ON THE MULTIGRADED HILBERT FUNCTION OF LINES AND RATIONAL CURVES IN MULTIPROJECTIVE SPACES

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Abstract. We study the multigraded Hilbert function of general configurations of lines in multiprojective spaces. In several cases we prove that this multigraded Hilbert function is the expected one. We make conjectures about other configurations and for small genus curves with a prescribed multidegree.

1. Introduction

Disclaimer: In this paper we only prove results on some general unions of lines in the Segre-Veronese embeddings of a multiprojective space Y, i.e. when we embed Y in a projective space by a complete linear system. More general smooth rational curves (or any curve with positive genus) appear only in the conjectures at the end of the introduction.

Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \ge 1$, $n_i > 0$, $1 \le i \le k$. The Segre embedding of Y is connected to tensors, while the Segre-Veronese embeddings of Y are connected to partially symmetric tensors ([23]). This is the main reason for the interest in multiprojective spaces in the last 20 years ([5, 6, 7, 12, 13, 14, 16, 22, 23, 26] are just a small sample, many other papers may be obtained from their bibliographies and from papers quoting them).

Let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection of Y onto its i-th factor. For any $i \in \{1, \dots, k\}$ set $Y_i := \prod_{h \ne i} \mathbb{P}^{n_h}$ with the convention that Y_1 is a single point if $k = 1$. Let $\eta_i : Y \to Y_i$ denote the projection (it is the map forgetting the i-th coordinate of the points of Y). For all $(d_1, \dots, d_k) \in \mathbb{Z}^k$ set $\mathcal{O}_Y(d_1, \dots, d_k) := \otimes_{i=1}^k \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(d_i))$. The line bundles $\mathcal{O}_Y(d_1, \dots, d_k)$, $(d_1, \dots, d_k) \in \mathbb{Z}^k$, form a $\mathbb{Z}$-basis of the abelian group $\mathrm{Pic}(Y)$. The Künneth formula gives $h^0(\mathcal{O}_Y(d_1, \dots, d_k)) = 0$ if some $d_i < 0$, $h^0(\mathcal{O}_Y(d_1, \dots, d_k)) = \prod_{i=1}^k \binom{n_i+d_i}{n_i}$ if $d_i \ge 0$ for all i and $h^1(\mathcal{O}_Y(d_1, \dots, d_k)) = 0$ if $d_i \ge -1$ for all i. For any $i \in \{1, \dots, k\}$ let $\mathcal{O}_Y(\varepsilon_i)$ be the line bundle $\mathcal{O}_Y(a_1, \dots, a_k)$ on Y with multidegree $(a_1, \dots, a_k)$ with $a_i = 1$ and $a_j = 0$ for all $j \ne i$. We have $h^0(\mathcal{O}_Y(\varepsilon_i)) = n_i + 1$. Let C be an integral projective curve and let $f : C \to Y$ be a morphism. The multidegree $(a_1, \dots, a_k) \in \mathbb{N}^k$ of the pair (C, f) is defined by the formula $a_i := \deg(f^*(\mathcal{O}_Y(\varepsilon_i)))$, $i = 1, \dots, k$. Note that $a_i = 0$ if and only if $\pi_i \circ f$ is a constant map, while if $a_i \ne 0$, then $a_i = \deg(\pi_i \circ f) \cdot \deg(\pi_i(f(C)))$, with the convention $\deg(\pi_i(f(C))) = 1$ if $n_i = 1$. An i-line of Y is a curve $L \subset Y$ with multidegree ε_i. The set of all i-lines is parametrized by an irreducible quasi-projective variety. If $k \ge 2$ two general i-lines are disjoint.

In this paper prove that general unions of an arbitrary number of i-lines have the " expected postulation " with respect to all line bundles on Y if we assume $n_i \ne 2$. We recall the meaning of " expected postulation " also called " maximal rank ".

Let X be a projective scheme, $\mathcal{L}$ a line bundle on X and $Z \subset X$ a closed subscheme of X. We say that Z has *maximal rank with respect to* $\mathcal{L}$ if the restriction map $H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}_{|Z})$ has maximal rank as a linear map, i.e. it is injective or surjective. When (as always in this paper) $h^1(X, \mathcal{L}) = 0$, Z has maximal rank with respect to $\mathcal{L}$ if and only if either $h^0(X, \mathcal{I}_Z \otimes \mathcal{L}) = 0$ or $h^1(X, \mathcal{I}_Z \otimes \mathcal{L}) = 0$.

We prove the following results.

Theorem 1.1. *Assume $n_1 = 1$. Fix integers $k \ge 2$, $t > 0$, and $(d_1, \dots, d_k) \in \mathbb{N}^k$. Let $T \subset Y$ be a general union of t 1-lines. Then T has maximal rank with respect to $\mathcal{O}_Y(d_1, \dots, d_k)$.*

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Theorem 1.2. *Assume $n_1 \geq 3$. Fix integers $k \geq 2$, $t > 0$, and $(d_1, \dots, d_k) \in \mathbb{N}^k$. Let $T \subset Y$ be a general union of t 1-lines. Then T has maximal rank with respect to $\mathcal{O}_Y(d_1, \dots, d_k)$.*

Question 1.3. *Is the case $n_1 = 2$ true, except for finitely many cases k, n_i , $2 \leq i \leq k$, and $(d_1, \dots, d_k)$?*

We expect a very short list of exceptional cases. See Example 2.9 for one easy case. We do not know other cases.

Conjecture 1.4. *Fix positive integers k and n_i, $1 \leq i \leq k$, such that $n_1 + \cdots + n_k \geq 3$. We conjecture the existence of $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that for all $(b_1, \dots, b_k) \in \mathbb{N}^k$ with $b_i \geq a_i$ for all i a general union $T \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of $b_1 + \cdots + b_k$ lines, exactly b_i of them being i-lines, has maximal rank with respect to all line bundles on Y.*

Conjecture 1.5. *Fix positive integers k and n_i, $1 \leq i \leq k$, such that $n_1 + \cdots + n_k \geq 3$. We conjecture the existence of an integer d (depending only on $k, n_1, \dots, n_k$) such that for all $(b_1, \dots, b_k) \in \mathbb{N}^k$ a general union $T \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of $b_1 + \cdots + b_k$ lines, exactly b_i of them being i-lines, has maximal rank with respect to all line bundles $\mathcal{O}_Y(d_1, \dots, d_k)$ with $d_i \geq d$ for all i.*

We are less sure about the following 3 stronger conjectures. Note that in Conjecture 1.6 (resp. Conjecture 1.7) we require that a (resp. d) does not depend on k and n_i, $1 \leq i \leq k$.

Conjecture 1.6. *We ask if there is an integer a with the following property. Fix positive integers k and n_i, $1 \leq i \leq k$, such that $n_1 + \cdots + n_k \geq 3$. For all $(b_1, \dots, b_k) \in \mathbb{N}^k$ with $b_i \geq a_i$ for all i a general union $T \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of $b_1 + \cdots + b_k$ lines, exactly b_i of them being i-lines, has maximal rank with respect to all line bundles on Y.*

Conjecture 1.7. *We ask if there is an integer d with the following property. Fix positive integers k and n_i, $1 \leq i \leq k$, such that $n_1 + \cdots + n_k \geq 3$. For all $(b_1, \dots, b_k) \in \mathbb{N}^k$ a general union $T \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of $b_1 + \cdots + b_k$ lines, exactly b_i of them being i-lines, has maximal rank with respect to all line bundles $\mathcal{O}_Y(d_1, \dots, d_k)$ with $d_i \geq d$ for all i.*

Conjecture 1.8. *Are Conjectures 1.4, 1.5, 1.6 or 1.7 true for smooth rational curves with prescribed multidegrees and/or curves with positive genus with respect to non-special line bundles ?*

The case of smooth rational curves should be easier to prove than the corresponding case for general disjoint unions of lines.

As in [20, 21] and papers inspired by the works of R. Hartshorne and A. Hirschowitz (see [24, 25] and references therein) the proof is inductive using a divisors D of the ambient variety Y (double induction, on the dimension of the variety Y and the " degree " of the line bundle). As in [20, 21] to prove Theorem 1.2 one defines several inductive statements (here called Assertions $A_{m,a,z}$ and $B_{m,a,z}$, each of them depending on the 3 integers m, a and z such that $3 \leq m \leq n_1$, $a \geq 0$ and $z > 0$) and the proof requires all of them to be completed, each of them being used for the proof of the other ones in higher dimensions. However, there is no logical needs for these inductive assumptions and different ones may be given. We think that our choice is efficient (many other choices would give longer proofs). As in [20, 21] the main technical problems come from the integers $h^0(\mathcal{L})$ for many line bundles $\mathcal{L}$ and the fact they are not divisible by the integer $h^0(C, \mathcal{L}_{|C})$, C a closed subscheme of Y we are intersested in, e.g. an i-line. Here we need several numerical lemmas. As in [20] the initial case for the dimension (here the case $n_1 = 3$ of Theorem 1.2) is a bit different. We do it in Section 3. To prove $A_{3,a,z}$ and $B_{3,a,z}$ we use the modified Asserions $\tilde{A}_{3,a,z}$ and $\tilde{B}_{3,a,z}$ proved at the end of Section 3.

The sundials from [20] were promoted to main actors in [8, 9, 10] and these papers (together with [20]) gave for instance [1, 3, 4, 17, 18, 27].

We work over an algebraically closed field with characteristic 0.

2. Preliminaries

Let X be a projective scheme, $Z \subset X$ a closed subscheme and $D \subset X$ an effective Cartier divisor of X. The *residual scheme of Z with respect to D* is the closed subscheme $\mathrm{Res}_D(Z)$ of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For any line bundle $\mathcal{L}$ on X the following sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z \cap D} \otimes \mathcal{L}_{|D} \to 0 \tag{1}$$

is exact and we will call it *the residual exact sequence of* D without mentioning $\mathcal{L}$ and Z.

A *tangent vector* (or a *tangent vector of the variety* W) is a closed zero-dimensional scheme $Z \subset W$ such that $\deg(Z) = 2$ and Z is connected. A tangent vector $v \subset Y$ is said to be *of type* i or a *tangent* i*-vector* (for some $i \in \{1, \dots, k\}$) if $\deg(\eta_i(v)) = 1$. Thus v is a tangent i-vector for some i if and only if $\nu(v)$ spans a line L contained in $\nu(Y)$, where ν is the Segre embedding of Y, and the type of v describes the ruling of Y containing L.

We say that a curve $C \subset Y$ is a *reducible* i*-conic*, $i \in \{1, \dots, k\}$, if it is a reduced and connected curve with 2 irreducible components, both of them being i-lines. Note that a reducible i-conic has a unique singular point. A reducible i-conic exists if and only if $n_i \ge 2$. The set of all reducible i-conics is parametrized by an integral quasi-projective variety. If $k \ge 2$ two general i-conics are disjoint.

Assume $n_i \ge 3$. We say that a scheme $Z \subset Y$ is an i*-sundial* if $C := Z_{\mathrm{red}}$ is a reducible i-conic, Z is the union of C and a tangent vector whose reduction is the singular point of C and, calling ν the Segre embedding of Y, the scheme $\nu(Z)$ spans a 3-dimensional linear space contained in Y. The latter condition is equivalent to require that $\eta_i(Z)$ is scheme-theoretically a point with its reduced structure. Note that an i-sundial exists if and only if $n_i \ge 3$. Let $Z \subset Y$ be an i-sundial. Then $h^0(\mathcal{O}_Z) = 2$ and $h^1(\mathcal{O}_Z) = 0$. The set of all i-sundials is parametrized by an integral quasi-projective variety.

A *planar* i*-double point* of Y is a connected degree 3 zero-dimensional scheme $Z \subset Y$ such that its Zariski tangent space has dimension 2 and $\deg(\eta_i(Z)) = 1$. The set of all planar i-double points is parametrized by an integral quasi-projective variety. Classically a planar double point is a double point of a plane (or of a smooth surface), sometimes embedded in a bigger projective variety. Our definition agrees with the classical one, but in our statements and proofs it is important to mention the ruling to which their plane belongs.

For any mutiprojective space $W = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$, any $(a_1, \dots, a_s) \in \mathbb{N}^s$, any linear subspace $V \subseteq H^0(\mathcal{O}_W(a_1, \dots, a_s))$ and any closed subscheme $Z \subset W$ set $V(-Z) := V \cap H^0(\mathcal{I}_Z(a_1, \dots, a_s))$.

For all positive integers n and d define the integers $a_{n,d}$ and $b_{n,d}$ by the relations

$$(d+1)a_{n,d} - b_{n,d} = \binom{n+d}{n}, \ 0 \le b_{n,d} \le d \tag{2}$$

Note that $a_{n,d} = \lceil \binom{n+d}{n}/(d+1) \rceil$.

For all positive integers n, d and z define the integers $a_{n,d,z}$ and $b_{n,d,z}$ by the relations

$$(d+1)a_{n,d,z} - b_{n,d,z} = z\binom{n+d}{n}, \ 0 \le b_{n,d,z} \le d \tag{3}$$

Note that $a_{n,d,z} = \lceil z\binom{n+d}{n}/(d+1) \rceil$, $a_{n,d} = a_{n,d,1}$ and $b_{n,d} = b_{n,d,1}$.

Remark 2.1. Let X be an integral projective variety with $\dim(X) > 0$, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ any linear subspace. Take a general $p \in X_{\mathrm{reg}}$ and a general tangent vector A of X at p. We have $\dim(H^0(\mathcal{I}_A \otimes \mathcal{L}) \cap V) = \max\{0, \dim(V) - 2\}$, because (in characteristic zero) any non-constant rational map $X \dashrightarrow \mathbb{P}^r$, $r \ge 1$, has non-zero differential at a general $p \in X_{\mathrm{reg}}$. See [15] for a more general result.

Remark 2.2. Let $T \subset Y$ be a union of t distinct 1-lines. Fix $(a_1, \dots, a_k) \in \mathbb{N}^k$ and integers a, b such that $a < t < b$.

(a) Assume $h^0(\mathcal{I}_T(a_1, \dots, a_k)) = 0$. Since $h^0(\mathcal{I}_Z(a_1, \dots, a_k)) = 0$ for every scheme $Z \supset T$, the semicontinuity theorem for cohomology gives $h^0(\mathcal{I}_B(a_1, \dots, a_k)) = 0$ for a general union of b 1-lines.

(b) Assume $h^1(\mathcal{I}_T(a_1, \dots, a_k)) = 0$. Let $A \subset T$ be any union of a connected components of T. Note that $h^1(\mathcal{I}_A(a_1, \dots, a_k)) = 0$.

Lemma 2.3. *Fix* $(a_1, \dots, a_k) \in \mathbb{N}^k$ *and a closed subscheme* $Z \subset Y$. *Assume* $h^0(\mathcal{I}_Z(a_1 - 1, a_2, \dots, a_k)) \le h^0(\mathcal{I}_Z(a_1, \dots, a_k)) - 2$ *and that the codimension* 1 *scheme-theoretic base locus* $\mathcal{B}$ *of* $|\mathcal{I}_Z(a_1, \dots a_k)|$ *does not contain a divisor of multidegree* $(a_1, b_2, \dots, b_k)$ *for some* $(b_2, \dots, b_k) \in \mathbb{N}^{k-1}$. *Then* $h^0(\mathcal{I}_{Z \cup v}(a_1, \dots, a_k)) = h^0(\mathcal{I}_Z(a_1, \dots, a_k)) - 2$ *for a general tangent vector of type* 1.

Proof. Since $\deg(v) = 2$, we have $h^0(\mathcal{I}_{Z\cup v}(a_1, \dots, a_k)) \ge h^0(\mathcal{I}_Z(a_1, \dots, a_k)) - 2$. Since we assumed $h^0(\mathcal{I}_Z(a_1 - 1, a_2, \dots, a_k)) \le h^0(\mathcal{I}_Z(a_1, \dots, a_k)) - 2$, $a_1 > 0$ and $h^0(\mathcal{I}_Z(a_1, \dots, a_k)) \ge$

2. Since v_{red} is a general point of Y, $h^0(\mathcal{I}_{Z\cup v}(a_1,\dots,a_k)) \ge h^0(\mathcal{I}_Z(a_1,\dots,a_k)) - 1$. Assume $h^0(\mathcal{I}_{Z\cup v}(a_1,\dots,a_k)) = h^0(\mathcal{I}_Z(a_1,\dots,a_k)) - 1$. Fix a general $D \in |\mathcal{I}_Z(a_1,\dots,a_k)|$. Since we assumed $h^0(\mathcal{I}_Z(a_1,\dots,a_k)) \ge 2$, there is at least one irreducible component D' of D not contained in $\mathcal{B}$. Thus $h^0(\mathcal{I}_{Z\cup\{p\}}(a_1,\dots,a_k)) = h^0(\mathcal{I}_Z(a_1,\dots,a_k)) - 1$ for a general $p \in D'$. By Remark 2.1 for a general $p \in D'$ we have $h^0(\mathcal{I}_{Z\cup E}(a_1,\dots,a_k)) = h^0(\mathcal{I}_{Z\cup\{p\}}(a_1,\dots,a_k))$, where $E := \eta_1^{-1}(\pi_1(p))$. Varying p we get that D' has multidegree $(0,c_2,\dots,c_k)$ for some $c_2,\dots,c_k$. Our assumption on $\mathcal{B}$ gives that $D \notin |\mathcal{O}_Y(a_1,\dots,a_k)|$, a contradiction. □

Remark 2.4. Fix a reducible conic $D \subset \mathbb{P}^r$, $r \ge 3$, and a 3-dimensional linear subspace U of $\mathbb{P}^r$ containing the plane M. Let o be the singular point of D. Let A be the union of D and of a degree 2 connected zero-dimensional scheme v with $v_{\mathrm{red}} = \{o\}$, $v \subset U$ and $v \nsubseteq M$. The scheme A is called a *sundial* in [10]. The scheme A is a flat limit of a family of pairs of disjoint lines contained in U ([20, Ex. 2.1] or [8, 9, 10]). The paper [20] inspired many papers (the use of unions of disjoint lines was brilliantly shown in [2]) and, after [8, 9, 10] its revival contains [3, 4, 17, 18, 19, 27]. Let $H \subset \mathbb{P}^r$ be a hyperplane such that $H \cap U = M$. We have $A \cap H = D$ (as schemes) and $\mathrm{Res}_H(A) = \{o\}$. In the same way we see that if $n_i \ge 3$ an i-sundial $Z \subset Y$ is a flat limit of a family of unions of two disjoint i-lines.

Remark 2.5. Fix $H \in |\mathcal{O}_Y(\varepsilon_i)|$. For any $p \in H$ there is an i-line $L \subset Y$ such that $H \cap L = \{p\}$. Thus for any linear subspace $V \subset H^0(\mathcal{O}_H(d_1,\dots,d_s))$ we have $\dim V(-H\cap L) = \max\{0, \dim V - 1\}$ for a general i-line $L \subset H$.

Remark 2.6. Fix $i \in \{1,\dots,k\}$. Any $p \in Y$ is contained in an i-line and hence any subset of cardinality t is contained in the union of at most t i-lines. Thus a general union $T \subset Y$ of i-lines has maximal rank with respect to every line bundle $\mathcal{O}_Y(a_1,\dots,a_k)$ with $a_i = 0$.

Lemma 2.7. *Let $T \subset Y$ be a finite union of i-lines. Fix $(d_1,\dots,d_k)$ such that $h^1(\mathcal{I}_T(d_1,\dots,d_k)) = 0$. Then $h^1(\mathcal{I}_T(d_1,\dots,d_k)) + m\varepsilon_j)) = 0$ for all $m > 0$ and all $j \in \{1,\dots,k\}$.*

Proof. The case $j \ne i$ is obvious, because $\mathcal{O}_Y(me_j)_{|T} \cong \mathcal{O}_T$ if $j \ne i$ and $\mathcal{O}_Y(me_j)$ has no base points.

Now assume $j = i$. It is sufficient to do the case $m = 1$. Since the case $k = 1$ is true by the Castelnuovo-Mumford lemma, we may assume $k \ge 2$. Fix $h \in \{1,\dots,k\} \setminus \{i\}$. Since T is a union of i-lines, $\pi_h(T)$ is finite. Thus there is $H \in |\mathcal{O}_Y(\varepsilon_h)|$ such that $H \cap T = \emptyset$. The long cohomology exact sequence associated to the following residual sequence of H

$$0 \to \mathcal{I}_T(d_1,\dots,d_k) \to \mathcal{I}_T(d_1,\dots,d_k) \otimes \mathcal{O}_Y(\varepsilon_h) \to \mathcal{O}_H(d_1,\dots,d_k) \to 0$$

proves the lemma. □

Lemma 2.8. *Assume $n_1 \ge 3$. Let $T \subset Y$ be a general union of za_{n_1,d_1} 1-lines. Then we have $h^1(\mathcal{I}_T(d_1,\dots,d_k)) = 0$.*

Proof. By the semicontinuity theorem for cohomology it is sufficient to find a union $T \subset Y$ of za_{n_1,d_1} 1-lines such that $h^1(\mathcal{I}_T(d_1,\dots,d_k)) = 0$. Fix a general $S \subset Y_1$ such that $\#S = z$. Let $E \subset \mathbb{P}^{n_1}$ be a general union of a_{n_1,d_1} lines. For each $o \in S$ let $T_o := E \cup \{o\} \subset \mathbb{P}^{n_1} \times \{o\}$ and $T := \cup_{o\in S} T_o$. Since S is general, T is the union of za_{n_1,d_1} disjoint 1-lines. For any connected component L of E fix $S_L \subset L$ with $\#S_L = d_1 + 1$. Set $S_1 := \cup_L S_L$. To prove the lemma it is sufficient to prove that $h^1(\mathcal{I}_{S_1\times S}(d_1,\dots,d_k)) = 0$. Note that $h^i(\mathbb{P}^{n_1}, \mathcal{I}_{S_1}(d_1)) = h^i(\mathbb{P}^{n_1}, \mathcal{I}_E(d_1))$, $i = 0,1$. Hence $h^1(\mathbb{P}^{n_1}, \mathcal{I}_{S_1}(d_1)) = 0$. Use the Künneth formula. □

Proof of Theorem 1.1: Since $k \ge 2$ for any $t > 0$ there are t pairwise disjoint 1-lines. Thus $h^0(\mathcal{O}_T(d_1,\dots,d_k)) = t(d_1+1)$. Set $z := \prod_{i=2}^k \binom{n_i+d_i}{n_i}$. Since $n_1 = 1$, $h^0(\mathcal{O}_Y(d_1,\dots,d_k)) = (d_1+1)z \equiv 0 \pmod{d_1+1}$. Thus T has maximal rank if and only if $h^0(\mathcal{I}_T(d_1,\dots,d_k)) = \max\{0, (d_1+1)(z-t)\}$. Equivalently, T has maximal rank if and only if $h^0(\mathcal{I}_T(d_1,\dots,d_k)) = \max\{0, (d_1+1)(t-z)\}$. Thus Remark 2.2 shows that it is sufficient to prove the case $t = z$.

To prove the theorem we use induction on the integer d_1, the case $d_1 = 0$ being true by Remark 2.6. Assume $d_1 > 0$ and that the theorem is true for the line bundle $\mathcal{O}_Y(d_1-1,d_2,\dots,d_k)$. Take $t = z$. Fix $o \in \mathbb{P}^1$ and set $H := \pi_1^{-1}(o) \in |\mathcal{O}_Y(\varepsilon_1)|$. For a general T we may assume that no connected component of T is contained in H and that $Z := T \cap H$ is a general union of z points of H. Thus $h^i(H,\mathcal{I}_{Z,H}(d_1,\dots,d_k)) = 0$, $i = 0,1$. By the inductive assumption $h^i(\mathcal{I}_T(d_1-1,d_2,\dots,d_k)) = 0$, $i = 0,1$. Use the residual exact sequence of H. $\square$

Example 2.9. Assume $k \ge 2$ and $n_1 = 2$. Fix an integer $d_1 \ge 2$ and take $d_i = 0$ for all $i \in \{2,\dots,k\}$. Set $\mathcal{L} := \mathcal{O}_Y(d_1,\dots,d_k)$. Note that $h^0(\mathcal{L}) = \binom{d_1+2}{2}$. Let $T \subset Y$ be a general union of t 1-lines. Since $k \ge 2$, T has t connected components and hence $h^0(T,\mathcal{L}_{|T}) = t(d_1+1)$. Since T is general $\pi_1(T)$ is a general union of t lines of $\mathbb{P}^2$. Thus $h^0(\mathcal{I}_T\otimes\mathcal{L}) = \binom{d_1-t}{2}$ and $h^1(\mathcal{I}_T\otimes\mathcal{L}) = t(d_1+1)-\binom{d_1-t+2}{2}$ if $t \le d_1$ and $h^0(\mathcal{I}_T\otimes\mathcal{L}) = 0$ for all $t > d_1$. Thus T has maximal rank if and only if either $t = 1$ or $t > d_1$.

3. First steps of the proof of Theorem 1.2

Set $z := \prod_{i=2}^{k}\binom{n_i+d_i}{n_i}$. By Remark 2.6 we may assume $d_1 > 0$. In this section we prove a few lemmas used for any $n_1 \ge 3$ and prove Theorem 1.2 for $n_1 = 3$.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Set $W := Q \times Y_1 \in |\mathcal{O}_{\mathbb{P}^3\times Y_1}(2\varepsilon_1)|$. We have $\mathrm{Pic}(W) \cong \mathrm{Pic}(Q)\times\mathrm{Pic}(Y_1) \cong \mathbb{Z}^{k+1}$. We take a basis $(1,0)$ and $(0,1)$ of $\mathrm{Pic}(Q)$ with $(1,0)$ and $(0,1)$ corresponding to the two rulings of Q and write $\mathcal{O}_W(e_1,e_2;a_2,\dots,a_k)$, $(e_1,e_2,a_2,\dots,a_k) \in \mathbb{Z}^{n+1}$, for the isomorphism classes of line bundles on W.

Lemma 3.1. *Assume $n_1 = 3$. Let $V \subset H^0(\mathcal{O}_W(e_1,e_2;a_2\dots,a_k))$ be a linear subspace. Assume $e_1+e_2 > 0$ and that either the base locus of V contains no divisor or the union of these divisors is an element $\Delta \in |\mathcal{O}_W(f_1,f_2;b_2,\dots,b_k)|$ with $e_1+e_2 > f_1+f_2$. Then $\dim V(-L\cap W) = \max\{0,\dim V-2\}$ for a general 1-line $L \subset Y$.*

Proof. If $\Delta \ne \emptyset$ we use $\mathcal{O}_W(e_1,e_2;a_2,\dots,a_k)(-\Delta)$ to reduce to the case $\Delta = \emptyset$. Thus from now on we assume that the base locus of V contains no divisor. Since a general $p \in W$ is contained in a general 1-line of Y, the case $\dim V \le 1$ is obvious. Assume $\dim V \ge 2$ and that the lemma fails for V. Fix a general $v \in V$ and set $D := \{v = 0\} \in |\mathcal{O}_W(a_1,\dots,a_k)|$. Write $D = \sum_{j=1}^{s} e_jD_j$, where e_j are positive integers, each D_j is an integral effective Cartier of the smooth variety W and $D_i \ne D_j$ for all $i \ne j$. Fix $j \in \{1,\dots,s\}$ and take a general $p \in D_j$. Since the base locus of V contains no divisor and p is general in D_j, we may assume $\dim V(-p) = \dim V - 1$. Let $A(p)$ be the set of all 1-lines containing p. The set $A(p)$ is isomorphic to a plane and exactly 2 elements of it are contained in W, say L_1, L_2, while a line $R \subset A(p)$ minus L_1, L_2 parametrizes the 1-lines tangent to W. For any $L \in A(p)\setminus R$ write $L\cap W = \{p,p_L\}$. Since we assumed that V is a counterexample to Lemma 3.1, p_L is contained in the base locus $\mathcal{B}_p$ of $V(-p)$. For a general $L \in A(p)$ we get that the Q-slice of W containing p is contained in $\mathcal{B}_p$. Since $\mathcal{B}_p \subseteq D$ and p is general in D_j, we get that for a general $q \in D_j$, D_j contains the Q-slice through p, i.e. that $D_j \in |\mathcal{O}_W(0,b_2,\dots,b_k)|$ for some $(b_2,\dots,b_k) \in \mathbb{N}^{k-1}$. Since this is true for all $j = 1,\dots,s$, we get $D \in |\mathcal{O}_W(0,0;c_2,\dots,c_k)|$ for some c_i, contradicting the assumption $e_1+e_2 > 0$. $\square$

Lemma 3.2. *Assume $n_1 \ge 3$. Fix $(d_1,\dots,d_k) \in \mathbb{N}^k$. If $b_{n_1,d_1} = 0$, then Theorem 1.2 is true for Y and $\mathcal{O}_Y(d_1,\dots,d_k)$.*

Proof. Note that $b_{n_1,d_1,z} = 0$. Thus $a_{n_1,d_1,z} = za_{n_1,d_1}$. By semicontinuity it is sufficient to find a union $T \subset Y$ of za_{n_1,d_1} lines such that $h^i(\mathcal{I}_T(d_1,\dots,d_k)) = 0$ for $i = 0,1$. Fix a general $S \subset Y_1$ such that $\#S = z$. Let $E \subset \mathbb{P}^{n_1}$ be a general union of a_{n_1,d_1} lines. For each $o \in S$ let $T_o := E\cup\{o\} \subset \mathbb{P}^{n_1}\times\{o\}$ and $T := \cup_{o\in S}T_o$. Since S is general, T is the union of za_{n_1,d_1} disjoint 1-lines. Thus it is sufficient to prove that $h^0(\mathcal{I}_T(d_1,\dots,d_k)) = 0$. Assume the existence of $G \in |\mathcal{I}_T(d_1,\dots,d_k)$. By [20] G contains $\mathbb{P}^{n_1}\times S$. Thus for each $p \in \mathbb{P}^{n_1}$ G vanishes on $\{p\}\times S$. Since $\#S = z$ and S is general, $h^0(Y_1,\mathcal{I}_S(d_2,\dots,d_k)) = 0$. Thus G vanishes on each $\{p\}\times Y_1$. Since this is true for all $p \in \mathbb{P}^{n_1}$, we get a contradiction. $\square$

Remark 3.3. We recall that $a_{d,3} = (d+3)(d+2)/6$ and $b_{3,d} = 0$ if $d \equiv 0,1 \pmod 3$ and that $a_{3,3h+2} =$ and $b_{3,3h+2} = h+1$ for all $h \in \mathbb{N}$ ([20, p. 173]). Thus Theorem 1.2 is true if $n_1 = 3$ and $d \equiv 0,1 \pmod 3$ (Lemma 3.2). Thus to prove the case $n_1 = 3$ we may assume $d_1 = 3h+2$ for some $h \in \mathbb{N}$.

Lemma 3.4. *Theorem 1.2 is true if $n_1 = 3$ and $d_1 = 3h+2$ for some $h \in \mathbb{N}$.*

Proof. By Remark 3.3 $h^i(\mathcal{I}_E(d_1,\dots,d_k)) = 0$, $i = 0,1$, for a general union $E \subset Y$ of $(h+1)(3h+2)/2$ 1-lines. Set $a := \lceil \binom{d_1+3}{3} z/(d_1+1) \rceil$ and $b := \lfloor \binom{d_1+3}{3} z/(d_1+1) \rfloor$. Let $A \subset W$ be a general union of a 1-lines and $B \subset W$ a general union of b 1-lines. By semicontinuity it is sufficient to prove that $h^0(\mathcal{I}_{E\cup A}(d_1,\dots,d_k)) = 0$ and that $h^1(\mathcal{I}_{E\cup B}(d_1,\dots,d_k)) = 0$. Since E is general, $E \cap W$ is a finite set and hence $\mathrm{Res}_W(E) = E$. Thus $\mathrm{Res}_W(E\cup A) = \mathrm{Res}_W(E\cup B) = E$. Since $h^i(\mathcal{I}_E(d_1,\dots,d_k)) = 0$, $i = 0,1$, the residual exact sequence of W shows that to prove the lemma it is sufficient to prove that $h^0(W, \mathcal{I}_{(E\cap W)\cup A,W}(d_1,\dots,d_k)) = 0$ and $h^1(W, \mathcal{I}_{(E\cap W)\cup B,W}(d_1,d_1;d_2,\dots,d_k)) = 0$. Since E is a general union of 1-lines, it would be sufficient to be able to apply Lemma 3.1. Certainly $a_1 = d_1 > 0$ by assumption. We need to exclude that at one of the steps a certain linear system has a very big divisor in its base locus. This is obvious by induction on d_1, even if the divisor has multidegree $(a_1,a_2;d_2,\dots,d_k)$ with $a_1 \ne a_2$, since it is sufficient to use that $a_1 + a_2 > 0$. □

Proof of Theorem 1.2 for $n_1 = 3$. Remark 3.3 and Lemma 3.4 prove the case $n_1 = 3$ of Theorem 1.2. □

To prove the case $n_1 > 3$ of Theorem 1.2 we also need the following Assertion $\tilde{A}_{3,a,z}$:

Assertion $\tilde{A}_{3,a,z}$: Assume $n_1 = 3$. There is $E \subset Y$ such that $E = A \cup B$, where A is a union of $a_{3,a,z} - 2b_{3,a,z}$ 1-lines, B is the union of $b_{3,a,z}$ reducible 1-conics with vertex contained in W and $h^i(\mathcal{I}_E(a,d_2,\dots,d_k)) = 0$, $i = 0,1$.

Note that $h^0(\mathcal{O}_E(a,d_2,\dots,d_k)) = z\binom{a+3}{3}$ if and only if $A \cap B = \emptyset$.

Lemma 3.5. *$\tilde{A}_{3,a,z}$ is true.*

Proof. If $b_{3,a,z} = 0$, then $\tilde{A}_{3,a,z}$ is true by the case $n_1 = 3$ of Theorem 1.2 just proved. Assume $b_{3,a,z} > 0$. By Remark 3.3 and Lemma 3.2 $a = 3h$ for some positive integer h. In particular $a \ge 3$ and $\tilde{A}_{3,a-2,z}$ is true. Fix a solution F of $\tilde{A}_{3,a-2,z}$, i.e. let $F \subset Y$ be a general union of $a_{3,a-2,z}$ 1-lines. Thus $F \cap W$ is general. Take a union $G \subset W$ of $a_{3,a,z} - a_{3,a-2,z}$ 1-lines which are general with the only restriction that $b_{3,a,z}$ of them contains a point of $F \cap W$. Set $E := F \cup G$. Use the residual exact sequence of W and that, by the generality of $E \cap W$, F has the same multigraded Hilbert function for W as a general union of $a_{3,a,z} - a_{3,a-2,z}$ 1-lines. □

4. The end of the proof of Theorem 1.2

In this section we conclude the proof of Theorem 1.2 using the case $n_1 = 3$ proved in Section 3. By Section 3 we may assume $n_1 \ge 4$ and that Theorem 1.2 is true for all multiprojective spaces with at least 2 factors and whose first factor has dimension m with $3 \le m \le n_1 - 1$.

Fix $H \in |\mathcal{O}_Y(\varepsilon_1)|$. By the inductive assumption Theorem 1.2 holds in H for all multidegrees.

For any $m \ge 3$, $a \ge 0$ and $z > 0$ we consider the following Assertions $A_{m,a,z}$, $A''_{m,a,z}$ and $B_{m,a,z}$:

Assertion $A_{m,a,z}$: There is a union $E = A \cup B \subset Y$ such that:

(1) A is a union of $a_{m,a,z} - 2b_{m,a,z}$ 1-lines;
(2) B is a union of $b_{m,a,z}$ reducible 1-conics, $\mathrm{Sing}(B) \subset H$, and $A \cap B = \emptyset$;
(3) $h^i(\mathcal{I}_E(a,d_2,\dots,s_k)) = 0$, $i = 0,1$.

Assertion $B_{m,a,z}$: There is a union $E = A \cup Z \subset Y$ such that

(1) A is a union of $a_{m,a,z} - 3$ 1-lines (case $b_{m,a,z} = 0$) or $a_{m,a,z} - 4$ 1-lines (case $b_{m,a,z} > 0$);
(2) Z is a union of a planar double 1-points, $A \cap Z = \emptyset$;
(3) $h^1(\mathcal{I}_E(a,d_2,\dots,s_k)) = 0$.

In the set-up of Assertion $B_{m,a,z}$ we write $\varepsilon(m,a,z) := 3$ if $b_{m,a,z} = 0$ and $\varepsilon(m,a,z) = a+4-b_{m,a,z}$ if $b_{m,a,z} > 0$. Note that always $\varepsilon(m,a,z) > 0$ and that $h^0(\mathcal{O}_E(a,d_2,\dots,d_k)) = z\binom{m+a}{m} - \varepsilon(m,a,z)$.

Remark 4.1. Fix $E = A \cup B$ satisfying conditions (1) and (2) of Assertion $A_{n_1,a,z}$. Since $h^0(\mathcal{O}_E(a,d_2,\dots,d_k)) = h^0(\mathcal{O}_Y(a,d_2,\dots,d_k)$, $h^0(\mathcal{I}_E(a,d_2,\dots,s_k)) = h^1(\mathcal{I}_E(a,d_2,\dots,d_k))$. Thus $h^0(\mathcal{I}_E(a,d_2,\dots,d_k)) = 0$ if and only if $h^1(\mathcal{I}_E(a,d_2,\dots,d_k)) = 0$.

The next lemma implies that $B_{m,a,z}$ is well-defined.

Lemma 4.2. *$a_{m,a-1,z} \ge 2a$ for all $a > 0$ and all $m \ge 3$.*

Proof. Assume $a_{m,a-1,z} \le 2a - 1$. We get $a(2a-1) \ge z\binom{m+a-1}{n_1}$. Since $z \ge 2$ and $m \ge 3$, we get a contradiction. □

By Remark 2.6 Theorem 1.2 is true if $d_1 = 0$. Thus we may assume $d_1 \geq 1$ and use induction on the integer d_1. Hence we may assume the theorem for all multidegrees $(b_1, \dots, b_k)$ with $b_1 < d_1$.

Lemma 4.3. *$B(3, a, z)$ is true for all $a \geq 1$ and all $z \geq 2$.*

Proof. Take Q and W as in Section 3.

(a) We first prove $B(3, 1, z)$. Since $b_{3,1} = 0$, $b_{3,1,z} = 0$ and $a_{3,1,z} = 2z$. Since $z \geq 2$, $a_{3,1,z} \geq 3$. Since $\varepsilon(3, a, z) > 0$ and $h^1(\mathcal{I}_Z(a, d_2, \dots, d_k)) = 0$, it is sufficient to use Lemma 3.1.

(b) Assume $a \geq 2$. Take $E = A \cup B$ satisfying $\tilde{A}_{3,a,z}$ (case $a \geq 3$) while for $a = 2$ take as E a general union of z 1-lines. In all cases $h^i(\mathcal{I}_E(a-2, d_2, \dots, d_k)) = 0$, $i = 0, 1$, and the singular points of E are contained in W.

(b1) Assume $b_{3,a-2,z} = b_{3,a,z} = 0$ and $a \geq 3$. Thus $B = \emptyset$. Let $Z \subset Y$ be a general union of a planar double 1-points with the only restriction that $Z_{\mathrm{red}} \subset W$ and that one of the connected components of Z is contained in W. Call S the reduction of the other connected components of Z. Since W is a smooth divisor of Y, $\mathrm{Res}_W(Z) = S$, i.e $\mathrm{Res}_W(Z)$ is a general union of $a-1$ points of W, and $Z \cap W$ is a general union of $a-1$ tangent vectors of type 1 of W and one planar double 1-point of W. Let $F \subset Y$ be a union of $a_{3,a-2,z} - 1$ connected components of E. Let $G \subset W$ be a general union of $a_{3,a,z} - a_{3,a-2} - 2$ 1-lines. Set $K := F \cup Z \cup G$. To prove this case it is sufficient to prove $h^1(\mathcal{I}_K(a, d_2, \dots, d_k)) = 0$. By the case $n_1 = 3$ of Theorem 1.2 $h^1(\mathcal{I}_F(a-2, d_2, \dots, d_k)) = 0$ (i.e. $h^0(\mathcal{I}_F(a-2, d_2, \dots, d_k)) = a-1$), and $h^0(\mathcal{I}_F(a-4, d_2, \dots, d_k)) = 0$. Since $h^0(\mathcal{I}_F(a-4, d_2, \dots, d_k)) = 0$, $h^0(\mathcal{I}_F(a-2, d_2, \dots, d_k)) = \#S$ and S is general, it is easy to check that $h^i(\mathcal{I}_{F\cup S}(a-2, d_2, \dots, d_k)) = 0$. Thus the residual exact sequence of W shows that it is sufficient to prove that $h^1(W, \mathcal{I}_{K\cap W,W}(a, a; d_2, \dots, d_k)) = 0$. Note that $h^0(\mathcal{O}_{K\cap W}(a, d_2, \dots, d_k)) = h^0(\mathcal{O}_W(a, a; d_2, \dots, d_k)) - 3$. By Lemma 2.3 it is sufficient to prove that $h^1(W, \mathcal{I}_{G\cup J}(a, a; d_2, \dots, d_k))$, where $J \subset W$ is a general planar 1-double point.

Claim 1: $h^0(\mathcal{O}_G(a, a; d_2, \dots, d_k)) \leq h^0(\mathcal{O}_W(a, a-1; d_2, \dots, d_k)) - 1$ if $a \geq 3$.

Proof of Claim 1: Since $h^0(\mathcal{O}_{\mathrm{Res}_W(K)}(a-2, d_2, \dots, d_k)) = z\binom{a+1}{3}$, $h^0(\mathcal{O}_W(a, a; d_2, \dots, d_k)) - h^0(\mathcal{O}_W(a, a-1; d_2, \dots, d_k)) = z(a+1)$ and $h^0(\mathcal{O}_{K\cap W}(a, d_2, \dots, d_k)) = h^0(\mathcal{O}_W(a, a; d_2, \dots, d_k)) - 3$, it is sufficient to check that $\#(F \cap W) + \deg(Z) \geq z(a+1) - 2$, i.e. $2a_{3,a-2,z} - 2 + 3a \geq z(a+1) - 2$. We have $a_{3,a-2,z} = z(a+1)a/6$. Thus all cases with $a \geq 3$ are covered.

We take $M \in |\mathcal{O}_W(0, 1; 0, \dots, 0)|$ and we degenerate J to a planar 1-double point contained in M and use the residual exact sequence of M in W.

(b2) Assume $b_{3,a-2,z} = 0$ and $b_{3,a,z} > 0$. Thus $a \equiv 0 \pmod 3$. In the proof of step (b1) we take as G a general union of $a_{3,a,z} - a_{3,a-2} - 3$ 1-lines.

(b3) Assume $b_{3,a-2,z} > 0$ and $b_{3,a,z} = 0$. Let T be a general union of 1-sundials with $T_{\mathrm{red}} = B$. Let $Z \subset W$ be a general union of a planar 1-double points. Let $F \subset W$ be a general union of $a_{3,a,z} - a_{3,a-2,z} - 3$ 1-lines. Set $K := A \cup T \cup Z \cup F$. Since $\mathrm{Res}_W(K) = E$ and $h^i(\mathcal{I}_E(a-2, d_2, \dots, d_k)) = 0$, the residual exact sequence of W shows that it is sufficient to prove $h^1(W, \mathcal{I}_{F\cup I,W}(a, a; d_2, \dots, d_k)) = 0$, where I is a general union of $a + b_{3,a,z}$ planar 1-double points. We take $M \in |\mathcal{O}_W(0, 1; 0, \dots, 0)|$ and we degenerate I to a planar 1-double point contained in M. To conclude as in step (b1) we only need to check that $h^0(\mathcal{O}_F(a, a-1; d_2, \dots, d_k)) \leq h^0(\mathcal{O}_W(a, a-1; d_2, \dots, d_k))$. Since $\varepsilon(3, a, z) = a + 4 - b_{3,a,z}$, it is sufficient to prove that $a + 4 - b_{3,a,z} + 2a_{3,a-2,z} + 3a \geq z(a+1)$, which is always true, because $b_{3,0,z} = 0$ and hence $a \geq 3$.

(b4) Assume $a = 2$. Note that $a_{3,0,z} = z$, $b_{3,0,z} = 0$, $a_{3,2,z} = \lceil 10z/3 \rceil$ and that $b_{3,2,z} = 3\lceil 10z/3 \rceil - 10z$. Thus $B = \emptyset$. Remark 2.6 gives $h^i(\mathcal{I}_A(0, d_2, \dots, d_k)) = 0$, $i = 0, 1$. Let $Z \subset Y$ be a general union of 2 double 1-points with the only restriction that $S := Z_{\mathrm{red}} \subset W$. Let $A' \subset Y$ be a union of $z-2$ connected components of W. Let $F \subset W$ be a general union of $\lceil 10z/3 \rceil - z - 1 - c$ 1-lines with $c = 0$ if $b_{3,2,z} = 0$, i.e. if $z \equiv 0 \pmod 3$, and $c = 1$ if $b_{3,2,z} > 0$. Set $K := A' \cup Z \cup F$. Since $\mathrm{Res}_W(K) = A' \cup S$, $h^1(\mathcal{I}_{A'}(0, d_2, \dots, d_k)) = 0$, $h^0(\mathcal{I}_{A'}(0, d_2, \dots, d_k)) = 0$, $h^0(\mathcal{O}_Y(-2, d_2, \dots, d_k)) = 0$ and S is general in W, we have $h^i(\mathcal{I}_{\mathrm{Res}_W(K)}(0, d_2, \dots, d_k)) = 0$. Since $Z \cap W$ is a general union of 2 tangent vectors of type 1, we may use Lemma 3.1. The residual exact sequence of W shows that it is sufficient to prove that $h^1(W, \mathcal{I}_F(2, d_2, \dots, d_k)) = 0$. Since $h^1(Q, \mathcal{I}_J(2, 2)) = 0$, where J is a general union of 3 elements of $\mathcal{O}_Q(1, 0)|$, it is sufficient to mimic the proof of Lemma 3.2 using that F has at most $3z$ connected components. $\square$

Lemma 4.4. *Assume $b_{n_1,a,z} \geq b_{n_1,a-1,z}$. Then $a_{n_1,a,z} - 2b_{n_1,a,z} \geq a_{n_1,a-1,z} - 2b_{n_1,a-1,z}$ for all $a \geq 1$.*

Proof. Subtracting the equation in (3) for the integer $a-1$ from the same equation for the integer a we get

$$a_{n_1,a-1,z} + (a+1)(a_{n_1,a,z} - a_{n_1,a-1,z}) - b_{n_1,a,z} + b_{n_1,a-1,z} = z\binom{n_1+a-1}{n_1-1}. \tag{4}$$

Assume $a_{n_1,a,z} - 2b_{n_1,a,z} \le a_{n_1,a-1,z} - 2b_{n_1,a-1,z} - 1$, i.e. $a_{n_1,a,z} - a_{n_1,a-1,z} \le 2b_{n_1,a,z} - 2b_{n_1,a-1,z} - 1$. Thus (4) gives $a_{n_1,a-1,z} + (2a+1)(b_{n_1,a,z} - b_{n_1,a-1,z}) - a - 1 \ge z\binom{n_1+a-1}{n_1}$. Since $b_{n_1,a,z} \le a$ and $b_{n_1,a_1,z} \ge 0$, we get $a_{n_1,a-1,z} + a(2a+1) - a - 1 \ge z\binom{n_1+a-1}{n_1-1}$. Since $0 \le b_{n_1,a-1,z} \le a-1$, $aa_{n_1,a-1,z} \le z\binom{n_1+a-1}{n_1} + a - 1$. Thus

$$z\binom{n_1+a-1}{n_1} + 2a^3 - 1 \ge az\binom{n_1+a-1}{n_1-1}. \tag{5}$$

Since $a\binom{n_1+a-1}{n_1-1} = \frac{(n_1+a-1)!}{(a-1)!(n_1-1)!}$, we get

$$2a^3 - 1 \ge z\frac{(n_1+a-1)!}{(a-1)!(n_1-1)!}(1 - 1/n_1). \tag{6}$$

For a fixed a the right hand side of (6) is an increasing function of n_1 and hence it is sufficient to disprove it for $n_1 = 4$. For $n_1 = 4$ (6) is

$$2a^3 - 1 \ge za(a+3)(a+2)(a+1)/8, \tag{7}$$

which is false for all $a \ge 1$, because $z \ge 2$. □

Remark 4.5. We have $a_{4,2,2} = 10$, $b_{4,2,2} = 0$, $a_{3,3,2} = 10$, $a_{4,3,2} = 18$ and $b_{4,3,2} = 1$. We have $a_{n_1,1,3} = \lceil 3(n_1+1)/2 \rceil$, $b_{n_1,1,3} = 0$ if n_1 is odd, $b_{n_1,1,3} = 1$ if n_1 is even, $a_{4,2,3} = 15$, $b_{4,2,3} = 0$, $a_{5,2,3} = 21$ and $b_{5,2,3} = 0$. Hence $b_{4,1,3} < b_{4,2,3}$ and $a_{5,2,3} - a_{5,1,3} = 12 = a_{4,2,3} - 3$. We have $a_{n_1,1,2} = n_1 + 1$, $b_{n_1,1,2} = 0$, $a_{n_1,2,2} = (n_1+2)(n_1+1)/3$ and $b_{n_1,2,2} = 0$ if $n_1 \equiv 0, 1 \pmod 3$ and $a_{n_1,2,2} = 3h^2 + 3h + 1$ and $b_{n_1,2,2} = 1$ if $n_1 = 3h$ for some integer $h > 0$. Thus $a_{4,2,2} - a_{4,1,2} = a_{3,2,2} - 2$, $a_{5,2,2} - a_{5,1,2} = a_{4,2,2} - 2$, $a_{6,2,2} - a_{6,1,2} = a_{5,2,2} - 2$, $a_{7,2,2} - a_{7,1,2} = a_{6,2,2} - 3$, and $a_{8,2,2} - a_{8,1,2} = a_{7,2,2} - 3$.

Lemma 4.6. *Assume $b_{n_1,a,z} \ge b_{n_1,a-1,z}$. Then $a_{n_1,a,z} - 2b_{n_1,a,z} - a_{n_1,a-1,z} + 2b_{n_1,a-1,z} \le a_{n_1-1,a,z} - 2a$ for all $a \ge 2$, except if $z = 2$, $a = 3$, $n_1 = 4$ or $a = 2$ (n_1, a, z) is one of the following triples: $(4, 3, 2)$ and and $(n_1, z) \in \{(4, 2), (5, 2), (6, 2), (7, 2), (8, 2), (4, 3), (5, 3)\}$.*

Proof. Assume $a_{n_1,a,z} - 2b_{n_1,a,z} - a_{n_1,a-1,z} + 2b_{n_1,a-1,z} \ge a_{n_1-1,a,z} - 2a + 1$, i.e. $a_{n_1,a,z} - a_{n_1,a-1,z} \ge 2b_{n_1,a,z} - 2b_{n_1,a-1,z} + a_{n_1-1,a,z} - 2a + 1$. From (4) we get

$$a_{n_1,a-1,z} + (a+1)a_{n_1-1,a,z} + (2a+1)(b_{n_1,a,z} - b_{n_1,a-1,z}) - (a+1)(2a-1) \le z\binom{n_1+a-1}{n_1-1}. \tag{8}$$

Since $(a+1)a_{n_1-1,a,z} \ge z\binom{n_1+a-1}{n_1-1}$ and $b_{n_1,a,z} \ge b_{n_1,a-1,z}$, (8) gives

$$a_{n_1,a-1,z} \le (a+1)(2a-1). \tag{9}$$

Note that the left hand side of (9) is an increasing function of n_1. Since $a_{n_1,a-1,z} = \rceil\binom{n_1+a-1}{n_1}\rceil$, the inequality (9) is false, unless either $a = 2$ and $(n_1, z) \in \{(4, 2), (5, 2), (6, 2), (7, 2), (8, 2), (4, 3), (5, 3)\}$ or $(n_1, a, z) = (4, 3, 2)$. These cases are discussed in Remark 4.5. □

Lemma 4.7. *Assume $n_1 \ge 4$ and $a \ge 3$. Then $a_{n_1,a,z} - a_{n_1,a-1,z} \le a_{n_1-1,a,z} - 4$.*

Proof. Assume $a_{n_1,a,z} - a_{n_1,a-1,z} \ge a_{n_1-1,a,z} - 3$. From (4) we get

$$a_{n_1,a-1,z} + (a+1)a_{n_1-1,a,z} - 3(a+1) + b_{n_1,a,z} - b_{n_1,a-1,z} \le z\binom{n_1+a-1}{n_1-1}. \tag{10}$$

Since $b_{n_1,a-1,z} \le a-1$, $b_{n_1,a,z} \ge 0$ and $(a+1)a_{n_1-1,a,z} \ge z\binom{n_1+a-1}{n_1}$, (10) gives $a_{n_1,a-1,z} \le 4a-3$ and hence

$$z\binom{n_1+a-1}{n_1} \le 4a^2 - 3a. \tag{11}$$

The left hand side of (11) is an increasing function of n_1. For $n_1 = 4$ to disprove (11) it is sufficient to prove the inequality

$$z(a+3)(a+2)(a+1) > 48(a-3). \tag{12}$$

Obviously (12) holds if $z(a+3)(a+2) \geq 48$, which is true because $z \geq 2$ and $a \geq 3$. $\square$

Lemma 4.8. *$A_{m,a,z}$ and $B_{m,a,z}$ are true for all a and all $m = 4, \dots, n_1$.*

Proof. By the inductive assumption we know $A_{m,a,z}$ if $m < n_1$. We will prove $A_{n_1,a,z}$ in step (b). In step (a) we will prove $B_{m,a,z}$ for all $m = 4, \dots, n_1$ using induction on m starting the induction with the case $m = 3$ proved in Lemma 4.3. We may assume that the case $a = 0$ of Theorem 1.2 is true by Remark 2.6. Thus for all statements we may assume that they are true for the integers $a' < a$. In step (b) we only use $B_{n_1-1,a,z}$ if $n_1 \geq 5$ and Lemma 4.3 if $n_1 = 4$.

(a) Fix an integer $m \in \{4, \dots, n_1\}$. We prove $B_{m,a,z}$ assuming that this assertion is true for smaller m', starting the induction with the case $m = 3$ proved as Lemma 4.3. For a fixed m we use induction on a, the case $a = 0$ being trivial. Set $Y' := \mathbb{P}^m \times \cdots \times \mathbb{P}^{n_k}$ and take $H' \in |\mathcal{O}_{Y'}(\varepsilon_1)|$. Remember that $\varepsilon(m,a,z) = 3$ if $b_{m,a,z} = 0$ and $\varepsilon(m,a,z) = a + 4 - b_{m,a,z} > 3$ if $b_{m,a,z} > 0$.

(a1) Assume $\varepsilon(m,a,z) \geq \varepsilon(m,a-1,z)$. Take a solution $A \cup Z$ for $B_{m,a-1,z}$ with Z a general union of $a-1$ double 1-points. Hence $h^1(\mathcal{I}_{A\cup Z}(a-1,d_2,\dots,d_k)) = 0$ and $h^0(\mathcal{I}_{A\cup Z}(a-1,d_2,\dots,d_k)) = \varepsilon(m,a-1,z)$. Recall that $\varepsilon(m,a-1,z) \geq 1$. Let $Z' \subset Y'$ be a general planar 1-double point with the restriction that $\{p\} := (Z')_{\mathrm{red}} \subset H'$. Thus $\{p\}$ is a general point of H'. Since $h^0(\mathcal{I}_{A\cup Z}(a-2,d_2,\dots,d_k)) = 0$ by the inductive assumption and p is general in H', $h^1(\mathcal{I}_{A\cup Z\cup\{p\}}(a-1,d_2,\dots,d_k)) = 0$. Let $F \subset H'$ be a general union of $a_{m,a,z} - a_{m,a-1,z} - c$ lines, where $c = 0$ if either $b_{m,a,z} = 0$ or $b_{m,a-1,z} > 0$ and $c = 1$ if $b_{m,a,z} > b_{m-1,a,z} = 0$. Set $K := A \cup Z \cup Z' \cup F$. Note that $\mathrm{Res}_{H'}(K) = A \cup Z \cup \{p\}$, $Z' \cap H'$ is a general union of a tangent 1-vector of H' and we may use Theorem 1.2 in H'. Thus to conclude using the residual exact sequence of H' it is sufficient to prove that $h^0(\mathcal{O}_F(a,d_2,\dots,d_k)) \leq z\binom{m+a-1}{m-1} - 2$. Since $a > 0$ and F has at most $a_{m,a,z} - a_{m,a-1,z} - c$ connected components, it is sufficient to use that $a_{m-1,a,z} < a_{m,a,z} - a_{m,a-1,z}$ (Lemma 4.7).

(a2) Assume $\varepsilon(a,m,z) < \varepsilon(m,a-1,z)$. Thus $b_{m,a-1,z} > 0$. Let $R \subset Y'$ be a general union of $a(m,a-1,z) - 1$ 1-lines. Thus $h^1(\mathcal{I}_R(a-1,d_2,\dots,d_k)) = 0$ and $h^0(\mathcal{I}_R(a-1,d_2,\dots,d_k)) = a - 1 - b_{m,a-1,z}$ by the inductive assumption on a if $m = n_1$ and the inductive assumption on n_1 for Theorem 1.2 if $m < n_1$. Take a general union $Z' \subset Y'$ of $a - 1 - b_{m,a-1,z}$ planar 1-double points with the only restriction that $S' := (Z')_{\mathrm{red}} \subset H'$. Let $Z'' \subset H'$ be a general union of $1 + b_{m,a-1,z}$ planar 1-double points. Note that $1 + b_{m,a-1,z} \leq a$. Let $T \subset H'$ be a general union of $a_{m,a,z} - 2 - c - a_{m,a,z}$ 1-lines with $c = 1$ if $b_{m,a,z} > 0$ and $c = 0$ if $b_{m,a,z} = 0$. Set $K := R \cup F \cup Z' \cup Z''$. By semicontinuity to prove $B_{m,a,z}$ it is sufficient to prove $h^1(\mathcal{I}_K(a,d_2,\dots,d_k)) = 0$. Note that $\mathrm{Res}_{H'}(K) = R \cup S'$. Since $h^1(\mathcal{I}_E(a-1,d_2,\dots,d_k)) = 0$, $h^0(\mathcal{I}_R(a-2,d_2,\dots,d_k)) = 0$, $h^0(\mathcal{I}_R(a-1,d_2,\dots,d_k)) = 0$ and S' is general in H', we have $h^i(\mathcal{I}_{R\cup S'}(a-1,d_2\dots,d_k)) = 0$, $i = 0,1$. Thus the residual exact of H' shows that it is sufficient to prove $h^1(H', \mathcal{I}_{K\cap H',H}(a,d_2,\dots,d_k)) = 0$. Note that $K \cap H' = (Z' \cap H') \cup Z'' \cup T$. Since $h^0(\mathcal{O}_{\mathrm{Res}_{H'}(K)}(a-1,d_2,\dots,d_k)) = z\binom{m+a-1}{m}$, $h^0(\mathcal{O}_{K\cap H'}(a,d_2,\dots,d_k)) = z\binom{m+a-1}{m-1} - \varepsilon(m-1,a,z)$. Thus by Lemma 3.1 it is sufficient to prove $h^1(H', \mathcal{I}_{Z''\cup T,H'}(a,d_2,\dots,d_k)) = 0$. Since Z'' has at most a connected components, by the inductive assumption on m it would be sufficient to prove that $a_{m,a,z} - 2 - c - a_{m,a,z} \leq a_{m-1,a,z} - c'$, where $c' = 3$ if $b_{m-1,a,z} = 0$ and $c' = 4$ if $b_{m-1,a,z} > 0$. In particular it is sufficient to have $a_{m,a,z}c - a_{m,a,z} \leq a_{m-1,a,z} - 2$. Use Lemma 4.7.

(b) By Remark 2.6 to prove $A_{n_1,a,z}$ we may assume $a > 0$ and that the lemma is true for all integers $a' < a$. Take $E = A \cup B$ satisfying $A_{n_1,a-1,z}$. Thus $h^i(\mathcal{I}_E(a-1,d_2,\dots,d_k)) = 0$, $i = 0,1$. By semicontinuity we may assume $\dim H \cap E = 0$ and that $E \cap H$ is a general union of $a_{n_1,a-1,z} - 2b_{n_1,a-1,z}$ points and $b_{n_1,a-1,z}$ tangent vectors of type 1.

(b1) Assume $b_{n_1,a,z} \geq b_{n_1,a-1,z}$. Lemma 4.4 gives $a_{n_1,a,z} - 2b_{n_1,a,z} \geq a_{n_1,a-1,z} - 2b_{n_1,a-1,z}$. Let $A' \subset H$ be a general union of $a_{n_1,a,z} - a_{n_1,a-1,z}$ 1-lines with the only restriction that $b_{n_1,a,z} - b_{n_1,a-1,z}$ of them contain a point of $A \cap H$; we use that $b_{n_1,a,z} - b_{n_1,a-1,z} \leq a_{n_1,a-1,z} - 2b_{n_1,a-1,z}$. Since $E \cap H$ is general in H, A' has the Hilbert function of $a_{n_1,a,z} - a_{n_1,a-1,z}$ general 1-lines of H'. Set $F := E \cup A'$. Since $\mathrm{Res}_H(F) = E$, the residual exact sequence of H shows that to prove $A_{n_1,a,z}$ it is sufficient to prove $h^i(H, \mathcal{I}_{(E\cap H)\cup A',H}(a,d_2,\dots,d_k)) = 0$, $i = 0,1$. By Lemma 2.3 it is sufficient to prove $h^1(H, \mathcal{I}_{A',H}(a,d_2,\dots,d_k)) = 0$. Note that $h^0(\mathcal{O}_{F\cap H}(a,d_2,\dots,d_k)) = z\binom{n_1+a-1}{n_1}$. Thus the inductive assumption on n_1 for Theorem 1.2 shows that it is sufficient to prove that $a_{n_1,a,z} - a_{n_1,a-1,z} < a_{n_1-1,a,z}$, which is true by Lemma 4.6.

(b2) Assume $b_{n_1,a,z} < b_{n_1,a-1,z}$. Write $B = B' \cup B''$ with B' union of $b_{n_1,a,z}$ connected components of B. Take as Z a general union of 1-sundials with $Z_{red} = B''$. Let $F \subset H$ be a general union of $a_{n_1,a,z} - a_{n_1,a-1,z}$ 1-lines. Set $G := A \cup B' \cup Z \cup F$. Note that $h^0(\mathcal{O}_G(a)) = z\binom{n_1+a}{n}$. By semicontinuity and Remark 2.4 it is sufficient to prove $h^i(\mathcal{I}_G(a, d_2, \dots, d_k)) = 0$, $i = 0, 1$. Since $\mathrm{Res}_H(G) = E$, the residual exact sequence of H shows that it is sufficient to prove $h^i(H, \mathcal{I}_{G\cap H,H}(a, d_2, \dots, d_k)) = 0$, $i = 0, 1$. Note that $h^0(\mathcal{O}_{G\cap H}(, d_2, \dots, d_k)) = z\binom{n_1+a-1}{n_1-1}$ and that $G \cap H$ is a general union of $a_{n_1,a,z} - a_{n_1,a-1,z}$ 1-lines of H. Since $a_{n_1,a,z} - a_{n_1,a-1,z} < a_{n_1-1,a,z}$ (Lemma 4.7), it is sufficient to use the inductive assumption on n_1. □

End of the proof of Theorem 1.2. Let a be the first non-negative integer such that $z\binom{n_1+a}{n_1} \ge t(a+1)$. By Lemma 2.7 to prove that T has maximal rank with respect to $\mathcal{O}_Y(x, d_2, \dots, d_k)$, $x \in \mathbb{N}$, it is sufficient to prove that $h^0(\mathcal{I}_T(a-1, d_2, \dots, d_k)) = 0$ and $h^1(\mathcal{I}_T(d_1, \dots, d_k)) = 0$.

(a) In this step we prove that $h^0(\mathcal{I}_T(a-1, d_2, \dots, d_k)) = 0$. Note that $t \ge a_{n_1,a-1,z}$. Take a solution $E = A \cup B$ of $A_{n_1,a-1,z}$ with B a union of $b_{n_1,a-1,z}$ reducible 1-conics. Thus $h^0(\mathcal{I}_E(a-1, d_2, \dots, d_k)) = 0$. Take a union Z of $a_{n_1,a-1,z}$ 1-sundials such that $Z_{\mathrm{red}} = B$. Obviously $h^0(\mathcal{I}_{A\cup Z}(a-1, d_2, \dots, d_k)) = 0$.

(b) In this step we prove that $h^1(\mathcal{I}_T(a, d_2, \dots, d_k)) = 0$. By the definition of the integer a we have $t \le a_{n_1,a,z}$ and $t < a_{n_1,a,z}$ if $b_{n_1,a,z} > 0$. If $b_{n_1,a,z} = 0$ it is sufficient to take as T a solution of $A_{n_1,a,z}$ whose existence is proved in Lemma 4.8 and, if necessary, discard $a_{n_1,a,z} - t$ of its connected components. Now assume $b_{n_1,a,z} > 0$. Increasing if necessary t we see that it is sufficient to do the case $t = a_{n_1,a,z} - 1$. Fix a solution $E = A \cup B$ of $A_{n_1,a-1,z}$ with B a general union of $b_{n_1,a-1,z}$ reducible 1-conics with singular point contained in H. Let $Z \supseteq B$ a general union of $b_{n_1,a,z}$ 1-sundials with $Z_{\mathrm{red}} = B$. Let $G \subset H$ be a general union of $a_{n_1,a,z} - 1 - a_{n_1,a-1,z}$ 1-lines. Set $F := A \cup Z \cup G$. Since Z is a flat limit of a family of $2b_{n_1,a-1,z}$ 1-lines (Remark 2.4), it is sufficient to prove that $h^1(\mathcal{I}_F(a, d_2, \dots, d_k)) = 0$. Since $\mathrm{Res}_H(F) = E$ and $h^i(\mathcal{I}_E(a-1, d_2, \dots, d_k)) = 0$, $i = 0, 1$, it is sufficient $H^1(H, \mathcal{I}_{F\cap H,H}(a, d_2, \dots, d_k)) = 0$. The scheme $F \cap H$ is a general union of G and $b_{n_1,a-1,z}$ planar 1-double points. Lemmas 4.6 and 4.7 imply $a_{n_1,a,z} - a_{n_1,a-1,z} - 1 \le a_{n_1-1,a,z} - 4$. Thus $B_{n_1-1,a,z}$ implies $H^1(H, \mathcal{I}_{F\cap H,H}(a, d_2, \dots, d_k)) = 0$. □

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