Injectivity of non-singular planar maps with one convex component

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Abstract

We prove that if a non-singular planar map $\Lambda \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ has a convex component, then $\Lambda$ is injective. We do not assume strict convexity.

Keywords: Local invertibility, global injectivity, non-strict convexity, Jacobian Conjecture.

1 Introduction

Let $\Omega$ be an open connected subset of $\mathbb{R}^n$. We say that $\Lambda : \Omega \to \mathbb{R}^n$ is locally injective (invertible) at $X \in \Omega$ if there exists a neighbourhoods $U_X \subset \Omega$ of $X$ and $V_{\Lambda(X)}$ of $\Lambda(X)$ such that the restriction $\Lambda : U_X \to V_{\Lambda(X)}$ is injective (invertible). If $\Lambda \in C^1(\Omega, \mathbb{R}^n)$, we denote by $J(X)$ the Jacobian matrix of $\Lambda$ at $X$. By the inverse function theorem, if $J(X)$ is non-singular then $\Lambda$ is locally injective at $X$. It is well-known that locally injective maps need not be globally injective, even if $J(X)$ is non-singular for all $X \in \Omega$, as in the case of the exponential map $\Lambda(x, y) = (e^x \cos y, e^x \sin y)$. Injectivity (invertibility) of locally injective (invertible) maps under suitable additional assumptions has been studied for a long time. In [14] it was conjectured that every polynomial map $\Lambda : \mathbb{C}^n \to \mathbb{C}^n$ with constant non-zero Jacobian determinant is globally invertible, with polynomial inverse. Such a problem, known as Jacobian Conjecture, was widely studied and inserted in a list of relevant problems in [20]. The Jacobian Conjecture was studied in several settings, even replacing $\mathbb{C}$ with other fields, but still remains unsolved for $n \geq 2$, [1, 4, 8, 24]. In [17] it was proved that asking for the determinant of $J(X)$ not to vanish is not sufficient to guarantee $\Lambda$ injectivity. After Pinchuk’s counterexample several papers appeared, dealing with injectivity of non-singular polynomial maps under suitable additional assumptions [5, 6, 7].

Injectivity appears also in connection to a global stability problem formulated in [15]. In this paper it was conjectured that if at any point $J(X)$ has

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eigenvalues with negative real parts then a critical point $O$ of the differential system

$$\dot{X} = \Lambda(X)$$

is globally asymptotically stable. Global asymptotic stability of (1) implies $\Lambda$ injectivity. In [16] it was proved that if $n = 2$, then the vice-versa is true, i.e. injectivity implies global asymptotical stability. Using such a result the conjecture was proved to be true for $n = 2$ [9, 10, 11]. On the other hand the conjecture does not hold in higher dimension, even for polynomial vector fields [2, 3].

Other additional conditions to get injectivity are growth conditions. A classical result in this field is Hadamard theorem [13], which states that if $\Lambda$ is proper, i.e. if $\Lambda^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}^n$, then $\Lambda$ is a bijection. Properness is ensured if $\Lambda$ is norm-coercive, that is if

$$\lim_{|X| \to +\infty} |\Lambda(X)| = +\infty. \quad (2)$$

Coerciveness requires all the component of $\Lambda$ to grow enough for (2) to hold. On the other hand coerciveness is not necessary in order to have injectivity, as the real map $x \mapsto \arctan x$ shows. In [18], studying planar maps $\Lambda(z) = (P(z), Q(z))$, injectivity was proved under a growth condition on just one component of $\Lambda$. In fact, if

$$\int_0^{+\infty} \inf_{|z|=r} |\nabla P(z)|dr = +\infty, \quad (3)$$

then $\Lambda$ is injective. As a consequence, if there exists $k > 0$ such that $|\nabla P(z)| \geq k$, then $\Lambda$ is injective.

Also in this paper, studying planar maps, we prove injectivity imposing a suitable condition on just one component. In fact, we prove that if one of the components $\Lambda(z) = (P(z), Q(z))$ is a non-strictly convex function, then $\Lambda(z)$ is injective. One of the steps in the proof is the same as in [18], since we prove the parallelizability of the Hamiltonian system

$$\begin{cases}
\dot{x} = P_y \\
\dot{y} = -P_x
\end{cases} \quad (4)$$

That is equivalent to prove the connectedness of the level sets of $P(z)$. The connectedness has been used in order to study maps injectivity or non-injectivity in [5, 7, 12, 17, 18]. We observe that the non-strict convexity of the function $P(z)$ implies the non-strict convexity of the orbits of (4), but the vice-versa is not true, as the exponential map shows. Hence injectivity cannot be proved assuming only the non-strict convexity of the orbits of (4).

For the special case of planar polynomial maps with constant, non-zero Jacobian determinant, the level set connectedness was considered in [21, 22, 23]. In [22, 23] was proposed an approach based on the commutativity of the Hamiltonian flows having $P(x, y)$ and $Q(x, y)$ as Hamiltonian functions, similarly to what done in [19].
2 Maps having one convex component

In order to introduce the proof of next theorem, we recall some properties of convex functions.

Proposition 1. Let \( f \in C^2(\mathbb{R}, \mathbb{R}) \), \( H \in C^2(\mathbb{R}^2, \mathbb{R}^2) \) be (non strictly) convex funtions. Then:

i) if \( f \) is non-constant then it is unbounded from above;

ii) if there exist \( u_1 < u_2 < u_3 \in \mathbb{R} \) such that \( f(u_1) = f(u_2) = f(u_3) \), then \( f \) is constant on the interval \([u_1, u_3] \);

iii) the restriction of \( H \) to every line is a convex one-variable function;

iv) sub-level sets of \( f \) and \( H \) are convex;

v) every level set of \( H \) at every point has a tangent line and lies entirely on one side of such a tangent.

vi) the intersection of a level set of \( H \) with any of its tangent lines is connected (a closed interval, in generalized sense).

In the proof of next theorem we consider the family of orbits of the differential system (4). A regular \( C^1 \) curve \( \sigma \) is said to be a section of (4) if it is transversal to (4) at every point of \( \sigma \). If \( \gamma \) is a non-trivial orbit, then for every \( z \in \gamma \) there exists a neighbourhood \( U_z \) of \( z \) and two open disjoint connected subsets \( U_z^+ \subset U_z \) lying on different sides of \( \gamma \), such that \( U_z = U_z^- \cup (\gamma \cap U) \cup U_z^+ \). If \( \sigma \) is a section of \( \gamma \) and \( \sigma \cap \gamma = \{z\} \), then there exist a neighbourhood \( U_z \) of \( z \) and two sub-curves \( \sigma^\pm \), called half-sections, such that \( \sigma^\pm = \sigma \cap U_z^\pm \).

Given a planar differential system without critical points, two orbits \( \gamma_1 \) and \( \gamma_2 \) are said to be inseparable if and only if there exist two half-sections \( \sigma_1 \) and \( \sigma_2 \) such that every orbit meeting \( \sigma_1 \) meets also \( \sigma_2 \) and vice-versa. It can be proved that if \( \gamma_1 \) and \( \gamma_2 \) are inseparable, then for every couple of points \( z_1 \in \gamma_1 \) and \( z_2 \in \gamma_2 \) there exist half-sections such that every orbit meeting \( \sigma_1 \) meets also \( \sigma_2 \) and vice-versa. In other words, the definition of inseparability does not depend on the choice of \( z_1 \) and \( z_2 \).

We denote by \( \phi(t, z) \) the local flow of (4). Since we deal with non-singular maps, such a system has no critical points. Its orbits are positively and negatively unbounded and separate the plane into two connected components. Every orbit is contained in a level set of \( P(z) \), even if in general level sets of \( P(z) \) do not reduce to a single orbit. In what follows we denote by \( A^o \) the interior of a set \( A \) and by \( \overline{A} \) its closure.

Theorem 1. Let \( \Lambda \in C^2(\mathbb{R}^2, \mathbb{R}^2) \) be a non-singular map. If one of its components is convex, then \( \Lambda \) is injective.
Proof. Possibly exchanging the components, we may assume \( P(z) \) to be convex. By lemma 2.2 and theorem 2.1 in [18], it is sufficient to prove that the level sets of \( P(z) \) are connected. By absurd, let us assume that a level set of \( P(z) = h \) is disconnected. As a consequence by lemma 2.2 in [18] the system (4) has a couple \( \gamma_1 \neq \gamma_2 \) of inseparable orbits. By continuity, \( P(z) \) assumes the same value on \( \gamma_1 \) and \( \gamma_2 \), say \( P(\gamma_1) = P(\gamma_2) = k \).

Let us consider two cases.

1) One among \( \gamma_1 \) and \( \gamma_2 \) is not a line. Assume \( \gamma_1 \) is not a line. Let \( \Gamma_1 \) be the closed convex set having \( \gamma_1 \) as boundary.

1.1) If \( \gamma_2 \subset \Gamma_1 \), then it is not a line, otherwise it would meet \( \gamma_1 \), contradicting uniqueness of solutions. Let \( z_1 \) be an arbitrary point of \( \gamma_1 \) and \( \tau_{z_2} \) be the line passing through \( z_1 \) and tangent to \( \gamma_2 \), existing by the convexity of \( \Gamma_2 \). Since \( \gamma_2 \) is not a line one can rotate \( \tau_{z_2} \) around \( z_1 \) until it meets \( \gamma_2 \) at two points \( z_1^1 \neq z_2^2 \). Let us call \( \tau^* \) such a line. Then \( \tau^* \) meets the level set \( P(z) = k \) at three distinct points, \( z_1, z_1^1, z_2^2 \). By proposition 1, (ii), \( P(z) \) is constant on the smallest segment \( \Sigma \) containing \( z_1, z_1^1, z_2^2 \). The set \( \gamma_1 \cup \Sigma \cup \gamma_2 \) is connected and contained in \( P(z) = k \), contradicting the fact that \( \gamma_1 \) and \( \gamma_2 \) are distinct connected components of \( P(z) = k \).

1.2) Let \( \gamma_2 \subset \Gamma_1 \). If \( \gamma_1 \subset \Gamma_2 \), then one can reply the argument of point 1.2), exchanging the role of \( \gamma_1 \) and \( \gamma_2 \).

1.3) Assume \( \gamma_1 \not\subset \Gamma_2 \) and \( \gamma_2 \not\subset \Gamma_1 \). Let \( D_1 \) be the subset of \( \gamma_1 \) consisting of its linear parts, i.e. half-lines and line segments. Since \( \gamma_1 \) is not a line, one has \( D_1 \neq \gamma_1 \). Let us choose arbitrarily \( z_1 \in \gamma_1 \setminus D_1 \) and let \( \tau_1 \) be the tangent line of \( \gamma_1 \) at \( z_1 \). By point 3) of Proposition 1 \( \gamma_1 \) lies on one side of \( \tau_1 \). One has \( \gamma_1 \cap \tau_1 = \{z_1\} \). Let \( \tau_{1}^z \) be the half-lines contained in \( \tau_1 \) having \( z_1 \) as extreme point, \( \tau_{1}^z \) tangent to the positive semi-orbit of \( z_1 \), \( \tau_{1}^z \) tangent to the negative semi-orbit of \( z_1 \). Let \( \Pi_1 \) the closed half-plane having \( \tau_1 \) as boundary and containing \( \gamma_1 \). For all \( \epsilon > 0 \) one has \( \phi(\pm \epsilon, z_1) \in \Pi_1^\circ \). Every such orbit meets \( \tau_1 \) at least at two points lying on distinct half-lines. As a consequence, \( z_1 \) is an isolated point of minimum of the restriction of \( P(z) \) to the line \( \tau_1 \). Hence \( \gamma_2 \) does not meet \( \tau_1 \).

By the inseparability of \( \gamma_1 \) and \( \gamma_2 \) there are half-sections \( \sigma_1 \) of \( \gamma_1 \) at \( z_1 \) and \( \sigma_2 \) of \( \gamma_2 \) at \( z_2 \) such that every orbit meeting \( \sigma_1 \) meets also \( \sigma_2 \) and vice-versa. One can take \( \sigma_1 \) and \( \sigma_2 \) small enough to have \( \overline{\sigma_1} \) and \( \overline{\sigma_2} \) compact, disjoint and such that \( \sigma_2 \cap \Pi_1 = \emptyset \).

There exist neighbourhoods \( U^\pm_{\epsilon} \) of \( \gamma_1(\pm \epsilon) \) such that \( U^\pm_{\epsilon} \subset \Pi^\circ \). By the continuous dependance on initial data there exists a neighbourhood \( U_1 \) of \( z_1 \) such that \( \phi(\pm \epsilon, U_1) \subset U^\pm_{\epsilon} \). This holds in particular for the points of \( \delta_1 = \sigma_1 \cap U_1 \), so that \( \phi(\pm \epsilon, \delta_1) \subset U^\pm_{\epsilon} \subset \Pi^\circ_1 \). \( \delta_1 \) is itself a half-section at \( z_1 \). For all \( z \in \delta_1 \), the orbit \( \phi(t, z) \) meets both \( \tau_{1}^z \) and \( \tau_{1}^z \), hence both half-lines contain points \( z^\pm \) such that \( P(z^-) = P(z^+) > P(z_1) \). Moreover, \( \phi(t, z) \) does not meet \( \tau_1 \) at a third point, since in that case, by point 2) of Proposition 1, \( P(z) \) would be constant on a segment of \( \gamma_1 \) containing \( z_1 \), contradiction. Hence, for all \( z \in \delta_1 \), both semi-orbits starting at \( z \) are definitively (resp. for \( t \to \pm \infty \)) contained in \( \Pi^\circ_1 \).

The set \( W = \phi([-\epsilon, \epsilon], \overline{\delta_1}) \) is compact. It is possible to take \( \overline{\delta_1} \) small enough
in order to have \( z_2 \notin W \cup \Pi_1 \) (otherwise \( z_2 = z_1 \)). By construction, every orbit starting at a point of \( \delta_1 \) is contained in the closed set \( W \cup \Pi_1 \). Let us denote by \( \delta_2 \) the part of \( \sigma_2 \) met by orbits starting at points of \( \delta_1 \). Since every point of \( \delta_2 \) lies on an orbit starting at \( \delta_1 \), the half-section \( \delta_2 \) is contained in \( W \cup \Pi_1 \). As a consequence, one has
\[
z_2 \in \delta_2 \subset W \cup \Pi,
\]
contradiction.

2) Assume both \( \gamma_1 \) and \( \gamma_2 \) to be lines. They are parallel, since otherwise they should meet at a point \( z_0 \) which should be a fixed point of (4), contradicting the nonsingularity of \( \Lambda \). Let \( \Sigma_{12} \) be the closed strip having boundary \( \gamma_1 \cup \gamma_2 \). Let \( \sigma \) be a line orthogonal to \( \gamma_1 \) and \( \gamma_2 \), and let us set \( z_1 = \gamma_1 \cap \sigma, z_2 = \gamma_2 \cap \sigma \), \( \sigma_{12} = \Sigma_{12} \cap \sigma \). The orbits \( \gamma_1 \) and \( \gamma_2 \) are inseparable, hence there exist open sub-sections \( \sigma_1 \) and \( \sigma_2 \) of \( \sigma_{12} \) such that \( z_1 \in \sigma_1, z_2 \in \sigma_2, \sigma_1 \cap \sigma_2 = \emptyset \) and every orbit meeting \( \sigma_1 \) meets \( \sigma_2 \), and vice-versa. Let \( \Phi_{12} \) be the union of the orbits meeting \( \sigma_1 \) and \( \sigma_2 \). Both \( \gamma_1 \) and \( \gamma_2 \) are contained in \( \partial \Phi_{12} \). The restriction of \( P(z) \) to the compact set \( \sigma_{12} \) is convex and non constant (because if it was constant \( \gamma_1 \), \( \gamma_2 \) and \( \sigma_{12} \) would be in \( P(z) = k \), contradiction). One has
\[
\max \{ P(z) : z \in \sigma_{12} \} = P(z_1) = P(z_2) = k.
\]

Let \( z_m \) a point of \( \sigma_{12} \) such that
\[
P(z_m) = \min \{ P(z) : z \in \sigma_{12} \} < P(z_1) = P(z_2) = k.
\]
The orbit starting at \( z_m \) is tangent to \( \sigma_{12} \) and lies entirely on one side of \( \sigma_{12} \). One has \( \nabla P(z_m) \perp \sigma_{12} \), with the vector \( \nabla P(z_m) \) pointing towards the half-strip \( \Sigma_{12}^+ \) not containing \( \phi(t, z_m) \). Let \( \eta \) be the line parallel to \( \gamma_1 \) and \( \gamma_2 \) passing through \( z_m \). The line \( \eta \) meets all the orbits passing through \( \sigma_1 \) and \( \sigma_2 \), hence the restriction of \( P(z) \) to \( \eta \) assumes every value belonging to \( [P(z_m), k] \). On the other hand, by proposition 1, \( i \), \( P(z) \) is unbounded from above on \( \eta \), hence there exists a point in \( z \in \eta \) such that \( P(z) = k \). Let \( z_{12} \) the point such that \( P(z_{12}) = k \), closest to \( z_m \). Then the orbit \( \phi(t, z_{12}) \) is inseparable from \( \gamma_1 \) and \( \gamma_2 \), since every orbit meeting \( \sigma_1 \) and \( \sigma_2 \) also meets \( \eta \) in a neighbourhood of \( z_{12} \). In other words, a suitable sub-segment \( \eta_{12} \) of \( \eta \) is a half-section of \( \phi(t, z_{12}) \) such that every orbit meeting \( \sigma_1 \) and \( \sigma_2 \) meets also \( \eta_{12} \), and vice-versa.

The orbit \( \phi(t, z_{12}) \) cannot be a line because in such a case either it would be parallel to \( \gamma_1 \) and \( \gamma_2 \), contradicting their inseparability, or transversal to them, implying the existence of two critical points, \( \gamma_1 \cap \gamma_{12} \) and \( \gamma_2 \cap \gamma_{12} \). Since \( \gamma_{12} \) is not a line point 1) applies.

A simple example of non-linear non-singular map with both non-strictly convex components is
\[
\Lambda(x, y) = (x + y + e^x, x + y + e^y).
\]
The Hamiltonian system of a non-strictly convex two-variables function has non-strictly convex orbits. The vice-versa is not true, as the function \( e^x \cos y \) shows.
In fact, the connected components of $e^x \cos y = 0$ are lines, and the connected components of $e^x \cos y = k \neq 0$ are strictly convex, since they are graphs of the one-variable functions

$$x = \ln \left( \frac{k}{\cos y} \right),$$

whose second derivative does not vanish. On the other hand the hessian matrix of $e^x \cos y$ is:

$$
\begin{pmatrix}
  e^x \cos y & -e^x \sin y \\
  -e^x \sin y & -e^x \cos y 
\end{pmatrix},
$$

whose Jacobian determinant is $-e^{2x} < 0$. In fact, the map $\Lambda(x, y) = (e^x \cos y, e^x \sin y)$ is not injective, even if both Hamiltonian systems of its components have non-strictly convex orbits.

### 3 Relationship to previous results

The key point in the proof of our main result is the level sets connectedness. This property has been already used in previous papers.

**References**


