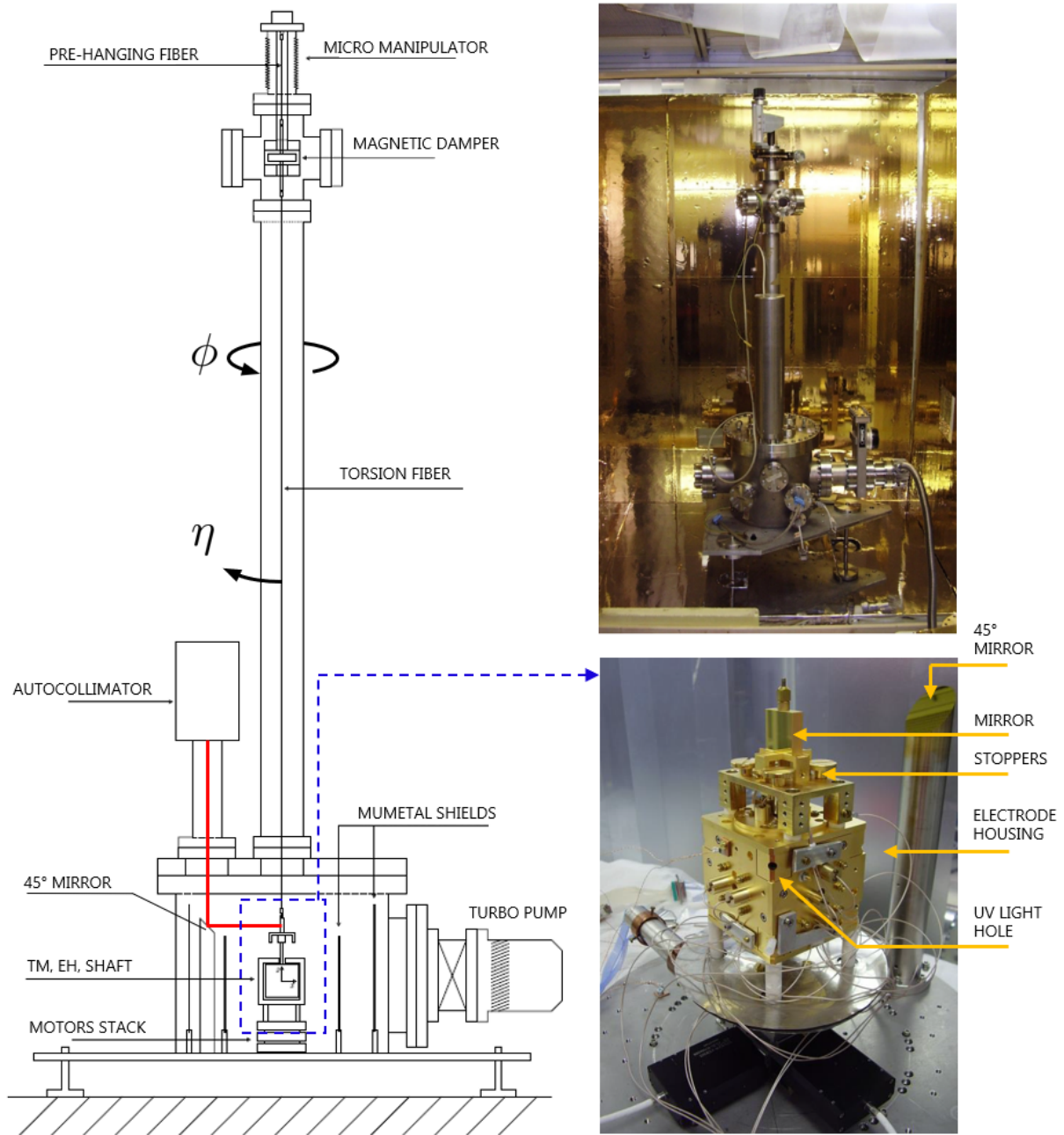


Figure 6.2: (Left) Sketch of the 1TM torsion pendulum, adapted from [48] and described in the text. It shows the important devices, but it is not a 1:1 reproduction of the real apparatus. (Top Right) Picture of the apparatus. (Bot right) Picture of the EH, with the shaft emerging on top.

trough injection of UV light with an optical fiber.

The EH is an early engineering model, so it is not the latest one that flew on LPF: the housing is gold-coated molybdenum, electrodes are gold-coated sapphire and the gaps along x , y , z are respectively 4 mm, 2.9 mm and 3.5 mm. We will see that this model is still good enough to perform specific tests, targeting for example actuation, since the geometry of electrodes and TM is the same.

- **The torsion fiber** is made of fused-silica, 40 μm radius and 1 m long. This is the same material used for the suspension of mirrors in ground-based detectors, and was chosen because of its low dissipation, i.e. high quality factor, that reaches values of $Q \approx 800\text{k}$. The thin fiber also allows a resonant frequency of $f_0 \approx 2\text{ mHz}$, meaning that the damping time is well above 5 years. Before this fiber, the pendulum used to have a tungsten one, but the thermal noise was reached and became the limiting factor to torque sensitivity, calling for a new, better performing one.
- **A vacuum chamber** contains inertial member an EH, and has a vacuum tube which contains the whole fiber. It is crucial to reach pressures $< 1 \times 10^{-7}\text{ mbar}$ to suppress viscous damping of the TM angular motion to negligible level with respect to thermal fiber noise. This is achieved through a turbo pump. The chamber sits on three, adjustable feet, placed on a concrete slab isolated from the rest of the laboratory; thus, human activity on neighboring slabs do not influence the pendulum tilt appreciably. However, especially temperature fluctuations cause the ground to tilt over hours time-scales, resulting in a substantial change of the $x - y$ equilibrium position of the TM, of order tens of μm . Effects causing cross-talk between $x \rightarrow \phi$ DOFs (called tilt-twist) can then be detrimental for the torque measurement. For this purpose, it is planned to substitute the chamber feet with piezo actuators and perform an active control of the chamber tilt at low-frequencies.
- The **suspension point** of the fiber is made of two stages: first, a 100 μm thick, 15 cm long tungsten fiber is connected to the vacuum chamber top, consisting in a vibration isolated micro-manipulator that can be manually operated to rise or rotate the fiber. After the thick fiber we have an aluminum disk, surrounded by toroidal magnets. These act as dampers for the *tilt* of the disk, through eddy current dissipation, but not the *twist*, given the symmetry of the disk. In this way, the swing motion of the pendulum has a decay time of approximately 70 s instead of several hours, and also acts as a filter to seismic vibrations of the suspension point.
- **An autocollimator** is positioned vertically on the chamber. This commercial device projects a wide laser spot on a surface: if this is tilted, the reflected beam has an angle with respect to the emitted one. The reflected light is collected again in the instrument and focused in a CCD camera, which then gives a *calibrated* measure of the reflection angle. The device position and light path can be seen in Figure 6.2; the autocollimator

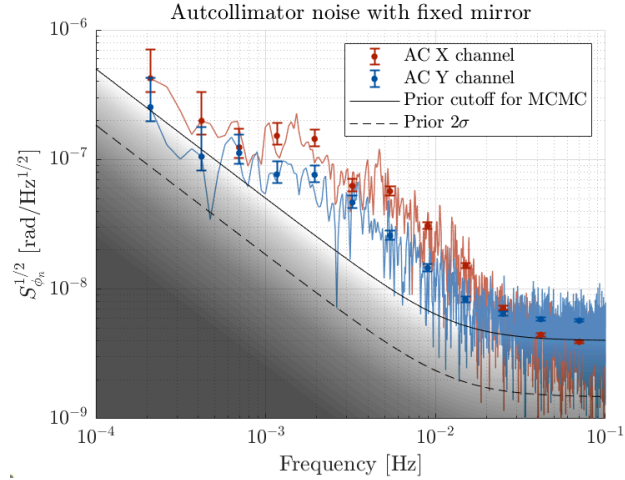
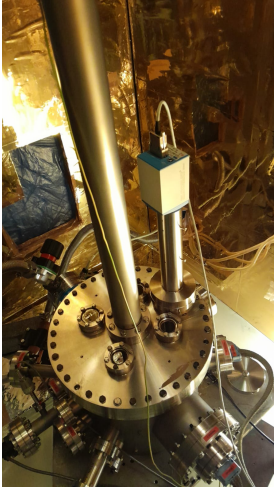


Figure 6.3: (Left) Picture of autocollimator positioned on the vacuum chamber. (Right) Noise in the AC angular readout for both axes. Later in the chapter, this measurement will be used to set a prior for parameters in some MCMC fits. The area that gets progressively more shaded represent the region in which a lower probability is assigned to the parameter: on the contrary, the non-shaded area represents where the parameter can explore freely any value.

is inserted in a cylindrical, vertical support, designed with very low tolerances and adjustable with some screws, in order to allow alignment with a 45° degrees mirror, which must then bounce the light on the shaft's mirror. The maximum range of the instrument is 5 mrad, therefore the TM must be aligned with the beam path well within this range, while also maintaining the best centering in the EH. A fine alignment of the optical lever also guarantees the suppression of cross-talk with other oscillation degrees of freedom. The model "Elcomat VARIO N 300/40" can measure with 25 nrad resolution on two axes: referring again to Figure 6.2 it measures ϕ on one axis and $\eta \propto x$ on the other. Readout noise of this device was measured fastening it to a granite table with a mirror mounted in front, and resulted in Figure 6.3. We can see that it is white at high frequencies ($S_{\phi_n} \approx 5 \text{ nrad}/\sqrt{\text{Hz}}$), but worsens at lower ones. The spectrum was observed to vary based on the conditions in which the autocollimator is placed, therefore we can not expect that this will be the exact readout noise of the device when mounted on the chamber.

Facility details, improvements and difficulties

Warning: In this section I will report some details regarding the facility. The apparatus was shut down in 2015 after almost 15 years of operations, and remained unused for 7 years: my first objective was to restore the pendulum to the performance it used to have before shut down. After restart, it suffered a series of major failures that showed up with an astonishing regularity, mainly due to aging of the instrumentation, often deleting part of the progress

Parameter	Description	Value
f_0	Resonance frequency	2.1 mHz
I	Moment of inertia	$4.31 \times 10^{-5} \text{ kgm}^2$
Q	Quality factor	$\approx 8 \times 10^5$
$\Gamma = (2\pi f_0)^2 I$	Stiffness	$7.5 \times 10^{-9} \text{ Nm/rad}$
P	Pressure	$< 4 \times 10^{-8} \text{ mbar}$
β	Gas damping ($4 \times 10^{-8} \text{ mbar}$)	$3 \times 10^{-12} \text{ Nm s/rad}$
r	Fiber radius	40 μm
	Fiber material	Fused silica

Table 6.1: Some parameters of the 1TM pendulum facility.

obtained in sensitivity up to that point. Facing them one by one, the facility is now renewed in many parts. However, reaching a stable level of sensitivity comparable with that observed 10 years ago was really the challenge of these years, and I think the current performance is very close, but I did not succeed completely; therefore, I need to include some details that may be of help to those who will attempt doing it again, or trying to improve the apparatus further. **The reader who is just interested in the measurements from this facility may skip to the next section.**

A top view of the facility can be found in Figure 6.4. The central vacuum chamber was already discussed in the previous section, here we discuss all the supporting instrumentation.

- **Thermal control:** The pendulum is caged in a chamber made of foam (in orange), to provide some isolation from the rest of the lab, which has a poor temperature stability of $\pm 0.5^\circ\text{C}$. The temperature inside the chamber is controlled through water circulation: some water flows in a copper coil placed in contact with the pendulum metal basement, some other water in a "custom" radiator equipped with fans to circulate air in the chamber. The radiator is placed on a scaffold which does not sit on the pendulum concrete slab, but outside, in order to not transfer vibrations. Also the turbo pump cooling fan was fastened to this scaffold, pointing towards the pump. Water is thermalized through a "Julabo" unit, which uses a PID with a thermometer placed on the pendulum basement as feedback.

A few other Pt-100 thermometers are placed in different parts of the apparatus to monitor the temperature: they are read by a digital multimeter, which has a serial connection to the acquisition computer.

- **Vacuum generation:** Despite being a high-vacuum system, the turbo pump is fastened to the main chamber using a viton o-ring and not a copper gasket, since we want to maintain electric isolation to avoid some ground loops detected in the past. A teflon sheet is also used between the flanges, to avoid accidental contacts; the screws are also wrapped in thin teflon. At the back of the turbo pump, a long vacuum tube connects to a secondary pump, which must establish order 0.1 mbar for the turbo to

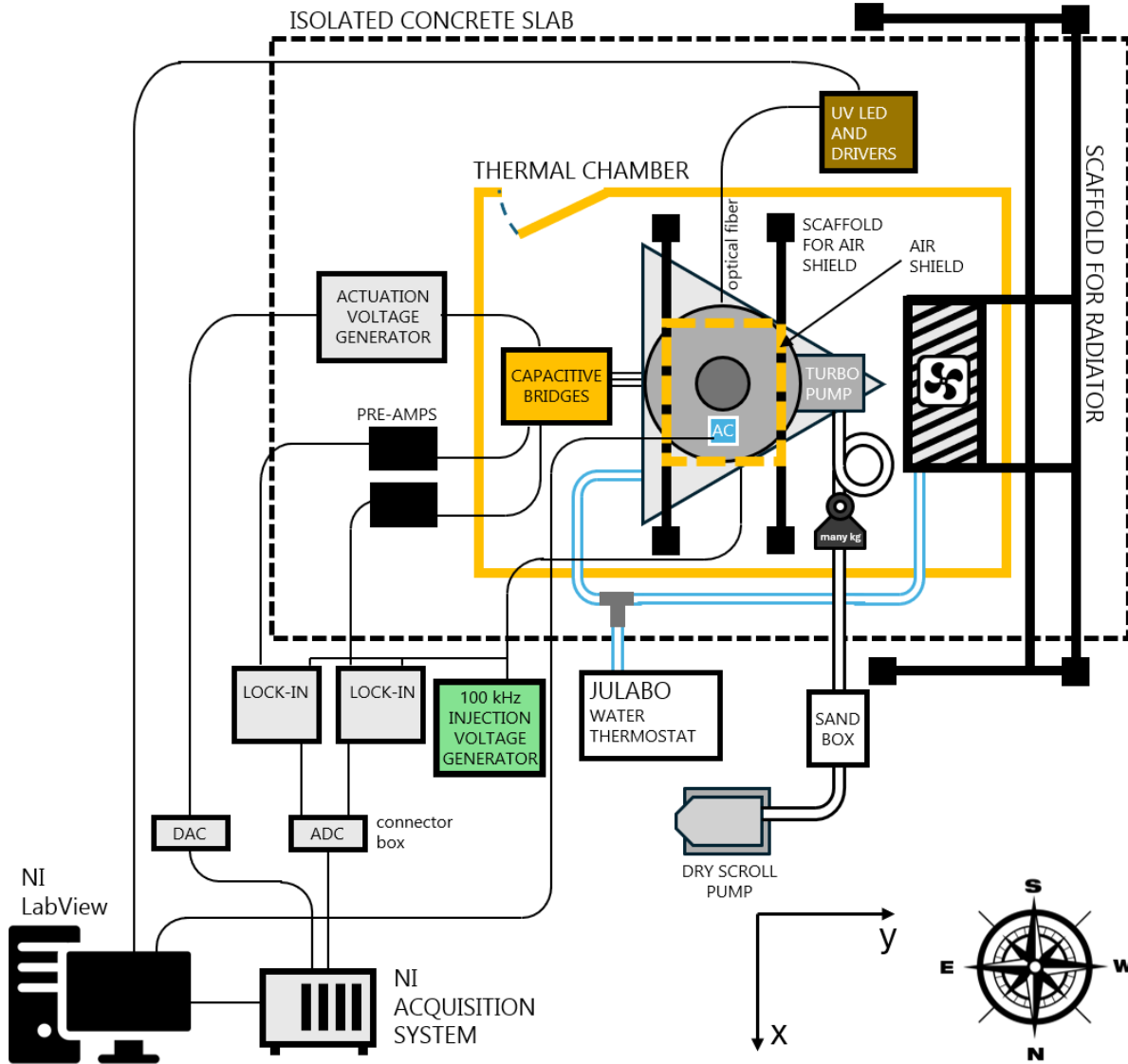


Figure 6.4: Schematic representation of the whole apparatus, described in the text. The central circle with a triangular basement represents the pendulum chamber in Figure 6.2. Not exactly to scale. The $x - y$ axes are the pendulum swing axes, the ϕ direction is anti-clockwise.

work properly. It is important that this vacuum tube does not transmit vibrations to the main chamber: these are suppressed in our apparatus by making the tube pass through a box full of sand, then the tube is again locked to the isolated concrete slab with a support and a heavy lead weight. It then runs a loop to leave the tube as soft as possible. I suggest to run a loop even between sand and lead weight in the future. With this system we were able to obtain pressures lower than 4×10^{-8} mbar.

- **The autocollimator** (AC) can be seen as a light blue square in Figure 6.4. This has a

direct serial connection with the acquisition computer and sends data at ≈ 30 Hz, read with a LabView routine. These are then not synchronous with the 10 Hz acquisition rate of the system: all the autocollimator 30 Hz data are anyway collected, averaged and re-interpolated to the system acquisition times in post-processing. Even not doing this, there are no effects at all at our frequencies of interest. The original autocollimator failed and the facility now mounts a new one.

- **Air currents shielding :** it is important to shield AC and vacuum tube from air currents generated by the fans in the chamber, otherwise it is likely to detect an increase in readout noise at low frequencies. For that, a foam shield was built to contain all above the main chamber in Figure 6.2, represented as an orange, dashed rectangle. This screen is not directly placed on the vacuum chamber but on a scaffold which transfers the weight on the floor. Also parts that can vibrate, like the tube between turbo and scroll pump, were shielded with some cardboard screens. Absolutely no fan must be mounted directly on the vacuum apparatus.
- **Acquisition system:** In addition to the optical lever, readout is performed by the capacitive bridges of the GRS, which also allow the application of actuation voltages. The system is a simplified version of the LISA and LPF FEE, and it is described in the next paragraph. To control which voltages to apply and read the ones coming from the sensing, there are a couple of ADC (PXIe-6341) and DAC (PXI-6704) boards, all part of a National Instruments system, connected to a computer which runs LabView. It allows acquisition of data and pendulum control at 10 Hz in quasi realtime, the main feature required from this system being the ability to measure continuously for very long periods of time (since we want to target low frequencies). During the last year I substituted the very old acquisition system with a new one and adapted the LabView routines of the 4TM torsion pendulum [30] to the 1TM facility.
- **UV Light:** it is necessary to control the TM potential and can also be subject of experiments. The facility was equipped a UV LED system provided by Roma Tor Vergata, including a pulsed LED current electronics drive. Furthermore, we also bought some optical fibers which can split the light 50-50, so I put in place a UV power measurement with a PMT (placed inside the thermal chamber for stability), which can be used to monitor light power (and phase!) during long measurements. A measurement campaign using LEDs will be discussed in Chapter 7.

Torsion pendulum capacitive sensing and actuation

The block diagram is presented in Figure 6.5. We start from the right of the Figure with a brief description of the sensing.

Sensing. A waveform generator supplies the $f_{inj} = 100$ kHz voltage, which is first fed to a transformer, in order to decouple the pendulum mass from that of the generator. The

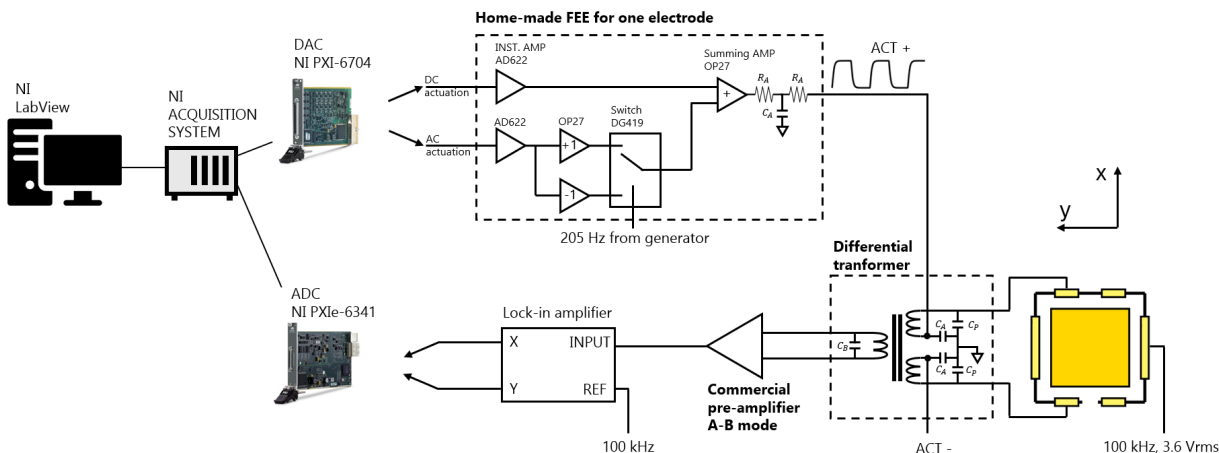


Figure 6.5: Block diagram of the capacitive sensing and actuation system used in the 1TM facility. (Boards images from NI website.)

wave then biases the TM through the injection electrodes: differential currents flow in the two primary coils of the differential transformer, generating a current in the secondary coil proportional to the TM displacement, as described in Chapter 3. This part of the system is very similar to the LISA and LPF. The first difference appears now: in LISA, the secondary current is converted to a voltage through a transimpedance amplifier. This home-made electronics was a first prototype and did not have this feature, so it just reads the voltage across the inductor with a commercial pre-amplifier. The downside of this is that the circuit is a resonant bridge with a width of few hundreds Hz, so the output of the pre-amp is maximized only if f_{inj} is tuned to the circuit resonance frequency. This is not a problem on its own, but becomes one when we think that we have 6 resonant bridges that must resonate all at the same frequency. Each resonance can be tuned adjusting the capacitance C_B in Figure 6.5, but is still a little annoying because each bridge can be subject to different drifts, with temperature for example. They work fine, but it is planned to update these bridges when there is time.

The output of the pre-amplifiers, which is still at 100 kHz, is fed to some lock-in amplifiers to demodulate, to then be sampled at 10 Hz by the ADC board in the acquisition system². These voltage outputs are converted to TM displacement and rotation through a calibration matrix, obtained by fitting to the optical lever calibrated outputs in dedicated calibration runs, in which large oscillations of the pendulum are excited on purpose.

Actuation waveform generator. Voltages to control the pendulum are generated through the circuit on top of Figure 6.5, still "home-made". It is possible to generate DC voltages

²The lock-in amplifiers are digital, so their outputs could be directly sent to the acquisition PC, without the need of an ADC to resample them. This last -maybe not optimal- procedure was adopted in the past as a simple solution for "real-time" sampling and control of the pendulum positions. The ADC sample all the lock-in outputs synchronously, instead of receiving un-synchronized data from all the different lock-ins. It is much easier then to also synchronously apply DAC voltages for pendulum control.

for all the 12 actuation electrodes, while AC is available only for x electrodes: these are not sines and cosines, but just square waves at 205 Hz filtered with simple low-passes. The generation happens as follows. We use DC voltages from a stable NI DAC board (PXI-6704), for a total of 16 (12 DC + 4 AC for x); while the DC commanded voltages $V_{DC,i}$ are straight buffered and sent to the electrodes, the AC voltages are obtained by feeding the DAC DC signals (which we call $V_{AC,i}$) to a switch, that alternates between ± 1 gain at 205 Hz, and then is summed to the DC part. The obtained voltages are finally injected in the capacitive bridges (Figure 3.3) after filtering, we indicate them as

$$V_i(t) = V_{AC,i} \text{filt_square_wave}(\omega_i t) + V_{DC,i} . \quad (6.14)$$

Since we know that the applied forces are $\propto \langle V_i^2(t) \rangle$, it is important to know the value of $\langle \text{filt_square_wave}^2(\omega_i t) \rangle = 0.86$, which was measured just by using the RMS function of the oscilloscope.

6.1.2 Pendulum torque estimation and noise floor

Now that we have presented the whole facility, we discuss its current sensitivity. We recall that our main interest is torque noise, which can be then converted to force noise by assuming a proper arm of the force under study: if it is actuation noise, then the lever arm of one electrode on the TM is approximately 1 cm. What we measure is the angle ϕ and from this we want to obtain the torque.

Let's assume that we want to do a purely noise run to estimate the pendulum sensitivity, so we do not apply anything at all, and let the pendulum swing freely, possibly with a small initial oscillation. The equation regulating the dynamics is (6.6), which we rewrite here

$$I\ddot{\phi} = -\Gamma(\phi - \phi_0) - \beta\dot{\phi} + N_{\text{ext}} + N , \quad (6.15)$$

where again $\phi_0 = \phi_0(t)$ is the equilibrium position, that can drift in time, N is the torque noise due to interaction between EH and TM, which we want to study, and N_{ext} are all the other disturbances. Unfortunately, we can not distinguish between these three contributions, because they all result in a torque! Therefore, we unify them in a single quantity $N_{\text{tot}}(t)$ and simplify to

$$N_{\text{tot}}(t) = I\ddot{\phi} + \Gamma\phi + \beta\dot{\phi} . \quad (6.16)$$

We know I from geometrical calculations . We can measure Γ by exciting a large oscillation: the damping is so low that the resulting motion is almost a perfect sine, which can be fitted and the resonance frequency extracted. We can also measure β by measuring the amplitude of the oscillation for days. Typical values for these parameters are given in Table 6.1. So to calculate the spectrum of $N_{\text{tot}}(t)$, one is tempted to first calculate the spectrum S_ϕ of the measured $\phi(t)$ and then use the pendulum transfer function to get

$$S_{N_{\text{tot}}}(\omega) = \frac{S_\phi(\omega)}{|H(\omega)|^2} . \quad \times \text{ We do not use this! } \times \quad (6.17)$$

In an ideal world, this works, but not in ours. Noise runs are often affected by transient events (in general, not only the pendulum), which can excite some oscillation of the pendulum, or cause a sudden change in equilibrium position. As a consequence, the $\phi(t)$ time-series is not purely noise, there can be signal inside, in particular the free dynamics of the the pendulum. When we Fourier transform this, *it will* compromise the noise spectrum $S_\phi(f)$, adding the contribution of the signals. Using (6.17) then leads to a poor estimation of the spectrum.

The solution in such cases is to subtract the signal from the time-series, and that is sort of what we do. We calculate $N_{\text{tot}}(t)$ directly in time-domain using (6.16), estimating $\dot{\phi}$ and $\ddot{\phi}$ with numerical methods: in this way, the pendulum dynamics is "subtracted" time by time. If there is a short transient, this is not part of pendulum simple dynamics, so it will show up in $N_{\text{tot}}(t)$ as a spike or something similar. But this will not cause a permanent signal that spoils the spectrum at all frequencies, and it can even be cut from the data if it is too big or of known origin. Only after we calculated $N_{\text{tot}}(t)$ we can compute its spectrum. Summarizing, the procedure to calculate torque is

$$\text{Measure } \Gamma, \beta, \phi(t) \rightarrow \text{Calculate } N_{\text{tot}}(t) \text{ with (6.16)} \rightarrow \text{Calculate } S_{N_{\text{tot}}} . \quad (6.18)$$

There is another advantage to this: Γ is not only determined by the fiber stiffness (see Table 6.1), but we said that if voltages are being applied to the electrodes, we also induce an electrostatic stiffness (see fore example Section 3.3.5). If the voltages are being changed at low frequency (for example in a modulation experiment) it can happen that $\Gamma = \Gamma(t)$. With the just proposed method, there is absolutely no problem with this, we just have to know $\Gamma(t)$ and the procedure works the same.

Pendulum noise floor

We can move on to discussing the pendulum torque sensitivity, which is reported in Figure 6.6. These measurements are just done by damping the pendulum to its current equilibrium position with the PID controller, then it is released with a residual oscillation of order $\sim 0.1 \mu\text{rad}$. A good measurement can be performed during an entire weekend, when nobody is around. The oscillation usually pumps up during the run, up to $1\text{-}2 \mu\text{rad}$, due to changes in equilibrium position drifting with temperature or just from the noisy torque. A typical time-series of ϕ and x measured through the optical readout is given at the top of Figure 6.6. We now discuss the feature of the torque spectrum.

- The solid blue line is the noise obtained at the beginning of 2024 with the optical readout, after the implementation of all the improvements discussed previously, except for the new acquisition system. This was the best noise floor I could ever obtain, highly comparable with the noise before 2015 ???. This performance was highly stable for 3 months, in which the measurement campaign described in Sec. 6.2 was carried out. After that, a long break happened because of the acquisition system failure and substitution, plus the failure of the air conditioning. After the pendulum was fully

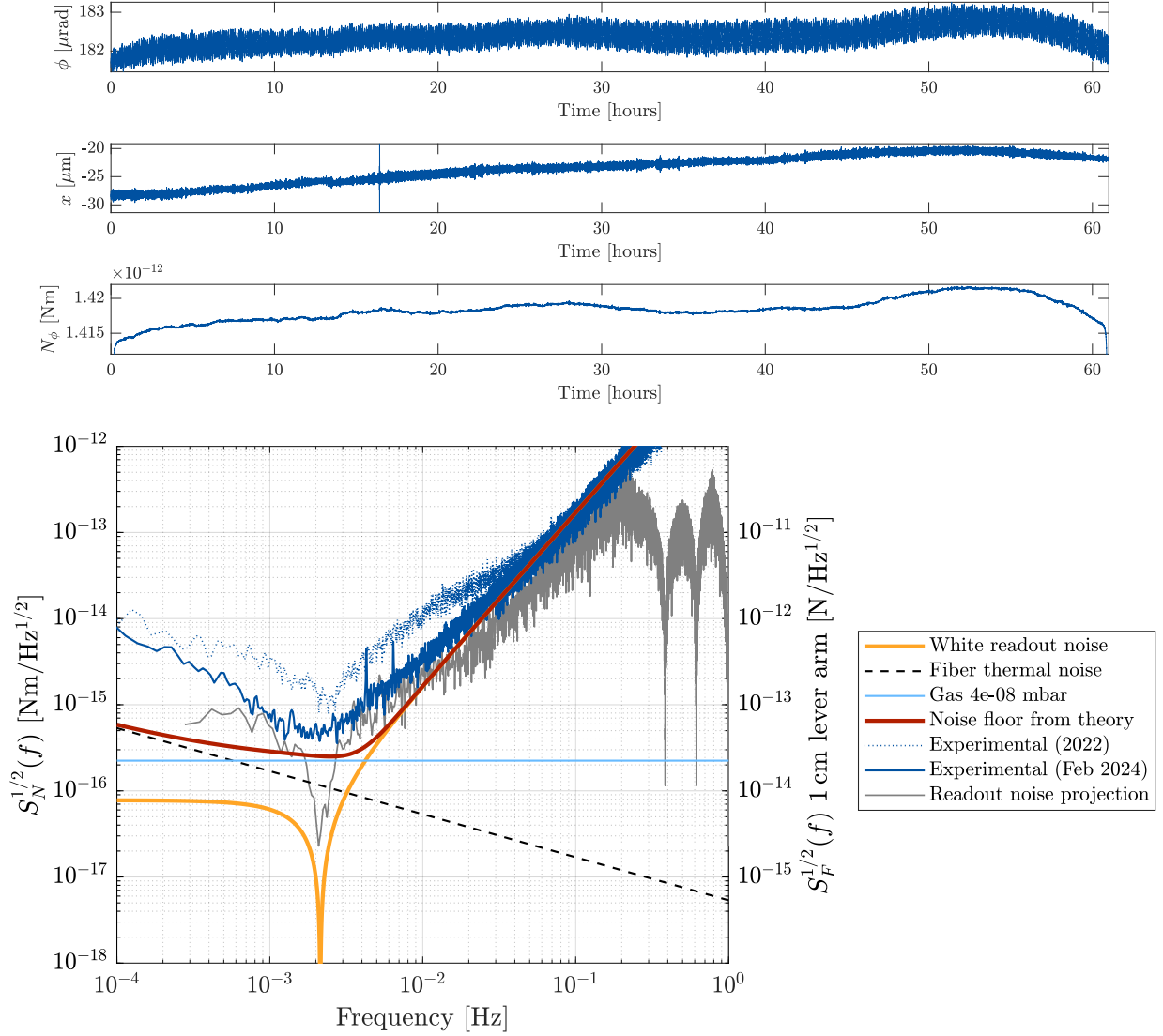


Figure 6.6: (Top) Time series of $\phi(t)$, $x(t)$ and reconstructed $N_\phi(1)$ (filtered at 1 mHz) during a sensitivity measurement run. The spike in $x(t)$ at ≈ 17 is not actually a spike, but a zoom would reveal that it has the typical shape of an earthquake. The time corresponds to a magnitude ML 3.5 Earthquake in Parma. Sometimes it is possible to see earthquakes from the whole planet if they are strong enough. (Bottom) Pendulum torque sensitivity (calculated from top $\phi(t)$) compared to theoretical models, described in the text. The right axis also shows the sensitivity to a torque caused by a force with 1 cm lever-arm, like actuation forces.

functional again, we measured a small degradation of the sensitivity of less than a factor ~ 1.5 with respect to Figure 6.6. The noise at resonance is unchanged, the degradation shows up at the neighboring frequencies: this can indicate a worsening of the readout noise. We are still trying to address this.

At approximately 4.2 mHz and 6.3 mHz, we can see two peaks. These are with all probability what we call "Autocollimator non-linearities", which cause distortions of the signal whenever a certain pixel of the CCD camera is crossed. Since $\phi(t)$ is a sinusoidal oscillation, it crosses these non-linearities twice every period, causing the appearance of harmonics. Sometimes it is possible to remove them; this is discussed in Appendix 9.5. At resonance, the sensitivity reached is $0.5\text{-}0.6 \text{ fNm}/\sqrt{\text{Hz}}$, corresponding to $50\text{-}60 \text{ fN}/\sqrt{\text{Hz}}$ for a 1 cm lever-arm force, as would be relevant for a noise source coupling through electronic noise on one or more X electrodes. Such a force, would cause an acceleration of a 2 kg LISA TM of $\approx 30 \text{ fms}^{-2}/\sqrt{\text{Hz}}$, just a factor 10 from the LISA requirement, and this is the great strength of the 1TM torsion pendulum, an unmatched instrument up to now in this matter.

- The blue dotted (almost shaded) line is instead the noise just after the pendulum was restarted, to show that there was an improvement over a large frequency band.
- The solid orange line is the theoretical model of readout noise, assuming a white noise of the autocollimator, and extrapolated from the f^2 trend of torque noise at high frequency. This level of noise is $S_{\phi_n} = (10 \text{ nrad}/\sqrt{\text{Hz}})^2$. First, we notice that the blue noise curve stops following the f^2 behavior for $f < 30 \text{ mHz}$, so there must be some excess noise. Is it readout noise or real forces? It is difficult to state it precisely, because to evaluate just readout noise, we would need to keep the TM perfectly still, something we can not do with our setup. However, I took the autocollimator measurement done on a table fastened to a static mirror exactly in front of the AC lenses, calculated the torque *as if* it was a measurement done with the pendulum, and obtained the gray curve of Figure 6.6. We see that it does not agree with the torque measurement at high frequencies, probably because of the different conditions in which the two measurements are realized. There could be more shot noise after the loss of some light in the optical path, stray light reflections, more vibrations from turbo pump and air currents. Still, it is unlikely that the AC can perform better than the gray line: this is a strong indication that around resonance we are most likely limited by AC noise. At resonance, instead, for sure we are not. At low frequencies, it is difficult to say. The analyses of the experiments proposed in this Chapter, may also help in answering these questions.
- As a black dashed line, the thermal noise from the dissipation in the torsional fiber is reported, calculated through (6.10) with a Q factor of 800k. Also the Brownian noise due to residual gas is included in formula (6.10), and obviously depends on the pressure. In particular, it can be calculated from theory for a cube placed in an infinite

volume of gas at fixed pressure P as

$$\beta_{\text{rot}}^{\infty} = P s^4 \left(1 + \frac{\pi}{12}\right) \sqrt{\frac{2M_m}{\pi k_b T}}, \quad (6.19)$$

with s the cube side and M_m the gas molecules mass, mainly water. However, it is well explained and experimentally demonstrated in [27] that Brownian noise is enhanced if the cube is placed in a constrained geometry, like the EH. In FIG. 5 of the same paper simulations are reported, and from that I estimated

$$\beta \approx 2\beta_{\text{rot}}^{\infty}. \quad (6.20)$$

This is the value used for the cyan curve in Figure 6.6.

- Finally the red solid line is just the sum of white readout noise and the Brownian noise, theoretically constituting the lowest noise floor possible. While we argued that some frequency ranges might be limited by the non-white part of the optical lever, there surely are contributions from other, unknown torque noise sources at very low frequencies and at resonance, since the theoretical Brownian noise do not seem to be quite big enough to be relevant in those bands.

Limits of the 1TM, motivations for a 4TM torsion pendulum

The main weakness of the 1TM torsion pendulum is that it can not measure linear forces. A force with no lever arm will not be seen by this device, and there are several effects which can do so, for example a temperature difference between to faces of the EH; or just think of the glitches detected on LPF, which were purely linear forces with no detectable arm [17]. Because of this intrinsic uncertainty, the 1TM is not the only instrument to test the LISA GRS. The role is shared with the 4TM torsion pendulum, which places 4 TMs in a + configuration, all at 10 cm from the fiber. One of the TMs is placed inside LISA EH, and any linear force acting on it induces a torsion of the fiber: this degree of freedom grants the same advantages discussed at the beginning of this Chapter. The price to pay is that the system is more complicated and noise sources are less easily rejected. Think that the 1TM inertial member is almost entirely contained and screened by the EH, while the + structure of the 4TM is much bigger and difficult to screen in all its parts. For these reasons, the instrument never reached the 1TM noise performance. If interested in this instrument, I point the reader to the theses of Davide Dal Bosco for the latest measurements [20] and of Giacomo Ciani for the facility development [30].

This said, the 1TM sensitivity is truly astonishing and allows to perform very precise tests, as we will see in what remains of this thesis. The facility can only measure torques, therefore it is ideal for high precision measurements of torque noise sources that also produce force noise (or can be configured to produce torque noise). Section 6.2 addresses exactly such a source. Furthermore, the pendulum can be used for precision measurements of other TM properties, for example the TM charge, which affect also the ϕ degree of freedom and not only x . This will be the topic of Chapter 7.

6.2 Enhanced actuation experiments

Chapter 4 was dedicated to the study of the performance of LPF electrostatic force actuation system, revealing that it is still the most important known source of noise, accounting for almost 40% of the observed acceleration noise. In particular, it caused a noise scaling with applied force.

This said, it is of fundamental importance to verify that the future LISA FEE satisfies the requirements with on-ground experiments before it flies. This was done also for LPF, with voltage measurements of gain fluctuations reported in [45], but the employed technique required actuation voltages from different TMs. For my Master's thesis [29] a technique to measure the exact combination of gain fluctuations entering LISA noise was demonstrated and proved to be highly suitable for the job: it also uses voltages of a single TM, allowing a simpler and fast testing procedure.

But what if there are other sources of force noise which fall out of our model and we do not detect with this homodyne, voltage measurement? Only a *force* measurement can remove this doubt, something that in LPF could only happen in space, through the observation of linear and angular acceleration noises caused by actuation and subject of Chapter 4.

In this Chapter instead, we demonstrate that **it is possible to use the 1TM torsion pendulum to directly observe torque noise from actuation**, indeed in a configuration sensitive to a combination of voltage fluctuations that is representative of the force noise for LISA. We can also check if it agrees with our gain fluctuations model. Obviously, we do not have the sensitivity to check this down to the level of LISA acceleration requirement, but with factor 10 increase in the forces we measure down to the gain stability levels needed for LISA.

In this case, observing torque instead of force noise is perfectly fine, since we know the nature and the arm-length of electrostatic actuation forces. In addition, we can safely use the current Electrode Housing present in the pendulum, even if it differs from the official one: this is because the applied forces are mainly linked to the voltage noise and just the geometry of the electrodes, which is actually the same. Therefore, the 1TM pendulum is perfect for this purpose.

Electrostatic actuation noise with 1TM

In Chapter 3, we described how ϕ actuation works for LISA. No net force is applied, however gain fluctuations α_i of the 4 x/ϕ electrode generate force noise given by (3.72), reported here

$$F_{x,\alpha} = \frac{1}{2} \left| \frac{\partial C_x^*}{\partial x} \right| [V_{1\phi}^2(\alpha_1 - \alpha_3) + V_{2\phi}^2(\alpha_2 - \alpha_4)] . \quad (6.21)$$

In the constant stiffness scheme, the maximum torques are achieved when $V_{2\phi} = 0$ and $V_{\phi} = V_{max}$, or vice-versa. The noise can then be entirely dominated by $\alpha_1 - \alpha_3$ or $\alpha_2 - \alpha_4$ in the extreme cases, it would be nice to study both the regimes.

In this situation, the torque would be

$$N_{\phi,\alpha} = \frac{1}{4} \left| \frac{\partial C_x^*}{\partial \phi} \right| [V_{1\phi}^2 - V_{2\phi}^2] + \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| [V_{1\phi}^2(\alpha_1 + \alpha_3) - V_{2\phi}^2(\alpha_2 + \alpha_4)] . \quad (6.22)$$

and this is what we would observe with the torsion pendulum. The first downside is that by setting for example $V_{1\phi} = 0$, we would measure the sum of α_i , so correlated fluctuations would not cancel. Secondly, we would apply a large torque to the TM, and keeping the pendulum in control range would require cumbersome and difficult operations.

To emulate a noise like (6.21) in the 1TM, it is more convenient to use x actuation: instead of applying a large torque to enhance actuation noise, we apply a large force, and it will not affect the pendulum motion, since completely driven by the much larger force of gravity. The torque obtained would be then

$$N_{\phi,\alpha} = \frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| [V_{1x}^2(\alpha_1 - \alpha_2) + V_{2x}^2(\alpha_3 - \alpha_4)] , \quad (6.23)$$

which is still not exactly equal to (6.21)³, but the difference $\alpha_1 - \alpha_2$ can be considered a good proxy for the quantity $\alpha_1 - \alpha_3$, since they come from nominally identical circuits. We can also study the regimes $V_{1x} = 0$ or $V_{2x} = 0$ without inducing any rotation of the pendulum.

To understand if the pendulum has enough sensitivity to detect this noise, we consider the simplified case of all uncorrelated gain fluctuations and $V_{1x} = V_{2x} = V_x$, obtaining

$$S_{N_{\phi,\alpha}} = 4 \left(\frac{1}{2} \left| \frac{\partial C_x^*}{\partial \phi} \right| V_x^2 \right)^2 S_{\alpha_{uc}} = 4 \left(I \frac{g_0}{R_\phi^*} \right)^2 S_{\alpha_{uc}} \quad (6.24)$$

As it was done in LPF, the clear strategy to reveal the underlying gain fluctuations is to increase the applied force g_0 , i.e V_x . In Figure 6.7, we show the expected noise observed on the 1TM, using $S_{\alpha_{uc}}$ according to its current requirement, given in (3.74), and a $V_x = 9$ V, corresponding to $g_0 = 4.7$ nm/s² for a 2 kg LISA TM. The right axis of the same Figure is normalized inverting Eq. (6.24) to $S_\alpha = S_{N_\phi}/4(\frac{1}{2}|\partial C_x^*/\partial \phi|V_x)^2$, in order to show the equivalent sensitivity of the pendulum in terms of gain fluctuations. The requirement at 1 mHz is ≈ 6 ppm/ $\sqrt{\text{Hz}}$, while the equivalent pendulum sensitivity is 3-4 ppm/ $\sqrt{\text{Hz}}$. The measurement is therefore pretty delicate.

Measurement demonstration with home-made electronics

Even if we do not yet have the LISA electronics, it was important to try this measurement to understand if there are further difficulties or things to be taken in account (oh, there are). To do this, we used the "home-made" electronics that are currently used to control the pendulum. The plan to perform this demonstration was the following:

³If we wanted, we could actually measure this by *physically* exchanging the FEE circuits driving electrodes 2 and 3!

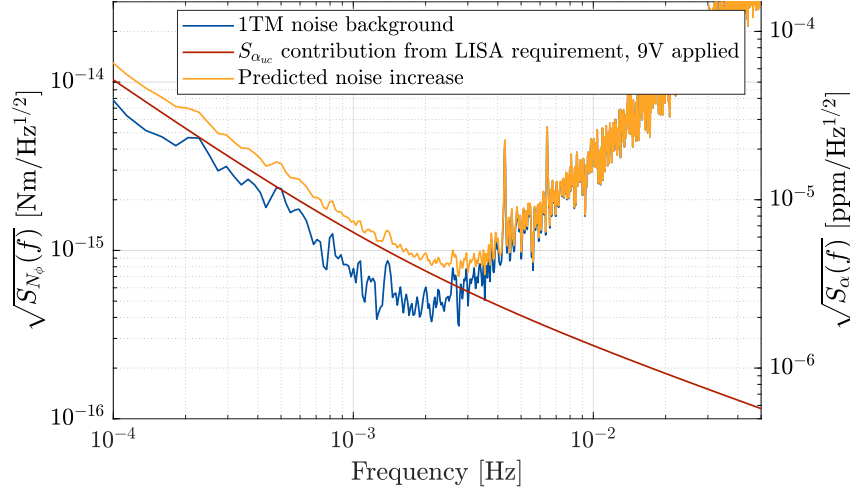


Figure 6.7: Expected noise increase in the 1TM torsion pendulum when applying 9 V, if LISA FEE is dominated by uncorrelated gain fluctuations at the level of requirement $S_{\alpha_{uc}}(f) = \left(50 \text{ ppm}/\sqrt{\text{Hz}}\right)^2 \left(\frac{0.1 \text{ mHz}}{f}\right)^2 + \left(4 \text{ ppm}/\sqrt{\text{Hz}}\right)^2 \left(\frac{1 \text{ mHz}}{f}\right)$. The right axis is instead normalized inverting Eq. (6.24), as described in the text.

1. **Torque measurements:** Measure the increase in pendulum noise in presence of large actuation, for different applied voltages. The voltages are chosen to resemble those of LISA, so AC and of opposite signs to maintain a neutral test mass; the difference is that they are filtered square waves, easily accounted for with a correction factor, and that we used different sign conventions, which do not matter for gain fluctuations. The expected torque (6.23) in this case is modified to

$$N_{\phi,\alpha} = 0.86 \left| \frac{\partial C_x^*}{\partial \phi} \right| \left[V_{1x}^2(\alpha_1 - \alpha_2) + V_{2x}^2(\alpha_3 - \alpha_4) \right], \quad (6.25)$$

2. **Gain noise estimation from torque:** From the torque measurements estimate the voltage fluctuations using our actuation noise model.
3. **Voltage measurements:** Measure directly voltage amplitude fluctuations (with a voltage measurement, not the pendulum) and compare the results with the estimates obtained from torque measurements.

We will see that for all the 3 bullets there were complications, that we try to introduce gradually.

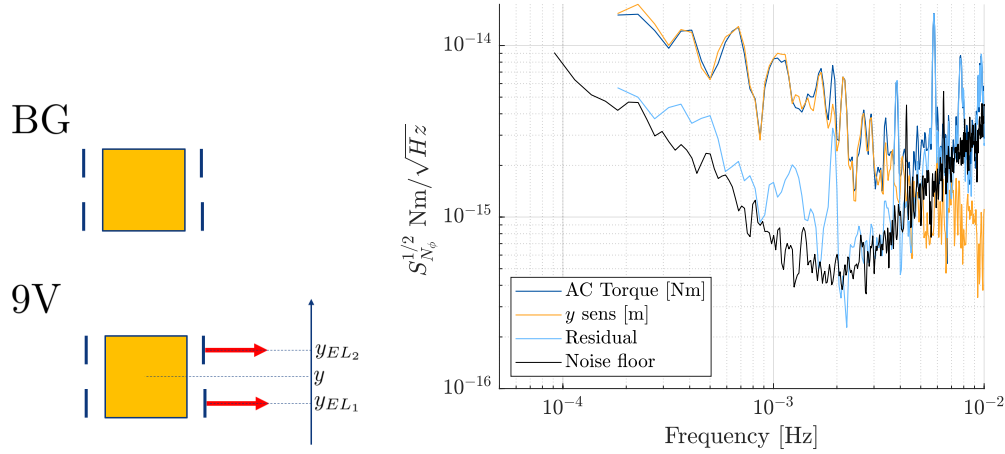


Figure 6.8: Experiment with just two electrodes. The dark blue PSD is the measured torque with optical readout, compared to the black noise floor of the pendulum. The orange PSD is that of $y(t)$ position, measured with the sensor, fit to the dark blue torque. The cyan line is the difference between dark blue and orange.

6.2.1 Actuation torque noise measurements

In February 2024, first trials of the measurement were done with just 2 electrodes, so $V_{2x} = 0$ and

$$N_{\phi,\alpha} = 0.86 \left| \frac{\partial C_x^*}{\partial \phi} \right| V_{1x}^2 (\alpha_1 - \alpha_2), \quad (6.26)$$

to see if we could resolve just the fluctuations $\alpha_1 - \alpha_2$. We detected a significant increase in noise, but it was soon discovered to be *fully* correlated with y position of the pendulum, completely hiding actuation noise. This is shown in Figure 6.8, together with a sketch of the TM and applied forces, which should explain the effect: a y variation changes the application point of the force, and therefore the arm-length.

$$N_\phi = (y - y_{EL1})F - (y_{EL2} - y)F \quad \rightarrow \quad \frac{dN_\phi}{dy} = 2F \quad (6.27)$$

For 9 V applied and our square wave actuation, $F = \frac{1}{2} |\partial C_x^* / \partial x| \langle V^2(t) \rangle \approx 7.7 \text{ nN}$. Do we recover the same number from the fit in Figure 6.8? Unfortunately, $y(t)$ comes from electrostatic sensing and we do not have an accurate calibration for this DOF. We can obtain a very rough estimate of this calibration by matching the spectrum of y to that of calibrated x , which in terms of pendulum tilt amounts to $\approx 0.7 \mu\text{rad}/\sqrt{\text{Hz}}$ at 1 mHz. After doing so, the fit coefficient recovered from Figure 6.8 gives $2F \approx 19 \text{ nN}$, which is not too far from the expected value of 15-16 nN. We could use the residuals from the fit to estimate actuation noise, but it would be a "dirty" way, vulnerable to fit procedure choice and to y noise conditions. Therefore, we did not proceed with 2 electrodes measurements. They would probably be possible with an active tilt compensation of the pendulum y position!

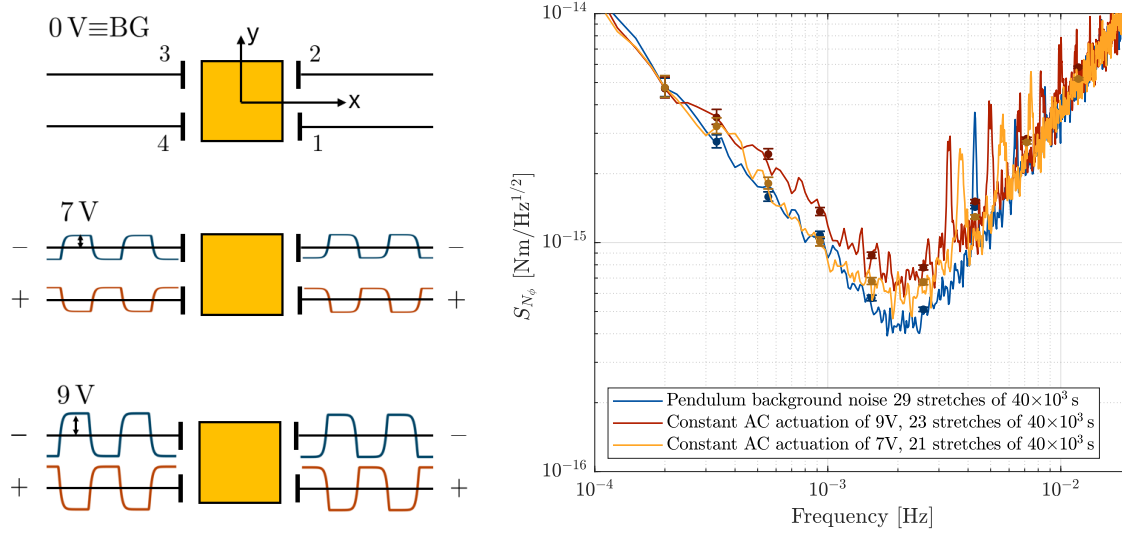


Figure 6.9: Torque noise in conditions of enhanced actuation, compared with pendulum background noise. They were realized by alternating one night background, one night 9V, one night background, etc... plus the weekends. Counting the acquisition system premonitory failures, this took almost two months: the performances were really stable and lead to a good resolution of actuation noise. No cross-talks were detected with other DOFs.

This effect is not present in the configuration in which 4 electrodes are used with the same force ($V_{1x} = V_{2x}$), because the arm-length change of one electrode is compensated by that of the electrode on the other side of the TM. The results of measurements with 7 V and 9 V (corresponding to $g_0=2.8$ and 4.7 nm/s^2 in flight) are reported in Figure 6.9, compared with pendulum background noise. It was checked that they do not exhibit detectable correlations with other DOFs, and we really seem to resolve an increase in noise scaling with the applied forces. However, the noise increase was mild, milder than Figure 6.7. The home-made electronics, with NI PXI-6704 DAC card, are actually stable enough to meet the LISA requirements at these frequencies so the runs needed a lot of integration time to clearly resolve the noise. The noise excess at resonance seems to increase with the applied voltage, as expected. Below that, the excess is significant only for the 9 V run. We notice two further facts

- There seems to be an increase in noise even *above* resonance.
- The peaks related to autocollimator non-linearities at multiples of the resonance frequency could not be removed, given the length of the measurement period (see Appendix 9.5 for details). We see that they change positions in the 3 spectra. This was expected, since the application of large voltages changes the stiffness of the pendulum,

and move its resonance peak. We can even calculate the expected shift,

$$\Gamma = \Gamma_{\text{fiber}} - \frac{1}{2} \left\langle \left[\left| \frac{\partial^2 C_x^*}{\partial \phi^2} \right| (V_1^2 + V_2^2 + V_3^2 + V_4^2) - \frac{2}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial \phi} \right|^2 (V_1 - V_2 + V_3 - V_4)^2 \right] \right\rangle \quad (6.28)$$

but it is always more accurate to directly measure the resonance frequency (this was done for the 3 configurations, finding $f_{0V} = 2.14 \text{ mHz}$, $f_{7V} = 1.86 \text{ mHz}$ and $f_{9V} = 1.66 \text{ mHz}$).

This experiment can not thus be treated exactly like the enhanced actuation runs of LPF in Chapter 4. There, we had four experiments $q=1,2,3,4$, described as

$$S_{\Delta g}^q = S_{\Delta g}^{ACT,q} + S_{\Delta g,bg} \quad (6.29)$$

with a background noise $S_{\Delta g,bg}$ common to all of them, almost fixed by the UURLA run. Here, instead, we know that the background noise of the pendulum likely has a relevant contribution from readout noise, so it can be described as

$$S_{bg}^{1TM} = S_{N_n}^{bg} + \frac{S_{\phi_n}}{|H(\omega)|}, \quad (6.30)$$

therefore, if we change the resonance frequency, we change $H(\omega)$ and in turn also the background noise. When we try to fit our actuation noise model to the data, we can not use a model like (6.29), we should instead use

$$S_{N_\phi}^q = S_{N_\phi}^{ACT,q} + S_{N_n}^{bg} + \frac{S_{\phi_n}}{|H_q(\omega)|} \quad (6.31)$$

in which the torque noise background is common to all the runs; also readout noise S_{ϕ_n} is, but it is weighted differently for each run by $H_q(\omega)$.

6.2.2 Gain fluctuations estimation from torque measurements

In the hypothesis that the resolved increase in torque noise is due to gain fluctuations, we now proceed to extract an estimate of them, using model (6.31), which takes in account the different readout noise contributions to torque noise in the three runs. Eq. (6.25) describes how gain fluctuations α_i impact torque noise; in our case, we are applying the same voltage $V_{x,q}(=0,7,9 \text{ V})$ to all the x/ϕ electrodes so it reduces to

$$\begin{aligned} N_{\phi,\alpha} &= 0.86 \left| \frac{\partial C_x^*}{\partial \phi} \right| V_{x,q}^2 [(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)] \\ \rightarrow S_{N_\phi}^{ACT,q} &= \left(0.86 \left| \frac{\partial C_x^*}{\partial \phi} \right| V_{x,q}^2 \right)^2 S_{(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)}(f) \end{aligned} \quad (6.32)$$

$S_{(\alpha_1-\alpha_2)+(\alpha_3-\alpha_4)}(f)$ represents the quantity that we want to recover. Now that the model is fully defined, we use the same Bayesian Monte Carlo used in Chapter 4 to estimate the posterior distributions of the quantities appearing in model (6.31) at each frequency (Minimally Correlated Frequencies MCF). We analyze frequencies up to ≈ 40 mHz, and unfortunately we have to cut out those between 3 and 10 mHz due to the presence of autocollimator nonlinearities, remaining with a total of 9 frequencies.

Results

The three torque noise runs in Figure 6.9 exhibit a very mild increase in noise, therefore the extraction of just actuation noise is expected to be difficult. In fact, we find some variability of the results depending on the priors we decide to use; after many trials, it was not difficult to realize that the variability was especially linked to the degeneracy between background "true" torque noise $S_{N_n}^{bg}(f)$ and readout noise $S_{\phi_n}(f)/|H_q(f)|$. As already explained, these quantities affect our experiments differently between the runs, so they should be resolvable. However, we will see that the experiments are pretty much dominated by actuation noise, so the differences between the two backgrounds end up being hidden and the fit is not sensitive to them.

We propose three analyses, which progressively inject more information on readout noise through specific prior choices.

1. A first fit in which we assume that the background noise of the pendulum does not actually have a strong contribution from readout noise, which is simply removed from the MCMC. At each of the 9 frequencies, we have then 1 background parameter and one for gain fluctuations, for a total of 18. Results for this fit are reported in the left panels of Figure 6.10. Obviously, the background run ends up completely fixing the torque noise floor, since it has no actuation noise, which will be fixed instead by the other two runs. The 68% confidence interval of the contribution from actuation noise is in red and it is well resolved between 0.3 and 3 mHz, constituting the dominant source.
2. A second fit is performed by putting a prior on $S_{\phi_n}(f)$: we approximate the trend of the autocollimator noise measured in Figure 6.3 and put it as a lower limit (using a smooth Gaussian to still allow for some freedom of the parameter, see App. 9.3, Eq. (9.30) for more information). So, we are somehow trying to inject the information that readout noise can not be much smaller than that. Still, we are not assuming any particular shape for $S_{\phi_n}(f)$, which can move independently at each frequency (9 parameters). We use the fiber thermal noise, plus damping, as a lower limit for true torque noise instead (9 parameters). Including the 9 gain noise parameters, we have a total of 27. Results are reported in the central panels of Figure 6.10. This more severe prior can seem to remove some degeneracy between $S_{N_n}^{bg}(f)$ in cyan and $S_{\phi_n}/|H_q(f)|$ in orange around resonance, but not at lower frequencies; the posteriors shapes are not reported, but they are often bimodal. Still, including both contributions, seems

to slightly increase the amount of actuation noise recovered by the fit (compare for example the red points at ≈ 1 mHz between the two 9V panels).

3. Given that the autocollimator noise in Figure 6.3 seems smooth, we also try to use an analytic function to describe readout noise:

$$S_{\phi_n}(f) = S_{\phi_n}^{\text{white}} + S_{\phi_n}^{\text{red}} \left(\frac{1 \text{ mHz}}{f} \right)^\xi \quad (6.33)$$

with $S_{\phi_n}^{\text{white}}$, $S_{\phi_n}^{\text{red}}$ and ξ as parameters, reducing the total number to 21. The results, in this case, clearly converge to a dominant readout noise and therefore also give a more accurate estimation of the other parameters. Notice that even in this case, true torque noise is never completely negligible (cyan points).

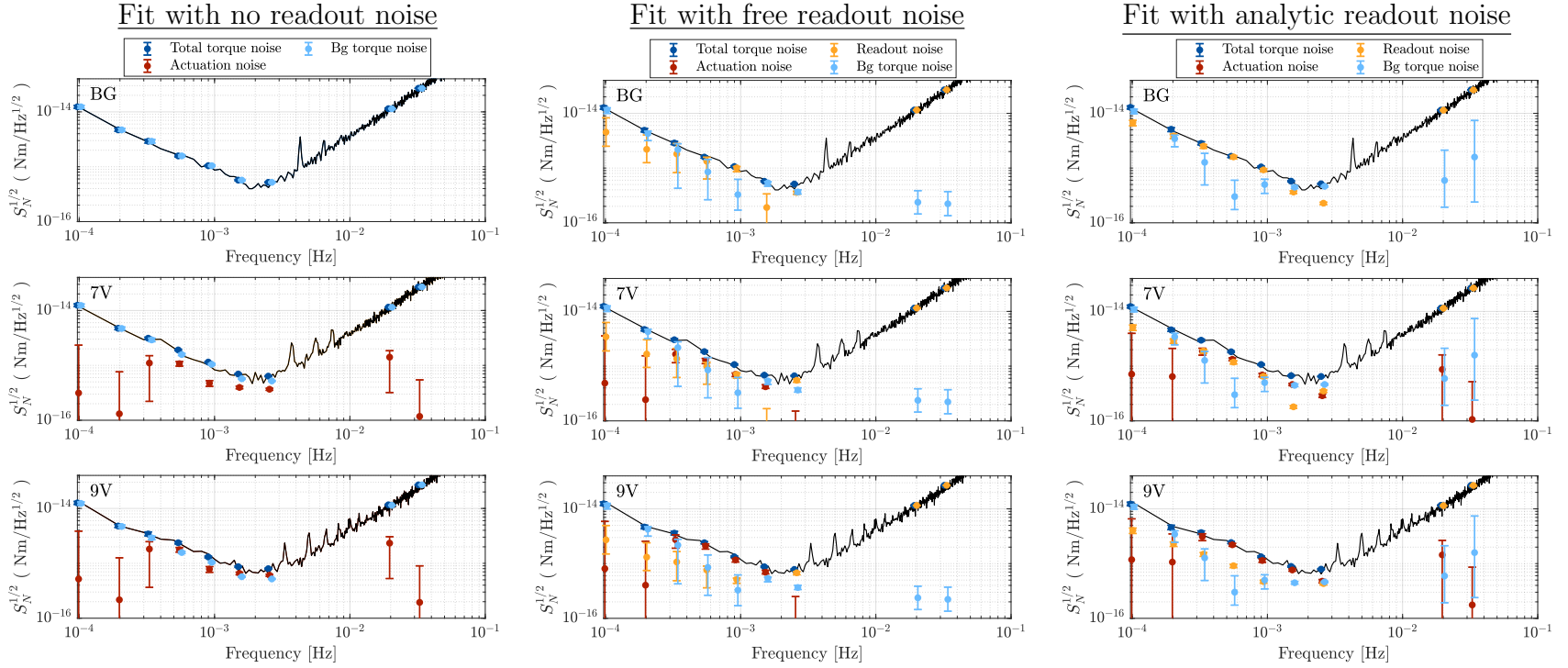


Figure 6.10: (Left panels) The three torque runs with enhances actuation, together with their posterior estimates (68% intervals) in case of a model which assumes negligible readout noise. Blue points called "total torque noise" just represent the noise reconstructed by the fit as the sum of all the model contribution: they are there to demonstrate that the fit is actually reconstructing the correct quantity. Other colors show the contribution from single noise components in the model written on top. The points are slightly shifted for better visualization. (Central panels) Exactly the same as left, but for a model which includes readout noise, limited below by a prior based on Autocollimator noise. (Right panels) A model assuming an analytic form for readout noise. Everything is explained in the text.

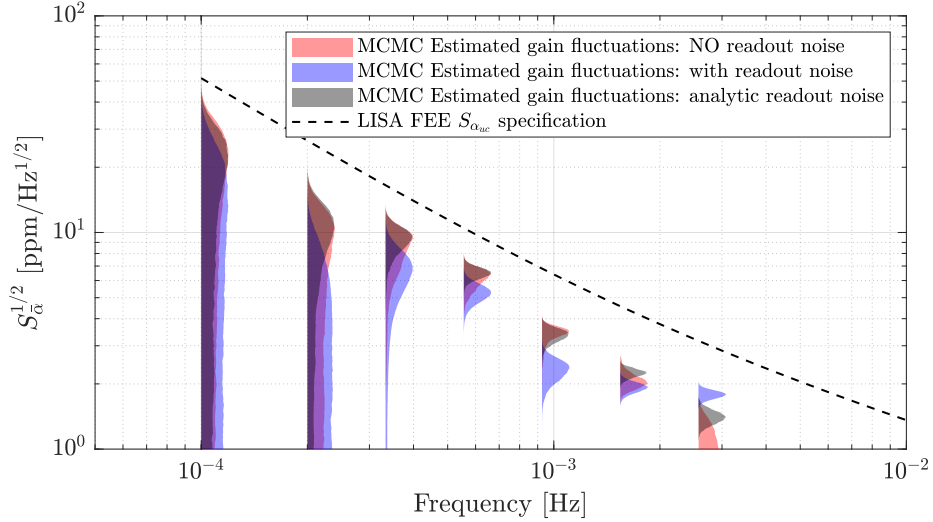


Figure 6.11: Average gain fluctuations per electrode $S_{\bar{\alpha}}$ recovered from the three different models of readout noise. The MCMC provides the posterior distributions for $S_{\bar{\alpha}}$ at each frequency, which are represented here as violin plots.

All these fits give valuable information but none of them can truly provide a final answer to the degeneracy between readout and torque noise. However, the gain fluctuations they extract are somehow solid, approximately consistent between the three fits. The posterior distributions of gain fluctuations extracted with the MCMC for the three fits are reported in Figure 6.11, where we defined the quantity

$$S_{\bar{\alpha}}(f) = \frac{S_{(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)}(f)}{4} \quad (6.34)$$

as the average fluctuations of one electrode, in order to compare with the LISA FEE requirement on $S_{\alpha_{uc}}$. We can see that the three different models lead to slight differences between the posterior estimates, which however almost always show overlap and less than 50% difference between their average values. At the two lowest frequency we do not detect a significant increase in noise, therefore the MCMC does not identify any contribution from actuation, providing instead relevant upper limits: these are still useful, since they clearly allow to state if the FEE under test passes it or not. For example, our home-made FEE would meet the LISA actuation stability requirement. At the other frequencies the observed noise excess leads to an estimation of gain fluctuations: at 1 mHz we could resolve an increase corresponding to $\approx 3 \text{ ppm}/\sqrt{\text{Hz}}$, demonstrating the capability of the pendulum to actually test down to the $\approx 6 \text{ ppm}/\sqrt{\text{Hz}}$ level required for the future LISA FEE. This is true down to 0.1 mHz as reported in Figure 6.11, so we can really perform the test over the whole frequency band relevant for this noise.

The final $S_{\bar{\alpha}}$ estimates suffer some uncertainties due to our incapability of removing the degeneracy between readout noise and true background torque noise. Between the three

analyses, the second one should be considered as reference, because it leaves the maximum freedom to parameters, while still imposing very reasonable constraints on readout noise. Even if the last one manages to distinguish between the background sources, as an experimentalist I find it difficult to trust it without a measurement to support it. A conclusive study of actuation would have needed other measurements. Anyway, both the analyses seem to strengthen the hypothesis that the instrument sensitivity has a relevant contribution from readout noise, something that was never considered before this campaign: more importantly, this would mean that the true torque noise of the pendulum could be even lower than $1 \text{ fNm}/\sqrt{\text{Hz}}$ in the band $0.5\text{-}3 \text{ mHz}$ and not only at resonance, leaving space for further improvements of the facility.

The analysis would have massively benefited from a better characterization of readout noise, and the campaign actually suggested a way to do it, discussed as a future plan in Appendix 9.6. It explains how using AC voltage on Y electrodes instead of X can induce a change in the pendulum stiffness without introducing actuation noise: such a change would then allow to observe directly a change in readout noise. This measurement is also suggested for the LISA FEE campaign.

In the analysis proposed up to now, we simply assessed to which gain fluctuations would the observed noise correspond, inferred that we are able to test down to the required level for LISA and identified readout noise as an important thing to characterize for the future test. However, in the following we present some direct measurements of the actuation voltages noise and evaluate if they match with the estimates recovered with the MCMCs, finding a somehow poor agreement. We bring evidence that a description of our home-made electronics in terms of just gain fluctuations might be insufficient.

6.2.3 Direct measurement of home-made electronics voltage fluctuations

The technique used to measure voltage gain fluctuations is simple. Let's take electrode 1 and 2, in LISA we describe their gain fluctuations as

$$\begin{aligned} V_1(t) &= [1 + \alpha_1(t)]V_{1x} \sin \omega_x t \\ V_2(t) &= -[1 + \alpha_2(t)]V_{1x} \sin \omega_x t \end{aligned} \tag{6.35}$$

If we simply sum them with a summing amplifier

$$V_1(t) + V_2(t) = [\alpha_1(t) - \alpha_2(t)]V_{1x} \sin \omega_x t \tag{6.36}$$

we can then obtain the time series of $[\alpha_1(t) - \alpha_2(t)]V_{1x}$ by demodulating this signal with a lock in amplifier (if a block diagram for this measurement is needed, it is exactly equal to Figure 5.6 for the measurement of actuation non-linearities of Chap. 5). Notice this is exactly the quantity appearing in the torque noise (6.23) and we should detect the noise

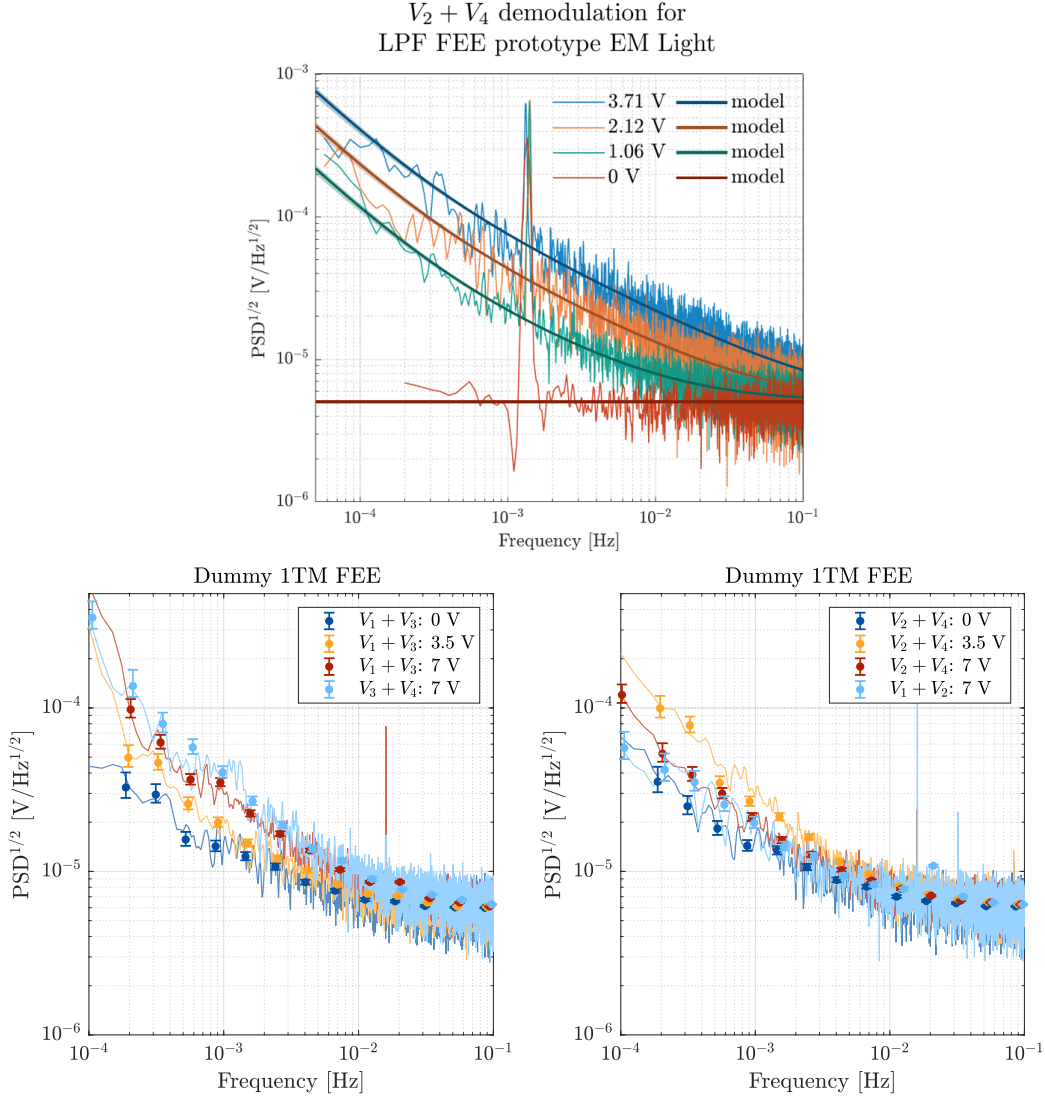


Figure 6.12: (Top) Voltage noise in the amplitude of the sum of two electrodes in LPF Prototype, noise scales with commanded amplitude, from [29]. The large peak at ~ 1 mHz is some other disturbance which can be ignored in this discussion. (Bottom) Same measurement performed on 1TM home-made electronics for both $V_1 + V_3$ and $V_2 + V_4$: no clear scaling with commanded amplitude is detected in the right plot. The noise seems more to change based on environmental conditions. The cyan curve, is instead a measurement of $V_1 + V_2$ and $V_3 + V_4$: this was the quantity that should have been measured from the start, but there was a labeling error. Still it seems to be consistent with the other measurements.

scaling with V_{1x} . As an example of such measurement, the top panel of Figure 6.12 shows the one performed on a LPF FEE prototype during my Master's thesis, just on the analogous quantity $V_2 + V_4$, in which a clear scaling with commanded amplitude is detected. For 0V applied, instead, we just measure white noise.

The bottom panels of Figure 6.12 show instead what was measured on the home-made electronics with the same technique⁴, simultaneously for two couples of electrodes, and exhibits a pretty different behavior. In the right plot, the scaling with V_{1x} is not that clear, and the zero amplitude noise run is not just white noise, almost at the same level of the other runs. In addition, notice that the 3.5 V run has significantly higher noise than the 7 V one: this is due to the presence of a step in the measured voltage time-series, which we do not think should be cut from the data, since happened during a perfectly quiet moment, and could have happen as well during the pendulum measurements. It was also found out that working in the vicinity of some cables connected to the home-made electronics, jumps were detected in the measurement, highlighting a strong sensitivity to environmental conditions. We could have expected the noise to be different from LISA FEE, since the way voltages are generated in the home-made electronics is completely different (see Figure 6.5 and related discussion for reference). A DC voltage coming from a DAC is modulated, meaning that if it is affected by low-frequency additive voltage noise, this will be amplitude-modulated in the actuator output signal, contributing to the effective gain noise. As a consequence, a better, simple model for the fluctuations of the **right plot** of Figure 6.12 would

$$V_i(t) = [V_{0i} + \delta V_{0i}(t)] \text{filt_square_wave}(\omega_i t) \quad (6.39)$$

where $\delta V_{0i}(t)$ are still amplitude fluctuations, uncorrelated between the electrodes, but are completely independent of the commanded V_{0i} . Therefore, each measurement run should in principle exhibit the same noise.

The **left plot** of Figure 6.12 complicates this picture, because it seems to exhibit some kind of scaling with V_{1x} . The physical origin is more difficult to explain; it could be a cross-talk between neighboring circuits or some kind of pick-up, but we are not sure. It can be seen that the spectra are related to the quantities $V_1 + V_3$ and $V_3 + V_4$, therefore we could be brought to think that V_3 could be the one responsible for this excess noise⁵.

⁴For the home-made electronics sum (6.36) looks like

$$V_1(t) + V_2(t) = [\alpha_1(t) - \alpha_2(t)] V_{1x} \text{filt_square_wave}(\omega_i t). \quad (6.37)$$

If we feed this to a lock-in, this measures the RMS of just the first harmonic component (which is almost not attenuated by the actuation filters), which is $4/\pi$ times the amplitude of the square wave, so for the in-phase component we must correct for these factors

$$\text{lock-in output} = \frac{1}{\sqrt{2}} [\alpha_1(t) - \alpha_2(t)] V_{1x} \frac{4}{\pi}. \quad (6.38)$$

⁵Two adder circuits were used simultaneously for this measurement, could it be a noise pick-up there? It seems not, because $V_1 + V_3$ and $V_3 + V_4$ measurements were done with different adder circuits.

We therefore have an uncertainty on how to model these actuation voltages. Probably the safest model would have to include both kinds of fluctuations

$$V_i(t) = [V_{0i} + \alpha_i V_{0i} + \delta V_{0i}(t)] \text{filt_square_wave}(\omega_i t) \quad (6.40)$$

but it would certainly not include transient events like the one of the 3.5 V run in Figure 6.12.

Note: In addition to the sources of confusion just discussed, these measurements were done a few months after the pendulum torque campaign, due to failure and substitution of the acquisition system. We managed to use the old DAC board employed in the campaign. During the substitution, cables and instruments were re-organized, so there is also a strong possibility that we completely changed the aforementioned sensitivity of the voltages to environmental conditions. Adding the failure of the department air conditioning in that period and the fact that there was a very limited amount of time to perform these measurements, they probably constitute the weakest part of this analysis.

Comparison of fluctuations from torque and voltage measurements

We now want to show a comparison between the measured voltage amplitude fluctuations of Figure 6.12 and the estimates obtained by the MCMC analysis of the torque runs: we will show the comparison only for the second fit presented, so the one that includes readout noise but does not describe it analytically.

With the lock-ins, we measured the quantities $V_1 + V_2 \equiv \delta V_1 + \delta V_2$ and $V_2 + V_4 \equiv \delta V_2 + \delta V_4$ (the measurement technique removes the big carrier and leaves only the fluctuations), so if we want to reconstruct the quantity that affected the pendulum $(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)$, given by (6.32), for the choice of actuation voltage signs in Figure 6.9, we combine the two simultaneous lock-in measurements as

$$(\delta V_1 + \delta V_2) - (\delta V_3 + \delta V_4) \stackrel{\text{If: gain fluct.}}{\equiv} [(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)] V_x \quad (6.41)$$

(in other words, we subtract the time series of the left and right plots of Figure 6.12) and calculate the PSD of this. To reconstruct the same quantity from the MCMC torque analysis, we use the MCMC chain for $S_{\bar{\alpha}}$ and build

$$S_{(\delta V_1 + \delta V_2) - (\delta V_3 + \delta V_4)}(f) = 4V_x^2 S_{\bar{\alpha}}(f) \quad (6.42)$$

This PSD calculated both from direct voltage measurements and from the MCMC is reported in Figure 6.13.

The two lowest frequencies are just not resolved by the MCMC fit, therefore we can not say much about the agreement. In the rest of the band we see that we seem to capture the order of magnitude of the fluctuations and their trend, but there are substantial discrepancies that are worth discussing:

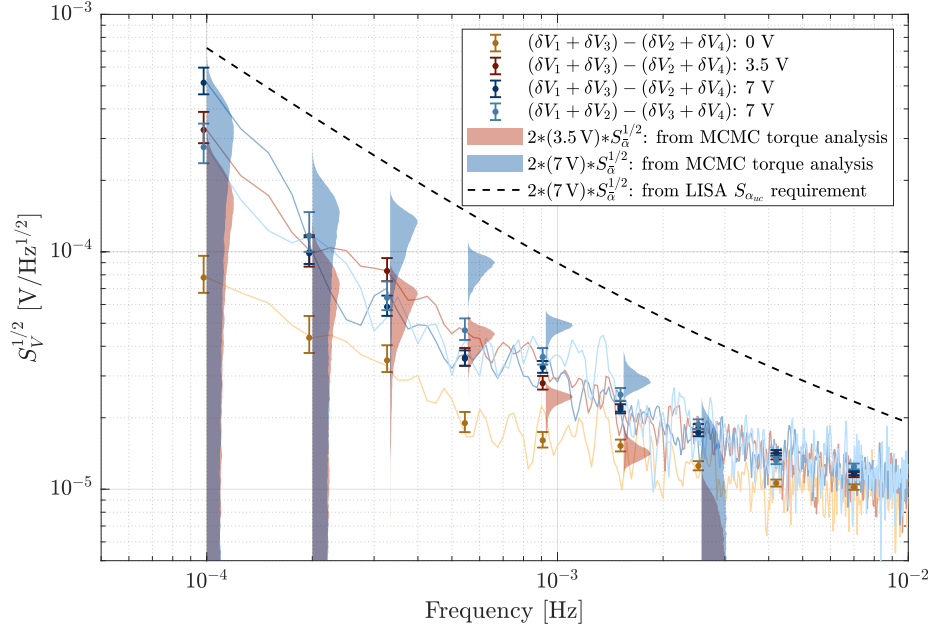


Figure 6.13: Comparison between direct voltage fluctuations measurements (as solid lines and points with error bars) and the estimation through torque noise measured with 1TM torsion pendulum (violin plots of their posterior distributions are used), just for the second MCMC fit described in Sec. 6.2.2, which includes readout noise (but not analytic). The real quantity to which the pendulum is sensitive, based on the choice of applied voltages, is $(\delta V_1 + \delta V_2) - (\delta V_3 + \delta V_4)$, however, we have only the cyan curve representing this when 7 V are applied. The other curves represent a different combination of electrodes, measured instead due to a labeling mistake in the lab: fortunately, they seem to be pretty compatible, therefore we do not expect the cyan quantity would have behaved differently at other voltages.

- The voltage amplitude noise reconstructed from the MCMC at 0.35 and 0.55 mHz are significantly higher than the direct voltage measurements, especially at 7 V. Figure 6.12 shows how the voltage noise can exhibit some important variability, sometimes jumps, or spikes; it could be that during the campaign there were particularly bad runs, which caused this excess. Nonetheless, this region of the spectrum has always been the most difficult to keep at a stable performance, and it is always the first that spoils when some inconvenience happens to the pendulum. Therefore, we can not exclude the presence of another contribution on top of actuation.
- Frequencies $\approx 1 - 1.5$ mHz are the ones in the vicinity of the pendulum resonance, where the instrument performances are the best and have always shown to be very stable. Here we find the best agreement with the home-made electronics voltage fluctuations.

Still, the scaling of the noise seems to not perfectly match⁶: it could be that at these voltages we start seeing the failure of a model which just considers gain fluctuations. A better comparison would have been possible if we had also a 9 V measurement with the lock-ins. We also tried to perform an MCMC fit that uses the more general model (6.40) for the actuation voltages: we do not report it here, because it gives very similar results to Figure 6.13. The fit always converges to a result that favors a description of the noise in terms of gain fluctuations, and any other "additive amplitude noise" is very poorly resolved.

- At ≈ 2.5 mHz instead the proposed fit does only give an upper limit; the analytic readout fit seems to be the one finding the best agreement with measurements. This frequency was the one that raised the suspect in Figure 6.9 that we could be dominated by readout noise, and we still think this could be the case: generally, sources of torque noise weaken at higher frequencies, while an analytic description of readout noise becomes more accurate.

Unfortunately we can not say much more about the agreement of the two estimates, also given the technical difficulties reported for the direct voltage measurements, which would have constituted a better benchmark if they were done immediately after the torque measurements, without changes in the facility. The order of magnitude of the recovered fluctuations seem to be correct, and the better agreement around pendulum resonance could be an indication that this was truly actuation noise. Even in the dramatic hypothesis that the observed increase in noise could have also contributions from other unidentified sources of noise, different from actuation (which however should also scale with V_x), this campaign gives the maximum impact that these can have on the measurement. If we look at Figure 6.11, the expected fluctuations if LISA FEE behaves according to the specification and is dominated by uncorrelated noise, are a more than a factor 2 higher, and should be resolved with no issues.

Conclusions

In this chapter the 1TM torsion pendulum facility was introduced. The instrument was restored after a shut down of approximately 7 years and renewed in many of its parts. The sensitivity to external torques acting on the TM was brought back to levels attained before the shut down, reaching 0.5-0.6 fNm/ $\sqrt{\text{Hz}}$ at resonance, corresponding to 50-60 fN/ $\sqrt{\text{Hz}}$ for a 1 cm lever-arm force, an unmatched performance in the on-ground testing of the LISA TM-EH system.

The facility is perfect to study forces with well known lever-arm, like those coming from electrostatic force actuation. In this context, an "exploratory" measurement campaign was

⁶Note: removing the "step" from the 3.5 V run discussed in Sec. 6.2.3, the scaling is resolved a bit better, but it would be sort of cheating.

conducted using some home-made electronics, to assess the pendulum capabilities of detecting actuation noise from the future LISA FEE. In particular, it is possible to observe a noise analogous to the combination of gain fluctuations entering the LISA GW measurement noise, without the need of particular model assumptions.

We conducted runs with enhanced actuation similar to the ones performed on LPF and subject of Chapter 4. We discovered that it is not possible to measure actuation noise coming from single couples of electrodes (that would be of interest for LISA), due to a cross-talk with the y degree of freedom. This measurement could be enabled by an active control of the pendulum tilt, an improvement planned for the future.

We could observe instead an increase in torque noise scaling with the applied torque itself, as expected from actuation noise, when using a combination of four electrodes. We could extract an estimate of the gain fluctuations that would be responsible of such increase in noise, finding at 1 mHz a level of $\approx 3 \text{ ppm}/\sqrt{\text{Hz}}$, demonstrating the capability of the pendulum to actually test down to the $\approx 6 \text{ ppm}/\sqrt{\text{Hz}}$ level required for the future LISA FEE. This holds for the whole frequency band of interest.

To obtain this result, a few different Bayesian analyses are proposed, which all give similar but not perfectly equal results; the uncertainty arises from our poor capability of distinguishing if readout noise has a relevant contribution to the pendulum noise floor. This degeneracy could be removed by performing measurements with AC actuation along the Y axis (instead of X), an implementation also envisioned for the near future.

Finally, we tried to understand if the gain noise recovered from the torque runs would match with a direct measurement of the home-made electronics voltage amplitude instabilities, carried out with a dedicated technique. The 1TM FEE deviates from the expected behavior of LISA FEE, and also exhibits a certain sensitivity to environmental conditions. In particular, in addition to a scaling with applied voltage, we detected an approximately $1/f^2$ noise with $10\text{-}30 \mu\text{V}/\sqrt{\text{Hz}}$ (per electrode) at 1 mHz independently of the commanded amplitude. Comparison between the direct voltage measurements and the estimates from the torque runs showed only a rough agreement. Maximum agreement is attained at the resonance frequency of the pendulum, where the performances are better and more stable. The measurements were conducted with several months of temporal delay and quite a few setbacks, therefore it is difficult to state precisely if the observed discrepancies are significant or not. Anyway, LISA FEE expected voltage fluctuations are quite higher than the levels measured here, therefore we should encounter no such difficulties in their resolution in the future.

Chapter 7

Continuous UV discharge: random charging noise characterization

This Chapter is a bit independent from the others, since it does not present a direct study electrostatic actuation, but moves to the problem of noise due to UV illumination in LISA. Still, the need to discharge the TM is driven by actuation, as in section 3.4.3 it was discussed that TM charge fluctuations can result in force noise, by interaction with surrounding DC electrostatic fields:

$$\delta F_x = -\frac{\delta Q_{TM}}{C_{\text{tot}}} \left| \frac{\partial C_i}{\partial x} \right| \Delta_{x,\text{DC}}. \quad (7.1)$$

The fluctuations $\delta Q_{TM}(t)$ in LISA will be mainly caused by the traffic of electrons induced by cosmic rays bombardment; in addition, if UV light is used continuously to keep the TM discharged, it will also contribute to the traffic of charges, adding at minimum a Poissonian shot noise, described as

$$S_{Q_{TM}}(f) \approx \frac{2e^2(\lambda_{\text{env}} + \lambda_{\text{UV}})}{(2\pi f)^2}. \quad (7.2)$$

We will see how the 1TM torsion pendulum can be used to measure this noise with great precision, how we used it to characterize the noise induced by LISA UV LEDs, and how this allowed us to verify our models of the UV discharge mechanism.

7.1 UV discharge mechanism and model

We start by discussing the working principle behind the TM charge control through UV light. A detailed treatment of this phenomenon can be found in Davide Dal Bosco's PhD thesis [20], here we try to introduce a simplified version of that, only to reach the fundamental quantities that we want to investigate.

The situation is depicted in Figure 7.1, where we consider a TM illuminated by UV light, ignoring other sources of charging (for example cosmic rays in LISA). Starting from the left drawing, we see that some light will bounce around and hit also the EH. So electrons will

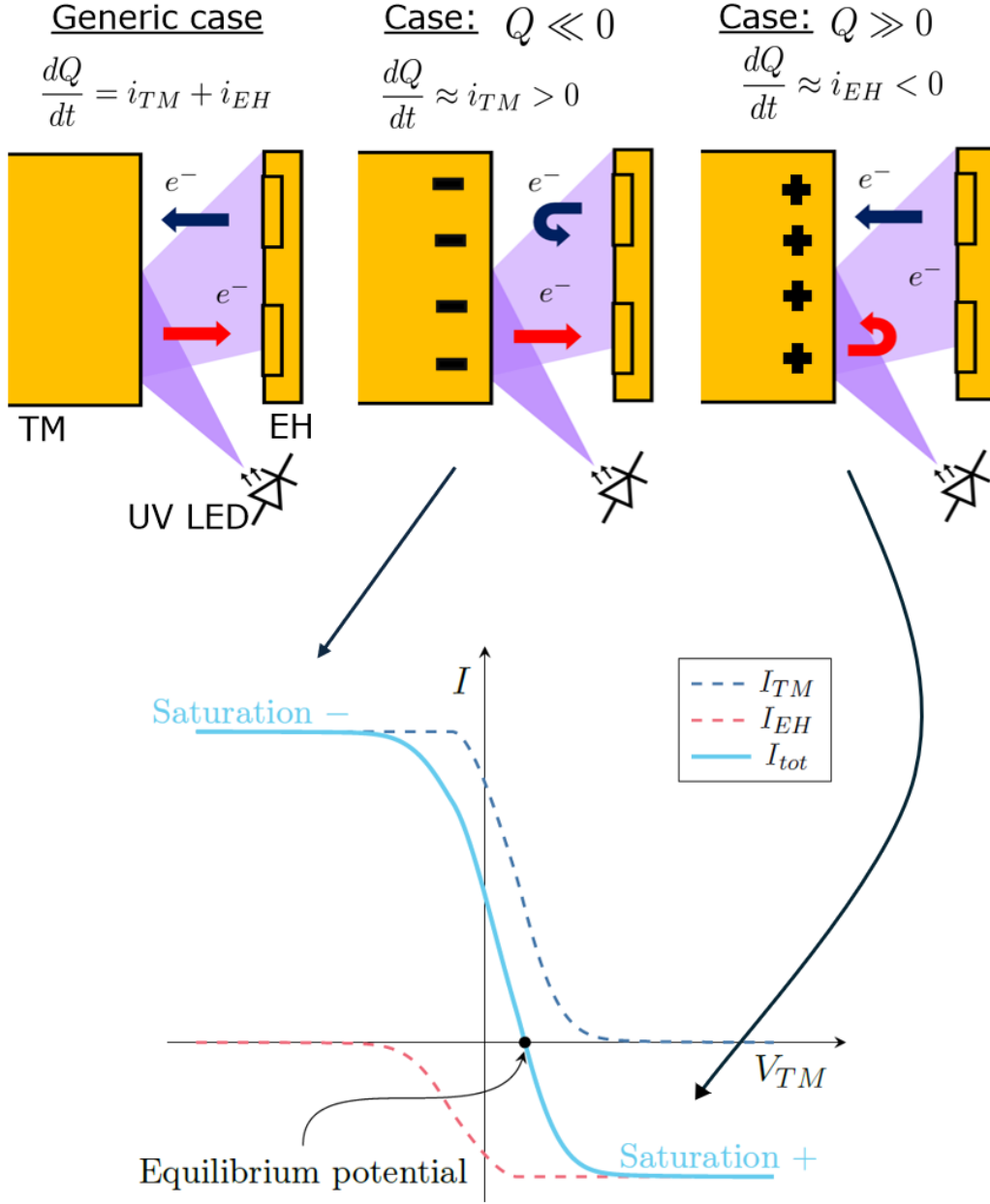


Figure 7.1: (Top) Sketch to illustrate the behavior of photocurrents based on TM voltage. (Bottom) Heuristic plot of the current leaving the TM, adapted from Davide Dal Bosco's thesis [20].

be extracted from both surfaces and the charging rate of the TM will be some kind of sum of them. However, if we look at the central sketch, if the TM happens to be very negatively charged ($V_{TM} \ll 0$), then the electrons emitted by the EH will not be able to overcome the repulsive force, and will be reflected back to the EH. In this situation then, the charging rate will be entirely dominated by only the current of electrons leaving the TM, that means $dQ_{TM}/dt > 0$. This regime is called *negative saturation* and it can be seen in the I vs V_{TM}

plot in Figure 7.1. The opposite happens is $V_{TM} \gg 0$, called *positive saturation*. Between the two saturation regimes, dQ_{TM}/dt must be some average between the two. Let us give a few comments:

- Given the signs of the currents at saturation, there must be a voltage V_{TM} for which $dQ_{TM}/dt = 0$! We call it *equilibrium potential* V_{eq} : why? Because if we imagine to have a highly negatively charged TM, then $dQ_{TM}/dt > 0$: if leave the light on, Q_{TM} will keep increasing, until $dQ_{TM}/dt = 0$, and then it will stop. V_{eq} is therefore a stable point for the system exposed to UV light.
- We understood that dQ_{TM}/dt is a strange function of V_{TM} . Except from this, on what does it depend? Surely on the voltages V_i of the surrounding electrodes, because they could deviate the e^- trajectories. Then it depends on how much light ends up on the surfaces: beam geometry (*b.g.*) and TM position ($\{x\}_i$) are then relevant. Finally, we know from the photoelectric effect that the photons have to surpass a certain energy threshold to be able to extract electrons, so also the spectrum $f(\lambda)$ of the UV light and the gold surface work function W will play a role. Finally, the photocurrents i_{TM} and i_{EH} are proportional to the UV power P_{UV} : while all the other dependencies can be difficult to model, this is the only certainty we have.

We describe all of these with a single quantity

$$\frac{dQ_{TM}}{dt} = eY P_{UV} \quad (7.3)$$

called **apparent yield**

$$Y = Y(V_{TM}, V_i, b.g., \{x\}_i, f(\lambda), W) \quad [\text{\#charges/s/nW}] \quad (7.4)$$

This is a difficult quantity to characterize, especially across different devices, for example two GRS that have been exposed to different degrees of contamination of the surfaces, or in which the UV beam geometry is different etc¹. However, the general shape is found experimentally to be always like Figure 7.1, and when studying a single device with a torsion pendulum (where the TM is not changing its position enough to modify the illumination pattern), we can reduce to

$$Y = Y(V_{TM}, V_i). \quad (7.6)$$

Note: As a consequence, also V_{eq} will depend on all these parameters!

¹In Davide Dal Bosco's thesis [20], a more general quantity $\mathcal{Y}$ is used: while Y is the number of charges per second, per unit power, $\mathcal{Y}$ is the number of charges per number of photons. The two definitions are linked, in case of perfectly monochromatic light with photon energy $h\nu$:

$$\mathcal{Y} = h\nu Y \quad (7.5)$$

- What are V_{TM} values in which saturations kick in? Well, it depends on how much energy the emitted electrons have. For the UV light used with peak wavelength 247 nm (4.89 eV) and a gold work function in the range 3.9-4.5 eV as observed for LPF TM/EH surfaces [14], emitted electrons have energy ~ 1 eV, so they can at maximum overcome a potential barrier of $V_{TM} \sim 1$ V, which is where saturations start.

7.1.1 Linearized apparent yield model

We said that if we simply turn on the UV light, at some point we will reach $V_{TM} = V_{eq}$, but can we predict $V_{TM}(t)$? This has a particularly simple shape if we put ourselves in the vicinity of V_{eq} and expand Y linearly

$$\frac{dQ_{TM}}{dt} = C_{tot} \frac{dV_{TM}}{dt} = \left[\underbrace{Y(V_{eq})}_{=0} + \left. \frac{dY}{dV_{TM}} \right|_{V_{eq}} (V_{TM} - V_{eq}) \right] eP_{UV} \quad (7.7)$$

Since $dY/dV_{TM}|_{V_{eq}}$ is negative (Figure 7.1), this is the simple equation of an exponential decay with solution

$$V_{TM}(t) = V_{eq} + (V_{TM}(t_0) - V_{eq})e^{-\frac{t}{\tau_{UV}}}, \quad \tau_{UV} = -\frac{1}{\left. \frac{dY}{dV_{TM}} \right|_{V_{eq}}} \frac{C_{tot}}{eP_{UV}}. \quad (7.8)$$

where we see that the discharging time constant $\tau_{UV} \propto 1/P_{UV}$ meaning that the more UV power we use, the faster the convergence to V_{eq} . With this new definition, Eq. (7.7) becomes

$$\frac{dV_{TM}}{dt} = -\frac{1}{\tau_{UV}}(V_{TM} - V_{eq}) \quad (7.9)$$

and we will see that most of the quantities regarding the dynamics of the system can be actually described in terms of τ_{UV} , without any knowledge of apparent yield or power used.

7.1.2 Fast discharge vs continuous discharge

So we learned that simply turning on a constant UV light, we will reach some V_{eq} , which are found to be of order hundreds of mV experimentally. But we want to be able to bring the TM at 0 V! There are two strategies to do this.

Fast discharge

Suppose we measured the TM voltage to be V_0 , we know V_{eq} and we want to bring the TM to a set-point voltage, V_{sp} . We can induce a change in TM potential by using the surrounding electrodes V_i , since we derived in the introduction that

$$V_{TM} = \frac{Q_{TM} + \sum_i C_i V_i}{C_{tot}} = V_0 + \frac{\sum_i C_i V_i}{C_{tot}} \quad (7.10)$$

therefore we can basically choose whatever voltage we want. We proceed as follows, guided by the sketch in Figure 7.2

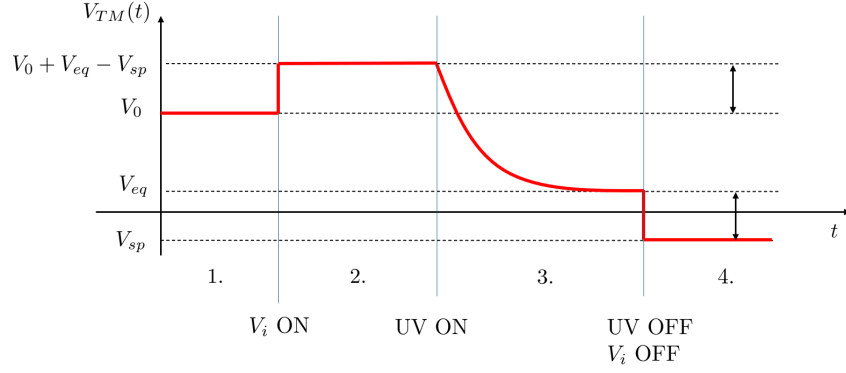


Figure 7.2: Fast discharge procedure, described in the text.

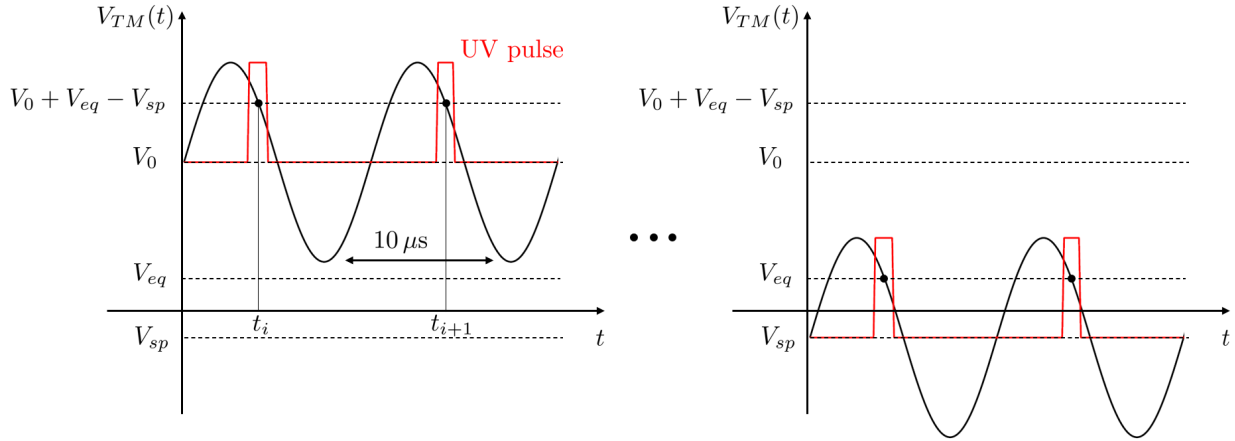


Figure 7.3: Pulsed continuous discharge procedure, described in the text.

1. TM initial voltage $V_{TM_1} = V_0$.
2. Choose V_i so that $V_{bias} = \sum_i C_i V_i / C_{tot} = V_{eq} - V_{sp}$. Then $V_{TM_2} = V_0 + V_{eq} - V_{sp}$.
3. Discharge with UV. Then $V_{TM_3} = V_{eq}$.
4. Set all $V_i = 0$! Then $V_{TM_4} = V_{TM_3} - V_{bias} = V_{sp}$.

So we can bring the TM to any V_{sp} by just choosing proper V_i and knowing V_{eq} (we do not even need to know V_0).

Note: We have to choose electrodes V_i which are far from the illumination spot, otherwise V_{eq} will change.

This technique was used in LPF: before a noise run of two weeks, the TM would be brought to -70 mV, so that with a cosmic ray charging rate of 25 e/s it would arrive at the end with 70 mV. The downside of such technique is that we really have to interrupt the measurement, apply large voltages to some electrodes, discharge quickly and then restart. In a GW observatory like LISA this is not ideal, since we would like to measure for much longer periods of time than two weeks.

Continuous pulsed discharge

The principle of this technique is the same as before: we want to use UV light when the TM is biased to $V_{eq} - V_{sp}$, so that when the bias is removed, the TM is automatically at voltage V_{sp} . The difference here is that instead of imposing a bias ourselves, we use one that is already on the TM.

In fact, up to now we talked about the use of DC UV illumination, and DC voltage biases. But we know that the TM voltage has actually a 100 kHz oscillating component, due to the *injection voltage*, used for sensing purposes. This is represented in Figure 7.3 and has expression

$$V_{TM}(t) = V_0 + \frac{C_{inj,tot} V_{inj} \sin(2\pi f_{inj} t)}{C_{tot}} = V_0 + V_{bias}(t) : \quad (7.11)$$

the second term here acts exactly like the bias in (7.10), and we set $V_{inj} = 5.1$ V to have a peak $|V_{bias}| \approx 0.6$ V. Now, let us see what happens if we illuminate the TM with short UV pulses at a fixed phase ϕ_{UV} with respect to the injection voltage, as again represented by the red line in Figure 7.3. This means that we illuminate at instants t_i such that $V_{bias} = C_{inj,tot} V_{inj} \sin \phi_{UV} / C_{tot}$. If we do this, as happened for the fast discharge, we know that the UV light will bring the TM voltage to equilibrium, that is $V_{TM}(t_i) = V_{eq}$. So where did the average voltage of the TM V_{sp} end up?

$$\begin{aligned} V_{TM}(t_i) &= V_{sp} + \frac{C_{inj,tot} V_{inj} \sin \phi_{UV}}{C_{tot}} = V_{eq} \\ \rightarrow V_{sp} &= V_{eq} - \frac{C_{inj,tot} V_{inj} \sin \phi_{UV}}{C_{tot}} \approx V_{eq} - (0.6 \text{ V}) \sin \phi_{UV} \end{aligned} \quad (7.12)$$

We see that by just specifying the phase ϕ_{UV} of the light, we can choose the average potential of the TM as we prefer. I cut it short, but I hope Figure 7.3 can clarify this procedure.

Some notes:

- Again, this process brings the DC TM voltage to V_{sp} independently of V_0 by just proper choice of light pulses. To choose a light pulse means to specify its phase ϕ_{UV} with respect to the injection voltage but also its duration $\Delta\phi_{UV}$. Therefore, we can think of rewriting the equation for the V_{TM} dynamics as

$$\frac{dV_{TM}}{dt} = -\frac{1}{\tau_{UV}} [V_{TM} - V_{sp}(V_{eq}, \phi_{UV}, \Delta\phi_{UV})]. \quad (7.13)$$

Note that $V_{eq} = V_{eq}(V_i)$, as discussed for Eq. (7.6), therefore we must be careful if other voltages are being applied to the electrodes.

- The procedure does not require to apply further biases or interruptions, so it can be really kept going during the whole LISA measurements, and **it will keep the mass permanently discharged** (if we set $V_{sp} = 0$).
- We can choose P_{UV} through *decimation* of the pulses, meaning we illuminate every N injection cycles.

- What matters is actually the TM average potential during illumination, therefore there are also other ways to perform this procedure that we will see later.
- The main limitation of this method is that the maximum injection bias is 0.6 V, so, if for some reason $|V_{eq}|$ happens to be > 0.6 V, from (7.12) we see that we can not use this method to bring the TM voltage to zero.

Effect of environmental charging and UV power stability

Up to now we discussed the illumination procedure in absence of any environmental charging. If this is present, the effect will be that of having an additional current added to the process, which charges the TM with a certain τ_{env} towards its own V_{eq}^{env} . On LPF for example the current was $e\lambda_{net}^{env} = +25\text{e/s}$ and $V_{eq}^{env} \approx 1$ V. We do not do the calculations (you can find everything in [20]), but in the case $\tau_{env} \gg \tau_{UV}$, this effect on our system is analogous to what a constant force does to an harmonic oscillator: it shifts its equilibrium position. Therefore, if we use some UV light settings to reach the setpoint V_{sp} , and we suddenly turn on some environmental charging, we will experience a change in equilibrium position

$$\Delta V_{eq} = \frac{\tau_{UV}}{\tau_{env}} V_{eq}^{env} = \frac{e\lambda_{net}^{env}}{C_{tot}} \tau_{UV} \quad (7.14)$$

This means that the final equilibrium position of the TM voltage will actually $\propto \tau_{UV} \propto P_{UV}^{-1}$. If the power P_{UV} fluctuates, than this will cause noise in the TM charge time series; without environmental charging, V_{TM} at equilibrium does not depend linearly on P_{UV} because the yield is zero, and this effect would not be relevant.

Note on 1TM: In our laboratory the environmental charging is negligible (<1 electron every 4 seconds from charge measurements with no UV light, see Figure 7.9). Therefore the dependence of P_{UV} should not be detected according to this model.

7.1.3 Continuous discharge related noise sources

So, let's suppose the TM is in free fall in space and we start measuring GWs. Are we really sure that we can keep the UV light on during the measurement without introducing some disturbances?

Any fluctuations $\delta Q_{TM}(t)$ will couple to stray electrostatic fields and can be a source of noise. If the UV pulses do not have a stable phase ϕ_{UV} , then V_{sp} will also oscillate. This will also happen for UV power instability in presence of environmental charging. But above all, the process of transfer of electrons exhibits *shot noise*. We can include this in (7.1) adding a noisy current i_{UV}

$$\frac{dV_{TM}}{dt} = -\frac{1}{\tau_{UV}} [V_{TM} - V_{sp}(V_i, \phi_{UV}, \Delta\phi_{UV}, P_{UV})] + \frac{i_{UV}}{C_{tot}}. \quad (7.15)$$

If we then move to the frequency domain

$$\tilde{V}_{TM} = \frac{1}{i\omega\tau_{UV} + 1} \left(\tilde{V}_{sp} + \frac{\tau_{UV}\tilde{i}_{UV}}{C_{tot}} \right), \quad (7.16)$$

which shows that all the involved fluctuations end up being low pass filtered at frequency $f_{UV} = 1/2\pi\tau_{UV}$.

Charge fluctuations due to UV phase noise

If the phase of the UV pulses is noisy with PSD $S_\phi(\omega)$, from Eq. (7.16) we have

$$S_{V_{TM}}(\omega) = \left(\frac{dV_{sp}}{d\phi} \right)^2 \frac{1}{1 + \omega^2\tau_{UV}^2} S_\phi(\omega), \quad (7.17)$$

and this is one of the quantities we studied with the 1TM torsion pendulum.

UV induced charge shot noise

With regard to the noise charge transfer instead we have

$$S_{V_{TM}}(\omega) = \left(\frac{\tau_{UV}}{C_{tot}} \right)^2 \frac{1}{1 + \omega^2\tau_{UV}^2} S_{i_{UV}}(\omega) \quad (7.18)$$

and for a purely Poissonian process we know that the noise PSD is white and equal to

$$S_{i_{UV}} = 2e^2\lambda_{eff} \quad (7.19)$$

where λ_{eff} is the total number of charges being exchanged per second (not the net flux, which is zero at equilibrium). It must be therefore related to the sum of the currents from TM and from EH, which compensate at V_{sp} .

$$\lambda_{eff} \simeq P_{UV}(|Y_{EH}(V_{sp})| + |Y_{TM}(V_{sp})|) \propto \frac{1}{\tau_{UV}} \quad (7.20)$$

Unfortunately we can never measure the two quantities separately, therefore it is very difficult to verify the correctness of this model. However, as $Y = 0$ in equilibrium, it must hold that $Y_{TM}(V_{sp}) = Y_{EH}(V_{sp})$. We can obtain a worst case, maximum noise scenario if we consider that the two surfaces can not emit more than their saturation values, that is $|Y_{TM}(V_{sp})| \leq |Y_{sat}^-|$ and $|Y_{EH}(V_{sp})| \leq |Y_{sat}^+|$. This leads to

$$\lambda_{eff} \leq 2P_{UV} \min(Y_{sat}^+, Y_{sat}^-). \quad (7.21)$$

Here we just gave a brief description, see [20] for a more detailed explanation. Observation of this noise was carried out with the 4TM torsion pendulum [20], which however was at the very limit of its sensitivity and it could barely test at the level of illumination envisioned for LISA ($\lambda_{eff} \approx 3000$ charges/s). Various λ_{eff} were measured up to $\lambda_{eff} \approx 1000$ charges/s for EH illumination and to $\lambda_{eff} \approx 10^4$ charges/s for TM illumination, finding a proportionality to P_{UV} , slightly biased with respect to (7.20). The 1TM torsion pendulum is a much more suitable instrument to measure this noise and verify its compatibility with our discharge model.

7.2 Measurement of discharge noise with torsion pendulum

The 1TM facility was equipped with the UV LED SETi 240 (produced by Sensor Electronic Technology, Inc., with a peak wavelength of 247.15 nm and FWHM of 10.28 nm [20]), driven by a current source developed in Roma Tor Vergata, allowing the generation of squared current pulses in phase with the 100 kHz injection voltage, like the ones sketched in Figure 7.3. The light enters a 50-50 splitting optical fiber that brings half of the light to the TM through a vacuum chamber feed-through, the other half to a PMT for power measurement. To regulate the total UV power delivered to the TM we decimate the number of UV pulses, ranging from pulses rates of 100 kHz to 25 Hz. LISA will have the possibility of injecting light both directly towards the TM and EH, with injectors called ISUKs. The 1TM facility just has one fiber pointing towards the TM: we will see that this is still sufficient for bipolar discharging of the TM around $V_{TM} = 0$, and thus sufficient for this illustrative test of continuous discharge and the associated charge noise.

Our objective is to measure the TM charge in time and see if we resolve an increase in noise due to UV illumination. Therefore, first of all, we need a way to measure the TM charge.

7.2.1 TM charge measurement and control

We measure the TM charge by modulating the x/ϕ electrodes voltages in order to induce a torque $N_\phi \propto V_{TM}$ at a frequency ω_{mod} , to then detect it by demodulation at ω_{mod} in post processing. This is easily seen if we report once again the full expression for the applied torque on the TM, expressed by Eq. (3.35) (neglecting V_{TM}^2 term because second order)

$$N_\phi = \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial \phi} V_i^2 - V_{TM} \sum_i^N \frac{\partial C_i}{\partial \phi} V_i + V_{TM} \frac{\partial (C\delta V)_{TM}^s}{\partial \phi} + \sum_i V_i \frac{\partial (C\delta V)_i^s}{\partial \phi}. \quad (7.22)$$

The second term is straight proportional to V_{TM} , therefore we can try to choose V_i such that the multiplying sum causes a large signal. Reminding also that

$$V_{TM} = \frac{Q_{TM} + \sum C_i V_i + (C\delta V)_{TM}^s}{C_{\text{tot}}} \quad (7.23)$$

If we choose

$$\text{Charge measurement:} \quad V_1 = -V_2 = V_3 = -V_4 = V_{\text{mod}} \sin \omega_{\text{mod}} t \quad (7.24)$$

(careful! This is a modulation of the DC actuation voltages, not the amplitude of AC waveforms) then we have

$$\begin{aligned}
 N_\phi = & -4 \underbrace{\left(\frac{Q_{TM} + (C\delta V)_{TM}^s}{C_{\text{tot}}} \right)}_{V_{TM_0}} \left| \frac{\partial C_x}{\partial \phi} \right| V_{\text{mod}} \sin \omega_{\text{mod}} t \\
 & + 2 \left(\left| \frac{\partial^2 C_x^*}{\partial \phi^2} \right| - \frac{8}{C_{\text{tot}}} \left| \frac{\partial C_x}{\partial \phi} \right|^2 \right) (\phi - \phi_0) V_{\text{mod}}^2 \sin^2 \omega_{\text{mod}} t \\
 & + \left| \frac{\partial C_x^*}{\partial \phi} \right| \left(\sum_i^4 \delta V_i(t) \right) V_{\text{mod}} \sin \omega_{\text{mod}} t \\
 & + \left(\frac{\partial (C\delta V)_1^s}{\partial \phi} - \frac{\partial (C\delta V)_2^s}{\partial \phi} + \frac{\partial (C\delta V)_3^s}{\partial \phi} - \frac{\partial (C\delta V)_4^s}{\partial \phi} \right) V_{\text{mod}} \sin \omega_{\text{mod}} t \quad (7.25) \\
 & + \text{other DC terms not } \propto V_{\text{mod}}
 \end{aligned}$$

1. The first line is a signal at ω_{mod} proportional to Q_{TM} , which is the one that we want to measure. Based on this, we demodulate $N_\phi(t)$ using $\sin \omega_{\text{mod}} t$ as reference, and obtain the demodulated amplitude $N_{\sin 1\omega}(t) = -4V_{TM_0} |\partial C_x / \partial \phi| V_{\text{mod}}$. Then our estimate of the TM voltage becomes

$$V_{TM}(t) = \frac{N_{\sin 1\omega}(t)}{-4 \left| \frac{\partial C_x}{\partial \phi} \right| V_{\text{mod}}} \quad (7.26)$$

We see that we will actually detect a quantity biased by the presence of stray biases $(C\delta V)_{TM}^s$. Now we will discuss the consequences of the other terms in 7.25.

2. The second line is a term $\propto \phi$ position of the pendulum, meaning that it softens the stiffness. Since the modulation changes in time, then the stiffness depends on time: we already discussed how we calculate $N_\phi(t)$ from $\phi(t)$ in presence of a variable stiffness, which we actually calculate with this formula. Finally, notice that we have $\phi - \phi_0$: here ϕ_0 is the pendulum position where the 4 capacitance derivatives of the 4 electrodes are equal, which usually does not correspond to the zero of the optical readout. As a consequence, this term will result in a signal at $2\omega_{\text{mod}}$ in N_ϕ , proportional to ϕ_0 . One can even estimate ϕ_0 by demodulating it in post-processing.
3. The third line involves a sum of $\delta V_i(t)$, which represent the low frequency voltage fluctuations present on each modulation electrode. We see that also this noise is modulated at ω_{mod} , therefore, it will directly add to our measurement of V_{TM_0} .
4. Finally, the last line shows the interaction of the modulation with the stray biases on the electrodes, and that they show up exactly at the frequency of our signal. We will see an example soon.

From the way V_{TM} is recovered through (7.26), we see that we want to detect a signal at ω_{mod} : therefore, we want to choose ω_{mod} such that it is placed in the region of maximum

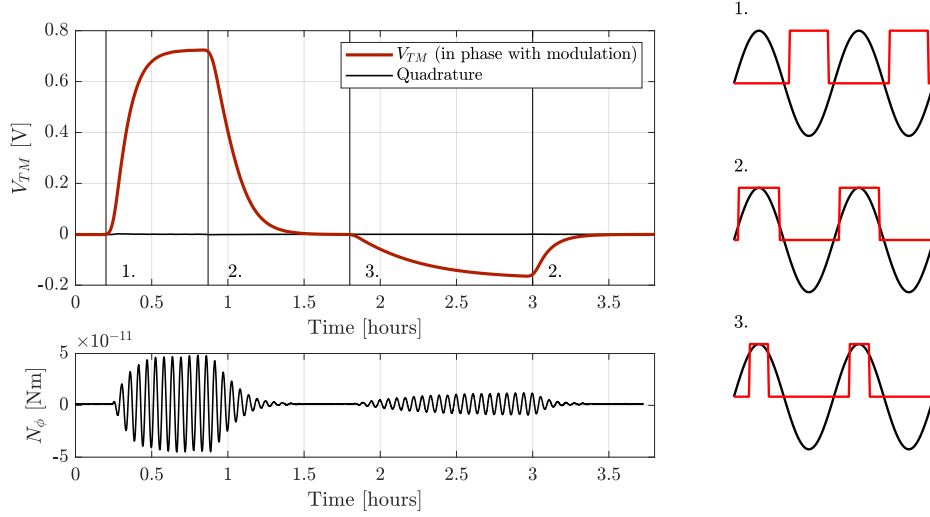


Figure 7.4: (Left top) Example of a charge measurement and demonstration of TM charge control through UV light pulses (real data). (Left bot) Torque signal $N_\phi(t)$ measured on the pendulum (filtered): demodulation of this signal using $V_{\text{mod}} \sin \omega_{\text{mod}} t$ as reference, yields the phase and quadrature components on top through Eq. (7.26). (Right) Sketches of the UV light patterns (red) with respect to the 100 kHz injection voltage (black) used to achieve the various discharges. All described in the text.

torque sensitivity of the pendulum, to maximize SNR (usually we go for 3 mHz). If at that frequency the pendulum noise floor is $S_N(\omega_{\text{mod}})$, then the background noise of the charge measurement is

$$S_{V_{TM}}(\omega) = \frac{S_N(\omega_{\text{mod}})}{\left(4 \left| \frac{\partial C_x}{\partial \phi} \right| V_{\text{mod}}\right)^2} + \frac{\left| \frac{\partial C_x^*}{\partial \phi} \right|}{\left| \frac{\partial C_x}{\partial \phi} \right|} \sum_i S_{\delta V_i}(\omega) \quad (7.27)$$

where the last term represents the PSD of actuation voltages noises discussed in bullet 3.

Therefore, we summarize the charge measurement procedure: we activate the four x/ϕ electrodes following (7.24), choosing ω_{mod} on the basis on how fast we want the measurement to be vs how precise. Then we choose V_{mod} , big for more resolution to low voltages, small if we do not want to induce too large pendulum oscillations, for example when the TM is very charged. We perform the experiment, we reconstruct $N_\phi(t)$ with a variable stiffness, and we finally demodulate in post processing at ω_{mod} to estimate V_{TM} . We will now show some examples.

Charge control with UV pulses

In Figure 7.4 a TM voltage measurement obtained through this procedure is reported. While the V_{TM} signal is in phase with the modulation, the demodulated quadrature should ideally be zero: we often use this as a sanity check (also in this case).

Figure 7.4 also wants to demonstrate how we can control the TM charge with the pulsed light. By just playing with the light pulses phase and duration we can basically decide the TM charge: even if we have just one illumination spot on the TM, and not one pointing the EH, we can induce charge variations in both directions. In the Figure we see that from a discharged TM we first move exponentially to a large voltage by illuminating while the 100 kHz injection voltage is negative. Here pulses of 4 μ s were used, as sketched on the right: using shorter pulsed would have brought the TM to even larger voltages.

The TM is then discharged to 0 V again by choosing light pulses such that the average TM potential during illumination equals V_{eq} , which for our system is ≈ 0.32 V. This procedure can be fine tuned through multiple iterations, but once the correct setting is found, it should always bring the TM to that potential, as demonstrated by the last discharge in Figure 7.4. Finally the step labeled "3." demonstrates that we can go negative for example also by reducing the illumination time, to focus on the moments in which the TM is more positive, at the cost of some UV power (a slightly larger discharge time constant can be noticed).

DC bias compensation

So again our final objective of this Chapter is to observe charging shot noise. We have to bring the TM to ≈ 0 V with UV light, keep the light ON and see if this induces some excess noise in the charge time-series: so the first step to do is just to measure the background noise when the UV light is not there. Figure 7.5 shows some first trials of monitoring the charge over long periods of time. During the shown measurement there was a huge temperature drift due to a problem in air conditioning system, which in turn caused a huge drift in the x^2 position of the pendulum. It is visible by naked eye that $V_{TM}(t)$ is correlated with $x(t)$ and a fit of the spectra confirms it. When such cross talks appears during a measurement that uses electrostatic forces, one must immediately think of DC stray biases, which were included in Eq. (7.25) exactly for this reason.

What we observe here is a $\partial N_{1\omega}/\partial x$, so from the last term in (7.25) we must have a not negligible contribution from

$$\begin{aligned} N_{\phi}^{\text{stray}} &= V_{\text{mod}} \cos \omega t \left(\left. \frac{\partial^2 (C\delta V)_1^s}{\partial \phi \partial x} \right|_0 - \left. \frac{\partial^2 (C\delta V)_2^s}{\partial \phi \partial x} \right|_0 + \left. \frac{\partial^2 (C\delta V)_3^s}{\partial \phi \partial x} \right|_0 - \left. \frac{\partial^2 (C\delta V)_4^s}{\partial \phi \partial x} \right|_0 \right) x \\ &\equiv V_{\text{mod}} \cos \omega t \left. \frac{\partial^2 C_x}{\partial \phi \partial x} \right| \Delta_{2x} x \end{aligned} \quad (7.28)$$

where again we introduced an effective voltage Δ_{2x} , normalized to the capacitance derivative that we will use to compensate it. In Section 3.3.3 we demonstrated that it is possible to compensate for this effect by inducing an equal and opposite one with the application of DC voltages. In fact, we can calculate that if these are present, they will also enter the

²We often identify the x DOF with the ground tilt η around the y GRS axis, since a change in η results in a translation $x = L\eta$ of the TM with respect to the GRS.

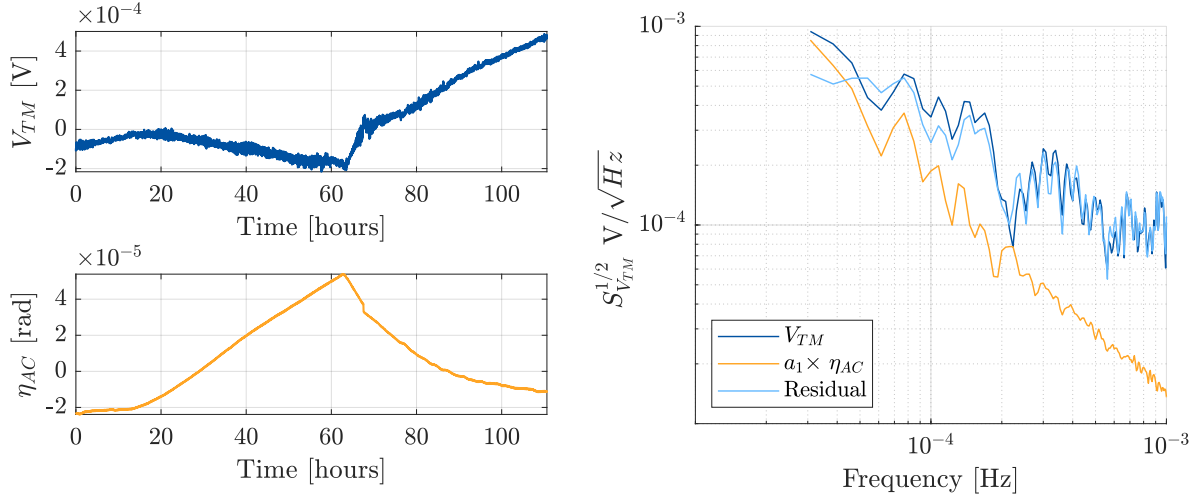


Figure 7.5: (Top left) Charge measurement time-series monitored for a very long period of time. (Bottom left) $\eta(\propto x)$ position of the pendulum, clearly showing a correlation with the upper plot. (Right) ASD of $V_{TM}(t)$ and fit of $\eta(t)$ to it. In cyan, the ASD of the subtraction: low frequencies are completely dominated by the cross-talk.

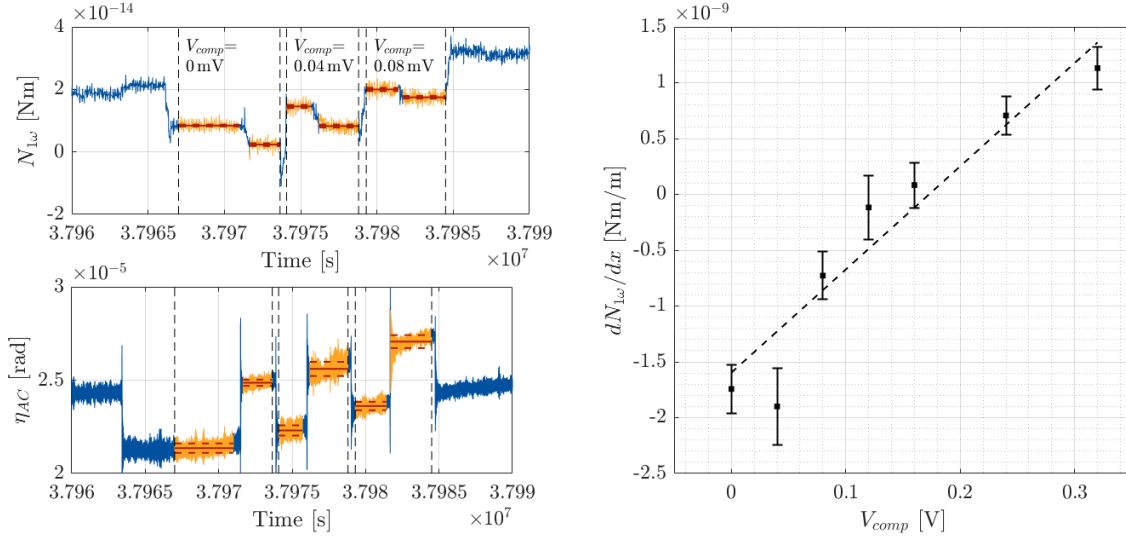


Figure 7.6: (Top left) $N_{1\omega}(t) \propto V_{TM}(t)$ monitored in time. The dashed lines delimit regions in which a certain compensation voltage is imposed: during that time, a variation of the x pendulum position (bottom left panel) is induced to then evaluate $\partial N_{1\omega}/\partial x$ as $\Delta N_{\phi,1\omega}/\Delta x$. This quantity is represented in the (right plot) together with its linear fit, for a few more points than just the ones on the left. It is important to use this "sudden variation technique", so that the estimate is not affected by the drifts that are visible in the left panels.

modulation

$$\begin{aligned}
 N_{\phi}^{V_{DC}} &= V_{\text{mod}} \cos \omega t \left(\left. \frac{\partial^2 C_1}{\partial \phi \partial x} \right|_0 V_{\text{DC},1} - \left. \frac{\partial^2 C_2}{\partial \phi \partial x} \right|_0 V_{\text{DC},2} + \left. \frac{\partial^2 C_3}{\partial \phi \partial x} \right|_0 V_{\text{DC},3} - \left. \frac{\partial^2 C_4}{\partial \phi \partial x} \right|_0 V_{\text{DC},4} \right) x \\
 &= V_{\text{mod}} \cos \omega t \left. \frac{\partial^2 C_x}{\partial \phi \partial x} \right| (V_{\text{DC},1} + V_{\text{DC},2} - V_{\text{DC},3} - V_{\text{DC},4}) x.
 \end{aligned} \tag{7.29}$$

All these terms together will determine the sensitivity of $N_{1\omega}$ (i.e. V_{TM}) to x

$$\frac{\partial N_{1\omega}}{\partial x} = V_{\text{mod}} \left. \frac{\partial^2 C_x}{\partial \phi \partial x} \right| \left(\Delta_{2x} + \underbrace{V_{\text{DC},1} + V_{\text{DC},2} - V_{\text{DC},3} - V_{\text{DC},4}}_{V_{\text{comp}}} \right) \tag{7.30}$$

In order to compensate just this effect while not introducing cross-talks with other degrees of freedom, the $V_{DC,i}$ must be chosen with a certain symmetry, here $V_{\text{DC},1} = V_{\text{DC},2} = -V_{\text{DC},3} = -V_{\text{DC},4}$. To find the V_{comp} that exactly balances Δ_{2x} , we have to measure $\partial N_{1\omega}/\partial x$ varying the compensation potential. This can be done by setting a certain DC voltage, then inducing a large x variation of the pendulum by placing some weights on the platform. Results of such procedure are reported in Figure 7.6. Linear interpolation gave as result that $V_{\text{comp}} = (170 \pm 20) \text{ mV}$, but the value was fine tuned in an iterative procedure to $V_{\text{comp}} = 216 \text{ mV}$, meany 54 mV per electrode.

7.2.2 Apparent yield measurement

Here a measurement of the apparent yield Y is presented, since it enters in many ways in the discharge model. From its definition (7.3), it can be calculated as

$$Y(V_{TM}) = \frac{C_{\text{tot}}}{eP_{UV}} \frac{dV_{TM}}{dt}. \tag{7.31}$$

To estimate dV_{TM}/dt at different V_{TM} we proceed as follows:

1. Charge the TM to a voltage as large as possible (let's say positive). With the fast discharge mechanism and the 1TM FEE voltages it is possible to reach a bit more than +1 V.
2. Turn on the pulsed UV light with the phase settings for which we want to obtain the Y curve. Choose the UV power so that it takes $\sim$ half a day to discharge completely, typically achieved with a time averaged UV power of 1 nW (realized for example with pulses of nominally 25 mA in the LED, 4 μ s and repetition rate of 6.25-12.5 kHz).
3. Measure $V_{TM}(t)$ and $P_{UV}(t)$ as it discharges. It will first discharge linearly when V_{TM} is big (since the apparent yield is constant at large saturation voltages), it will then start to approach the set-point exponentially.

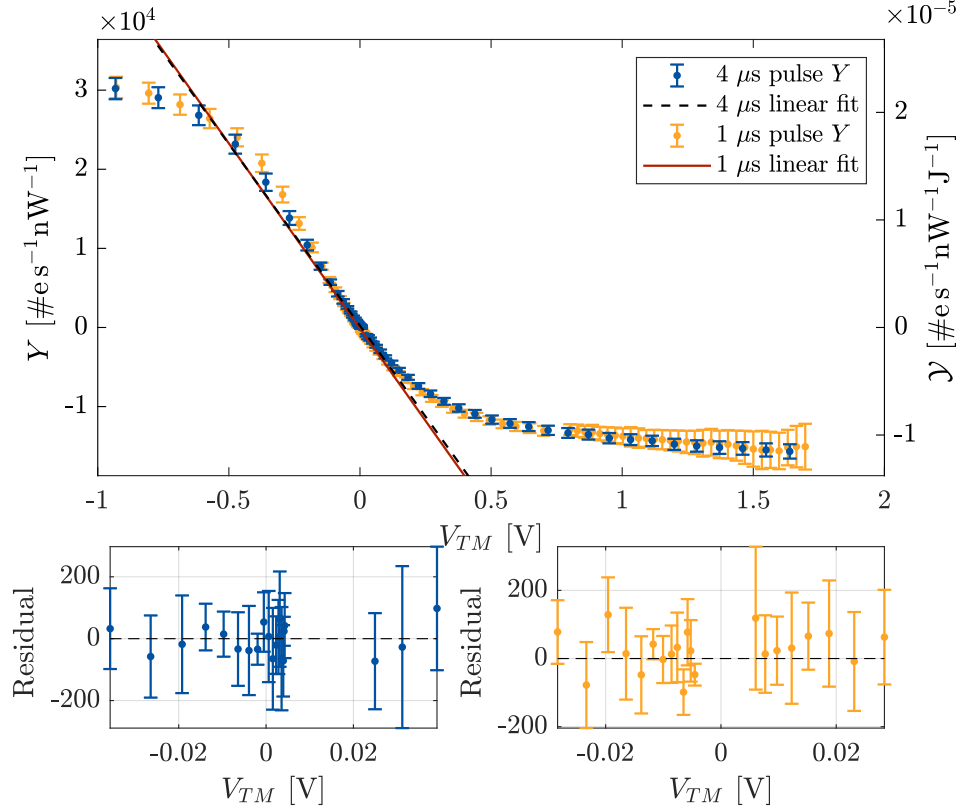


Figure 7.7: (Top) Apparent Yield measurement for two configurations of UV pulses, one using $4\ \mu\text{s}$ illumination in phase with the injection voltage peaks, the other using $1\ \mu\text{s}$ pulses: both were chosen to result in $V_{sp} \approx 0$. The right axis is shown in units as [20]. (Bottom) Residuals of the linear fits in the regions around $V_{sp} \approx 0$.

4. Divide the time series $V_{TM}(t)$ in stretches of equal length. For each stretch evaluate $\langle V_{TM} \rangle$ and $\langle dV_{TM}/dt \rangle$.
5. Use (7.31) to obtain the AY.
6. Repeat from point 1. but charging the TM negative.

In Figure the result of such measurement is reported for two configurations of UV light. The apparent yield curve exhibits the same qualitative behavior discussed previously, with two saturation regimes, the negative one being larger in modulus than the positive, as expected from illumination of the TM. Illumination of the EH would have lead to the opposite behavior. Also shown are linear fits to a subset of points, $\pm 40\ \text{mV}$ around the equilibrium potential, with the respective residuals. This gives a value of $dY/dV_{TM} = (-4.6 \pm 0.2) \times 10^{13}\ \#e/s/nW/V$ for both the curves. We will compare this number with other estimates obtained through other methods in the following sections.

The time-averaged power measurement is performed *simultaneously* to illumination of the

TM, with a splitting fiber, a PMT + amplifier + lock-amplifier. The lock-in is locked to the 100 kHz injection voltage and demodulates higher harmonics of the UV pulses: this allows to not only measure the average illumination power, but also monitor the phase of the light pulses. For more details see App. 9.7. Probably the most important source of uncertainty in this measurement is knowing the calibration of the power measurement with the PMT dP/dV : this then would reflect in a systematic multiplicative factor error in Figure 7.7, but the ratio between powers of different measurements should be accurate. At confirmation of this, the orange curve in Figure 7.7 is realized in 3 pieces, all performed with different UV powers. While the measurement of dV_{TM}/dt have completely different values in the three pieces, they perfectly align after normalization to Y with P_{UV} measurements.

7.2.3 UV induced charge noise measurement campaign

A measurement campaign was carried out to observe noise in Q_{TM} induced by continuous, pulsed illumination, and in particular to observe if there is excess charge noise beyond the simple poissonian model presented in Eq. (7.18). One campaign was carried out with pulses of 1 μ s in 2022, the other with 4 μ s pulses: this last was done to be consistent with the previous 4TM measurements [20]. The choice of 4 μ s centered on the injection voltage peaks minimizes the impact of light phase fluctuations on the noise (see Section 7.2.4), and allows application of larger UV power. Here only this campaign will be reported. The 1 μ s pulse campaign was carried out with worse pendulum noise performance and in absence of apparatus temperature control; nonetheless, it gave results consistent with what will be presented here. To observe the noise related to UV charging, the UV phase is set in order to have $V_{TM} \approx 0$ as much as possible. Then, the light should automatically keep V_{TM} constant. In these conditions $V_{TM}(t)$ is measured using a large modulation of $V_{mod} = 5$ V at 3 mHz and it is registered for a long period of time. An example of the typical spectrum during such a measurement can be found in Figure 9.13, in App. 9.5. We first perform a run with the UV light off, then a series of runs with increasing UV power. Figure 7.9 reports all the runs, together with their PSD. Each run is characterized by its discharge time τ , which was measured for each of them by inducing a discharge and fitting to an exponential function. The τ derived in this way are shown in Figure 7.8, in a plot which also shows the UV power measured simultaneously with the PMT. The two quantities in our model are inversely proportional, as expressed by (7.8), reported here

$$P_{UV} = -\frac{1}{\left.\frac{dY}{dV_{TM}}\right|_{V_{eq}}} \frac{C_{tot}}{e\tau_{UV}} = A\tau^B. \quad (7.32)$$

By fitting the data in Figure 7.8, we extract $B = (-0.94 \pm 0.01)$: my hypothesis on why it is not exactly -1 is that the discharges were performed a bit outside of the purely linear regime of Y (you can see they reach 100 mV in Figure 7.8, where the apparent yield linear approximation starts to fail in Figure 7.7), resulting in a systematic slight overestimation of τ . The tension could be solved by using less wide discharges.

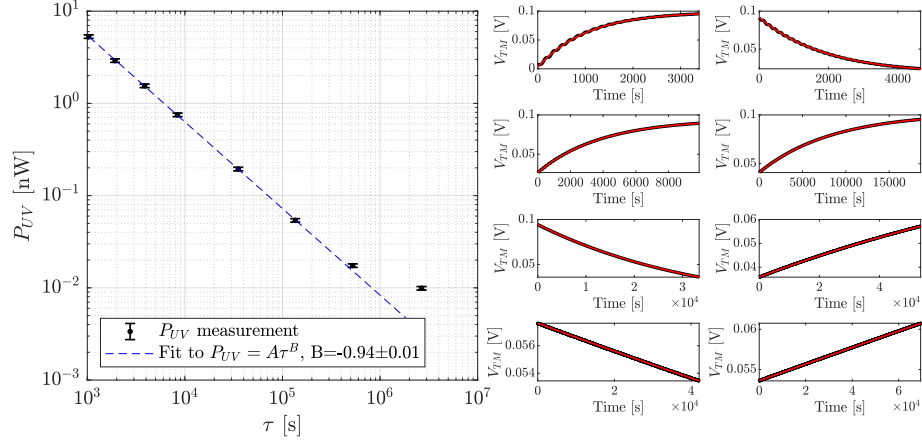


Figure 7.8: (Left) Time-averaged UV power versus discharge time constant τ , fitted to an inversely proportional law. The last two points deviate from the trend of the others: this was due to an error, a too large lock-in full scale, which resulted in poor resolution of P_{UV} . These two points are excluded from the fit, but the τ measurements should be correct, since they are extracted from measurement of exponential decays. (Right) In black, the measured exponential decay of $V_{TM}(t)$ while going toward equilibrium for different UV powers. In red, the fitted exponentials from which τ estimates are extracted.

The slope instead depends on dY/dV_{TM} , so from the fit parameter A we extract $dY/dV_{TM} = (-5.9 \pm 0.6) \times 10^4 \text{ \#e/s/nW}$, which is compatible at 2σ with the estimate obtained in Section 7.2.2. This is a confirmation of the relation between UV power and discharge time through apparent yield.

Moving on to the actual noise runs in Figure 7.9, it is visible by eye that the time series with UV light are noisier than the one with no UV light³; this obviously reflects in an increase in the PSD, which however seem to flatten at low frequencies for the highest UV power runs (it is pretty evident in the red run). This behavior is actually expected if we look at the noise model (7.18), which highlights that more power also means a higher lowpass filter frequency of the noise (it will be more clear when we show a fit to the model). This behavior was never observed so accurately before this campaign.

³It may be noticed that some of the largest oscillations have periods of approximately one day. Many things in the lab show the day/night modulation due to temperature, so it is difficult to point out exactly which phenomenon is causing the behavior here. Since our analysis focuses on the hours scale, this behavior may affect only the lowest frequency in this work.

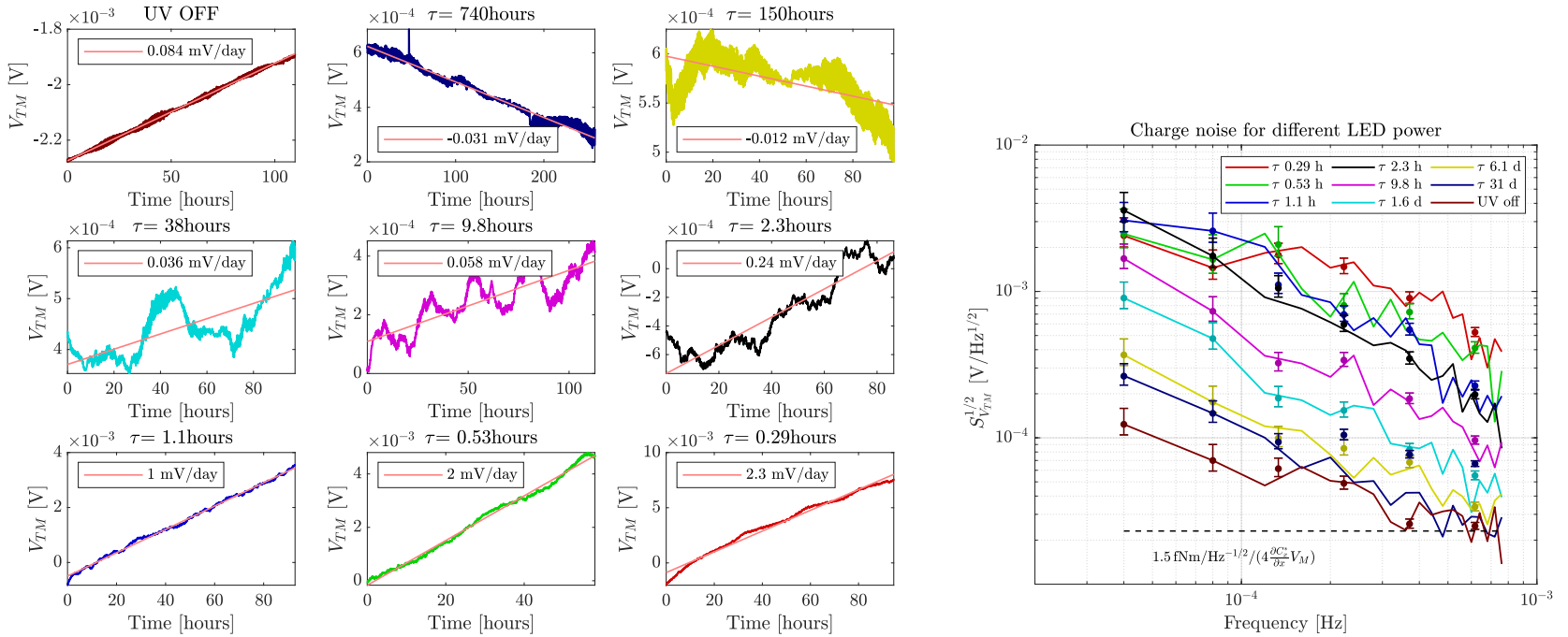


Figure 7.9: (Left) Time series of $V_{TM}(t)$ monitored for a long period of time in presence of UV light of different intensities. A linear trend is also fitted to the data, represented as a pink line, and its slope reported in the legend. (Right) ASDs of all the time series on the left, with corresponding colors. They are shown up to 0.8 mHz: after that they start being heavily affected by the Blackman-Harris filter used in the demodulation of the signal. As a black dashed line, the background noise fixed by the pendulum sensitivity at 3 mHz, expressed by in Eq. (7.27), discussed in App. 9.8. The "UV OFF" run shows an increase in noise toward low frequencies, which is roughly compatible with the voltage noise of the 1TM electronics. As stated before, the V_{TM} signal is in phase with the modulation, while we should find nothing in the quadrature: this is indeed observed and it is reported in App. 9.8. **Note:** the $\tau = 740$ hours run is very long and is reported to show the drift on such timescales. However, to estimate the spectrum we only use the first and last 60 hours, corresponding to two weekends. The working days are often affected by small transients, which can be discarded but we did not want to manipulate the data too much. The initial spike corresponds instead to a nearby Earthquake, so the corresponding stretch was removed from the analysis.

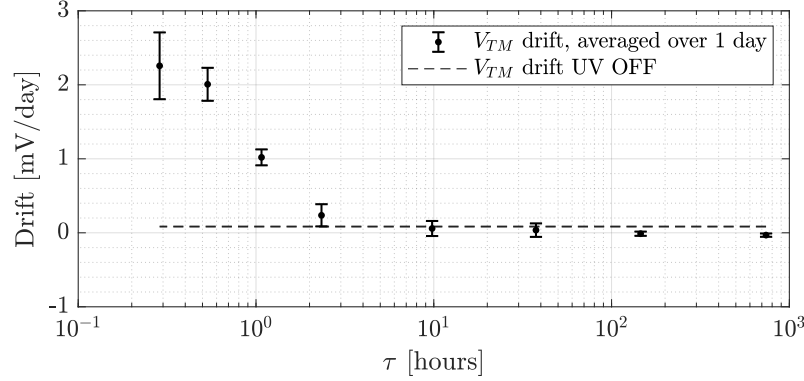


Figure 7.10: Estimate of the drift of equilibrium voltage as a function of the discharge time $\tau \propto 1/P_{UV}$, obtained from each run of Figure 7.9. Uncertainties are estimated by fitting a linear model to 1 day of data, then repeating the fit on an interval shifted by 2 hours, iterating for the whole length of the data. All the estimates obtained are averaged and the error is the standard deviation. This procedure averages out periodic daily behaviors.

Drift in the equilibrium voltage

We stated that choosing the phase of UV pulses will bring the TM to a voltage and keep it constant. So why do the runs in Figure 7.9 exhibit drifts? This is a true mystery of this measurement. It can be seen that even the UV OFF run exhibits a drift of 0.08 mV/day (corresponding to ≈ 1 electron every 4 s), which was found consistent also in other UV OFF runs. Then, the drifts as a function of τ are reported in Figure 7.10. The runs with a very weak UV light seem to exhibit a negative drift, but this could be just due to the fact that with such a long τ , the equilibrium potential was never actually reached, and we are just seeing the ultra slow exponential decay towards that. More interesting are the high power runs (low τ), since they seem to exhibit a drift increasing with power! It is not correlated with any other signal in the apparatus and the drift is always positive. We have no quantitative explanation for this behavior up to now. It seems that this was not observed in measurements with the LPF "replica" EH in the 4TM pendulum [20]. The measurements shown there are corrected for another kind of drift in x pendulum position, and seem to always remain within 1 mV for almost a week. Even without the subtraction of the just mentioned x drift, they remain within 2 mV independently of P_{UV} . In the 1TM campaign instead we observe drifts of 10 mV over 4 days at max power (≈ 5 nW).

The main hypothesis is that something inside the EH is capturing some of the electrons extracted by the UV light, it is charging up very slowly and its increasing voltage is biasing the TM. Another hypothesis is that we are modifying the properties of the surfaces with UV light (we are "cooking" them). Neither of these explain the observed drift at UV OFF, which corresponds to 1 electron every 5 seconds leaving the TM. May it be worth to place a particle counter inside the chamber to see if it could be related to atmospheric muons?

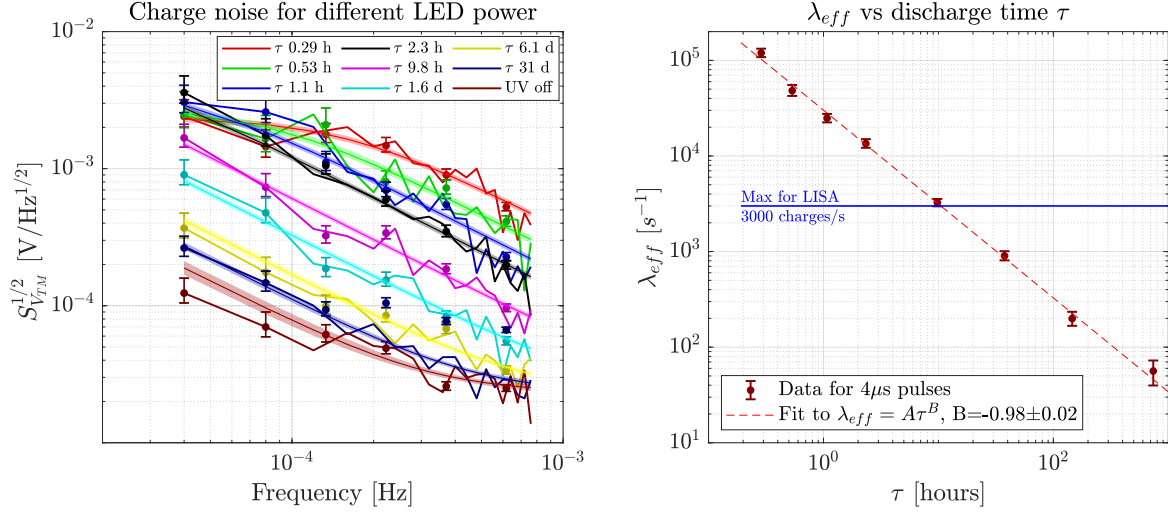


Figure 7.11: (Left) The same spectra of Figure 7.9, with the addition of the result of MCMC fitting to the discharge noise model (7.33). One value of λ_{eff} is extracted for each spectrum, so they are almost independent estimates (only the background is common to all of them). (Right) λ_{eff} obtained by the fits, plotted against the discharge time. In blue, the maximum level of charge traffic envisioned for LISA.

Maybe, but still it does not explain why this is not observed on the 4TM.

What if this drift is actually real and it is on LISA? It is planned to use $\tau \approx 1$ day (see next Section for explanation), corresponding to a drift < 0.1 mV/day in our measurements. The fixed goal of the continuous discharge strategy is to keep the TM within ± 5 mV for one month: therefore, we think such a drift would still be compatible with such goal.

Fit to discharge noise model

The spectra in Figure 7.9 are fitted to analytic model (7.18), reported here in a more complete way

$$S_{V_{TM}}(\omega) = \left(\frac{\tau}{C_{tot}} \right)^2 \frac{2e^2 \lambda_{eff}}{1 + \omega^2 \tau^2} + S_{V_{TM}}^{UV\ off}(\omega). \quad (7.33)$$

We use the measured values of τ (they are not parameters to be extracted from the fit) and $S_{V_{TM}}^{UV\ off}(\omega)$ as a background noise for the measurement, also described as $1 + 1/f^2$ spectrum. With the usual MCMC fit we extract the values of λ_{eff} (absolute number of charges exchanged per second) for each run. Results of the fit are reported in Figure 7.11, together with the various λ_{eff} extracted. It is now easier to appreciate the expected low pass filter behavior that the PSD should exhibit when large powers are used, and how this matches the observations. Overall, the data seems to be compatible with a model that just assumes charge oscillation due to shot noise from the photoelectrons.

The behavior of λ_{eff} with τ is described by (7.20), which we rewrite here

$$\lambda_{eff} \simeq \frac{C_{tot}}{e \left| \frac{dY}{dV_{TM}} \right|_{V_{eq}}} \frac{|Y_{EH}(V_{sp})| + |Y_{TM}(V_{sp})|}{\tau} = A\tau^B \quad (7.34)$$

Fitting to the data on the right of Figure 7.11, coefficient $B = -0.98 \pm 0.02$ confirms the expected behavior of the noise with power. The coefficient A instead should give information on the sum of EH and TM yields absolute values, quantity that we can not access through other measurements. Using the value measured from Figure 7.7 for $\frac{dY}{dV_{TM}}|_{V_{eq}}$, here we find

$$|Y_{EH}(V_{sp})| + |Y_{TM}(V_{sp})| = \frac{Ae \left| \frac{dY}{dV_{TM}} \right|_{V_{eq}}}{C_{tot}} = (20 \pm 4) \times 10^3 \#e/s/nW \quad (7.35)$$

This is in agreement with our upper limit for this quantity $2 \min(Y_{sat}^+, Y_{sat}^-) \approx 30 \times 10^3 \#e/s/nW$, estimated to be roughly $15 \times 10^3 \#e/s/nW$ from the positive saturation of Figure 7.7. Also represented in Figure 7.11 is the maximum allowed level of charge fluctuations for LISA, corresponding to $\lambda \approx 3000$ charges/s. We want to use a UV light which minimizes τ to have the fastest control of TM charge, in order to answer big charging events like SEPs: with the device in this analysis a good compromise would be to use a $\tau \approx 1$ day, corresponding to a time-averaged UV power of ≈ 0.1 nW on the TM. We already mentioned that this number is the same envisioned for LISA.

7.2.4 Contributions from phase fluctuations

In the previous section it was assumed that the observed noise was only due to shot noise, here we consider the possibility of charge noise correlated with oscillations in the UV light pulses phase $\delta\phi(t)$. The expression for this noise is given by (7.17), which highlights that a filtered version of $\delta\phi(t)$ will affect V_{TM} . What is instead directly proportional to phase noise is

$$\frac{dV_{TM}}{dt} = -\frac{1}{\tau_{UV}} \left(V_{TM} - V_{sp}(\phi_0) - \left. \frac{\partial V_{sp}}{\partial \phi} \right|_{\phi_0} \delta\phi(t) \right) \quad (7.36)$$

derived from (7.15). Therefore, here we report an analysis of the correlation between TM voltage derivative and phase oscillations during the noise runs described in the previous section. First we comment on two possible charge control strategies:

- **Short pulses:** this is the method explained in the introduction, using short pulses to illuminate the TM only when it has a precise voltage. A sketch can be found at the top right of Figure 7.12: the bottom panel instead shows how the equilibrium voltage $V_{sp}(\phi_0)$ changes with phase in this case. It follows the shape of the injection voltage, and at the ϕ_0 we chose to discharge the TM, we measured $\partial V_{sp}/\partial \phi \approx 450$ mV/rad.

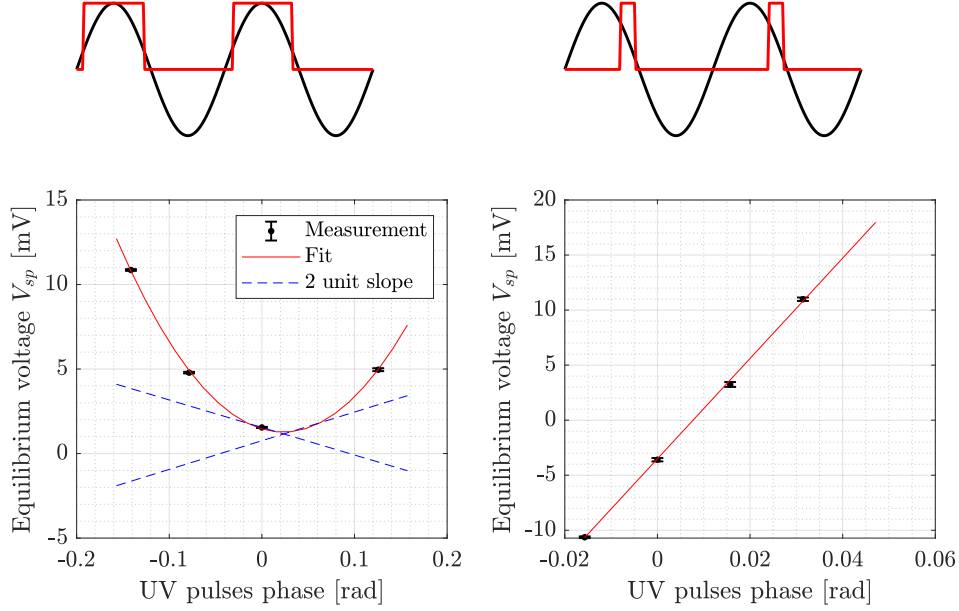


Figure 7.12: (Top) Sketches of the two illumination strategies. (Bottom) Measurement of $V_{eq}(\phi_0)$ in the two cases. The left panel experimental points are fit to a parabola, while the right panel ones to a straight line. From the parabola fit, we estimate the possible maximum slopes, as described in the text.

- **Long pulses:** these instead are always centered on the sinusoid peak and the equilibrium voltage is chosen by setting a different illumination time, in order to average over a larger or smaller fraction of the injection voltage. The measurement campaign was carried out in this conditions with $\approx 4 \mu\text{s}$ illumination. Since we are at the sinusoid peak (Figure 7.12), we are at a local minimum for $V_{sp}(\phi_0)$, where the slope should be minimized. Assuming that during the measurement we could have committed an error of 50 ns (the UV driving electronics allow to change the phase of 25 ns units) in centering the pulses to the injection peak, we take as a rough estimate of the *maximum* slope $\partial V_{sp}/\partial \phi \approx 20 \text{ mV/rad}$, reduced by more than a factor ten with respect to the 1 μs strategy.

We anticipate that the correlation found was very low, for both cases. Therefore, we only discuss the "short pulses" case, in which the sensitivity to phase noise is maximized, and finally give a few comments on the "long pulses" runs. Figure 7.13 reports a run that uses the maximum power we can shine with the current setup. During the first day of the measurement, the UV phase is changed by 25 ns units, four times: it is possible to see that also V_{TM} then exponentially decays towards four different equilibrium voltages, which are the same reported in Figure 7.12. The following days of the measurement, highlighted by a blue area, are instead just meant to measure noise. Part of the time series of the left panels of Figure 7.13 are also highlighted in red: these are the days during the weekend and are

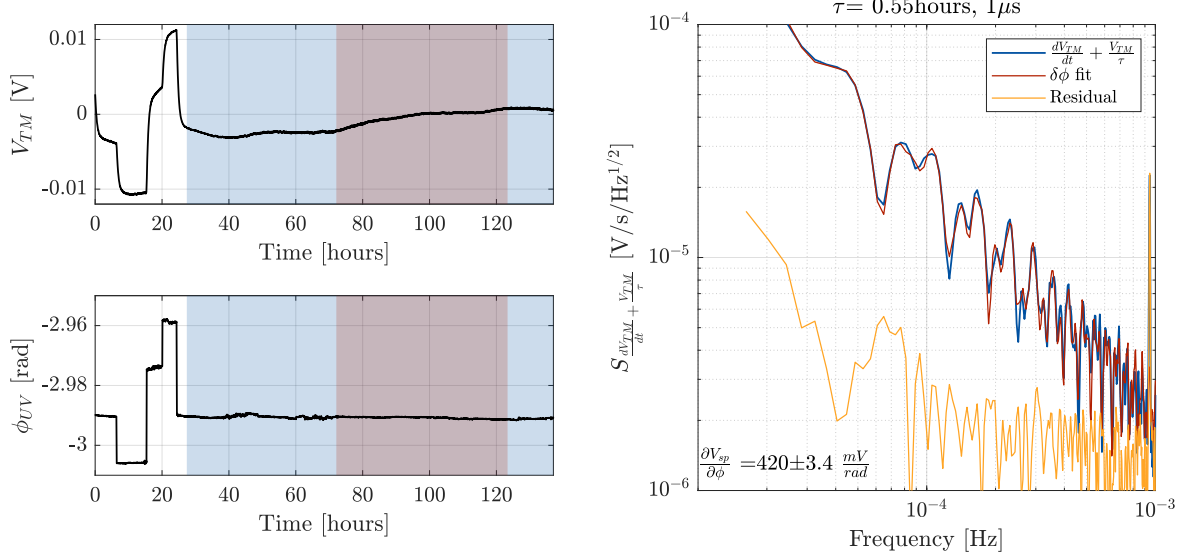


Figure 7.13: (Top left) $V_{TM}(t)$ time series during illumination with UV pulses of $1 \mu s$. (Bottom left) $\phi_{UV}(t)$ measured simultaneously with a lock-in amplifier, as described in App.9.7. The blue region highlights the time dedicated to V_{TM} noise measurement, while the red one is only during the week end, where the phase is clearly more quiet. (Right) ASD of the quantity $v(t)$ described by Eq. (7.37) in blue, which includes the "big" variations at the time-series beginning; in red, ASD of $\phi_{UV}(t)$ in the bottom left panel, fitted to $v(t)$ as described in the text, and yielding an estimate $\partial V_{sp}/\partial \phi = (420 \pm 4) \text{ mV/rad}$. In orange, the residuals of the fit, roughly compatible for $f > 0.1 \text{ mHz}$ with $v(t)$ calculated just using the data highlighted by the blue area (so excluding the big signals).

visibly less disturbed than the previous ones. We will discuss how we use this data in a moment.

To evaluate the correlation between the two time series we try to perform a subtraction in frequency domain. We can rewrite (7.36), as

$$v(t) = \frac{dV_{TM}}{dt} + \frac{V_{TM}}{\tau_{UV}} = \alpha \delta\phi(t) + v_n(t) \quad (7.37)$$

where we set $V_{sp}(\phi_0) \approx 0$, $\alpha = \frac{1}{\tau_{UV}} \partial V_{sp}/\partial \phi$ and added a generic noise $v_n(t)$, accounting for all the other noises (for example the shot noise of previous section) except the phase one. We do not know this $v_n(t)$ and our objective is to determine it, together with α . We have instead $\delta\phi(t)$ and we can construct $v(t)$ using as τ_{UV} the measured one. So we invert the previous equation and move to the frequency PSD domain

$$v_n(t) = v(t) - \alpha \delta\phi(t) \quad \rightarrow \quad S_{v_n}(\omega) = S_v(\omega) + \alpha S_{\delta\phi}(\omega) \alpha^\dagger - 2\text{Re}\{\alpha S_{v,\delta\phi}(\omega)\} \quad (7.38)$$

where S_v , $S_{\delta\phi}$ are the PSD and $S_{v,\delta\phi}$ the cross spectrum of the respective time series, which can be calculated from the experimental data at the MCF on which we want to run the

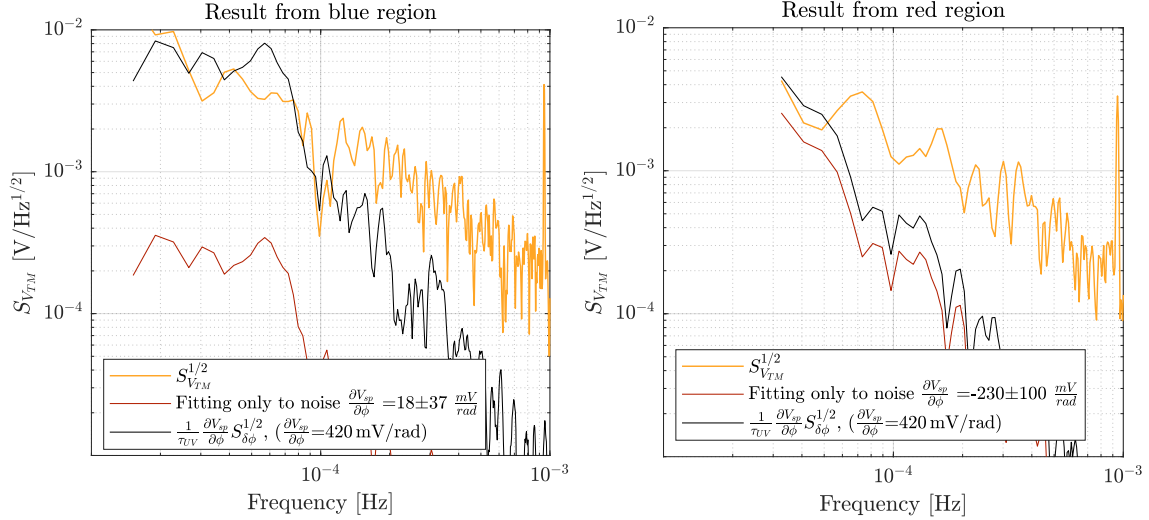


Figure 7.14: (Orange) ASD of V_{TM} during UV illumination with 1 μ s pulses. (Black) Expected contribution from phase fluctuations, calculated from Eq. ((7.17)) and using $\partial V_{sp}/\partial\phi = (420 \pm 4)$ mV/rad, obtained from a MCMC fit to the run with large induced signals. (Red) Contribution identified by running the fit described in the text only on the noise part of the run, without signals. (Left-Right) The two figures refer to the time series restricted to the blue and red regions of Figure 7.13.

analysis. Then, an MCMC is performed on the α parameter to calculate $S_{v_n}(\omega)$, constrained by a likelihood function that correctly describes the distribution of a PSD variable like $S_{v_n}(\omega)$ ⁴. An example of result of such subtraction is shown on the right of Figure 7.13, where the whole signals on the left are used in the analysis. Much of the spectrum shown is dominated by the huge signals at the beginning, which however allow a precise evaluation of the α parameter and consequently of $\partial V_{sp}/\partial\phi = (420 \pm 4)$ mV/rad. This result should be much more accurate than what can be evaluated from the slope of the data in right Figure 7.12, since those can be affected by drifts and other transients. This example is also meant to demonstrate the effectiveness of the analysis.

We now focus on a piece of the V_{TM} time-series in which there is no signal, only noise, to assess if phase noise could have had a relevant impact: we anticipate the conclusion and then try to motivate it. From the estimated $\partial V_{sp}/\partial\phi = (420 \pm 4)$ mV/rad and the measured $S_{\delta\phi}$ noise, it seems like it should contribute an important part to V_{TM} , however, we do not find any relevant correlation. We are strongly convinced that this is due to an overestimation of $S_{\delta\phi}$, that gets additional noise somewhere in the measurement chain PMT - Amplifier - Lock-in. This additional noise hides the true UV light phase fluctuations therefore no correlation can be found with V_{TM} .

To show this, in Figure 7.14 the spectrum of V_{TM} is reported in yellow for the two red

⁴This kind of subtraction called "time series decorrelation" was employed widely in the analysis of LPF data, and was improved to a new method in [52], also extended to an arbitrary number of time series.

and blue regions in Figure 7.13. This is compared with the expected contribution of phase fluctuations given by Eq. (7.17) in black. The blue region is affected by some disturbances during working days: if these were true phase fluctuations, they should be resolvable in the V_{TM} noise (the black spectrum is even higher than the yellow one). Instead, if we re-run the whole analysis presented before (Eq. (7.38)) we do not find any correlation at all: the result of the fit is represented in red. We also checked by inspection of the cross-coherence and using an alternative de-correlation procedure, namely the LTPDA function `flscov`, used for LPF data analysis, ending to the same conclusion.

Exactly the same analysis is repeated for the red region in 7.13 and reported in right Figure 7.14, hoping that removing the working days disturbances we might really resolve true phase fluctuations. The black curve shows that indeed phase noise should be resolvable again, but when we run the de-correlation algorithm and check the cross coherence, we find uncertain results. In the proposed Figure, the fit seems to identify a non-negligible correlation at very low-frequencies, which however leads to a wrong and poor estimation of $\partial V_{sp}/\partial\phi = (-230 \pm 100) \text{mV/rad}$. Re-running the analysis estimating with a different number of stretches leads almost always to completely different results, which is another reason for believing that we are actually not resolving any correlation. What we lack in this whole discussion is an estimate of the "background noise" of the phase noise measurement, which would allow us to assess if we are resolving any "true" UV phase fluctuation, but such a measurement is not trivial to perform and we do not have it here.

This analysis was repeated for all the runs at our disposal, included the "long-pulses" ones, always finding negligible correlations. In any case, the black curves of Figure 7.14 constitute truly an upper limit to the noise in the case of $1 \mu\text{s}$ pulses: for the $5 \mu\text{s}$ ones, we expect these upper limits to be reduced at least of a factor 10, as discussed previously. Therefore, even if we can state that V_{TM} fluctuations due to phase were not observed, we are completely sure that they were absolutely negligible during the campaign presented in Figure 7.2.3.

Conclusions

In this chapter we presented the observation of TM charge shot noise, induced by the traffic of photoelectrons extracted by continuous UV light pulses. Noisy oscillations of TM charge can couple to DC stray voltages in the EH and cause acceleration noise, therefore they must be kept under a certain level.

This noise had already been observed with the 4TM torsion pendulum using LISA LEDs, which was however at the limit of its sensitivity and could barely test this source of noise to the level envisioned for LISA, also finding some small discrepancies with our models. The enhanced sensitivity of the 1TM facility allowed a more precise characterization of this noise as a function of the UV power used.

First of all, the observed noise shows very good agreement with our models, which link many of the relevant quantities to the apparent yield curve and its saturation regimes. The noise

model was tested from regimes of 10^5 charges/s down to < 100 charges/s, and confirmed the possibility of discharging the TM with a typical time of ~ 1 day, without exceeding maximum traffic level of 3000 charges/s required for LISA. No excess was observed above the $1/f^2$ expected Poissonian noise across the band 0.05-1 mHz.

With the continuous pulsed UV illumination, it is possible to choose the TM potential by just setting an appropriate phase of the light pulses with respect to the injection voltage. As a consequence, noise in the pulses phase should also cause fluctuations in the TM charge. We tried to observe these by monitoring the light pulses phase during the measurements. We do not seem to observe relevant correlations in any of the runs performed, first of all because the maximum UV power at our disposal did not allow to enhance the effect of phase fluctuations (scales as $\propto P_{UV}$) enough to be much higher than the shot noise (scales as $\propto \sqrt{P_{UV}}$). Secondly, because the UV phase measurement is probably affected by some other noise in the detection chain, which seems to be dominant over true phase fluctuations. In any case, we could put upper limits to the noise which demonstrate with certainty that the noise runs presented in the campaign had negligible contributions from phase fluctuations and are fully compatible with just photoelectrons shot noise.

Finally, drifts of order 1-2 mV/day were observed in the equilibrium voltage during UV illumination with high time-averaged power ($> \text{nW}$). We could not identify the source of these drifts, which were not observed in analogous measurements performed with the LPF replica EH in the 4TM pendulum (maximum drifts observed of order 0.5 mV over 4-5 days). Anyway, if this effect was present on LISA, it is not expected to be relevant with the envisioned level of UV power.

Chapter 8

Important results summary

In this thesis we explored various aspects of the electrostatic control force system for LISA Test Masses, describing why and how it works, how it performed on LPF and how we plan to experimentally test it for LISA.

The system proved to be compatible with the $3 \text{ fms}^{-2}/\sqrt{\text{Hz}}$ free-fall environment, even in the more challenging LPF conditions with x actuation. In Chapter 4 LPF enhanced actuation runs were studied: we found very good agreement with a model describing the noise in terms of uncorrelated gain fluctuations of the 8 relevant x/ϕ electrodes, with average levels of $7 \text{ ppm}/\sqrt{\text{Hz}}$ at 1 MHz and $40 \text{ ppm}/\sqrt{\text{Hz}}$ at 0.1 MHz . Also very good agreement was found with pre-flight ground measurements. According to this model, actuation gain fluctuations account for 20-40% of the total differential acceleration noise observed during LPF noise runs, classifying as the largest known contribution. The remaining noise is currently unknown, but we know it must be a source which does not result in any lever arm (unlike actuation). Through the analysis of other LPF experiments with a charged TM, also fluctuations in the x electrostatic field were characterized and found completely compatible with those associated with FEE low frequency additive noise. This instead was a minor contribution to LPF observed acceleration noise.

In the effort of understanding the LPF mysterious low frequency noise excess, in Chapter 5 we investigated the presence of possible non-linearities in the control forces, due to distortions of the applied voltage waveforms. Through simulations, we found that $1\text{-}2 \text{ }\mu\text{V}$ errors in the waveform amplitudes could explain a large fraction of LPF noise. Measurements on a LPF FEE copy confirmed that an already known digitization error had been corrected to the sub- μV level, but revealed the presence of further, electrode dependent non-linearities. Through a careful study of simulations and analytic calculations, we understood that the truly important quantity for this noise source is the Differential Non Linearity in the waveform effective amplitude, which seemed to be $< 1 \text{ }\mu\text{V}$ for the FEE under test, leading to a not dominant contribution to LPF low frequency noise. Finally, we understood that these non linearities get converted to acceleration noise by drifts in the commanded forces/torques: for LISA, it is suggested to keep the commanded ϕ torque drift lower than $2 \text{ prad/s}^2/\text{day}$ to

have complete suppression of this noise. The sources of these drifts should be mainly related to TM charging and are easily compensated with the application of DC biases. Also a noisy torque command can lead to acceleration noise, but it is negligible if the commanded torque noise requirement is respected. In the future, LISA FEE will probably have to be tested to meet a $\text{DNL} < 1 \mu\text{V}$, but the strategy to do it is yet to be identified.

Chapter 6 introduced the 1TM torsion pendulum facility, which was restored and brought back to a torque sensitivity reaching $0.5\text{-}0.6 \text{ fNm}/\sqrt{\text{Hz}}$ at resonance. Given the importance of TM control forces stability for LISA low-frequency sensitivity, it is important to test the FEE before flight, and we plan to also use the 1TM facility for this purpose. The advantage of a torque noise measurement is that it tests directly the LISA observable, being therefore independent of any model we might have for the electrode voltage noises. The Chapter demonstrates that the facility has the capability to test actuation stability down to the requirement of $\approx 6 \text{ ppm}/\sqrt{\text{Hz}}$ at 1 mHz. This was done using some home-made electronics and observing an increase in pendulum noise floor proportional to the torque authority used for four electrodes. Measuring torque noise from a couple of electrodes would also be of interest, but this is currently not possible due to the presence of a cross-talk with the test mass y -position: this is a further motivation for the implementation of an active tilt control of the pendulum in the near future. Complication in the data analysis for the recovery of gain fluctuations were encountered due to poor knowledge of the contribution of optical lever readout noise to the observed noise. This degeneracy can be removed with further experiments which employ AC actuation on Y electrodes, an improvement currently being implemented. Finally, the recovered level of fluctuations ($(3 \pm 1) \text{ ppm}/\sqrt{\text{Hz}}$ at 1 mHz) was compared with a direct measurement of voltage noise: the two measurements only show a rough agreement, but were affected by several problems, thus the results are not easily interpreted.

Chapter 7 describes a measurement campaign conducted with the 1TM facility, targeting the noisy charge fluctuations induced by the use of continuous pulses of UV light, a strategy envisioned for LISA to keep the TM discharged. The campaign completes a wider study carried out with the 4TM torsion pendulum, testing this source of noise down to the levels of 100 charges/s transfers, thus exploring also the level envisioned for LISA of 3000 charges/s. Perfect agreement was found with our discharge model, linking the UV power used to the discharge time and Apparent Yield of the sensor under investigation. The observed noise is compatible with a purely Poissonian process: other possible sources of noise like fluctuations in the UV pulses phase were found to be completely negligible. Finally, a drift of 1-2 mV/day in TM equilibrium voltage was observed during UV illumination with high time-averaged power ($> \text{nW}$). We could not identify a cause for this phenomenon, but it was not observed in other facilities and would be completely irrelevant for LISA.

Chapter 9

Appendix

9.1 Appendix A - PSD of delayed signals

Given a noise $s(t)$, this is stationary if its autocorrelation $R_{ss}(\tau) = \langle s(t)s(t+\tau) \rangle$ does not depend on t , meaning that you obtain the same result by calculating this quantity choosing any time as a starting point. The Power Spectral Density of $s(t)$ is then just the Fourier transform of $R_{ss}(\tau)$:

$$S_{ss}(\omega) = \int_{-\infty}^{+\infty} R_{ss}(\tau) e^{i\omega\tau} d\tau \quad (9.1)$$

What is then the PSD of a signal $s'(t) = s(t) + s(t-2T)$? We start by calculating the autocorrelation

$$\begin{aligned} R_{s's'}(\tau) &= \langle [s(t) + s(t-2T)] [s(t+\tau) + s(t-2T+\tau)] \rangle \\ &= + \langle s(t)s(t+\tau) \rangle + \langle s(t)s(t-2T+\tau) \rangle \\ &\quad + \langle s(t-2T)s(t+\tau) \rangle + \langle s(t-2T)s(t-2T+\tau) \rangle \end{aligned} \quad (9.2)$$

the first and the last term are just $R_{ss}(\tau)$ since the noise is stationary, so if we now take the Fourier transform, they will just give one $S_{ss}(\omega)$ by definition (9.1). We calculate instead the FT of the second term

$$\begin{aligned} &\int_{-\infty}^{+\infty} \langle s(t)s(t-2T+\tau) \rangle e^{i\omega\tau} d\tau = \\ &= \int_{-\infty}^{+\infty} \langle s(t)s(t+\tau') \rangle e^{i\omega\tau'} e^{i\omega 2T} d\tau' = e^{i\omega 2T} S_{ss}(\omega). \end{aligned} \quad (9.3)$$

Analogously, the third term gives $e^{-i\omega 2T} S_{ss}(\omega)$, and summarizing

$$S_{s's'}(\omega) = (2 + 2 \cos 2\omega T) S_{ss}(\omega) = 4 \cos^2(\omega T) S_{ss}(\omega). \quad (9.4)$$

Instead, for a noise $s''(t) = s(t) - s(t-2T)$, we have

$$S_{s''s''}(\omega) = (2 - 2 \cos 2\omega T) S_{ss}(\omega) = 4 \sin^2(\omega T) S_{ss}(\omega). \quad (9.5)$$

9.2 Appendix B - Shot noise at the photodiode to phase noise

The arrival times of the photons are completely uncorrelated between each other, therefore this constitutes a Poissonian process. Considering a beam of average power $\bar{P}$, the rate of arrival of photons is given by

$$\Lambda = \frac{\bar{P}}{h\nu} = \frac{\bar{P}\lambda}{hc}, \quad (9.6)$$

and the auto-covariance of the process is

$$C(t, t') = (h\nu)^2 \Lambda \delta(t - t') \quad (9.7)$$

which means a white power spectral density of amplitude

$$S_{\text{shot}} = (h\nu)^2 \Lambda = \frac{hc}{\lambda} \bar{P}. \quad (9.8)$$

Ok, the photodiode measures power, but we are interested in the phase of the laser, how does this noise propagate on that? Let's see it with a simple calculation. We said many times that the phase/frequency measurement is performed by interfering the local laser with the one coming back from the round trip. We write their electric fields

$$E_{em}(t) = E_{em} \sin[\omega t], \quad E_{back}(t) = E_{back} \sin[\omega t + \phi(t)], \quad (9.9)$$

where we supposed that the incoming beam has gained a phase that we want to measure, and maintains a constant amplitude. On the photodiode, by simple algebra and discarding all the terms that oscillate at optical frequencies (averaged by the photodiode response)

$$\langle |E_{em} + E_{back}|^2 \rangle = \frac{E_{em}^2 + E_{back}^2}{2} + E_{em} E_{back} \cos \phi(t) \quad (9.10)$$

For the sake of simplicity, we suppose $\phi(t) = \pi/2 + \delta\phi(t)$, which basically means studying the power response to small phase variations where we have maximum sensitivity.

$$\langle |E_{em} + E_{back}|^2 \rangle = \frac{E_{em}^2 + E_{back}^2}{2} + E_{em} E_{back} \delta\phi(t) \quad (9.11)$$

To transform to power, we assume to have a perfect photodiode **that detects all the light** and recall that the intensity of the light is related to the electric field by $I = c\epsilon_0 |E|^2 / 2 = P/\text{Area}$. Then by a simple multiplication of the above formula we obtain

$$P(t) = \frac{P_{em} + P_{back}}{2} + \sqrt{P_{em} P_{back}} \delta\phi(t), \quad (9.12)$$

which means that the power measured is on average the sum of the two laser power, plus a small oscillation proportional to the phase we are interested in. We discard the DC part

and we want to cast this equation in a form that allows to asses S_{IFO} : therefore we need to convert the phase to a length by recalling that 2π phase corresponds to one λ of the laser ($x = \lambda\phi/2\pi$)

$$x(t) = \frac{\lambda}{2\pi} \delta\phi(t) = \frac{\lambda}{2\pi} \frac{P(t)}{\sqrt{P_{em}P_{back}}} \quad (9.13)$$

Finally, if the power measurement is affected by shot noise $n_P(t)$ with PSD given by (9.8), we can add it to Eq. (9.13) by substituting $P(t) \rightarrow P(t) + n_P(t)$ and we find that the noise in x is then

$$S_{IFO}(\omega) = \left(\frac{\lambda}{2\pi}\right)^2 \frac{S_{shot}}{P_{em}P_{back}} = \left(\frac{\lambda}{2\pi}\right)^2 \frac{hc(P_{em} + P_{back})}{\lambda P_{em}P_{back}} \quad (9.14)$$

For example, in the case that $P_{back} \ll P_{em}$, it is immediately evident that $S_{IFO} \propto 1/P_{back}$, which means that the noise amplitude improves as $1/\sqrt{P}$, a known characteristic of shot noise, but also that the power of the weaker beam is the limiting factor for this noise.

9.3 Appendix C - Bayesian parameter estimation of electrode noises

This appendix is dedicated to the detailed description of how the actuation gain fluctuation model for LPF, introduced in Sec. 4.1.1, is "fitted" to the LPF enhanced actuation runs with a Bayesian Markov Chain Monte Carlo algorithm. The objective is the parameter estimation of the spectrum S_α of all the noise generators appearing in the model, correlated and uncorrelated.

We introduce a distinction between the experimental PSD, calculated from the actual data, and the "model" PSD, or let's say "theoretical" one: this is the one that we calculate from our model. Obviously, we would like to find the model PSD that better describes the observed experimental PSD. To put it in more Bayesian terms, we want to calculate the posterior distributions of the PSDs that describe gain fluctuations at each electrode, conditioned by the fact that we observed certain experimental realization of them. This distinction should become clearer following this Appendix. We indicate with

$\hat{S}$ the experimental PSD, with S the "model" PSD.

Computing the experimental PSD

Let us first see how $\hat{S}$ is computed from the experimental data. We have 4 different experiments, for which we have linear acceleration $\Delta g(t)$ and angular acceleration $\Delta\gamma_\phi(t)$. These are ~ 2 day long time series sampled every $T = 0.1$ s, and to get an estimate of the PSD we usually divide them in stretches. To account for all of these, we introduce the following notation

- $\Delta g(t)[n, q, s]$ is the $\Delta g(t)$ n -th sample of the s -th stretch of the q -th experiment. Same for $\Delta\gamma_\phi[n, q, s]$.

- $q=1,2,3,4$ represents the 4 experiments, ordered as UURLA, Nominal, Big, Big+off. $n \in [0, N - 1]$ where N is the length of each stretch. $s \in [0, N_s(q) - 1]$ where $N_s(q)$ is the number of stretches in which that experiment was divided (since they have different lengths).

Given one stretch, we can estimate the *modified periodogram* as in [52], for example for Δg as

$$X_{\Delta g}[k, q, s] = \sqrt{\frac{T}{N}} \sum_{n=0}^{N-1} \Delta g[n, q, s] w[n] e^{-i2\pi k n/N} \equiv X_{\Delta g}(f = k/NT, q, s), \quad (9.15)$$

with an analogous definition for $X_{\Delta \gamma_\phi}$ and where $w[n]$ is a spectral window, commonly employed in the estimation of PSDs. We will always use the Blackman-Harris window.

By grouping the modified periodograms in the complex vector

$$\mathbf{X}_s = \sqrt{2} \begin{bmatrix} X_{\Delta g}(f, q, s) \\ X_{\Delta \gamma_\phi}(f, q, s) \end{bmatrix}, \quad (9.16)$$

the experimental estimate $\hat{S}$ of the *single-sided cross spectral density* (CPSD) matrix can be expressed as

$$\hat{\mathbf{S}}(f, q) = \begin{bmatrix} \hat{S}_{\Delta g}(f, q) & \hat{S}_{\Delta g, \Delta \gamma_\phi}(f, q) \\ \hat{S}_{\Delta g, \Delta \gamma_\phi}^*(f, q) & \hat{S}_{\Delta \gamma_\phi}(f, q) \end{bmatrix} = \langle \mathbf{X}_s \mathbf{X}_s^\dagger \rangle_s \quad (9.17)$$

where $\langle \rangle_s$ indicates an average over all the $N_s(q)$ periodograms obtained by the $N_s(q)$ different stretches. Notice that this is the estimate of the CPSD at a single frequency f , to evaluate it at a different one we could decide to start from scratch and choose completely different stretches. Therefore, we can generalize $N_s(q) \rightarrow N_s(f, q)$: one must just be careful not to introduce correlations in the estimation of $\hat{\mathbf{S}}$ at different frequencies with this game. In this thesis the stretches are chosen in two ways

1. **Standard Welch method:** [68] the same number of stretches is chosen for estimating $\hat{\mathbf{S}}$ at all frequencies, which so depends only on q , $N_s = N_s(q)$. Using 110×10^3 s, 50% overlapping stretches, we have 3 for UURLA and 2 for the other experiments. Using a Blackman-Harris window, sufficiently uncorrelated estimates of $\hat{\mathbf{S}}$ are at frequencies $n f_{\min}$, with $f_{\min} = 4/NT$ and n an integer [64]. PSDs estimated in this way are plotted as continuous lines.
Data computed with this method are used for the fits that assume smooth functions to describe the noise spectra.
2. **Minimally correlated frequencies method:** this technique is used to calculate $\hat{\mathbf{S}}$ with a number of data stretches that increases with frequency, having therefore $N_s = N_s(f, q)$. The frequencies, following Ref. [7, Supp.mat] and [63], are

$$f_1 = \frac{4}{NT}, f_2 = 2f_1, f_3 = \left(\frac{5}{3}\right) f_2, f_4 = \left(\frac{5}{3}\right) f_3, \dots \quad (9.18)$$

This provides points which are equally spaced in $\log f$, exhibit minimal correlation, with precision improving at higher frequencies. This is a useful tool for reducing the amount of frequencies entering the analysis while still maintaining the spectrum shape and correct statistics. It is particularly suited for "smooth" spectra (with no sharp peaks).

Estimates like these ones are reported in figures in this thesis as single points: sometimes they are presented with error bars. While the reported central point is just the result of (9.17), the error-bars are calculated following [63], in the case of a PSD variable (not cross spectrum). If we take a generic experimental PSD $\hat{S}$ calculated with N stretches as an example, its upper and lower quantiles are

$$\hat{S}_{\min}(f) = \frac{\hat{S}(f)}{\Gamma_{\text{reg}}^{-1}\left[N, \frac{1-c}{2}\right]}, \quad \hat{S}_{\max}(f) = \frac{\hat{S}(f)}{\Gamma_{\text{reg}}^{-1}\left[N, \frac{1+c}{2}\right]} \quad (9.19)$$

where $\Gamma_{\text{reg}}^{-1}[a, b]$ is the inverse of the regularised Gamma function ("upper" tail), and c is the confidence interval of choice. In this work $c = 68\%$ is used. Eq. (9.19) is slightly different from the result presented in [63], which assumes a uniform prior on the PSD. It was later agreed that a prior uniform in its logarithm is more suitable for such a scale parameter, and the formulas presented here are the result of that assumption. Data computed with this method are used to estimate the electrode noise at each single frequency, without assuming that the spectra can be described through smooth functions of frequency.

Model for enhanced actuation runs

The model for gain instability noise is discussed in Sec. 4.1.1. Here we describe the complete model for the four actuation runs and its parameters. We can recast the quantities in the model for actuation noise (4.10) in a form like (9.17), so

$$\mathbf{S}^{ACT}(f, q) = \begin{bmatrix} S_{\Delta g}^{ACT}(f, q) & S_{\Delta g, \Delta \gamma_\phi}^{ACT}(f, q) \\ S_{\Delta g, \Delta \gamma_\phi}^{ACT}(f, q) & S_{\Delta \gamma_\phi}^{ACT}(f, q) \end{bmatrix}, \quad (9.20)$$

Note that the off-diagonal terms (cross-correlations) are assumed to be real: a non-zero imaginary part would be associated with the presence of common signals which appear delayed in one of the two time series. In the hypothesis that net torques and forces arise from the same forces acting locally, then any force-torque correlation would be free of delays and thus result in a real value of the cross-spectrum. If we then also introduce the coefficients matrix and noise generators matrix

$$\mathbf{a}(q) = \begin{bmatrix} a & b \\ a_1 & b_1 \\ a_2 & b_2 \\ a_{11} & b_{11} \\ \dots & \dots \\ a_{24} & b_{24} \end{bmatrix}, \quad \mathbf{S}_\alpha(f) = \begin{bmatrix} S_\alpha & & & & \\ & S_{\alpha_1} & & & \\ & & S_{\alpha_2} & & \\ & & & S_{\alpha_{11}} & \\ 0 & & & & \dots & \\ & & & & & S_{\alpha_{24}} \end{bmatrix}, \quad (9.21)$$

actuation noise model (4.10) is then rewritten as

$$\mathbf{S}^{ACT}(f, q) = \mathbf{a}^T(q) \mathbf{S}_\alpha(f) \mathbf{a}(q). \quad (9.22)$$

$\mathbf{a}(q)$ is a 11×2 matrix, containing all the known sensitivity coefficients, depending only on the actuation forces (index q), while $\mathbf{S}_\alpha(f)$ is an 11×11 matrix which contains the actuators gain noise PSDs at frequency f on the diagonal, which we want to extract by fitting the model to the experimental data.

Now, the four runs do not only contain actuation noise, especially UURLA, therefore we introduce a common background for all of them. The full model is then

$$\mathbf{M}(f, q) = \mathbf{S}^{ACT}(f, q) + \mathbf{S}^{bg}(f). \quad (9.23)$$

where

$$\mathbf{S}^{bg}(f) = \begin{bmatrix} S_{\Delta g}^{bg}(f) & \xi(f) \sqrt{S_{\Delta g}^{bg}(f) S_{\Delta \gamma_\phi}^{bg}(f)} \\ \xi(f) \sqrt{S_{\Delta g}^{bg}(f) S_{\Delta \gamma_\phi}^{bg}(f)} & S_{\Delta \gamma_\phi}^{bg}(f) \end{bmatrix}. \quad (9.24)$$

This background matrix allows for an arbitrary residual background for both Δg and $\Delta \gamma_\phi$, while also including one for the cross correlation through the parameter $\xi(f) \in [-1, 1]$ multiplied by $\sqrt{S_{\Delta g}^{bg}(f) S_{\Delta \gamma_\phi}^{bg}(f)}$, which is the maximum possible correlation. This background allows the maximum freedom, so no particular assumptions are taken, except for a real cross-correlation, and we will marginalize over these parameters in the end to just look at the gain fluctuations.

Note: the noise generator matrix (9.21) also includes the term S_α , which was argued to be indistinguishable from a background in Sec. 4.1. For this reason, it is effectively included in (9.24).

This means that in this model we have 10 S_{α_i} and $S_{\alpha_{ij}}$, the analysis objectives, plus 3 backgrounds $\xi(f)$, $S_{\Delta g}^{bg}$, $S_{\Delta \gamma_\phi}^{bg}$, for a total of 13 noise generators.

Likelihood function

We indicate with $\hat{\mathbf{D}}$ the collection of experimental data with which we want to perform the parameter estimation. For example, if we are interested in evaluating the 13 parameters from the 4 experiments at a single frequency the collection would be $\hat{\mathbf{D}}(f) = \{\hat{\mathbf{S}}(f, q)\}_q$. If we want to analyze all the frequencies at once, we would instead group $\hat{\mathbf{D}} = \{\hat{\mathbf{S}}(f, q)\}_{q,f}$.

We can write down the posterior distribution of our model parameters conditioned by the observed data using Bayes theorem

$$p(\mathbf{M}|\hat{\mathbf{D}}) = \frac{L(\hat{\mathbf{D}}|\mathbf{M})P(\mathbf{M})}{P(\hat{\mathbf{D}})}. \quad (9.25)$$

where $L(\hat{\mathbf{D}}|\mathbf{M})$ is the Likelihood function, $P(\mathbf{M})$ the model parameters prior and $P(\hat{\mathbf{D}})$ the evidence ¹.

The likelihood function is obtained following Refs. [52, 36]. Considering a single realization of $\mathbf{X}_s$ as defined in (9.16), this follows a 2-variate complex Gaussian probability distribution

$$p_{f,q}(\mathbf{X}_s|\mathbf{M}) = \frac{1}{\pi^2|\mathbf{M}|} \exp \left\{ -\mathbf{X}_s^\dagger \mathbf{M}^{-1} \mathbf{X}_s \right\}. \quad (9.27)$$

Then, in [36] it is shown that the CPSD matrix defined in (9.17) is distributed according to a *Complex Wishart distribution*

$$p_{f,q}(\hat{\mathbf{S}}|\mathbf{M}) = \frac{N_s^{N_s} |\hat{\mathbf{S}}|^{N_s-2}}{\tilde{\Gamma}_2(N_s) |\mathbf{M}|^{N_s}} \text{etr} \left[-N_s \mathbf{M}^{-1} \hat{\mathbf{S}} \right] \quad (9.28)$$

where $\text{etr}[\cdot] = \exp(\text{tr}[\cdot])$ and $\tilde{\Gamma}$ is the multivariate complex Gamma function. Summarizing, this is the probability of observing a certain value for matrix $\hat{\mathbf{S}}$ given a theoretical CPSD $\mathbf{M}$, at a single frequency, using N_s periodograms of Δg and $\Delta \gamma_\phi$.

9.3.1 Parameter estimation from four actuation experiments at a single frequency

A frequency by frequency analysis of the spectra is useful because it does not assume any particular analytical function to describe the spectra. It is not smart to analyze singularly all the frequencies obtained with the Welch method, because they are a lot and with large uncertainties, which would force us to then average the results over frequency bands. That is why performing the parameter estimation on the minimally correlated frequencies introduced in this Appendix is very useful: this is done in this section. Our data-set at a single frequency is then $\hat{\mathbf{D}}(f) = \{\hat{\mathbf{S}}(f, q)\}_q$. The probability of observing this is

$$L_f(\hat{\mathbf{D}}|\mathbf{M}) = \prod_q p_{f,q}(\hat{\mathbf{S}}|\mathbf{M}) \quad (9.29)$$

With this likelihood a Markov Chain Monte Carlo (MCMC) based on Metropolis-Hastings sampling algorithm was implemented. This allowed us to estimate the posterior distributions of our model parameters that we recall are 13. We perform this analysis at 8 minimally correlated frequencies. The results are discussed in the following.

¹There was a suggestion that the expression for Bayes theorem proposed here and in [13] is not completely accurate, since it seems a posterior distributions on the space of *different models* that could describe the data, while here we have a single model, and we get the posterior distribution of *its parameters*. A more accurate expression would have been

$$p(\{S_{\alpha_i}, S_{\alpha_{ij}}, \mathbf{S}^{bg}\}|\hat{\mathbf{D}}) = \frac{L(\hat{\mathbf{D}}|\{S_{\alpha_i}, S_{\alpha_{ij}}, \mathbf{S}^{bg}\})P(\{S_{\alpha_i}, S_{\alpha_{ij}}, \mathbf{S}^{bg}\})}{P(\hat{\mathbf{D}})}. \quad (9.26)$$

Thanks to Valentina Fusaro for the discussion!

First attempts and priors choice

A number of $13 \times 8 = 108$ parameters is too large to be commented or reported at all. In this section some relevant examples will be discussed in order to explain our further choices in the analysis.

Our first attempt consisted of using $\{S_{\alpha_i}, S_{\alpha_{ij}}, S_{\Delta g}^{bg}, S_{\Delta \gamma \phi}^{bg}, \xi\}$ directly as the random walkers in the MCMC. We want to be as agnostic as possible and not bias the posterior with any prior knowledge: this is achieved by using the so called Jeffreys prior for the parameters [37]. For a PSD, which is a scale parameter, a uniform prior on log PSD is the true uninformative one, which was chosen for all the parameters².

Consequences of a log-uniform priors. Let's compare a uniform prior with a log-uniform one. In the first case, the PSD is strictly $\in [0, +\infty]$. So we set 0 as a lower bound for all the parameters. There is actually no need for an upper bound to this priors in our algorithm, since in our model (9.23), all the parameters enter with a positive sign. Their sum can not exceed the values measured experimentally so their Monte Carlo chains never diverge to $+\infty$: the ultimate limit is that of just one generator explaining all the noise and the others being zero.

In the case of a log-uniform prior, log PSD can range instead from $-\infty$ to $+\infty$. While the reasoning that none of these parameters can diverge to $+\infty$ still holds, now there is nothing to prevent them from going to $-\infty$. For an experimental data set like ours, where there is not enough data to constrain all the model parameters, this happens with 100% probability if a lower limit is not set for the parameters, and the MCMC chain never converges. To put it in more physical terms, if one electrode is completely dominant in the noise with $100 \text{ ppm}/\sqrt{\text{Hz}}$, it does not matter if the other electrodes exhibit 1, 0.01 or $10^{-20} \text{ ppm}/\sqrt{\text{Hz}}$: in any of these cases, they will never contribute significantly to the total observed noise, which in turn means that the likelihood is completely insensitive to their value. As a consequence, their behavior in the Markov chain is just driven by the prior, not constrained by the likelihood at all.

It was therefore mandatory to set a lower "cutoff" $\bar{S}_{\text{co}}$ in the parameters priors. In order to set these in a quantitative way and leave as much freedom to the fit as possible, we used some physically motivated lower bounds. For the S_{α_i} , the stability of the voltage reference component LT1021 reported on the datasheet is $(130 \text{ nV}/\sqrt{\text{Hz}})^2 [1 + (10 \text{ Hz}/f)]$ on a voltage of 10 V: the ratio of these two quantities gives an estimate of the possible level of correlated fluctuations. This is already a very conservative lower limit which could never impact the measurements, but we anyway take 1/100 of this as $\bar{S}_{\text{co}}$. We assume this level also for the uncorrelated fluctuations, which is orders of magnitude below the levels observed with ground measurements.

For $S_{\Delta g}^{bg}, S_{\Delta \gamma \phi}^{bg}$ we take instead 1/100 of the observed Brownian noise level in LPF. This an

²While still learning these techniques, I used just a uniform prior on the PSD, resulting in an evident bias in the posteriors. The fit was reconstructing more noise than the observed one.

extremely well understood quantity [12] and it is sure to be a lower limit for the noise background. Finally, the priors cutoffs are not implemented as a sharp edges but are smoothed out with a Gaussian in the logarithm. This has absolutely no physical meaning, it just helps with the convergence, since less good step proposals are discarded just because one of the parameters fell out of the edge. Summarizing, the priors on the PSDs look like this:

$$P(S) \propto \begin{cases} \exp \left[-\frac{(\log S - \log \bar{S}_{co})^2}{2\sigma_{co}^2} \right] & \text{if } S < \bar{S}_{co} \\ 1 & \text{if } S > \bar{S}_{co} \end{cases} \quad (9.30)$$

where σ_{co} is set to 1 and it can be seen that there is no upper bound at all. Some parameters posteriors in the analysis are limited by this lower cutoff, so we will recognize it in what follows.

First attempt results

In Figure 9.1 we report the posterior distributions of the model parameter logarithms for a single frequency 0.12 mHz, obtained from the MCMC. One frequency is enough to understand the behavior. The plots on the diagonal represent the marginalized distribution over all the other parameters, so effectively the posterior of the single parameter. The off diagonal plots are instead the 2D distributions of couples of parameters. We proceed with a few comments:

- The parameters posteriors are undeniably ugly. Let us look at the top left plot, S_{α_1} which shows the typical behavior of many parameters. It exhibits a sharp peak around a value, then it has a constant (in log) tail extending until the prior cutoff, where the Gaussian smoothening can be seen. This is exactly due to the previously discussed log-prior behavior. Our experiments do not constrain all the parameters due to the limited amount of runs and also degeneracy between some A, B, C coefficients. When S_{α_1} becomes so small that it does not contribute to the total noise anymore, its random walk just reproduces the prior shape, a constant.
- From the plot below the top-left, one can spot that S_{α_1} is very correlated with S_{α_1} . If one is big, the other is small. This behavior is also evident for $S_{\alpha_{21}}$ and $S_{\alpha_{24}}$, for example. This is another suggestion of the presence of degeneracies in the model, and that maybe a different model with reduced parameters would give "better" results.
- Another symptom appears when we calculate S^{ACT} of Eq. (9.22), which is one of our main interests, since it can give an estimate of how much actuation noise is contributing to LPF total noise. The posterior of this quantity is shown in Figure 9.2, at all the mcf frequencies, just for Δg . The distributions are certainly well peaked around non-zero values, do not show divergences, but are bimodal or worse. This suggests that the posterior landscape has local minima.

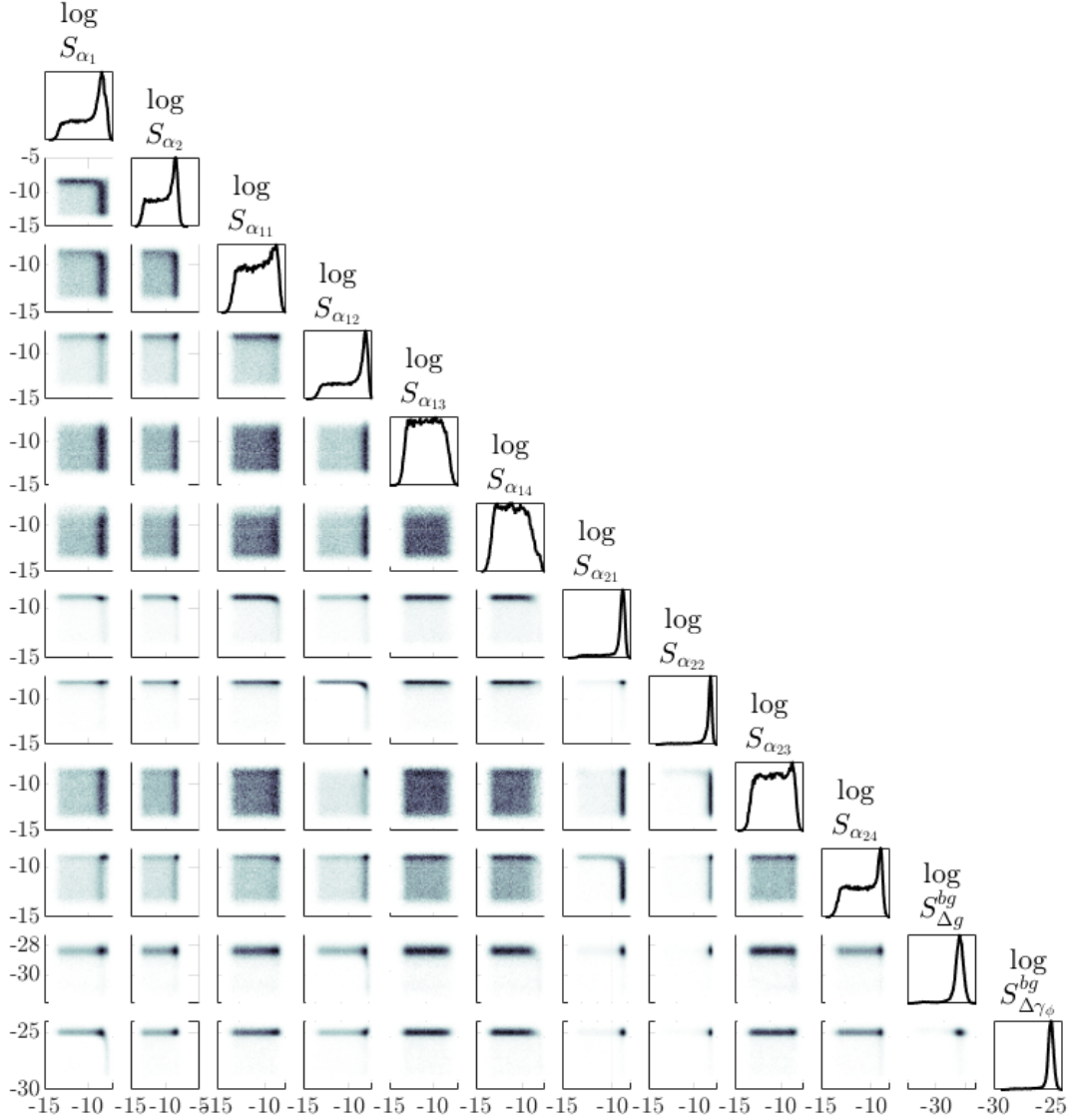


Figure 9.1: On the diagonal, plots of the distributions of the Monte Carlo chains for the model parameters at 1.2 mHz. The off diagonal are the 2D scatter plots of couples of parameters chains.

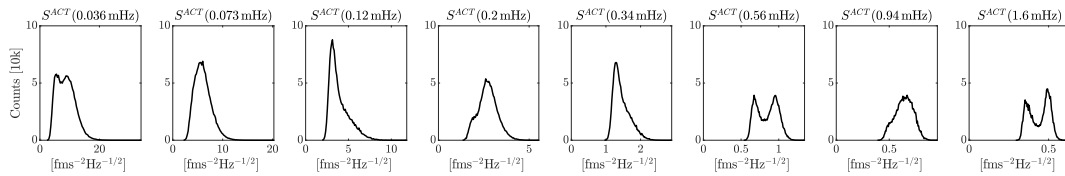


Figure 9.2: Reconstructed posterior $S_{\Delta g}^{ACT}$ through eq. (9.22) at all the investigated frequencies.

One could clearly live with these results. There is no objective problem, since the total Δg noise is really sensitive to all the described generators, and the degeneracies and bimodalities probably arise from a lack of data. We still remained not very satisfied about the result of Figure 9.2, since it was very important also for other analyses in [12]. So many other attempts were done to "regularize" these results. Considerable effort was spent in trying to simplify the model by reducing the number of parameters. However, none of the attempts lead to satisfying results: the combinations of parameters that seemed to be important to reproduce the data were not that intuitive and physically justifiable.

The idea behind a Bayesian approach is that we start from some hypothesis on the distributions, i.e. the priors: then, the more experimental data we add, the better we progressively modify our initial hypotheses towards the "real" distribution. Even from a completely wrong prior, with infinite data we will eventually converge there. However, the wronger is our prior choice, the more data is needed to actually converge to the correct posterior. The feeling here was that to allow maximum freedom to the parameters, we were still pretty far from converging.

The most disturbing fact about this priors choice was that it really allows for one generator, for example $S_{\alpha_{11}}$, to be 100 ppm/ $\sqrt{\text{Hz}}$, while $S_{\alpha_{22}}$ can simultaneously be 0.0001 ppm/ $\sqrt{\text{Hz}}$. It is true that they are related to different electrodes, but they come from nominally identical circuits: we expect them to be different but not *that many* orders of magnitude. In the next subsection it is discussed how this physical hypothesis was implemented in the model.

Model re-parametrization

Instead of putting a log-uniform prior on all $\{S_{\alpha_i}, S_{\alpha_{ij}}\}$, we re-parametrize them as

$$\begin{aligned}
 S_{\alpha_1} &= S_{\alpha_C}(1 + \mu_c) \\
 S_{\alpha_2} &= S_{\alpha_C}(1 - \mu_c) \\
 S_{\alpha_{11}} &= S_{\alpha_{UC}}(1 + \mu_{\pm})(1 + \mu_{tb+})(1 + \mu_1) \\
 S_{\alpha_{12}} &= S_{\alpha_{UC}}(1 + \mu_{\pm})(1 - \mu_{tb+})(1 + \mu_2) \\
 S_{\alpha_{13}} &= S_{\alpha_{UC}}(1 - \mu_{\pm})(1 - \mu_{tb-})(1 + \mu_3) \\
 S_{\alpha_{14}} &= S_{\alpha_{UC}}(1 - \mu_{\pm})(1 + \mu_{tb-})(1 + \mu_4) \\
 S_{\alpha_{21}} &= S_{\alpha_{UC}}(1 + \mu_{\pm})(1 + \mu_{tb+})(1 - \mu_1) \\
 S_{\alpha_{22}} &= S_{\alpha_{UC}}(1 + \mu_{\pm})(1 - \mu_{tb+})(1 - \mu_2) \\
 S_{\alpha_{23}} &= S_{\alpha_{UC}}(1 - \mu_{\pm})(1 - \mu_{tb-})(1 - \mu_3) \\
 S_{\alpha_{24}} &= S_{\alpha_{UC}}(1 - \mu_{\pm})(1 + \mu_{tb-})(1 - \mu_4).
 \end{aligned} \tag{9.31}$$

where all parameters μ_m represent differences between electrodes, are constrained to the interval $\mu_m \in [-1, 1]$ and have a uniform (flat) prior. For instance, $\mu_{\pm} = 0$ would imply that the $+X$ and $-X$ actuators are equally noisy, while $\mu_{\pm} = \pm 1$ implies that the total

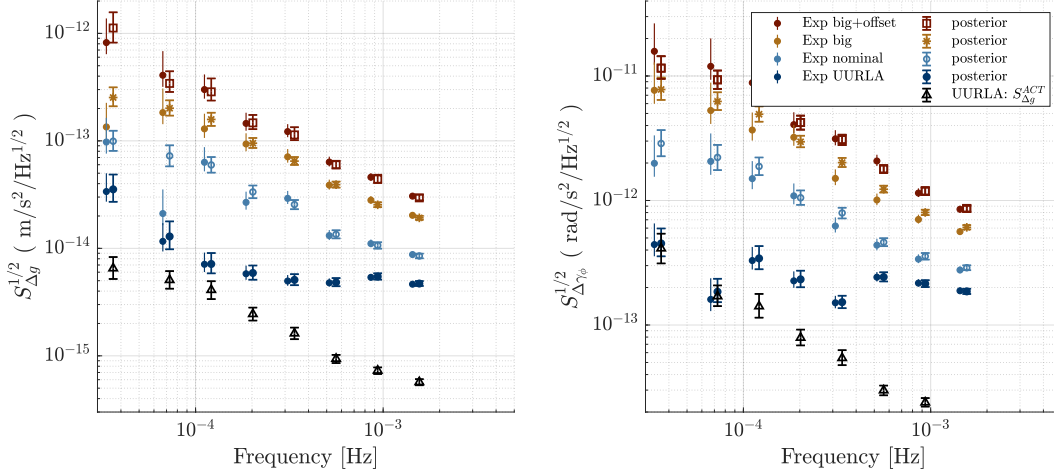


Figure 9.3: Experimental data with cap-less error bars as from (9.19), compared to the 68% symmetric intervals from the MCMC posterior, represented with capped error bars. In black triangles, the posterior estimates of S^{ACT} for UURLA, the amount of noise from gain fluctuations. Also the contribution from white noise was added.

sum of the uncorrelated gain noise comes from the $+X$ (or $-X$) actuators, with the other group completely noiseless. The parameters S_{α_C} and $S_{\alpha_{UC}}$ represent instead the *average* noise of the electrodes, and on these we keep a log-uniform prior. This re-parametrization then leaves the freedom to the order of magnitude of the noises, but disfavors the appearance of a many orders of magnitude difference between them. The result of the fit is compared to experimental data in Figure 9.3, while the Monte Carlo chains for the frequency 1.2 mHz are reported in Figure 9.4 to compare them with the previous case. The physical discussion of the results is left in the main Chapter 4.

- There are no evident "bananas" in the cross correlations as before. $S_{\alpha_{UC}}$ is very well constrained as well as the backgrounds, which however still need the lower cutoff to ensure the MCMC convergence.
- $\mu_{\pm}$ is very polarized towards $+1$, meaning that we find positive electrodes to be noisier than the negative ones. Also μ_{tb+} (which expresses the difference between the couple of electrodes 11, 21 and the couple 12, 22) seems to be constrained by the MCMC, probably thanks to the cross-spectrum in the analysis. Other parameters like μ_1, μ_2 , are better constrained at higher frequencies where there are more periodograms, but not at low frequency. μ_3, μ_4 are instead almost always constant and completely uninformative. This was expected, since they represent the difference between corresponding electrodes of different TMs (for example μ_4 for 14 and 24), and we already discussed that this difference was not enhanced by the experiments. In general, parameters regarding positive electrodes are better resolved because enhanced by the big+offset run.

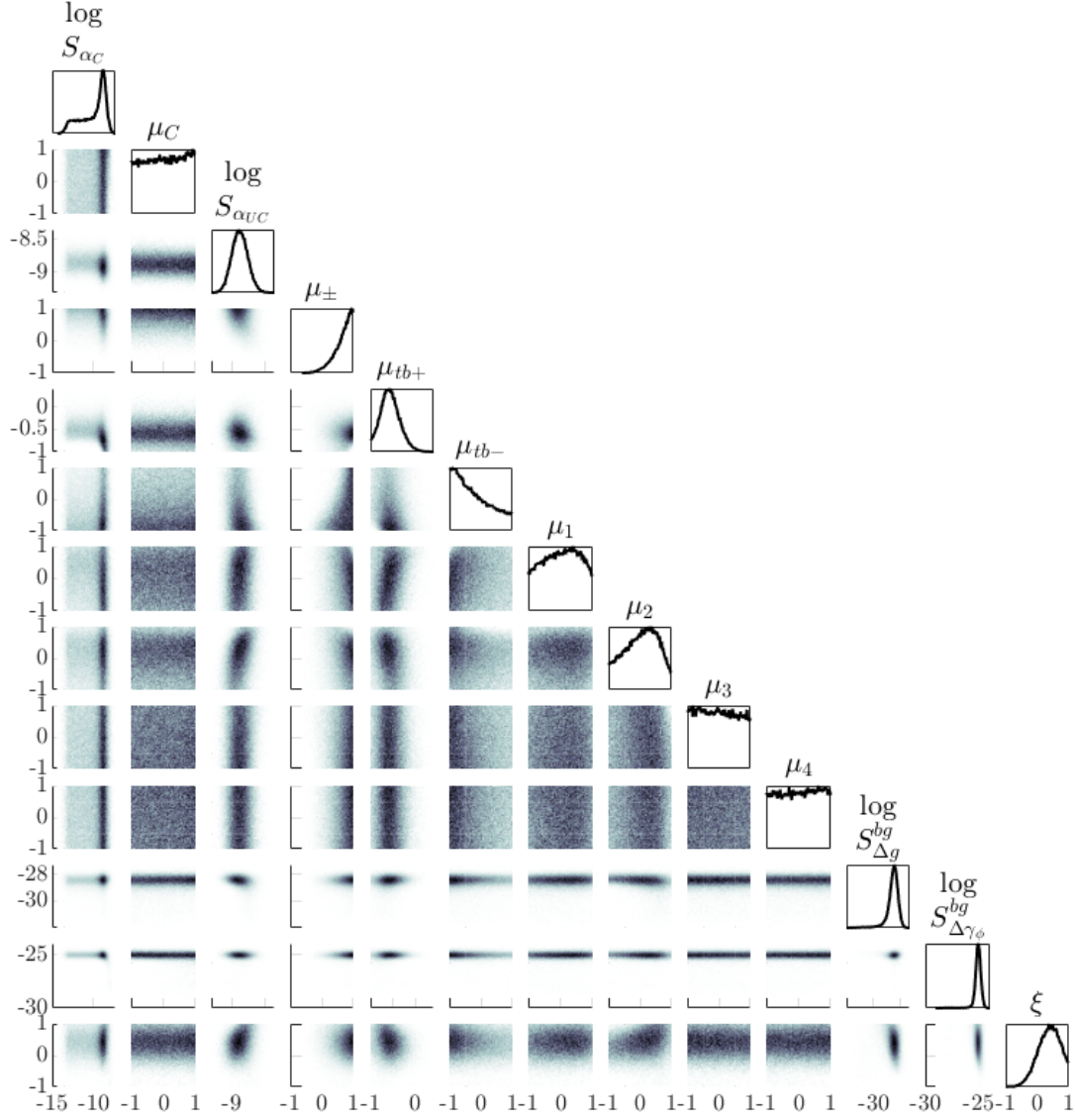


Figure 9.4: Monte Carlo chains for the re-parametrized model parameters at 1.2 mHz.

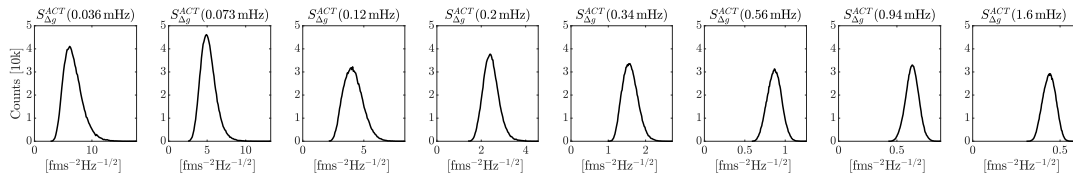


Figure 9.5: Reconstructed posterior $S_{\Delta g}^{\text{ACT}}$ through eq. (9.22) at all the investigated frequencies. These are the posteriors used for the black points in Figure 9.3 left.

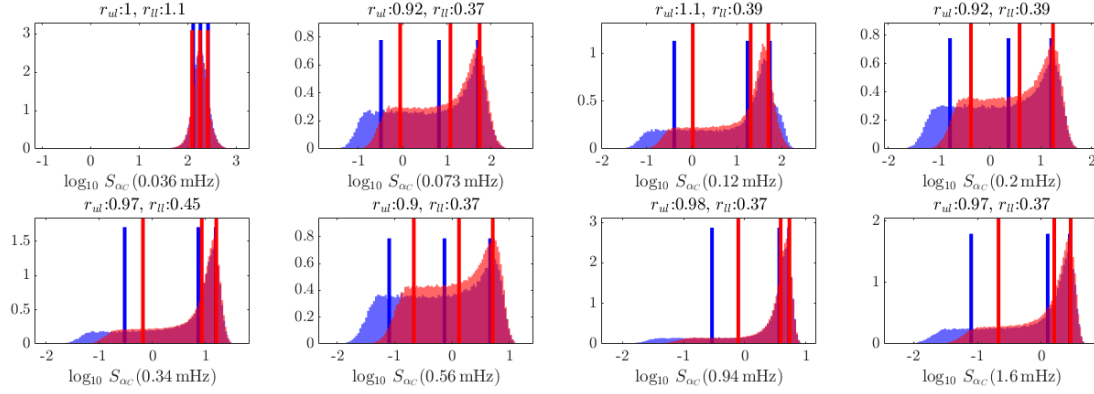


Figure 9.6: Behavior of log-uniform tail of S_{α_C} posterior. The red posterior is obtained with the priors described in the text, the blue one is obtained by dividing *all* parameters priors by a factor 10. If $r_{ll} < 1/2$ (the ratio between blue lower limit and red), the lower limit is considered not meaningful and the error-bars are dash-dotted.

- S_{α_C} instead still shows the log-uniform tail, and no difference is found between TM₁ and TM₂ (μ_C is constant). Still it gives an important upper limit on the parameter which is not worth ignoring. In presence of important tails like S_{α_C} in Figure 9.6, if the cutoff ($\bar{S}_{co}$) is lowered of a factor 10, the estimated lower limit ($S_{ll}(\bar{S}_{co})$) quickly behaves almost exactly as if the distribution was uniform (in log), decreasing of $\sim 86\%$ in linear scale. We adopted a rule of thumb (Chiavegato's 3dB per decade rule) by marking **with a dash-dot line** all the parameters estimates that obey

$$r_{ll} \equiv \frac{S_{ll}(\bar{S}_{co}/10)}{S_{ll}(\bar{S}_{co})} < \frac{1}{2}. \quad (9.32)$$

This is a visual indication for the reader to recognize parameters that do not really have a meaningful lower limit. For these parameters, the distribution upper limit is instead much less sensitive to the prior cutoff choice, varying typically of a few percents and never more than 10%, even for the most extreme cases. This process should be better understood by looking at Figure 9.6.

- Figure 9.5 shows the posteriors of $S_{\Delta g}^{ACT}$, which resulted much more regular than before.

Posterior predictive check

In this Bayesian Monte Carlo approach we used the experimental data to infer the distribution of model parameters: we can informally say that we went from $\hat{\mathbf{S}}$ to $\mathbf{M}$. Now, is our model any good? Or, does it reproduce the experimental data? One could look at Figure 9.3 and say that the error bars between the model and experimental data overlap, but it is not very quantitative. So we proceeded with a posterior predictive check. This is basically

doing the reverse process of what we did before: derive what would be the distribution of $\hat{\mathbf{S}}_{ppc}$ if $\mathbf{M}$ is the underlying model ("ppc" stands for "posterior predictive check"). Then see if 68% of the real experimental data $\hat{\mathbf{S}}$ fall in the 68% intervals of $\hat{\mathbf{S}}_{ppc}$.

To do this, for each accepted MCMC step, we calculate $\mathbf{M}$ and we would like to use (9.27) to sample $N_s(q, f)$ periodograms $\mathbf{X}_s$ and calculate a value for $\hat{\mathbf{S}}_{ppc}$ by averaging over the stretches with (9.17): by doing this for all the MCMC chain steps, we construct its posterior.

Sampling from (9.27) is not straightforward. A way to do this is diagonalizing $\mathbf{M}^{-1}$, since it is symmetric and positive definite, as

$$p_{f,q}(\mathbf{X}_s|\mathbf{M}) = \frac{1}{\pi^2|\mathbf{M}|} \exp \left\{ - \underbrace{\mathbf{X}_s^\dagger \mathbf{V}^{-1}}_{\mathbf{P}^\dagger} \underbrace{\mathbf{V} \mathbf{M}^{-1} \mathbf{V}^{-1}}_{\mathbf{D}} \underbrace{\mathbf{V} \mathbf{X}_s}_{\mathbf{P}} \right\}. \quad (9.33)$$

This 2-variate complex Gaussian distribution can be equivalently sampled by extracting real and imaginary parts of $\mathbf{P}$ from a 4-variate (real) Gaussian distribution [36]. To put it in more physical terms, $\mathbf{M}$ being non diagonal means that Δg and $\Delta \gamma_\phi$ are correlated, so are their periodograms. By diagonalizing $\mathbf{M}^{-1}$ we are choosing 2 combinations of Δg and $\Delta \gamma_\phi$ that are uncorrelated. For each of these new noises, real and imaginary parts of their periodograms are uncorrelated and with variance given by half the value of their spectrum (see for example the Appendix of [64], Eq. (A4)). It is easier to summarize all of this with an operative procedure:

1. For a step of the Markov Chain, calculate $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$.
2. Diagonalize it by finding the matrix $\mathbf{V}$ so that $\mathbf{D} = \mathbf{V} \mathbf{M}^{-1} \mathbf{V}^{-1}$, with $\mathbf{D}$ diagonal.
3. If we indicate $\mathbf{P} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, first of all we can sample P_1 and P_2 independently since $\mathbf{D}$ diagonal. More in detail, we can also sample $Re[P_1]$ and $Im[P_1]$ independently, from a Gaussian distribution that has $\sigma^2 = D_{11}/2$. Depending on the frequency and experiment we are simulating, we sample $N_s(q, f)$ of these values, and we do the same for P_2 , using $\sigma^2 = D_{22}/2$ as variance.
4. We go back to the original periodograms using $\mathbf{X}_s = \mathbf{V}^{-1} \mathbf{P}$. We will have $N_s(q, f)$ of them, so we use (9.17) to calculate $\hat{\mathbf{S}}_{ppc}$.
5. By doing the same for all the MCMC steps, we get an equivalent number of samples for $\hat{\mathbf{S}}_{ppc}$, from which we can get its distribution.

The results of such procedure are reported in Figure 9.7. From such a plot one can really appreciate the difference between the uncertainties on $\hat{\mathbf{S}}_{ppc}$ and the estimate of the underlying ASD ($\mathbf{M}$). Even if $\mathbf{M}$ had no uncertainty, $\hat{\mathbf{S}}_{ppc}$ would due to the finite number of stretches of finite length.

Out of the total 96 experimental points, a fraction of 88% fall in the posterior predictive

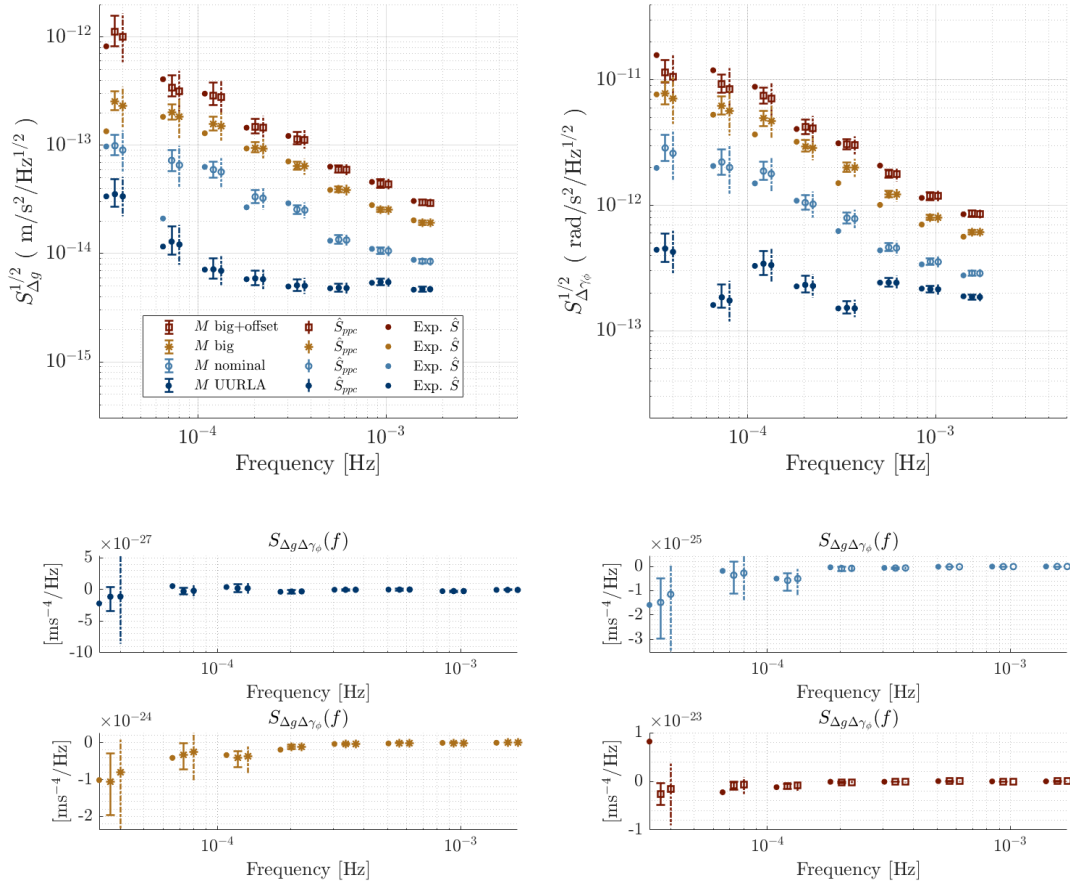


Figure 9.7: Results of posterior predictive check. On top ASD of Δg and $\Delta \gamma_\phi$. The four plots at the bottom are Cross spectral densities for each experiment. To preserve the sign we are forced to use a linear scale, so they are difficult to visualize. As simple dots, experimental data. As error bars with caps, the result of fitting procedure, so the estimated we got for the model **M**. As dashed error bars without caps, the 68% in which an experimental result should fall in the hypothesis that model **M** holds.

check intervals (to exclude the possibility that 1 experiment is contributing 1% and the others 99.9%, all the ratios are reported in Table 9.1). Ideally it should have been 68%, and this suggests that there is probably a slight over-fitting: each frequency has 4 experiments and 3 experimental channels (acceleration, torque and cross), and is fitted with 13 parameters. Anyhow, we can certainly state we are reproducing the experimental data.

Fraction of experimental points falling in prediction					
	UURLA	Nominal	Big	Big+Offset	Total
$S_{\Delta g}$	1	0.875	0.75	1	0.906
$S_{\Delta\gamma\phi}$	1	0.875	0.5	0.875	0.813
$S_{\Delta g\Delta\gamma\phi}$	1	1	0.875	0.75	0.902
Total	1	0.917	0.708	0.875	0.875

Table 9.1

9.3.2 Parameter estimation from four actuation experiments with analytic functions

The observed spectra at single frequencies are pretty smooth, therefore we also performed an analysis involving all the frequencies at once and using some analytic functions to describe the spectra. These were chosen just looking at the spectra shapes, not based on particular physical reasons: for example the actuation dominated runs seem to whiten at low frequencies, so we used some kind of low-pass like function plus $1/f$ terms. Also the backgrounds were described looking at their shape.

For this analysis we used the data calculated with the Welch method and cut out the frequencies between 0.4 and 0.58 mHz for the presence of a known bump in UURLA torque due to SC rotation. The experimental points are then 52 frequencies for each experiment, for a total of 648. Using analytical functions the number of parameters reduces greatly, so this kind of fit was already working fine before we introduced the re-parametrization (9.31), but we use it also here for consistency. This said, the chosen function for the correlated fluctuations are

$$\begin{aligned}
 S_{\alpha_1}(f) &= \frac{\epsilon_C}{f^2 + f_{cut}^2}(1 + \mu_{C_{2f}}) + \frac{\rho_C}{f}(1 + \mu_{C_{1f}}) \\
 S_{\alpha_1}(f) &= \frac{\epsilon_C}{f^2 + f_{cut}^2}(1 - \mu_{C_{2f}}) + \frac{\rho_C}{f}(1 - \mu_{C_{1f}})
 \end{aligned} \tag{9.34}$$

which should suggest the logic with which also the other generators are constructed: the "difference parameters" μ are introduced for each analytic function used, to allow for maximum freedom. f_{cut} is the "low-pass" roll-off frequency and that instead is common for all

the analytic functions. The uncorrelated generators are

$$\begin{aligned}
 S_{\alpha_{11}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 + \mu_{\pm 2f})(1 + \mu_{tb+2f})(1 + \mu_{12f}) + \frac{\rho_{UC_{1f}}}{f} (1 + \mu_{\pm 1f})(1 + \mu_{tb+1f})(1 + \mu_{11f}) \\
 S_{\alpha_{12}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 + \mu_{\pm 2f})(1 - \mu_{tb+2f})(1 + \mu_{22f}) + \frac{\rho_{UC_{1f}}}{f} (1 + \mu_{\pm 1f})(1 - \mu_{tb+1f})(1 + \mu_{21f}) \\
 S_{\alpha_{13}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 - \mu_{\pm 2f})(1 - \mu_{tb-2f})(1 + \mu_{32f}) + \frac{\rho_{UC_{1f}}}{f} (1 - \mu_{\pm 1f})(1 - \mu_{tb-1f})(1 + \mu_{31f}) \\
 S_{\alpha_{14}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 - \mu_{\pm 2f})(1 + \mu_{tb-2f})(1 + \mu_{42f}) + \frac{\rho_{UC_{1f}}}{f} (1 - \mu_{\pm 1f})(1 + \mu_{tb-1f})(1 + \mu_{41f}) \\
 S_{\alpha_{21}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 + \mu_{\pm 2f})(1 + \mu_{tb+2f})(1 - \mu_{12f}) + \frac{\rho_{UC_{1f}}}{f} (1 + \mu_{\pm 1f})(1 + \mu_{tb+1f})(1 - \mu_{11f}) \\
 S_{\alpha_{22}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 + \mu_{\pm 2f})(1 - \mu_{tb+2f})(1 - \mu_{22f}) + \frac{\rho_{UC_{1f}}}{f} (1 + \mu_{\pm 1f})(1 - \mu_{tb+1f})(1 - \mu_{21f}) \\
 S_{\alpha_{23}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 - \mu_{\pm 2f})(1 - \mu_{tb-2f})(1 - \mu_{32f}) + \frac{\rho_{UC_{1f}}}{f} (1 - \mu_{\pm 1f})(1 - \mu_{tb-1f})(1 - \mu_{31f}) \\
 S_{\alpha_{24}}(f) &= \frac{\epsilon_{UC_{2f}}}{f^2 + f_{cut}^2} (1 - \mu_{\pm 2f})(1 + \mu_{tb-2f})(1 - \mu_{42f}) + \frac{\rho_{UC_{1f}}}{f} (1 - \mu_{\pm 1f})(1 + \mu_{tb-1f})(1 - \mu_{41f})
 \end{aligned} \tag{9.35}$$

and finally the backgrounds

$$\begin{aligned}
 S_{\Delta g}^{bg}(f) &= \frac{\epsilon_{\Delta g}^{bg}}{f^2} + \rho_{\Delta g}^{bg} \\
 S_{\Delta \gamma_\phi}^{bg}(f) &= \frac{\epsilon_{\Delta \gamma_\phi}^{bg}}{f} + \rho_{\Delta \gamma_\phi}^{bg} f^2 \\
 S_{\Delta g \Delta \gamma_\phi}^{bg}(f) &= \xi \sqrt{S_{\Delta g}^{bg}(f) S_{\Delta \gamma_\phi}^{bg}(f)}
 \end{aligned} \tag{9.36}$$

All of this looks very menacing but results in a total of 26 parameters in total for 648 data points. For comparison, the single frequency fit was basically having 104 parameters for 96 data points.

The fitting procedure is exactly the same as before, it is just important to state that using all the frequencies at once, the likelihood must be

$$L(\hat{\mathbf{D}}|\mathbf{M}) = \prod_{q,f} p_{f,q}(\hat{\mathbf{S}}|\mathbf{M}). \tag{9.37}$$

We do not comment the results of the chains in detail as before because it does not seem interesting for the reader. We just include the result of the fit in Figure 9.8 and the posterior predictive check in Figure 9.9. In Table 9.2 instead we report the fraction of experimental points that fall in the predicted interval, which gives a total result of 70%, an indication for us that this model is indeed very good at describing the data.

Fraction of experimental points falling in prediction

	UURLA	Nominal	Big	Big+Offset	Total
$S_{\Delta g}$	0.7	0.7	0.69	0.78	0.72
$S_{\Delta \gamma_\phi}$	0.63	0.69	0.76	0.74	0.7
$S_{\Delta g \Delta \gamma_\phi}$	0.61	0.7	0.81	0.61	0.69
Total	0.65	0.7	0.75	0.71	0.7

Table 9.2: Posterior predictive check results for analytic functions fit to all frequencies.

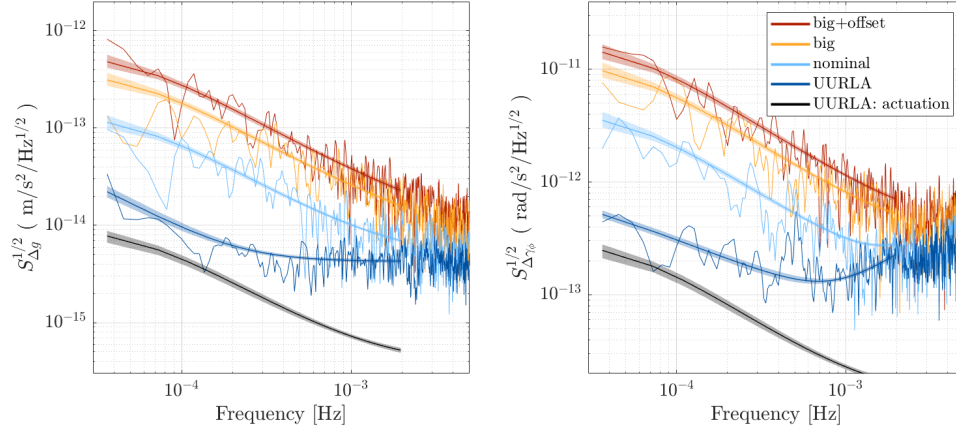


Figure 9.8: Results of the parameter estimation. The thin, broken lines are the experimental data while the smooth areas represent the 68% interval of the posteriors. In black the contribution of actuation noise to UURLA.

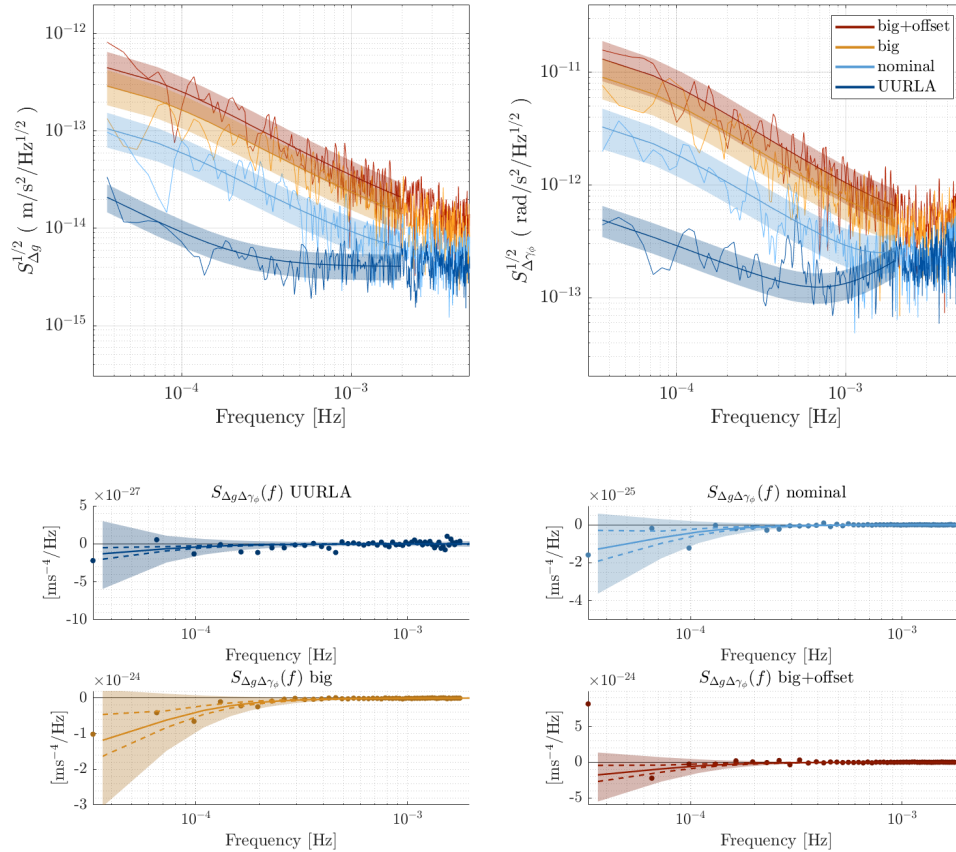


Figure 9.9: Areas are the predicted 68% interval in which the experimental data should fall if our model holds. Bottom plots are for cross correlations; single points are experimental data, dashed lines are the posterior on M .

9.3.3 Details for low-frequency voltage noise parameter estimation

The fitting procedures, likelihoods and conventions are the same used in the previously discussed gain fluctuations.

In the main text it is explained that we perform a fit without any particular assumptions on $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$, the other assuming that they are dominated by uncorrelated fluctuations in the electrode voltages. The model for the first one is

$$\mathbf{M}(f, q) = V_{TM}^2(q) \begin{bmatrix} \frac{1}{M^2} \frac{\partial C_X}{\partial x}^2 S_{\Delta(\Delta_x)} & \frac{1}{MI} \frac{\partial C_X}{\partial \phi} \frac{\partial C_X}{\partial x} \xi_V \sqrt{S_{\Delta(\Delta_x)} S_{\Delta(\Delta_\phi)}} \\ \frac{1}{MI} \frac{\partial C_X}{\partial \phi} \frac{\partial C_X}{\partial x} \xi_V \sqrt{S_{\Delta(\Delta_x)} S_{\Delta(\Delta_\phi)}} & \frac{1}{I^2} \frac{\partial C_X}{\partial \phi}^2 S_{\Delta(\Delta_\phi)} \end{bmatrix} + \mathbf{S}^{bg}(f) \quad (9.38)$$

where $\mathbf{S}^{bg}(f)$ is the same as that in (9.23). We further include the assumption that $S_{\Delta(\Delta_x)}$ and $S_{\Delta(\Delta_\phi)}$ are not expected to differ by orders of magnitude, so

$$\begin{aligned} S_{\Delta(\Delta_x)} &= S_{\Delta}(1 + \mu_V) \\ S_{\Delta(\Delta_\phi)} &= S_{\Delta}(1 - \mu_V) \end{aligned} \quad (9.39)$$

The parameters then are S_{Δ} , μ_V and $\xi_V \in [-1, 1]$ for the cross-correlation, plus the 3 background parameters.

For the second fit, we can summarize the PSD model (4.24) similarly to (9.23) as

$$\mathbf{M}(f, q) = \mathbf{v}^T(q) \mathbf{S}_{\Delta(\Delta)}(f) \mathbf{v}(q) + \mathbf{S}^{bg}(f) \quad (9.40)$$

where $\mathbf{S}^{bg}(f)$ is the same as that in (9.23), while

$$\begin{aligned} \mathbf{v}(q) &= V_{TM}(q) \begin{bmatrix} \frac{\partial C_X}{\partial x}/M & \frac{\partial C_X}{\partial \phi}/I \\ -\frac{\partial C_X}{\partial x}/M & \frac{\partial C_X}{\partial \phi}/I \end{bmatrix} \\ \mathbf{S}_{\Delta(\Delta)}(f) &= \begin{bmatrix} S_{\Delta(\Delta_{14})}(f) & 0 \\ 0 & S_{\Delta(\Delta_{23})}(f) \end{bmatrix} \end{aligned} \quad (9.41)$$

Finally, to be consistent with the previous approach, we include again the hypothesis of "nominally identical" electrodes, writing

$$\begin{aligned} S_{\Delta(\Delta_{14})} &= S_{\Delta}(1 + \mu_b) \\ S_{\Delta(\Delta_{23})} &= S_{\Delta}(1 - \mu_b) \end{aligned} \quad (9.42)$$

At each minimally correlated frequency, we use the same MCMC described in the previous section with likelihood (9.29) to estimate the distributions of the parameters S_{Δ} , $S_{\Delta g}^{bg}$, $S_{\Delta \gamma \phi}^{bg}$ with a prior uniform in the logarithm, while μ_b and ξ (expressing the correlation between backgrounds as in (9.23)) with a uniform prior on $[-1, 1]$.

We also perform a fit describing the relevant quantities as smooth functions of frequency. While backgrounds are written as in (9.36), we write the other quantities as sums of white,

$1/f$ and $1/f^2$ noise, that with the usual re-parametrization become

$$\begin{aligned} S_{\Delta(\Delta_{14})} &= \frac{\epsilon}{f^2}(1 + \mu_{2f}) + \frac{\rho}{f}(1 + \mu_{1f}) + \eta(1 + \mu_w) \\ S_{\Delta(\Delta_{23})} &= \frac{\epsilon}{f^2}(1 - \mu_{2f}) + \frac{\rho}{f}(1 - \mu_{1f}) + \eta(1 - \mu_w) \end{aligned} \quad (9.43)$$

and MC estimate ϵ , ρ , η and the μ parameters. The results seem to indicate that only the $1/f$ component is significantly different from zero.

9.4 Appendix D - Analytic models for conversion of voltage non-linearities to acceleration noise from a noisy command

Here we give a justification for the formulas given in Sec. 5.4.4.

As stated multiple times, the force fluctuations due to voltage errors δV on a single electrode are

$$\delta g(t) = \frac{1}{2M} \frac{\partial C^*}{\partial x} V(t) \delta V(t). \quad (9.44)$$

The applied voltage $V(t)$ is determined by how much torque we need through

$$V(t) = \sqrt{\frac{I(\gamma_{\phi_0} + \gamma_{\phi_c}(t))}{\frac{\partial C^*}{\partial \phi}}} \quad (9.45)$$

Typically in LPF $\gamma_{\phi_c}(t)$ can have a DC value, $\bar{\gamma}_{\phi_c}$ comparable with γ_{ϕ_0} ; the non-DC part $\delta\gamma_{\phi_c}(t)$ instead, always has oscillations of much smaller entity, therefore we can expand as follow

$$V(t) \approx \sqrt{\frac{I(\gamma_{\phi_0} + \bar{\gamma}_{\phi_c})}{\frac{\partial C^*}{\partial \phi}}} \left[1 + \frac{1}{2} \frac{\delta\gamma_{\phi_c}(t)}{\gamma_{\phi_0} + \bar{\gamma}_{\phi_c}} \right] \quad (9.46)$$

For simplicity, we assume that $\bar{\gamma}_{\phi_c} = 0$. This makes all electrodes applying the same average torque and we can treat them all at once. we can neglect the time dependent part of $V(t)$ in (9.44) and rewrite it as

$$\delta g(t) = \frac{1}{2M} \frac{\partial C^*}{\partial x} \sqrt{\frac{I\gamma_{\phi_0}}{\frac{\partial C^*}{\partial \phi}}} \delta V(t) \quad \rightarrow \quad S_g^{1/2} = \frac{1}{2M} \frac{\partial C^*}{\partial x} \sqrt{\frac{I\gamma_{\phi_0}}{\frac{\partial C^*}{\partial \phi}}} S_{\delta V}^{1/2} \quad (9.47)$$

The acceleration spectrum is therefore only linked to the non linearities, which change in time because the command is being changed: we try to estimate $S_{\delta V}^{1/2}$.

From Eq. (9.46) we write the relation between $V(t)$ and $\gamma_{\phi_c}(t)$ spectra

$$S_V^{1/2} = \sqrt{\frac{I}{4 \frac{\partial C^*}{\partial \phi} \gamma_{\phi_0}}} S_{\gamma_{\phi_c}}^{1/2} \quad (9.48)$$

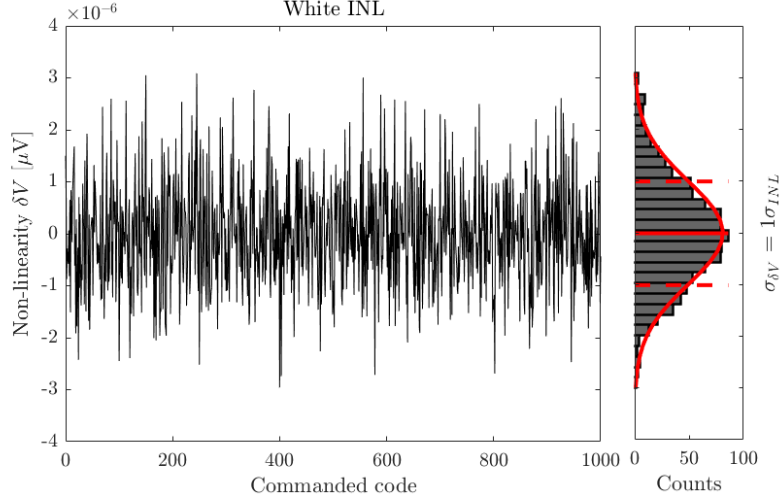


Figure 9.10

Now, **if we assume $S_{\gamma_{\phi_c}}$ is white**, also S_V is, and for a white spectrum we have $S_{\text{white}} = 2\sigma^2/f_s$, where f_s is the sampling frequency and σ is the standard deviation of the noise in time domain. Using this, we find that

$$\sigma_V = \sqrt{\frac{I}{4 \frac{\partial C^*}{\partial \phi} \gamma_{\phi_0}}} \frac{S_{\gamma_{\phi_c}}^{1/2}}{\sqrt{2/f_s}} \quad (9.49)$$

where, we repeat, σ_V is the standard deviation of $V(t)$, the command, not the non-linearities $\delta V(t)$! Is it possible to link this quantity σ_V with the amplitude of non-linearities noise?

We tried to answer this question with a physical approach, that starts by realizing that

$$n = \frac{\sigma_V}{V_{\text{LSB}}} \quad (9.50)$$

gives an estimate of the number n of different commands used. For each of these n commands, we have a value of non-linearity. If $V(t)$ is white, than it means that at each time, we choose at random one of these n non-linearities. **If then $n \gg 1$, and if we know the distribution of these non-linearities**, than at each instant of time it will be like extracting a random sample from that distribution³. The condition $n \gg 1$ is expressed in Eq. (5.52).

So what is the probability distribution of non-linearities?

White INL

For the white INL case, it is pretty simple, since δV is a zero mean gaussian with width σ_{INL} . An example of INL generated for 1000 commands, and the corresponding distribution,

³If $n \gg 1$ is not respected, this approximation fails, since we are always choosing between a few non-linearities

Appendix D - Analytic models for conversion of voltage non-linearities to acceleration noise from a noisy command

is reported in Figure 9.10. Therefore, the resulting noise of picking one random sample from this distribution at each time, would be

$$S_{\delta V}^{1/2} = \sigma_{INL} \sqrt{2/f_s}, \quad (9.51)$$

which must then be inserted in Eq. (9.47) to obtain the acceleration noise presented in (5.54) (almost, see following). This result is completely independent on the level of torque noise $S_{\gamma_{\phi c}}^{1/2}$ and V_{LSB} : as a consequence, the only way to increase it is by increasing $\gamma_{\phi 0}$.

In Eq. (5.54) it is possible to notice a further factor $\sqrt{2/3}$, which we omitted from Eq. (9.51) to focus on the physical idea. The factor is there due to the presence of the Sigma-Delta loop: the linear interpolation performed by the loop transforms a sampled, white process in something more complicated and filtered: therefore, we must expect that the σ of a white process and its filtered version must differ by some correction factors⁴ In particular, we have that the effect of the Sigma-Delta on the spectrum is

$$S_{\delta V}^*(f^*) = 2\sigma_{INL}^2 V_{LSB} \rightarrow S_{\delta V}^*(f^*) = 2\sigma_{INL}^2 V_{LSB} \text{sinc}^4(\pi V_{LSB} f^*). \quad (9.52)$$

The standard deviation of such noise is not just σ_{INL} anymore. It is possible to calculate the correct one by exploiting the property of the spectrum

$$\begin{aligned} \sigma_{\delta V}^2 &= \int_0^\infty S_{\delta V}^*(f^*) df^* = \int_0^\infty 2\sigma_{INL}^2 V_{LSB} \text{sinc}^4(\pi V_{LSB} f^*) df^* \\ &= \frac{2\sigma_{INL}^2}{\pi} \underbrace{\int_0^\infty \text{sinc}^4 x dx}_{=\pi/3} = \frac{2}{3} \sigma_{INL}^2 \end{aligned} \quad (9.53)$$

So, Eq. (9.51) can be corrected as

$$S_{\delta V}^{1/2} = \sqrt{\frac{2}{3}} \sigma_{INL} \sqrt{2/f_s}, \quad (9.54)$$

giving the final result (5.54).

White DNL

This case is more difficult, since the distribution is not a simple Gaussian: actually, one cannot even write down a distribution for these non-linearities, since they constitute a "non-stationary process". Let's say we generate n samples of white DNL by sampling them from a Normal distribution $\mathcal{N}(0, \sigma_{DNL})$ (obtaining something exactly equal to Figure 9.10), and label each sample $\delta V_{DNL,k}$. We can then obtain the non-linearity δV_k of the k -th command by

$$\delta V_k = \sum_{l=1}^k \delta V_{DNL,l}. \quad (9.55)$$

⁴Note: while the PSD* is not white anymore, the PSD (of $\delta V(t)$) still is, since we still pick random samples at each time, based on the randomness of $V(t)$!

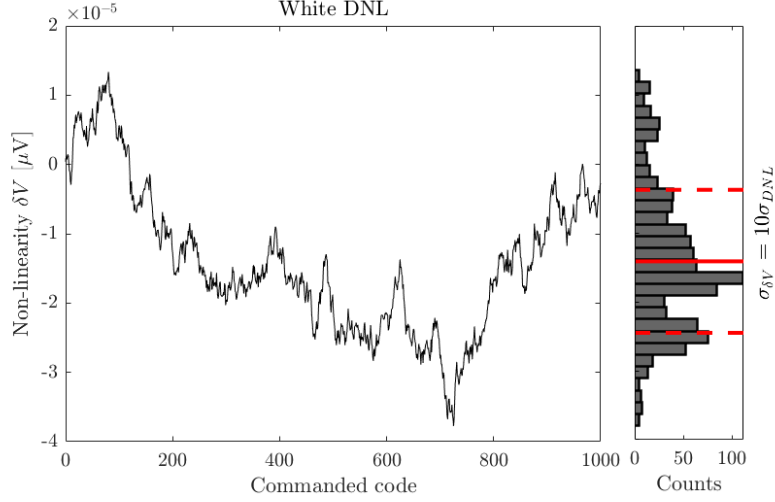


Figure 9.11

A visual example of this can be found in Figure 9.11: notice in particular how the distribution of the samples is not Gaussian at all and has a standard deviation much larger than just σ_{DNL} . Still, we can give a rough estimate of the dispersion of the non-linearities by realizing that eq. (9.55) is nothing more than the definition of a random walk with Gaussian step. A well known property of these is that the distribution of n -th point over many repetitions of the random walk is $\mathcal{N}(0, \sqrt{n}\sigma_{\text{DNL}})$. Therefore, *on average*, the n -th point will be found at a distance $\sqrt{n}\sigma_{\text{DNL}}$ from the initial point. So, what is the average amplitude of the non-linearities between command 0 and n ? We take the average of the variances of all points between 0 and n

$$\sigma_{\delta V}^2 = \frac{1}{n} \sum_{k=0}^n k \sigma_{\text{DNL}}^2 = \frac{\sigma_{\text{DNL}}^2}{n} \frac{1}{2} n(n+1) = \frac{\sigma_{\text{DNL}}^2}{2} (n+1) \approx \frac{\sigma_{\text{DNL}}^2}{2} n. \quad (9.56)$$

The following is the final, approximate reasoning that leads to the result (5.55) in Sec. 5.4.4. Since we assumed $V(t)$ is white and Gaussian, at each time we pick a random sample from this Gaussian, which corresponds to one δV . So again, it is like choosing one random value from the distribution of Figure 9.11. If the distribution happens to be particularly non-uniform, there will be for sure a dependence on its shape, in fact, simulations can give pretty scattered results. But *on average*, we are picking from a distribution that has width given by (9.56): the spectrum of the resulting noise is white and we write it in terms of its time-domain standard deviation

$$S_{\delta V}^{1/2} = \frac{\sigma_{\text{DNL}}}{\sqrt{2}} \sqrt{n} \sqrt{2/f_s}, \quad (9.57)$$

then by using (9.49) and (9.50) we finally obtain

$$S_{\delta V}^{1/2} = \frac{\sigma_{\text{DNL}}}{\sqrt{2}} \sqrt{\frac{1}{V_{\text{LSB}}} \sqrt{\frac{I}{4 \frac{\partial C^*}{\partial \phi} \gamma_{\phi_0}}} S_{\gamma_{\phi_c}}^{1/2} \sqrt{2/f_s}}. \quad (9.58)$$

which is equivalent to (5.55). The reasoning is solid in addressing the quantities entering the formula, and we think it captures correctly the trends σ_{DNL} , $V_{\text{LSB}}^{-1/2}$, $\gamma_{\phi_0}^{1/4}$, $S_{\gamma_{\phi_c}}^{1/4}$ and $f_s^{-1/4}$; however, given the roughness of guesses (9.50) and (9.56) we expected to be wrong of a multiplicative factor, which instead seems to be reproducing the simulations pretty well. We think this is just by chance and other corrections are expected, for example from the Sigma-Delta, which we never even mentioned in this case (it would probably be a small correction, since it affects high frequencies and not the lower ones, which are the true cause of large δV fluctuations).

9.5 Appendix E - Autocollimator non-linearities

Here we try to give an explanation to the 2 peaks that appear in the 1TM pendulum noise floor, in Figure 6.6 at multiples of the resonance frequency. These peak get worse if the oscillation is large; furthermore, if a modulated force is applied to the pendulum, a forest of peaks appears, since all the peaks at the modulation frequency are added, and also their beat-notes with the other peaks related to the pendulum resonance. This is a clear indication that there must be some non-linearities in the readout, that we indicate as $\delta\phi(\phi)$. We think these could be related to systematic errors in the CCD camera pixels of the autocollimator. We can estimate them with a smart trick, summarized in Figure 9.12. We excite an oscillation of the pendulum in $\phi \in [\phi_{\min}, \phi_{\max}]$, the region in which we want to estimate the nonlinearities. The obtained motion is practically a perfect sine (top plot in Figure 9.12): we divide it in half periods and flip the time-series (in time!) obtaining the red curve. In this way, we have a time series in which ϕ explores from $\phi_{\max}$ to $\phi_{\min}$ multiple times. We fit each red half period with a sine function, so, the residual of the fit will contain both noise

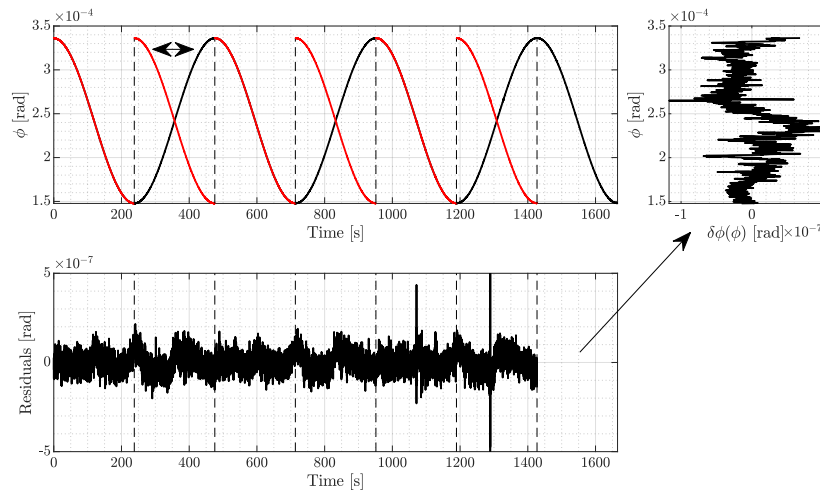


Figure 9.12: Algorithm to estimate AC nonlinearities, described in the text. This is real data, included the sinusoids in top panel.

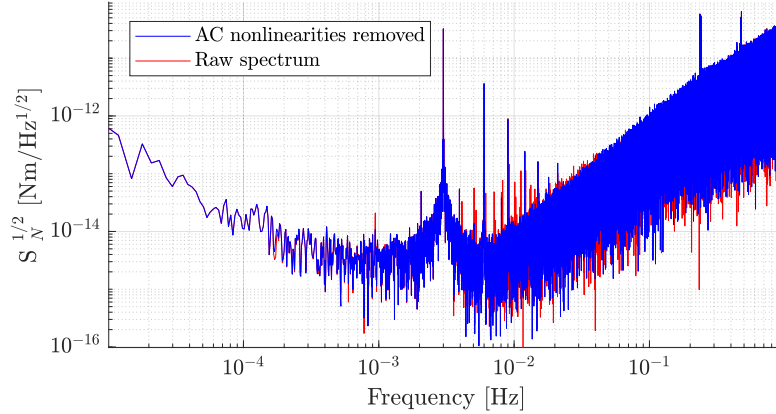


Figure 9.13: Example of removal of AC non-linearities during an experiment in which a very large modulation at 3 mHz is present.

and non-linearities, repeated multiple times in the bottom plot of Figure 9.12. However, since the nonlinearities are systematic, we can estimate them by averaging over all the half-periods, operation that will instead suppress the uncorrelated readout noise. The result of the average is then reported in top right panel of Figure 9.12: this is just an example with a few periods, we usually use more and get a more resolved result, for example those reported in top right Figure 9.14.

They seem to be of order of several tens of nrad, which is actually compatible with the instrument resolution of 25 nrad, and they show up as some clearly detectable "spikes". In addition to these, there seem to be also smoother drifts. With these nonlinearities at hand, we can *correct* measurements performed in that range by subtracting them! An example is shown in Figure 9.13, where a very large modulation is present during a charge measurement: this subtraction strategy then eliminates a large part of the peaks, leaving only the harmonics of the main peak, expected to be there in this particular measurement.

So why didn't I show a corrected version of the background noise in Figure 6.6? Because this procedure seems to work very well on short runs, while I can't seem to make it work on long runs (order days). While some things are subtracted, the low frequency peaks just don't seem to be subtracted well enough to completely disappear during purely noise runs. They are not a major problem for the measurements in this thesis, therefore I could not dedicate more than a few weeks of work to understand this, but still I tried to give an explanation. I noticed that measuring the non-linearities in the same range after a few days used to give slightly different results: my hypothesis is that if the nonlinearities are related to the CCD camera pixels used, it can not be just $\delta\phi(\phi)$ but something like $\delta\phi(\phi, \eta)$, since the AC has two axes. η is the angle measured by the AC which is directly proportional to the x position of the TM. During one period of ϕ oscillation, the pendulum swings hundreds of times along x , so we are detecting an average on AC non-linearities over the x swing around $\langle x \rangle$. However, $\langle x \rangle$ drifts over hours because of laboratory ground deformations, as shown in

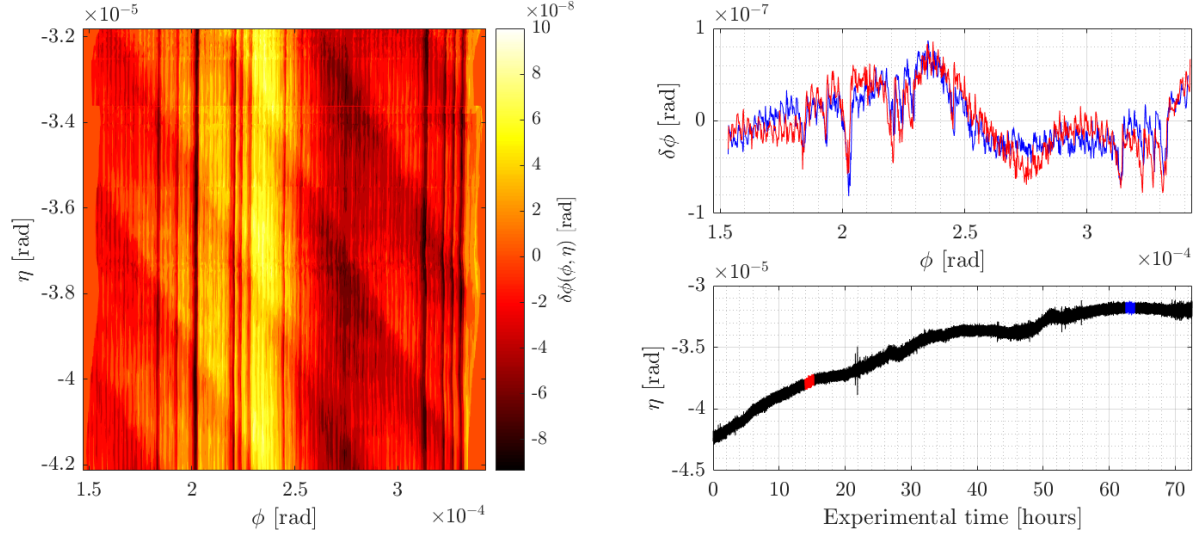


Figure 9.14: (Left) 2D map of AC non-linearities $\delta\phi(\phi, \eta)$. The color indicates the the nonlinearity magnitude. (Right top) Comparison between non-linearities at different x average positions of the pendulum. (Right bot) $\eta \propto x$ coordinate of the pendulum during the measurement used to obtain the 2D map. The regions highlighted in red and blue are those used to estimate the non linearities of the top-right plot.

bottom right Figure 9.14, therefore we could expect a change in the non-linearities. Figure 9.14 shows a long measurement in which I monitored the evolution of non-linearities during a drift in $\eta \propto x$ pendulum position, creating a map of $\delta\phi(\phi, \eta)$. We see that there are some differences also with η variations, especially in the smooth trends: the spikes instead seem to be somehow more constant.

I tried to construct a routine that can use such map to correct the data, but it is really unpractical: one would need to construct one of these maps for each η and ϕ range the pendulum is at that moment. η is not under control of actuation forces, since it is driven by gravity, and to create such map one has to hope for a drift to happen. In a few words, without a way to actively control η it is too difficult to realize these maps consistently. An active control of η would actually solve the problem at the root, since we could keep the pendulum at a fixed value η_0 ; this is a further reason to implement a piezo feet tilt control, which we plan for the future.

9.6 Appendix F - Suggested measurement for pendulum readout noise characterization

We stated that the reason for which we can not distinguish between readout and true torque noise in these measurements is that their differences are actually hidden by actuation noise.

Appendix F - Suggested measurement for pendulum readout noise characterization

Can we then induce the same effect on the background *without* inducing also an increase in actuation torque noise? Yes: we can use the Y electrodes.

Y electrodes are oriented horizontally and can not be used to induce a ϕ rotation of the TM actively. This is expressed by the fact that $\partial C_Y / \partial x$ is zero at zeroth order. Still, it is not zero at second order. We can give the expansion of a Y electrode capacitance as we did in Sec. 3.3.4

$$C_5 = C_y + \left| \frac{\partial C_y}{\partial y} \right| y + \left| \frac{\partial C_y}{\partial \theta} \right| \theta + \frac{1}{2} \left| \frac{\partial^2 C_y}{\partial y^2} \right| y^2 + \frac{1}{2} \left| \frac{\partial^2 C_y}{\partial \phi^2} \right| \phi^2 + \frac{1}{2} \left| \frac{\partial^2 C_y}{\partial \theta^2} \right| \theta^2 \dots$$

$$\dots + \left| \frac{\partial^2 C_y}{\partial \theta \partial y} \right| y \theta + \left| \frac{\partial^2 C_y}{\partial \theta \partial z} \right| z \theta - \left| \frac{\partial^2 C_y}{\partial \phi \partial x} \right| \phi x \quad (9.59)$$

and we see it has a non-zero ϕ^2 term. This means that if a Y electrode is active, it will contribute to the stiffness of the pendulum.

Now we should see where this is going: we activate all the 4 Y electrodes with a large voltage and induce a change in pendulum stiffness, even higher than the one achieved with X electrodes, since the gaps are smaller. But we are not applying any torque at first order! Therefore actuation noise will not affect the pendulum, and we should only see the effect of transfer function change.

We tried performing such measurements to complete the analysis of Chapter 6 but we did not complete this campaign due to a series of technical interruptions. First, we encountered a problem that we could not understand at first, then the acquisition system failed. After its substitution, even if we understood how to address the problems, the pendulum noise floor had changed, therefore it could not be compared anymore with the X actuation measurements. It would have been necessary to re-run the whole campaign and this was not compatible with the timelines.

The initial problem encountered regards stray electrostatic fields and merits discussion. The 1TM home-made FEE has AC actuation only for the X electrodes, while can only apply DC voltages on Y and Z electrodes. We immediately noticed that using the Y electrodes like this, the pendulum torque noise was dominated by a cross-talk with x degree of freedom, analogous to the situation observed in Figure 6.8, hiding again the possible change in readout noise.

In Section 3.3.3, one of the paragraphs is called "The need for AC actuation" for a reason. If DC voltages are used, it is sure that these will couple with stray voltage patches and, between other things, cause forces that depend on relative electrode-patch position, i.e. the TM position. So we have two solutions for this:

1. Just implement Y AC waveforms. The home-made electronics boards are ready to host it, and when we will have the LISA FEE EM, this will be already available. This is the cleanest way of solving the problem.
2. If we still wanted to use DC actuation, we would have to compensate the DC stray biases. Again in Section 3.3.3 an idea was given on how to do it, we develop the

Appendix F - Suggested measurement for pendulum readout noise characterization

calculations a little bit for future reference in the next section (I think it helps to understand the effect and can be useful for people in the laboratory). The main weakness of this procedure is that we are never sure if, removed the dominant cross-talk, there will be another one under that. Unfortunately then, if we found a cross-talk with y , that could not be compensated. For these reasons I would always choose solution 1.

Concluding, the final message of this Appendix is that when tests will be performed on the LISA FEE to characterize actuation noise, in addition to the runs with X AC actuation presented through the chapter, it will be important to perform runs with just Y AC actuation: this will allow the distinction between readout and true torque noise, enabling a much better characterization of actuation noise. It would also be very useful to characterize the pendulum noise floor.

DC bias compensation for Y electrodes actuation noise measurement

Here we discuss the procedure to eliminate a cross talk with x observed during a torque measurement which applies DC voltages to Y electrodes. We start from equation (3.43), reported here for torque without the terms involving V_{TM} considered very small:

$$\begin{aligned} N_\phi &= \frac{1}{2} \sum_i^N \frac{\partial C_i^*}{\partial \phi} V_i^2 + \sum_i^N V_i \frac{\partial (C\delta V)_i^s}{\partial \phi} + \sum_i^N V_i \frac{\partial C_i^*}{\partial \phi} V_{DC,i} \\ &= N_\phi^{\text{command}} + N_\phi^{\text{stray}} + N_\phi^{\text{comp}} \end{aligned} \quad (9.60)$$

We use only Y electrodes, so we set for example $V_5 = -V_6 = V_7 = -V_8 = V_y$, and all the others to zero. The first term N_ϕ^{command} is the one that we use to decide which force we want to apply: if we use Y electrodes, we discussed that it can only result in stiffness terms and it won't help us in the compensation, therefore we drop it. The third term N_ϕ^{comp} is the one we use to compensate stray biases, choosing appropriate $V_{DC,i}$, while the second term N_ϕ^{stray} represents the stray interaction we want to cancel. Let us expand it at first order

$$N_\phi^{\text{stray}} = \sum_i \frac{\partial (C\delta V)_i^s}{\partial \phi} \bigg|_0 V_i + \sum_{i,p} \frac{\partial^2 (C\delta V)_i^s}{\partial \phi \partial p} \bigg|_0 V_i p. \quad (9.61)$$

Disregarding the first term, which is just a DC torque, if experimentally what we observe is a cross-talk with x it means that we have non-negligible term in the last sum of form

$$\begin{aligned} N_\phi^{\text{stray}} &= \sum_i \frac{\partial^2 (C\delta V)_i^s}{\partial \phi \partial x} \bigg|_0 V_i x. \\ &= V_y \left(\frac{\partial^2 (C\delta V)_5^s}{\partial \phi \partial x} \bigg|_0 - \frac{\partial^2 (C\delta V)_6^s}{\partial \phi \partial x} \bigg|_0 + \frac{\partial^2 (C\delta V)_7^s}{\partial \phi \partial x} \bigg|_0 - \frac{\partial^2 (C\delta V)_8^s}{\partial \phi \partial x} \bigg|_0 \right) x. \end{aligned} \quad (9.62)$$

So we have a quantity in the brackets which we do not know, because it depends on stray voltage patches, which causes a signal $\propto x$ and V_y . One good thing of having this large

parasitic signal is that we can measure it: we can do it either by fitting the $x(t)$ time series to the $N_\phi(t)$, or by measuring the quantity $\partial N_\phi / \partial x$, for example by inducing a large, sudden x variation by placing a heavy weight on the pendulum platform, in order to tilt it (this is usually the most effective way). Then, if we want to cancel this interaction, we have to find an analogous term in N_ϕ^{comp} : if we look at (9.59), we see that the y actuation electrodes actually have a term $\propto x$. If we apply small DC biases, this will also result in a stiffness similar to (9.62). We calculate it:

$$\begin{aligned} N_\phi^{\text{comp}} &= V_y \left(-\frac{\partial^2 C_y^*}{\partial \phi \partial x} \Big|_0 V_{DC,5} + \frac{\partial^2 C_y^*}{\partial \phi \partial x} \Big|_0 V_{DC,6} + \frac{\partial^2 C_y^*}{\partial \phi \partial x} \Big|_0 V_{DC,7} - \frac{\partial^2 C_y}{\partial \phi \partial x} \Big|_0 V_{DC,8} \right) x = \\ &= V_y \left| \frac{\partial^2 C_y^*}{\partial \phi \partial x} \right| (-V_{DC,5} + V_{DC,6} + V_{DC,7} - V_{DC,8}) x. \end{aligned} \quad (9.63)$$

Now every quantity of this expression is under our control! So if we want the sum of (9.62) and (9.63) to vanish, we must choose the DC voltages so that

$$\left| \frac{\partial^2 C_y^*}{\partial \phi \partial x} \right| (-V_{DC,5} + V_{DC,6} + V_{DC,7} - V_{DC,8}) = -\frac{1}{V_y} \frac{\partial N_\phi}{\partial x} \quad (9.64)$$

Looks complicated, probably it is, but it works. Another application of this method can be also found in Chapter 7.

Just a notation reference. In papers and laboratory, since expression (9.62) can be compensated by application of a proper combination of Y DC voltages, it is usually expressed as follows

$$\begin{aligned} N_\phi^{\text{stray}} &= V_y \left(\frac{\partial^2 (C\delta V)_5^s}{\partial \phi \partial x} \Big|_0 - \frac{\partial^2 (C\delta V)_6^s}{\partial \phi \partial x} \Big|_0 + \frac{\partial^2 (C\delta V)_7^s}{\partial \phi \partial x} \Big|_0 - \frac{\partial^2 (C\delta V)_8^s}{\partial \phi \partial x} \Big|_0 \right) x \\ &= V_y \left| \frac{\partial^2 C_y^*}{\partial \phi \partial x} \right| \Delta_\theta x. \end{aligned} \quad (9.65)$$

Basically, it consists in describing the whole stray bias contribution by a single voltage Δ_θ , which is normalized to one Y electrode.

9.7 Appendix G - UV power measurement

The LED Seti 240 light is sent through a splitting fiber for which we measured that 60.7% and 39.3% ends up at the two ends. The "larger amount" of light is sent to the the TM through a vacuum feed-through and vacuum optical fiber, which add an attenuation $\alpha_{ff} = 0.46$. The 39.3% goes on a PMT + amplifier, which converts the incident power to a voltage. This was calibrated against another calibrated photodiode on the 4TM facility, finding a conversion coefficient $dP/dV = -(2.95 \pm 0.13) \times 10^{-7} \text{W/V}$ (for the light of the same UV LED). This calibration accounts for "attach-detach" errors, that are power measurement variations

when the optical fiber is detached and re-attached from the PMT. These have shown to be dominant errors and are estimated by just attaching and detaching the fiber several times and taking the standard deviation of the different power measurements at each trial, also removing a dark current background.

So the signal we generate with the LED is a time series $P(t)$, a series of pulses of amplitude P_0 , length T_p and at a rate of $1/T$. Then, the PMT will convert this to an analogous series of pulses $V(t)$, which will have amplitude $V_0 = (dV/dP)P_0$. This signal has non-zero average, and first-harmonic at $f_1 = 1/T$, but also has all the other harmonics. Its Fourier expansion is

$$V(t) = a_0 + \sum_{i=1} a_n \cos(n\omega_n t + \phi(t)) \quad (9.66)$$

where $\phi(t)$ represents the (noisy) phase of the pulses with respect to the injection voltage. The Fourier coefficients are

$$a_0 = V_0 \frac{T_p}{T}, \quad a_n = V_0 \frac{2}{n\pi} \sin\left(n\pi \frac{T_p}{T}\right) \quad (9.67)$$

The pulses can be generated at a maximum of 100 kHz: lower frequencies are generated by "decimation", skipping a certain number of injection cycles. A decimation of N means that $f_1 = 100 \text{ kHz}/N$, but this in turn means that the signal $V(t)$ N -th harmonic is at 100 kHz. So, we can demodulate it with a lock-in amplifier, locked to the injection voltage. The two outputs of the lock-in will be

$$\begin{aligned} X(t) &= \frac{a_N}{\sqrt{2}} \cos \phi(t) \\ Y(t) &= -\frac{a_N}{\sqrt{2}} \sin \phi(t) \end{aligned} \quad (9.68)$$

From this we can estimate first of all the phase of the UV light

$$\phi(t) = -\tan^{-1} \left(\frac{Y(t)}{X(t)} \right). \quad (9.69)$$

Secondly we can also estimate $\langle V(t) \rangle$ by using the relation between a_0 and a_N .

$$\langle V(t) \rangle = a_0 = \frac{a_0}{a_N} a_N = \frac{n\pi T_p}{2T \sin\left(n\pi \frac{T_p}{T}\right)} \sqrt{2(X^2(t) + Y^2(t))} \quad (9.70)$$

We can finally calculate the power sent to the TM accounting for all the corrections

$$P_{TM} = \alpha_{ff} \left(\frac{0.607}{0.393} \right) \left(\frac{dP}{dV} \right) \langle V(t) \rangle \quad (9.71)$$

9.8 Appendix H - Demodulated quadrature of charge measurement runs

Here we report the noise observed in phase and quadrature demodulation during a charge measurement. While the phase should be proportional to V_{TM} and therefore also be affected by its noise, the quadrature should instead find no signal at all and exhibit a white PSD fixed by the torque noise floor of the pendulum. Both are reported for each run of the campaign in Figure 9.15. The quadrature seems indeed white for almost all the runs; the high power ones instead show larger noise (especially $\tau = 0.53$ hours) due to the presence of transients spikes, which however are not big enough to impact the red spectrum. The quadrature noise observed is roughly explained with a background torque noise of the pendulum at 3 mHz of $1.5 \text{ fNm}/\sqrt{\text{Hz}}$. A typical performance of the pendulum at the time of this measurement campaign was actually a little better, $1 \text{ fNm}/\sqrt{\text{Hz}}$, as reported in Figure 9.16. However, the large modulation of 5 V used for the measurement induces huge oscillations of the pendulum, which always result in a small degradation of the noise floor.

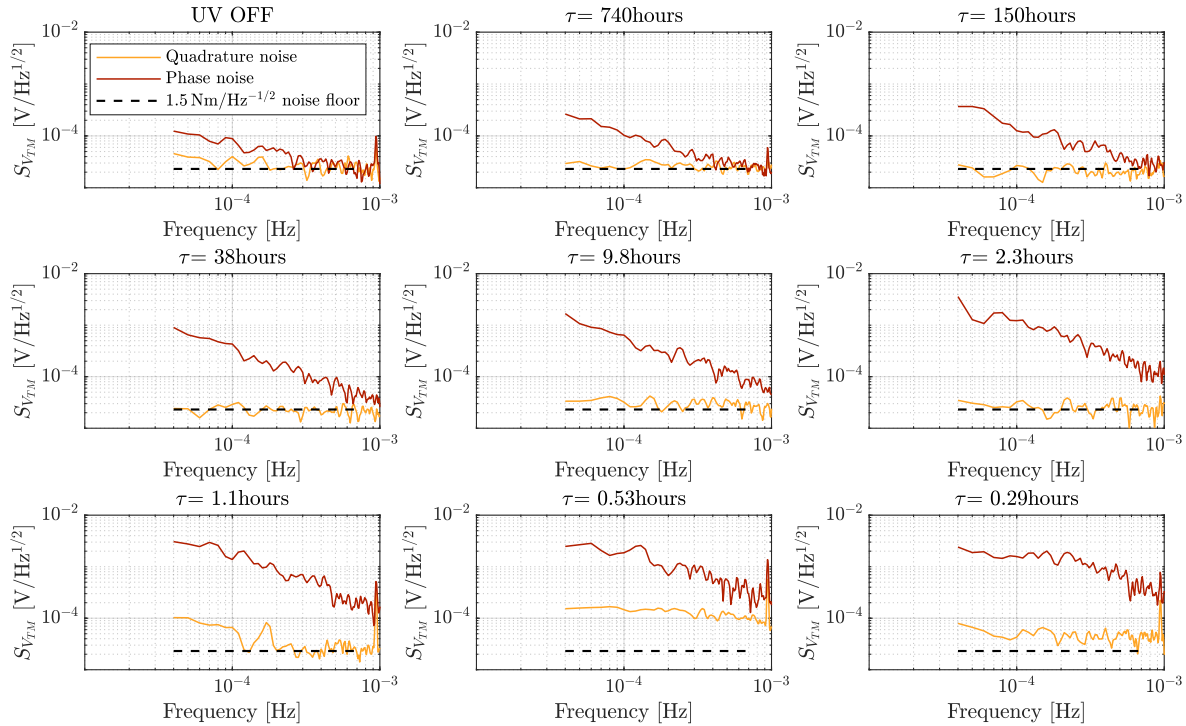


Figure 9.15: ASD of V_{TM} for all the noise runs presented in the campaign of Sec. 7.2.3. (Red) Demodulation of the signal in phase. (Orange) Demodulation of the quadrature signal. Discussed in the text.

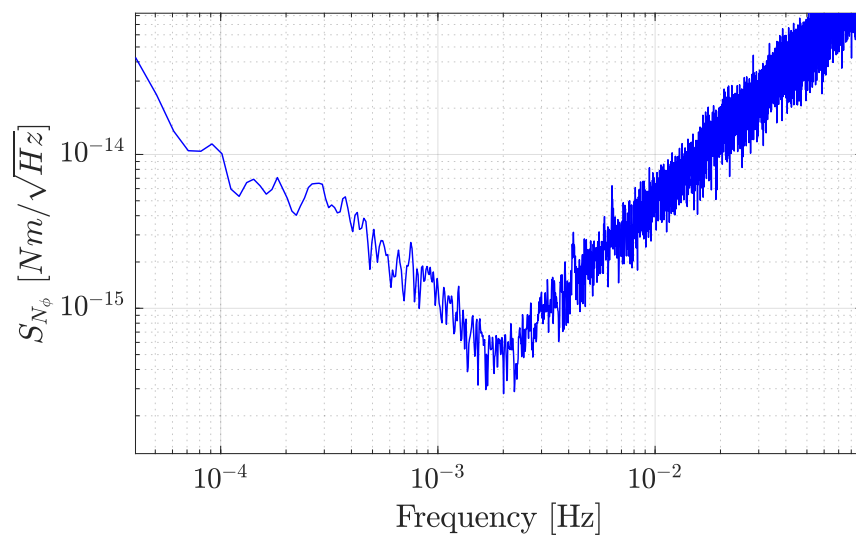


Figure 9.16: Typical pendulum torque noise floor at the time of UV induced charge noise campaign. The performance shown in Chap. 6 was not yet obtained at this time.

Acknowledgments

First of all, my deepest thanks go to my supervisor Bill. I have a sincere admiration for you and your genius, for me you represent what it truly means to be a scientist. This work would not have been possible without your guidance, the curiosity and the passion you transmit. And thank you for trusting in me, it was a true pleasure for me to work at your side.

Thanks to the entire Experimental Gravitation group, starting from the lab people. Antonella, a watchful eye on the essence of people. Teodoro, who taught me to be patient. Francisco, whose presence always lifted the spirits. Renato, a friend always ready to help. Rita, of incredible resilience, perseverance and attention to the human side of people.

Special thanks to my companions of struggles and successes, Davide Dal Bosco and Francesco Venturelli. Davide, I admire your vast knowledge, your clarity and precision in communication and science, your effort in understanding ourselves and courage to do new things. I have a lot to learn from you. Francesco, having you by my side in the lab has always given me serenity, especially late in the evenings. I have never seen you being brought down by adversities (something which I am not very good at, instead); I admire your perseverance and uniqueness, I think this will lead you to a bright future.

Thanks to Stefano Vitale, the person that started my interest for LISA. I remember the excitement that I was feeling during his lectures: I still feel it when he presents one of his new studies. The opportunity to collaborate with him was an honor for me.

I am incredibly thankful to Lorenzo Sala. First of all for the help he gave me with LPF analyses at the beginning and through all my PhD studies. Then, for the fantastic research we made together during the last year: it was an amazing feeling to work with a smart and reliable person like you, reasoning in front of a whiteboard and see the puzzle pieces come together. I had been missing this feeling from the Bachelor's years.

Thanks to Francesco Dimiccoli, for the reciprocal understanding, the laughs and the help with an impossible knot to untangle (i.e. a LabView code). Thanks to the other members of the group Carlo Zanoni, Daniele Vetrugno, Valerio Ferroni, Giacomo Ciani, especially for the helpful discussions on how to tackle a life in science. Thanks to Karine Frisinghelli and her ability to "get things done". Thanks to the Engineering colleagues Matteo Tomasi, Davide Vignotto, Edoardo Dalla Ricca, Abraham Gelan, Francesco Marzari and Daniele Bortoluzzi for the laughs and the hard work they do.

I also want to thank the people that worked on the 1TM torsion pendulum and have given me the possibility to work with such a wonderful instrument: David Tombolato, Daniele

Nicolodi, Hai-Bo Tu, Giuliana Russano.

Thanks to the LISA team of INFN/Roma Tor Vergata, especially Fulvio De Persio, Giuseppe Masciantonio and Massimo Bassan, for providing the prototype CMD electronics and the UV-LEDs.

Also thanks to Luigi Ferraioli, Davor Mance and the ETHZ group for providing the LPF FEE prototype EM Light and for the fruitful discussions regarding the measurements carried out on it for this thesis.

Thanks to the "Laboratori didattici" staff, Mauro Hueller, Gigi Minati, Mariano di Mauro, Maria Teresa Lopez-arias Montenegro and Chiara Piotto, always cheering me up.

I would now like to thank the LISA group in Hannover. Professor Gerhard Heinzel for giving me the opportunity of spending one month at the Albert Einstein Institute, to visit and carry out research, which for me was very fruitful. In this context, I thank Christoph Gentemann, fellow bearer of the pendulum legacy and curse. It was a pleasure to work with you. Thanks to all the people of the Institute, which welcomed me warmly, especially the Italian community and Alvis Pizzella. I want to thank Gudrun Wanner, who was extremely kind and supportive to me, listening to my doubts and fears on a life in science. Finally, I can not forget of Marie-Sophie Hartig and Lennart Wissel. These two were incredible. Marie is a source of constant inspiration, with a determination and initiative that I have never seen in anyone else. Lennart, a wonderful and smart person that treated me with a care that I thought could not exist.

Coming back to Trento, I would like to thank my officemates. Sophie Bini, Riccardo Nicolaidis, Greta Brianti and Francesco Rossi for the support, the laughs and the gossips, which were essential vent valves during difficult PhD times.

Special credit must be given to my flatmates, with which I spent a huge amount of time and shared the struggles of the PhD life. Giovanni, there is no need for me to write anything: I still hope to find you there when I come home. Maria, always ready to listen and make me feel that people with a soul and empathy exist. Giorgio, you crazy friend, a thread is somehow linking us as we go through life, and I hope it continues to do so. Stefano, I loved to find in you so many things that resonated with me and the laughs we had.

Voglio ringraziare profondamente Valentina. Grazie per non avermi mai fatto sentire solo nonostante i miei sbalzi di umore. Per avermi ascoltato davvero con l'intenzione di capirmi. Per aver sopportato parte del mio veleno e per avermi dato in cambio un angolo di serenità. Per questo sarò sempre in debito.

Papà Carlo e Mamma Antonella. Per ventotto anni mi avete considerato come una persona di valore. Non avete *mai* minimamente dubitato nelle mie capacità, e il vostro unico desiderio nei miei confronti era quello che fossi libero, e felice. Nulla può rendermi più fiero di questo, e mi ritengo infinitamente fortunato ad aver trovato voi due. Senza di voi non sarei mai arrivato sino a qui.

Bibliography

- [1] B. P. Abbott et al. “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral”. In: *Phys. Rev. Lett.* 119 (16 Oct. 2017), p. 161101. DOI: 10.1103/PhysRevLett.119.161101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.119.161101>.
- [2] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. In: *Phys. Rev. Lett.* 116 (6 Feb. 2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.116.061102>.
- [3] R. Abbott et al. “GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo during the Second Part of the Third Observing Run”. In: *Phys. Rev. X* 13 (4 Dec. 2023), p. 041039. DOI: 10.1103/PhysRevX.13.041039. URL: <https://link.aps.org/doi/10.1103/PhysRevX.13.041039>.
- [4] Klaus Abich et al. “In-Orbit Performance of the GRACE Follow-on Laser Ranging Interferometer”. In: *Phys. Rev. Lett.* 123 (3 July 2019), p. 031101. DOI: 10.1103/PhysRevLett.123.031101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.123.031101>.
- [5] Pau Amaro-Seoane et al. *Laser Interferometer Space Antenna*. 2017. arXiv: 1702.00786 [astro-ph.IM].
- [6] F. Antonucci et al. “Interaction between Stray Electrostatic Fields and a Charged Free-Falling Test Mass”. In: *Phys. Rev. Lett.* 108 (18 Apr. 2012), p. 181101. DOI: 10.1103/PhysRevLett.108.181101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.108.181101>.
- [7] M. Armano et al. “Beyond the Required LISA Free-Fall Performance: New LISA Pathfinder Results down to 20 μHz ”. In: *Phys. Rev. Lett.* 120 (6 Feb. 2018), p. 061101. DOI: 10.1103/PhysRevLett.120.061101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.120.061101>.
- [8] M. Armano et al. “Calibrating the system dynamics of LISA Pathfinder”. In: *Phys. Rev. D* 97 (12 June 2018), p. 122002. DOI: 10.1103/PhysRevD.97.122002. URL: <https://link.aps.org/doi/10.1103/PhysRevD.97.122002>.

- [9] M. Armano et al. “Capacitive sensing of test mass motion with nanometer precision over millimeter-wide sensing gaps for space-borne gravitational reference sensors”. In: *Phys. Rev. D* 96 (6 Sept. 2017), p. 062004. DOI: 10.1103/PhysRevD.96.062004. URL: <https://link.aps.org/doi/10.1103/PhysRevD.96.062004>.
- [10] M. Armano et al. “Charge-Induced Force Noise on Free-Falling Test Masses: Results from LISA Pathfinder”. In: *Phys. Rev. Lett.* 118 (17 Apr. 2017), p. 171101. DOI: 10.1103/PhysRevLett.118.171101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.118.171101>.
- [11] M. Armano et al. “Charging of free-falling test masses in orbit due to cosmic rays: Results from LISA Pathfinder”. In: *Phys. Rev. D* 107 (6 Mar. 2023), p. 062007. DOI: 10.1103/PhysRevD.107.062007. URL: <https://link.aps.org/doi/10.1103/PhysRevD.107.062007>.
- [12] M. Armano et al. “In-depth analysis of LISA Pathfinder performance results: Time evolution, noise projection, physical models, and implications for LISA”. In: *Phys. Rev. D* 110 (4 Aug. 2024), p. 042004. DOI: 10.1103/PhysRevD.110.042004. URL: <https://link.aps.org/doi/10.1103/PhysRevD.110.042004>.
- [13] M. Armano et al. “Nano-Newton electrostatic force actuators for femto-Newton-sensitive measurements: System performance test in the LISA Pathfinder mission”. In: *Phys. Rev. D* 109 (10 May 2024), p. 102009. DOI: 10.1103/PhysRevD.109.102009. URL: <https://link.aps.org/doi/10.1103/PhysRevD.109.102009>.
- [14] M. Armano et al. “Precision charge control for isolated free-falling test masses: LISA pathfinder results”. In: *Phys. Rev. D* 98 (6 Sept. 2018), p. 062001. DOI: 10.1103/PhysRevD.98.062001. URL: <https://link.aps.org/doi/10.1103/PhysRevD.98.062001>.
- [15] M. Armano et al. “Sensor Noise in LISA Pathfinder: In-Flight Performance of the Optical Test Mass Readout”. In: *Phys. Rev. Lett.* 126 (13 Apr. 2021), p. 131103. DOI: 10.1103/PhysRevLett.126.131103. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.126.131103>.
- [16] M. Armano et al. “Sub-Femto- g Free Fall for Space-Based Gravitational Wave Observatories: LISA Pathfinder Results”. In: *Phys. Rev. Lett.* 116 (23 June 2016), p. 231101. DOI: 10.1103/PhysRevLett.116.231101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.116.231101>.
- [17] M. Armano et al. “Transient acceleration events in LISA Pathfinder data: Properties and possible physical origin”. In: *Phys. Rev. D* 106 (6 Sept. 2022), p. 062001. DOI: 10.1103/PhysRevD.106.062001. URL: <https://link.aps.org/doi/10.1103/PhysRevD.106.062001>.

- [18] Stanislav Babak, Martin Hewitson, and Antoine Petiteau. *LISA Sensitivity and SNR Calculations*. 2021. arXiv: 2108.01167 [astro-ph.IM]. URL: <https://arxiv.org/abs/2108.01167>.
- [19] L Barsotti, S Gras, and p Fritschel. “Updated Advanced LIGO sensitivity design curve”. In: (Apr. 2018). URL: <https://dcc.ligo.org/LIGO-T1800044/public>.
- [20] Davide Dal Bosco. “Torsion Pendulum Testing of the LISA Charge Management System”. PhD thesis. University of Trento, Apr. 2023.
- [21] Nico Brandt and Walter Fichter. “Revised electrostatic model of the LISA Pathfinder inertial sensor”. In: *Journal of Physics: Conference Series* 154.1 (Mar. 2009), p. 012008. DOI: 10.1088/1742-6596/154/1/012008. URL: <https://dx.doi.org/10.1088/1742-6596/154/1/012008>.
- [22] L. Carbone et al. “Achieving Geodetic Motion for LISA Test Masses: Ground Testing Results”. In: *Phys. Rev. Lett.* 91 (15 Oct. 2003), p. 151101. DOI: 10.1103/PhysRevLett.91.151101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.91.151101>.
- [23] Sean M. Carroll. *Spacetime and Geometry*. Cambridge University Press, July 2019. ISBN: 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7.
- [24] Eleonora Castelli. “LISA Pathfinder noise performance results: disturbances in the sub-mHz frequency band and projection to LISA”. PhD thesis. University of Trento.
- [25] A Cavalleri et al. “Direct force measurements for testing the LISA Pathfinder gravitational reference sensor”. In: *Classical and Quantum Gravity* 26.9 (Apr. 2009), p. 094012. DOI: 10.1088/0264-9381/26/9/094012. URL: <https://doi.org/10.1088/0264-9381/26/9/094012>.
- [26] A. Cavalleri et al. “A new torsion pendulum for testing the limits of free-fall for LISA test masses”. In: *Classical and Quantum Gravity* 26.9 (Apr. 2009), p. 094017. ISSN: 0264-9381. DOI: 10.1088/0264-9381/26/9/094017. URL: <https://doi.org/10.1088/0264-9381/26/9/094017>.
- [27] A. Cavalleri et al. “Increased Brownian Force Noise from Molecular Impacts in a Constrained Volume”. In: *Phys. Rev. Lett.* 103 (14 Sept. 2009), p. 140601. DOI: 10.1103/PhysRevLett.103.140601. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.103.140601>.
- [28] V. Chiavegato et al. *Actuation Nonlinearities*. Tech. rep. LISA-UTN-INST-TN-0028. Oct. 2024.
- [29] Vittorio Chiavegato. “Experimental Investigation of Force Noise from Electrostatic Force Actuation in the LISA Gravitational Wave Observatory”. MA thesis. University of Trento, 2021.

- [30] Giacomo Ciani. “Free-fall of LISA test masses: a new torsion pendulum to test translational acceleration”. PhD thesis. University of Trento, 2008.
- [31] Monica Colpi et al. *LISA Definition Study Report*. 2024. arXiv: 2402.07571 [astro-ph.CO]. URL: <https://arxiv.org/abs/2402.07571>.
- [32] Peter Zweifel D. Mance Li Gan. *IS-FEE PFM ACTUATION STABILITY CHARACTERIZATION*. Tech. rep. S2-ETH-TN-3029. May 2010.
- [33] LISA Instrument Group ESA. “Design Description, Lisa Payload Definition Document, ESA-L3-EST-INST-DD-001”. In: (2.1 May 2018).
- [34] L. Ferraioli. *GRS actuation model*. Tech. rep. LISA-ETH-INST-TN-006. May 2019.
- [35] E Fitzsimons, J Martino, and D Vetrugno. *LISA performance model and Error Budget*. Tech. rep. LISA-LCST-INST-TN-003. Aug. 2023.
- [36] N. R. Goodman. “Statistical Analysis Based on a Certain Multivariate Complex Gaussian Distribution (An Introduction)”. In: *The Annals of Mathematical Statistics* 34.1 (1963), pp. 152–177. DOI: 10.1214/aoms/1177704250. URL: <https://doi.org/10.1214/aoms/1177704250>.
- [37] Jeffreys Harold. “An invariant form for the prior probability in estimation problems”. In: *Proc. R. Soc. Lond. A* 186: 453–461 (1945). URL: <http://doi.org/10.1098/rspa.1946.0056>.
- [38] Olaf Hartwig and Martina Muratore. “Characterization of time delay interferometry combinations for the LISA instrument noise”. In: *Phys. Rev. D* 105 (6 Mar. 2022), p. 062006. DOI: 10.1103/PhysRevD.105.062006. URL: <https://link.aps.org/doi/10.1103/PhysRevD.105.062006>.
- [39] Seong Hyeon Hong and John W. Conklin. “Finding the suitable drag-free acceleration noise level for future low-low satellite-to-satellite tracking geodesy missions”. In: *Advances in Space Research* 63.1 (2019), pp. 32–50. ISSN: 0273-1177. DOI: <https://doi.org/10.1016/j.asr.2018.07.022>. URL: <https://www.sciencedirect.com/science/article/pii/S0273117718305842>.
- [40] M Hueller et al. “Torsion pendulum facility for ground testing of gravitational sensors for LISA”. In: *Classical and Quantum Gravity* 19.7 (Mar. 2002), p. 1757. DOI: 10.1088/0264-9381/19/7/372. URL: <https://dx.doi.org/10.1088/0264-9381/19/7/372>.
- [41] Mauro Hueller. “Geodesic motion of LISA test masses: development and testing of drag-free position sensors”. PhD thesis. University of Trento, 2003.
- [42] Greg Kopp and Judith L. Lean. “A new, lower value of total solar irradiance: Evidence and climate significance”. In: *Geophysical Research Letters* 38.1 (2011). DOI: <https://doi.org/10.1029/2010GL045777>. eprint: <https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/2010GL045777>. URL: <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2010GL045777>.

- [43] Kopp, Greg. “Magnitudes and timescales of total solar irradiance variability”. In: *J. Space Weather Space Clim.* 6 (2016), A30. DOI: 10.1051/swsc/2016025. URL: <https://doi.org/10.1051/swsc/2016025>.
- [44] Bryant D. Loomis, R. S. Nerem, and S. B. Luthcke. “Simulation study of a follow-on gravity mission to GRACE”. In: *Journal of Geodesy* 86.5 (May 2012), pp. 319–335. ISSN: 1432-1394. DOI: 10.1007/s00190-011-0521-8. URL: <https://doi.org/10.1007/s00190-011-0521-8>.
- [45] W.J. Weber M. Davor. *Technical note: IS-FEE Actuation Stability Characterization*. Tech. rep. S2-ETH-TN-3027. Feb. 2008, pp. 70–73.
- [46] D. V. Martynov et al. “Sensitivity of the Advanced LIGO detectors at the beginning of gravitational wave astronomy”. In: *Phys. Rev. D* 93 (11 June 2016), p. 112004. DOI: 10.1103/PhysRevD.93.112004. URL: <https://link.aps.org/doi/10.1103/PhysRevD.93.112004>.
- [47] N. Meshksar et al. “Analysis of the accuracy of actuation electronics for the laser interferometer space antenna”. In: *Review of Scientific Instruments* 91.9 (Sept. 2020), p. 095003. ISSN: 0034-6748. DOI: 10.1063/5.0018536. eprint: https://pubs.aip.org/aip/rsi/article-pdf/doi/10.1063/5.0018536/14801678/095003_1_online.pdf. URL: <https://doi.org/10.1063/5.0018536>.
- [48] Daniele Nicolodi. “Femto Newton level testing of free fall on ground”. PhD thesis. University of Trento, Mar. 2011.
- [49] Anna M. Nobili and Alberto Anselmi. *On the first test of the Weak Equivalence Principle in low Earth orbit*. 2024. arXiv: 2302.06400 [gr-qc]. URL: <https://arxiv.org/abs/2302.06400>.
- [50] Travis Robson, Neil J Cornish, and Chang Liu. “The construction and use of LISA sensitivity curves”. In: *Classical and Quantum Gravity* 36.10 (Apr. 2019), p. 105011. DOI: 10.1088/1361-6382/ab1101. URL: <https://doi.org/10.1088/1361-6382/ab1101>.
- [51] Giuliana Russano. “A torsion pendulum ground test of the LISA Pathfinder Free-fall mode”. PhD thesis. University of Trento, 2015.
- [52] Lorenzo Sala. “Residual test mass acceleration in LISA Pathfinder: in-depth statistical analysis and physical sources”. PhD thesis. University of Trento, 2023. DOI: https://dx.doi.org/10.15168/11572_384049.
- [53] Peter R. Saulson. “Thermal noise in mechanical experiments”. In: *Phys. Rev. D* 42 (8 Oct. 1990), pp. 2437–2445. DOI: 10.1103/PhysRevD.42.2437. URL: <https://link.aps.org/doi/10.1103/PhysRevD.42.2437>.
- [54] Bernard Schutz. *A First Course in General Relativity*. 2nd ed. Cambridge University Press, 2009. DOI: 10.1017/CB09780511984181.

- [55] Thomas S. Schwarze et al. “Picometer-Stable Hexagonal Optical Bench to Verify LISA Phase Extraction Linearity and Precision”. In: *Phys. Rev. Lett.* 122 (8 Feb. 2019), p. 081104. DOI: 10.1103/PhysRevLett.122.081104. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.122.081104>.
- [56] Alberto Sesana. “Prospects for Multiband Gravitational-Wave Astronomy after GW150914”. In: *Phys. Rev. Lett.* 116 (23 June 2016), p. 231102. DOI: 10.1103/PhysRevLett.116.231102. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.116.231102>.
- [57] Massimo Tinto, Sanjeev Dhurandhar, and Dishari Malakar. “Second-generation time-delay interferometry”. In: *Phys. Rev. D* 107 (8 Apr. 2023), p. 082001. DOI: 10.1103/PhysRevD.107.082001. URL: <https://link.aps.org/doi/10.1103/PhysRevD.107.082001>.
- [58] David Tombolato. “A laboratory study of force disturbances for the LISA Pathfinder free fall demonstration mission”. PhD thesis. University of Trento, 2008.
- [59] P. Touboul et al. “CHAMP, GRACE, GOCE Instruments and Beyond”. In: *Geodesy for Planet Earth*. Ed. by Steve Kenyon, Maria Christina Pacino, and Urs Marti. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 215–221. ISBN: 978-3-642-20338-1.
- [60] Pierre Touboul et al. “MICROSCOPE Mission: First Results of a Space Test of the Equivalence Principle”. In: *Phys. Rev. Lett.* 119 (23 Dec. 2017), p. 231101. DOI: 10.1103/PhysRevLett.119.231101. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.119.231101>.
- [61] M. Tse et al. “Quantum-Enhanced Advanced LIGO Detectors in the Era of Gravitational-Wave Astronomy”. In: *Phys. Rev. Lett.* 123 (23 Dec. 2019), p. 231107. DOI: 10.1103/PhysRevLett.123.231107. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.123.231107>.
- [62] S Vitale et al. “Estimate of force noise from electrostatic patch potentials in LISA Pathfinder”. In: *Classical and Quantum Gravity* 41.19 (Aug. 2024), p. 195009. DOI: 10.1088/1361-6382/ad7089. URL: <https://dx.doi.org/10.1088/1361-6382/ad7089>.
- [63] S. Vitale. *Minimum-error, low-correlation, power spectral density estimation with proper error statistics*. Tech. rep. S2-UTN-TN-3130 1.1. 2017.
- [64] Stefano Vitale et al. “Data series subtraction with unknown and unmodeled background noise”. In: *Phys. Rev. D* 90 (4 Aug. 2014), p. 042003. DOI: 10.1103/PhysRevD.90.042003. URL: <https://link.aps.org/doi/10.1103/PhysRevD.90.042003>.
- [65] V. Ferroni W.J.Weber. *Analysis of DC force requirements, gravitational interfaces, and possible apportioning schemes*. Tech. rep. LISA-UTN-INST-TN-006. June 2019.

- [66] William Joseph Weber et al. “Application of LISA Gravitational Reference Sensor Hardware to Future Intersatellite Geodesy Missions”. In: *Remote Sensing* 14.13 (2022). ISSN: 2072-4292. DOI: 10.3390/rs14133092. URL: <https://www.mdpi.com/2072-4292/14/13/3092>.
- [67] J. M. Weisberg and J. H. Taylor. *Relativistic Binary Pulsar B1913+16: Thirty Years of Observations and Analysis*. 2004. arXiv: astro-ph/0407149 [astro-ph]. URL: <https://arxiv.org/abs/astro-ph/0407149>.
- [68] P. Welch. “The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms”. In: *IEEE Transactions on Audio and Electroacoustics* 15 (1967), pp. 70–73.