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PhD Thesis

Geometry and Dynamics of Nonholonomic Affine Mechanical Systems

Candidate:
Paolo Eugenio Petit Valdés Villarreal

Advisor:
Prof. Nicola Sansonetto

Co-advisor:
Prof. Luis C. García Naranjo

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Abstract

In this Thesis we study two types of mechanical nonholonomic systems, namely systems with linear constraints and lagrangian with a linear term in the velocities, and nonholonomic systems with affine constraints and lagrangian without a linear term in the velocities. For the former type of systems we construct an almost-Poisson bracket using elements related to a riemannian metric induced by the kinetic energy, and we show that under certain conditions gauge momenta exist. For the latter type of systems, we focus on the ones possessing a *Noether symmetry*. To everyone of these systems we associate an equivalent system of the former type, and we exhibit the procedure to relate them and their gauge momentum. As a test case for the theory, we analyze the system of a heavy ball rolling without slipping on a rotating surface of revolution: we elucidate that also in this framework the so-called Routh integrals are related to symmetries, we give conditions for boundedness of the motions. In the particular case the surface of revolution is an inverted cone we characterize the qualitative behavior of the motions.

Notations and Assumptions

Along the Thesis we assume that all manifolds and maps are smooth, except when differently stated. Summation over repeated indices or Einstein summation convention is used.

We also use the following notation extensively. We introduce it to make a clearer presentation.

$C^\infty(Q)$ – the set of smooth real valued functions defined on the manifold Q.

$\mathfrak{X}(Q)$ – the set of smooth vector field on the manifold Q.

$\Gamma(\mathcal{D})$ – the set of smooth sections of a vector bundle D, where $\mathcal{D}$ is the total space.

$\Omega^k(Q)$ – the set of smooth k-forms on the manifold Q.

$X^\ell : D^* \longrightarrow \mathbb{R}$ – the fiberwise linear function associated to $X \in \Gamma(\mathcal{D})$.

$\langle \cdot, \cdot \rangle$ – the pairing between a vector bundle and its dual bundle.

$\langle \cdot, \cdot \rangle_g$ – the pairing induced by a riemannian metric g on a manifold.

*The scientist does not study nature because it is useful to do so.
He studies it because he takes pleasure in it, and he takes
pleasure in it because it is beautiful.*

Henri Poincaré

Donde dejo mi sombrero, ahí está mi casa.

Proverb

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Introduction

The purpose of this Thesis is to study certain classes of nonholonomic mechanical systems from a geometric and dynamical point of view. A good understanding in the subjects of ordinary differential equations, differential geometry and Lie groups is preferred, as well a good knowledge in analytical mechanics is rather useful to comprehend the structure of the dynamics of the considered mechanical systems.

In the subsequent paragraphs we present a brief historical passage trying to elucidate the importance of mechanics in mathematics, and vice versa, since both areas of study have been nurturing symbiotically along their existence. Nowadays Geometric Mechanics is a branch of Mechanics and Mathematics in which a vast range of theories and applications live together. Even though this Thesis is about nonholonomic mechanics, the machinery developed on the classical, or more precisely unconstrained, setting impacts directly on the techniques and approaches used to analyze nonholonomic systems. The reader can consult for example [1, 4, 3, 84, 95] for more historical and theoretical information about classical mechanics.

Mechanical systems have accompanied humanity since the beginnings of civilization with the inclined ramp, lever and pulley as ones of their most simple and characteristic examples. On the nonholonomic side the wheel is arguably the first man-made example of such kind, since in a rough terrain situation it does not slips. The systematic study of mechanics as a branch of physics may have begun with the ancient Greeks, being Aristotle and Archimedes the main representatives.

Later in time, between the 16th and 17th century, a major breakthrough came with Galileo Galilei[1] and Johannes Kepler. In this period the study of celestial mechanics was vastly developed and the research approach was, in a sense, from a qualitative perspective. For example Galilean relativity is the precursor of inertial reference frames and as we can see in Kepler's laws of planetary motion, global information of central body systems is obtained.

The analytical study of mechanics from 17th to 19th century arrived with Isaac Newton, Leonhard Euler, Giuseppe Luigi Lagrange, Jean Le Rond d'Alembert, Adrien-Marie Legendre, Pierre-Simon Laplace and William Rowan Hamilton, among many others. Newton's second Law famously stated as

$$F = ma,$$

is the cornerstone of mechanics, where all mechanical theories converge as starting point. Our approach to study nonholonomic systems is based on Euler-Lagrange equations and d'Alembert principle. We first present both concepts and then precise the meaning of nonholonomic constraints and their equations of motion. It is well known that Euler-Lagrange equations are equivalent to Hamilton's principle [1, 4]. Let $q : [t_0, t_1] \subset \mathbb{R} \to Q$ be a (at least) twice differentiable curve in a n-dimensional smooth manifold Q, a variation of the curve $q(t)$ is a family of curves $q(t, \varepsilon)$, parametrized by $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

[1] In [4] the author uses modern language to expose Galileo's ideas on relativity of reference frames.

$\varepsilon_0 > 0$, such that $q(0) = q(0,\varepsilon)$, $q(0) = q(0,\varepsilon)$, $q(t) = q(t,0)$. The derivative

$$\delta q(t) := \left.\frac{dq(t,\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \tag{1}$$

is called an infinitesimal displacement of the variation of the path $q(t)$. Hamilton's principle states that the trajectories of the system are precisely the extrema of the action integral

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{t_0}^{t_1} L(q,\dot{q})dt = 0,$$

for any smooth variation of $q(t)$. This principle is proven to be equivalent to Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1,...,n. \tag{2}$$

In mechanics the smooth function $L : TQ \to \mathbb{R}$ is called the *lagrangian*, often encountered in the literature as $L = T - V \circ \tau_Q$, where T and V are the kinetic and potential energies of the system respectively. In more general cases a gyroscopic or magnetic term γ^ℓ is considered: $L = T + \gamma^\ell - V \circ \tau_Q$. In lifted local coordinates[2] $(q,\dot{q})$ the lagrangian reads[3]

$$L(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j + \gamma_i(q)\dot{q}^i - V(q), \tag{3}$$

where $g = g_{ij} dq^i \otimes dq^j$ is a riemannian metric and $\gamma = \gamma_i dq^i$ is a 1-form on Q. Before continuing we clarify the terminology and notation we use throughout the thesis: we denote by $L_0 = T - V \circ \tau_Q$ a lagrangian composed by kinetic minus potential energy and say it is a *natural* lagrangian or of natural type. On the other hand if a lagrangian also contains a gyroscopic or magnetic term we denote it by $L = T + \gamma^\ell - V \circ \tau_Q$ and say it is a gyroscopic or mechanical lagrangian or of gyroscopic type. Newton and Euler-Lagrange formulations opened the doors to study mechanical systems in not only a case by case fashion, but also to develop general results and techniques that gives previsions that can be tested by experiments. Contemporaneously the translation of the principle of virtual work from statics into dynamics, carried by d'Alembert, was also developed (our approach essentially relies in both Euler-Lagrange equations and d'Alembert principle). This principle states that in a constrained mechanical system the infinitesimal variations δq of the curve q must satisfy the constraints. Roughly speaking nonholonomic systems are mechanical systems with constraints in the velocities which are not derived from position restraints. On the other hand constraints just depending on the position are called holonomic. Suppose that a mechanical system with lagrangian L as in (3) has $n - r$ nonholonomic linear constraints that we locally write as

$$S_{\alpha j}(q)\dot{q}^j = 0, \quad \alpha = 1,...,n-r. \tag{4}$$

At the same time constraints (4) define $n - r$ 1-forms on Q

$$S_{\alpha j}(q)dq^j, \quad \alpha = 1,...,n-r,$$

whose point-wise kernel defines a constant rank distribution $\mathcal{D}$ on Q called constraint distribution. As a note, constraints (4) are holonomic if and only if there exist $n - r$ smooth functions F_α on Q such that

$$S_{\alpha j} = \frac{\partial F_\alpha}{\partial q^j}, \quad \alpha = 1,...,n-r,$$

[2] Lifted coordinates are in general local coordinates in TQ induced by a local coordinate chart of the manifold Q.

[3] We use the convention of summation over repeated indices.

and then (4) can be reformulated by integration as

$$F_\alpha(q) = 0, \quad \alpha = 1, ..., n - r.$$

Now a curve $t \mapsto q(t)$ on Q is a solution of a nonholonomic problem if and only if satisfies Lagrange-d'Alembert equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_\alpha S_{\alpha i}(q), \quad i = 1, ..., n, \tag{5}$$

where the multipliers λ_α are determined by derivation of the constraint equations (4).

The synthetic expression of equations of motion (5) was worked out by Otto Hölder, but through history its validity has been a theme of confusion and discussion, see e.g. [87, 12, 36]. It is to be remarked that the work of Sergey A. Chaplygin [31] settled down the misunderstandings on the correct form for the nonholonomic equations of motion.

Many of the examples in nonholonomic mechanics are related to rigid bodies, for an ample variety of examples see [93, 92, 87, 91, 40, 12]. Euler describe the motion of a rigid body with just its angular velocity Ω in body representation by his marvelous equation

$$I\dot{\Omega} = I\Omega \times \Omega.$$

Euler also studied rolling bodies without sliding ([50] is perhaps the first scientific study of a nonholonomic system) and spinning rigid bodies (tops, which are the precursors of the gyroscope), these two subjects are in essence the cornerstone of this Thesis.

Rolling bodies without slipping were studied extensively by Edward J. Routh [93] and S. A. Chaplygin [30]. See [87, 19, 95, 12, 64, 63, 41] for some recent references. The rolling with out sliding constraint for rigid bodies require that the contact point between the rigid body and the surface has zero velocity, this condition is geometrically stated as

$$v_m + (v_c \times \omega) = 0,$$

where v_m and v_c are the velocities of the rigid body's center of mass and the contact point respectively and ω is the spatial angular velocity of the body. In Chapter 5 we analyze the particular case of an homogeneous ball rolling without slipping in a rotating surface of revolution.

The gyroscope was invented by Leon Foucault as an alternative[4] way to demonstrate Earth's rotation [60]. This mechanism proved to have several applications in navigation and engineering, to the point that some smartphones have an electronic realization of it. Spinning rigid bodies or spinning tops and the gyroscopic forces arising were studied for instance by Felix Klein [73] and Harold Crabtree [37].

It is known [3, 84, 86] that in unconstrained mechanical systems with cyclic variables the routhian has the form of (3), in rigid body dynamics cyclic variables can be realized by attaching rotors to the system. Rotors, hence a lagrangian with linear terms in the velocities, have proven to be very useful in stabilization and guidance techniques in Control Theory [13, 100, 72]. Inspired by this we present the local ideas of nonholonomic Routh reduction in the next section[5].

Nonholonomic Routh reduction

We present a nonholonomic version of Routh reduction, this serves as a motivation and a generator of physical examples where the lagrangian can have a linear term in the velocities. Consider a nonholonomic system with linear constraints and natural lagrangian on a configuration manifold $\mathcal{M} = Q \times K$, where Q is a n-dimensional manifold and K an abelian Lie group[6]. Let $(q^j, \theta^J) \in Q \times K$ be coordinates on $\mathcal{M}$ and a natural

[4]The first was Foucault's pendulum.

[5]For an in depth historical exposition of nonholonomic systems we refer the reader to [42, 23].

[6]Recall that finite dimensional abelian Lie groups are isomorphic to a product of the form $\mathbb{R}^k \times (\mathbb{S}^1)^r$.

lagrangian $\mathcal{L} : TM \to \mathbb{R}$ such that θ^J are cyclic variables, that is

$$\mathcal{L}(q, \dot{q}, \dot{\theta}) = \frac{1}{2}\mathcal{G}_{ij}(q)\dot{q}^i\dot{q}^j + \mathcal{G}_{iJ}(q)\dot{q}^i\dot{\theta}^J + \frac{1}{2}\mathcal{G}_{IJ}(q)\dot{\theta}^I\dot{\theta}^J - \tilde{V}(q).$$

Additionally suppose that the constraints are independent of θ and $\dot{\theta}$ and are given in terms of q and $\dot{q}$ by (4). That is if the constraints are written as

$$S_{\alpha j}(q,\theta)\dot{q}^j + S_{\alpha(n+J)}(q,\theta)\dot{\theta}^J,$$

then $S_{\alpha(n+J)}(q,\theta) = 0$ and $\frac{\partial S_{\alpha j}}{\partial \theta^J} = 0$, for all $j = 1, \dots, n$, $J = 1, \dots, l$ and $\alpha = 1, \dots, n-r$.

This scenario is realized in a system of rigid bodies, with l fixed rotors, in which the nonholonomic constraints just affect the rigid bodies, for examples see [22, 20].

Then the Lagrange-d'Alembert equations of the system are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^j}\right) - \frac{\partial \mathcal{L}}{\partial q^j} &= \lambda_\alpha S_{\alpha j}(q), \qquad j = 1, \dots, n, \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}^J}\right) &= 0, \qquad J = 1, \dots, l, \end{aligned} \tag{6}$$

where the multipliers λ_α are uniquely determined by the constraints (4). Then the momenta associated to the cyclic variables

$$p_J = \frac{\partial \mathcal{L}}{\partial \dot{\theta}^J} = \mathcal{G}_{iJ}(q)\dot{q}^i + \mathcal{G}_{IJ}(q)\dot{\theta}^I, \qquad J = 1, \dots, l,$$

are clearly first integrals. Let us consider the restriction of the system (6) to a level set of these integrals by setting $p_J = \mu_J$ for some fixed $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{R}^l$. Along such level set we may express[7]

$$\dot{\theta}^J = \mathcal{G}^{IJ}(q)(\mu_I - \mathcal{G}_{iI}(q)\dot{q}^i), \qquad J = 1, \dots, l, \tag{7}$$

and we may therefore eliminate $\dot{\theta}$ from the first set of equations in (6) to obtain a reduced system involving q and $\dot{q}$ only. As is well known (see e.g. [83]), such elimination is conveniently done in terms of the classical Routhian function $R^\mu = R^\mu(q, \dot{q})$ defined by

$$R^\mu(q, \dot{q}) := \left[\mathcal{L}(q, \dot{q}, \dot{\theta}) - \mu_J\dot{\theta}^J\right]_{p_J = \mu_J},$$

with the convention that in the right hand side $\dot{\theta}$ is written in terms of $(q, \dot{q})$ as in (7). One remarkable property of the Routhian is

$$\frac{d}{dt}\left(\frac{\partial R^\mu}{\partial \dot{q}^j}\right) - \frac{\partial R^\mu}{\partial q^j} = \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^j}\right) - \frac{\partial \mathcal{L}}{\partial q^j}\right]_{p_J = \mu_J}, \qquad j = 1, \dots, l,$$

where we again think of $\dot{\theta} = \dot{\theta}(q, \dot{q})$ in the right hand side, so the dynamics are determined by the routhian and l parameters. Therefore, if we write $L(q, \dot{q}) := R^\mu(q, \dot{q})$, the reduced system can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = \lambda_\alpha S_{\alpha j}(q), \qquad j = 1, \dots, n, \tag{8}$$

together with the nonholonomic constraints (4).

Now note that (8) can be interpreted as the equations of motion of a nonholonomic system on the configuration manifold Q, with constraints given by (4) and with lagrangian

[7] As usual, we denote by $\mathcal{G}^{IJ}$ the entries of the inverse matrix of the block $\mathcal{G}_{IJ}$.

$L = R^\mu : TQ \to \mathbb{R}$. As in the unconstrained case the function L has a linear term in the velocities. Indeed (see e.g. [84]), $L = R^\mu$ is given by (3) with

$$g_{ij} = \mathcal{G}_{ij} - \mathcal{G}_{Ii}\mathcal{G}^{IJ}\mathcal{G}_{Jj}, \qquad \gamma_j = \mathcal{G}_{Ij}\mathcal{G}^{IJ}\mu_J, \qquad V = \tilde{V} + \frac{1}{2}\mathcal{G}^{IJ}\mu_I\mu_J. \tag{9}$$

In Appendix A we present an intrinsic construction of the nonholonomic Routh reduction to show that it is a global procedure, where we give a geometric explanation of equalities (9) and an interpretation of the condition that the gyroscopic 1-form $\gamma = \mathcal{G}_{Ij}\mathcal{G}^{IJ}\mu_J\, dq^j$ is not necessarily closed.

Main Contributions

Almost-Poisson formulation

Dirac structures have proved to be the general framework for constrained mechanical systems [97, 71, 67]. Using such formalism in principle we can construct an *almost-Poisson* structure on nonholonomic systems regardless of the kind of constraints and the type of lagrangian function. The almost-Poisson structure for nonholonomic systems with linear constraints and natural lagrangian (kinetic minus potential energy) is well documented [96, 28, 44, 12]. In this Thesis we focus on the case of a linear constrained nonholonomic system with a gyroscopic lagrangian. This type of systems have not attracted very much attention and the literature is not abundant. In [76, 74] some examples with gyroscope or rotors are considered, and in [20] the authors give an almost-Poisson structure for some classical examples with a gyroscope (in this research direction se also [48]).

In Chapter 3 we present an intrinsic construction of an almost-Poisson bracket for nonholonomic systems on a configuration manifold Q with linear constraints and gyroscopic lagrangian. This Chapter is based on a project in collaboration with J. C. Marrero, D. Martín de Diego and L. García-Naranjo. The main contribution we add to the existing results in the field is the construction of an almost-Poisson bracket for nonholonomic systems with linear constraints and mechanical lagrangian, for the formulation we use elements given by the kinetic energy metric instead of more sophisticated elements coming from symplectic or Dirac geometry. This material is developed in Section 3.3. The construction of an almost-Poisson bracket can also be important for hamiltonization [65, 16, 6, 5, 64] of this type of nonholonomic systems. As noted before the appearance of such almost-Poisson brackets in the literature exists just for certain examples but to our knowledge not in a general setting. Our construction is made in the dual bundle $\mathcal{D}^*$ instead of the image of $\mathcal{D}$ (recall $\mathcal{D}$ is the constraint distribution) under the legendre transform of the lagrangian L. This choice is made because the former bundle is a vector bundle and the latter is an affine subbundle of T^*Q, in this way $\mathcal{D}^*$ serves as the phase space for the almost hamiltonian description of the equations of motion, nevertheless the affine nature translates into the almost-Poisson bracket. We prove that the bracket is linear if and only if the 1-form γ (related to the gyroscopic term of the lagrangian) is closed, i.e. $d\gamma = 0$. We also prove that Jacobi identity is satisfied if and only if the constraints are holonomic, or equivalently if the distribution $\mathcal{D}$ is integrable.

We also consider the scenario where symmetries are present. We perform reduction of the bracket to the quotient space $\mathcal{D}^*/G$ via standard construction [78, 84, 38], the reduced bracket turns out to be again of affine nature and linear if and only if $d\gamma(X, Y) = 0$, for all equivariant vector fields X, Y on Q.

Role of symmetry in nonholonomic systems

In unconstrained mechanical systems as wel as in nonholonomic mechanics symmetries have proven to be fundamental to decrease the degrees of freedom of the system: influential works on this direction are [77, 14, 15]. A general reduction process by means of symmetries is treated in [11, 81, 27, 29], when the systems have a Poisson structure [78], and in the special case the configuration space is a Lie group [66]. In holonomic systems the existence of a symmetry guarantees the presence of first integrals (Noether's Theorem), however in nonholonomic systems this is not directly the case: if the symmetry are related with the constraints then a nonholonomic Noether Theorem can be stated [94, 82, 54]. A possible generalization is given by gauge-symmetries which may generate conserved quantities [10, 53, 58, 7, 64, 9, 8] (see in particular [8] for a recent detailed study of this aspect). In the particular case of nonholonomic systems with affine constraints the energy is in general no longer conserved, but in case has a Noether symmetry is present then the so-called *Moving energy* is a first integral of the system [56, 57].

In Chapter 3 we also present a way to construct a system with gyroscopic lagrangian from a natural lagrangian, this construction is somehow artificial but in some cases is physically meaningful [57]. The formulation goes as follows: suppose we are given a nonholonomic system with linear constraints as in (4) and natural[8] lagrangian L_0, a vector field N on Q and a real parameter $\nu \in \mathbb{R}$. We define a mechanical[9] lagrangian L_ν, defined as $L_\nu(v_q) = L_0(v_q + N_q)$, $v_q \in T_qQ$. Our contributions are presented in Section 3.5 and are the following: assume that the momentum p_Z^0 generated by a vector field Z on Q is a first integral for the system with lagrangian L_0, then under certain conditions we *extend* Z to a vector field Z_ν on Q such that its associated momentum [10] $p_{Z_\nu}^\nu$ is a first integral for the system with lagrangian L_ν and $p_{Z_\nu}^0 = p_Z^0$.

Dynamical behavior of a ball rolling without slipping in a rotating surface of revolution

The nonholonomic system of a homogeneous ball rolling without slipping on a rotating surface of revolution is well known and has been studied by several authors [93, 30, 19, 57, 41]. The case in which the surface is a plane [31, 17] is by far the most studied, other particular cases such as the cylinder [56], cone [21] and paraboloid [20, 41] are studied as well. This family of nonholonomic systems are known to have a $SO(3)\times SO(2)$ symmetry group which can be used to perform reduction. Additionally these systems posses three functionally independent first integrals, the Routh integrals [41] and the moving energy [57]. The authors in [41] study the qualitative dynamics for a generic superquadratic surface's profile f (a function $f : \mathbb{R} \to \mathbb{R}$ is said to be superquadratic if satisfies the limit $\lim_{r\to\infty} \frac{f(r)}{r^2} = \infty$). In [57] using Routh integrals and the so-called moving energy the authors prove that all motions are bounded independently of the surface rotation.

By means of the theory developed in Chapters 3 and 4 we prove that these type of systems admit two gauge momenta affine in the velocitites which are $SO(3) \times SO(2)$-invariant first integrals functionally dependent with Routh integrals, the nature of Routh integrals was missing in the literature. Furthermore we analyze the nature of gauge sections generating such first integrals and we see that in general they (its tangent lift) are not symmetries of the lagrangian.

In Section 5.6 we focus on the particular case where the surface of revolution is an inverted cone. We analyze the reduced dynamics and we see that the rotation of the surface has a stabilizing effect, in the sense that if the surface is rotating then all motions are bounded. We furthermore analyze the reduced system restricted to the level sets of the Routh integrals, which is proven to be a lagrangian system of dimension 2 [41] with the moving energy restricted to the level sets as the energy of the system. We

[8] Kinetic minus potential energy.

[9] Kinetic minus potential energy plus a gyroscopic term.

[10] A vector field Z generates the momentum $p_Z^\nu := \langle \frac{\partial L_\nu}{\partial \dot q}, Z\rangle$, for the nonholonomic system with lagrangian L_ν.

qualitatively characterize the reduced equilibrium and in particular we prove that there exist asymptotic and quasiperiodic motions in both when the surface is still and rotating.

Structure of the Thesis

The Thesis is divided into three parts, two theoretical and one dynamical. The first part is composed by Chapters 1 and 2, in which the background material is introduced. In Chapter 1 we expose the general theory of nonholonomic systems with symmetries, with constraints affine in the velocities and gyroscopic lagrangians. We derive the equations of motion in both lifted bundle coordinates and quasivelocities. Chapter 2 compresses the theory for linear constrained with natural lagrangian systems and serves as a test case. Particular descriptions of what is done in Chapter 1 are given, the construction of an almost-Poisson structure, and the pertinent reduction process in the case of symmetries are presented.

The second part is composed by Chapters 3 and 4. Chapter 3 starts with the generalization of the almost-Poisson structure presented in Chapter 2 to nonholonomic systems with gyroscopic lagrangians, and it ends with the characterization and existence conditions of first integrals affine in the velocities which are gauge momentum, Propositions 3.11, 3.12 and 3.13 and Theorem 3.14. Chapter 4 is devoted to the relations between affine constrained systems with natural lagrangian possessing a Noether symmetry and linear constrained systems with gyroscopic lagrangian. These relations are constructed in terms of the equations of motion and on the gauge momentum perspective, the main result is Theorem 4.4.

The third part is Chapter 5, where a systematic dynamical analysis of the system formed by a heavy homogeneous ball rolling without slipping in a rotating surface of revolution is given, existence and nature of three functionally independent first integrals is derived, Theorem 5.4. In the particular case in which the surface is an inverted cone, analysis and classification of the relative equilibria and reduced motions are also carried out, Propositions 5.8 and 5.10.

There are two Appendices A and B. In Appendix A we present an intrinsic construction of the (abelian) nonholonomic Routh reduction, we opted to include this material since it is a procedure which in principle yields several possible examples of the kind treated in Chapter 3. In Appendix B we include a brief introduction of Poisson manifold, emphasizing the properties of the canonical Poisson bracket on the cotangent bundle of a smooth manifold.

Chapter 1

Background of nonholonomic systems

This Chapter is intended to present an introduction to the study of nonholonomic systems. In this work a nonholonomic system is determined by three objects: the configuration space, the lagrangian and the constraints. Along this chapter we consider the most general scenario that is a lagrangian of *gyroscopic* type, namely given by kinetic minus potential energy plus a generalized gyroscopic term, and the constraints are *affine*, that is linear non-homogeneous, in the velocities.

To compute the equations of motion of a nonholonomic system we use Lagrange-d'Alembert approach (with reaction forces), and its derivation is done locally. We follow the exposition of [87, 15, 51]

1.1 Geometry of nonholonomic systems

A nonholonomic system is determined by a triple $(Q, L, \mathcal{M})$, where Q is a n-dimensional smooth manifold called *configuration space*, L is the *lagrangian*, which is a smooth function $L : TQ \to \mathbb{R}$ on the state space played by the tangent bundle of the configuration space. We consider L to be of a gyroscopic lagrangian, meaning that

$$L = T + \gamma^\ell - V \circ \tau_Q \tag{1.1}$$

where $T : TQ \to \mathbb{R}$ is the fiberwise quadratic form associated to a riemannian metric g on Q, g is called the *kinetic energy metric*, $\gamma^\ell : TQ \to \mathbb{R}$ a smooth function linear in the velocities, associated to a differential 1-form γ on Q, by the relation $\gamma^\ell(v_q) = \langle \gamma(q), v_q \rangle$ for all $v_q \in T_qQ$ and $q \in Q$ and $V : Q \to \mathbb{R}$ a smooth function. From a physical perspective, T corresponds to the kinetic energy, γ^ℓ to a generalized gyroscopic energy and V to the potential energy of the system. Finally, $\mathcal{M}$ is the *constraint manifold*: an affine subbundle of the tangent bundle TQ modeled over a vector subbundle $\mathcal{D}$, with the property that $\mathcal{D}$ is not the tangent bundle of any submanifold of Q, that is $\mathcal{D}$ is not an integrable distribution. In other words, every fiber $\mathcal{M}_q$ of $\mathcal{M}$ is an affine subspace of T_qQ and $\mathcal{D}_q$ is the vector subspace of T_qQ that models $\mathcal{M}_q$. This means that locally, there exists a vector field $Y \in \mathfrak{X}(Q)$ on Q such that

$$\mathcal{M}_q = \{v_q \in T_qQ \,:\, v_q = Y_q + w_q,\, w_q \in \mathcal{D}_q\} \tag{1.2}$$

for all $q \in Q$. The affine distribution $\mathcal{M}$ has *rank* r at $q \in Q$ if $\mathcal{D}$ has *rank* r at $q \in Q$. We often write $\mathcal{M} = Y + \mathcal{D}$ or $\mathcal{M}_q = Y_q + \mathcal{D}_q$, to formally highlight the affine structure of $\mathcal{M}$. The subbundles $\mathcal{M}$ and $\mathcal{D}$ are at the same time submanifolds of the tangent bundle TQ, in the rest of the work we make no distinction in notation between both representations. As a submanifold $\mathcal{M}$ is described by $n-r$ functionally independent and affine in the velocities smooth functions $f_1,...,f_{n-r}$ on TQ, called the *constraint functions*, such that

$$\mathcal{M} = \bigcap_{\alpha=1}^{n-r} f_\alpha^{-1}(0).$$

Let $(q,\dot{q})$ be lifted bundle coordinates in TQ, since the functions $f_1,...,f_{n-r}$ are affine in the velocities we can construct the 1-forms

$$\chi^\alpha = \frac{\partial f_\alpha}{\partial \dot{q}^i} dq^i, \quad \alpha = 1,...,n-r,$$

and therefore we have relations $f_\alpha(q,\dot{q}) = (\chi^\alpha)^\ell(q,\dot{q}) + f_\alpha(q,0)$, $\alpha = 1,...,n-r$. Equivalently, the subbundle $\mathcal{M}$ can be thought as a regular affine distribution modeled over a regular non-integrable distribution[1] $\mathcal{D}$ where

$$\mathcal{D}_q = \bigcap_{\alpha=1}^{n-r} \ker \chi_q^\alpha.$$

One of the thesis' objectives is to give a hamiltonian-like description of a nonholonomic system $(Q,L,\mathcal{M})$, to this end we need to introduce objects to relate the tangent and cotangent bundles of Q and as well for the vector bundle $\mathcal{D}$ and its dual $\mathcal{D}^*$. This theory is mainly used in Chapters 2 and 3, but it helps to remark certain properties of $(Q,L,\mathcal{M})$.

Consider the orthogonal decomposition

$$TQ = \mathcal{D} \oplus \mathcal{D}^\perp, \tag{1.3}$$

induced by the kinetic energy metric g, where $\mathcal{D}^\perp \subset TQ$ is the subbundle whose fibers are the orthogonal complement of $\mathcal{D}_q$ in T_qQ. In symbols

$$\mathcal{D}_q^\perp = \{v_q \in T_qQ \,:\, \langle v_q, w_q\rangle_g = 0, \, \forall w_q \in \mathcal{D}_q\}.$$

For a nice expression in coordinates, let $\{X_i\}_{i=1}^n$ be an adapted local frame of $\mathfrak{X}(Q)$ associated to the decomposition (1.3), with $X_a \in \Gamma(\mathcal{D})$ and $X_\alpha \in \Gamma(\mathcal{D}^\perp)$, $a = 1,...,r$, $\alpha = r+1,...,n$. For all $v_q \in T_qQ$ we have $v_q = v^i X_i(q)$, such frame induces local coordinates (q,v) on TQ, note that (q^i, v^a) and (q^i, v^α) are local coordinates for the vector bundles $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively, we use this coordinates in the following. Associated to the submanifolds $\mathcal{D}$ and $\mathcal{D}^\perp$ of TQ we have the inclusions

$$i_\mathcal{D} : \mathcal{D} \hookrightarrow TQ, \quad i_{\mathcal{D}^\perp} : \mathcal{D}^\perp \hookrightarrow TQ. \tag{1.4}$$

In local coordinates (q^i, v^a) and (q^i, v^α), of $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively, the inclusions (1.4) read

$$i_\mathcal{D}(q, v^a) = (q, v^a, 0), \qquad i_{\mathcal{D}^\perp}(q, v^\alpha) = (q, 0, v^\alpha).$$

The decomposition (1.3) of TQ also induces the projectors of TQ onto $\mathcal{D}$ and $\mathcal{D}^\perp$

$$P_\mathcal{D} : TQ \to \mathcal{D}, \quad P_{\mathcal{D}^\perp} : TQ \to \mathcal{D}^\perp, \tag{1.5}$$

related to the inclusions (1.4) by $P_\mathcal{D} \circ i_\mathcal{D} = Id_\mathcal{D}$ and $P_{\mathcal{D}^\perp} \circ i_{\mathcal{D}^\perp} = Id_{\mathcal{D}^\perp}$. In (q^i, v^a) and (q^i, v^α) coordinates

$$P_\mathcal{D}(q, v^a, v^\alpha) = (q, v^a), \qquad P_{\mathcal{D}^\perp}(q, v^a, v^\alpha) = (q, v^\alpha).$$

[1]The language of distributions helps in the case the distribution $\mathcal{D}$ is not regular, see for example [35].

Since the inclusions (1.4) and projectors (1.5) are bundle morphisms we can consider their dual, see [80]

$$\begin{aligned} i_{\mathcal{D}}^* : T^*Q \to \mathcal{D}^*, \quad i_{\mathcal{D}^\perp}^* : T^*Q \to (\mathcal{D}^\perp)^*, \\ P_{\mathcal{D}}^* : \mathcal{D}^* \hookrightarrow T^*Q, \quad P_{\mathcal{D}^\perp}^* : (\mathcal{D}^\perp)^* \hookrightarrow T^*Q, \end{aligned} \tag{1.6}$$

where $\mathcal{D}^*$ and $\left(\mathcal{D}^\perp\right)^*$ are the dual vector bundles to $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively.

Observation 1.1.1. The mentioned inclusions are monomorphisms and the projectors are epimorphisms so their dual are epimorphisms and monomorphisms respectively. As a consequence $\mathcal{D}^*$ is not only an abstract vector bundle on Q, but it is a subbundle of T^*Q.

A direct implication of the tangent bundle splitting (1.3) is the following presentation of an affine subbundle of TQ.

Proposition 1.1. *Let (Q, g) be a riemmanian manifold and $\mathcal{M}$ an affine subbundle of TQ then, locally, there exists a unique $\xi \in \Gamma(\mathcal{D}^\perp)$ such that $\mathcal{M} = \xi + \mathcal{D}$. Moreover the vector field ξ is the orthogonal projection of any $Y \in \Gamma(\mathcal{M})$ onto $\mathcal{D}^\perp$, i.e. $\xi = i_{\mathcal{D}^\perp}\left(P_{\mathcal{D}^\perp}(Y)\right)$.*

Proof. We know that locally an affine bundle $\mathcal{M}$ can be written as $\mathcal{M} = Y + \mathcal{D}$ and a point $v_q \in T_qQ$ is in $\mathcal{M}_q$ iff $Y_q - v_q \in \mathcal{D}_q$.

Using equations (1.4) and (1.5) we can write $Y = i_{\mathcal{D}^\perp}\left(P_{\mathcal{D}^\perp}(Y)\right) + i_{\mathcal{D}}\left(P_{\mathcal{D}}(Y)\right)$, then define $\xi = i_{\mathcal{D}^\perp}\left(P_{\mathcal{D}^\perp}(Y)\right)$, by construction $Y_q - \xi_q \in \mathcal{D}_q$ therefore $\mathcal{M}_q = Y_q + \mathcal{D}_q = \xi_q + \mathcal{D}_q$ for all points q in Q.

□

We consider three ways to relate the tangent and cotangent bundle of Q, one is given by the riemannian metric g, other by the Legendre transformation induced by the lagrangian L and the third is given by a choice of a local frame of $\mathfrak{X}(Q)$ and considering its dual frame on $\Omega^1(Q)$. First, the kinetic energy metric induces the bundle isomorphism

$$\flat_g : TQ \to T^*Q,$$

defined by $\langle \flat_g(v_q), w_q\rangle := \langle v_q, w_q\rangle_g$ for all $v_q, w_q \in T_qQ$ and $q \in Q$. Using the isomorphism $\flat_g$ the lagrangian L can be rewritten as $L = T + \flat_g(N)^\ell - V$, where N is the unique vector field on Q such that $\gamma = \flat_g(N)$. Moreover, consider the[2] *annihilator* subbundles $\mathcal{D}^\circ$ and $(\mathcal{D}^\perp)^\circ$ of the cotangent bundle T^*Q, whose fibers are

$$\begin{aligned} \mathcal{D}_q^\circ &= \left\{\beta_q \in T_q^*Q \,:\, \langle \beta_q, v_q\rangle = 0,\ \forall v_q \in \mathcal{D}_q\right\}, \\ (\mathcal{D}^\perp)_q^\circ &= \left\{\beta_q \in T_q^*Q \,:\, \langle \beta_q, v_q\rangle = 0,\ \forall v_q \in \mathcal{D}_q^\perp\right\}, \end{aligned} \tag{1.7}$$

we then get $\mathcal{D}_q^* \cong (\mathcal{D}^\perp)_q^\circ$ and $\mathcal{D}_q^\circ \cong (\mathcal{D}^\perp)_q^*$. Then we have the following vector bundle isomorphisms

$$\mathcal{D}^* \cong (\mathcal{D}^\perp)^\circ, \quad (\mathcal{D}^\perp)^* \cong \mathcal{D}^\circ. \tag{1.8}$$

The second way of relating the tangent and cotangent bundle of Q is by using the fiber derivative of L, see [46, 84],

Definition 1.1. The fiber derivative of $L \in C^\infty(TQ)$ is the bundle transformation, over the identity in Q, $\mathbb{F}L : TQ \to T^*Q$ defined by

$$\langle \mathbb{F}L(v_q), w_q\rangle = \left.\frac{d}{dt}\right|_{t=0} L(v_q + tw_q), \quad v_q, w_q \in T_qQ.$$

[2]This representation of the 1-form α is useful in some context such as in Proposition 1.10 or in Section 4.2.

In local bundle coordinates $(q, \dot{q})$

$$\langle \mathbb{F}L(q,v), w\rangle = \langle \frac{\partial L}{\partial \dot{q}}(q,v), w\rangle, \qquad v, w \in T_qQ.$$

In mechanics literature [1, 4, 84, 95] $\frac{\partial L}{\partial \dot{q}^i}$ is commonly referred as the *ith-generalized momenta* and is denoted as p_i, $i = 1, .., n$.

Observation 1.1.2. $\mathbb{F}L$ is a bundle isomorphism if and only if $Hess(L|_{T_qQ})$ is non singular for all $q \in Q$. If the lagrangian is of mechanical type, natural or gyroscopic, this hypothesis is always satisfied.

In our case, that is when L is a gyroscopic lagrangian, we have a more explicit expression for $\mathbb{F}L$

$$\mathbb{F}L(v_q) = \flat_g(v_q) + \gamma_q, \quad \forall v_q \in T_qQ. \tag{1.9}$$

Clearly $\mathbb{F}L$ is an affine bundle transformation if and only if the 1-form γ is non zero.

The third option is of local nature, let $\{X_i\}_{i=1}^n$ be a local frame of Q then we can consider its *dual frame* [80] $\{\chi^i\}_{i=1}^n$ defined as[3]

$$\langle \chi^i(q), X_j(q)\rangle = \delta_{ij}, \quad i, j = 1, ..., n.$$

This assignation induces a local bundle diffeomorphism between TQ and T^*Q, it is given fiberwise by $v^i X_i(q) \mapsto p_i \chi^i(q)$, where $p_i = v^i$, and we can obtain a coordinate description of (1.6):

$$i_{\mathcal{D}^*}(q, p_a, p_\alpha) = (q, p_a), \qquad i_{(\mathcal{D}^\perp)^*}(q, p_a, p_\alpha) = (q, p_\alpha).$$
$$P_{\mathcal{D}^*}(q, p_a) = (q, p_a, 0), \qquad P_{(\mathcal{D}^\perp)^*}(q, p_\alpha) = (q, 0, p_\alpha).$$

1.2 Local form of the equations of motion

To compute the equations of motion of a nonholonomic system $(Q, L, \mathcal{M})$ we follow the treatment of [51]. We assume the validity of d'Alembert principle of ideal constraints: the reaction forces that the constraints can exert annihilate the *virtual displacements*, that are infinitesimal variations of curves satisfying the constraints. Consider a smooth curve $q : I \to Q$, where $I \subset \mathbb{R}$ is an open interval, take $q_0 \in Q$ and $s_0 \in I$, such that $q(s_0) = q_0$ and $\dot{q}(s) \in \mathcal{M}_{\gamma(s)}$, then Lagrange-d'Alembert principle ensures that $\delta q(s) \in D_{q(s)}$, where δq is a smooth infinitesimal displacement of the variation of the curve q (see (1)), and the reaction force is then a function $R : TQ \to \mathcal{D}^\circ$, trivially $\langle R, \delta q\rangle = 0$ [84, 36].

We perform the calculation of the equations of motion of the nonholonomic system $(Q, L, \mathcal{M})$ in lifted bundle coordinates $(q, \dot{q})$, where $q = (q^1, ..., q^n)$ and $\dot{q} = (\dot{q}^1, ..., \dot{q}^n)$. The equations of motion have the familiar expression of Lagrange-d'Alembert equations [2]

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\bigg|_{\mathcal{M}} = R|_{\mathcal{M}}, \tag{1.10}$$

where the reaction force $R : TQ \to \mathcal{D}^\circ$ is a smooth function on the state space. Roughly speaking the reaction force arises from the constraints and d'Alembert principle and is not an external force to the system. Our approach to obtain the nonholonomic equations of motion is local, so we need to give coordinate expressions for the lagrangian and constraints. The lagrangian L writes as

$$L(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q} + \gamma(q) \cdot \dot{q} - V(q), \tag{1.11}$$

[3] Where $\delta_{ij} = 0$ iff $i \neq j$ and $\delta_{ij} = 1$ iff $i = j$.

where $A(q)$ is a non singular, symmetric and positive definite matrix that represents the riemannian metric g, i.e. $A_{ij} = \langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \rangle_g$. $\gamma(q) \in \mathbb{R}^n$ is the coordinate representation of the 1-form $\gamma \in \Omega^1(Q)$ and $\cdot$ represents the classic dot product in $\mathbb{R}^n$. The constraint manifold $\mathcal{M} = Y + \mathcal{D}$ is a rank r affine subbundle of TQ each fiber $\mathcal{M}_q$ can be defined as the kernel of an affine, linear non-homogeneous, system of equations that we write as

$$S(q)\dot{q} + s(q) = 0, \tag{1.12}$$

with $S(q)$ a full rank $(n-r) \times n$ matrix and $s(q) \in \mathbb{R}^{n-r}$. Since $Y_q \in \mathcal{M}_q$ we get $s(q) = -S(q)Y_q$, and

$$\begin{aligned} \mathcal{D}_q &= \{\dot{q} \in T_qQ \,|\, S(q)\dot{q} = 0\}\,, \\ \mathcal{M}_q &= \{\dot{q} \in T_qQ \,|\, S(q)(\dot{q} - Y_q) = 0\}\,. \end{aligned}$$

By d'Alembert principle the reaction force R is a linear combination of the rows of S, then there exist $n-r$ functions $\lambda = (\lambda_1, ..., \lambda_{n-r})$ of $(q, \dot{q})$ called the *Lagrange multipliers*, such that $R = S^T\lambda$. All the ingredients are set, we just need to determine λ, rewrite (1.10) using (1.11) to obtain a coordinate expression

$$A(q)\ddot{q} + \eta(q, \dot{q}) = S^T\lambda \tag{1.13}$$

where $\eta_i(q, \dot{q}) = \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j - \frac{\partial L}{\partial q^i}$, $i = 1, ..., n$.

Observation 1.2.1. Consider the vector function $\eta = (\eta_1, ..., \eta_n)$, using the local form of L (1.11), the contribution of the 1-form γ on the equations of motion is the function $((D\gamma)^T - D\gamma)\dot{q}$ which is precisely the coordinate expression for $i_{\dot{q}}d\gamma$, where the matrix $(D\gamma)_{ij} = \frac{\partial \gamma_i}{\partial q^j}$. So if $d\gamma = 0$ then the gyroscopic term does not play a role in the motions of the system.

Now we derive the constraint equation (1.12) along a curve and get

$$S(q)\ddot{q} + \sigma(q, \dot{q}) = 0, \tag{1.14}$$

with $\sigma_l(q, \dot{q}) = \frac{\partial S_{lj}}{\partial q^k}\dot{q}^j\dot{q}^k + \frac{\partial s_l}{\partial q^k}\dot{q}^k$, $l = 1, ..., n-r$. Using relations (1.13) and (1.14) we get

$$SA^{-1}S^T\lambda - SA^{-1}\eta = -\sigma.$$

Now the matrix $SA^{-1}S^T$ is invertible since S has rank $n-r$ then

$$\lambda = (SA^{-1}S^T)^{-1}(SA^{-1}\eta - \sigma), \tag{1.15}$$

and the reaction force in coordinates is therefore given by

$$R = S^T(SA^{-1}S^T)^{-1}(SA^{-1}\eta - \sigma). \tag{1.16}$$

Observation 1.2.2. From equation (1.16) note that the reaction force R restricted to $\mathcal{M}_q$ is not in general a linear function in the velocities.

Then the Lagrange-d'Alembert equations write

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\Bigg|_{\mathcal{M}} = S^T(SA^{-1}S^T)^{-1}(SA^{-1}\eta - \sigma)|_{\mathcal{M}}. \tag{1.17}$$

This system of equations define a vector field $X_{nh} \in \mathfrak{X}(\mathcal{M})$ on $\mathcal{M}$ which determine the dynamics of the nonholonomic system $(Q, L, \mathcal{M})$, see [2] for the linear and [56] for the affine constrained cases.

Observation 1.2.3. The function λ depends on the choice of S, but this choice ensures that each of the constraints equations (1.12) is a first integral of the system. This implies that the equations of motion are well defined on its level sets, which their intersection is the submanifold $\mathcal{M}$.

Another important topic in mechanical systems is energy conservation, in Proposition 1.3 we give conditions on when the energy is conserved, we use the fiber derivative $\mathbb{F}L$ to define the lagrangian energy

Definition 1.2. Consider a lagrangian L on a manifold Q. The lagrangian energy $E_L : TQ \to \mathbb{R}$ is the smooth function defined by

$$E_L(v_q) = \langle \mathbb{F}L(v_q), v_q\rangle - L(v_q), \quad \forall v_q \in T_qQ.$$

In bundle coordinates $(q, \dot{q})$ of TQ the lagrangian energy is represented as

$$\begin{aligned} E_L(q,\dot{q}) &= \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L(q,\dot{q}) \\ &= \frac{1}{2}\dot{q} \cdot A(q)\dot{q} + V(q), \end{aligned} \tag{1.18}$$

note that the gyroscopic part does not play any role in the lagrangian energy.

We derived the nonholonomic equations of motion locally,we now prove that this is a globally defined vector field on $\mathcal{M}$. We do it in coordinates, nevertheless the non-holonomic vector field has intrinsic nature, some references for this construction are [46, 43, 36].

Proposition 1.2. *The form of equations of motion (1.17) is invariant under change of coordinates. Equivalently, let (U, q) and $(\tilde{U}, \tilde{q})$ be two coordinate charts in Q such that $U = \phi(\tilde{U})$ and $q = \phi(\tilde{q})$, then $q(t) = \phi(\tilde{q}(t))$ is an integral curve if and only if $\tilde{q}(t)$ is.*

Proof. First note that if $q = \phi(\tilde{q})$ then $\dot{q} = T\phi\dot{\tilde{q}}$, where $T\phi$, in vector notation, is the jacobian matrix of ϕ, additionally we denote $\phi(\tilde{q}) = (\phi_1(\tilde{q}), ..., \phi_n(\tilde{q}))$. We define the lagrangian and constraints in coordinates $\tilde{q} = (\tilde{q}^1, ..., \tilde{q}^n)$ as

$$\begin{aligned} &\tilde{L}(\tilde{q}, \dot{\tilde{q}}) = L(\phi(\tilde{q}), T\phi\dot{\tilde{q}}), \\ &\tilde{S}(\tilde{q})\dot{\tilde{q}} + \tilde{s}(\tilde{q}) = S(\phi(\tilde{q}))T\phi(\dot{\tilde{q}}) + s(\phi(\tilde{q})), \end{aligned}$$

then in vector notation

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}} \cdot \tilde{A}(\tilde{q})\dot{\tilde{q}} + \tilde{\gamma}(\tilde{q}) \cdot \dot{\tilde{q}} - \tilde{V}(\tilde{q}),$$

where the following relations hold

$$\begin{aligned} &\tilde{A} = T\phi^T(A \circ \phi)T\phi, \quad \tilde{\gamma} = T\phi^T(\gamma \circ \phi), \quad \tilde{V} = V \circ \phi. \\ &\tilde{S} = (S \circ \phi)T\phi, \quad \tilde{s} = s \circ \phi. \end{aligned}$$

Now we relate the functions $\tilde{\eta}$ and $\tilde{\sigma}$ with η and σ respectively. Recall from lagrangian mechanics the following relation on change of coordinates of Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}} - \frac{\partial \tilde{L}}{\partial \tilde{q}} = T\phi^T\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) \circ \phi$$

A straightforward but cumbersome computation using the chain rule shows

$$\begin{aligned} \tilde{\sigma}_l(\tilde{q}, \dot{\tilde{q}}) &= \frac{\partial \tilde{S}_{lj}}{\partial \tilde{q}^k}\dot{\tilde{q}}^j\dot{\tilde{q}}^k + \frac{\partial \tilde{s}_l}{\partial \tilde{q}^k}\dot{\tilde{q}}^k \\ &= \sigma(\phi(\tilde{q}), T\phi\dot{\tilde{q}}) + S_{li}(\phi(\tilde{q}))\frac{\partial^2 \phi_i}{\partial \tilde{q}^j \partial \tilde{q}^k}\dot{\tilde{q}}^j\dot{\tilde{q}}^k, \end{aligned}$$

where $l = 1, ..., n - r$. And

$$\tilde{\eta}_i(\tilde{q}, \dot{\tilde{q}}) = \frac{\partial^2 \tilde{L}}{\partial \tilde{q}^j \partial \dot{\tilde{q}}^i} \dot{\tilde{q}}^j - \frac{\partial \tilde{L}}{\partial \tilde{q}^i}$$
$$= (T\phi^T)_{ij} \eta_j(\phi(\tilde{q}), T\phi\dot{\tilde{q}}) + (T\phi^T)_{ij} A(\phi(\tilde{q}))_{jk} \frac{\partial^2 \phi_k}{\partial \tilde{q}^m \partial \tilde{q}^n} \dot{\tilde{q}}^m \dot{\tilde{q}}^n,$$

where the index $i = 1, ..., n$. Therefore we derive the equality $\tilde{S}\tilde{A}^{-1}\tilde{\eta} - \tilde{\sigma} = SA^{-1}\eta - \tilde{\sigma}$, and the reaction forces in both coordinate systems are related by

$$\tilde{R} = \tilde{S}^T \left[(\tilde{S}\tilde{A}^{-1}\tilde{S}^T)^{-1} (\tilde{S}\tilde{A}^{-1}\tilde{\eta} - \tilde{\sigma}) \right]$$
$$= T\phi^T S^T (SA^{-1}S^T)^{-1} (SA^{-1}\eta - \sigma) \circ \phi$$
$$= T\phi^T (R \circ \phi)$$

□

Remark 1.2.1. A consequence of the above proposition is that if we have two nonholonomic systems $(Q, L, \mathcal{M})$ and $(\tilde{Q}, \tilde{L}, \tilde{\mathcal{M}})$ and a diffeomorphism $\phi : \tilde{Q} \to Q$ such that $(T\phi)^*(L) = \tilde{L}$ and $T\phi(\tilde{\mathcal{M}}) = \mathcal{M} \circ \phi$, then the nonholonomic vector fields X^Q_{nh} and $X^{\tilde{Q}}_{nh}$ are $T\phi|_{\tilde{\mathcal{M}}}$-related.

1.3 Reaction annihilator distribution

We now introduce the so called *reaction annihilator* distribution $\mathcal{R}^\circ$, first introduced in [54]. The importance of $\mathcal{R}^\circ$ arises from two main situations: on one hand the energy is not generally conserved in nonholonomic systems with affine constraints, and $\mathcal{R}^\circ$ plays a crucial role in this fact as made explicitly in Proposition 1.3, on the other hand on the conservation of momentum (see Section 1.6).

To construct the reaction annihilator distribution $\mathcal{R}^\circ$ we first consider the fibered subset $\mathcal{R} \subset T^*Q$ defined as follows: let $R(q, \mathcal{M}_q) \subseteq \mathcal{D}^\circ_q$ be defined as

$$R(q, \mathcal{M}_q) = \{ R(q, v) \, : \, v \in \mathcal{M}_q \} \, .$$

Then $\mathcal{R}$ is the disjoint union over $q \in Q$ of the sets $R(q, \mathcal{M}_q)$

$$\mathcal{R} = \bigsqcup_{q \in Q} R(q, \mathcal{M}_q) \subseteq T^*Q.$$

Finally, the reaction-annihilator distribution is conformed fiberwise by the annihilator of the fibers of the reaction distribution $\mathcal{R}$

$$\mathcal{R}^\circ = \bigsqcup_{q \in Q} R(q, \mathcal{M}_q)^\circ,$$

where $R(q, \mathcal{M}_q)^\circ$ is the annihilator of the set $R(q, \mathcal{M}_q)$. $R(q, \mathcal{M}_q)^\circ$ is a linear subspace of T_qQ, and so $\mathcal{R}^\circ$ is a distribution on Q, but not necessarily of constant rank nor smooth. However by construction we have the inclusion $\mathcal{D} \subseteq \mathcal{R}^\circ$.

Observation 1.3.1. We anticipate here that we encounter three behaviors for $\mathcal{R}^\circ$ in Chapter 5 when we study the system of a ball rolling without slipping in a surface of revolution.

- $\mathcal{R}^\circ = \mathcal{D}$; this happens when the surface is a cone.
- $\mathcal{D} \subsetneq \mathcal{R}^\circ$; this happens when the surface is a vertical cylinder.

- $\mathcal{R}^\circ = TQ$; this is the case of the plane.

Even more, in the case of the cylinder $\mathcal{M} \subseteq \mathcal{R}^\circ$, this aspect is relevant in the following Proposition 1.3.

The reaction annihilator distribution helps to place an obstruction for energy conservation, we now present conditions on when the lagrangian energy, of an affine nonholonomic system, is a first integral.

Proposition 1.3. *[51] Consider a nonholonomic system $(Q, L, \mathcal{M})$ with affine constraints $\mathcal{M} = Y + \mathcal{D}$. The lagrangian energy $E_{L,\mathcal{M}} = E_L|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ is a first integral of the nonholonomic system $(Q, L, \mathcal{M})$ if and only if $Y \in \Gamma(\mathcal{R}^\circ)$.*

Proof. Using the invariance on the structure of Lagrange-d'Alembert equations we prove this result in coordinates. Let $(q(t), \dot{q}(t))$ be a solution curve in $\mathcal{M} = \xi + \mathcal{D}$. Then the local expression of equations of motion (1.17) and lagrangian energy (1.18) imply

$$\begin{aligned}\frac{d}{dt}E_{L,\mathcal{M}} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q}\\ &= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\dot{q}\\ &= R(q,\dot{q})\cdot\dot{q},\end{aligned}$$

since $\dot{q} \in \mathcal{M}_q$ then we can write $\dot{q} = \xi_q + w_q$, with $w_q \in \mathcal{D}_q$ therefore $\frac{d}{dt}E_{L,\mathcal{M}} = 0$ iff $R(q,\dot{q}) \cdot \xi_q = 0$, i.e. $\xi \in \Gamma(\mathcal{R}^\circ)$.

□

1.4 Quasi-velocities and Hamel-d'Alembert equations

It is well known that in lagrangian mechanics tangent lifts of diffeomorphisms of the configuration space preserve the structure of Lagrange equations, see [92, 91, 4]. However from a mathematical perspective, we are dealing with the vector bundle TQ, so tangent lifts of diffeomorphisms are not the only ones which preserve the bundle structure of TQ. From a physics viewpoint, the velocities induced by the coordinates might not be well suited to analyze a given system, for example in rigid body dynamics it is useful to use the angular velocity to write the equations of motion instead of lifted bundle coordinates [4, 84]. The other kind of diffeomorphisms that we consider and preserve the vector bundle structure of TQ are the bundle automorphism of TQ over the identity of Q. In natural bundle coordinates $(q, \dot{q})$ correspond to transformations $(q, \dot{q}) \mapsto (q, \Phi(q)\dot{q})$, with $\Phi : Q \to GL(\mathbb{R}^n)$. Such bundle transformations correspond, on each fiber, to a basis change transformation, so if $X_1, ..., X_n$ is a local frame of $\mathfrak{X}(Q)$, then there exist smooth functions $B_{ij} : Q \to \mathbb{R}$, $i, j = 1, ..., n$ such that

$$X_i = B_{ij}\frac{\partial}{\partial q^j}, \quad i = 1, ..., n. \tag{1.19}$$

We denote by $B = (B_{ij})$ the matrix associated to the frame change. Let $v = (v^1, ..., v^n)$ be the local fiber coordinates induced by the frame $\{X_i\}_{i=1}^n$, then

$$\dot{q}^i = B_{ji}v^j, \quad i = 1, ..., n, \tag{1.20}$$

or in matrix notation $\dot{q} = B^T v$. If the local frame $\{X_i\}_{i=1}^n$ is not a coordinate frame[4], that is if $[X_i, X_j] \neq 0$ for some i, j, then the fiber coordinates v^i are the so-called *quasi-velocities* [87, 77, 15, 39]. We pay attention to the (local) structure functions $C_{ij}^m : Q \to$

[4]Non coordinate frames are also called nonholonomic frames.

$\mathbb{R}$ associated to a nonholonomic frame, because they are useful in the computation of directional derivatives in terms of the local frame $\{X_i\}_{i=1}^n$ [80]. The structure functions C_{ij}^m are defined by relations

$$[X_i, X_j] = C_{ij}^m X_m,$$

We can give explicit expressions for C_{ij}^m in terms of B, its inverse and derivatives. From (1.19) and the Jacobi-Lie bracket in coordinates we then have

$$\begin{aligned}
[X_i, X_j] &= \left[B_{ik}\frac{\partial}{\partial q^k}, B_{jl}\frac{\partial}{\partial q^l}\right] \\
&= \left(B_{ik}\frac{\partial B_{jl}}{\partial q^k} - B_{jk}\frac{\partial B_{il}}{\partial q^k}\right)\frac{\partial}{\partial q^l} \\
&= \left(B_{ik}\frac{\partial B_{jl}}{\partial q^k} - B_{jk}\frac{\partial B_{il}}{\partial q^k}\right)B^{lm}X_m, \quad i,j,k,l,m = 1,...,n,
\end{aligned}$$

here B^{lm} denotes the $lm-$entry of the inverse B^{-1} of B. So

$$C_{ij}^m = \left(B_{ik}\frac{\partial B_{jl}}{\partial q^k} - B_{jk}\frac{\partial B_{il}}{\partial q^k}\right)B^{lm}. \tag{1.21}$$

This functions are crucial in the determination of Hamel-d'Alembert equations, done in Subsection 1.4.2. More precisely they are related to the so-called *transpositional symbols* in Hamel equations (see for example [87, 77, 15]), that is the equations of motion (1.13) in terms of quasi-velocities.

1.4.1 Constraints

In various examples the nonholonomic constraints are obtained in terms of quasi-velocities or physically adapted coordinates [87, 12], and we follow this approach in Subsection 5.1.4. The constraints (5.6) are easily written in quasi-velocities using (1.20), by which we can write $S\dot{q} = SB^T v$. Therefore by definition the matrix $\tilde{S} = SB^T$, the constraints in quasi-velocities write as

$$\tilde{S}_{\alpha i} v^i + s_\alpha, \quad \alpha = 1, ..., n - r.$$

A powerful application of quasi-velocities realizes when one chooses $X_1, ..., X_r$ as a local frame of $\Gamma(\mathcal{D})$ and $X_{r+1}, ..., X_n$ as a local frame of $\Gamma(\mathcal{D}^\perp)$. If $Y \in \Gamma(\mathcal{D}^\perp)$ is such that $\mathcal{M} = Y + \mathcal{D}$, then the constraint equations in quasi-velocities can be written as $v^\alpha = Y^\alpha$, $\alpha = n - r, ..., n$, where Y^i are the components of the vector field $Y \in \Gamma(\mathcal{D}^\perp)$ written in terms of the local frame X_i, $i = 1, ..., n$.

1.4.2 Hamel-d'Alembert equations

Hamel equations are Lagrange-d'Alembert equations of motion (1.17) expressed in quasi-velocities or, equivalently, in terms of moving frames, for a more in depth exposition see [87, 39, 77, 56]. Its coordinate derivation is done by applying of the chain rule. Let $\tilde{L}(q, v) = L(q, \dot{q}(q, v))$ be the lagrangian written in quasi-velocities then

$$\tilde{L}(q, v) = v \cdot \tilde{A}(q)v + \tilde{\gamma}(q) \cdot v - V(q),$$

where $\tilde{A}(q)$ and $\tilde{\gamma}(q)$ are the kinetic energy metric and the gyroscopic 1-form, respectively, in v coordinates and they are related to $A(q)$ and $\gamma(q)$ by

$$\tilde{A} = BAB^t, \quad \tilde{\gamma} = \gamma B^t. \tag{1.22}$$

Proposition 1.4. *Consider the nonholonomic system $(Q, L, \mathcal{M})$. The Hamel-d'Alembert equations in quasivelocities (q, v) are*

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^i} = B_{ik}\frac{\partial \tilde{L}}{\partial q^k} - C_{il}^m \frac{\partial \tilde{L}}{\partial v^m} v^l + B_{ik}\tilde{R}_k. \tag{1.23}$$

Proof. We first notice

$$\frac{\partial \tilde{L}}{\partial v^i} = B_{ij}\frac{\partial L}{\partial \dot{q}^j} \quad \text{and} \quad \frac{\partial \tilde{L}}{\partial q^i} = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^j}\frac{\partial B_{kj}}{\partial q^i} v^k, \tag{1.24}$$

where again we think $\dot{q}$ as a function of q and v. Finally to express the reaction force R in terms of quasi-velocities, set $\tilde{\sigma}(q, v) = \sigma(q, \dot{q}(q, v))$ and $\tilde{\eta}(q, v) = \eta(q, \dot{q}(q, v))$ then their expressions in such fiber coordinates are

$$\tilde{\sigma}_a = \dot{q}^k \frac{\partial}{\partial q^k}\left(S_{aj}\dot{q}^j + s_a\right) = B_{ik}v^i\left(\frac{\partial \tilde{S}_{am}}{\partial q^k} v^m + \frac{\partial s_a}{\partial q^k}\right) \quad a = 1, ..., n-r.$$

And

$$\tilde{\eta}_i = B^{ik}\left(\frac{\partial^2 \tilde{L}}{\partial v^s \partial v^k}\frac{\partial B^{rs}}{\partial q^j}\dot{q}^j \dot{q}^r + \frac{\partial^2 \tilde{L}}{\partial q^j \partial v^k}\dot{q}^j\right) + \left(\frac{\partial B^{ik}}{\partial q^j} - \frac{\partial B^{kj}}{\partial q^i}\right)\frac{\partial \tilde{L}}{\partial v^k}\dot{q}^j, \quad i = 1, ..., n.$$

Then $\tilde{R}(q, v) = R(q, \dot{q}(q, v))$ is defined by

$$\tilde{R} = B^{-1}\tilde{S}^t \left(\tilde{S}\tilde{A}^{-1}\tilde{S}^t\right)^{-1}\left(\tilde{S}\tilde{A}^{-1}B\tilde{\eta} - \tilde{\sigma}\right).$$

Before computing Hamel-d'Alembert equations use (1.13) to get

$$\begin{aligned}\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^i} &= B_{ik}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} + \frac{\partial L}{\partial \dot{q}^k}\frac{\partial B_{ik}}{\partial q^j}\dot{q}^j \\ &= B_{ik}\left(\frac{\partial L}{\partial q^k} + R_k\right) + \frac{\partial L}{\partial \dot{q}^k}\frac{\partial B_{ik}}{\partial q^j}\dot{q}^j.\end{aligned}$$

Now using relations (1.24) and the structure coefficients expression (1.21) we substitute to get the Hamel-d'Alembert equations

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^i} = B_{ik}\frac{\partial \tilde{L}}{\partial q^k} - C_{il}^m \frac{\partial \tilde{L}}{\partial v^m} v^l + B_{ik}\tilde{R}_k.$$

□

1.5 Symmetries of nonholonomic systems

In unconstrained mechanical systems, the presence of symmetries plays a crucial role in finding first integrals, Noether's Theorem, see for example [92, 84, 4], being the cornerstone that relates symmetries with conserved quantities. With the arise of geometric mechanics different types of reduction, such as symplectic or Poisson [85, 101, 83], and ways to find constants of motion were developed. However the case of nonholonomic systems is not so simple and completely understood, nevertheless for some particular nonholonomic systems analogous ideas to those of unconstrained mechanics serve as an inspiration (we treat such cases in Chapter 2 additionally see [81, 94, 53, 44, 59], the linear case has been better understood bt [8]). However for general nonholonomics systems, e.g. with affine constraints, the question if first integrals arise from symmetries remain unsolved.

The main objective of this Section is to introduce group actions and reduction in nonholonomic systems. Later, on Sections 1.6, 2.5 and 3.5 we use symmetries to construct functions which are candidates to be first integrals.

Consider the nonholonomic system $(Q, L, \mathcal{M})$, and let G be a Lie group that acts on Q by a *free* and *proper* action, denote such action by $\Psi : G\times Q \to Q$, and $T\Psi : G\times TQ \to TQ$ be the associated lifted action. And denote by $\Psi_h : Q \to Q$ and $T\Psi_h : TQ \to TQ$ the corresponding diffeomorphisms related to the group element $h \in G$.

Definition 1.3. Consider a Lie group G acting on a smooth manifold Q. The orbit of an element q in Q, denoted by Orb_G, is

$$Orb_G(q) = \{p \in Q \,:\, \exists h \in G \text{ s.t. } p = \Psi(h, q)\}\,.$$

The orbit of a point $q \in Q$, $Orb_G(q)$, is a closed submanifold of Q. Denote by $T_q Orb_G$ the tangent space of $Orb_G(q)$ at q. The distribution $TOrb_G$ is on each point spanned by the infinitesimal vector fields at q.

Let $\rho : Q \to \overline{Q} := Q/G$ be the quotient map associated to the group action, it defines a principal bundle [62]. Similarly, we denote by $\rho_{TQ} : TQ \to \overline{TQ} := TQ/G$ the quotient map associated to the lifted action of G on TQ. And by properties of quotient maps we have the bundle isomorphism $T(Q/G) \cong TQ/G$, as a consequence $T_qQ \cong (TQ/G)_q$ as vector spaces. To give a coordinate description we choose adapted coordinates (x^d, y^u) to the principal bundle $\rho : Q \to \overline{Q}$ such that $\rho(x^d, y^u) = (x^d)$. For the tangent case additionally to the adapted coordinates we consider an equivariant frame $\{X_i\}_{i=1}^n$ of $\mathfrak{X}(Q)$ then TQ has coordinates (x^d, y^u, v^i) and $\rho_{TQ}(x^d, y^u, v^i) = (x^d, v^i)$.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. To each vector $\xi \in \mathfrak{g}$, we associate the infinitesimal generator $\xi^Q \in \mathfrak{X}(Q)$ defined by

$$\xi^Q(q) = \left.\frac{d}{ds}\right|_{s=0} \Psi_{exp(s\xi)}(q),$$

where $\exp : \mathfrak{g} \to G$ is the Lie group exponenial map.

Proposition 1.5. *The map $\mathfrak{g} \to \mathfrak{X}(Q)$, sending an element of the Lie algebra $\mathfrak{g}$ to its infinitesimal generator, $\xi \mapsto \xi^Q$, is a Lie algebra anti-homomorphism,*

$$\left[\xi^Q, \zeta^Q\right] = -\left[\xi, \zeta\right]^Q.$$

Consider a given nonholonomic system (Q, L, M) and a Lie group acting on the configuration manifold Q, we care about when certain objects and functions being invariant under the group action.

Definition 1.4.
- Let $f : Q \to \mathbb{R}$ be a smooth function on Q, we say that f is G-invariant if $\Psi_h^* \circ f = f$ for all $h \in G$.
- Let $F : TQ \to \mathbb{R}$ be a smooth function, we say that F is G-invariant if $T\Psi_h^* \circ F = F$ for all $h \in G$.
- Let P be a distribution on Q, we say P is G-invariant if $T_q\Psi_h(P_q) = P_{\Psi_h(q)}$, for all $h \in G$ and $q \in Q$.
- Let $Z \in \mathfrak{X}(Q)$ be a vector field on Q, Z is called G-equivariant if $\Psi_h^* Z = Z$, for all $h \in G$.

Observe that if the constraint submanifold $\mathcal{M}$ is given by constraint functions $F_l : TQ \to \mathbb{R}$, $l = 1, ..., n - r$, then $\mathcal{M}$ is $G-$invariant if and only if the functions F_l, $l = 1, ..., n - r$, are G-invariant.

Definition 1.5. We say that a nonholonomic system $(Q, L, \mathcal{M})$ is invariant with respect to a Lie group G action if and only if the lagrangian L and the constraint manifold $\mathcal{M}$ are G-invariant. In this case we also say the system $(Q, L, \mathcal{M})$ is G-invariant or G-symmetric.

As we see in the next Proposition, the G-invariance of a nonholonomic system as in Definition 1.5 implies several consequences in the geometry of the system.

Proposition 1.6. *Let $(Q, L, \mathcal{M})$ be a G-symmetric nonholonomic system, with $L = T + \gamma^\ell - V\tau_Q$ a gyrsocopic lagrangian, let g be the kinetic energy meric induced by L, and $\mathcal{M} = \xi + \mathcal{D}$ with $\xi \in \Gamma(\mathcal{D}^\perp)$ then*

1. *Ψ_h is an isometry of (Q, g), for all $h \in G$.*
2. *The 1-form $\gamma \in \Omega^1(Q)$ and the potential energy function $V \in C^\infty(Q)$ are G-invariant, hence the lagrangian energy E_L is also G-invariant.*
3. *The vector field $\xi \in \Gamma(\mathcal{D}^\perp)$ is G-equivariant.*
4. *The distribution $\mathcal{D}^\perp$ is G-invariant.*

Proof. The proof of 1. and 2. are rely on comparing the homogeneity degrees, on the velocities, of L and $L \circ T\Psi$. That is, let $v_q \in T_qQ$ and $h \in G$, then

$$\begin{aligned} L(T\Psi_h(v_q)) &= \langle T\Psi_h(v_q), T\Psi_h(v_q)\rangle_g + \langle \gamma(\Psi_h(q), T\Psi_h(v_q)\rangle - V(\Psi_h(q)) \\ &= \langle v_q, v_q\rangle_g + \langle \gamma, v_q\rangle - V(q) \\ &= L(v_q), \end{aligned}$$

this implies the following relations $\langle T\Psi_h(v_q), T\Psi_h(v_q)\rangle_g = \langle v_q, v_q\rangle_g$, $\langle \gamma(\Psi_h(q), T\Psi_h(v_q)\rangle = \langle \gamma, v_q\rangle$ and $V(\Psi_h(q)) = V(q)$, for all $q \in Q$ and $v_q \in T_qQ$. Therefore

$$T\Psi_h^* T = T, \quad \Psi_h^*\gamma = \gamma \quad \text{and} \quad \Psi_h^* V = V,$$

the riemannian metric g is G-invariant since the quadratic form, T, associated to it is. And the lagrangian energy E_L is clearly G-invariant since T and V are.

3. and 4. follow from the following assertion, the distribution $\mathcal{D}$ is G-invariant, i.e. $T\Psi_h(\mathcal{D}) = \mathcal{D} \circ \Psi_h$. To prove it note that $T\Psi_h$ is a linear bundle isomorphism, then $T\Psi_h(\mathcal{D}_q)$ is a vector subspace of $T_{\Psi_h(q)}Q$, and by hypothesis $T\Psi_h(\mathcal{M}_q) = \mathcal{M}_{\Psi_h(q)}$ then $T\Psi_h(\xi_q + w_q) - \xi_{\Psi_h(q)} \in D_{\Psi_h(q)}$, for all $w_q \in \mathcal{D}_q$, this implies $T\Psi_h(\mathcal{D}_q) \subseteq D_{\Psi_h(q)}$ for every $q \in Q$, hence $\mathcal{D}$ is G-invariant. Using 1. we clearly get that $\mathcal{D}^\perp$ is also a G-invariant distribution, and $T\Psi_h(\xi_q + w_q) - \xi_{\Psi_h(q)} = 0$. □

The following result is a direct consequence of the invariance of a nonholonomic system.

Proposition 1.7. *[12] Let $(Q, L, \mathcal{M})$ be a G-invariant nonholonomic system. Then the nonholonomic vector field X_{nh} of $(Q, L, \mathcal{M})$ is G-invariant and so it defines a vector field $\overline{X_{nh}}$ on $\mathcal{M}/G$ which coincides with the nonholonomic vector field of the system $(Q/G, l, \mathcal{M}/G)$, where $L = l \circ \rho_{TQ}$.*

Remark 1.5.1. Apart from the previous Proposition, we consider different kinds of reduction and restriction of the equations of motion to invariant submanifolds related to symmetries. In particular, in Sections 2.4 and 3.4 we present reduction of the *almost-Poisson bracket* of nonholonomic systems. Such formulation is given on the cotangent bundle T^*Q in which we consider the dual action $T\Psi^* : G \times T^*Q \to T^*Q$ of $T\Psi$ defined by

$$\langle T\Psi_h^*(\beta_q), v_{\Psi_{h^{-1}}(q)}\rangle = \langle \beta_q, T\Psi_h(v_{\Psi_{h^{-1}}(q)})\rangle. \tag{1.25}$$

The dual action $T\Psi^*$ inherits the properties of $T\Psi$, so it is a free and proper smooth action.

Definition 1.6. Suppose there is a given free and proper action of a Lie group G on a smooth manifold Q, with quotient map $\rho : Q \to Q/G$. The vertical subbundle $\mathcal{V} \subset TQ$ is defined fiberwise as

$$\mathcal{V}_q = \ker T_q\rho.$$

In [14, 59] the authors use the vertical bundle $\mathcal{V}$ to perform reduction using a principal connection adapted to the action, one of the main differences between the two approaches is the *dimension assumption*

Definition 1.7. We say the dimension assumption holds for a nonholonomic system $(Q, L, \mathcal{M})$ if

$$TQ = \mathcal{V} + \mathcal{M}.$$

Observation 1.5.1. As proven in [14], if the dimension assumption is valid then the *nonholonomic connection* [12, 14], coincides with a principal connection and one can project the nonholonomic dynamics into $\mathcal{M}/G$.

1.6 Momentum generated by a vector field

Noether theorem for unconstrained mechanical systems gives conditions on when certain functions, linear on the velocities are first integrals, so the importance of such functions is well established.

Definition 1.8. Let $Z \in \mathfrak{X}(Q)$ a vector field on Q and $L : TQ \to \mathbb{R}$ be a lagrangian function on TQ. The momentum generated by Z is the function $p_Z : TQ \to \mathbb{R}$ on TQ defined by

$$p_Z(v_q) = \langle \mathbb{F}L(v_q), Z_q \rangle.$$

When L is a lagrangian of gyroscopic type then the momentum p_Z in bundle coordinates $(q^i, \dot{q}^i)$ is

$$\begin{aligned} p_Z(q, \dot{q}) &= Z \cdot A(q)\dot{q} + Z \cdot \gamma \\ &= Z^i A_{ij}\dot{q}^j + Z^i\gamma^i. \end{aligned}$$

If X is a vector field on TQ a natural question when one has a vector field X is to see if there are any invariant submanifolds along the flow of X. The existence of such submanifolds is important because the dynamics can be restricted to it. One way to find such submanifolds is to look for smooth functions F on TQ, such that $\mathcal{L}_X F = 0$. If this condition holds we say that F is a *first integral* of the vector field X. By definition of Lie derivative, see for e.g. [80], it is easily seen that the level sets of the first integral F are invariant under the flow of X. This notion is general, but here we give a specific definition for a nonholonomic system $(Q, L, \mathcal{M})$.

Definition 1.9. Let X_{nh} be the nonholonomic vector field of $(Q, L, \mathcal{M})$ and $F : \mathcal{M} \to \mathbb{R}$. We say F is a first integral of X_{nh} if

$$\mathcal{L}_{X_{nh}} F = 0.$$

The conservation of a momentum or when a momentum is a first integral is related to the following Proposition, the proof of which is given in coordinates, since that way is a straightforward computation, however it has intrinsic nature as we can see from Porposition 1.9.

Proposition 1.8 ([56]). *Let $Z \in \mathfrak{X}(Q)$ be a vector field on Q and p_Z the momentumassociated to Z then the derivative of p_Z along a curve of the nonholonomic system $(Q, L, \mathcal{M})$ is*

$$\frac{d}{dt}p_Z = Z^{TQ}[L] + \langle R, Z \rangle.$$

Proof. The proof is given in bundle lifted coordinates $(q,\dot q)$ of TQ

$$\begin{aligned}\frac{d}{dt}p_Z(q,\dot q) &= \frac{d}{dt}\langle\frac{\partial L}{\partial \dot q}, Z_q\rangle\\ &= \langle\frac{d}{dt}\frac{\partial L}{\partial \dot q}, Z_q\rangle + \langle\frac{\partial L}{\partial \dot q}, \frac{d}{dt}Z_q\rangle\\ &= \langle\frac{\partial L}{\partial q} + R(q,\dot q), Z_q\rangle + \langle\frac{\partial L}{\partial \dot q}, \frac{\partial Z}{\partial q}\dot q\rangle\\ &= Z^{TQ}[L] + \langle R(q,\dot q), Z_q\rangle.\end{aligned}$$

□

Recall the following result from [54], which is clearly inspired from Proposition 1.8. We include a proof for completeness.

Proposition 1.9. *[54] Consider a nonholonomic system $(Q,L,\mathcal{M})$ where L is a gyroscopic lagrangian and $\mathcal{M}$ is an affine distribution on Q. Let $Z\in\mathfrak{X}(Q)$ then any two of the following conditions imply the third.*

1. *$Z\in\Gamma(R^\circ)$. Where R° is the reaction force annihilator distribution.*
2. *$Z^{TQ}[L]\,|_{\mathcal{M}}=0$.*
3. *$p_Z|_{\mathcal{M}}$ is a first integral. Where $p_Z(v_q)=\langle\mathbb{F}L(v_q),Z_q\rangle$.*

Proof. We may prove this in coordinates. Let $(q(t),\dot q(t))$ be an integral curve of X_{nh}, then

$$\frac{d}{dt}p_Z(q,\dot q) = \frac{d}{dt}\frac{\partial L}{\partial \dot q}\cdot Z_q + \frac{\partial L}{\partial \dot q}\cdot\frac{d}{dt}Z_q = (\frac{\partial L}{\partial q} + R(q,\dot q))\cdot Z_q + \frac{\partial L}{\partial \dot q}\cdot\frac{d}{dt}Z_q,$$

since $(\frac{d}{dt}Z_q)^i = \frac{\partial Z^i}{\partial q^j}\dot q^j$ we get

$$\frac{d}{dt}p_Z(q(t),\dot q(t)) = Z^{TQ}(L)(q,\dot q) + R(q,\dot q)\cdot Z_q.$$

The above expression has three monomials, each one is related to one of the conditions and the result clearly follows. □

We now give a characterization of when two vector fields $Z_1,Z_2\in\mathfrak{X}(Q)$ have the same associated momentum restricted to $\mathcal{M}$, generalizing a result in [2], where the authors give a characterization in the case of linear constraints and natural lagrangians.

Proposition 1.10. *Two vector fields Z_1,Z_2 on Q define the same momentum, $p_{Z_1}\big|_{\mathcal{M}}$, $p_{Z_2}\big|_{\mathcal{M}}$, on $\mathcal{M}$ if and only if the following conditions are satisfied.*

$$\begin{aligned}&Z_1 - Z_2\in\Gamma(\mathcal{D}^\perp),\\ &\langle Z_1, Y+N\rangle_g = \langle Z_2, Y+N\rangle_g.\end{aligned}$$

Proof. Suppose $p_{Z_1}\big|_{\mathcal{M}} = p_{Z_2}\big|_{\mathcal{M}}$ then

$$\begin{aligned}p_{Z_1}|_{\mathcal{M}}(v_q) &= \langle Z_1(q), v_q\rangle_{g(q)} + \langle Z_1(q), N(q)\rangle_{g(q)}\\ &= \langle Z_2(q), v_q\rangle_{g(q)} + \langle Z_2(q), N(q)\rangle_{g(q)}\\ &= p_{Z_2}|_{\mathcal{M}}(v_q),\quad \forall v_q\in\mathcal{M}_q \text{ and } q\in Q.\end{aligned}$$

Since the zero section is a section of $\mathcal{D}$ we have

$$\langle Z_1, Y + N\rangle_g = \langle Z_2, Y + N\rangle_g.$$

Then

$$\langle Z_1(q), v_q\rangle_g - \langle Z_2(q), v_q\rangle_g = \langle Z_1(q) - Z_2(q), v_q\rangle_g = 0, \quad \forall v_q \in \mathcal{D}_q,$$

this happens if and only if $Z_1 - Z_2 \in \Gamma(\mathcal{D}^\perp)$.

□

1.6.1 Gauge momenta

Assume there's a Lie group G acting on the nonholonomic system $(Q, L, \mathcal{M})$ then we can consider a special type of momenta which are generated by sections of $TOrb$. Note that such sections need not be infinitesimal generators of the action.

Definition 1.10. Let $Z \in \Gamma(TOrb)$ be a vector field tangent to the group orbits. We say that Z is a *gauge-section*.

- Z is called a horizontal gauge-symmetry if $Z \in \Gamma(\mathcal{D})$ and $Z^{TQ}(L)|_{\mathcal{M}} = 0$.
- Z is called a gauge-symmetry if $Z^{TQ}(L)|_{\mathcal{M}} = 0$.

Moreover we say that the momentum p_Z generated by Z is a (horizontal) gauge-momentum.

Later on we investigate when a *gauge-momentum* is a first integral, the results depend on the kind of nonholonomic system we are dealing with. For nonholonomic systems with linear constraints and natural Lagrangian there is a known result commonly called *nonholonomic Noether theorem*, see [94, 55], but for other kind of nonholonomic systems there is not an analogous result, nevertheless we give partial results about this matter.

Chapter 2

Linear constrained systems with natural lagrangian

Nonholonomic systems with linear constraints and natural lagrangian are by far the most studied in nonholonomic mechanics (a non exhaustive list of recent books includes [87, 12, 24, 36, 92, 40, 91]). The aim of this Chapter is to present basic aspects and some relevant result to use them as test case and inspiration for the forthcoming chapters. We use the same language and notation of Chapter 1 in order to relate and particularize the theory in an easier way.

As one can expect from this Section's title we consider a constraint distribution which is a vector subbundle $\mathcal{M} = \mathcal{D}$ of TQ, the constraint functions defining $\mathcal{M}$ (1.12) are linear homogeneous in the velocities, and the lagrangian L_0 is of natural type and then writes as $L_0 = T - V \circ \tau_Q$, where T and V are the kinetic and potential energy, respectively. Thus a triple $(Q, L_0, \mathcal{D})$ defines a nonholonomic system.

2.1 Local Lagrange-d'Alembert equations

Let Q be a n-dimensional smooth manifold and consider lifted coordinates $(q, \dot{q})$ on its tangent bundle TQ. Using the same notation as in (1.11), consider a natural lagrangian L_0

$$L_0 = T - V \circ \tau_Q, \quad \text{in coordinates} \quad L_0(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q} - V(q). \tag{2.1}$$

Recall that A is the kinetic energy matrix in the coordinates $(q, \dot{q})$, (we are assuming that no gyroscopic forces are included i.e. the 1-form γ in (1.1) vanishes). The constraint submanifold $\mathcal{M} = \mathcal{D}$ is a linear subbundle of TQ (because $Y = 0$ see (1.2)). In coordinates the constraints equations are

$$S(q)\dot{q} = 0,$$

where the matrix S is such that $\mathcal{D}_q = \ker S(q)$ as in (1.12). Then the Lagrange-d'Alembert equations in coordinates are

$$\left(\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}} - \frac{\partial L_0}{\partial q} \right)\Big|_{\mathcal{D}} = S^t(SA^{-1}S^t)^{-1}(SA^{-1}\eta - \sigma)|_{\mathcal{D}}, \tag{2.2}$$

where $\eta_i = \frac{\partial^2 L_0}{\partial q^j \partial \dot{q}^i}\dot{q}^j - \frac{\partial L_0}{\partial q^i}, \quad i = 1, ..., n$ and $\sigma_a = \frac{\partial S_{aj}}{\partial q^i}\dot{q}^i\dot{q}^j, \quad a = 1, ..., n - r$.

Remark 2.1.1. The equations of motion (2.2) are quadratic in the velocities, therefore the system is reversible, i.e. if $t \mapsto q(t)$ is an integral curve of X_{nh} then so $t \mapsto q_-(t) = q(-t)$ is [1].

To prove this assertion note that

$$L_0(q_-(t), \dot{q}_-(t)) = L_0(q_-(t), -\dot{q}_-(t))$$

and that the constraint distribution $\mathcal{D}$ is also invariant under this diffeomorphism

$$S(q_-(t))\dot{q}_-(t) = -S(q(-t))\dot{q}(-t) = 0, \quad \forall t.$$

Then $\eta(q(-t), -\dot{q}(-t)) = \eta(q(-t), \dot{q}(-t))$ and $\sigma(q(-t), -\dot{q}(-t)) = \sigma(q(-t), \dot{q}(-t))$ therefore equations (2.2) are invariant under the tangent lift of the diffemorphism $t \mapsto -t$.

Proposition 1.3 guarantees the conservation of the lagrangian energy $E_{L_0,\mathcal{D}}$, see Definition 1.2, since $Y = 0$ obviously lies in the reaction-annihilator distribution R° [56, 75].

2.2 Quasi-velocities and Hamel-d'Alembert equations

We give here the expressions of Hamel equations in the case a given nonholonomic system has linear constraints. Consider the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$ of the tangent bundle induced by the kinetic energy metric g. Let $\{X_i\}_{i=1}^n$ be a local frame of TQ such that $X_a \in \Gamma(\mathcal{D})$ and $X_\alpha \in \Gamma(\mathcal{D}^\perp)$, where the index $a = 1, ..., r$ and $\alpha = r+1, ..., n$. Denote by $v = (v^1, ..., v^n)$ the quasi-velocities associated to this frame. The kinetic energy matrix A in this coordinates reads

$$\tilde{A} = BAB^t = \begin{pmatrix} A_{\mathcal{D}} & 0 \\ 0 & A_{\mathcal{D}^\perp} \end{pmatrix},$$

where B is the frame change matrix as in (1.19), and $A_{\mathcal{D}}$, $A_{\mathcal{D}^\perp}$ are the block matrices corresponding to the restriction of the kinetic energy metric g to $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. The lagrangian $\tilde{L}_0$ in quasi-velocities writes as

$$\tilde{L}_0(q, v) = \frac{1}{2} v_{\mathcal{D}} \cdot A_{\mathcal{D}}(q) v_{\mathcal{D}} + \frac{1}{2} v_{\mathcal{D}^\perp} \cdot A_{\mathcal{D}^\perp}(q) v_{\mathcal{D}^\perp} - V(q),$$

where $v = v_{\mathcal{D}} + v_{\mathcal{D}^\perp}$ is the decomposition of v defined by $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$. The constraints equations in quasi-velocities adapted to the constraints become

$$v^\alpha = 0, \quad \alpha = r+1, ..., n.$$

Observation 2.2.1. This representation of the constraint equations is not exclusive for the orthogonal decomposition of TQ, but for any frame adapted to a decomposition $TQ = \mathcal{D} \oplus W$, where W is a subbundle of TQ.

Following (1.23), let $L_{0,c} := \tilde{L}|_{\mathcal{D}}$ be the restriction of the lagrangian $\tilde{L}_0$ to $\mathcal{D}$, then the Hamel-d'Alembert equations [12] are

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L_{0,c}}{\partial v^a} = B_{ak}\frac{\partial L_{0,c}}{\partial q^k} - C_{al}^m \frac{\partial L_{0,c}}{\partial v^m} v^l, \quad a, l, m = 1, ..., r, \\ &\frac{dq^i}{dt} = B_{ai} v^a, \end{aligned} \tag{2.3}$$

and together with the constraint equations $v^\alpha = 0$, $\alpha = r+1, ..., n$, they characterize the dynamics.

Observation 2.2.2. By d'Alembert principle the reaction force restricted to the constraint manifold vanishes, i.e. $R|_{\mathcal{D}} = 0$.

2.3 Almost hamiltonian formulation

In this Section we introduce the analogous of Hamilton equations for nonholonomic systems using an *almost*-Poisson bracket[1] in $\mathcal{D}^*$ the dual vector bundle to $\mathcal{D}$. For this exposition we follow [11, 96, 28, 44]. Many authors [78, 27, 12] construct the almost-Poisson bracket in $\mathcal{D}^*$ by using the canonical symplectic structure of T^*Q, and inclusions and projectors defined by it. Instead we prefer to use elements related to the riemannian metric structure of Q to define the bracket in the bundle $\mathcal{D}^*$.

2.3.1 Legendre transformation and dynamics in $\mathcal{D}^*$

One of the main obstructions to construct a Poisson bracket in $\mathcal{D}^*$ is its representation as vector bundle inside the cotangent bundle T^*Q. We use here the Legendre transformation induced by the lagrangian L_0 and the orthogonal decomposition on the tangent bundle TQ induced by the kinetic energy metric to define the necessary geometric elements to construct a bundle isomorphism between $\mathcal{D}$ and $\mathcal{D}^*$.

The *Legendre transformation and dynamics in* $\mathcal{D}^*$ is referred to the fiber derivative $\mathbb{F}L$, which in this particular case is a linear bundle isomorphism, so we can perform a push-forward of the dynamics for $\mathcal{D}$ onto $\mathcal{D}^*$. Once again consider the orthogonal decomposition of $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$ induced by the metric g. Using the bundle isomorphism $\mathbb{F}L$ this decomposition induces the decomposition $T^*Q = (\mathcal{D}^\perp)^\circ \oplus \mathcal{D}^\circ$ of the cotangent bundle, where $(\mathcal{D}^\perp)^\circ, \mathcal{D}^\circ$ are the annihilator distributions of $\mathcal{D}^\perp$ and $\mathcal{D}$ respectively. Moreover observe that $\mathcal{D}^*$ is isomorphic to $(\mathcal{D}^\perp)^\circ$[2].

To compute the equations of motion in $\mathcal{D}^*$, let (q, v) be local coordinates in $\mathcal{D}$, with $q = (q^1, ..., q^n)$ and $v = (v^1, ..., v^r)$ as in Section 2.2, then (q, p) are local coordinates in $\mathcal{D}^*$, $p = (p_1, ..., p_r)$, with respect to the local co-frame $\{\mathbb{F}L_{0,c}(X_a)\}_{a=1}^r$ where

$$p_a = \frac{\partial L_{0,c}}{\partial v^a} = (A_\mathcal{D})_{ab} v_b, \quad a, b = 1, ..., r, \tag{2.4}$$

are the fiber coordinates called *quasi-momenta*. Here we use $L_{0,c}$ instead of L_0 because the constraints in this setup are $v^\alpha = 0$, $\alpha = r+1, ..., n$ so $\frac{\partial L_{0,c}}{\partial v^a} = \frac{\partial L_0}{\partial v^a}\big|_\mathcal{D}$.

Since $\mathbb{F}L_{0,c} : \mathcal{D} \to \mathcal{D}^*$ is a diffeomorphism, we can consider the pushforward $(\mathbb{F}L_{0,c})_*(X_{nh}) \in \mathfrak{X}(\mathcal{D}^*)$ of this vector field is related to the constrained hamiltonian $H_c : \mathcal{D}^* \to \mathbb{R}$, defined by $H_c = (E_{L_0} \circ \mathbb{F}L_{0,c}^{-1})$. In bundle adapted coordinates (q, p)

$$H_c(q, p) = \frac{1}{2} p \cdot A_\mathcal{D}^{-1} p + V(q), \tag{2.5}$$

this implies

$$\frac{\partial H_c}{\partial q^i} = -\frac{\partial L_{0,c}}{\partial q^i}, \qquad \frac{\partial H_c}{\partial p_a} = v^a. \tag{2.6}$$

Using (2.3) and (2.6) we obtain the equations of motion in $\mathcal{D}^*$:

$$\frac{dq^i}{dt} = B_{ai} \frac{\partial H_c}{\partial p_a}, \qquad \frac{dp_a}{dt} = -Baj \frac{\partial H_c}{\partial q^j} - C_{al}^m \frac{\partial H_c}{\partial p_l} p_m, \tag{2.7}$$

where $a, m, l = 1, ..., r$. We denote the dynamical vector field in $\mathcal{D}^*$ by X_{nh}^*, and its coordinate expression is

$$X_{nh}^* = B_{ai} \frac{\partial H_c}{\partial p_a} \frac{\partial}{\partial q^i} - \left(B_{aj} \frac{\partial H_c}{\partial q^j} + C_{al}^m \frac{\partial H_c}{\partial p_l} p_m \right) \frac{\partial}{\partial p_a}. \tag{2.8}$$

Recall that C_{ij}^k are the structure coefficients (1.21) and B is the matrix to pass from velocities to quasi-velocities (1.19) related to the local frame $\{X_i\}_{i=1}^n$

[1] Meaning that Jacobi identity is not satisfied.

[2] From linear algebra we have $(\mathcal{D}^\perp)_q^\circ \cong \mathcal{D}_q^*$ and $\mathcal{D}_q^\circ \cong (\mathcal{D}^\perp)_q^*$

2.3.2 Almost Poisson bracket in $\mathcal{D}^*$

There are different approaches to construct the almost-Poisson bracket in $\mathcal{D}^*$, such as the symplectic formulation,[11], or the Dirac bracket approach [71, 67], here we follow the treatment of [28, 44]2. As already anticipated at the beginning of this Section the main difference between our approach and the others just mentioned is the use of elements associated to the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$ given by the kinetic energy metric g on Q. To construct the almost-Poisson bracket we use the canonical Poisson bracket $\{\cdot,\cdot\}$ in T^*Q, with Π the associated bi-vector field (see Appendix B for more details).

Definition 2.1. Let $(Q, L_0, \mathcal{D})$ be a nonholonomic system and $\mathcal{D}^*$ the dual bundle of $\mathcal{D}$. We define the bracket in $\mathcal{D}^*$, $\{\cdot,\cdot\}_{\mathcal{D}^*} : C^\infty(\mathcal{D}^*) \times C^\infty(\mathcal{D}^*) \to C^\infty(\mathcal{D}^*)$ by

$$\{F_1, F_2\}_{\mathcal{D}^*} = \{F_1 \circ i^*_{\mathcal{D}}, F_2 \circ i^*_{\mathcal{D}}\} \circ P^*_{\mathcal{D}}, \quad F_1, F_2 \in C^\infty(\mathcal{D}^*),$$

where $i^*_{\mathcal{D}} : T^*Q \to \mathcal{D}^*$ and $P^*_{\mathcal{D}} : \mathcal{D}^* \hookrightarrow T^*Q$ are the projector and the inclusion defined in (1.6).

Observation 2.3.1. There is a correspondence between fiberwise linear functions on $\mathcal{D}^*$ and vector fields on $\mathcal{D}$. This fact is based on the isomorphism (of finite dimensional vector spaces) $\mathcal{D}_q \cong \mathcal{D}^{**}_q$ for all $q \in Q$.

Proposition 2.1. *Let $X^\ell, Y^\ell \in C^\infty(\mathcal{D}^*)$ and $f, h \in C^\infty(Q)$. Then*

$$\{X^\ell, Y^\ell\}_{\mathcal{D}^*} = -(P_{\mathcal{D}}\,[i_{\mathcal{D}} \circ X, i_{\mathcal{D}} \circ Y])^\ell, \quad \{X^\ell, f \circ \pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} = -X(f) \circ \pi_{\mathcal{D}^*},$$
$$\{f \circ \pi_{\mathcal{D}^*}, h \circ \pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} = 0.$$

Where $\pi_{\mathcal{D}^*} : \mathcal{D}^* \to Q$ is the bundle projection.

Proof. First recall the following, since the Poisson bracket on T^*Q is linear, for $W_1, W_2 \in \mathfrak{X}(Q)$ and $f, h \in C^\infty(Q)$ we have

$$\left\{W_1^\ell, W_2^\ell\right\} = -\left[W_1, W_2\right], \quad \left\{W_1^\ell, f \circ \pi_Q\right\} = -W_1(f) \circ \pi_Q, \quad \left\{f \circ \pi_Q, h \circ \pi_Q\right\} = 0.$$

Also, $i_{\mathcal{D}} : \mathcal{D} \hookrightarrow TQ$ and $P_{\mathcal{D}} : TQ \to \mathcal{D}$ are vector bundle morphisms, hence the dual morphism $i^*_{\mathcal{D}}$ and $P^*_{\mathcal{D}}$ are too and $\pi_Q \circ P^*_{\mathcal{D}} = \pi_{\mathcal{D}^*}$.

Let $X \in \Gamma(\mathcal{D})$ and $\beta \in \Omega^1(Q)$, then

$$\begin{aligned}(X^\ell \circ i^*_{\mathcal{D}})(\beta) &= X^\ell(i^*_{\mathcal{D}}(\beta)) \\ &= \langle i^*_{\mathcal{D}}(\beta), X\rangle \\ &= \langle \beta, i_{\mathcal{D}}(X)\rangle \\ &= (i_{\mathcal{D}} \circ X)^\ell(\beta).\end{aligned}$$

By the same kind of argument we get

$$W_1^\ell \circ P^*_{\mathcal{D}} = (P_{\mathcal{D}} \circ W_1)^\ell.$$

The result then follows from the above observations and Definition 2.1. □

For a coordinate description, let $\{X_a, X_\alpha\}$, $a = 1, ..., r$ and $\alpha = r+1, ..., n$, be a local frame of $\mathfrak{X}(Q)$ adapted to the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$, that is $X_a \in \Gamma(\mathcal{D})$, $X_\alpha \in \Gamma(\mathcal{D}^\perp)$, and $X_i = B_{ij}\frac{\partial}{\partial q^j}$. Let $\{\chi^a, \chi^\alpha\}$ be the associated dual frame. Using Proposition 2.1 we obtain the expressions for the coordinate functions (q^i, p_a)

$$\begin{aligned}&\left\{q^i, q^j\right\}_{\mathcal{D}^*} = 0, \qquad \left\{q^i, p_a\right\}_{\mathcal{D}^*} = B_{ia}, \\ &\{p_a, p_b\}_{\mathcal{D}^*} = C^c_{ab} p_c.\end{aligned} \tag{2.9}$$

Then the bivector field $\Pi_{\mathcal{D}^*}$ in coordinates reads

$$\Pi_{\mathcal{D}^*} = B_{ia}\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_a} - \frac{1}{2}C^c_{ab}p_c\frac{\partial}{\partial p_a}\wedge\frac{\partial}{\partial p_b}. \tag{2.10}$$

We now prove that this bracket is almost-Poisson and compatible with the dynamics, that is the equations of motion X^*_{nh} are given by the (almost)-hamiltonian vector field associated to the restricted hamiltonian H_c.

Proposition 2.2. *Let $(Q, L_0, \mathcal{D})$ be a nonholonomic system and $\mathcal{D}^*$ the dual bundle of $\mathcal{D}$ with almost-Poisson bracket $\{\cdot,\cdot\}_{\mathcal{D}^*}$ defined in 2.1. Then the bracket $\{\cdot,\cdot\}_{\mathcal{D}^*}$ has the following properties*

1. *It is $\mathbb{R}$-bilinear and skew-symmetric.*
2. *Leibniz rule is satisfied on each entry.*
3. *Jacobi identity holds if and only if the distribution $\mathcal{D}$ is involutive.*

Proof. 1. and 2. follow from Definition 2.1, and the properties of the Poisson bracket in T^*Q.

To prove 3., let $Y, Z \in \Gamma(\mathcal{D})$, and $X_{Y^\ell} = \{\cdot, Y\}_{\mathcal{D}^*}$ be the hamiltonian vector field related to the function $Y^\ell \in C^\infty(\mathcal{D}^*)$. Then by Proposition 2.1 for every $f \in C^\infty(Q)$

$$X_{Y^\ell}(f\circ\pi_{\mathcal{D}^*}) = -Y(f)\circ\pi_{\mathcal{D}^*},$$

which is equivalent to say that the vector fields X_{Y^ℓ} and Y are $\pi_{\mathcal{D}^*}$-related, this imply

$$T\pi_{\mathcal{D}^*}[X_{Y^\ell}, X_{Z^\ell}] = [Y, Z]\circ\pi_{\mathcal{D}^*}.$$

. Now assume Jacobi identity is satisfied, then

$$\begin{aligned} X_{-(P_{\mathcal{D}}[Y,Z])^\ell} &= X_{\{Y^\ell, Z^\ell\}_{\mathcal{D}^*}} \\ &= -[X_{Y^\ell}, X_{Z^\ell}], \end{aligned}$$

so $[X_{Y^\ell}, X_{Z^\ell}]$ and $P_{\mathcal{D}}[Y,Z]$ are $\pi_{\mathcal{D}^*}$-related therefore $[Y,Z] = P_{\mathcal{D}}[Y,Z]$.

If $\mathcal{D}$ is involutive then $[Y,Z] = P_{\mathcal{D}}[Y,Z]$, using the fact that the bracket in $\mathcal{D}^*$ is linear and proposition 2.1 Jacobi identity follows. □

Observation 2.3.2. The obstruction of the bracket $\{\cdot,\cdot\}_{\mathcal{D}^*}$ to be a Poisson bracket is related to the non-integrability of the (constraint) distribution $\mathcal{D}$, so in general in non-holonomic mechanics such bracket is not Poisson.

At last, we relate the equations of motion in $\mathcal{D}^*$ and the hamiltonian vector field of H_c.

Theorem 2.3. *Consider a nonholonomic system $(Q, L_0, \mathcal{D})$, $\mathcal{D}^*$ the dual bundle of $\mathcal{D}$ with almost-Poisson bracket $\{\cdot,\cdot\}_{\mathcal{D}^*}$ defined in 2.1 and $H_c = E_{L_0}\circ\mathbb{F}L_{0,c} : \mathcal{D}^* \to \mathbb{R}$. The equations of motion (2.7) in $\mathcal{D}^*$ are (almost) hamiltonian with respect to the almost-Poisson bracket in $\mathcal{D}^*$ and the hamiltonian function H_c:*

$$X^*_{nh}(F) = \{F, H_c\}_{\mathcal{D}^*}, \quad \forall F \in C^\infty(\mathcal{D}^*).$$

Proof. We may prove this result in adapted coordinates (q,p), $q = (q^1, ..., q^n)$ and $p = (p_1, ..., p_r)$, as in subsection 2.3.1. It suffices to show $X^*_{nh}(q^i) = \{q^i, H_c\}_{\mathcal{D}^*}$ and $X^*_{nh}(p_a) = \{p_a, H_c\}_{\mathcal{D}^*}$. Since $dH_c = \frac{\partial H_c}{\partial q^i}dq^i + \frac{\partial H_c}{\partial p_a}dp_a$, then using equations (2.9) we get

$$\begin{aligned} \{q^i, H_c\}_{\mathcal{D}^*} &= B_{ia}\frac{\partial H_c}{\partial p_a}, \\ \{p_a, H_c\}_{\mathcal{D}^*} &= -B_{ia}\frac{\partial H_c}{\partial q^i} - C^c_{ab}p_c\frac{\partial H_c}{\partial p_b}. \end{aligned}$$

Which coincides with the coordinate expression (2.8) of X^*_{nh}. □

Theorem 2.3 implies that equations of motion (2.7) in $\mathcal{D}^*$ can be written in matrix form

$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_a \end{pmatrix} = \begin{pmatrix} 0 & B_{ib} \\ -B_{aj} & -C^c_{ab} p_c \end{pmatrix} \begin{pmatrix} \frac{\partial H_c}{\partial q^j} \\ \frac{\partial H_c}{\partial p_b} \end{pmatrix}. \tag{2.11}$$

2.4 Symmetries and reduction of the almost Poisson bracket

In Section 1.5 symmetries in nonholonomic systems are introduced in a general way, in this Section we use them to reduce the almost-Poisson structure in $\mathcal{D}^*$ [78, 94, 27].

Let $(Q, L_0, \mathcal{D})$ be a nonholonomic system with a G-symmetry, as in Definition 1.5. Let $\Psi : G \times Q \to Q$ denote the action of G on Q and

$$\rho : Q \to \overline{Q} = Q/G,$$

the quotient map, which is a G-principal bundle. As stated in Section 1.5 the action Ψ extends to TQ by using its tangent lift, $T\Psi$, and since $\mathcal{D}$ is G-invariant, the lifted action restricts to $\mathcal{D}$. Moreover we can consider the dual $T\Psi_g^* : T^*Q \to T^*Q$ action on T^*Q. Then we have the principal bundle $\rho_{T^*Q} : T^*Q \to \overline{T^*Q} = T^*Q/G$. Using the invariance of $\mathcal{D}$ and of L_0 (and hence of $\mathbb{F}L_0$) we also get that $\mathcal{D}^* \subset T^*Q$ is G-invariant, and we can consider the principal bundle $\rho_{\mathcal{D}^*} : \mathcal{D}^* \to \overline{\mathcal{D}^*} = \mathcal{D}^*/G$. Consequently the vector field $X^*_{nh} \in \mathfrak{X}(\mathcal{D}^*)$ can be projected to the vector field $\overline{X^*_{nh}} \in \mathfrak{X}\left(\overline{\mathcal{D}^*}\right)$, by means of the quotient map $\rho_{\mathcal{D}^*}$. Since $\mathcal{D}$ and $\mathcal{D}^*$ are G-invariant the vector bundle morphisms $i^*_{\mathcal{D}}$ and $P^*_{\mathcal{D}}$ are G-equivariant and thus they induce the vector bundle morphisms

$$\overline{i^*_{\mathcal{D}}} : \overline{T^*Q} \to \overline{\mathcal{D}^*}, \quad \overline{P^*_{\mathcal{D}}} : \overline{\mathcal{D}^*} \to \overline{T^*Q}. \tag{2.12}$$

Definition 2.2. Let $\{\cdot,\cdot\}_{\overline{\mathcal{D}^*}} : C^\infty(\overline{\mathcal{D}^*}) \times C^\infty(\overline{\mathcal{D}^*}) \to C^\infty(\overline{\mathcal{D}^*})$ be the bracket in $\overline{\mathcal{D}^*}$ defined as

$$\{F_1, F_2\}_{\overline{\mathcal{D}^*}} = \left\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}} \circ \overline{P^*_{\mathcal{D}}}, \quad F_1, F_2 \in C^\infty(\overline{\mathcal{D}^*}).$$

On the other hand the nonholonomic bracket is G-invariant in the following sense, if $F, H \in C^\infty(\mathcal{D}^*)$ are G-invariant functions, then so $\{F, H\}_{\mathcal{D}^*}$ is. We conclude

Corollary 2.4. *The quotient map $\rho_{\mathcal{D}^*} : \mathcal{D}^* \to \overline{\mathcal{D}^*}$ is an almost-Poisson morphism, i.e.*

$$\{F_1, F_2\}_{\overline{\mathcal{D}^*}} \circ \rho_{\mathcal{D}^*} = \{F_1 \circ \rho_{\mathcal{D}^*}, F_2 \circ \rho_{\mathcal{D}^*}\}_{\mathcal{D}^*}, \quad F_1, F_2 \in C^\infty\left(\overline{\mathcal{D}^*}\right). \tag{2.13}$$

Proof. To prove this claim we use the fact that the quotient map $\rho_{T^*Q} : T^*Q \to \overline{T^*Q}$ is a Poisson map (see Appendix B) and relations

$$\rho_{T^*Q} \circ P^*_{\mathcal{D}} = \overline{P^*_{\mathcal{D}}} \circ \rho_{\mathcal{D}^*}, \quad i^*_{\mathcal{D}} \circ \rho_{T^*Q} = \rho_{\mathcal{D}^*} \circ \overline{i^*_{\mathcal{D}}},$$

given by the equivariance of the morphisms $i^*_{\mathcal{D}}$, $P^*_{\mathcal{D}}$. By definitions 2.1 and 2.2 we get

$$\begin{aligned} \{F_1 \circ \rho_{\mathcal{D}^*}, F_2 \circ \rho_{\mathcal{D}^*}\}_{\mathcal{D}^*} &= \{F_1 \circ \rho_{\mathcal{D}^*} \circ i^*_{\mathcal{D}}, F_2 \circ \rho_{\mathcal{D}^*} \circ i^*_{\mathcal{D}}\} \circ P^*_{\mathcal{D}} \\ &= \left(\left\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}} \circ \rho_{T^*Q}\right) \circ P^*_{\mathcal{D}} \\ &= \left(\left\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}} \circ \overline{P^*_{\mathcal{D}}}\right) \circ \rho_{\mathcal{D}^*} \\ &= \{F_1, F_2\}_{\overline{\mathcal{D}^*}} \circ \rho_{\mathcal{D}^*} \end{aligned}$$

□

The bracket in $\overline{\mathcal{D}^*}$ inherits the following properties

Proposition 2.5. *The bracket* $\{\cdot,\cdot\}_{\overline{\mathcal{D}^*}}$ *has the following properties:*

1. *It is* $\mathbb{R}$*-bilinear and skew-symmetric.*
2. *Leibniz rule is satisfied on each entry.*

Proof. Bilinearty and skew-symmetry follow directly from the definition of the reduced bracket. To prove Leibniz rule note that $Id_{\overline{\mathcal{D}^*}} = \overline{i^*_{\mathcal{D}}} \circ \overline{P^*_{\mathcal{D}}}$, then

$$\begin{aligned}\{F_1F_2, F_3\}_{\overline{\mathcal{D}^*}} &= \left\{(F_1F_2)\circ\overline{i^*_{\mathcal{D}}}, F_3\circ\overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}}\circ\overline{P^*_{\mathcal{D}}}\\ &= \Big((F_1\circ\overline{i^*_{\mathcal{D}}})\left\{F_2\circ\overline{i^*_{\mathcal{D}}}, F_3\circ\overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}}\\ &\quad +(F_2\circ\overline{i^*_{\mathcal{D}}})\left\{F_1\circ\overline{i^*_{\mathcal{D}}}, F_3\circ\overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}}\Big)\circ\overline{P^*_{\mathcal{D}}}\\ &= F_1\{F_2,F_3\}_{\overline{\mathcal{D}^*}} + F_2\{F_1,F_3\}_{\overline{\mathcal{D}^*}}.\end{aligned}$$

∎

Recall the set bijections $C^\infty(Q)^G = C^\infty(\overline{Q})$, $C^\infty(\mathcal{D}^*)^G = C^\infty(\overline{\mathcal{D}^*})$ and $\Gamma(\mathcal{D})^G = \Gamma(\overline{\mathcal{D}})$, then the next Proposition is a straightforward computation which follows from Proposition 2.1

Proposition 2.6. *Let* $X^\ell, Y^\ell \in C^\infty(\overline{\mathcal{D}^*})$ *and* $f, h \in C^\infty(\overline{Q})$. *Then*

$$\{X^\ell, Y^\ell\}_{\overline{\mathcal{D}^*}} = -(\overline{P_{\mathcal{D}}}\,[X,Y])^\ell, \quad \{X^\ell, f\circ\pi_{\overline{\mathcal{D}^*}}\}_{\overline{\mathcal{D}^*}} = -X(f)\circ\pi_{\overline{\mathcal{D}^*}},$$
$$\left\{f\circ\pi_{\overline{\mathcal{D}^*}}, h\circ\pi_{\overline{\mathcal{D}^*}}\right\}_{\overline{\mathcal{D}^*}} = 0.$$

Observation 2.4.1. Even if the bracket defined in $\mathcal{D}^*$ is not a Poisson bracket, the bracket in $\overline{\mathcal{D}^*}$ might satisfy Jacobi identity. For example this is trivially the case if the manifold $\overline{\mathcal{D}^*}$ has dimension 2.

To obtain a coordinate description of the reduced bracket we use adapted coordinates (x^d, y^u, p_a) to the fibration $\rho_{\mathcal{D}^*} : \mathcal{D}^* \to \overline{\mathcal{D}^*}$. We evaluate the coordinate functions x^d and p_a

$$\begin{aligned}&\left\{x^d, r^k\right\}_{\overline{\mathcal{D}^*}} = 0, \qquad \left\{x^d, p_a\right\}_{\overline{\mathcal{D}^*}} = B_{da},\\ &\{p_a, p_b\}_{\overline{\mathcal{D}^*}} = C^c_{ab}p_c,\end{aligned} \tag{2.14}$$

Now we can give an expression for the reduced dynamics

Theorem 2.7. *The reduced dynamics in* $\overline{\mathcal{D}^*}$ *are given*

$$\overline{X^*_{nh}}(F) = \left\{F, \overline{H_c}\right\}_{\overline{\mathcal{D}^*}}, \quad F\in C^\infty(\overline{\mathcal{D}^*}). \tag{2.15}$$

Where $\overline{H_c}\circ\rho_{\mathcal{D}^*} = H_c$.

Proof. The constrained hamiltonian H_c is G-invariant since L_c, $\mathbb{F}L_c$ are G-invariant functions then there exists $\overline{H_c} : \overline{\mathcal{D}^*} \to \mathbb{R}$ such that $H_c = \overline{H_c}\circ\rho_{\mathcal{D}^*}$. Using Theorem 2.3, Corollary 2.4, and the fact that the vector fields X^*_{nh} and $\overline{X^*_{nh}}$ are $\rho_{\mathcal{D}^*}$-related we get

$$\begin{aligned}\overline{X^*_{nh}}(f)\circ\rho_{\mathcal{D}^*} &= X^*_{nh}(f\circ\rho_{\mathcal{D}^*})\\ &= \{f\circ\rho_{\mathcal{D}^*}, H_c\}_{\mathcal{D}^*}\\ &= \left\{f\circ\rho_{\mathcal{D}^*}, \overline{H_c}\circ\rho_{\mathcal{D}^*}\right\}_{\mathcal{D}^*}\\ &= \left\{f, \overline{H_c}\right\}_{\overline{\mathcal{D}^*}}\circ\rho_{\mathcal{D}^*}\\ &= X_{\overline{H_c}}(f)\circ\rho_{\mathcal{D}^*},\end{aligned}$$

for every $f \in C^\infty(\overline{\mathcal{D}^*})$. The quotient map $\rho_{\mathcal{D}^*}$ is a surjective sumbersion therefore $\overline{X^*_{nh}} = X_{\overline{H_c}}$. ∎

As a consequence, the dynamics in $\overline{\mathcal{D}^*}$ can be written in matrix form

$$\begin{pmatrix} \dot{r}^m \\ \dot{p}_a \end{pmatrix} = \begin{pmatrix} 0 & B_{mb} \\ -B_{ak} & -C^c_{ab} p_c \end{pmatrix} \begin{pmatrix} \frac{\partial \overline{H_c}}{\partial q^k} \\ \frac{\partial \overline{H_c}}{\partial p_b} \end{pmatrix}. \tag{2.16}$$

2.5 Symmetries and momenta

Along this Section we assume there is a free and proper action Ψ of a Lie group G on Q (for a more general exposition see [11], where the authors consider non free actions). We focus on two aspects related with the presence of symmetries, on one hand on the existence of first integrals and on the other hand reduction. Here only the former is treated and the latter has been already presented in Sections 1.5 and 2.4 using the almost-Poisson formulation of the dynamics.

In this scenario the fiber derivative $\mathbb{F}L : TQ \to T^*Q$ is a linear bundle isomorphism which induces the isomorphism $\mathbb{F}L_0|_{\mathcal{D}}$ between $\mathcal{D}$ and its dual $\mathcal{D}^*$. First we give the particular case of Proposition 1.10

Proposition 2.8. *[2] Two vector fields Z_1, Z_2 on Q define the same linear function, $p_{Z_1}\big|_{\mathcal{D}}$, $p_{Z_2}\big|_{\mathcal{D}}$, on $\mathcal{D}$ if and only if*

$$Z_1 - Z_2 \in \Gamma(\mathcal{D}^\perp).$$

Using Proposition 1.9 we get the following result.

Theorem 2.9. *Suppose $Z \in \mathfrak{X}(Q)$ generates a first integral of the unconstrained system. Then, $p_Z|_{\mathcal{D}}$ is a first integral of X_{nh} if and only if $Z \in R^\circ$.*

Proof. If $Z \in \mathfrak{X}(Q)$ generates a first integral of the unconstrained system then $Z^{TQ}[L] = 0$, then by Proposition 1.9 the result follows. □

Observation 2.5.1. We observe that if $Z \in \mathfrak{X}(Q)$ generates a first integral $p_Z|_{\mathcal{D}}$ of X_{nh}, then by Proposition 1.10 the orthogonal projection $Z_{\mathcal{D}} \in \Gamma(\mathcal{D})$ of Z onto $\mathcal{D}$, is such that $p_Z|_{\mathcal{D}} = (p_{Z_{\mathcal{D}}})|_{\mathcal{D}}$. This mean that all first integrals linear in the velocities admit a generator which is horizontal, i.e. by sections of the distribution $\mathcal{D}$.

A special instance of Theorem 2.9 is the following result, which sometimes is referred to as the Nonholonomic Noether theorem see [11, 14, 82, 54, 55]

Theorem 2.10 (Nonholonomic Noether Theorem)**.** *Consider a nonholonomic systems $(Q, L_0, \mathcal{D})$ with a G-symmetry. If there exists $\xi \in \mathfrak{g}$ such that $\xi^Q \in \Gamma(\mathcal{D})$, then $p_\xi|_{\mathcal{D}}$ is a first integral of (2.2).*

We stress the fact that in the case the intersection $\Gamma(\mathcal{D}) \cap \Gamma(TOrb)$ is just the zero section, then there are no conserved quantities of such kind. In [8] the authors extend previous results proving existence theorems for the so called horizontal gauge momenta. The following results help us to state similar existence theorems in Section 3.5. We present here such results without a proof.

Definition 2.3. Let Q be a smooth manifold endowed with a riemannian metric g and a smooth distribution S on Q. The metric g is said to be strong S-invariant if the following relation holds for all $Y_i \in \Gamma(S)$, $i = 1, 2, 3$.

$$\langle Y_1, [Y_2, Y_3] \rangle_g = -\langle Y_3, [Y_2, Y_1] \rangle_g.$$

Definition 2.4. Let Q be a smooth manifold and $\mathcal{D}, S \subset TQ$ be smooth distributions. A distribution $H \subseteq \mathcal{D}$ is said to be S-orthogonal if $H = S^\perp \cap \mathcal{D}$.

Definition 2.5. Let Q be a smooth manifold, $\mathcal{D} \subseteq TQ$ a distribution and G a Lie groups acting on Q. Consider the distribution $S \subset TQ$ constructed pointwise by $S_q := \mathcal{D}_q \cap \mathcal{V}_q$, where $\mathcal{V}$ is the vertical space of the action. The distribution $\mathfrak{g}_S \subset Q \times \mathfrak{g}$ is defined by

$$(\mathfrak{g}_S)_q := \{\xi \in \mathfrak{g} \,:\, \xi_Q(q) \in S_q\}\,.$$

Definition 2.6. We say that a nonholonomic system $(Q, L, \mathcal{D})$ with a symmetry group G satisfies conditions $\mathcal{A}$ if

1. The action of G is free and proper.
2. The dimension assumption is satisfied.
3. The bundle $\mathfrak{g}_S \to Q$ is trivial.
4. The dimension of the quotient space Q/G is 1.

Theorem 2.11 ([8])**.** *Consider a nonholonomic system $(Q, L, \mathcal{D})$ with a Lie group G acting on it. Suppose conditions $\mathcal{A}$ are satisfied and with a S-orthogonal horizontal space H. Even more assume that the kinetic energy metric is strong invariant on S and that*

$$\langle X, [Z, X]\rangle_g = 0,$$

with X a G-equivariant vector field on Q, which is a section of H, and for all $Z \in \Gamma(S)$. Then there exist $k = rank(S)$ first integrals of X_{nh} which are G-invariant horizontal gauge momenta, additionally they are functionally independent.

Chapter 3

Linear constrained systems with gyroscopic type lagrangian

This Chapter is based on a project in collaboration with J. C. Marrero, D. Martín de Diego and L. García-Naranjo. The scope of the Chapter is to investigate the geometric properties of nonholonomic systems with linear constraints and gyroscopic lagrangians. For our understanding this type of systems, also with holonomic constraints, serve to model and control a rigid body with rotors [100, 22, 12, 61], or a rigid body under the influence of a magnetic field [20, 102, 48]. Nevertheless in the nonholonomic scenario their geometric aspects, for e.g. the almost-Poisson formulation, had not been fully explored. To our knowledge the general Dirac structure [71] has not been particularized and analyzed in this case.

3.1 Local Lagrange-d'Alembert equations

Let Q be a given n dimensional smooth manifold and consider lifted coordinates $(q, \dot{q})$ in its tangent bundle TQ. Using the same notation as in (1.11), consider a gyroscopic lagrangian L [1]

$$L = T + \gamma^{\ell} - V, \quad \text{in coordinates} \quad L(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q} + \gamma(q) \cdot \dot{q} - V(q). \tag{3.1}$$

The constraint submanifold $\mathcal{M} = \mathcal{D}$ is a linear subbundle of TQ. In bundle coordinates $(q, \dot{q})$ the constraints equations are

$$S(q)\dot{q} = 0,$$

where the matrix S on Q is such that $\mathcal{D}_q = \ker S(q)$ as in (1.12), that is the fibers of $\mathcal{D}$ are given point by point as the kernel of S at that point. Then the Lagrange-d'Alembert equations in coordinates are

$$\left.\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\right|_{\mathcal{D}} = S^T(SA^{-1}S^T)^{-1}(SA^{-1}\eta - \sigma)|_{\mathcal{D}}. \tag{3.2}$$

Where $\eta_i = \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j - \frac{\partial L}{\partial q^i}, \quad i = 1, ..., n$ and $\sigma_a = \frac{\partial S_{aj}}{\partial q^i}\dot{q}^i\dot{q}^j, \quad a = 1, ..., n-r.$

[1] Recall that T is the kinetic energy, γ is the 1-form which corresponds to the gyroscopic energy and V is the potential energy.

Proposition 1.3 ensures that the lagrangian energy restricted to $\mathcal{D}$ $E_L|_{\mathcal{D}}$ (see Definition 1.2) is a constant of motion.

3.2 Quasi-velocities and Hamel-d'Alembert equations

We give here coordinate expressions of Hamel equations. Consider the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^{\perp}$ of the tangent bundle induced by the kinetic energy metric g. Let $\{X_i\}$ be a local frame of $\mathfrak{X}(Q)$ adapted to the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^{\perp}$, i.e. $X_a \in \Gamma(\mathcal{D})$ and $X_\alpha \in \Gamma(\mathcal{D}^{\perp})$, and $v = (v^1, ..., v^n)$ be the fiber coordinates induced by this frame. The kinetic energy matrix A in this coordinates is

$$BAB^t = \begin{pmatrix} A_{\mathcal{D}} & 0 \\ 0 & A_{\mathcal{D}^{\perp}} \end{pmatrix}.$$

where B is the frame change matrix as in (1.19) and $A_{\mathcal{D}}$, $A_{\mathcal{D}^{\perp}}$ are the block matrices corresponding to the restriction of the metric g to $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. The lagrangian $\tilde{L}$ in quasi-velocities writes as

$$\tilde{L}(q,v) = \frac{1}{2} v_{\mathcal{D}} \cdot A_{\mathcal{D}}(q) v_{\mathcal{D}} + \frac{1}{2} v_{\mathcal{D}^{\perp}} \cdot A_{\mathcal{D}^{\perp}}(q) v_{\mathcal{D}^{\perp}} + \gamma_{\mathcal{D}} \cdot v_{\mathcal{D}} + \gamma^{\perp} \cdot v_{\mathcal{D}^{\perp}} - V(q),$$

where $v = v_{\mathcal{D}} + v_{\mathcal{D}^{\perp}}$ is the decomposition of v associated to $TQ = \mathcal{D} \oplus \mathcal{D}^{\perp}$, $\gamma_{\mathcal{D}} := P^*_{\mathcal{D}} \circ i^*_{\mathcal{D}} \circ \gamma$ and $\gamma^{\perp} := P^*_{\mathcal{D}^{\perp}} \circ i^*_{\mathcal{D}^{\perp}} \circ \gamma$ (the dual morphisms $P^*_{\mathcal{D}}, i^*_{\mathcal{D}}, P^*_{\mathcal{D}^{\perp}}$ and $i^*_{\mathcal{D}^{\perp}}$ are defined in (1.6)). The constraints equations in quasi-velocities adapted to the constraints become $v_{\mathcal{D}^{\perp}} = 0$ or equivalently $v^\alpha = 0, \quad \alpha = r+1, ..., n$.

Following equations (1.23) the Hamel-d'Alembert equations are

$$\begin{aligned} &\frac{dq^i}{dt} = B_{ai} v^a, \quad a,b,c = 1,...,r, \\ &\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^a} = B_{ak}\frac{\partial \tilde{L}}{\partial q^k} - C^c_{ab}\frac{\partial \tilde{L}}{\partial v^c} v^b - C^\alpha_{ab}\frac{\partial \tilde{L}}{\partial v^\alpha} v^b, \\ &v^\alpha = 0, \quad \alpha = r+1,...,n. \end{aligned} \tag{3.3}$$

Let $L_c := \tilde{L}|_{\mathcal{D}}$ be the lagrangian restricted to $\mathcal{D}$, in coordinates

$$L_c(q^i, v^a) = \frac{1}{2} g_{ab} v^a v^b + \gamma_a v^a - V(q).$$

Then equations of motion (3.3) in $\mathcal{D}$ take the following expression

$$\begin{aligned} &\frac{dq^i}{dt} = B_{ai} v^a, \\ &\frac{d}{dt}\frac{\partial L_c}{\partial v^a} = B_{ak}\frac{\partial L_c}{\partial q^k} - C^c_{ab}\frac{\partial L_c}{\partial v^c} v^b - C^\alpha_{ab}\gamma_\alpha v^b, \quad a,b,c = 1,...,r, \; \alpha = r+1,...,n. \end{aligned} \tag{3.4}$$

3.3 Almost hamiltonian formulation

In this Section we introduce and give a precise description of the *almost*-Poisson bracket in $\mathcal{D}^*$ compatible with the dynamics for a nonholonomic system with linear constraints and gyroscopic type lagrangian. We decide to use $\mathcal{D}^*$ as phase space instead of the affine bundle $\mathbb{F}L(\mathcal{D})$ because the latter is in general an affine subbundle of T^*Q, as a consequence the almost-Poisson bracket in $\mathcal{D}^*$ recovers the affine character. As already done in the previous Chapter we use elements related to the kinetic energy metric. To our understanding this particular description is new.

3.3.1 Equivalence between $\mathbb{F}L(\mathcal{D})$ and $\mathcal{D}^*$ and dynamics in $\mathcal{D}^*$

In Subsection 2.3.1 the type of nonholonomic systems considered helped to relate the fiber bundles $\mathbb{F}L(\mathcal{D})$ and $\mathcal{D}^*$. In the case of a nonholonomic system with gyroscopic lagrangian the fiber bundle $\mathbb{F}L(\mathcal{D})$ is affine, nevertheless the vector bundle for which $\mathbb{F}L(\mathcal{D})$ is modeled over is the annihilator distribution $(\mathcal{D}^\perp)^\circ$ of $\mathcal{D}^\perp$ and it is isomorphic to $\mathcal{D}^*$. We use this fact to construct an isomorphism between $\mathbb{F}L(\mathcal{D})$ and $\mathcal{D}^*$. First note that[2] $\mathbb{F}L(u_q) = \flat_g(u_q) + \gamma_q$, for all $u_q \in T_qQ$ therefore the fibers of the affine bundle $\mathbb{F}L(\mathcal{D})$ are

$$\mathbb{F}L(\mathcal{D})_q = \left\{\flat_g(u_q) + \gamma_q \in T_q^*Q \,:\, u_q \in \mathcal{D}_q\right\},$$

and we can then write

$$\mathbb{F}L(\mathcal{D}) = (\mathcal{D}^\perp)^\circ + \gamma.$$

Thanks to to the splitting $T^*Q = (\mathcal{D}^\perp)^\circ \oplus \mathcal{D}^\circ$ of the cotangent bundle of Q, we can write $\gamma = \gamma_{\mathcal{D}} + \gamma^\perp$, and since $\gamma_{\mathcal{D}} \in \Gamma(\mathcal{D}^\perp)^\circ$, then we can refine the previous expression of $\mathbb{F}L(\mathcal{D})$ to

$$\mathbb{F}L(\mathcal{D}) = (\mathcal{D}^\perp)^\circ + \gamma^\perp.$$

The 1-form $\gamma^\perp \in \Gamma(\mathcal{D}^\circ)$ plays an important role on the affine nature in the (almost) hamiltonian formulation of the dynamics as we see later on this Section.

Using the dual morphisms (1.6), we define the affine bundle morphism between $\mathbb{F}L(\mathcal{D})$ and $\mathcal{D}^*$ by

$$\Phi := i_{\mathcal{D}}^*|_{\mathbb{F}L(\mathcal{D})} : \mathbb{F}L(\mathcal{D}) \longrightarrow \mathcal{D}^*. \tag{3.5}$$

Note that Φ is clearly an affine bundle morphism, since $i_{\mathcal{D}}^*$ is a vector bundle transformation and $\mathbb{F}L(\mathcal{D})$ is an affine subbundle of the cotangent bundle T^*Q. The expression of Φ in coordinates (q^i, p_a, p_α) adapted to the cotangent bundle splitting $T^*Q = (\mathcal{D}^\perp)^\circ \oplus \mathcal{D}^\circ$ is

$$\Phi(q^i, p_a, p_\alpha + \gamma_\alpha) = i_{\mathcal{D}}^*(q^i, p_a, p_\alpha + \gamma_\alpha) = (q^i, p_a).$$

Proposition 3.1. *The affine bundle morphism $\Phi : \mathbb{F}L(\mathcal{D}) \to \mathcal{D}^*$ is in fact an isomorphism with inverse*

$$\Phi^{-1} = P_{\mathcal{D}}^* + \gamma^\perp \circ \pi_{\mathcal{D}^*}.$$

Proof. Recall the following identities $(P_{\mathcal{D}} \circ i_{\mathcal{D}})^* = Id_{\mathcal{D}^*}$ and $(P_{\mathcal{D}^\perp} \circ i_{\mathcal{D}})^* = 0$. Let $\beta_q \in \mathcal{D}_q^*$

$$\begin{aligned}\Phi(P_{\mathcal{D}}^*(\beta_q) + \gamma^\perp(q)) &= (i_{\mathcal{D}}^* \circ P_{\mathcal{D}}^*)(\beta_q) + i_{\mathcal{D}}^*(\gamma^\perp(q)) \\ &= \beta_q + i_{\mathcal{D}}^*\,(P_{\mathcal{D}^\perp}^*(i_{\mathcal{D}^\perp}^*(\gamma_q))) \\ &= \beta_q.\end{aligned}$$

Now let $w_q \in \mathcal{D}_q$, then $\flat_g(w_q) + \gamma^\perp(q) \in \mathbb{F}L(\mathcal{D})$

$$\begin{aligned}P_{\mathcal{D}}^*\left(\Phi(\flat_g(w_q) + \gamma^\perp(q))\right) + \gamma^\perp(q) &= (P_{\mathcal{D}}^* \circ i_{\mathcal{D}}^*)(\flat_g(w_q)) + (P_{\mathcal{D}}^* \circ i_{\mathcal{D}}^*)(\gamma^\perp(q)) + \gamma^\perp(q) \\ &= \flat_g(w_q) + \gamma^\perp(q),\end{aligned}$$

since $\gamma^\perp = (P_{\mathcal{D}^\perp}^* \circ i_{\mathcal{D}^\perp}^*)(\gamma)$. Therefore $\Phi^{-1} = P_{\mathcal{D}}^* + \gamma^\perp \circ \pi_{\mathcal{D}^*}$, and it is clearly an affine bundle morphism. □

Using Φ we define the isomorphism between $\mathcal{D}$ and $\mathcal{D}^*$

$$\mathbb{F}L_c := \Phi \circ \mathbb{F}L \circ i_{\mathcal{D}} = \mathcal{D} \to \mathcal{D}^*, \tag{3.6}$$

equivalently $\mathbb{F}L_c = i_{\mathcal{D}}^* \circ \mathbb{F}L \circ i_{\mathcal{D}}$.

[2] Recall that $\flat_g : TQ \to T^*Q$ is the musical isomorphism related to the kinetic energy metric g, $\flat_g(u_q)(v_q) = \langle u_q, v_q\rangle_g$, for all $u_q, v_q \in T_qQ$.

Observation 3.3.1. Actually, if we consider the fiber derivative of the constrained lagrangian $L_c : \mathcal{D} \to \mathbb{R}$, it coincides with $\mathbb{F}L_c$.

In coordinates $\mathbb{F}L_c$ writes

$$\mathbb{F}L_c(q^i, v^a) = (q^i, A_{\mathcal{D}} v_{\mathcal{D}} + \gamma_{\mathcal{D}}) = (q^i, g_{ab} v^b + \gamma_a),$$

and its inverse reads

$$(\mathbb{F}L_c)^{-1}(q^i, p_a) = (q^i, A_{\mathcal{D}}^{-1}(p - \gamma_{\mathcal{D}})) = (q^i, g^{ab}(p_b - \gamma_b)).$$

We now define the constrained hamiltonian $H_c : \mathcal{D}^* \to \mathbb{R}$, and the dynamic equations X^*_{nh} in D^*:

$$H_c := E_L \circ i_{\mathcal{D}} \circ (\mathbb{F}L_c)^{-1}, \qquad X^*_{nh} := \mathbb{F}L_{c*}(X_{nh}).$$

In order to compute the expressions in coordinates we first note

$$\frac{\partial H_c}{\partial q^i} = -\frac{\partial L_c}{\partial q^i}, \qquad \frac{\partial H_c}{\partial p_a} = v^a.$$

Let $p = (p_1, ..., p_r)$ and $\gamma_{\mathcal{D}} = (\gamma_1, ..., \gamma_r)$, then we have

$$\begin{aligned} H_c(q^i, p_a) &= \frac{1}{2} p \cdot A_{\mathcal{D}}^{-1} p - \gamma_{\mathcal{D}} \cdot A_{\mathcal{D}}^{-1} p + \frac{1}{2}\gamma_{\mathcal{D}} \cdot A_{\mathcal{D}}^{-1}\gamma_{\mathcal{D}} + V(q), \\ X^*_{nh}(q^i, p_a) &= B_{ai}\frac{\partial H_c}{\partial p_a}\frac{\partial}{\partial q^i} - \left(B_{aj}\frac{\partial H_c}{\partial q^j} + (C^c_{ab} p_c + C^\alpha_{ab}\gamma_\alpha)\frac{\partial H_c}{\partial p_b}\right)\frac{\partial}{\partial p_a}. \end{aligned} \tag{3.7}$$

Furthermore, we relate the constrained and unconstrained dynamics with the elements just introduced.

Proposition 3.2. *Let $H = E_L \circ \mathbb{F}L^{-1} : T^*Q \to \mathbb{R}$ be the hamiltonian, and X_H, $X_{H_c \circ i^*_{\mathcal{D}}} \in \mathfrak{X}(T^*Q)$ the hamiltonian vector fields of H and $H_c \circ i^*_{\mathcal{D}}$ respectively, then*

1. $H_c \circ \Phi = H|_{\mathbb{F}L(\mathcal{D})}$,
2. $X^*_{nh} \circ \Phi = Ti^*_{\mathcal{D}} \circ X_H|_{\mathbb{F}L(\mathcal{D})}$,
3. $X^*_{nh} \circ \Phi = Ti^*_{\mathcal{D}} \circ X_{H_c \circ i^*_{\mathcal{D}}}|_{\mathbb{F}L(\mathcal{D})}$.

Even more, $X_H|_{\mathbb{F}L(\mathcal{D})} = X_{H_c \circ i^*_{\mathcal{D}}}|_{\mathbb{F}L(\mathcal{D})}$.

Proof. 1. Follows from the next equalities, $\Phi = i^*_{\mathcal{D}}|_{\mathbb{F}L(\mathcal{D})} = i^*_{\mathcal{D}} \circ (\mathbb{F}L \circ i_{\mathcal{D}})$, $H|_{\mathbb{F}L(\mathcal{D})} = H \circ (\mathbb{F}L \circ i_{\mathcal{D}})$ and $H_c = E_L \circ i_{\mathcal{D}} \circ (i^*_{\mathcal{D}} \circ \mathbb{F}L \circ i_{\mathcal{D}})^{-1}$, then we have $H_c \circ \Phi = E_L \circ i_{\mathcal{D}}$ and $H|_{\mathbb{F}L(\mathcal{D})} = E_L \circ \mathbb{F}L^{-1} \circ (\mathbb{F}L \circ i_{\mathcal{D}}) = E_L \circ i_{\mathcal{D}}$.

We do the proofs of 2. and 3. in adapted coordinates (q^i, p_a, p_α) of T^*Q. The hamiltonian vector fields X_H, $X_{H_c \circ i^*_{\mathcal{D}}}$ in such coordinates have the following expressions (see Appendix B)

$$\begin{aligned} X_H &= B_{ji}\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^i} - \left(B_{ij}\frac{\partial H}{\partial q^j} + C^k_{ij}\frac{\partial H}{\partial p_j} p_k\right)\frac{\partial}{\partial p_i}, \quad i, j, k = 1, ..., n, \quad a = 1, ..., r, \\ X_{H_c \circ i^*_{\mathcal{D}}} &= B_{ai}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_a}\frac{\partial}{\partial q^i} - \left(B_{ij}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial q^j} + C^k_{ia}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_a} p_k\right)\frac{\partial}{\partial p_i}. \end{aligned}$$

Let $\beta_q = (q^i, p_a, \gamma_\alpha) \in \mathbb{F}L(\mathcal{D})$, from the coordinate expressions of H and $H_c \circ i^*_{\mathcal{D}}$ we note that $\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_\alpha} = 0$, $\frac{\partial H}{\partial p_\alpha}(\beta_q) = 0$, $\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_a} = \frac{\partial H}{\partial p_a}$, and $\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial q^j}(\beta_q) = \frac{\partial H}{\partial q^j}(\beta_q)$, then

$$\begin{aligned} X_H(\beta_q) &= X_{H_c \circ i^*_{\mathcal{D}}}(\beta_q) \\ &= B_{ai}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_a}\frac{\partial}{\partial q^i} - \left(B_{ij}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial q^j} + C^c_{ib}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_b} p_c + C^\alpha_{ib}\frac{\partial H_c \circ i^*_{\mathcal{D}}}{\partial p_b}\gamma_\alpha\right)\frac{\partial}{\partial p_i}. \end{aligned}$$

Since the projector $i^*_{\mathcal{D}}$ has the coordinate expression $i^*_{\mathcal{D}}(q^j, p_a, p_\alpha) = (q^j, p_a)$, we can conclude

$$T_{\beta_q} i^*_{\mathcal{D}}(X_{H_c \circ i^*_{\mathcal{D}}}(\beta_q)) = B_{ai}\frac{\partial H_c}{\partial p_a}\frac{\partial}{\partial q^i} - \left(B_{aj}\frac{\partial H_c}{\partial q^j} + (C^c_{ab}p_c + C^\alpha_{ab}\gamma_\alpha)\frac{\partial H_c}{\partial p_b}\right)\frac{\partial}{\partial p_a},$$

which is precisely the expression of $X_n h^*(\Phi(\beta_q))$, see equation (3.7). □

3.3.2 Affine almost-Poisson bracket in $\mathcal{D}^*$

To extend to the gyroscopic case the construction of the almost-Poisson bracket in $\mathcal{D}^*$, we need the canonical Poisson bracket $\{\cdot,\cdot\}$ in T^*Q.

Definition 3.1. The map $\{\cdot,\cdot\}_{\mathcal{D}^*} : C^\infty(\mathcal{D}^*) \times C^\infty(\mathcal{D}^*) \to C^\infty(\mathcal{D}^*)$ defined by

$$\{F_1, F_2\}_{\mathcal{D}^*} = \{F_1 \circ i^*_{\mathcal{D}}, F_2 \circ i^*_{\mathcal{D}}\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi^*_{\mathcal{D}}), \quad F_1, F_2 \in C^\infty(\mathcal{D}^*).$$

defines a bracket in $\mathcal{D}^*$.

Observation 3.3.2. If $\gamma^\perp = 0$, then we recover the bracket introduced in Definition 2.1, so this affine bracket is truly a generalization of the bracket defined in Section 2.3.

As in Section 2.3 we now compute the expressions for basic and linear (in the momenta) functions, to prove basic properties and give coordinate description it is sufficient to do so because of the vector bundle structure of $\mathcal{D}^*$ and the linearity of the Poisson bracket in T^*Q.

Proposition 3.3. *Let $X^\ell, Y^\ell \in C^\infty(\mathcal{D}^*)$ and $f, h \in C^\infty(Q)$. Then*

$$\{X^\ell, Y^\ell\}_{\mathcal{D}^*} = -(P_{\mathcal{D}}[X,Y])^\ell + d\gamma^\perp(X,Y), \quad \{X^\ell, f\circ\pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} = -X(f)\circ\pi_{\mathcal{D}^*},$$
$$\{f\circ\pi_{\mathcal{D}^*}, h\circ\pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} = 0.$$

Proof. Recall observations made in the proof of Proposition 2.1, using Definition 3.1 we obtain

$$\begin{aligned}
\{X^\ell, Y^\ell\}_{\mathcal{D}^*} &= \left\{X^\ell \circ i^*_{\mathcal{D}}, Y^\ell \circ i^*_{\mathcal{D}}\right\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= \left\{(i_{\mathcal{D}} \circ X)^\ell, (i_{\mathcal{D}} \circ Y)^\ell\right\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= -\left[(i_{\mathcal{D}} \circ X), (i_{\mathcal{D}} \circ Y)\right]^\ell \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= -(P_{\mathcal{D}}\left[(i_{\mathcal{D}} \circ X), (i_{\mathcal{D}} \circ Y)\right])^\ell - \gamma^\perp\left[(i_{\mathcal{D}} \circ X), (i_{\mathcal{D}} \circ Y)\right],
\end{aligned}$$

using Cartan's magic formula and $\gamma^\perp(i_{\mathcal{D}}(X)) = \gamma^\perp(i_{\mathcal{D}}(Y)) = 0$ we get $d\gamma^\perp(X,Y) = -\gamma^\perp[X,Y]$, then the result follows.

Similarly, for $X \in \Gamma(\mathcal{D})$ and $f \in C^\infty(Q)$ we have

$$\begin{aligned}
\{X^\ell, f\circ\pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} &= \left\{X^\ell \circ i^*_{\mathcal{D}}, f\circ\pi_{\mathcal{D}^*}\circ i^*_{\mathcal{D}}\right\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= \left\{(i_{\mathcal{D}} \circ X)^\ell, f\circ\pi_Q\right\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= -(i_{\mathcal{D}} \circ X)(f)\circ\pi_Q \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= -(i_{\mathcal{D}} \circ X)(f)\circ\pi_{\mathcal{D}^*} \\
&= -X(f)\circ\pi_{\mathcal{D}^*}.
\end{aligned}$$

At last, a simple computation shows for $f, h \in C^\infty(Q)$

$$\begin{aligned}
\{f\circ\pi_{\mathcal{D}^*}, h\circ\pi_{\mathcal{D}^*}\}_{\mathcal{D}^*} &= \{f\circ\pi_{\mathcal{D}^*}\circ i^*_{\mathcal{D}}, h\circ\pi_{\mathcal{D}^*}\circ i^*_{\mathcal{D}}\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= \{f\circ\pi_Q, h\circ\pi_Q\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= 0.
\end{aligned}$$

□

In the following we prove two properties about the bracket defined in 3.1: it is an almost-Poisson bracket and the vector field X^*_{nh} is compatible with the dynamics, that is the (almost) hamiltonian vector field associated to the restricted energy H_c (Theorem 3.5).

Proposition 3.4. *The affine bracket $\{\cdot,\cdot\}_{\mathcal{D}^*}$ has the following properties*

1. *It is $\mathbb{R}$-bilinear and skew-symmetric.*
2. *Leibniz rule is satisfied on each entry.*
3. *Jacobi identity holds if and only if the constraint distribution $\mathcal{D}$ is involutive.*
4. *The bracket in $\mathcal{D}^*$ is affine, and it is linear if and only if $d\gamma^\perp(X,Y)=0$, for all $X,Y\in\Gamma(\mathcal{D})$.*

Proof. 1. and 2. follow directly from the bracket's definition, $i^*_{\mathcal{D}}\circ P^*_{\mathcal{D}} = Id_{\mathcal{D}^*}$ and $\pi_{\mathcal{D}^*} = \pi_Q\circ(P^*_{\mathcal{D}}+\gamma^\perp\circ\pi_{\mathcal{D}^*})$.

To prove 3., let $Y,Z\in\Gamma(\mathcal{D})$, and $X_{Y^\ell}=\{\cdot,Y\}_{\mathcal{D}^*}$ be the hamiltonian vector field related to the function $Y^\ell\in C^\infty(\mathcal{D}^*)$. Then, by Proposition 3.3 for every $f,g\in C^\infty(Q)$ we get

$$\begin{aligned}&X_{g\circ\pi_{\mathcal{D}^*}}(f\circ\pi_{\mathcal{D}^*})=0,\\&X_{Y^\ell}(f\circ\pi_{\mathcal{D}^*})=-Y(f)\circ\pi_{\mathcal{D}^*},\end{aligned}$$

the second equation is equivalent to say that the vector fields X_{Y^ℓ} and Y are $\pi_{\mathcal{D}^*}$-related, if $Z\in\Gamma(\mathcal{D})$ this imply

$$T\pi_{\mathcal{D}^*}\left[X_{Y^\ell},X_{Z^\ell}\right]=[Y,Z]\circ\pi_{\mathcal{D}^*}.$$

Now we get

$$\begin{aligned}X_{\{Y^\ell,Z^\ell\}_{\mathcal{D}^*}}&=X_{-(P_{\mathcal{D}}[Y,Z])^\ell+d\gamma^\perp(Y,Z)\circ\pi_{\mathcal{D}^*}}\\&=X_{-(P_{\mathcal{D}}[Y,Z])^\ell}+X_{d\gamma^\perp(Y,Z)\circ\pi_{\mathcal{D}^*}},\end{aligned}$$

then $X_{\{Y^\ell,Z^\ell\}_{\mathcal{D}^*}}(f\circ\pi_{\mathcal{D}^*})=X_{-(P_{\mathcal{D}}[Y,Z])^\ell}(f\circ\pi_{\mathcal{D}^*})=-P_{\mathcal{D}}\left[Y,Z\right](f)\circ\pi_{\mathcal{D}^*}$

Suppose Jacobi identity is satisfied, then $X_{\{Y^\ell,Z^\ell\}_{\mathcal{D}^*}}=[Y,Z]$ so $[X_{Y^\ell},X_{Z^\ell}]$ and $P_{\mathcal{D}}\left[Y,Z\right]$ are $\pi_{\mathcal{D}^*}$-related therefore $[Y,Z]=P_{\mathcal{D}}\left[Y,Z\right]$.

If $\mathcal{D}$ is involutive then $[Y,Z]=P_{\mathcal{D}}\left[Y,Z\right]$, which implies $d\gamma^\perp([Y,Z])=0$ using proposition 3.3 Jacobi identity follows.

Finally, the statement of 4. means that if $F,K\in C^\infty(\mathcal{D}^*)$ are affine functions then so its bracket $\{F,K\}_{\mathcal{D}^*}$ is. If F,K are affine functions then there exist vector fields $Y,Z\in\Gamma(\mathcal{D})$ and functions $f,k\in C^\infty(Q)$ such that $F=Y^\ell+f\pi_{\mathcal{D}^*}$ and $K=Z^\ell+k\pi_{\mathcal{D}^*}$, using Proposition 3.3 and 1) we get

$$\begin{aligned}\{F,K\}_{\mathcal{D}^*}&=\left\{Y^\ell+f\circ\pi_{\mathcal{D}^*},Z^\ell+k\circ\pi_{\mathcal{D}^*}\right\}_{\mathcal{D}^*}\\&=\left\{Y^\ell,Z^\ell\right\}_{\mathcal{D}^*}+\left\{Y^\ell,k\circ\pi_{\mathcal{D}^*}\right\}_{\mathcal{D}^*}+\left\{f\circ\pi_{\mathcal{D}^*},Z^\ell\right\}_{\mathcal{D}^*}\\&=-(P_{\mathcal{D}}\left[Y,Z\right])^\ell+d\gamma^\perp(Y,Z)\circ\pi_{\mathcal{D}^*}-Y(k)\circ\pi_{\mathcal{D}^*}+Z(f)\circ\pi_{\mathcal{D}^*},\end{aligned}$$

which is clearly an affine function. If $d\gamma^\perp(Y,Z)=0$ for all $Y,Z\in\Gamma(\mathcal{D})$ then the same computation with $f=k=0$ proves the assertion. □

The coordinate expressions for the functions (q^i,p_a) are

$$\begin{aligned}&\left\{q^i,q^j\right\}_{\mathcal{D}^*}=0,\qquad\left\{q^i,p_a\right\}_{\mathcal{D}^*}=B_{ia},\\&\{p_a,p_b\}_{\mathcal{D}^*}=-C^c_{ab}p_c-C^\alpha_{ab}\gamma_\alpha,\end{aligned}\tag{3.8}$$

and the bivector field $\Pi_{\mathcal{D}^*}$ related to the almost-Poisson bracket in coordinates is

$$\Pi_{\mathcal{D}^*} = B_{ia}\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_a} - \frac{1}{2}(C^c_{ab}p_c + C^\alpha_{ab}\gamma_\alpha)\frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial p_b}. \tag{3.9}$$

At last, we relate the equations of motion in $\mathcal{D}^*$ and the hamiltonian vector field of H_c.

Theorem 3.5. *The equations of motion in $\mathcal{D}^*$ (3.7) are (almost) hamiltonian with respect to the almost-Poisson bracket in $\mathcal{D}^*$ and hamiltonian function H_c,*

$$X^*_{nh}(F) = \{F, H_c\}_{\mathcal{D}^*}, \quad \forall F \in C^\infty(\mathcal{D}^*).$$

Proof. Use Proposition 3.2 and $\Phi^{-1} = P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}$. Let $\alpha_q \in \mathcal{D}^*_q$ and $\beta_q \in \mathbb{F}L(\mathcal{D})$, such that $\alpha_q = \Phi(\beta_q)$

$$\begin{aligned}
X^*_{nh}(F)(\alpha_q) &= \langle dF(\alpha_q), X^*_{nh}(\alpha_q)\rangle \\
&= \langle dF(\alpha_q), T_{\beta_q} i^*_{\mathcal{D}} X_{H_c \circ i^*_{\mathcal{D}}}(\beta_q)\rangle \\
&= \langle (T_{\beta_q} i^*_{\mathcal{D}})^* dF(\alpha_q), X_{H_c \circ i^*_{\mathcal{D}}}(\beta_q)\rangle \\
&= \langle (T_{\beta_q} i^*_{\mathcal{D}})^* d(F \circ i^*_{\mathcal{D}})(\beta_q), X_{H_c \circ i^*_{\mathcal{D}}}(\beta_q)\rangle \\
&= X_{H_c \circ i^*_{\mathcal{D}}}(F \circ i^*_{\mathcal{D}})(\Phi^{-1}(\alpha)).
\end{aligned}$$

Since $X_{H_c \circ i^*_{\mathcal{D}}}$ is the hamiltonian vector field of the function $H_c \circ i^*_{\mathcal{D}}$, with respect to the canonical bracket in T^*Q, we conclude

$$\begin{aligned}
X^*_{nh}(F)(\alpha_q) &= X_{H_c \circ i^*_{\mathcal{D}}}(F \circ i^*_{\mathcal{D}})(\Phi^{-1}(\alpha)) \\
&= \{F \circ i^*_{\mathcal{D}}, H_c \circ i^*_{\mathcal{D}}\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*})(\alpha) \\
&= \{F, H_c\}_{\mathcal{D}^*}(\alpha_q),
\end{aligned}$$

where last equality is given by Definition 3.1. □

Theorem 3.5 implies that equations of motion (3.7) in $\mathcal{D}^*$ can be written in matrix form

$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_a \end{pmatrix} = \begin{pmatrix} 0 & B_{ib} \\ -B_{aj} & -C^c_{ab}p_c - C^\alpha_{ab}\gamma_\alpha \end{pmatrix} \begin{pmatrix} \frac{\partial H_c}{\partial q^j} \\ \frac{\partial H_c}{\partial p_b} \end{pmatrix}. \tag{3.10}$$

3.4 Symmetries and reduction of the (affine) almost-Poisson bracket

In Section 1.5, symmetries in nonholonomic systems are introduced. Along this Section we use them to reduce to the quotient space the almost-Poisson structure defined in the phase space $\mathcal{D}^*$.

Consider a nonholonomic system $(Q, L, \mathcal{D})$ with a G-symmetry, and let $\Psi : G\times Q \to Q$ be its action on Q. Denote by

$$\rho : Q \to \overline{Q} = Q/G,$$

the quotient map, which is the projection of a G-principal bundle. This G-action extends to TQ by the tangent lift, $T\Psi$ and since $\mathcal{D}$ is G-invariant, the lifted action restricts to $\mathcal{D}$. Let $T\Psi^*_g : T^*Q \to T^*Q$ denote the dual action on T^*Q, so we have the principal bundle $\rho_{T^*Q} : T^*Q \to \overline{T^*Q} = T^*Q/G$. Using the invariance of $\mathcal{D}$ and L, and hence of $\mathbb{F}L$, we also get that $\mathcal{D}^* \subset T^*Q$ is G-invariant, then we can consider the principal bundle $\rho_{\mathcal{D}^*} : \mathcal{D}^* \to \overline{\mathcal{D}^*} = \mathcal{D}^*/G$. Consequently the dynamical vector field $X^*_{nh} \in \mathfrak{X}(\mathcal{D}^*)$ can be projected to the vector field $\overline{X^*_{nh}} \in \mathfrak{X}(\overline{\mathcal{D}^*})$, by means of the quotient map $\rho_{\mathcal{D}^*}$. The vector bundle morphisms $i^*_{\mathcal{D}}$ and $P^*_{\mathcal{D}}$ are G-equivariant, since $\mathcal{D}$ and $\mathcal{D}^*$ are G-invariant, so they induce the following vector bundle morphisms

$$\overline{i^*_{\mathcal{D}}} : \overline{T^*Q} \to \overline{\mathcal{D}^*}, \quad \overline{P^*_{\mathcal{D}}} : \overline{\mathcal{D}^*} \to \overline{T^*Q}. \tag{3.11}$$

Proposition 1.6 opens the path to define a reduced bracket in $\overline{\mathcal{D}^*}$.

Definition 3.2. Let $\{\cdot,\cdot\}_{\overline{\mathcal{D}^*}} : C^\infty(\overline{\mathcal{D}^*}) \times C^\infty(\overline{\mathcal{D}^*}) \to C^\infty(\overline{\mathcal{D}^*})$ be the bracket in $\overline{\mathcal{D}^*}$ defined as

$$\{F_1, F_2\}_{\overline{\mathcal{D}^*}} = \left\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\right\}_{\overline{T^*Q}} \circ (\overline{P^*_{\mathcal{D}}} + \overline{\gamma^\perp} \circ \pi_{\overline{\mathcal{D}^*}}), \quad F_1, F_2 \in C^\infty(\overline{\mathcal{D}^*}).$$

On the other hand the nonholonomic bracket is G-invariant in the following sense, if $F_1, F_2 \in C^\infty(\mathcal{D}^*)$ are G-invariant functions then so $\{F_1, F_2\}_{\mathcal{D}^*}$ is. We conclude

Corollary 3.6. *The quotient map $\rho : \mathcal{D}^* \to \overline{\mathcal{D}^*}$ is an almost Poisson bracket transformation, i.e.*

$$\{F_1, F_2\}_{\overline{\mathcal{D}^*}} \circ \rho_{\mathcal{D}^*} = \{F_1 \circ \rho_{\mathcal{D}^*}, F_2 \circ \rho_{\mathcal{D}^*}\}_{\mathcal{D}^*}, \quad F_1, F_2 \in C^\infty\left(\overline{\mathcal{D}^*}\right). \tag{3.12}$$

Proof. To prove this claim we use the fact that the quotient map $\rho_{T^*Q} : T^*Q \to \overline{T^*Q}$ is a Poisson map (see Appendix B) and relations

$$\rho_{T^*Q} \circ P^*_{\mathcal{D}} = \overline{P^*_{\mathcal{D}}} \circ \rho_{\mathcal{D}^*}, \quad i^*_{\mathcal{D}} \circ \rho_{T^*Q} = \rho_{\mathcal{D}^*} \circ \overline{i^*_{\mathcal{D}}}, \quad \rho_{T^*Q} \circ \gamma^\perp = \overline{\gamma^\perp} \circ \rho_Q,$$

given by the equivariance of the morphisms $i^*_{\mathcal{D}}$, $P^*_{\mathcal{D}}$ and the 1-form $\gamma^\perp$. By definitions 3.1 and 3.2 we get

$$\begin{aligned}
\{F_1 \circ \rho_{\mathcal{D}^*}, F_2 \circ \rho_{\mathcal{D}^*}\}_{\mathcal{D}^*} &= \{F_1 \circ \rho_{\mathcal{D}^*} \circ i^*_{\mathcal{D}}, F_2 \circ \rho_{\mathcal{D}^*} \circ i^*_{\mathcal{D}}\} \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= \left(\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\}_{\overline{T^*Q}} \circ \rho_{T^*Q}\right) \circ (P^*_{\mathcal{D}} + \gamma^\perp \circ \pi_{\mathcal{D}^*}) \\
&= \left(\{F_1 \circ \overline{i^*_{\mathcal{D}}}, F_2 \circ \overline{i^*_{\mathcal{D}}}\}_{\overline{T^*Q}} \circ (\overline{P^*_{\mathcal{D}}} + \overline{\gamma^\perp} \circ \pi_{\overline{\mathcal{D}^*}})\right) \circ \rho_{\mathcal{D}^*} \\
&= \{F_1, F_2\}_{\overline{\mathcal{D}^*}} \circ \rho_{\mathcal{D}^*}
\end{aligned}$$

□

The bracket in $\overline{\mathcal{D}^*}$ inherits the following properties from those of the bracket in the dual bundle $\mathcal{D}^*$

Proposition 3.7. *The bracket $\{\cdot,\cdot\}_{\overline{\mathcal{D}^*}}$ has the following properties:*

1. *It is $\mathbb{R}$-bilinear and skew-symmetric.*

2. *Leibniz rule is satisfied on each entry.*

3. *The bracket is fiberwise linear if and only if $d\gamma^\perp(X, Y) = 0$, for all G-equivariant vector fields $X, Y \in \mathfrak{X}(Q)$.*

Proof. 1. is proved directly using Corollary 3.6 and Proposition 3.4. The computations to prove 2. are analogous to those performed in Proposition 2.5, with the extra relation $\overline{i^*_{\mathcal{D}}} \circ \overline{\gamma^\perp} = 0$.

3. follows from the fact that there is a bijection between equivariant sections of $\mathcal{D}$, $\Gamma(\mathcal{D})^G$, and sections of $\overline{Q} \to \overline{\mathcal{D}}$, $\Gamma(\overline{\mathcal{D}})$. Then $d\gamma^\perp(X, Y) = \overline{d\gamma^\perp}(X, Y) \circ \rho_Q$ and clearly $\{\cdot,\cdot\}_{\overline{\mathcal{D}^*}}$ is a linear bracket if and only if $\overline{d\gamma^\perp} = 0$. □

The next proposition is a consequence of Corollary 3.6 and Proposition 3.3, it gives expressions for the reduced bracket of fiberwise linear functions in $\overline{\mathcal{D}^*}$ and basic functions on $\overline{Q}$.

Proposition 3.8. *Let $X^\ell, Y^\ell \in C^\infty(\overline{\mathcal{D}^*})$ and $f, h \in C^\infty(\overline{Q})$. Then*

$$\{X^\ell, Y^\ell\}_{\overline{\mathcal{D}^*}} = -(\overline{P_{\mathcal{D}}}\,[X,Y])^\ell + \overline{d\gamma^\perp}(X,Y)\circ\pi_{\overline{\mathcal{D}^*}}, \quad \{X^\ell, f\circ\pi_{\overline{\mathcal{D}^*}}\}_{\overline{\mathcal{D}^*}} = -X(f)\circ\pi_{\overline{\mathcal{D}^*}},$$

$$\left\{f\circ\pi_{\overline{\mathcal{D}^*}}, h\circ\pi_{\overline{\mathcal{D}^*}}\right\}_{\overline{\mathcal{D}^*}} = 0.$$

To obtain a coordinate description of the reduced bracket we use adapted coordinates (x^d, y^u, p_a) to the fibration $\rho_{\mathcal{D}^*} : \mathcal{D}^* \to \overline{\mathcal{D}^*}$, that is $\rho_{\mathcal{D}^*}(x^d, y^u, p_a) = (x^d, p_a)$. We evaluate the coordinate functions x^d and p_a

$$\begin{aligned} \{x^d, x^k\}_{\overline{\mathcal{D}^*}} &= 0, \qquad \{x^d, p_a\}_{\overline{\mathcal{D}^*}} = B_{da}, \\ \{p_a, p_b\}_{\overline{\mathcal{D}^*}} &= -C^c_{ab}p_c - C^\alpha_{ab}\gamma_\alpha, \end{aligned} \tag{3.13}$$

Now we give an (almost) Poisson formulation for the reduced dynamics.

Theorem 3.9. 1. *The constrained hamiltonian H_c is $G-$invariant, so there exists $\overline{H_c} \in C^\infty(\overline{\mathcal{D}^*})$ such that $\overline{H_c}\circ\rho_{\mathcal{D}^*} = H_c$.*

2. *The reduced dynamics in $\overline{\mathcal{D}^*}$ are given by*

$$\overline{X^*_{nh}}(F) = \left\{F, \overline{H_c}\right\}_{\overline{\mathcal{D}^*}}, \quad F \in C^\infty(\overline{\mathcal{D}^*}).$$

Proof. The constrained hamiltonian H_c is G-invariant since L, $\mathbb{F}L_c$ are G-invariant functions, then there exists $\overline{H_c} : \overline{\mathcal{D}^*} \to \mathbb{R}$ such that $H_c = \overline{H_c}\circ\rho_{\mathcal{D}^*}$. Using Theorem 3.5, Corollary 3.6, and the fact that the vector fields X^*_{nh} and $\overline{X^*_{nh}}$ are $\rho_{\mathcal{D}^*}$-related we get

$$\begin{aligned} \overline{X^*_{nh}}(F)\circ\rho_{\mathcal{D}^*} &= X^*_{nh}(F\circ\rho_{\mathcal{D}^*}) \\ &= \{F\circ\rho_{\mathcal{D}^*}, H_c\}_{\mathcal{D}^*} \\ &= \left\{F\circ\rho_{\mathcal{D}^*}, \overline{H_c}\circ\rho_{\mathcal{D}^*}\right\}_{\mathcal{D}^*} \\ &= \left\{F, \overline{H_c}\right\}_{\overline{\mathcal{D}^*}}\circ\rho_{\mathcal{D}^*}, \end{aligned}$$

for every $F \in C^\infty(\overline{\mathcal{D}^*})$. Since the quotient map $\rho_{\mathcal{D}^*}$ is a surjective sumbersion we have proven the statement. ◻

A consequence of Theorem 3.9 the dynamics in $\overline{\mathcal{D}^*}$ can be written in matrix form as

$$\begin{pmatrix} \dot r^m \\ \dot p_a \end{pmatrix} = \begin{pmatrix} 0 & B_{mb} \\ -B_{ak} & -C^c_{ab}p_c - C^\alpha_{ab}\gamma_\alpha \end{pmatrix} \begin{pmatrix} \frac{\partial \overline{H_c}}{\partial q^k} \\ \frac{\partial \overline{H_c}}{\partial p_b} \end{pmatrix}. \tag{3.14}$$

3.5 Symmetries and affine momenta

In this Section we generalize, in a sense to be defined, what is done in Section 2.5, to this end we consider a family of nonholonomic systems $(Q, L_\nu, \mathcal{D})$ parametrized by a real parameter $\nu \in \mathbb{R}$, such that the lagrangian $L_\nu : TQ \to \mathbb{R}$ is defined as

$$L_\nu(v_q) = L_0(v_q + \nu N_q),$$

where $N \in \mathfrak{X}(Q)$ is any vector field on Q, and for $\nu = 0$ the lagrangian L_0 is natural, i.e. $L_0 = T - V\circ\tau_Q$. Observe that the lagrangian is of gyroscopic type except when

$\nu = 0$, in such case we have a nonholonomic system with linear constraints and natural lagrangian as in Chapter 2.

The procedure of generalization is the following: under the assumption that the nonholonomic system $(Q, L_0, \mathcal{D})$ admits a first integral which is a momentum generated by a vector field we show that under certain conditions such vector field properly deformed in a way that it generates a first integral for every $\nu \neq 0$. To present a precise mathematical statement we introduce the following notation.

- Denote by X^{ν}_{nh} the nonholonomic vector field in $\mathcal{D}$ of $(Q, L_\nu, \mathcal{D})$.
- Denote by $p^{\nu}_Z : TQ \to \mathbb{R}$ the momentum associated to a vector field Z on Q with respect to the lagrangian $L_\nu : TQ \to \mathbb{R}$.

Given $Z \in \mathfrak{X}(Q)$ such that $\mathcal{L}_{X^0_{nh}} p^0_Z = 0$, we look for a family of vector fields on Q Z_ν such that $\mathcal{L}_{X^{\nu}_{nh}} p^{\nu}_{Z_\nu} = 0$, $p^0_{Z_\nu} = p^0_Z$.

The generalization mentioned above goes as follows: let $v_q \in T_qQ$ then

$$L_\nu(v_q) = L_0(v_q) + \nu \langle N_q, v_q \rangle_g + \frac{\nu^2}{2} \|N_q\|^2_g .$$

Observation 3.5.1. As noted before, L_ν contains a linear term in the velocities, to relate such term with the previous notation we have

$$\gamma^\ell = (\nu \flat_g(N))^\ell .$$

Consider the nonholonomic system $(Q, L_\nu, \mathcal{D})$ and let $Z \in \mathfrak{X}(Q)$ be a vector field on Q, then the momenta $p^{\nu}_Z : TQ \to \mathbb{R}$ generated by Z is defined as

$$p^{\nu}_Z(v_q) = \langle \mathbb{F}L_\nu(v_q), Z(q) \rangle = \langle v_q, Z_q \rangle_g + \nu \langle N_q, Z_q \rangle_g . \tag{3.15}$$

Proposition 1.10 characterize this kind of functions restricted to $\mathcal{D}$, indeed.

Corollary 3.10. *Two vector fields Z_1, Z_2 on Q define the same affine function, $p^{\nu}_{Z_1}\big|_{\mathcal{D}}$, $p^{\nu}_{Z_2}\big|_{\mathcal{D}}$, on $\mathcal{D}$, if and only if the following conditions are satisfied.*

$$Z_1 - Z_2 \in \Gamma(\mathcal{D}^\perp), \qquad \nu \langle Z_1, N \rangle_g = \nu \langle Z_2, N \rangle_g .$$

We now define in a more precise way what we mean by an extension of a momenta generator $Z \in \Gamma(\mathcal{D})$ for the system $(Q, L_0, \mathcal{D})$.

Definition 3.3. Consider a given family of nonholonomic systems $(Q, L_\nu, \mathcal{D})$. Let $Z \in \Gamma(\mathcal{D})$, we say $\tilde{Z} \in \mathfrak{X}(Q)$ is an *extension* of Z if and only if the following two conditions are satisfied

1. $\tilde{Z} - Z \in \Gamma(\mathcal{D}^\perp)$,
2. $p^0_Z|_{\mathcal{D}} = p^0_{\tilde{Z}}|_{\mathcal{D}}$.

Since we want $p^{\nu}_{\tilde{Z}}|_{\mathcal{D}}$ to be a first integral of $(Q, L_\nu, \mathcal{D})$, we can refine the choice of $\tilde{Z}$ by using Proposition 1.10. So we can consider $\tilde{Z} \in \Gamma(\mathcal{D} + L\xi)$, where $L\xi$ is the TQ subbundle with fibers $L\xi_q = \operatorname{span}\{\xi_q\}$ and ξ_q is the orthogonal projection of N_q onto $\mathcal{D}^\perp$, with respect to the kinetic energy metric. Note that a section $\tilde{Z} \in \Gamma(\mathcal{D} + L\xi)$ can be written as $\tilde{Z} = Z + f\xi$, where $Z \in \Gamma(\mathcal{D})$ and $f \in C^\infty(Q)$.

Our purpose to find an extension $\tilde{Z} = Z + f\xi$ of Z such that the function $p^{\nu}_{\tilde{Z}}$ is a first integral of X^{ν}_{nh} translates on when the function f exists. In the general case the existence of f is related to the existence and uniqueness of solutions of a suitable differential equation. Denote by $X^{\nu}_{nh}\mathfrak{X}(\mathcal{D})$ the nonholonomic vector field of $(Q, L_\nu, \mathcal{D})$

and let $Z \in \Gamma(\mathcal{D})$ be the horizontal generator of a first integral $p^0_Z|_{\mathcal{D}}$ of X^0_{nh}, then the conditions on the existence of f are related to the existence and uniqueness of solutions of the following equation

$$\begin{aligned} 0 &= \mathcal{L}_{X^\nu_{nh}} p^\nu_{Z+f\xi}|_{\mathcal{D}} = \mathcal{L}_{X^\nu_{nh}} p^\nu_Z|_{\mathcal{D}} + f\mathcal{L}_{X^\nu_{nh}} p^\nu_\xi|_{\mathcal{D}} + p^\nu_\xi|_{\mathcal{D}}\mathcal{L}_{X^\nu_{nh}} f \\ &= \mathcal{L}_{X^\nu_{nh}} p^\nu_Z|_{\mathcal{D}} + \nu f \mathcal{L}_{X^\nu_{nh}} \|\xi\|^2_g + \nu \|\xi\|^2_g \mathcal{L}_{X^\nu_{nh}} f. \end{aligned} \tag{3.16}$$

We now study the existence and uniqueness of the function f for three particular scenarios.

The following Proposition is in a sense a generalization of the Nonholonomic Noether Theorem.

Proposition 3.11. *Consider a given nonholonomic systems $(Q, L_\nu, \mathcal{D})$ and a vector field $Z \in \Gamma(\mathcal{D})$. If $Z^{TQ}[L_0] = 0$ and $[Z, N] = 0$, then $p^\nu_Z|_{\mathcal{D}}$ is a first integral of X^ν_{nh}. In particular in this case $f \equiv 0$.*

Proof. By Proposition 1.9 we have that $p^0_Z|_{\mathcal{D}}$ is a first integral of X^0_{nh} and by the same proposition we just need to see that $Z^{TQ}[L_\nu]|_{\mathcal{D}} = 0$ for all $\nu \in \mathbb{R}$, and this is clearly true since by hypothesis Z preserves the kinetic energy metric and $(\Phi^Z_t)^*(N) = N$. Therefore the 1-form $\flat_g(N)$ and the basic function $\|N\|^2_g$ are preserved by the flow of the vector field Z. □

Observation 3.5.2. Let $\gamma \in \Omega^1(Q)$ and $\gamma^\ell \in C^\infty(TQ)$ be the associated linear function. Let $X \in \mathfrak{X}(Q)$ and $v_q \in T_qQ$ then $(\mathcal{L}_X\gamma)^\ell = \mathcal{L}_{X^{TQ}}\gamma^\ell$. Let Z be a given vector field on Q such that $Z^{TQ}[L_\nu] = 0$ this implies $\mathcal{L}_{Z^{TQ}}\flat_g(N)^\ell = 0$ and by the above observation the 1-form $\flat_g(N)$ is invariant under the flow of Z.

In the next result one of the hypothesis is the vector field N is in $\mathcal{D}$, this implies that the almost-Poisson structure associated to $(Q, L_\nu, \mathcal{D})$ is in fact linear, and a first integral of the system $(Q, L_0, \mathcal{D})$ can be easily translated into a first integral when $\nu \neq 0$.

Proposition 3.12. *Consider a given family of nonholonomic systems $(Q, L_\nu, \mathcal{D})$. Let $Z \in \Gamma(\mathcal{D})$, such that $p^0_Z|_{\mathcal{D}}$ is a first integral of X^0_{nh}, and $N \in \Gamma(\mathcal{D})$. If $[Z, N] = 0$, then $p^\nu_Z|_{\mathcal{D}}$ is a first integral of X^ν_{nh}.*

Proof. Proposition 1.9 and the hypothesis imply that Z preserves the kinetic energy metric in $\mathcal{D}$. And we just need to verify

$$Z^{TQ}[L_\nu]|_{\mathcal{D}} = 0,$$

to see that the above assertion is true it is sufficient to prove

$$\mathcal{L}_Z\flat_g(N)|_{\mathcal{D}} = 0 \quad \text{and} \quad \mathcal{L}_Z\|N\|^2_g = 0.$$

Let $v_q \in \mathcal{D}_q$ then

$$\begin{aligned} (\mathcal{L}_Z\flat_g(N))_q(v_q) &= \left.\frac{d}{dt}\right|_{t=0} \Phi^{Z*}_t\flat_g(N)(v_q) = \left.\frac{d}{dt}\right|_{t=0} \flat_g(N)_{\Phi^Z_t(q)}(T_q\Phi^Z_t v_q) \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle N_{\Phi^Z_t(q)}, T_q\Phi^Z_t v_q\rangle_g = \left.\frac{d}{dt}\right|_{t=0} \langle T_q\Phi^Z_t N_q, T_q\Phi^Z_t v_q\rangle_g \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle N_q, v_q\rangle_g = 0. \end{aligned}$$

And

$$\begin{aligned}\mathcal{L}_Z \|N\|_g^2 &= \left.\frac{d}{dt}\right|_{t=0} \Phi_t^{Z*} \|N\|_g^2 = \left.\frac{d}{dt}\right|_{t=0} \|N\|_g^2 \circ \Phi_t^Z \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle N \circ \Phi_t^Z, N \circ \Phi_t^Z \rangle_g = \left.\frac{d}{dt}\right|_{t=0} \langle T\Phi_t^Z N, T\Phi_t^Z N \rangle_g \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle N, N \rangle_g = 0\end{aligned}$$

□

For the following result we assume that all nonholonomic systems $(Q, L_\nu, \mathcal{D})$ define the same vector field, hence a first integral of $(Q, L_0, \mathcal{D})$ is a first integral for all systems with $\nu \neq 0$. We use this fact to retrieve the expressions in the case the first integral is a gauge momenta.

Proposition 3.13. *Consider a family of nonholonomic systems $(Q, L_\nu, \mathcal{D})$. Let $Z \in \Gamma(\mathcal{D})$, such that $p_Z^0|_\mathcal{D}$ is a first integral of X_{nh}^0. If $d(\flat_g(N)) = 0$ and $\mathcal{L}_Z \|N\|_g^2 = 0$ then $p_{\tilde{Z}}^\nu|_\mathcal{D}$ is a first integral of X_{nh}^ν, where $\tilde{Z} = Z - \frac{\langle N,Z\rangle}{\langle N,\xi\rangle}\xi$.*

Proof. Assume the gyroscopic 1-form is closed, i.e. $d(\flat_g(N)) = 0$, then the vector fields defined by $(Q, L_\nu, \mathcal{D})$ and $(Q, L_0 + \frac{\nu^2}{2}\|N\|_g^2, \mathcal{D})$ are the same. By proposition 1.9 and by hypothesis it is obvious that

$$Z^{TQ}\left[L_0 + \frac{\nu^2}{2}\|N\|_g^2\right]\Big|_\mathcal{D} = 0.$$

Then $p_Z^0|_\mathcal{D}$ is a first integral of X_{nh}^ν and we just need to find the function f for which

$$\begin{aligned}p_z^0|_\mathcal{D} &= p_{Z+f\xi}^\nu|_\mathcal{D} \\ &= p_z^0|_\mathcal{D} + \nu\langle N, Z\rangle_g + f p_\xi^0|_\mathcal{D} + \nu f \langle N, \xi\rangle_g \\ &= p_z^0|_\mathcal{D} + \nu\langle N, Z\rangle_g + \nu f \langle N, \xi\rangle_g.\end{aligned}$$

Then $f = -\frac{\langle N,Z\rangle}{\langle N,\xi\rangle}$ does the trick.

□

3.5.1 Existence of first integrals coming from symmetry

Now we focus on a nonholonomic systems $(Q, L_\nu, \mathcal{D})$ with a Lie group G acting on Q, moreover assume the action of is free and proper. For the next result we assume the following conditions are satisfied

H1. L_0 is G-invariant.

H2. $\mathcal{D}$ is G-invariant.

H3. $N = \zeta_Q$, with $\zeta \in \mathfrak{g}$.

H4. $[N, \eta_Q] = 0$ for all $\eta \in \mathfrak{g}$.

H5. $Z \in \Gamma(\mathcal{D} \cap TOrb_G)$.

H6. Z is G-equivariant.

H7. dim Q/G=1.

Conditions H1-H4 imply that $(Q, L_\nu, \mathcal{D})$ is a G-symmetric nonholonomic system. Conditions H3 and H4 say that ζ is in the center of the Lie algebra $\mathfrak{g}$, and by condition H6 we have $[Z, N] = 0$. Finally, conditions H1-H2 imply the G-invariance of the distribution $\mathcal{D}^\perp$ and by H4 N is G-equivariant, then ξ is also G-equivariant. If $f : Q \to \mathbb{R}$ is a G-invariant function, then the momentum $p^\nu_{Z+f\xi}$ is an invariant function under the lifted action of G on TQ. So the existence and uniqueness of the function f is related to solving an ordinary differential equation obtained by H7 and (3.16), in the case H7 is not satisfied then one must solve a partial differential equation.

Theorem 3.14. *Consider a nonholonomic system $(Q, L_\nu, \mathcal{D})$ and a vector field Z on Q. Suppose conditions H1-H7 hold and furthermore $p^0_Z|_\mathcal{D}$ is a first integral of X^0_{nh}, then there exists a G-invariant function $f \in C^\infty(Q)$ such that $p^\nu_{Z+f\xi}|_\mathcal{D}$ is a first integral of X^ν_{nh}.*

Proof. We prove this result locally and then globalize it . In this scenario the quotient map $\rho : Q \to Q/G$ is a surjective submersion so in a suitable charts $U \subseteq Q$ and $\rho(U) \subseteq Q/G$ it can be thought as the projection $(r, x) \mapsto r$, furthermore we have $C^\infty(U)^G = \rho^* C^\infty(\rho(U))$ and $\Omega^1(U)^G = \rho^*\Omega^1(\rho(U))$; in other words G-invariant functions on U just depend on the variable r and G-invariant 1-forms can be written as $h(r)dr$, where $h : U \to \mathbb{R}$ is G-invariant. Let $f \in C^\infty(Q)$ be G-invariant. Recall equation (3.16) that is

$$\mathcal{L}_{X^\nu_{nh}} p^\nu_{Z+f\xi}|_\mathcal{D} = \mathcal{L}_{X^\nu_{nh}} p^\nu_Z|_\mathcal{D} + \nu f \mathcal{L}_{X^\nu_{nh}} \|\xi\|^2_g + \nu \|\xi\|^2_g \mathcal{L}_{X^\nu_{nh}} f.$$

Note that for any G-invariant function $h \in C^\infty(U)$ we have $\mathcal{L}_{X^\nu_{nh}} h(r)|_\mathcal{D} = h'(r)\mathcal{L}_{X^\nu_{nh}} r|_\mathcal{D}$. The function $\|\xi\|^2_g$ is G-invariant since the metric is G-invariant and ξ is equivariant, hence we rewrite $\|\xi\|^2_g$ as $a(r)$. By construction we have the following identity

$$\mathcal{L}_{X^\nu_{nh}} r|_\mathcal{D} = \dot{r}|_\mathcal{D}.$$

The only term left to analyze is $\mathcal{L}_{X^\nu_{nh}} p^\nu_{Z+f\xi}|_\mathcal{D}$. Since $p^0_Z|_\mathcal{D}$ is a first integral of X^0_{nh}, $Z \in \Gamma(\mathcal{D})$ and proposition 1.9 we have

$$\mathcal{L}_{X^\nu_{nh}} p^\nu_Z|_\mathcal{D} = Z^{TQ}\left[L_\nu\right]|_\mathcal{D} + \langle R_\nu|_\mathcal{D}, Z\rangle = \nu Z^{TQ}\left[\flat_g(N)^\ell\right]|_\mathcal{D}.$$

Observation 3.5.2 gives

$$Z^{TQ}\left[\flat_g(N)^\ell\right]|_\mathcal{D} = \mathcal{L}_Z \flat_g(N)^\ell|_\mathcal{D}.$$

Clearly $\mathcal{L}_Z \flat_g(N)$ is a G-invariant 1-form so locally $(\mathcal{L}_Z \flat_g(N))_{(r,x)} = b(r)dr$ for a certain function b, in consequence $\mathcal{L}_Z \flat_g(N)^\ell(q, \dot{q}) = b(r)\dot{r}$. Then

$$\mathcal{L}_{X^\nu_{nh}} p^\nu_{Z+f\xi}|_\mathcal{D} = \nu \mathcal{L}_{X^\nu_{nh}} r|_\mathcal{D} \left(b(r) + a'(r)f(r) + a(r)f'(r)\right).$$

Even though equation (3.16) is of intrinsic nature we need to relate the differential equation $\mathcal{L}_{X^\nu_{nh}} p^\nu_{Z+f\xi}|_\mathcal{D}$ in two coordinate charts (U, r) and $(\tilde{U}, \tilde{r})$ of the base space Q/G, and a diffeomorphism $\phi : \tilde{U} \to U$, $r = \phi(\tilde{r}$. The function f is globally defined, we have $\tilde{f} = f \circ \phi$, the vector field ξ is $G-$equivariant and the metric is G-invariant then the restriction of $\|\xi\|^2_g$ to both charts are related by $\tilde{a} = a \circ \phi$, at last the 1-form $\mathcal{L}_Z \flat_g(N)$ is also G-invariant then the restrictions to U and $\tilde{U}$ are related by the pull-back of ϕ, that is $\tilde{b}(\tilde{r})d\tilde{r} = \phi^*(b(r)dr) = b(\phi(\tilde{r}))\phi' d\tilde{r}$. Then

$$\tilde{b}(\tilde{r}) + \tilde{a}'(\tilde{r})\tilde{f}(\tilde{r}) + \tilde{a}(\tilde{r})\tilde{f}'(\tilde{r}) = \phi'(b(r) + a'(r)f(r) + a(r)f'(r)) \circ \phi,$$

therefore the differential equation is globally defined. The ordinary differential equation $b(r) + a'(r)f(r) + a(r)f'(r) = 0$ is non degenerate if and only if ξ is a non vanishing vector field, i.e. $a(r) \neq 0$, this is a topological obstruction of Q.

□

Chapter 4

Affine constrained systems with a Noether Symmetry

In this Chapter we study a special kind of nonholonomic systems with affine constraints and natural lagrangians: the ones that are endowed with a Noether symmetry (in [57] the authors also consider a vector field with equivalent definition). In contrast to Chapters 2 and 3 we do not investigate the almost-hamiltonian nature but we do explore the relations, induced by the Noether symmetry, with nonholonomic systems with linear constrains and gyroscopic lagrangians. The material related to Noether symmetries is based on [57] and on a work in progress in collaboration with J.C. Marrero, D. Martín de Diego and L. García-Naranjo.

The local expression of the equations of motion for nonholonomic systems with affine constraints and natural lagrangian can be obtained in lifted coordinates (1.17) and quasivelocities (1.23) by considering the 1-form γ to be identically zero.

4.1 Time dependent diffeomorphisms

A key idea is to use the flow of a Noether symmetry to construct a time dependent diffeomorphism and get an equivalent nonholonomic system with linear constraints. The theory involved is somehow standard and it is very similar to the techniques used to investigate non-autonomous ordinary differential equations, we refer the reader to [80, 32], and for a detailed exposition on time-dependent nonholonomic systems see [43].

Let Q be a n dimensional smooth manifold and $F : \mathbb{R} \times Q \to Q$ a time dependent diffeomorphism, that is for every time $t \in \mathbb{R}$ the function $F_t := F(t, \cdot) : Q \to Q$ is a diffeomorphism. We extend F to the product manifold

$$\Psi : \mathbb{R} \times Q \longrightarrow \mathbb{R} \times Q, \quad \Psi := \mathrm{Id}_{\mathbb{R}} \times F_t,$$

where $\mathrm{Id}_{\mathbb{R}}$ is the identity function of $\mathbb{R}$. Note that Ψ is also a diffeomorphism. Moreover we consider the tangent lift $T\Psi$ of Ψ

$$T\Psi : T\mathbb{R} \times TQ \to T\mathbb{R} \times TQ, \quad T\Psi(\varepsilon_t, v_q) = \left(\varepsilon_t, T_qF_t(v_q) + \frac{\partial F}{\partial t}(\varepsilon_t, v_q)\right).$$

The restriction of $T\Psi$ to the submanifold $\mathbb{R} \times \{1\} \times TQ$, i.e. when $\varepsilon_t = 1$, is called the $1-$*jet prolongation* [43], of F_t and we denote it by $\Lambda : \mathbb{R} \times TQ \to \mathbb{R} \times TQ$

$$\Lambda(t, v_q) = \left(t, T_qF_t(v_q) + \frac{\partial F_q}{\partial t}\right).$$

Let $\rho_{TQ} : \mathbb{R} \times TQ \to TQ$ denote the projection onto the second factor TQ, the map

$$\tilde{F} : \mathbb{R} \times TQ \longrightarrow TQ, \quad \text{by } \tilde{F} := \rho_{TQ} \circ \Lambda,$$

is a time dependent diffeomorphism that induces the diffeomorphism $\tilde{F}_t := \tilde{F}(t, \cdot) : TQ \to TQ$ given by $\tilde{F}_t = TF_t + \frac{\partial F}{\partial t}$.

4.2 Equivalence of affine constrained with natural lagrangian and linear constrained with gyroscopic lagrangian nonholonomic systems

Let $(Q, L, \mathcal{M})$ be a nonholonomic system with affine constraints, and $L = T - V \circ \tau_Q$ a natural lagrangian. Using the same notation as in (1.11), we consider the lagrangian L in lifted bundle coordinates $(q, \dot{q})$

$$L(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q} - V(q).$$

Recall that locally the affine distribution $\mathcal{M} = \xi + \mathcal{D}$, where $\xi \in \Gamma(\mathcal{D}^\perp) \subset \mathfrak{X}(Q)$ is a vector field on Q and $\mathcal{D}$ is a linear non-integrable distribution over Q.

Definition 4.1. Let $N \in \mathfrak{X}(Q)$ be a vector field on Q and $\Phi_t^N : Q \to Q$ the associated flow at time t. The vector field N is a Noether symmetry of the nonholonomic system $(Q, L, \mathcal{M})$, if the following conditions are satisfied

1. $N \in \Gamma(\mathcal{M})$,
2. $\mathcal{D}$ is Φ_t^N-invariant, i.e. $T_q\Phi_t^N(\mathcal{D}_q) = \mathcal{D}_{\Phi_t^N(q)}$, for all $q \in Q$,
3. $N^{TQ}(L) = 0$ or equivalently L is invariant with respect to the tangent lift of Φ_t^N.

Remark 4.2.1. If N is a Noether symmetry, then by Proposition 1.6 the kinetic energy metric g is also preserved by the tangent lift of Φ_t^N, that is N is a Killing vector field for it. This fact implies that $\mathcal{D}^\perp$ is also invariant under Φ_t^N. As a consequence the vector field ξ is equivariant with respect to the action induced by the flow, in other words $[\xi, N] = 0$.

Let $(Q, L, \mathcal{M})$ be an affine constrained nonholonomic system with a Noether symmetry, we aim to construct a nonholonomic system equivalent to $(Q, L, \mathcal{M})$ for which the constraint distribution is linear. In the following we use the same notation as in Section 4.1. Denote by $F_t := \Phi_{-t}^N$ for all $t \in \mathbb{R}$, the inverse of the Noether symmetry flow, then F_t induces the 1-jet prolongation

$$\Lambda : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times TQ,$$

which is a diffeomorphism. Now, since the Noether symmetry $N \in \Gamma(\mathcal{M})$ is a section of $\mathcal{M}$, let $v_q \in \mathcal{M}_q$ so that $v_q = N_q + w_q$, with $w_q \in \mathcal{D}_q$. Then

$$\begin{aligned} \Lambda(t, v_q) &= \left(t, TF_t(v_q) + \frac{\partial F(t,q)}{\partial t}\right) \\ &= \left(t, T\Phi_{-t}^N(v_q) + \frac{\partial \Phi_{-t}^N(q)}{\partial t}\right) \\ &= \left(t, T\Phi_{-t}^N(w_q + N_q) - N_{\Phi_{-t}^N(q)}\right) \\ &= \left(t, T\Phi_{-t}^N(w_q)\right). \end{aligned} \tag{4.1}$$

Note that $T\Phi^N_{-t}(w_q) \in D_{\Phi^N_{-t}(q)}$ since the distribution $\mathcal{D}$ is invariant under the flow of N, therefore

$$\Lambda\big|_{\mathbb{R}\times\mathcal{M}} : \mathbb{R}\times\mathcal{M} \longrightarrow \mathbb{R}\times\mathcal{D},$$

and consequently

$$\tilde{F}_t\big|_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{D},$$

is a diffeomorphism for all $t \in \mathbb{R}$.

Observation 4.2.1. We observe that by construction, for every $v_q \in T_qQ$

$$\tilde{F}_t(v_q) = T_qF_t(v_q) - N_{F_t(q)},$$

which in general defines an affine bundle morphism.

We now define the function $\tilde{L} := L \circ \tilde{F}_t^{-1}$. Since the lagrangian L is TF_t-invariant we get

$$\begin{aligned}\tilde{L}(v_q) &= L(T_qF_t(v_q) + N_{F_t(q)}) = L(v_q + N_q)\\ &= L(v_q) + \langle N_q, v_q\rangle_g + \frac{1}{2}\left\|N_q\right\|_g^2,\end{aligned} \tag{4.2}$$

that ensures the function $\tilde{L}$ to be time independent. In bundle coordinates $(q, \dot{q})$ we have

$$\tilde{L}(q,\dot{q}) = \frac{1}{2}\dot{q}\cdot A(q)\dot{q} + N(q)\cdot A(q)\dot{q} + \frac{1}{2}N(q)\cdot A(q)N(q) - V(q).$$

We can then consider the nonholonomic system with linear constraints and gyroscopic lagrangian $(Q, \tilde{L}, \mathcal{D})$. To relate both systems $(Q, L, \mathcal{M})$ and $(Q, \tilde{L}, \mathcal{D})$, we first define the extended dynamics which are the nonholonomic vector fields associated to $(\mathbb{R}\times Q, L\circ\rho_{TQ}, \mathbb{R}\times\mathcal{M})$ and $(\mathbb{R}\times Q, \tilde{L}\circ\rho_{TQ}, \mathbb{R}\times\mathcal{D})$[1]. In symbols, the "extensions" are related as follows: let $X^{\mathcal{M}}_{nh} \in \mathfrak{X}(\mathcal{M})$ and $X^{\mathcal{D}}_{nh} \in \mathfrak{X}(\mathcal{D})$ be the nonholonomic vector fields associated to $(Q, L, \mathcal{M})$ and $(Q, \tilde{L}, \mathcal{D})$ respectively. The vector fields $X^{\mathbb{R}\times\mathcal{M}}_{nh} \in \mathfrak{X}(\mathbb{R}\times\mathcal{M})$ and $X^{\mathbb{R}\times\mathcal{D}}_{nh} \in \mathfrak{X}(\mathbb{R}\times\mathcal{D})$ denote the extended vector fields associated to $(Q, L, \mathcal{M})$ and $(Q, \tilde{L}, \mathcal{D})$

$$X^{\mathbb{R}\times\mathcal{M}}_{nh} = X^{\mathcal{M}}_{nh} + \frac{\partial}{\partial t}, \qquad X^{\mathbb{R}\times\mathcal{D}}_{nh} = X^{\mathcal{D}}_{nh} + \frac{\partial}{\partial t}.$$

Proposition 4.1. *The extended dynamics of $(Q, L, \mathcal{M})$ and $(Q, \tilde{L}, \mathcal{D})$ are equivalent, that is they are related by the 1-jet prolongation, Λ, of the diffeomorphism F_t, i.e.*

$$\Lambda_*|_{\mathbb{R}\times\mathcal{M}}\left(X^{\mathcal{M}}_{nh} + \frac{\partial}{\partial t}\right) = X^{\mathcal{D}}_{nh} + \frac{\partial}{\partial t}.$$

Moreover we have

$$\Lambda_*\big|_{\mathbb{R}\times\mathcal{M}}(X^{\mathcal{M}}_{nh}) = X^{\mathcal{D}}_{nh} + N^{TQ}\big|_{\mathbb{R}\times\mathcal{D}}.$$

Proof. The first claim of the proof follows from Observation 1.2.1 considering the nonholonomic systems $(\mathbb{R}\times Q, L\circ\rho_{TQ}, \mathbb{R}\times\mathcal{M})$ and $(\mathbb{R}\times Q, \tilde{L}\circ\rho_{TQ}, \mathbb{R}\times\mathcal{D})$, where $\rho_{TQ} : \mathbb{R}\times TQ \to TQ$ is the fiber projection, and the diffeomorphism relating both systems is $\Lambda|_{\mathbb{R}\times\mathcal{M}}$.

To prove the second relation note that $\frac{\partial T\Phi^N_{-t}}{\partial t} = -N^{TQ}$, then from equation (4.1) and we get

$$\Lambda_*\big|_{\mathbb{R}\times\mathcal{M}}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} - N^{TQ}\big|_{\mathcal{D}}.$$

Using the linearity property of the derivative and the first part of this Proposition we conclude

$$\Lambda_*\big|_{\mathbb{R}\times\mathcal{M}}(X^{\mathcal{M}}_{nh}) = X^{\mathcal{D}}_{nh} + N^{TQ}\big|_{\mathbb{R}\times\mathcal{D}}.$$

□

[1] The "extensions" are the same as in the non-autonomous case of ordinary differential equations [32]

4.2.1 Moving energy

An important consequence of Proposition 4.1 is the construction of the so called *moving energy* first introduced in [57] and, then the concept was generalized in [51]. To motivate the definition of moving energy we first introduce the concept of kinematically interpretable moving energy, and afterwards present the general definition with a characterization on when it is a first integral.

Definition 4.2. Let $(Q, L, \mathcal{M})$ be a nonholonomic systems with affine constraints, natural lagrangian and Noether symmetry N. The kinematically interpretable moving energy $E_{L,N} : \mathcal{M} \to \mathbb{R}$ of the nonholonomic system is the function

$$E_{L,N} := E_{\tilde{L}} \circ \tilde{F}_t|_{\mathcal{M}}.$$

Where $E_{\tilde{L}}$ is the lagrangian energy of $\tilde{L}$.

We observe that the moving energy is time independent, i.e. $E_{\tilde{L}} \circ \tilde{F}_t|_{\mathcal{M}} = E_{\tilde{L}} \circ \tilde{F}_{t'}|_{\mathcal{M}}$, for all $t, t' \in \mathbb{R}$. To see this first note that $E_{\tilde{L}} = T + V \circ \tau_Q - \frac{1}{2}\left\|N\right\|_g^2 \circ \tau_Q$ and recall that the kinetic energy metric g and the potential energy V are invariant with respect to the flow Φ_t^N. Let $v_q \in \mathcal{M}_q$ then

$$\begin{aligned} E_{L,N}(v_q) &= T(v_q) + V(q) - \flat_g(N)^\ell(v_q) \\ &= T(v_q) - \langle N_q, v_q \rangle_g + V(q) \\ &= E_L(v_q) - p_N(v_q). \end{aligned}$$

Since for a linear constrained nonholonomic system the lagrangian energy is always preserved we can use the equivalence of the vector fields $X_{nh}^{\mathcal{M}}$ and $X_{nh}^{\mathcal{D}}$ to get the following result.

Theorem 4.2. *[57] The moving energy $E_{L,N}$ is a first integral of the nonholonomic vector field $X_{nh}^{\mathcal{M}}$.*

Proof. The lagrangian energy restricted to $\mathcal{D}$, $E_{\tilde{L},\mathcal{D}}$ is a first integral of $X_{nh}^{\mathcal{D}}$, since $(\tilde{Q}, \tilde{L}, \mathcal{D})$ is a nonholonomic system with linear constraints, then because it is time-independent, its trivial extension to $\mathbb{R} \times \mathcal{D}$, is also a first integral of $X_{nh}^{\mathcal{D}} + \frac{\partial}{\partial t}$, therefore by Proposition 4.1 the function $\Lambda^*\big|_{\mathbb{R}\times\mathcal{M}}(E_{\tilde{L},\mathcal{D}})$, because it is time-independent, is a first integral of $X_{nh}^{\mathcal{M}}$. In symbols it is easily proven using the Lie derivative, on one hand

$$\begin{aligned} \mathcal{L}_{X_{nh}^{\mathcal{D}} + \frac{\partial}{\partial t}} E_{\tilde{L},\mathcal{D}} &= \mathcal{L}_{X_{nh}^{\mathcal{D}}} E_{\tilde{L},\mathcal{D}} + \frac{\partial E_{\tilde{L},\mathcal{D}}}{\partial t} dt \left(\frac{\partial}{\partial t}\right) \\ &= \mathcal{L}_{X_{nh}^{\mathcal{D}}} E_{\tilde{L},\mathcal{D}} \\ &= 0, \end{aligned}$$

and on the other hand

$$\begin{aligned} 0 &= \Lambda^*\big|_{\mathbb{R}\times\mathcal{M}} \left(\mathcal{L}_{X_{nh}^{\mathcal{D}} + \frac{\partial}{\partial t}} E_{\tilde{L},\mathcal{D}}\right) \\ &= \mathcal{L}_{\Lambda_*^{-1}\big|_{\mathbb{R}\times\mathcal{D}}(X_{nh}^{\mathcal{D}} + \frac{\partial}{\partial t})} (\Lambda^*\big|_{\mathbb{R}\times\mathcal{D}}(E_{\tilde{L},\mathcal{D}})) \\ &= \mathcal{L}_{X_{nh}^{\mathcal{M}} + \frac{\partial}{\partial t}} E_{L,N} \\ &= \mathcal{L}_{X_{nh}^{\mathcal{M}}} E_{L,N} + \frac{\partial E_{L,N}}{\partial t} dt \left(\frac{\partial}{\partial t}\right) \\ &= \mathcal{L}_{X_{nh}^{\mathcal{M}}} E_{L,N}. \end{aligned}$$

□

For the generalization of the kinematically interpretable moving energy we consider a nonholonomic system $(Q, L, \mathcal{M})$ with affine contraints, $\mathcal{M} = Y + \mathcal{D}$, and gyroscopic lagrangian. Given a vector field Z on Q define

$$E_{L,Z} := E_L - p_Z : TQ \longrightarrow \mathbb{R}.$$

Definition 4.3. A function $f : \mathcal{M} \to \mathbb{R}$ is a moving energy of the system $(Q, L, \mathcal{M})$ if there exists a vector field $Z \in \mathfrak{X}(Q)$ such that $f = E_{L,Z}\big|_{\mathcal{M}}$.

The next characterization is not exhaustive, in the sense there could exist vector fields Z for which $E_{L,Z}\big|_{\mathcal{M}}$ is a first integral of $(Q, L, \mathcal{M})$, but not satisfying the other conditions.

Proposition 4.3. *[51] Any two of the following statements imply the third.*

1. $Z - Y \in \Gamma(R^\circ)$.
2. $Z^{TQ}[L]\,|_{\mathcal{M}} = 0$.
3. $E_{L,Z}\big|_{\mathcal{M}}$ *is a first integral of* $(Q, L, \mathcal{M})$.

Proof. We perform the prove in coordinates, it is elementary and similar to that of Proposition 1.9. By Proposition 1.3 we have

$$\frac{d}{dt} E_L|_{\mathcal{M}} = R \cdot Y.$$

On the other side Proposition 1.9 implies

$$\frac{d}{dt} p_Z|_{\mathcal{M}} = Z^{TQ}[L] + R \cdot Z.$$

Combining both equations

$$\frac{d}{dt} E_{L,Z}\big|_{\mathcal{M}} = R \cdot (Y - Z) - Z^{TQ}[L].$$

And the result follows. □

4.3 Gauge momenta relations in the presence of a Noether symmetry

The next result relates the momentum of a vector field Z on Q in both nonholonomic systems $(Q, L, \mathcal{M})$ and $(Q, \tilde{L}, \mathcal{D})$, when they are related by a Noether symmetry.

Theorem 4.4. *Let $Z \in \mathfrak{X}(Q)$ be a vector field on Q such that $[Z, N] = 0$. The momentum $p_Z\big|_{\mathcal{M}}$ is a first integral of $X_{nh}^{\mathcal{M}}$ if and only if $\tilde{p}_Z\big|_{\mathcal{D}}$ is a first integral of $X_{nh}^{\mathcal{D}}$. Where*

$$p_Z(v_q) = \langle \mathbb{F}L(v_q), Z_q \rangle, \qquad \tilde{p}_Z(v_q) = \langle \mathbb{F}\tilde{L}(v_q), Z_q \rangle.$$

In order to prove the Theorem 4.4 we first prove some properties which are helpful.

Proposition 4.5. *Let $Z \in \mathfrak{X}(Q)$ be a vector field on Q, such that $[Z, N] = 0$. Then $p_Z = \tilde{F}_t^* \tilde{p}_Z$.*

Proof. Let $v_q \in T_qQ$, $\tilde{F}_t(v_q) = T_qF_t(v_q) - N_{F_t(q)}$. Then

$$\begin{aligned}\tilde{p}_Z \circ \tilde{F}_t(v_q) &= \tilde{p}_Z(T_qF_t(v_q) - N_{F_t(q)}) \\ &= \langle T_qF_t(v_q) - N_{F_t(q)}, Z_{F_t(q)}\rangle_g + \langle N_{F_t(q)}, Z_{F_t(q)}\rangle_g \\ &= \langle T_qF_t(v_q), T_qF_t(Z_q)\rangle_g \\ &= p_Z(v_q).\end{aligned}$$

Where we used $(F_t)_*Z = Z$.

□

As a consequence the momenta p_Z and $\tilde{p}_Z$ associated to the vector field Z restricted to $\mathcal{M}$ and $\mathcal{D}$ respectively are related by the diffeomorphism $\tilde{F}_t$, i.e.

$$p_Z\big|_{\mathcal{M}} = \tilde{F}_t\big|_{\mathcal{M}}^* \tilde{p}_Z\big|_{\mathcal{D}}.$$

Moreover if $[N, Z] = 0$ and $p_Z\big|_{\mathcal{M}}$ is a first integral of $X_{nh}^{\mathcal{M}}$, then we show that the Lie derivative is zero $\mathcal{L}_{N^{TQ}|_{\mathcal{D}}}\tilde{p}_Z\big|_{\mathcal{D}} = 0$. This claim is true because the tangent lift N^{TQ} is tangent to $\mathcal{D}$, so it makes sense to consider the restriction $N^{TQ}\big|_{\mathcal{D}}$. Now we prove the following general result.

Proposition 4.6. *Let $H \in C^\infty(TQ)$ be a smooth function on TQ and $X, Y \in \mathfrak{X}(Q)$ be two commuting vector fields on Q, i.e. $[X, Y] = 0$. If $\mathcal{L}_{Y^{TQ}}[H] = 0$, then the function $p_X^H : TQ \to \mathbb{R}$ defined as $p_X^H(v_q) = \langle \mathbb{F}H(v_q), X_q\rangle$ is a first integral of Y^{TQ}.*

Proof. It suffices to prove that p_X^H is $T\Phi_t^Y$-invariant for all t and then we apply the definition of Lie derivative. Let $v_q \in T_qQ$ then

$$\begin{aligned}p_X^H \circ T\Phi_t^Y(v_q) &= \frac{d}{dt}\bigg|_{t=0} H\left(T\Phi_t^Y(v_q) + tX_{\Phi_t^Y(q)}\right) \\ &= \frac{d}{dt}\bigg|_{t=0} H\left(T\Phi_t^Y(v_q) + tT\Phi_t^Y(X_q)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} H\left(T\Phi_t^Y(v_q + tX_q)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} H(v_q + tX_q) \\ &= p_X^H(v_q).\end{aligned}$$

Where we used the fiberwise linearity of $T\Phi_t^Y$ and the $T\Phi_t^Y$-invariance of H, which is given in the hypothesis.

□

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. We just do one implication the other one is proven in the same fashion.

Suppose $p_Z\big|_{\mathcal{M}}$ is a first integral of $X_{nh}^{L,\mathcal{M}}$, in symbols it is $\mathcal{L}_{X_{nh}^{L,\mathcal{M}}}(p_Z\big|_{\mathcal{M}}) = 0$. Then

$$\begin{aligned}0 &= (\tilde{F}_t\big|_{\mathcal{M}})_*\left(\mathcal{L}_{X_{nh}^{L,\mathcal{M}}}(p_Z\big|_{\mathcal{M}})\right) \\ &= \mathcal{L}_{(\tilde{F}_t|_{\mathcal{M}})_*(X_{nh}^{L,\mathcal{M}})}\left((\tilde{F}_t\big|_{\mathcal{M}})_*(p_Z\big|_{\mathcal{M}})\right) \\ &= \mathcal{L}_{X_{nh}^{\tilde{L},\mathcal{D}} + N^{TQ}|_{\mathcal{D}}}(p_Z\big|_{\mathcal{M}} \circ \tilde{F}_t^{-1}\big|_{\mathcal{M}}) \\ &= \mathcal{L}_{X_{nh}^{\tilde{L},\mathcal{D}}}(\tilde{p}_Z\big|_{\mathcal{D}}) + \mathcal{L}_{N^{TQ}|_{\mathcal{D}}}(\tilde{p}_Z\big|_{\mathcal{D}}) \\ &= \mathcal{L}_{X_{nh}^{\tilde{L},\mathcal{D}}}(\tilde{p}_Z\big|_{\mathcal{D}}).\end{aligned}$$

□

Observation 4.3.1. Using Theorem 4.4 we can recover the results of Section 3.5 in terms of nonholonomic systems with affine constraints and natural lagrangian.

Chapter 5

Ball rolling without slipping on a turning surface of revolution

5.1 Introduction to the system

In this Chapter we analyze the system of a homogeneous ball that rolls without slipping in a rotating surface of revolution.

The system consist on three parts, first we use the kinematics on the Lie group $SO(3)$ to represent the attitude of the ball, and since we deal with a symmetric and homogeneous ball we may only use the angular velocity of the ball [1]. Second, the ball's movement along the surface is given by its center of mass coordinates. And third, the nonholonomic constraint is the rolling without slipping condition.

We make the following assumptions on the physical properties of the ball, we assume that the ball has unitary mass $m = 1$, and radius $r = 1$, that do not change the qualitative behavior of the system but simplify the computations.

The surface of revolution in which the ball's center of mass is constrained to move along is parametrized by a function $f : \mathbb{R} \to \mathbb{R}$. Without loss of generality we assume $f(0) = 0$, then the surface is given by

$$\mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) \mapsto (x, y, f(\sqrt{x^2 + y^2})).$$

Note that the function f just depends on the radius, then the position of the center of mass in polar is given by $(r, \beta) \mapsto (r, \beta, f(r))$. To investigate the dynamics we use polar coordinates since they are better suited to put in evidence the symmetry of the system. As usual, the relation between cartesian and polar coordinates is given by

$$x = r \cos\beta, \quad y = r \sin\beta, \quad r \geq 0,\ \beta \in [0, 2\pi)\,. \tag{5.1}$$

5.1.1 Rigid body dynamics

The system is formed by a homogeneous ball moving along a surface of revolution under the action of gravity and of nonholonomic constraints. To represent the orientation of the ball we first consider two orthonormal coordinate frames: an *inertial frame* $\{e_1, e_2, e_3\}$ and a *body frame* $\{E_1, E_2, E_3\}$ rigidly attached to the ball and that rotates with it. The orientation of the rigid body at time t is given by the attitude matrix $\mathcal{R}(t) \in SO(3)$, which is precisely the change of basis matrix between the inertial and body frames. The *angular velocity vector* with respect to the inertial frame $\{e_1, e_2, e_3\}$ is denoted by

[1] Euler equations for the rigid body [84].

$\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ and $\Omega = (\Omega_x, \Omega_y, \Omega_z) \in \mathbb{R}^3$ denotes the one with respect to the body frame $\{E_1, E_2, E_3\}$. The coordinate representations of the angular velocity ω and Ω respectively correspond to the right and left trivialisations of the velocity vector $\dot{\mathcal{R}} \in T_{\mathcal{R}}SO(3)$, namely [84]:

$$\hat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}, \qquad \hat{\omega} = \dot{\mathcal{R}}\mathcal{R}^{-1},$$

where $\hat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$ is the well known *hat* isomorphism, to each vector $v \in \mathbb{R}^3$ associates the skew-symmetric matrix $\hat{v} \in \mathfrak{so}(3)$, characterized by the following condition $\hat{v}u = v \times u$, for all $u \in \mathbb{R}^3$, '$\times$' denotes the standard vector product in $\mathbb{R}^3$. In coordinates

$$v = (v_1, v_2, v_3) \mapsto \hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

The angular velocity vectors in space and body representations are related by $\omega = \mathcal{R}\Omega$. In matrix notation $\hat{\omega} = \mathcal{R}\hat{\Omega}\mathcal{R}^{-1} = \mathrm{Ad}_{\mathcal{R}^{-1}}(\hat{\Omega})$, with $\mathrm{Ad} : SO(3) \to GL(\mathfrak{so}(3))$ the adjoint representation of the Lie group $SO(3)$.

We use the *inertia tensor* I in body coordinates to obtain an isomorphism between $\mathfrak{so}(3)$ and its dual $\mathfrak{so}^*(3)$, the *angular momentum vector* is then expressed in the body frame as $M = I\Omega \in \mathbb{R}^3$, where the inertia tensor I is a 3×3 symmetric, positive definite matrix, which encodes the mass distribution of the body. In our case (the ball is homogeneous of mass and radius equal to 1) the inertia tensor I is a diagonal matrix with moment of inertia $k = \frac{2}{5}$. The "rotational" kinetic energy of the ball is $\frac{1}{2}M \cdot \Omega = \frac{1}{2}I\Omega \cdot \Omega$, where '$\cdot$' denotes the standard scalar product in $\mathbb{R}^3$. For more details on the subject we refer the reader to [4, 84].

5.1.2 Quasivelocities

The configuration space is $Q = SO(3) \times \mathbb{R}^2$, endowed with local coordinates $(\phi, \theta, \psi, r, \beta)$, where $(r, \beta) \in \mathbb{R}^{>0} \times [0, 2\pi)$ are polar coordinates and $0 < \varphi, \psi < 2\pi$, $0 < \theta < \pi$ are Euler angles in accordance with the *x-convention* (see e.g. [84]). We parametrize a matrix $\mathcal{R} \in SO(3)$ as

$$\mathcal{R} = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\varphi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}, \tag{5.2}$$

According to this convention the angular velocity in space coordinates ω reads:

$$\omega = \begin{pmatrix} \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\sin\theta \\ \dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{pmatrix}. \tag{5.3}$$

The matrix $B : Q \to GL(\mathbb{R}^n)$ to pass from velocities to quasivelocities $v = B(q)\dot{q}$ is

$$B = \begin{pmatrix} 0 & \cos\varphi & \sin\theta\sin\varphi & 0 & 0 \\ 0 & \sin\varphi & -\sin\theta\cos\varphi & 0 & 0 \\ 1 & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To obtain an explicit expression of the inertial frame $\left\{e_1, e_2, e_3, \frac{\partial}{\partial r}, \frac{\partial}{\partial \beta}\right\}$ in terms of the frame induced by the coordinates we need to consider the inverse matrix B^{-1}.

The quasivelocities just introduced in TQ we denote them as (q, v), where $q = (\phi, \theta, \psi, r, \beta)$ and $v = (\omega, \dot{r}, \dot{\beta})$.

5.1.3 Lagrangian

The lagrangian of the system is of natural type, meaning that it is kinetic minus potential energy. Due to the nature of the system we can split the kinetic energy into two parts, one is related to the rotation of the ball and the other with the velocity of its center of mass. As we already noted the rotational kinetic energy of the ball is $\frac{k}{2}\omega \cdot \omega$. And the kinetic energy related to the center's of mass velocity is

$$\frac{1}{2}\left(\dot{r}\cos\beta - r\dot{\beta}\sin\beta, \dot{r}\sin\beta + r\dot{\beta}\cos\beta, \dot{r}f'(r)\right)\cdot\left(\dot{r}\cos\beta - r\dot{\beta}\sin\beta, \dot{r}\sin\beta + r\dot{\beta}\cos\beta, \dot{r}f'(r)\right).$$

Then the total kinetic energy of the system is equal to $T(q,v) = \frac{k}{2}\omega\cdot\omega + \frac{1}{2}(\dot{r}^2 + r^2\dot{\beta}^2) + \frac{1}{2}f'(r)^2\dot{r}^2$, and the associated kinetic energy matrix in quasivelocities is

$$A = \frac{1}{2}\begin{pmatrix} k & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 1+f'(r)^2 & 0 \\ 0 & 0 & 0 & 0 & r^2 \end{pmatrix}.$$

The potential energy V of the system is given by the weight applicated to the ball's center of mass, $V(r) = gf(r)$, where g is the constant of gravity. The lagrangian of the system is then the smooth function $L_0 : TQ \to \mathbb{R}$ defined by

$$L_0(q,v) = T(q,v) - V(q) = \frac{k}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) + \frac{1}{2}(F^2(r)\dot{r}^2 + r^2\dot{\beta}^2) - gf(r), \tag{5.4}$$

where $F^2(r) = 1 + f'(r)^2$.

5.1.4 Nonholonomic Constraints

The nonholonomic constraint considered for this system is the rolling without slipping condition. Let P be the point in space representing the ball's center of mass, v_P its velocity and N the outwards normal vector to the surface. The non slipping condition writes as

$$v_P + \omega \times N - \Omega e_z \times (N + P) = 0. \tag{5.5}$$

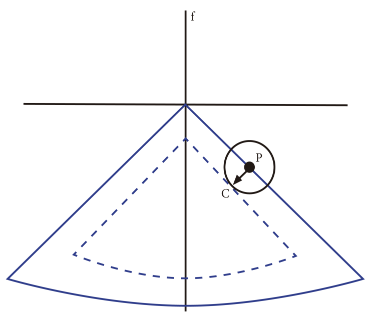


Figure 5.1: Cross section

Remark 5.1.1. Notation disclaimer, from here to the end of the Chapter we use the symbol Ω to represent the angular velocity of the surface of revolution. It should not give place to confusion since the angular velocity of the ball in body representation is not used again.

In polar coordinates and spatial angular velocity we have, $P = (r\cos\beta, r\sin\beta, f(r))$, $v_P = \frac{d}{dt}P$, $N = (\frac{f'}{F}\cos\beta, \frac{f'}{F}\sin\beta, \frac{-1}{F})$, $e_z = (0,0,1)$, and then the constraint condition (5.5) reads

$$
\begin{aligned}
-\frac{\omega_y}{F} - \frac{f'}{F}\sin\beta\omega_z + \cos\beta\dot{r} - r\sin\beta\dot{\beta} + \Omega\sin\beta\left(r + \frac{f'}{F}\right) &= 0,\\
\frac{\omega_x}{F} + \frac{f'}{F}\cos\beta\omega_z + \sin\beta\dot{r} + r\cos\beta\dot{\beta} - \Omega\cos\beta\left(r + \frac{f'}{F}\right) &= 0.
\end{aligned} \tag{5.6}
$$

The constraint functions are defined as

$$
\begin{aligned}
f_1(q,v) &= \frac{\omega_x}{F} + \frac{f'}{F}\cos\beta\omega_z + \sin\beta\dot{r} + r\cos\beta\dot{\beta}, \quad s_1(r,\beta) = -\Omega\cos\beta\left(r + \frac{f'}{F}\right),\\
f_2(q,v) &= -\frac{\omega_y}{F} - \frac{f'}{F}\sin\beta\omega_z + \cos\beta\dot{r} - r\sin\beta\dot{\beta}, \quad s_2(r,\beta) = \Omega\sin\beta\left(r + \frac{f'}{F}\right).
\end{aligned} \tag{5.7}
$$

And allow us to define the 8-dimensional constraint submanifolds $\mathcal{M}$ and $\mathcal{D}$ of TQ

$$
\begin{aligned}
\mathcal{M} &:= (f_1 + s_1)^{-1}(0) \cap (f_2 + s_2)^{-1}(0),\\
\mathcal{D} &:= f_1^{-1}(0) \cap f_2^{-1}(0).
\end{aligned}
$$

As stated in Section 1.1, constraints linear in the velocities can be rephrased in terms of 1-forms over Q, so in Euler angles we can rewrite the linear part of the constraints as

$$
\begin{aligned}
&\frac{f'}{F}\cos\beta d\phi + \frac{\cos\phi}{F}d\theta + \frac{1}{F}\left(\sin\theta\sin\phi + f'\cos\theta\cos\beta\right)d\psi + \sin\beta dr + r\cos\beta d\beta,\\
-&\frac{f'}{F}\sin\beta d\phi - \frac{\sin\phi}{F}d\theta + \frac{1}{F}\left(\sin\theta\cos\phi - f'\cos\theta\sin\beta\right)d\psi + \cos\beta dr - r\sin\beta d\beta.
\end{aligned} \tag{5.8}
$$

We define the following vector fields on Q

$$
\begin{aligned}
X_1 &= F\frac{\partial}{\partial\phi} - \frac{f'}{r}\frac{\partial}{\partial\beta},\\
X_2 &= \left(-1 + f'\cot\theta\sin(\beta-\phi)\right)\frac{\partial}{\partial\phi} + f'\cos(\beta-\phi)\frac{\partial}{\partial\theta} - f'\csc\theta\sin(\beta-\phi)\frac{\partial}{\partial\psi},\\
X_3 &= F\cot\theta\cos(\beta-\phi)\frac{\partial}{\partial\phi} - F\sin(\beta-\phi)\frac{\partial}{\partial\theta} - F\csc\theta\cos(\beta-\phi)\frac{\partial}{\partial\psi} + \frac{\partial}{\partial r},
\end{aligned} \tag{5.9}
$$

by direct computation we can see that X_1, X_2, X_3 form a frame for $\mathcal{D}$. Aided by the kinetic energy metric we construct the following frame for the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$

$$
\begin{aligned}
X_4 &= \frac{r}{kF}\left(f' + \cot\theta\sin(\beta-\phi)\right)\frac{\partial}{\partial\phi} + \frac{r}{kF}\cos(\beta-\phi)\frac{\partial}{\partial\theta} - \frac{r}{kF}\csc\theta\sin(\beta-\phi)\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\beta},\\
X_5 &= -\frac{F}{k}\cot\theta\cos(\beta-\phi)\frac{\partial}{\partial\phi} + \frac{F}{k}\sin(\beta-\phi)\frac{\partial}{\partial\theta} + \frac{F}{k}\csc\theta\cos(\beta-\phi)\frac{\partial}{\partial\psi} + \frac{\partial}{\partial r}.
\end{aligned} \tag{5.10}
$$

Using the constraint 1-forms (5.8) it is straight forward to verify that the vector field $\xi \in \mathfrak{X}(Q)$ such that $\mathcal{M}_q = \xi_q + \mathcal{D}_q$ and $\xi \in \Gamma(\mathcal{D}^\perp)$ is

$$
\xi = \Omega\mu\left(1 + \frac{f'}{rF}\right)X_4, \tag{5.11}
$$

where the constant $\mu = \frac{k}{k+1} = \frac{2}{7}$.

In matrix notation, the matrices $S(q)$ and $s(q)$ are

$$
S(r,\beta) = \begin{pmatrix} \frac{1}{F} & 0 & \frac{f'}{F}\cos\beta & \sin\beta & r\cos\beta\\ 0 & -\frac{1}{F} & -\frac{f'}{F}\sin\beta & \cos\beta & -r\sin\beta \end{pmatrix}, \qquad s(r,\beta) = \begin{pmatrix} -\Omega\cos\beta\left(r + \frac{f'}{F}\right)\\ \Omega\sin\beta\left(r + \frac{f'}{F}\right) \end{pmatrix}.
$$

5.1.5 Symmetries

The system has two kinds of symmetries, one comes from the ball's homogeneity and the other from the rotational symmetry of the surface. In mathematical terms these symmetries are described by an action of the Lie group $G = SO(3) \times SO(2)$ on $Q = SO(3) \times \mathbb{R}^2$ defined as follows. Let $(K, H) \in G$ and $(R, p) \in Q$ then

$$(K, H) \cdot (R, p) = (\tilde{H}RK, Hp),$$

where

$$\tilde{\cdot} : SO(2) \hookrightarrow SO(3),$$

is the inclusion given by the rotation around the z-axis. In matrix notation

$$\tilde{H} = \begin{pmatrix} H & 0_{2\times1} \\ 0_{1\times2} & 1 \end{pmatrix},$$

with $0_{2\times1} \in M_{2\times1}(\mathbb{R})$ and $0_{1\times2} \in M_{1\times2}(\mathbb{R})$ are the zero matrices, and

$$H = \begin{pmatrix} \cos\Omega & -\sin\Omega \\ \sin\Omega & \cos\Omega \end{pmatrix}.$$

Using the matrix representation (5.2) and polar coordinate relations (5.1) we get that the action of $SO(2)$ on Q written in coordinates is

$$(\phi, \theta, \psi, r, \beta) \mapsto (\phi + \Omega, \theta, \psi, r, \beta + \Omega). \tag{5.12}$$

The infinitesimal generators of this action are

$$\begin{aligned} \xi_1^Q &= \cos\theta\sin\psi\frac{\partial}{\partial\phi} + \cos\psi\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\psi}, & \xi_3^Q &= \frac{\partial}{\partial\psi}, \\ \xi_2^Q &= \csc\theta\cos\psi\frac{\partial}{\partial\phi} - \sin\psi\frac{\partial}{\partial\theta} - \cot\theta\cos\psi\frac{\partial}{\partial\psi}, & \xi_4^Q &= \Omega\frac{\partial}{\partial\phi} + \Omega\frac{\partial}{\partial\beta}. \end{aligned} \tag{5.13}$$

The vector fields ξ_i^Q on Q, $i = 1, 2, 3$ are the infinitesimal generators of the $SO(3)$ action and ξ_4^Q is associated to the $SO(2)$ action.

Proposition 5.1. *The vector fields $\{X_i\}_{i=1}^5$ are equivariant with respect to the action of the group G.*

Proof. It is sufficient to prove

$$\left[X_i, \xi_a^Q\right] = 0, \quad i = 1, 2, 3, 4, 5,\ a = 1, 2, 3, 4.$$

This can be seen directly by doing the computations. We used mathematica for such end.

□

The following Corollary is a direct implication of the above result

Corollary 5.2. *The distribution $\mathcal{D}$ is G-invariant.*

Observation 5.1.1. Note that $X_1, X_2, X_4 \in \Gamma(TOrb_G)$, in fact X_1, X_2 generate $\mathcal{D} \cap TOrb_G$ and X_4 generates $\mathcal{D}^\perp \cap TOrb_G$. Even more, ξ is the orthogonal projection of ξ_4^Q onto $\mathcal{D}^\perp$ and is equivariant.

The action of G on $SO(3)$ comes from a linear action, so the lifted action of G on $TSO(3)$ is

$$(K,H)\cdot(R,\dot{R}) = (\tilde{H}RK, \tilde{H}\dot{R}K),$$

the action of G translates to the Lie algebra $\mathfrak{so}(3)$ as

$$\hat{\omega} \mapsto Ad_{\tilde{H}}\hat{\omega},$$

and equivalently we get

$$\omega \mapsto \tilde{H}\omega.$$

Therefore the action of G on TQ in quasivelocities writes

$$\begin{aligned}(\phi,\theta,\psi,r,\beta,&\omega_x,\omega_y,\omega_z,\dot{r},\dot{\beta})\\ &\mapsto (\phi+\Omega,\theta,\psi,r,\beta+\Omega,\omega_x\cos\Omega-\omega_y\sin\Omega,\omega_x\sin\Omega+\omega_y\cos\Omega,\omega_z,\dot{r},\dot{\beta}).\end{aligned}$$

Using the above coordinate representation of the action we see, by direct computation, that the lagrangian L_0 (5.4) is G-invariant.

Remark 5.1.2. The infinitesimal generator ξ_4^Q is a Noether symmetry of the nonholonomic system $(Q,L,\mathcal{M})$.

We just proved the following result

Proposition 5.3. *The nonholonomic system $(Q,L_0,\mathcal{M})$ is G-symmetric. Hence the vector field X_{nh}^{Ω} on $\mathcal{M}$ is G-invariant.*

5.1.6 Equations of motion

To compute the equations of motion we follow the treatment developed in Section 1.4. We use (q,v) as coordinates on TQ, where $q=(\phi,\theta,\psi,r,\beta)$ and $v=(\omega_x,\omega_y,\omega_z,\dot{r},\dot{\beta})$, and the quasivelocities expression of L_0, A, S and s are given in the previous subsections. Then the equations of motion of the system are

$$\begin{aligned}
\dot{q} &= B^{-1}v,\\
\ddot{r} &= -\frac{f'f''}{F^2}\dot{r}^2+\frac{(1+\mu f'^2)r}{F^2}\dot{\beta}^2+\frac{\mu f'}{F}\dot{\beta}\omega_z-\frac{\mu\Omega}{F^2}(f'+rF)\dot{\beta}-\frac{\gamma f'}{F^2},\\
\ddot{\beta} &= \dot{r}\left(-\left(\frac{2}{r}+\frac{\mu f'f''}{F^2}\right)\dot{\beta}-\frac{\mu f''}{rF}\omega_z+\mu\Omega\left(\frac{1}{r}+\frac{f''}{rF}+\frac{f'f''}{F^2}\right)\right),\\
\dot{\omega}_z &= -\frac{(1-\mu)}{F^2}\dot{r}\left(\frac{rf'^2f''}{F}\dot{\beta}+f'f''\omega_z-\frac{\Omega f'}{F}(F^2+rf'f''+f''F)\right),\\
\omega_x &= -f'\cos\beta\omega_z-F\sin\beta\dot{r}-rF\cos\beta\dot{\beta}+\Omega\cos\beta\,(rF+f'),\\
\omega_y &= -f'\sin\beta\omega_z+F\cos\beta\dot{r}-rF\sin\beta\dot{\beta}+\Omega\sin\beta\,(rF+f'),
\end{aligned}\tag{5.14}$$

where $\mu=\frac{k}{k+1}$ and $\gamma=\frac{g}{k+1}$. Note that in our case $\mu=\frac{2}{7}$, because the ball is homogeneous.

We denote by $X_{nh}^{\Omega}\in\mathfrak{X}(\mathcal{M})$ the vector field defined by the equations of motion (5.14).

5.1.7 Moving Energy

The moving energy $E_{L_0,Y}:\mathcal{M}\to\mathbb{R}$ associated to the Noether symmetry $Y=\xi_4^Q$ is the restriction to $\mathcal{M}$ of the function $E:TQ\to\mathbb{R}$:

$$\begin{aligned}
E &= T+V\circ\tau_Q-\flat_g(Y)^{\ell},\\
E(q,v) &= \frac{1}{2}F^2\dot{r}^2+\frac{1}{2}r^2\dot{\beta}^2+\frac{1}{2}k(\omega_x^2+\omega_y^2+\omega_z^2)+gf-\Omega(r^2\dot{\beta}+k\omega_z).
\end{aligned}$$

For more details on the construction of the moving energy see Subsection 4.2.1.

Its restriction to $\mathcal{M}$ is

$$\begin{aligned} E_{L_0,Y} &= \frac{1}{2}(k+1)F^2\dot{r}^2 + \frac{1}{2}r^2(1+kF^2)\dot{\beta}^2 + \frac{1}{2}kF^2\omega_z^2 + krf'F\dot{\beta}\omega_z \\ &\quad - \Omega r(r+krF^2+kf'F)\dot{\beta} - \Omega k(F^2+rf'F)\omega_z + \frac{1}{2}\Omega^2 k\left(r^2F^2+f'(f'+2rF)\right) \\ &\quad + gf, \end{aligned}$$

where we used (5.6) for explicit expressions of ω_x and ω_y as functions of $(r,\beta,\dot{r},\dot{\beta},\omega_z)$, and observed that $\omega_x^2+\omega_y^2$ doesn't depend on β.

Proposition 4.3 imply that the moving energy $E_{L_0,Y}$ is a constant of motion of the nonholonomic system $(Q,L_0,\mathcal{M})$.

5.2 Invariant Polynomials and Routh integrals

Along this Section we use polar coordinates and angular velocity to define the invariant polynomials, we use them to compute Routh integrals. The $SO(2)$ action over $Q = SO(3)\times\mathbb{R}^2$ is free, proper and linear, nevertheless since the action of $SO(2)$ on $\mathbb{R}^2$ is linear, then it is not free. To avoid dealing with singular reduction we do not consider any motion of the vector field X_{nh}^{Ω} on $\mathcal{M}$ passing through the vertex of the surface of revolution, in other words we restrict the action to $SO(3)\times\mathbb{R}^2\setminus\{0,0\}$. We proceed as follows, the action of $SO(3)$ on Q is free and proper in both factors, and since the constraint manifold $\mathcal{M}$ is $G-$invariant then we can consider the quotient manifold$\mathcal{M}_5 := \mathcal{M}/SO(3)$ which is smooth. Consider coordinates $(r,\beta,\omega_z,\dot{r},\dot{\beta})$ in $\mathcal{M}_5$. A set of $SO(2)$-invariant polynomials on this variables is

$$\begin{aligned} p_0 &= \frac{\dot{r}^2+r^2\dot{\beta}^2}{2}, \quad p_1 = \frac{r^2}{2}, \quad p_2 = r\dot{r}, \quad p_3 = r^2\dot{\beta}, \\ p_4 &= \omega\cdot N = -rf'\dot{\beta} - F\omega_z + \Omega f'\left(r+\frac{f'}{F}\right), \end{aligned} \tag{5.15}$$

where N is the normal vector to the surface. Then the quotient manifold $\mathcal{M}_5/SO(2)$ can be represented as the semi-algebraic variety

$$\tilde{\mathcal{M}}_4 = \left\{(p_0,p_1,p_2,p_3,p_4)\in\mathbb{R}^5 : 4p_0p_1 = p_2^2+p_3^2,\ p_0,\ p_1\geq 0\right\}$$

and we may restrict the analysis to

$$\mathcal{M}_4 = \left\{(p_0,p_1,p_2,p_3,p_4)\in\mathbb{R}^5 : 4p_0p_1 = p_2^2+p_3^2,\ p_0,\ p_1 > 0\right\}$$

.

For a discussion about invariant polynomials and singular reduction of nonholonomic systems, i.e. when the Lie group action is proper but not free, see [11].

We use invariant polynomials to compute the so called Routh first integrals in a constructive manner, and then analyze the system restricted to the levels sets of the first integrals. First we rewrite the equations of motion in terms of invariant polynomials. To this end we introduce the smooth function

$$\Psi : \mathbb{R}\to\mathbb{R},$$

such that $\Psi(p_1) = f(r)$ see [52] for details. Observe that the following relations hold

$$f'(r) = \sqrt{2p_1}\Psi'(p_1), \quad f''(r) = \Psi'(p_1) + 2p_1\Psi''(p_1),$$

then the equations of motion for invariant polynomials write

$$\begin{aligned}
\dot{p}_0 &= p_2\mathcal{F}^2\left(\left(\mu\Psi''p_3p_4-\gamma\Psi'-2\Psi'^2-\Psi'\Psi''p_2^2\right)+\Omega\mu\left(\Psi'^2+\mathcal{F}\Psi''\right)p_3\right)\\
\dot{p}_1 &= p_2,\\
\dot{p}_2 &= \mathcal{F}^2\left(2p_0-2\gamma\Psi'p_1-\mu\Psi'p_3p_4-2\Psi'\Psi''p_1p_2^2\right)-\Omega\mu\left(1+\Psi'\mathcal{F}\right)\mathcal{F}^2p_3,\\
\dot{p}_3 &= p_2\left(G_3p_4+\Omega g_3\right),\\
\dot{p}_4 &= p_2\left(G_4p_3+\Omega g_4\right),
\end{aligned} \tag{5.16}$$

where $\mathcal{F}=\frac{1}{\sqrt{1+2p_1\Psi'^2}}$, $G_3=\mu\mathcal{F}^2\left(\Psi'+2p_1\Psi''\right)$, $G_4=\mathcal{F}^2\left(\Psi'^3-\Psi''\right)$, $g_3=\mu\left(1+\left(\Psi'+2p_1\Psi''\right)\mathcal{F}^3\right)$, $g_4=\mathcal{F}^2\left(1+\mathcal{F}\Psi'\right)\left(\Psi'+2p_1\Psi''\right)$.

The moving energy (5.1.7) in $\mathcal{M}_5$ written in terms of invariant polynomials reads

$$\mathcal{E}=\frac{\Psi'^2}{2}p_2^2+\frac{\mu}{2}p_4^2+p_0+\Omega\left(\mu\mathcal{F}p_4-p_3\right)+\Omega^2\mu p_1\left(1-\mathcal{F}^2\Psi'^2\right)+\gamma\Psi. \tag{5.17}$$

5.2.1 Routh integrals

We here follow the approach and construction of the Routh integrals given in [41]. There one can find proofs about functional independence between the two Routh integrals and the moving energy. As we already anticipated, Routh integrals are two first integrals of the system, they arise as solutions of the following system of ordinary differential equations

$$\begin{aligned}
\dot{p}_3 &= p_2\left(G_3p_4+\Omega g_3\right),\\
\dot{p}_4 &= p_2\left(G_4p_3+\Omega g_4\right).
\end{aligned} \tag{5.18}$$

Thanks to the equation $\dot{p}_1 = p_2$ we can perform a change of variables and time reparametrization on (5.18), and then consider the linear, non-autonomous and non-homogeneous system of differential equations

$$u'=\mathcal{G}u+g \tag{5.19}$$

where $u=u(p_1)$, here the '$\prime$' symbol denotes derivative with respect to p_1 and

$$\mathcal{G}=\begin{pmatrix}0 & G_3\\ G_4 & 0\end{pmatrix} \quad\text{and}\quad g=\begin{pmatrix}g_3\\ g_4\end{pmatrix}.$$

To solve this differential equation we use the variation of parameters method [32]. Let U be a fundamental matrix solution, at zero, of

$$U'=\mathcal{G}U$$

and u an integral curve of (5.19) such that $u(0)=(0,0)$ then define the function

$$J=\begin{pmatrix}J_1\\ J_2\end{pmatrix}=U(p_1)^{-1}\left(\begin{pmatrix}p_3\\ p_4\end{pmatrix}-\Omega u(p_1)\right), \tag{5.20}$$

J_1 and J_2 are the so called Routh integrals. A straightforward computation shows that these functions are first integrals of (5.16).

To obtain dynamical results of the nonholonomic system $(Q, L, \mathcal{M})$ we analyze the vector field in $\mathcal{M}_4$ restricted to the level sets of the Routh integrals $J=(J_1,J_2)$, where we can express the variables p_3, p_4 as functions of p_1 and of the value of J. Even more, the moving energy (5.17) restricted to the level sets $J^{-1}(j_1, j_2)$ plays the role of a natural lagrangian of a second order differential equation (this fact can be shown by direct computation). Denote by $\mathcal{E}_j := \mathcal{E}|_{J^{-1}(j)}$, $j \in \mathbb{R}^2$, and write $\mathcal{E}_j = \mathcal{T} - \mathcal{V}_j$ with

$$\begin{aligned} \mathcal{T}(p_1, p_2) &= \frac{1}{4p_1\mathcal{F}^2}p_2^2, \\ \mathcal{V}_j(p_1) &= \frac{\mu}{2}p_4^2 + \frac{1}{4p_1}p_3^2 + \Omega\left(\mu\mathcal{F}p_4 - p_3\right) + \Omega^2\mu p_1\left(1 - \mathcal{F}^2\Psi'^2\right) + \gamma\Psi, \end{aligned} \tag{5.21}$$

where p_3 and p_4 are thought as functions of (p_1, j_1, j_2) and the function $\mathcal{V}_j$ is called the *effective potential*.

5.3 First integrals coming from horizontal gauge momenta of the static case $\Omega = 0$.

It is known that in the static case, $\Omega = 0$, the system admits two horizontal gauge momenta [103, 70, 19, 52, 58]. Here we prove that their generators can be extended to obtain two first integrals also when the surface is uniformly rotating. To do this we use an algorithm to produce a first integral and then we compare them with the Routh integrals obtained above.

First we consider a vector field $Z = h_1(r)X_1 + h_2(r)X_2 + h_3(r)\xi$ on Q, where $h_1, h_2, h_3 \in C^\infty(Q)$ are smooth G-invariant functions, and consider the associated linear function $p_Z|_\mathcal{M}$ restricted to $\mathcal{M}$. Recall that $X_1(q), X_2(q) \in \mathcal{D}_q \bigcap T_qOrb\,G$, $\forall q \in Q$. We now search for conditions on the functions h_1, h_2, h_3 such that

$$\mathcal{L}_{X^\Omega_{nh}}\, p_Z|_\mathcal{M} = 0,$$

recall X^Ω_{nh} is the vector field on $\mathcal{M}$ of the nonholonomic system $(Q, L_0, \mathcal{M})$. To simplify the computations we introduce the following variables

$$\tilde{\omega}_x = \cos\beta\omega_x - \sin\beta\omega_y, \qquad \tilde{\omega}_y = \sin\beta\omega_x + \cos\beta\omega_y. \tag{5.22}$$

In this variables the constraints can re-write as

$$\dot{r} = \frac{\tilde{\omega}_y}{F}, \qquad \dot{\beta} = -\frac{\tilde{\omega}_x}{rF} - \frac{f'\omega_z}{rF} + \Omega\left(1 + \frac{f'}{rF}\right). \tag{5.23}$$

The following Theorem is a consequence of Theorem 3.14, this fact is elucidated in Subsection 5.4.2, nevertheless we present a constructive proof which is useful as a possible algorithm to compute first integrals.

Theorem 5.4. *The nonholonomic system $(Q, L_0, \mathcal{M})$ has two functionally independent G-invariant gauge momenta.*

Proof. Let $Z = h_1(r)X_1 + h_2(r)X_2 + h_3(r)\xi$ then the function $p_Z|_\mathcal{M} : \mathcal{M} \to \mathbb{R}$ in quasivelocities is written as

$$\begin{aligned} p_Z|_\mathcal{M}(r, \tilde{\omega}_x, \omega_z) =& f'\left(kh_2 + \frac{h_1}{F}\right)\tilde{\omega}_x + \left(h_1\left(\frac{f'^2}{F} + kF\right) - kh_2\right)\omega_z - \Omega h_1 f'\left(r + \frac{f'}{F}\right) \\ &+ \mu\Omega^2 h_3\left(r^2 + \frac{f'^2}{F^2} + \frac{2rf'}{F}\right). \end{aligned} \tag{5.24}$$

To compute the Lie derivative $\mathcal{L}_{X^{\Omega}_{nh}}\, p_Z|_{\mathcal{M}}$ in such coordinates first note

$$\begin{aligned}\dot{\omega}_x &= \frac{\tilde{\omega}_y}{(k+1)rF^5}\left[\left(rf'f''-(k+1)F^4\right)\tilde{\omega}_x-\left((k+1)f'F^4+rf''\right)\omega_z\right.\\ &\qquad\left.+\Omega\left(rf''+rF^3+(k+1)F^4(f'+rF)\right)\right],\\ \dot{\omega}_z &= \frac{f'\tilde{\omega}_y}{(k+1)F^5}\left[f'f''\tilde{\omega}_x-f''\omega_z+\Omega(f''+F^3)\right].\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}\, p_Z|_{\mathcal{M}} =& \frac{\tilde{\omega}_x\tilde{\omega}_y}{rF^2}\left[rf'(h_1'+kFh_2')+(rf''-f')(h_1+kFh_2)\right]\\ &+\frac{\tilde{\omega}_y\omega_z}{rF^2}\left[r(kF^2+f'^2)h_1'-krFh_2'+f'((k+1)rf''-f')h_1-kFf'^2h_2\right]\\ &+\frac{\Omega(f'+rF)}{rF^2}\tilde{\omega}_y\left[-rf'Fh_1'+(rf''-f')h_1+kFf'h_2\right]\\ &+\frac{\Omega^2k(f'+rF)}{(k+1)F^5}\tilde{\omega}_y\left[F(rF+f')h_3'+2(F^3+f'')h_3\right],\end{aligned} \tag{5.25}$$

since we want to impose the condition $\frac{d}{dt}\, p_Z|_{\mathcal{M}}=0$, we solve the system for the quadratic terms in the velocities. This leads to solve the following system of ordinary differential equations for h_1, h_2

$$\begin{aligned}h_1' &= \frac{1}{(k+1)rf'F^2}\left[(G_1-krf'^2f'')h_1+kFG_2h_2\right],\\ h_2' &= \frac{1}{(k+1)rf'F^3}\left[G_2h_1+F(kG_1-rf'^2f'')h_2\right].\end{aligned} \tag{5.26}$$

Where $G_1=F^2(f'-rf'')$ and $G_2=f'F^2-rf''$. It is a non autonomous linear ordinary differential equations so it has two independent solutions, each of such solutions give place to the differential equation for h_3

$$h_3' = -\frac{1}{\Omega rF^2(rF+f')}\left[FG_2h_1+F^2(kf'F^2+rf'')h_2+2\Omega r(F^3+f'')h_3\right], \tag{5.27}$$

this is a linear non homogeneous, non autonomous differential equation so it always has a solution. Each solutions give place to an equivariant vector field Z so the linear function generated by it is $G-$invariant, and by construction the restriction of p_Z to $\mathcal{M}$ is a first integral.

□

Observation 5.3.1. To characterize when the vector field Z is a $\mathcal{D}$ or R^0 gauge symmetry we note the following. If the function h_3 is not identically zero then by construction $Z\notin\Gamma(\mathcal{D})$, but in the case where $h_3\,\|\xi\|_g^2$ is a constant function then we can consider $\tilde{Z}=h_1X_1+h_2X_2$ as the generator and obviously the gauge momenta generated by $\tilde{Z}$ is still a first integral and it is functionally dependent to $p_Z|_{\mathcal{M}}$, we encounter this situation in Section 5.6. Now, to see under which conditions $Z\in\Gamma(R^0)$ we have the following alternatives: on one hand if $h_3\xi\in\Gamma(R^0)$ clearly $Z\in\Gamma(R^0)$, on the other hand if $Z^{TQ}(L_0)\big|_{\mathcal{M}}=0$ we use Proposition 1.9 to obtain that Z is a R^0-gauge symmetry.

Theorem 5.5. *The two Routh integrals and the two gauge momenta obtained in Theorem 5.4 are pairwise functionally dependent.*

Proof. Consider the constraint space $\mathcal{M}$, recall that $\dim \mathcal{M} = 8$. Then $\dim \mathcal{M}/G = 4$, where $G = SO(3)\times SO(2)$. Furthermore, the moving energy $E_{L_0,Y}$ is G-invariant then the nonholonomic system can only posses two more functionally independent first integrals (because it is a non-zero vector field), therefore it must happen that Routh integrals are a functional combination of the first integrals found in Theorem 5.4. □

5.4 Equivalent gyroscopic system

This Section is devoted to exemplify the theory developed in Chapters 3 and 4. Along this Section use the notation and elements introduced in the current Chapter.

5.4.1 Equivalence using the Noether symmetry

We use the Noether symmetry $Y = \xi_4^Q$ (see Remark 5.1.2) of the nonholonomic system $(Q, L_0, \mathcal{M})$ to linearize the system as presented in Section 4.2. We use quasi-velocities (q, v), $q = (\phi, \theta, \psi, r, \beta)$, $v = (\omega_x, \omega_y, \omega_z, \dot r, \dot\beta)$ defined in (5.3). The lagrangian L_0 is defined in (5.4) and the constraint manifold $\mathcal{M}$ in (5.6).

The flow $\Phi_t^Y : Q \to Q$ of the vector field $Y = \Omega\frac{\partial}{\partial\phi} + \Omega\frac{\partial}{\partial\beta}$ on Q in coordinates $(\phi, \theta, \psi, r, \beta)$ reads

$$\Phi_t^Y(\phi, \theta, \psi, r, \beta) = (\phi + t\Omega, \theta, \psi, r, \beta + t\Omega).$$

Therefore the 1-jet prolongation $\Lambda : \mathbb{R} \times TQ \to \mathbb{R} \times TQ$ of the function $F_t = \Phi_{-t}^Y$ in quasivelocities has the expression

$$\Lambda(t, (\phi, \theta, \psi, r, \beta), (\omega_x, \omega_y, \omega_z, \dot r, \dot\beta)) = (t, (\phi - t\Omega, \theta, \psi, r, \beta - t\Omega), (\omega_x, \omega_y, \omega_z - \Omega, \dot r, \dot\beta - \Omega))$$

and

$$\tilde F_t(\phi, \theta, \psi, r, \beta, \omega_x, \omega_y, \omega_z, \dot r, \dot\beta) = (\phi - t\Omega, \theta, \psi, r, \beta - t\Omega, \omega_x, \omega_y, \omega_z - \Omega, \dot r, \dot\beta - \Omega).$$

Then the lagrangian $L_\Omega = \tilde F_t^{-1} \circ L_0$ of the nonholonomic system (Q, L_Ω, D) writes as

$$L_\Omega(q, v) = \frac{F^2(r)}{2}\dot r^2 + \frac{r^2}{2}\dot\beta^2 + \frac{k}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) + \Omega\left(k\omega_z + r^2\dot\beta\right) + \frac{\Omega^2}{2}\left(k + r^2\right) - gf(r). \tag{5.28}$$

Constraint equations (5.6) using $\tilde F_t$ translate to

$$\begin{aligned} -\frac{\omega_y}{F} - \frac{f'}{F}\sin\beta\omega_z + \cos\beta\dot r - r\sin\beta\dot\beta = 0, \\ \frac{\omega_x}{F} + \frac{f'}{F}\cos\beta\omega_z + \sin\beta\dot r + r\cos\beta\dot\beta = 0. \end{aligned} \tag{5.29}$$

The matrix S, such that $D_q = \ker S_q$, in this coordinates is

$$S(r, \beta) = \begin{pmatrix} \frac{1}{F} & 0 & \frac{f'}{F}\cos\beta & \sin\beta & r\cos\beta \\ 0 & -\frac{1}{F} & -\frac{f'}{F}\sin\beta & \cos\beta & -r\sin\beta. \end{pmatrix}$$

Using Proposition 1.4 we get the equations of motion of the nonholonomic system

(Q, L_Ω, D)

$$\begin{aligned}
\dot q &= B^{-1}v,\\
\ddot r &= -\frac{f'f''}{F^2}\dot r^2 + \frac{(1+\mu f'^2)r}{F^2}\dot\beta^2 + \frac{\mu f'}{F}\dot\beta\omega_z + \frac{\Omega r}{(k+1)F^2}(2+kF^2)\dot\beta + \frac{\Omega\mu f'}{F} + \frac{\Omega^2 r - \gamma f'}{F^2},\\
\ddot\beta &= \dot r\left(-\left(\frac{2}{r} + \frac{\mu f' f''}{F^2}\right)\dot\beta - \frac{\mu f''}{rF}\omega_z - \frac{\Omega}{r}(2-\mu)\right),\\
\dot\omega_z &= -\frac{(1-\mu)}{F}\dot r\left(\frac{r f'^2 f''}{F^2}\dot\beta + \frac{f'f''}{F}\omega_z - \Omega f'\right),\\
\omega_x &= -f'\cos\beta\omega_z - F\sin\beta\dot r - rF\cos\beta\dot\beta,\\
\omega_y &= -f'\sin\beta\omega_z + F\cos\beta\dot r - rF\sin\beta\dot\beta,
\end{aligned} \tag{5.30}$$

where $\mu = \frac{k}{k+1}$ and $\gamma = \frac{g}{k+1}$. Denote by $X_{nh}^{\mathcal{D}}$ the vector field defined by equations (5.30).

The tangent lift of Y is

$$Y^{TQ} = \Omega\left(e_3 + \frac{\partial}{\partial\beta}\right),$$

then it is easy to check that $(\Lambda|_{\mathcal{M}})_*(X_{nh}^{\Omega}) = X_{nh}^{\mathcal{D}} + Y^{TQ}|_{\mathcal{D}}$.

Note that the two generators of the gauge momenta obtained in Theorem 5.4 satisfy the hypothesis (by construction they commute with the Noether symmetry Y) of Theorem 4.4.

5.4.2 Almost-Poisson bracket

We construct the almost-Poisson bracket for the nonholonomic system $(Q, L_\Omega, \mathcal{D})$ using the theory developed in Section 3.3.

Consider the lagrangian $L_0 : TQ \to \mathbb{R}$, given in quasi-velocities by

$$L_0(q,v) = \frac{F^2(r)}{2}\dot r^2 + \frac{r^2}{2}\dot\beta^2 + \frac{k}{2}(\omega_x^2+\omega_y^2+\omega_z^2) - gf(r).$$

Furthermore consider the vector field on Q

$$Y = \Omega e_3 + \Omega\frac{\partial}{\partial\beta}.$$

Now we define the lagrangian with gyroscopic term

$$\begin{aligned}
L_\Omega(q,v) &= L_0(q, v + Y_q)\\
&= \frac{F^2(r)}{2}\dot r^2 + \frac{r^2}{2}\dot\beta^2 + \frac{k}{2}(\omega_x^2+\omega_y^2+\omega_z^2) + \Omega\left(k\omega_z + r^2\dot\beta\right) + \frac{\Omega^2}{2}\left(k+r^2\right) - gf(r),
\end{aligned} \tag{5.31}$$

so the 1-form γ associated to the gyroscopic term is $\gamma = \Omega k e^3 + \Omega r^2 d\beta$. Observe that lagrangians (5.28) and (5.31) coincide and the notation used is consistent.

Recall the frame $\{X_1, X_2, X_3\}$ of $\mathcal{D}$ (5.9) and $\{X_4, X_5\}$ of $\mathcal{D}^\perp$ (5.10). Then we get the coefficients of the change of frame matrix ρ associated to the frame $\{X_1, X_2, X_3, X_4, X_5\}$

$$\begin{aligned}
&\rho_{11} = F, && \rho_{15} = -\tfrac{f'}{r}, && \rho_{34} = 1,\\
&\rho_{21} = -1 + f'\cot\theta\sin(\beta-\phi), && \rho_{22} = f'\cos(\beta-\phi), && \rho_{23} = -f'\csc\theta\sin(\beta-\phi),\\
&\rho_{31} = F\cot\theta\cos(\beta-\phi), && \rho_{32} = -F\sin(\beta-\phi), && \rho_{33} = -F\csc\theta\cos(\beta-\phi),
\end{aligned}$$

and $\rho_{12} = \rho_{13} = \rho_{14} = \rho_{24} = \rho_{25} = \rho_{35} = 0$.

The velocities $(v_1, v_2, v_3, v_4, v_5)$ defined $\{X_i\}_{i=1}^5$ relate to (5.3) as

$$\begin{aligned}
v_1 &= \frac{\mu}{f'}\left(\cos\beta\omega_x + \sin\beta\omega_y\right) + \mu\omega_z - \frac{rF}{(k+1)f'}\dot\beta,\\
v_2 &= \frac{\mu + f'^2}{f'F^2}\left(\cos\beta\omega_x + \sin\beta\omega_y\right) - \frac{1}{(k+1)F^2}\omega_z - \frac{r}{(k+1)f'F}\dot\beta\\
v_3 &= \frac{\mu}{F}\left(\cos\beta\omega_y - \sin\beta\omega_x\right) + \frac{1}{k+1}\dot r,\\
v_4 &= \frac{\mu}{rF}\left(\cos\beta\omega_x + \sin\beta\omega_y\right) + \frac{\mu f'}{rF}\omega_z + \mu\dot\beta,\\
v_5 &= -\frac{\mu}{F}\left(\cos\beta\omega_y - \sin\beta\omega_x\right) + \mu\dot r.
\end{aligned}$$

Conditions $v_4 = 0$ and $v_5 = 0$ are equivalent to the constraints functions (5.29). Along the constraint $\mathcal{D}$ the quasivelocities read

$$v_1 = -\frac{rF}{f'}\dot\beta, \quad v_2 = -\omega_z - \frac{rF}{f'}\dot\beta, \quad v_3 = \dot r.$$

Then it is easy to calculate the restricted lagrangian $L_{\Omega,c} : \mathcal{D} \to \mathbb{R}$, in these quasivelocities reads

$$\begin{aligned}
L_{\Omega,c} =& \frac{1}{2}\left(k + 1 - \frac{1}{F^2}\right)v_1^2 + \frac{1}{2}kF^2v_2^2 + \frac{1}{2}(k+1)v_3^2 - kv_1v_2 + \Omega\left(k - \frac{rf'}{F}\right)v_1 - \Omega k v_2\\
&+ \Omega^2\frac{k + r^2}{2} - gf.
\end{aligned}$$

Hence the quasi momenta p_1, p_2, p_3 are

$$\begin{aligned}
p_1 &= (k + 1 - \frac{1}{F^2})v_1 - kv_2 + \Omega\left(k - \frac{rf'}{F}\right),\\
p_2 &= kF^2v_2 - kv_1 - \Omega k,\\
p_3 &= (k+1)v_3.
\end{aligned}$$

And the restricted hamiltonian $H_c : \mathcal{D}^* \to \mathbb{R}$ is calculated to be

$$\begin{aligned}
H_c =& \frac{F^2}{2(k+1)f'^2}p_1^2 + \frac{\mu + f'^2}{2kf'^2F^2}p_2^2 + \frac{1}{2(k+1)F^2}p_3^2 + \frac{1}{(k+1)f'^2}p_1p_2\\
&+ \Omega\left(\frac{rF}{(k+1)f'} - \mu\right)p_1 + \Omega\frac{f' + rF}{(k+1)f'F^2}p_2 - \frac{\Omega^2\mu}{2}\left(\frac{f'}{F} + r\right) + gf.
\end{aligned}$$

Using the kinetic energy metric we calculate the dual basis of $\{X_i\}_{i=1}^5$ defined in (5.9) and (5.10)

$$\begin{aligned}
\chi^1 =& \mu d\phi + \frac{\mu\cos(\beta - \phi)}{f'}d\theta + \frac{\mu}{f'}\left(f'\cos\theta - \sin\theta\sin(\beta - \phi)\right)d\psi - \frac{rF}{(k+1)f'}dr,\\
\chi^2 =& \frac{(\mu + f'^2)\cos(\beta - \phi)}{f'F^2}d\theta - \frac{1}{k+1}\left(\frac{\cos\theta}{F^2} + \left(\frac{k}{f'} + \frac{f'}{F^2}\right)\sin\theta\sin(\beta - \phi)\right)d\psi\\
&+ \frac{1}{(k+1)F^2}d\phi - \frac{r}{(k+1)f'F}d\beta,\\
\chi^3 =& -\frac{\mu\sin(\beta - \phi)}{F}d\theta - \frac{\mu\sin\theta\cos(\beta - \phi)}{F}d\psi + \frac{1}{k+1}dr,\\
\chi^4 =& \frac{\mu f'}{rF}d\phi + \frac{\mu\cos(\beta - \phi)}{rF}d\theta + \frac{\mu}{rF}\left(f'\cos\theta - \sin\theta\sin(\beta - \phi)\right)d\psi + \mu d\beta,\\
\chi^5 =& \frac{\mu\sin(\beta - \phi)}{F}d\theta + \frac{\mu\sin\theta\cos(\beta - \phi)}{F}d\psi + \mu dr.
\end{aligned}$$

Hence the 1-form γ written in the above frame writes

$$\gamma = \Omega\left(k - \frac{rf'}{F}\right)\chi^1 - \Omega k \chi^2 + \Omega r\left(r + \frac{f'}{F}\right)\chi^4,$$

and this yields an expression for $\gamma^\perp$

$$\gamma^\perp = \Omega r\left(r + \frac{f'}{F}\right)\chi^4.$$

To compute the structure coefficients C^i_{ab} related to the frame $\{X_i\}_{a=1}^3$ of $\Gamma(\mathcal{D})$ we note

$$\begin{aligned}
[X_1, X_2] =& -\frac{\mu f'}{rF^2}(f' + rF)X_3 + \frac{\mu f'}{rF^2}(f' + rF)X_5,\\
[X_1, X_3] =& \left(\frac{1}{r} + \frac{\mu F}{f'} - \frac{f''}{(k+1)f'F^2}\right)X_1 + \left(\frac{1}{r} - \frac{1}{(k+1)f'F} + \frac{F}{f'} - \frac{f''}{(k+1)f'F^4}\right)X_2\\
&+ \frac{\mu}{r}\left(1 + \frac{f''}{F^3}\right)X_4,\\
[X_2, X_3] =& -\frac{\mu}{f'}\left(F^3 + f''\right)X_1 - \left(\frac{\mu F}{f'} + \frac{\mu + f'^2}{f'F}f''\right)X_2 - \left(\frac{\mu F^2}{r} + \frac{\mu f''}{rF}\right)X_4.
\end{aligned}$$

Then these are

$$\begin{aligned}
&C^3_{12} = -\frac{\mu f'}{rF^2}(f' + rF), \quad C^5_{12} = \frac{\mu f'}{rF^2}(f' + rF), \quad C^1_{13} = \frac{1}{r} + \frac{\mu F}{f'} - \frac{f''}{(k+1)f'F^2},\\
&C^2_{13} = \frac{1}{r} - \frac{1}{(k+1)f'F} + \frac{F}{f'} - \frac{f''}{(k+1)f'F^4}, \quad C^4_{13} = \frac{\mu}{r}\left(1 + \frac{f''}{F^3}\right),\\
&C^1_{23} = -\frac{\mu}{f'}\left(F^3 + f''\right), \quad C^2_{23} = -\left(\frac{\mu F}{f'} + \frac{\mu + f'^2}{f'F}f''\right), \quad C^4_{23} = -\left(\frac{\mu F^2}{r} + \frac{\mu f''}{rF}\right).
\end{aligned}$$

Now we have all the elements to obtain an expression for the almost-Poisson bracket in accordance to Subsection 3.3.2. We get

$$\begin{aligned}
\{p_1, p_2\}_{\mathcal{D}^*} =& \frac{\mu f'}{rF^2}(f' + rF)p_3,\\
\{p_1, p_3\}_{\mathcal{D}^*} =& -\left(\frac{1}{r} + \frac{\mu F}{f'} - \frac{f''}{(k+1)f'F^2}\right)p_1 - \left(\frac{1}{r} - \frac{1}{(k+1)f'F} + \frac{F}{f'} - \frac{f''}{(k+1)f'F^4}\right)p_2\\
&- \frac{\Omega\mu}{F^4}(F^3 + f'')(f' + rF),\\
\{p_2, p_3\}_{\mathcal{D}^*} =& \frac{\mu}{f'}\left(F^3 + f''\right)p_1 + \left(\frac{\mu F}{f'} + \frac{\mu + f'^2}{f'F}f''\right)p_2 + \frac{\Omega\mu}{F^2}(F^3 + f'')(f' + rF),
\end{aligned}$$

and denote by $q = (\phi, \theta, \psi, r, \beta)$ therefore $\left\{p_a, q^i\right\}_{\mathcal{D}^*} = \rho_{ai}$. We then have an expression for the almost-Poisson bracket in $\mathcal{D}^*$.

5.5 Dynamical consequences for particular surfaces of revolution

5.5.1 Bounded from above profiles

In this part we assume the surface is at rest, i.e. $\Omega = 0$, otherwise stated.

Proposition 5.6. *Let $\Omega = 0$ and $f : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function.*

1. *If f is bounded from above, then the vector field X^0_{nh} (5.14) has unbounded motions with $\dot\beta = 0$.*

2. *Assume that $\lim_{r\to\infty} f' < 0$. If there is a reduced motion, in $J^{-1}(j)$, for which $\lim_{r\to\infty} -f'r^3\dot\beta^2 - Fr^2\dot\beta\omega_z > 0$, then there exists an unbounded motion with $\dot\beta \neq 0$.*

Where again we think $\dot\beta$ and ω_z as functions of r, j_1, j_2 by means of the Routh integrals.

Proof. 1. Consider the level set $\mathcal{M}_0 := J^{-1}(0,0)$, we know it is an invariant set and on it we have that $\dot\beta = 0$, $\omega_z = 0$. On $\mathcal{M}_0$ the equations of motion are

$$\dot r = \dot r, \quad \dot v_r = -\frac{\gamma f'}{F^2} - \frac{f'f''}{F^2}\dot r^2,$$

which is a lagrangian system with $\ell_0 = \frac{1}{2}F^2(r)\dot r^2 - \gamma f(r)$ being the lagrangian. It is known that for this kind of systems if the potential part of the lagrangian, in this case it is γf, is bounded from above then there are unbounded solutions. By reconstruction we obtain an unbounded solution of the unreduced system.

2. We first note that condition $f' < 0$ is equivalent to $\Psi' < 0$ since $\Psi'(p_1) = \frac{f'(r)}{r}$ and $r > 0$. Using (5.15) we get the relation $p_3(r,\dot\beta)p_4(r,\dot\beta,\omega_z) = -f'r^3\dot\beta^2 - Fr^2\dot\beta\omega_z$. We then prove the claim using invariant polynomials and Routh integrals.

Consider the effective potential $\mathcal{V}_j$ (5.21), since $\Omega = 0$

$$\mathcal{V}_j = \gamma\Psi + \frac{p_3^2}{4p_1} + \frac{\mu}{2}p_4^2$$

where we understand p_3 and p_4 as functions of p_1, j_1 and j_2, see Subsection 5.2.1. Since

$$\begin{pmatrix} p_3 \\ p_4 \end{pmatrix} = U\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, \quad U' = GU$$

we get that

$$\frac{\partial p_3}{\partial p_1} = G_3 p_4, \quad \frac{\partial p_4}{\partial p_1} = G_4 p_3.$$

Which leads to

$$\frac{\partial \mathcal{V}_j}{\partial p_1} = \gamma\Psi' + \mu\Psi'\left(\frac{1}{p_1} + \Psi'^2\right)p_3p_4 - \frac{p_3^2}{4p_1^2}.$$

So if there is a reduced motion for which $\lim_{p_1\to\infty} p_3p_4 > 0$ then $\lim_{p_1\to\infty}\frac{\partial\mathcal{V}_j}{\partial p_1} < 0$ and since the reduced system is a 2 dimensional lagrangian system with potential $\mathcal{V}_j$, which is a decreasing function, there is an unbounded motion. Because $\lim_{p_1\to\infty} p_3p_4 > 0$, then $\lim_{p_1\to\infty} p_3 \neq 0$ which implies $\lim_{r\to\infty}\dot\beta \neq 0$ since $p_3 = r^2\dot\beta$.

□

Observation 5.5.1. For a general profile f it is hard to prove the existence of reduced motions agreeing statement 2. of Proposition 5.6. To see that condition $\lim_{p_1\to\infty} p_3p_4 > 0$ is not empty, for all profiles f, we show there exist such a reduced motion for the particular case of the downward cone $f(r) = -br$, see Section 5.6 for details on this particular case. In this case we have explicit expressions for the Routh integrals and lead to

$$p_3(p_1, j_1, j_2) = j_1, \qquad p_4(p_1, j_1, j_2) = \frac{bj_1}{\sqrt{2p_1}} + j_2,$$

and by direct computation we get

$$p_3p_4 = j_1j_2 + \frac{bj_1^2}{\sqrt{2p_1}}.$$

Therefore when j_1 and j_2 have the same sign the desired reduced motion exists. The product p_3p_4 as a function of p_1, j_1, j_2 has not definite but when we consider the limit as p_1 tends to infinity it does.

5.5.2 Mexican hat

In this section we study a special type of profiles which are defined in parts using bump functions or mullifiers[2] to "glue" two or more functions. To illustrate the mechanism we first present an example of such kind of construction.

Example. Let $f, h : \mathbb{R} \to \mathbb{R}$ be smooth functions and define

$$f(r) = \begin{cases} e^{-1/(1-\varepsilon r^2)} + h(r), & \text{if } r < \frac{1}{\sqrt{\varepsilon}} \\ h(r), & \text{if } r \geq \frac{1}{\sqrt{\varepsilon}} \end{cases}$$

where $\varepsilon \in \mathbb{R}^{>0}$ is a parameter. Note that f is smooth if the function h is and that their derivatives coincide in the interval

$$\frac{d^n f}{dr^n} = \frac{d^n h}{dr^n}, \quad \text{at } r \geq \frac{1}{\sqrt{\varepsilon}}.$$

This implies that equations of motion for the surface of revolution with profile h change significantly when $r < \frac{1}{\sqrt{\varepsilon}}$ and $r \geq \frac{1}{\sqrt{\varepsilon}}$.

Definition 5.1. Consider a smooth real function $h : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a h-hat function if there exists a positive real number $r_0 > 0$ such that $f(r) = h(r)$ for all $r > r_0$.

Proposition 5.7. *Consider a smooth real function h and let f be a h-hat function.*

1. *If the integral curves of equations of motion (5.14) with profile h are all bounded then the motions of the system with profile f are all bounded.*

2. *Let $\Omega = 0$. If the function h is bounded from above then equations of motion (5.14) with profile f have an unbounded motion.*

Proof. 1. Suppose that the motions of the system with profile h are all bounded, this property is independent of initial conditions. If f is a h-hat profile we have 4 scenarios for its motions. First if the initial conditions for a motion has $r < r_0$ it has two options either there exists a time t_0 such that $r(t_0) \geq r_0$ or for all t we have $r(t) < r_0$, if the latter happens then that motion is bounded and for the former we have the other two possibilities either there exists a time $t_1 > t_0$ such that $r(t_1) < r_0$, for this case use a recursive argument, or for all $t > t_1$ we have $r(t) \geq r_0$ this case corresponds to the hypothesis that motions for the profile h are bounded. Note that if the initial condition is such that $r \geq r_0$ then we can apply similar arguments.

2. Now assume $\Omega = 0$ and the function h is bounded from above, then by Proposition 5.6 the system with profile f has an unbounded motion.

□

5.6 Example: Downward cone

We focus to the particular case when the profile $f(r) = -br$, with parameter $b > 0$, the surface generated by f corresponds to a downward cone (without its vertex). The

[2]The existence of bump functions is assured by partitions of unity in a smooth manifold [47, 80].

parameter $b > 0$ regulates the amplitude of the cone. We investigate the dynamics of the system and in particular we look for bounded, unbounded, periodic and asymptotic motions. In [41] the authors give general conditions to analyze the stability of reduced equilibria, since here we treat a particular case we can say more about it. For example we add the possible scenarios regarding quantity and types of reduced equilibria with their dependence on the parameters.

Consider the profile function $f(r) = -br$, $b > 0$. In invariant polynomials it reads $\Psi(p_1) = -b\sqrt{2p_1}$. Since the cone with its vertex is not a differentiable manifold its vertex is not considered, so in this case $Q = SO(3) \times \left(\mathbb{R}^2 \setminus \{0,0\}\right)$, and we parametrize it by Euler angles and polar coordinates. We use quasivelocities (q, v) in $TSO(3)$ as in 5.1.2, where $q = (\phi, \theta, \psi, r, \beta)$ and $v = (\omega_x, \omega_y, \omega_z, \dot{r}, \dot{\beta})$. The lagrangian L (5.4) reads as

$$L(q, v) = \frac{k}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) + \frac{1}{2}(F^2\dot{r}^2 + r^2\dot{\beta}^2) + gbr,$$

where $F^2 = b^2 + 1$. Using constraint equations (5.5) we write ω_x and ω_y as functions of $(r, \beta, \omega_z, \dot{r}, \dot{\beta})$

$$\begin{aligned} \omega_x &= -F\sin\beta\dot{r} + \cos\beta(-rF\dot{\beta} + b\omega_z) + \Omega\cos\beta(rF - b), \\ \omega_y &= F\cos\beta\dot{r} + \sin\beta(-rF\dot{\beta} + b\omega_z) + \Omega\sin\beta(rF - b), \end{aligned} \tag{5.32}$$

then the submanifold $\mathcal{M}$ of TQ can be represented in quasi-velocities as

$$\mathcal{M} = \left\{(q, v) \in TQ \,:\, \omega_x = \omega_x(r, \beta, \omega_z, \dot{r}, \dot{\beta}),\, \omega_y = \omega_y(r, \beta, \omega_z, \dot{r}, \dot{\beta})\right\}.$$

And the equations of motion for the inverted cone are

$$\begin{aligned} \dot{q} &= Bv, \\ \dot{v}_r &= \frac{(\mu b^2 + 1)r}{F^2}\dot{\beta}^2 - \frac{b\mu}{F}\dot{\beta}\omega_z + \mu\Omega\left(\frac{b}{F} - r\right)\dot{\beta} + \frac{\gamma b}{F^2}, \\ \dot{v}_\beta &= \frac{-2}{r}\dot{r}\dot{\beta} + \frac{\mu\Omega}{r}\dot{r}, \\ \dot{\omega}_z &= \frac{b\Omega(\mu - 1)}{F}\dot{r}, \end{aligned} \tag{5.33}$$

together with the constraint equations (5.32).

The moving energy and Routh integrals are

$$\begin{aligned} \frac{1}{k+1}E_{L,Y}(r, \dot{r}, \dot{\beta}, \omega_z) &= \frac{F^2}{2}\dot{r}^2 + \frac{\mu b^2 + 1}{2}r^2\dot{\beta}^2 + \frac{\mu F^2}{2}\omega_z^2 - \mu brF\dot{\beta}\omega_z \\ &\quad + \Omega\left[\mu F(br - F)\omega_z + (\mu br(F - br) - r^2)\dot{\beta}\right] + \frac{\mu\Omega^2}{2}(rF - b)^2 - \gamma br, \\ J_1(r, \dot{\beta}) &= \frac{r^2}{2}\left(2\dot{\beta} - \mu\Omega\right), \\ J_2(r, \omega_z) &= -F\,\omega_z + b\Omega\left((\mu - 1)r + \frac{b}{F}\right). \end{aligned}$$

Where $\gamma = \frac{g}{k+1}$, $\mu = \frac{k}{k+1}$, note that in this case $F = \sqrt{b^2 + 1}$.

We now present the two gauge symmetries constructed by means of Theorem 5.4.

$$\begin{aligned} Z_1 &= \frac{1}{k+1}X_1 - \frac{1}{k(k+1)F}X_2 - \frac{e^2}{(k+1)(rF - b)^2}\xi \\ Z_2 &= \frac{r}{(k+1)bF}X_1 + \frac{r}{(k+1)bF^2}X_2 + \frac{r(2b - rF)}{2\Omega\,(b - rF)^2}\xi \end{aligned}$$

Note that

$$-\frac{e^2}{(k+1)(rF-b)^2}\left\|\xi\right\|_g^2 = -\frac{k\Omega^2 e^2}{(k+1)^2F^2},$$

is a constant function. Therefore $\tilde{Z}_1 = \frac{1}{k+1}X_1 - \frac{1}{k(k+1)F}X_2$ is a horizontal gauge symmetry. For the vector field Z_2 on Q by direct computation we see that $Z_2^{TQ}[L] \neq 0$ so it is not a R^0-gauge momentum. The first integrals generated by $\tilde{Z}_1, Z_2$ are

$$\begin{aligned} p_{\tilde{Z}_1}\big|_{\mathcal{M}} &= F\omega_z + \Omega\frac{b}{k+1}\left(r - \frac{b}{F}\right), \\ p_{Z_2}\big|_{\mathcal{M}} &= r^2\left(\dot{\beta} - \Omega\frac{k}{2(k+1)}\right). \end{aligned}$$

5.6.1 Reduced equations of motion

To analyze the qualitative behavior of the motions we use the reduced system and the fact that it is a second order ordinary differential equation, with langrangian energy being equal to the reduced moving energy. Therefore the qualitative properties are intimately related with the effective potential V_j, and then by reconstruction arguments we obtain information about unreduced motions.

We first project the equations of motion to $\mathcal{M}_4 = \mathcal{M}/G$

$$\begin{aligned} \dot{r} &= \dot{r}, \\ \ddot{r} &= \frac{(\mu b^2+1)r}{F^2}\dot{\beta}^2 - \frac{b\mu}{F}\dot{\beta}\omega_z + \mu\Omega\left(\frac{b}{F} - r\right)\dot{\beta} + \frac{\gamma b}{F^2}, \\ \ddot{\beta} &= \frac{-2}{r}\dot{r}\dot{\beta} + \frac{\mu\Omega}{r}\dot{r}, \\ \dot{\omega}_z &= \frac{b\Omega(\mu-1)}{F}\dot{r}. \end{aligned} \tag{5.34}$$

Observe that since all the first integrals are G-invariant then they pass to the quotient as first integrals of the reduced system.

We restrict equations (5.34) to the level sets of the Routh integrals $J = (J_1, J_2)$. To this end we write $\dot{\beta}$ and ω_z as functions of r, j_1 and j_2, where $j = (j_1, j_2)$ are regular values of the Routh integrals $J = (J_1, J_2)$

$$\dot{\beta}(r, j_1, j_2) = \frac{j_1}{r^2} + \frac{\mu\Omega}{2}, \quad \omega_z(r, j_1, j_2) = \Omega\left(\frac{b(\mu-1)}{F}r + \frac{b^2}{F^2}\right) - \frac{j_2}{F}. \tag{5.35}$$

Then the reduced equations of motion are the first two element of (5.34)

$$\begin{aligned} \dot{r} &= \dot{r}, \\ \ddot{r} &= \frac{(\mu b^2+1)r}{F^2}\dot{\beta}^2 - \frac{b\mu}{F}\dot{\beta}\omega_z + \mu\Omega\left(\frac{b}{F} - r\right)\dot{\beta} + \frac{\gamma b}{F^2} \end{aligned} \tag{5.36}$$

considering $\dot{\beta}$ and ω_z as in (5.35). Denote by $E_j(r,\dot{r}) = E_{L,Y}\left(r, \dot{r}, \dot{\beta}(r, j_1, j_2), \omega_z(r, j_1, j_2)\right)$ the moving energy restricted to $J^{-1}(j)$

$$E_j = T + V_j \circ \tau_{\mathcal{M}_4},$$

where $\tau_{\mathcal{M}_4} : \mathcal{M}_4 \to Q/G$ is the bundle projection and V_j is called the effective potential.

The explicit expressions of the functions T and V_j are

$$\begin{aligned} T(r,\dot{r}) =& \frac{F^2}{2}\dot{r}^2, \\ V_j(r) =& -\gamma br + \frac{\mu}{2}\left(j_2 + \frac{b}{r}j_1\right)^2 + \frac{j_1^2}{2r^2} + \Omega\left[\left(\frac{\mu}{F} - \frac{b\mu^2}{2}r\right)\left(j_2 + \frac{b}{r}j_1\right) + \frac{\mu-2}{2}j_1\right] \\ &+ \Omega^2\left[\frac{(b^2\mu+1)}{8}\mu^2r^2 - \frac{b\mu}{2F}\left(\frac{b}{F} + \mu r\right)\right]. \end{aligned}$$

The reduced equations of motion (5.34) restricted to $J^{-1}(j)$ form a system of ordinary differential equations of second order with E_j as first integral, then to study the qualitative properties of such system it suffices to analyze the effective potential V_j. We split the investigation of the effective potential into two cases, when the surface is at rest and $\Omega = 0$, and when the surface is rotating $\Omega \neq 0$.

To obtain the number and type of the critical points of V_j we use Descartes' rule of signs and the discriminant, denoted by d_Ω, of the polynomial

$$\begin{aligned} P_\Omega = r^3\frac{dV_j}{dr} =& \Omega^2\mu^2\frac{b^2\mu+1}{4}r^4 - \left(\gamma b + \Omega\frac{b\mu^2 j_2}{2} + \Omega^2\frac{b\mu^2}{2F}\right)r^3 \\ &- \left(b\mu j_1 j_2 + \Omega\frac{\mu b j_1}{F}\right)r - (b^2\mu+1)j_1^2. \end{aligned} \tag{5.37}$$

We emphasize the dependency of the functions d_Ω and P_Ω on the real parameter Ω, since there is a substantial difference between the cases $\Omega = 0$ and $\Omega \neq 0$.

Case $\Omega = 0$. If the surface is not rotating, i.e. $\Omega = 0$, the effective potential reads

$$V_j(r) = -\gamma br + \frac{(\mu b^2+1)j_1^2}{2r^2} + \frac{b\mu j_1 j_2}{r} + \frac{\mu j_2^2}{2}.$$

The function V_j has two main asymptotic behaviors, if $j_1 \neq 0$ then $\lim_{r\to 0} V_j = \infty$ and $\lim_{r\to\infty} V_j = -\infty$, and if $j_1 = 0$ then $\lim_{r\to 0} V_j = \frac{\mu j_2^2}{2}$ and $\lim_{r\to\infty} V_j = -\infty$.

Using Proposition 5.6, we can conclude that the downwards cone has unbounded motions.

Proposition 5.8. *Consider reduced equations of motion (5.36) and $\Omega = 0$, then there are three possibilities for the reduced equilibria.*

i) There are no equilibrium if one of the following conditions is satisfied, $j_1 j_2 \geq 0$ or $d_0 < 0$.

ii) There is one equilibrium, a cusp, if and only if the following two conditions are satisfied, $d_0 = 0$ and $j_1 j_2 < 0$.

iii) There are two equilibria, one of elliptic type and one saddle, if and only if the following two conditions are satisfied, $d_0 > 0$ and $j_1 j_2 < 0$.

Proof. Since $r > 0$, to compute the zeros of $\frac{dV_j}{dr}$ is the same as to compute the zeros of $r^3\frac{dV_j}{dr}$ which is the following degree 3 polynomial, P_0, on r

$$P_0 = -\gamma br^3 - b\mu j_1 j_2 r - (\mu b^2+1)j_1^2.$$

Let d_0 be the discriminant of P_0, then we have the following three alternatives

- $d_0 = 0$ if and only if $j_1 = -\frac{4b^2\mu^3 j_2^3}{27\gamma(\mu b^2+1)^2}$,

- $d_0 > 0$ if and only if one of the following two conditions hold $\left\{j_1 > 0 \text{ and } j_1 < -\frac{4b^2\mu^3 j_2^3}{27\gamma(\mu b^2+1)^2}\right\}$ or $\left\{j_1 < 0 \text{ and } j_1 > -\frac{4b^2\mu^3 j_2^3}{27\gamma(\mu b^2+1)^2}\right\}$,
- $d_0 < 0$ if and only if one of the following two conditions hold $\left\{j_1 > 0 \text{ and } j_1 > -\frac{4b^2\mu^3 j_2^3}{27\gamma(\mu b^2+1)^2}\right\}$ or $\left\{j_1 < 0 \text{ and } j_1 < -\frac{4b^2\mu^3 j_2^3}{27\gamma(\mu b^2+1)^2}\right\}$.

i) If we consider P_0 as a real variable polynomial then by Descartes' rule of signs P_0 has only one negative root if $j_1 \neq 0$, and if $j_1 = 0$ then there are no positive roots, this implies that there are no equilibria.

ii) If $d_0 = 0$ and $j_1 j_2 < 0$ then V_j has just one critic point which correspond to an almost saddle.

iii) If $d_0 > 0$ and $j_1 j_2 < 0$ then V_j has two critic points a minimun and a maximum which correspond to a center and a saddle point of the reduced system.

□

Remark. In polar coordinates the inequality $d_0 \geq 0$ is written as

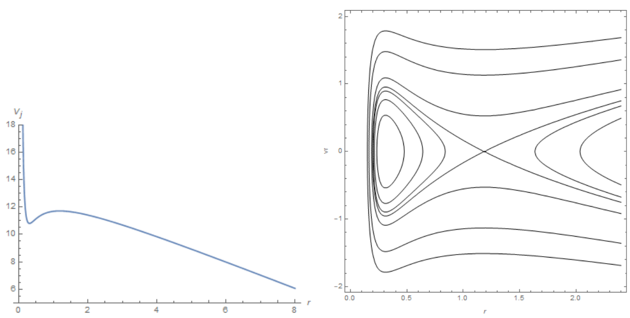
$$d_0 = 0 \text{ iff } r^2\dot{\beta} = \frac{4b^2(b^2+1)^{3/2}\mu^3}{27\gamma(\mu b^2+1)^2}\omega_z^3,$$

$$d_0 > 0 \text{ iff } \left\{\dot{\beta} > 0 \text{ and } r^2\dot{\beta} < \frac{4b^2(b^2+1)^{3/2}\mu^3}{27\gamma(\mu b^2+1)^2}\omega_z^3\right\},$$

$$d_0 > 0 \text{ iff } \left\{\dot{\beta} < 0 \text{ and } r^2\dot{\beta} > \frac{4b^2(b^2+1)^{3/2}\mu^3}{27\gamma(\mu b^2+1)^2}\omega_z^3\right\}.$$

So we have that case *ii*) implies that $\dot{\beta}$ and ω_z have opposite signs and case *iii*) implies that $\dot{\beta}$ and ω_z have the same sign.

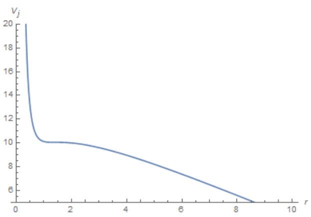
Proposition 5.8 gives conditions on the behavior of the reduced motions, this conditions can be shown as the qualitative structure of the effective's potential graph and phase space and bifurcation diagrams. We add some graphs in order complement with visual representations. All graphs are done with the following values for the parameters $\gamma = b = 1$, $\mu = \frac{2}{7}$ and $\Omega = 0$.



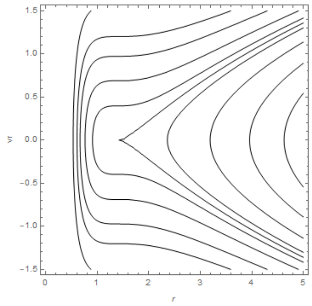
(a) Graph of the effective potential.
(b) Level sets of the reduced energy.

Figure 5.2: $j_1 = \frac{10000}{15309}$, $j_2 = -10$. Reduced system with 2 equilibria.

In figures 5.2 and 5.3 we have heteroclinic orbits so there are asymptotic motions towards an equilibrium point, this in the unreduced system traslates as motions that go asymptotically to a certain height of the cone. Along these motions the angular velocities

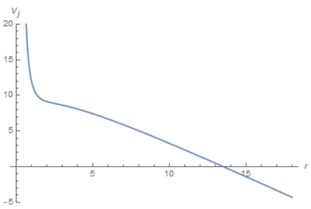


(a) Graph of the effective potential.

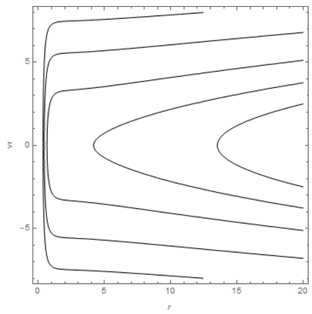


(b) Level sets of the reduced energy.

Figure 5.3: $j_1 = \frac{32000}{15309}$, $j_2 = -10$. Reduced system with 1 equilibrium.



(a) Graph of the effective potential.



(b) Level sets of the reduced energy.

Figure 5.4: $j_1 = \frac{60000}{15309}$, $j_2 = -10$. Reduced system with no equilibria.

$\dot{\beta}$ is bounded and $\omega_z \equiv -\frac{j_2}{\sqrt{b^2+1}}$, this is due to the fact that the reduced equilibria have coordinate r non zero and expressions given by (5.35).

In the scenario of Figure 5.4 the motions are unbounded and the angular velocity $\dot{\beta}$ has $\lim_{r\to\infty} \dot{\beta} = 0$, see (5.35).

The shadowed region of Figure 5.5 corresponds to values of j_1, j_2 on which there are two equilibria. The red curve is when there is just one equilibrium.

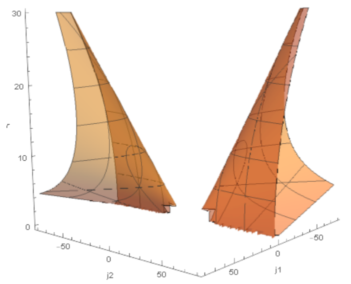


Figure 5.6: Critical points of V_j, j_1, j_2 are variables.

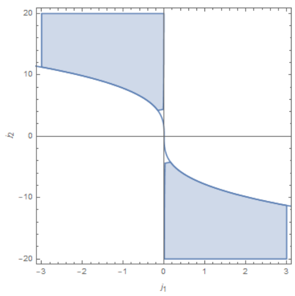


Figure 5.5: Bifurcation diagram of j_1 and j_2.

Case $\Omega \neq 0$.

Proposition 5.9. *Consider the reduced equations of motion (5.36) and let $\Omega \neq 0$. Then the reduced motions are bounded for all values $j \in \mathbb{R}^2$.*

Proof. The highest degree monomial of r in V_j is $\frac{(b^2\mu+1)}{8}\mu^2\Omega^2 r^2$, it is positive for every $\Omega \neq 0$ then we have

$$\lim_{r\to\infty} V_j = \infty,$$

so all the level sets of $E_j = T + V_j$ are bounded this implies that all motions of the reduced system are bounded.

□

Proposition 5.10. *Suppose $\Omega \neq 0$ then we have the following possibilities for the reduced equilibria. We treat the cases $j_1 = 0$ and $j_1 \neq 0$ separately. Assume $j_1 \neq 0$ then*

i) There are two equilibrium points, one of elliptic type and one cusp if and only if the following three conditions hold

$$j_2 > -\frac{2\gamma}{\mu^2\Omega} - \frac{\Omega}{\sqrt{b^2+1}},\ j_1\left(j_2 + \frac{\Omega}{\sqrt{b^2+1}}\right) < 0 \text{ and } d_\Omega = 0.$$

ii) There are three equilibrium points, two elliptics and one saddle point if and only if the following three conditions hold

$$j_2 > -\frac{2\gamma}{\mu^2\Omega} - \frac{\Omega}{\sqrt{b^2+1}},\ j_1\left(j_2 + \frac{\Omega}{\sqrt{b^2+1}}\right) < 0 \text{ and } d_\Omega > 0.$$

iii) Otherwise there is just one equilibrium point of elliptic type.

iv) Suppose $j_1 = 0$. There is one elliptic type equilibrium point if and only if

$$j_2 > -\frac{2\gamma}{\mu^2\Omega} - \frac{\Omega}{\sqrt{b^2+1}},$$

otherwise there are no equilibrium points.

Recall that d_Ω is the discriminant of the polynomial P_Ω.

Proof. First we prove $i), ii)$ and $iii)$ so assume $j_1 \neq 0$. To prove $i)$ and $ii)$ observe the following, extend the definition of P_Ω to a real variable polynomial, then using Descartes' rule of signs to analyze the positive roots of P_Ω one can see that the only possibility for this polynomial to have more than one positive root is when

$$j_2 > -\frac{2\gamma}{\mu^2\Omega} - \frac{\Omega}{\sqrt{b^2+1}} \text{ and } j_1\left(j_2 + \frac{\Omega}{\sqrt{b^2+1}}\right) < 0.$$

Now use the same result to analyze $P_\Omega(-x)$, with the above assumptions on j_1 and j_2, we get that there is just one negative root. Now if $d_\Omega = 0$ then there should be a multiple root and since it can't be a complex one then P_Ω must have three positive roots (counting multiplicity) but $\lim_{r\to\infty} V_j = \infty$ then V_j is forced to have one minimum and a degree 2 critic point. If $d_\Omega > 0$ all the roots of P_Ω must be real and different so V_j has two minima and one maximum. Analyzing the phase portrait we prove *i*) and *ii*).

To prove *iii*) again we use Descartes' rule of signs and observe that there is always one and only one positive root which corresponds to a minimum of V_j.

iv) Let $j_1 = 0$ then

$$\lim_{r\to 0} V_j = \frac{\mu}{2}\left(j_2^2 + \frac{2j_2\Omega}{\sqrt{b^2+1}} - \frac{b^2\Omega^2}{b^2+1}\right), \lim_{r\to\infty} V_j = \infty$$

$$P_\Omega(r) = \frac{\mu b^2+1}{4}\mu^2\Omega^2 r - \frac{b}{2}\left(2\gamma + \mu^2\Omega\left(j_2 + \frac{\Omega}{\sqrt{b^2+1}}\right)\right).$$

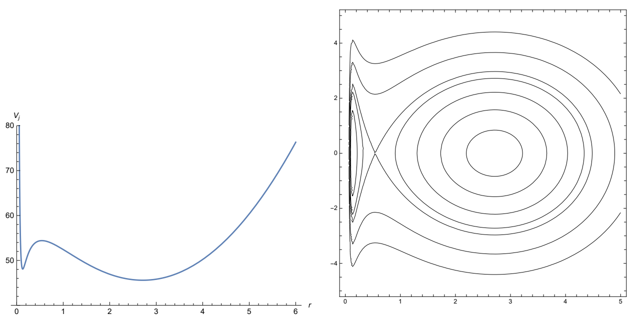
so there is just one critic point if and only if

$$j_2 > -\frac{2\gamma}{\mu^2\Omega} - \frac{\Omega}{\sqrt{b^2+1}}$$

□

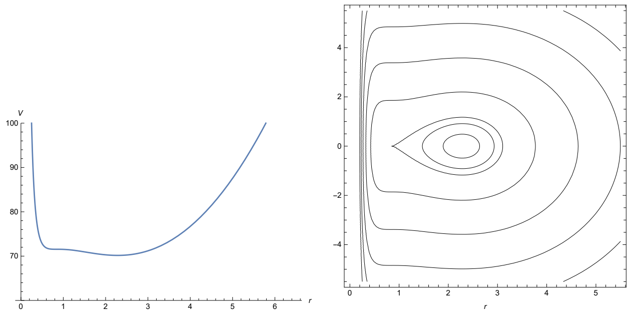
Observation. We analyze the polynomial P_Ω because its roots are the non zero critic points of V_j. In fact $P_\Omega(r)$ in upward cone case corresponds to $P_\Omega(-r)$ on the downward cone case.

Proposition 5.10 gives conditions on the behavior of the reduced motions, this conditions can be shown as the qualitative structure of the effective's potential graph and phase space and bifurcation diagrams. We add some graphs in order give visual representations. For following graphs we consider the values for the parameters $\gamma = b = 1$, $\mu = \frac{2}{7}$, and $\Omega = 15$.



(a) Graph of the effective potential. (b) Level sets of the reduced energy.

Figure 5.7: $j_1 = -0.7$, $j_2 = 15$. Reduced system with three equilibria.

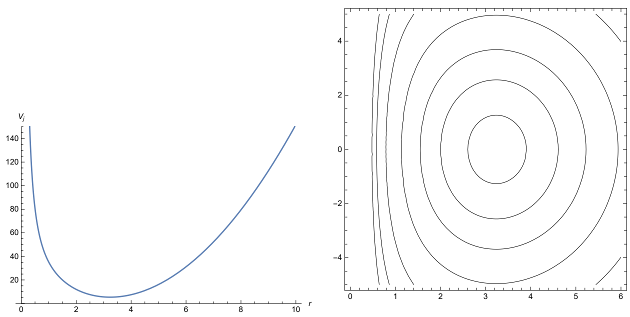


(a) Graph of the effective potential.

(b) Level sets of the reduced energy.

Figure 5.8: $j_1 \approx -2.95$, $j_2 = 15$. Reduced system with two equilibria.

In figures 5.7 and 5.8 we have heteroclinic orbits so there are asymptotic motions towards an equilibrium point, this in the unreduced system traslates as motions that go asymptotically to a certain height of the cone. Along these motions the angular velocities $\dot{\beta}$ and ω_z are bounded, this is due to the fact that the reduced equilibria have coordinate r non zero and the expressions given by (5.35).



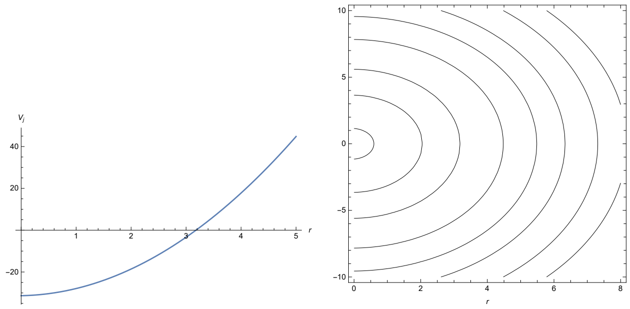
(a) Graph of the effective potential.

(b) Level sets of the reduced energy.

Figure 5.9: $j_1 = 3$, $j_2 = 15$. Reduced system with just one equilibrium.

In the scenario of Figure 5.9 have an elliptic type behavior on the reduced system which by reconstruction arguments this means that motions are trapped on a height strip.

The case of Figure 5.10 is an odd setting because $r = 0$ is not defined, but in this case the limit $\lim_{r \to 0} V_j < \infty$, since the limit of the reduced motions when r goes to 0 exists it is not clear what kind of motions we have because they aren't asymptotic to the vertex but in theory, after analytic extension, must reach it. The angular velocities $\dot{\beta} \equiv \frac{\mu\Omega}{2}$ and ω_z are bounded but can change signs, see expression (5.35).



(a) Graph of the effective potential. (b) Level sets of the reduced energy.

Figure 5.10: $j_1 = 0$, $j_2 = -13$. Reduced system with no equilibria.

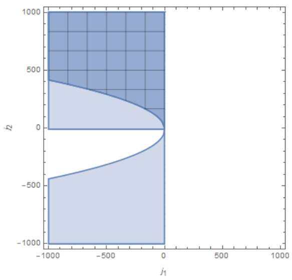


Figure 5.11: Bifurcation diagram of j_1 and j_2.

The squared region of Figure 5.11 represents the values of (j_1, j_2), for which the reduced system has three equilibria, the blue line is when it has two equilibria.

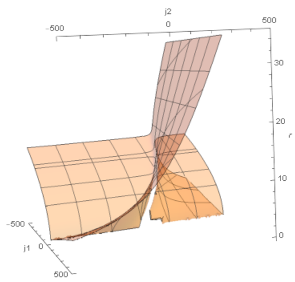


Figure 5.12: Critical points of V_j, j_1, j_2 are variables.

Chapter 6

Conclusions

The interesting behavior of nonholonomic systems with affine constraints or with mechanical lagrangian comes from their affine nature, this feature prevents such systems to have an evident (almost) Poisson formulation, and in general linear functions in the velocities are not first integrals. In this Chapter we give an overview of the work and future perspectives. We separate in three directions the contributions made in this Thesis.

Overview and future perspectives

Almost-Poisson bracket

In Chapter 2 we recall a known construction of an almost-Poisson bracket for linear constrained systems and natural lagrangian, then in Chapter 3 we presented a generalization for linear constrained nonholonomic systems with gyroscopic lagrangian.

Our contribution consists in the intrinsic construction and coordinate representation of an *affine* almost-Poisson bracket for this kind of systems. The formulation uses elements related to the kinetic energy metric and the canonical bracket on the cotangent bundle of a smooth manifold, moreover particular emphasis is made on the role that plays the gyroscopic 1-form for the bracket failing to be linear. Furthermore in the case symmetries are present we give a standard reduction procedure for the almost-Poisson bracket, the affine nature of the reduced bracket is again related to the gyroscopic 1-form. This material is developed in Sections 3.3 and 3.4.

Future perspectives

- We are working on a construction of this type of almost-Poisson bracket for affine constrained nonholonomic systems.
- We also aim to study gauge transformations for this affine almost-Poisson brackets and use them as a technique for hamiltonization (generalization in the spirit of [6, 64]).
- In the case the almost-Poisson system is invariant under a Lie group action to determine when such symmetries give place to Casimir functions of the bracket.

First integrals affine in the velocities

In Chapter 2 we treat nonholonomic systems with linear constraints and natural lagrangian, we presented known results on the existence and classification of momentum generated by a vector field which is also a first integral of the system. Inspired by these results in Chapter 3 we classify and introduce momentum generated by a vector field in the setting of nonholonomic systems with gyroscopic lagrangian.

The new theoretical contribution consists on the characterization of momentum generated by a vector field which are first integrals of certain nonholonomic systems with gyroscopic lagrangian. The construction uses a momentum of the underlying system (with natural lagrangian). The scenario when the system has symmetries is also treated, that is when a momentum is in fact a gauge momentum, this material is developed in Section 3.5.

In Chapter 4 we relate affine constrained systems with natural lagrangian and linear constrained systems with gyroscopic lagrangian, in the case the former system posses a Noether symmetry. We present conditions on when the momentum of both systems are related, and moreover when a vector field generates a momentum which is a first integral in both systems (Section 4.3).

Future perspectives

- As a future research project we pretend to investigate the relations of momentum affine in the velocities with the almost-Poisson structure, namely to determine when such momentum are Casimir functions of the bracket.
- Another interesting path is to give conditions on the existence of momentum which are first integrals for affine constrained systems not possessing a Noether symmetry.

Dynamics of a ball rolling without slipping in a rotating surface of revolution

In Chapter 5 we analyze the system of a homogeneous ball rolling without slipping in an uniformly rotating surface of revolution, the formulation of the equations of motions is based on [51, 41]. We analyzed the system from a geometric and dynamical point of view. Our contributions are in several directions: we prove the existence of two gauge momenta and are functionally dependent to them. As a result Corollary we elucidated the nature of the Routh integrals which was missing, (Section 5.3). We give an almost-Poisson bracket for this system using the theory developed in Sections 3.3 and 4.2. On the dynamical side we gave conditions on the existence of unbounded motions. Moreover when the surface is an inverted cone we analyze qualitatively and characterize all the motions (for both cases when the surface is at rest and rotating), namely we prove the existence of bounded, unbounded (for the static case), asymptotic and quasi-periodic motions.

Future perspectives

- To define an equivalent almost-Poisson bracket without the need to linearize the system, relate it to the existing reduced brackets [41] and explore its properties, and to see if the gauge momenta are Casimir for it.
- Investigate if the rotation of the surface stabilizes the system for general profiles, that is when the surface rotates then all motions are bounded.

Appendix A

The geometry of the nonholonomic Routh reduction

A.1 Intrinsic Routh reduction

This Appendix is devoted to the intrinsic construction of the nonholonomic Routh reduction presented in the Introduction. Consider a given configuration manifold $M = Q \times K$, with Q a n-dimensional manifold and K a $l-$dimensional abelian Lie group and a mechanical lagrangian $\mathcal{L} : TM \to \mathbb{R}$ which induces a riemannian metric $\mathcal{G}$ on M. The group K acts on M by left/right translations on the K-factor so the manifold M is trivially a principal K-bundle, with projection $\rho : M \to M/K = Q$. The nonholonomic constraint is given by a regular non integrable distribution $\tilde{\mathcal{D}} = \mathcal{D} \times TK$, where $\mathcal{D}$ is a non integrable regular distribution over Q. Furthermore we assume $\mathcal{L}$ is a K-invariant function and $\tilde{\mathcal{D}}$ is clearly K-invariant. Under this assumptions we can consider two principal connections $\Lambda, \mathcal{A} : TM \to \mathfrak{k}$, where Λ is the connection associated to the trivial principal bundle structure, $\mathcal{A}$ is the mechanical connection, see for e.g.[12], and $\mathfrak{k}$ is the Lie algebra of the group K. Note that both connections have as vertical space $VP_p = \ker T_p\rho$, $\forall p \in \mathcal{M}$, but their horizontal spaces may be different, in fact

$$\ker \Lambda_p = T_qQ \times \{0\} \quad \text{and} \quad \ker \mathcal{A}_p = (VP_p)^{\perp}.$$

We consider the horizontal lift $^h : TQ \to TM$ associated to $\mathcal{A}$ which is uniquely characterized by $T\rho \circ {}^h = id_{TQ}$ and $\mathcal{A} \circ {}^h = 0$, [29]. Using the horizontal lift we can define the orthogonal projections $hor, ver : TM \to TM$ as

$$hor = {}^h \circ T\rho \quad \text{and} \quad ver = id_{TM} - hor.$$

For each $\mu \in \mathfrak{k}^*$ consider the 1-form Λ_μ on $\mathcal{M}$ defined as

$$\Lambda_\mu(w_p) = \langle \mu, \Lambda(w_p) \rangle, \quad \forall w_p \in TM.$$

And its metric equivalent vector field $Z_\mu = \sharp_{\mathcal{G}}(\Lambda_\mu)$.

With all of the above elements and for every $\mu \in \mathfrak{k}^*$ the routhian $R^\mu : TQ \to \mathbb{R}$ is defined as

$$R^\mu(v_q) = (\mathcal{L} - \Lambda_\mu^\ell)(v_q^h + ver(Z_\mu)).$$

Using $\Lambda_\mu^\ell(ver(Z_\mu)) = \langle Z_\mu, ver(Z_\mu) \rangle_{\mathcal{G}} = \langle ver(Z_\mu), ver(Z_\mu) \rangle_{\mathcal{G}}$, we obtain

$$R^\mu(v_q) = \frac{1}{2} \left\| v_q^h \right\|_{\mathcal{G}}^2 - \Lambda_\mu^\ell(v_q^h) - \frac{1}{2} \left\| ver(Z_\mu) \right\|_{\mathcal{G}}^2 - \tilde{V}. \tag{A.1}$$

Letting $L = R^\mu$, $\gamma^\ell = \Lambda^\ell_\mu \circ h$ and $V = \frac{1}{2}\|ver(Z_\mu)\|^2_{\mathcal{G}} + \tilde{V}$ we obtain the same expression as in (9).

Remark. Even though Λ_μ is a closed 1-form on M, $\gamma \in \Omega^1(Q)$ need not be closed.

The non holonomic vector field of $(\mathcal{L} - \Lambda^\ell_\mu, \mathcal{M}, \tilde{\mathcal{D}})$, restricted to $hor(\tilde{\mathcal{D}}) + ver(Z_\mu)$, projects on the one of the system $(R^\mu, Q, \mathcal{D})$ because

$$T\rho(\tilde{\mathcal{D}}) = \mathcal{D}, \quad \tilde{\mathcal{D}} = \{v^h_q + ver(Z_\mu) \in TM \mid v_q \in \mathcal{D},\ \mu \in \mathfrak{k}^*\}.$$

A.2 Local expressions

Let $(q^j, \theta^J) \in Q \times K$ be coordinates on M, $j = 1, ..., n$, $J = 1, ..., l$. The principal bundle projection $\rho : \mathcal{M} \to Q$ in coordinates is $\rho(q, \theta) = q$. Since $\mathcal{L}$ is K-invariant then

$$\mathcal{L}(q, \dot{q}, \dot{\theta}) = \frac{1}{2}\mathcal{G}_{ij}(q)\dot{q}^i\dot{q}^j + \mathcal{G}_{iJ}(q)\dot{q}^i\dot{\theta}^J + \frac{1}{2}\mathcal{G}_{IJ}(q)\dot{\theta}^I\dot{\theta}^J - \tilde{V}(q).$$

On $\mathfrak{k}$ we choose the basis $\{o_J\}$, $J = 1, ..., l$, induced by the coordinates θ^J and on $\mathfrak{k}^*$ the dual basis $\{o^J\}$ associated to it. So we have

$$\mathcal{A}(q, \dot{q}, \theta, \dot{\theta}) = (\dot{\theta}^J + \mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j)o_J, \quad \Lambda(q, \dot{q}, \theta, \dot{\theta}) = \dot{\theta}^J o_J.$$

The horizontal lift in coordinates is

$$(q, \dot{q})^h = \dot{q}^j \frac{\partial}{\partial q^j} - \mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j \frac{\partial}{\partial \theta^J}.$$

Then the projections *hor* and *ver* write as

$$hor(q, \dot{q}, \theta, \dot{\theta}) = \dot{q}^j \frac{\partial}{\partial q^j} - \mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j \frac{\partial}{\partial \theta^J},$$
$$ver(q, \dot{q}, \theta, \dot{\theta}) = (\dot{\theta}^J + \mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j)\frac{\partial}{\partial \theta^J}.$$

The 1-form Λ_μ is $\Lambda_\mu(q, \theta) = \mu_J d\theta^J$, then

$$\Lambda^\ell_\mu(q, \dot{q}, \theta, \dot{\theta}) = \mu_J \dot{\theta}^J,$$
$$Z_\mu(q, \theta) = \mu_I \left(\tilde{\mathcal{G}}^{jI}\frac{\partial}{\partial q^j} + \tilde{\mathcal{G}}^{IJ}\frac{\partial}{\partial \theta^J}\right),$$
$$ver(Z_\mu(q, \theta)) = \mu_I \left(\tilde{\mathcal{G}}^{IJ} + \mathcal{G}^{JN}\mathcal{G}_{jN}\tilde{\mathcal{G}}^{jI}\right)\frac{\partial}{\partial \theta^J}.$$

Where $\tilde{\mathcal{G}} = \mathcal{G}^{-1}$. We note that $\mathcal{G}^{IJ} = \tilde{\mathcal{G}}^{IJ} + \mathcal{G}^{JN}\mathcal{G}_{jN}\tilde{\mathcal{G}}^{jI}$, then

$$ver(Z_\mu(q, \theta)) = \mu_I \mathcal{G}^{IJ}\frac{\partial}{\partial \theta^J}.$$

Using (A.1) the routhian $R^\mu : TQ \to \mathbb{R}$ in coordinates is represented as

$$\begin{aligned} R^\mu(q, \dot{q}) =& \frac{1}{2}\mathcal{G}_{ji}\dot{q}^j\dot{q}^i - \mathcal{G}_{iJ}\mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j\dot{q}^i + \frac{1}{2}\mathcal{G}_{IJ}\left(\mathcal{G}^{JN}\mathcal{G}_{jN}\dot{q}^j\right)\left(\mathcal{G}^{IS}\mathcal{G}_{iS}\dot{q}^i\right) \\ &+ \mu_J\mathcal{G}^{JI}\mathcal{G}_{jI}\dot{q}^j - \frac{1}{2}\mathcal{G}_{IJ}\left(\mu_N\mathcal{G}^{NI}\right)\left(\mu_S\mathcal{G}^{SJ}\right) - \tilde{V} \\ =& \frac{1}{2}\left(\mathcal{G}_{ij} - \mathcal{G}_{Ii}\mathcal{G}^{IJ}\mathcal{G}_{Jj}\right)\dot{q}^i\dot{q}^j + \mathcal{G}_{Ij}\mathcal{G}^{IJ}\mu_J\dot{q}^j - \frac{1}{2}\mathcal{G}^{IJ}\mu_I\mu_J - \tilde{V}. \end{aligned}$$

Appendix B

Poisson brackets

In this Appendix we present a brief and direct to the point introduction to Poisson manifolds, for a historical and more in depth exploration on the subject we refer the reader to [101, 4, 84, 38] and references therein.

B.1 Brief introduction to Poisson manifolds

Definition B.1. A Poisson manifold $(P, \{\cdot,\cdot\})$ is conformed by a smooth manifold P and a Poisson bracket $\{\cdot,\cdot\}$, which is a function

$$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P),$$

with the following properties

1. Skew-symmetric $\{f, g\} = -\{g, f\}$.
2. $\mathbb{R}$-bilinear $\{\lambda f + g, h\} = \lambda \{f, h\} + \{g, h\}$.
3. Leibniz rule $\{fh, g\} = h\{f, g\} + f\{h, g\}$.
4. Jacobi identity $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$.

For all functions $f, g, h \in C^\infty(P)$ and $\lambda \in \mathbb{R}$.

If Jacobi identity is not satisfied we say that $\{\cdot,\cdot\}$ is an almost-Poisson bracket.

As is standard, after defining the objects of study the functions that preserve them are introduced.

Definition B.2. Let $(P_1, \{\cdot,\cdot\}_1)$ and $(P_2, \{\cdot,\cdot\}_2)$ be two Poisson manifolds. A smooth map $\Phi : P_1 \to P_2$ is called Poisson if for all $f, g \in C^\infty(P_2)$

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi.$$

The hamiltonian formulation of the equations of motion in a mechanical system is a fundamental idea, in geometric mechanics more specifically in the Poisson setting it is defined as follows

Definition B.3. Let $(P, \{\cdot,\cdot\})$ be a Poisson manifold and $H \in C^\infty(P)$ a smooth function. The hamiltonian vector field $X_H \in \mathfrak{X}(P)$ associated to H is defined as

$$X_H(f) = \{f, H\}, \quad \forall f \in C^\infty(P).$$

Then we have that hamilton equation of motion with hamiltonian H is given by

$$\dot{f} = \{f, H\}.$$

Furthermore, let ϕ_t be the flow of X_H then

$$\frac{d}{dt}F \circ \phi_t = d_{\phi_t}F(X_H) = X_H(F \circ \phi_t) = \{F \circ \phi_t, H\}. \tag{B.1}$$

A direct consequence of Definition B.3 is the relation between the sets of smooth functions and hamiltonian vector fields.

Proposition B.1. *The assignation $H \mapsto X_H$ is a Lie algebra antihomomorphism, i.e.*

$$X_{\{F,H\}} = -[X_F, X_H]$$

Proof. By definition it is clearly a bilinear and skew-symmetric map. To prove that it preserves the Lie bracket we use Jacobi identity and the equality $X_H(f) = \{f, H\}$. Let $f \in C^\infty(P)$ then

$$\begin{aligned} X_{\{F,H\}}(f) &= \{f, \{F, H\}\} = -\{H, \{f, F\}\} - \{F, \{H, f\}\} \\ &= X_H(X_F(f)) - X_F(X_H(f)) = -[X_F, X_H](f), \end{aligned}$$

since this is for all functions f we get the desired result.

□

Because for every function $f \in C^\infty(P)$ there is a linear derivation related to it, concretely the hamiltonian vector field X_f, and $\{g, f\} = X_f(g) = dg(X_f)$, then the function $\{g, f\}$ just depends on the differentials dg and df, therefore there is a bivector field $\Pi \in \bigwedge^2(TP)$ such that

$$\{f, g\} = \Pi(df, dg).$$

Observation B.1.1. A bivector field $\Pi \in \bigwedge^2(TP)$ is said to be Poisson if $[\Pi, \Pi] = 0$, where the considered bracket is the Schouten bracket, this condition is equivalent to the Jacobi identity being satisfied, see [84, 38].

Let $(x^1, ..., x^n)$ be local coordinates on the manifold P, then, locally, the Poisson bracket is determined by the functions

$$\left\{x^i, x^j\right\} = \Pi_{ij} \quad i, j = 1, ..., n. \tag{B.2}$$

So we obtain the coordinate expression for the bivector field Π

$$\Pi = \Pi_{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

The Poisson bivector field has such local representation because, locally, T^*P is generated by the differentials dx^i. Using the local expression of Π we get

$$\{f, g\} = \Pi_{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

The bivector field Π induces a vector bundle morphism $\Pi^\sharp : T^*P \to TP$ given by the property

$$\Pi(df, dg) = \langle df, \Pi^\sharp(dg)\rangle,$$

and as a consequence a hamiltonian vector field can be represented as $X_f = \Pi^\sharp(df)$.

Definition B.4. The rank of a Poisson bracket at a point $p \in P$ is the rank of the function

$$\Pi_p^\sharp : T_p^* P \to T_p P.$$

Two functions are said to be in involution or Poisson commute if their bracket is zero, this implies that each function is a first integral of the other ones hamiltonian vector field, or equivalently.

Proposition B.2. *Let F, G be two smooth functions. $\{F, G\} = 0$ if and only if the flows of X_F and X_G commute.*

Among all functions there are special ones which are first integrals for all hamiltonian vector fields.

Definition B.5. A function $F \in C^\infty(P)$ is called a Casimir for the Poisson bracket if and only if

$$\{F, G\} = 0 \quad \forall G \in C^\infty(P).$$

Proposition B.3. *Let $\Phi : P_1 \to P_2$ be a Poisson map and a function $H \in C^\infty(P_2)$. If $\gamma(t)$ is an integral curve of the hamiltonian vector field $X_{H\circ\Phi}$ then $\Phi \circ \gamma(t)$ is an integral curve of X_H.*

Proof. Let $F \in C^\infty(P_2)$ then

$$\frac{d}{dt} F \circ \Phi(\gamma(t)) = \frac{d}{dt}(F \circ \Phi)(\gamma(t)) = \{F \circ \Phi, H \circ \Phi\}_1 (\gamma(t)) = \{F, H\}_2 (\Phi(\gamma(t))),$$

so $\frac{d}{dt} F \circ \Phi(\gamma(t)) = (X_H(F)) (\Phi \circ \gamma(t))$ for all functions $F \in C^\infty(P_2)$. Therefore $\Phi \circ \gamma$ is an integral curve of X_H.

□

Proposition B.4. *The flow of a hamiltonian vector field is a Poisson map.*

Proof. Let ϕ_t be the flow of the hamiltonian vector field X_H and $F, G \in C^\infty(P)$ smooth functions. Define the function $f = \{F \circ \phi_t, G \circ \phi_t\} - \{F, G\} \circ \phi_t$ then using bilinearity of the Poisson bracket and equation (B.1) we get

$$\begin{aligned} \frac{d}{dt} f &= \frac{d}{dt} \{F \circ \phi_t, G \circ \phi_t\} - \frac{d}{dt} \{F, G\} \circ \phi_t \\ &= \{\{F \circ \phi_t, H\}, G \circ \phi_t\} + \{F \circ \phi_t, \{G \circ \phi_t, H\}\} - \{\{F, G\} \circ \phi_t, H\} \end{aligned}$$

using Jacobi identity on the first two

$$\frac{d}{dt} f = \{f, H\} = X_H(f),$$

the function f being a solution of the ordinary differential equation is equivalent to write $f(t, p) = f(0, \phi_t(p))$ but $f(0, p) = 0$ for all points p, therefore ϕ_t is a Poisson map.

□

Definition B.6. Let $(P_1, \{\cdot, \cdot\}_1)$ and $(P_2, \{\cdot, \cdot\}_2)$ be Poisson manifolds and $i : P_2 \hookrightarrow P_1$ an injective immersion. $(P_2, \{\cdot, \cdot\}_2)$ is a Poisson submanifold if and only if i is a Poisson map.

The next theorem is fundamental because it locally characterizes the structure of a Poisson manifold.

Theorem B.5 (Splitting theorem). *[101] Let x_0 be an element on a Poisson manifold P. Then there exists a neighborhood U of x_0 and a diffeomorphism $\phi = \phi_S \times \phi_N : U \to S \times N$, where S is a symplectic manifold with $\dim S = rank \Pi^\sharp_{x_0}$ and the induced Poisson bracket in N has rank zero at $\phi_N(x_0)$.*

The proof of the splitting Theorem B.5 is similar to the one of Darboux's Theorem, see [84, 26]. In fact if the Poisson structure is of constant rank in the neighborhood U we obtain a similar coordinate description.

Corollary B.6. *Suppose the Poisson structure has constant rank around x_0. Then there exists a coordinate chart of x_0 $(U, (q, p, y))$, with $q = (q^1, ..., q^k)$, $p = (p^1, ..., p^k)$ and $y = (y^1, ..., y^r)$, and the coordinate functions satisfy $\left\{q^i, q^j\right\} = \left\{p^i, p^j\right\} = \left\{q^i, y^a\right\} = \left\{p^i, y^a\right\} = \left\{y^a, y^b\right\} = 0$, with $i, j = 1, ..., k$ and $a, b = 1, ..., r$.*

B.1.1 Linear Poisson brackets

Let V be a finite dimensional vector space endowed with a *linear* Poisson bracket $\{\cdot, \cdot\}$, i.e. the bracket of linear functions is again a linear function, this condition makes the dual space V^* into a Lie algebra (with the Poisson bracket as Lie bracket). The Poisson bracket on V is given as follows. Let $f, g \in C^\infty(V)$ and $v \in V$ then

$$\{f, g\}(v) = \langle [df(v), dg(v)], v\rangle, \tag{B.3}$$

where $\langle \cdot, \cdot\rangle$ is the pairing between V and its dual V^*. This construction can be used when we consider a Lie algebra $\mathfrak{g}$ (take $\mathfrak{g} = V^*$), in such case the resulting Poisson bracket defined by equation (B.3) in $\mathfrak{g}^*$ is called Lie-Poisson bracket. For a coordinate description let $X_1, ..., X_r \in \mathfrak{g}$ be a basis, with structure coefficients $[X_i, X_j] = C^k_{ij} X_k$ and $x^1, ..., x^r : \mathfrak{g} \to \mathbb{R}$ the coordinate functions associated to such basis. Then the Poisson bracket of the coordinate functions is $\left\{x^i, x^j\right\} = C^k_{ij} x^k$, and for functions $f, g \in C^\infty(\mathfrak{g})$ we get

$$\{f, g\} = \Pi_{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad \text{where } \Pi_{ij} = C^k_{ij} x^k.$$

We can extend the idea of a linear Poisson bracket in the vector bundle scenario by requiring that on each fiber the restriction of the Poisson bracket is linear, for example the canonical Poisson bracket on T^*Q is linear, where Q is a smooth manifold.

B.2 Poisson brackets and symmetry

Suppose there is a Lie group G acting free and properly on a Poisson manifold P, furthermore assume that for every $g \in G$ the diffeomorphism $\Psi_g : P \to P$ is a Poisson map, then

Definition B.7. Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold and G a Lie group acting free and proper on P. We say that the Poisson structure is G-invariant if for every $g \in G$ the diffeomorphism $\Psi_g : P \to P$ is a Poisson map.

The above definition is equivalent to ask for the invariance of the bivector field Π associated to the Poisson bracket, i.e. $(\Psi_g)_* \Pi = \Pi$.

Observation B.2.1. In the setting of Definition B.7 it implies that the set of smooth G-invariant functions $C^\infty(P)^G$ is closed under the bracket, $\left\{C^\infty(P)^G, C^\infty(P)^G\right\} \subseteq C^\infty(P)^G$.

Theorem B.7. *Let G be a Lie group acting free and properly on a Poisson manifold P such that the Poisson bracket is G-invariant. Then there is a unique Poisson structure on the manifold P/G such that the quotient map $\rho : P \to P/G$ is a Poisson map.*

Proof. First we prove uniqueness of the reduced Poisson bracket in P/G follows from Observation B.2.1 and the bijection between $C^\infty(P)^G$ and $C^\infty(P/G)$, in other words for every G-invariant function $F \in C^\infty(P)^G$ there exist a unique function $f \in C^\infty(P/G)$ such that $F = f \circ \rho$, we apply this reasoning to the function $\{f \circ \rho, k \circ \rho\}_P \in C^\infty(P)^G$, where $f, k \in C^\infty(P/G)$ and by hypothesis we have $\{f \circ \rho, k \circ \rho\}_P = \{f, k\}_{P/G} \circ \rho$, since ρ is surjective then the reduced bracket is uniquely determined.

The bracket just defined is clearly our candidate, the above argument also proves that it is well defined. To prove that it is in fact Poisson is a straightforward computation involving the properties of the Poisson bracket in P, the quotient map ρ being Poisson and the bijection between $C^\infty(P)^G$ and $C^\infty(P/G)$.

□

In the case a function $H \in C^\infty(P)$ is G-invariant then there exists a function $h \in C^\infty(P/G)$ such that $H = h \circ \rho$ and the hamiltonian vector fields $X_H \in \mathfrak{X}(P)$ and $X_h \in \mathfrak{X}(P/G)$ are ρ-related, in symbols $T\rho X_H = X_h \circ \rho$, to prove this assertion we show that $X_H(f \circ \rho) = X_h(f) \circ \rho$, for every function $f \in C^\infty(P/G)$

$$\begin{aligned} X_H(f \circ \rho) &= \{f \circ \rho, H\}_P = \{f \circ \rho, h \circ \rho\}_P \\ &= \{f, h\}_{P/G} \circ \rho = X_h(f) \circ \rho. \end{aligned}$$

B.3 Poisson structure on T^*Q

Let Q be a smooth manifold. It is well known that the cotangent bundle T^*Q has a canonical symplectic structure constructed as follows. Consider the cotangent and tangent bundle projections

$$\pi_Q : T^*Q \to Q, \quad \tau_{T^*Q} : T(T^*Q) \to T^*Q,$$

using the bundle projections we define the tautological or Liouville 1-form $\theta \in \Omega^1(T^*Q)$ as

$$\theta(X) := \langle \tau_{T^*Q}(X), T\pi_Q(X) \rangle, \quad \forall X \in \mathfrak{X}(T^*Q).$$

And the symplectic form $\omega \in \Omega^2(T^*Q)$ on the cotangent bundle T^*Q is

$$\omega = d\theta.$$

For all the local descriptions along the section we use the (canonical) coordinates $(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$ on T^*Q, where q are local coordinates on Q and the fiber coordinates p are induced by the differentials, $dq^1, ..., dq^n$ in $\Omega^1(Q)$.

Then a local representation of the Liouville 1-form θ in the mentioned coordinates is

$$\theta = p_i dq^i,$$

and therefore the symplectic form ω is given by

$$\omega = dq^i \wedge dp_i.$$

As in a Poisson manifold we define the, symplectic, hamiltonian vector field $X_H \in \mathfrak{X}(T^*Q)$ of a function $H \in C^\infty(T^*Q)$ to be the unique vector field satisfying

$$i_{X_H}\omega = -dH.$$

Locally we have

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Using the symplectic form and hamiltonian vector fields, we can define a Poisson bracket on T^*Q, the construction goes as follows. Let $f, g \in C^\infty(T^*Q)$ then the Poisson bracket is defined as

$$\{f, g\} := \omega(X_f, X_g). \tag{B.4}$$

The prove that such bracket is in fact Poisson are straightforward computations, all properties but Jacobi identity are easily verified using the inherited properties of the symplectic form ω. Jacobi identity for this Poisson bracket follows from the symplectic form being closed, i.e. $d\omega = 0$.

The bivector field Π associated to such Poisson structure locally writes

$$\Pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

Since the Poisson bracket in T^*Q is linear we can easily compute the coordinate expression for it in generalized coordinates. Let $\{X_i\}_{i=1}^n \subset \mathfrak{X}(Q)$ be a local frame such that $X_i = B_{ij}\frac{\partial}{\partial q^j}$, where $B_{ij} : Q \to \mathbb{R}$ are smooth function, associated to this frame are the structure constant $C_{ij}^k : Q \to \mathbb{R}$. If $\tilde{p} = (\tilde{p}_1, ..., \tilde{p}_n)$ are the fiber coordinates, in T^*Q, induced by the dual frame of $\{X_i\}_{i=1}^n$ then the bivector field Π has the local expression

$$\Pi = B_{ji}\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial \tilde{p}_j} - \frac{1}{2}C_{ij}^k \tilde{p}_k \frac{\partial}{\partial \tilde{p}_i} \wedge \frac{\partial}{\partial \tilde{p}_j}.$$

Observation B.3.1. In the case ω is not closed but it is non degenerate formula (B.4) still makes sense to define an almost-Poisson bracket.

Observation B.3.2. The above construction of a Poisson bracket using a symplectic form proves that all symplectic manifolds are Poisson, the converse is not true see [85, 101].

Poisson structures derived from symplectic ones have special properties such as the followings. The symplectic form ω induces a vector bundle isomorphism $\omega^\flat : T(T^*Q) \to T^*(T^*Q)$ given by $\omega^\flat(X) = i_X\omega$. The fact that it is an isomorphism is due to the non-degeneracy of ω. Moreover we have $(\omega^\flat)^{-1} = \Pi^\sharp$, this implies that $\Pi^\sharp$ is of maximal constant rank at every point of T^*Q.

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