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On large and small torsion pairs

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Contents

Introduction

Torsion pairs were introduced by Dickson in 1966 as a generalisation of the concept of torsion abelian group to arbitrary abelian categories. Using torsion pairs, we can divide complex abelian categories in smaller parts which are easier to understand.

The central topic of this thesis is the theory of torsion pairs in the category of finitely generated modules over an artin algebra.

These torsion pairs were thoroughly investigated in the last years from different viewpoints:

In [1] it was shown that some nice torsion pairs, the functorially finite ones, can be parametrized by means of particular objects: the support $\tau-$tilting modules. These objects are a generalisation of classical (1-)tilting modules and admit a particularly nice theory of mutation.

Using the $\tau-$tilting approach, in [31], it was proven a fundamental connection between torsion pairs and bricks, that is modules whose endomorphism ring is a division ring. In the same work, the authors introduced the concept of $\tau-$tilting finite algebras, a class of finite-dimensional algebras Λ with only a finite number of support $\tau-$tilting modules: it was shown that for these algebras all the torsion classes in $\Lambda\text{-}\mathrm{mod}$ are functorially finite and that an algebra is $\tau-$tilting finite if and only if it admits only a finite number of torsion classes in $\Lambda\text{-}\mathrm{mod}$.

More recently, the complete lattice structure of the set of torsion classes was investigated in [32], where it was shown to have several peculiar characteristics, such as complete semi-distributivity, and once again it was shown the important role of bricks: they can be used to label the arrows of the Hasse quiver of the lattice of torsion classes. In fact, they control the covering relation between torsion classes.

This role of bricks as labels of the covering relation, is further investigated in [11] and [12], with particular emphasis on the interplay between torsion classes and wide subcategories, and in [17] through the concept of minimal extending modules.

A uniform approach to the study of torsion pairs, wide subcategory and the more general ICE-closed subcategories is proposed in [35], where monobricks, special sets of bricks with a poset structure, allow for a complete classification of torsion pairs in abelian length categories. In the same paper we also find an interesting conjecture relating $\tau-$tilting infiniteness with the existence of an infinite semibrick.

Among the most recent results on the structure of bricks over $\tau-$tilting finite algebras we find in [59] the proof of a modern analogue of the first Brauer-Thrall conjecture: a finite-dimensional algebra is $\tau-$tilting finite if and only if there is a bound on the length of its finitely generated bricks. In the same paper the authors present a possible analogue of the second Brauer-Thrall conjecture: a finite-dimensional algebra is $\tau-$tilting infinite if and only if there exists an integer d such that there is an infinite family of non-isomorphic bricks of length d. This last question is still open.

Inspired by the process which led to the solution of the classical second Brauer-Thrall conjecture for finite-dimensional algebras through the classification of minimal representation-infinite algebras, in [50] the authors introduced minimal $\tau-$tilting infinite algebras. In the same paper, we also find a proof of the modern first conjecture using a geometrical approach.

Notice that the approaches to the study of torsion pairs and bricks proposed in the quoted papers were mostly not taking advantage of the influence that infinitely generated modules have on the "small" ones. In this thesis we will focus on the interplay between large and small torsion pairs.

In particular, a fundamental observation is that the lattice of torsionfree classes in the category of finitely generated modules is isomorphic to the lattice of definable torsionfree classes in the category of (all) modules, see Theorem 1.3.28. As every definable torsionfree class is cogenerated by a cosilting module, which is essentially a not necessarily compact support $\tau^{-1}-$tilting module, we can access in this large setting to some tools which, in the finitely generated world, are only available for functorially-finite pairs.

Moreover, passing to large modules fixes the lack of completeness which some results exhibit in the category of finitely generated modules. As an example consider the existence of monobricks without maximal elements [35].

Further motivation for this "large modules" approach can be found in [6] where the authors developed a general concept of mutation for not necessarily compact cosilting objects. This operation controls certain inclusions of torsion classes: let $\mathbf{t} \subseteq \mathbf{u}$ be torsion classes in the category of finitely generated modules over an artin algebra Λ. Then we have cosilting modules C_T, C_U such that $\mathbf{t} = {}^{\perp_0}C_T \cap \Lambda\text{-mod}$ and $\mathbf{u} = {}^{\perp_0}C_U \cap \Lambda\text{-mod}$.

We have that $\mathbf{t}^{\perp_0} \cap \mathbf{u}$ is a wide subcategory of $\Lambda\text{-mod}$ (i.e. it is closed under kernels, cokernels and extensions) if and only if the cosilting module C_T is obtained as a mutation of C_U.

While working with arbitrary modules brings in a good amount of complications, we hope to convince the reader that this approach can produce results worth of the additional strain.

Structure of the thesis

Chapter 1 is a collection of most of the notions needed in the rest of the thesis: this includes both some basic categorical notions, like torsion pairs and localising subcategories, and some more specialised material regarding purity, silting and cosilting modules and $\tau-$tilting theory. This chapter does not contain any new result.

Chapter 2 is centred around the notion of torsionfree, almost torsion modules with respect to some torsion pair, as introduced in [3].

After showing some compatibility results with the finitely generated version of this concept, the minimal extending modules of [17], we proceed to prove that cosilting torsionfree classes are determined by their torsionfree, almost torsion modules.

In the final part of the chapter, using the existence of locally maximal torsion classes in the category of finitely generated modules on a $\tau-$tilting infinite algebra, we prove the following characterisation of $\tau-$tilting finiteness:

Theorem (2.4.4)**.** *An artin algebra Λ is $\tau-$tilting finite if and only if every brick over Λ is finitely generated.*

Chapter 3 revolves around the notion of wide subcategory. In section 3.1, we revisit the constructions associating wide subcategories with torsion pairs, discovered in [43] for hereditary rings and extended in [48] to the general case.

Given any torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ we have that

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \text{ for all } T \in \mathcal{T}, f : T \to X, \ker(f) \in \mathcal{T}\}$$
$$\beta(\mathcal{F}) := \{X \in \mathcal{F} \mid \text{ for all } F \in \mathcal{F}, f : X \to F, \operatorname{coker}(f) \in \mathcal{F}\}$$

are wide subcategories of $\mathcal{A}$. After recalling some known properties of these subcategories, we notice that the simple objects of $\alpha(\mathcal{T})$ are precisely the torsion, almost torsionfree objects and that the simples of $\beta(\mathcal{F})$ are the torsionfree, almost torsion objects with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

We conclude the section focusing on the case where the abelian category is the module category $R\text{-}\mathrm{Mod}$ of some left noetherian ring and the torsion pair is cogenerated by a cosilting module. In Proposition 3.1.12 we give a compatibility result: the wide subcategories $\widetilde{\alpha}(\mathbf{t}), \widetilde{\beta}(\mathbf{f})$ obtained from the restriction $(\mathbf{t}, \mathbf{f}) = (\mathcal{T} \cap R\text{-}\mathrm{mod}, \mathcal{F} \cap R\text{-}\mathrm{mod})$ are exactly the intersections of the large ones $\alpha(\mathcal{T}), \beta(\mathcal{F})$ with $R\text{-}\mathrm{mod}$.

In section 3.2 we give an alternative description of the wide subcategories in terms of the approximation sequence $0 \to C_1 \to C_0 \to E(R)$ induced by every cosilting module C, where $E(R)$ is an injective cogenerator. If we fix $\mathcal{T} = {}^{\perp_0}C$ and $\mathcal{F} = \mathrm{Cogen}(C)$ then by Theorem 3.2.1 and Theorem 3.2.13 we have that

$$\alpha(\mathcal{T}) = {}^{\perp_{1h}}C_0 \cap \mathcal{T}$$
$$\beta(\mathcal{F}) = {}^{\perp_0}C_1 \cap \mathcal{F}$$

These categories are essentially perpendicular categories, as we can see more clearly in the case where C is a cotilting module. In fact under this stronger assumption we have $\alpha(\mathcal{T}) = {}^{\perp_{0,1}}C_0$ and $\beta(\mathcal{F}) = {}^{\perp_{0,1}}C_1$.

These new descriptions allow us to prove that $\beta(\mathcal{F})$ is always a coreflective subcategory of the module category, while $\alpha(\mathcal{T})$ is a wide subcategory closed under coproducts, which is coreflective in the noetherian case. Moreover, we can show that the assignment associating to a definable torsionfree class $\mathcal{F}$ the wide subcategory $\beta(\mathcal{F})$ is injective. In Proposition 3.2.32 we also characterise $\tau-$tilting finiteness in terms of the class of wide subcategories closed under coproducts.

In the final section 3.3 we give further applications of these results: first we show that non-trivial minimal cosilting torsion pairs always admit a mutation, in the sense of [6]. Secondly, we explore the connection between minimal silting and minimal cosilting modules and Ext-orthogonal pairs, as introduced in [46], over an hereditary left artinian ring.

Chapter 4 presents some ongoing work on mutation of torsion pairs.

In [11], Asai gives bijections between the set of functorially finite torsion pairs and certain simple-minded collections in the derived category. These collections are precisely the simple objects of the length heart corresponding to a support $\tau-$tilting module.

A first objective of the chapter is to extend this bijection to arbitrary torsion pairs. However, as the heart obtained from a non-compact cosilting module is only locally finitely presented, it is not determined by its simple objects. The idea is to replace simple objects with indecomposable injectives. As indecomposable injective objects in the heart of a cosilting t-structure correspond with indecomposable pure-injective objects in the derived category $\mathrm{D}(R)$ and with indecomposable pure-injective modules, our constructions will involve the Ziegler spectrum. This is a topological space whose points are isomorphism classes of indecomposable pure-injective objects and whose closed sets are in bijection with definable subcategories of the module category.

Inspired by [1] and [19], in section 4.1 we give two possible notions: maximal rigid systems in the derived category and cosilting pairs in the module category. We prove that both these objects are in bijection, up to a suitable notion of equivalence, with cosilting modules over an artin algebra.

The aim of section 4.2 is to find a way to compute explicitely cosilting mutation in the sense of [6] in terms of basic operations on cosilting pairs, mimicking the finite length case from [1].

This requires a heavy machinery, involving the logical-topological notion of neg-isolated point of the Ziegler spectrum, studied in [3] for cotilting modules. To adapt the results to the cosilting case we need a further discussion of characteristic bricks, in particular of torsion, almost torsionfree modules.

We are finally able to prove that mutation is possible and well-behaved when performed at the so called “special” and “very critical” points of the cosilting pair which are in bijection with torsion, almost torsionfree and torsionfree, almost torsion modules respectively.

The final result of the chapter is Theorem 4.2.43 which describes explicitely the mutation operation in terms of exchange of elements of cosilting pairs.

Finally, the Appendix contains a brief overview of some pathologies happening in the lattice of torsion pairs over the wild Kronecker algebra $k\mathcal{K}_3$.

Notation

Whenever we consider an operator applied to some class $\mathcal{S}$ consisting of a single element X, we will usually write X instead of $\{X\}$.

Unless specified, R is a ring, A is a (left) artinian ring, Λ is an artin algebra and modules will be left modules.

- $(\mathcal{A}, \mathcal{B})$: the category of additive functors and natural transformations, where $\mathcal{A}$, $\mathcal{B}$ are additive categories.
- $\operatorname{add}(\mathcal{S})$: the class of objects isomorphic to direct summands of finite direct sums of objects in $\mathcal{S}$
- $\operatorname{Add}(\mathcal{S})$: the class of objects isomorphic to direct summands of direct sums of objects in $\mathcal{S}$
- $\operatorname{cogen}(\mathcal{S})$: the class of objects isomorphic to a subobject of some finite sum of objects in $\mathcal{S}$
- $\operatorname{Cogen}(\mathcal{S})$: the class of objects isomorphic to a subobject of some product of objects in $\mathcal{S}$
- $\operatorname{Cogen}_*(\mathcal{S})$: the class of objects isomorphic to a pure subobject of some product of objects in $\mathcal{S}$
- $\operatorname{filt}(\mathcal{S})$: the extension closure of the set $\mathcal{S}$
- $\operatorname{Filt}(\mathcal{S})$: the class of objects admitting a transfinite filtration by objects in $\mathcal{S}$
- $\operatorname{gen}(\mathcal{S})$: the class of objects isomorphic to a quotient of some finite sum of objects in $\mathcal{S}$
- $\operatorname{Gen}(\mathcal{S})$: the class of objects isomorphic to a quotient of some direct sum of objects in $\mathcal{S}$.
- $\operatorname{Prod}(\mathcal{S})$: the class of objects isomorphic to direct summands of products of objects in $\mathcal{S}$
- $\mathbf{F}(\mathcal{S})$: the smallest torsionfree class containing some subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{G}$
- $\widetilde{\mathbf{F}}(\mathcal{S})$: the smallest torsionfree class containing some subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ with artinian objects
- $\mathbf{T}(\mathcal{S})$: the smallest torsion class containing some subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{G}$
- $\widetilde{\mathbf{T}}(\mathcal{S})$: the smallest torsion class containing some subcategory $\mathcal{S}$ of an abelian category $\mathcal{N}$ with noetherian objects

- ${}^{\perp_I}\mathcal{S}$: for $I \subseteq \mathbb{Z}$ and $\mathcal{S} \subseteq \mathcal{A}$, with $\mathcal{A}$ abelian, the subcategory consisting of the objects $X \in \mathcal{A}$, such that $\mathrm{Ext}^i_{\mathcal{A}}(X, S) = 0$, for all $i \in I$, $S \in \mathcal{S}$.

  Analogously, if $\mathcal{S} \subseteq \mathcal{T}$ with $\mathcal{T}$ triangulated with shift functor Σ, the subcategory consisting of the objects $X \in \mathcal{T}$, such that $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i S) = 0$, for all $i \in I$, $S \in \mathcal{S}$.

- $\mathcal{S}^{\perp_I}$: for $I \subseteq \mathbb{Z}$ and $\mathcal{S} \subseteq \mathcal{A}$, with $\mathcal{A}$ abelian, the subcategory consisting of the objects $X \in \mathcal{A}$, such that $\mathrm{Ext}^i_{\mathcal{A}}(S, X) = 0$, for all $i \in I$, $S \in \mathcal{S}$.

  Analogously, if $\mathcal{S} \subseteq \mathcal{T}$ with $\mathcal{T}$ triangulated with shift functor Σ, the subcategory consisting of the objects $X \in \mathcal{T}$, such that $\mathrm{Hom}_{\mathcal{T}}(S, \Sigma^i X) = 0$, for all $i \in I$, $S \in \mathcal{S}$.

- $\varinjlim \mathcal{S}$: the direct limit closure of $\mathcal{S}$ in some cocomplete category

- R-mod : the category of finitely presented left modules over a ring R.

- mod-R : the category of finitely presented right modules over a ring R.

- R-Mod : the category of left modules over a ring R.

- Mod-R : the category of right modules over a ring R.

- $\langle \mathcal{S} \rangle$: for a subcategory $\mathcal{S}$ of some Grothendieck category, the smallest definable subcategory containing $\mathcal{S}$.

- **TPair**(R) : For a ring R, the collection of torsion pairs in R-Mod.

- **tpair**(R) : For a left coherent ring R, the collection of torsion pairs in R-mod.

- **Tors**(R) : For a ring R, the collection of torsion classes in R-Mod.

- **tors**(R) : For a left coherent ring R, the collection of torsion classes in R-mod.

- **ftors**(A) : For an artin algebra A, the collection of functorially finite torsion classes in A-mod.

- **Torf**(R) : For a ring R, the collection of torsionfree classes in R-Mod.

- **torf**(R) : For a left coherent ring R, the collection of torsionfree classes in R-mod.

- **ftorf**(A) : For an artin algebra A, the collection of functorially finite torsionfree classes in A-mod.

- **Cosilt**(R) : For a ring R, the collection of torsionfree classes in R-Mod cogenerated by a cosilting module, or equivalently the collection of torsion pairs whose torsionfree class satisfies this condition.

- **Silt**(R) : For a ring R, the collection of torsion classes in R-Mod generated by a silting module, or equivalently the collection of torsion pairs whose torsion class satisfies this condition.

- $\mathbf{Wide}(R)$: For a ring R, the collection of wide subcategories of R-Mod.
- $\mathbf{wide}(R)$: For a left coherent ring R, the collection of wide subcategories of R-mod.
- $\mathbf{Wide}_{\coprod}(R)$: For a ring R, the collection of wide subcategories of R-Mod closed under coproducts.
- $\mathcal{C}_\sigma$: the class $\left\{X \in R - \text{Mod} \,\middle|\, \text{Hom}_R(X, \sigma) \text{ is surjective}\right\}$ for a given $R-$module homomorphism σ
- $\mathcal{D}_\sigma$: the class $\left\{X \in R - \text{Mod} \,\middle|\, \text{Hom}_R(\sigma, X) \text{ is surjective}\right\}$ for a given $R-$module homomorphism σ
- $\text{A}(\mathcal{T})$: the subcategory $\left\{X \in \mathcal{A} \,\middle|\, \text{for all } T \in \mathcal{T}, f : T \to X, \ker(f) \in \mathcal{T}\right\}$ for a torsion class $\mathcal{T}$ in an abelian category $\mathcal{A}$
- $\alpha(\mathcal{T})$: $\text{A}(\mathcal{T}) \cap \mathcal{T}$
- $\text{B}(\mathcal{F})$: the subcategory $\left\{X \in \mathcal{A} \,\middle|\, \text{for all } F \in \mathcal{F}, f : X \to F, \text{coker}(f) \in \mathcal{F}\right\}$ for a torsionfree class $\mathcal{F}$ in an abelian category $\mathcal{A}$
- $\beta(\mathcal{F})$: $\text{B}(\mathcal{F}) \cap \mathcal{F}$

Chapter 1

Preliminaries

All the categories we consider are assumed to be locally small. All the subcategories will be full and replete, we will not distinguish a subcategory from the corresponding class of objects. We tacitly assume that all the abelian categories we deal with are well-powered.

None of the results appearing in this chapter is original.

1.1 Torsion pairs

Fix an abelian category $\mathcal{A}$.

Definition 1.1.1. Let $\mathcal{T}$, $\mathcal{F}$ be subcategories of $\mathcal{A}$. Then $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if:

(i) For all $F \in \mathcal{F}$, for all $T \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$.

(ii) For all $M \in \mathcal{A}$ there is a short exact sequence:

$$0 \to T \to M \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

In this case, $\mathcal{T}$ is a *torsion class* and $\mathcal{F}$ is a *torsionfree class.*

Remark 1.1.2. *Notice that the concept of torsion pair is self-dual, while that of torsion class is dual to that of torsionfree class.*

The short exact sequences provided by a torsion pair are unique up to isomorphism and functorial (under the axiom of choice).

Proposition 1.1.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair and $M \in \mathcal{A}$. Then the short exact sequence $0 \to \mathrm{t}_{\mathcal{T}} M \to M \to M/(\mathrm{t}_{\mathcal{T}} M) \to 0$ provided by the definition of a torsion pair is unique up to isomorphism.*

Moreover, the assignments $M \mapsto \mathrm{t}_{\mathcal{T}} M$ and $M \mapsto M/(\mathrm{t}_{\mathcal{T}} M)$ can be extended to endofunctors on $\mathcal{A}$. The functor $\mathrm{t}_{\mathcal{T}}$ is known as the torsion radical *and is right adjoint to the inclusion functor $\mathcal{T} \to \mathcal{A}$.*

We will usually omit the subscripts in the approximation sequences, unless they are needed to avoid ambiguities.

The classes in a torsion pair determine each other:

Proposition 1.1.4. *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\mathcal{A}$*, then:*

(i) $T \in \mathcal{T}$ *if and only if for all* $F \in \mathcal{F}$, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$.

(ii) $F \in \mathcal{F}$ *if and only if for all* $T \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$.

Under some additional hypotheses, torsion and torsionfree classes can be described in term of closure conditions:

Proposition 1.1.5. *Let* $\mathcal{A}$ *be a (well-powered) abelian category. Let* $\mathcal{T}$ *be a subcategory of* $\mathcal{A}$*. Assume that for every object* $M \in \mathcal{A}$ *the poset of subobjects of* M *belonging to* $\mathcal{T}$ *admits a maximal element. Then* $\mathcal{T}$ *is a torsion class if and only if it is closed under extensions and quotients.*

Proof. Any torsion class is closed under extensions and epimorphic images. Assume now that $\mathcal{T}$ has such closure properties. It is enough to show that $(\mathcal{T}, \mathcal{T}^{\perp_0})$ is a torsion pair.

Since, by definition, we have no non-zero morphism between this two subcategories, we have to show that every object $M \in \mathcal{A}$ fits in a short exact sequence of the right shape.

By hypothesis, M admits some maximal subobject in $\mathcal{T}$, but since $\mathcal{T}$ is closed under finite direct sums and epimorphic images, this maximal subobject, say $\mathrm{t}_{\mathcal{T}} M$, must be unique, thus a maximum.

Consider the short exact sequence $0 \to \mathrm{t}_{\mathcal{T}} M \to M \to M/\mathrm{t}_{\mathcal{T}} M \to 0$, assume we have a map from $\mathcal{T}$ to $M/\mathrm{t}_{\mathcal{T}} M$ which we might assume injective, using again that $\mathcal{T}$ is closed under quotients.

This gives rise to the pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{t}_{\mathcal{T}} M & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{t}_{\mathcal{T}} M & \longrightarrow & M & \longrightarrow & M/\mathrm{t}_{\mathcal{T}} M & \longrightarrow & 0 \end{array}$$

Since $\mathcal{T}$ is closed under extension, $P \in \mathcal{T}$, but then by maximality of tM, we must have $\mathrm{t}_{\mathcal{T}} M = P$ and $T = 0$. So that $M/\mathrm{t}_{\mathcal{T}} M \in \mathcal{T}^{\perp_0}$. □

As a corollary of the previous proposition and of its dual we obtain the following more practical result:

Corollary 1.1.6. *Let* $\mathcal{G}$ *be a complete and cocomplete abelian category. Then a subcategory* $\mathcal{T} \subseteq \mathcal{G}$ *is a torsion class if and only if it is closed under extensions, quotients and coproducts.*

Dually, a subcategory $\mathcal{F} \subseteq \mathcal{G}$ *is a torsionfree class if and only if it is closed under extensions, submodules and products.*

Proof. It is enough to show the existence of maximal subobjects in $\mathcal{T}$ for any $M \in \mathcal{G}$. This is simply the join (i.e the sum) of all the subobjects of M lying in $\mathcal{T}$. (This is an object in $\mathcal{T}$ using the closure under coproducts and quotients). □

Remark 1.1.7. *Let $\mathcal{C}$ be a class of objects in a complete and co-complete abelian category $\mathcal{A}$.*

We have the following two torsion pairs: the torsion pair generated *by $\mathcal{C}$, defined as $({}^{\perp_0}(\mathcal{C}^{\perp_0}), \mathcal{C}^{\perp_0})$ and the torsion pair* cogenerated *by $\mathcal{C}$, defined as $({}^{\perp_0}\mathcal{C}, ({}^{\perp_0}\mathcal{C})^{\perp_0})$.*

We will use the symbols $\mathbf{T}(\mathcal{C})$ (resp. $\mathbf{F}(\mathcal{C})$) for the torsion class generated by $\mathcal{C}$ (resp. the torsionfree class cogenerated by $\mathcal{C}$).

We also have a small version of the corollary above:

Corollary 1.1.8. *Let $\mathcal{N}$ be an abelian category such that all the objects of $\mathcal{N}$ are noetherian.*

Then a subcategory $\mathbf{t} \subseteq \mathcal{N}$ is a torsion class if and only if it is closed under extensions and quotients.

Dually we have:

Corollary 1.1.9. *Let $\mathcal{A}$ be an abelian category such that all the objects of $\mathcal{A}$ are artinian.*

Then a subcategory $\mathbf{f} \subseteq \mathcal{A}$ is a torsionfree class if and only if it is closed under extensions and submodules.

This also yields:

Corollary 1.1.10. *Let $\mathcal{L}$ be an abelian category where all the objects have finite length.*

Then a subcategory $\mathbf{t} \subseteq \mathcal{L}$ is a torsion class if and only if it is closed under extensions and quotients and a subcategory $\mathbf{f} \subseteq \mathcal{L}$ is a torsionfree class if and only if it is closed under extensions and submodules.

Remark 1.1.11. *In the case of length categories, we use the symbols $\widetilde{\mathbf{T}}(\mathcal{C})$ and $\widetilde{\mathbf{F}}(\mathcal{C})$ for the smallest torsion and torsionfree class generated by some subcategory $\mathcal{C}$.*

The proof of the existence of the two classes is the same as in the case of a complete and cocomplete abelian category.

We conclude with a lemma dealing with couples of torsion pairs, for which we will use the following notation:

Definition 1.1.12. Let $\mathcal{C}, \mathcal{D}$ be two subcategories of some abelian category $\mathcal{A}$.

We denote by $\mathcal{C} \star \mathcal{D}$ the full subcategory of $\mathcal{A}$ whose objects are the M with a short exact sequence:

$$0 \to C \to M \to D \to 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Lemma 1.1.13. *Let $\mathcal{A}$ be an abelian category and let $(\mathcal{T}, \mathcal{F})$, $(\mathcal{T}', \mathcal{F}')$ be two torsion pairs in $\mathcal{A}$. Assume $\mathcal{F} \subseteq \mathcal{F}'$, then $\mathcal{F}' = (\mathcal{T} \cap \mathcal{F}') \star \mathcal{F}$.*

Proof. One of the two inclusions is trivial, as every torsionfree class is closed under extensions. For the other, let $M \in \mathcal{F}'$. The approximation sequence $0 \to \mathrm{t}_{\mathcal{T}} M \to M \to M/\mathrm{t}_{\mathcal{T}} M \to 0$, with respect to $(\mathcal{T}, \mathcal{F})$, expresses M as an extension of $(\mathcal{T} \cap \mathcal{F}')$ and $\mathcal{F}$. □

1.1.1 An explicit construction in well-behaved categories

In this section R will be an arbitrary ring.

As we have shown, in a complete and cocomplete abelian category it is possible to form a minimal torsion and a minimal torsionfree class containing any given subcategory.

In the case of module categories, it is possible to give an explicit description of the smallest torsion class generated by some category.

To this aim, we will need the concept of a transfinite extension. For an extensive treatment of this topic and further applications to approximation theory we refer to [39, Sections 6.1].

Definition 1.1.14. Let μ be an ordinal, $(A_\alpha \mid \alpha \leq \mu)$ a sequence in R-Mod. Let $(f_{\alpha\beta}) : A_\alpha \to A_\beta \mid \alpha \leq \beta \leq \mu)$ be a sequence of monomorphisms, such that $\mathcal{A} = \{(A_\alpha, (f_{\alpha\beta}))\}$ is a direct system of modules.

Then $\mathcal{A}$ is a *continuous direct system* if $A_0 = 0$ and $A_\beta = \varinjlim_{\alpha<\beta} A_\alpha$ for all limit ordinals $\beta \leq \mu$.

Given a subcategory $\mathcal{C} \subseteq R$-Mod we say that a module M is $\mathcal{C}-$filtered if there exists a continuous directed system $\mathcal{A} = \{(A_\alpha, (f_{\alpha\beta})), \alpha \leq \beta \leq \mu\}$ with $A_\mu = M$ such that $\operatorname{coker} f_{\alpha\alpha+1} \in \mathcal{C}$ for all $\alpha < \mu$.

We denote by $\operatorname{Filt}(\mathcal{C})$ the class of $\mathcal{C}-$filtered modules. The symbol $\operatorname{filt}(\mathcal{C})$ denotes the subclass of $\operatorname{Filt}(\mathcal{C})$ of modules with a finite $\mathcal{C}-$filtration.

Proposition 1.1.15 ([10, Lemma 3.2])**.** *Let $\mathcal{C} \subseteq R$-Mod, then:*

$$\mathbf{T}(\mathcal{C}) = \operatorname{Filt}\operatorname{Gen}(\mathcal{C})$$

For torsion pairs in abelian length categories we have well-known explicit descriptions for both the torsion and torsionfree class:

Proposition 1.1.16. *([48, Lemma 3.1]) Let $\mathcal{L}$ be an abelian length category, $\mathcal{C} \subseteq \mathcal{L}$. Then:*

$$\widetilde{\mathbf{T}}(\mathcal{C}) := \operatorname{filt}\operatorname{gen}(\mathcal{C})$$
$$\widetilde{\mathbf{F}}(\mathcal{C}) := \operatorname{filt}\operatorname{cogen}(\mathcal{C})$$

1.2 Bireflective subcategories and ring epimorphisms

In this section we recall the correspondence between bireflective subcategories of module categories and ring epimorphisms due to Gabriel and de la Peña [37].

For the basic categorical results in this section we refer to [23, Section 3.5].

Definition 1.2.1. Let $\mathcal{C}$ be a full and replete subcategory of some category $\mathcal{D}$. Let $i : \mathcal{C} \to \mathcal{D}$ be the inclusion functor. Then:

(i) $\mathcal{C}$ is a *reflective* subcategory if i admits a left adjoint $r : \mathcal{D} \to \mathcal{C}$. The functor r is then called the *reflection*.

(ii) $\mathcal{C}$ is a *coreflective* subcategory if i admits a right adjoint $c : \mathcal{D} \to \mathcal{C}$. The functor c is then called the *coreflection*.

(iii) A reflective and coreflective subcategory is called *bireflective*.

Reflective and coreflective subcategories have good properties with respect to limits and colimits:

Proposition 1.2.2. *Let $\mathcal{D}$ be a complete category and $\mathcal{C}$ a reflective subcategory of $\mathcal{D}$.*
Then $\mathcal{C}$ itself is a complete category and, more specifically, $\mathcal{C}$ is closed under limits in $\mathcal{D}$.

The dual result tells that co-reflective subcategories are closed under co-limits. We also have:

Proposition 1.2.3. *Let $\mathcal{D}$ be a cocomplete category and $\mathcal{C}$ a reflective subcategory of $\mathcal{D}$.*
Then $\mathcal{C}$ itself is a cocomplete category and, more specifically, colimits in $\mathcal{C}$ are computed as reflections of the corresponding colimits in $\mathcal{D}$.

We now recall the notion of ring epimorphism:

Definition 1.2.4. Let R, S be rings. A ring homomorphism $f : R \to S$ is a *ring epimorphism* if for every ring T and every pair of ring homomorphisms $h_1, h_2 : S \to T$, we have $h_1 \circ f = h_2 \circ f$ if and only if $h_1 = h_2$.

Examples 1.2.5. *(i) Every surjective ring homomorphism is trivially a ring epimorphism.*

(ii) Let R be a commutative ring, S a multiplicative subset of R, then the localisation $R \to R_S$ is a ring epimorphism. A special case is the inclusion of the integers in the rational numbers.

(iii) Let k be a field. Then the inclusion of the triangular matrix algebra $T_n(k)$ into the corresponding matrix algebra $M_n(k)$ is a ring epimorphism. This is a special case of the universal localisation construction.

The following is a module-theoretic characterisation of ring epimorphisms:

Proposition 1.2.6. *Let R, S be rings. $f : R \to S$ a ring homomorphism. Then f is a ring epimorphism if and only if the restriction of scalars functor $f^* : S\text{-}\mathrm{Mod} \to R\text{-}\mathrm{Mod}$ is fully-faithful.*

Remark 1.2.7. *Let $f : R \to S$ be a ring epimorphism. In this case S-Mod can be identified with a full subcategory of R-Mod. Notice that the restriction of scalars functor has both a left and a right adjoint, thus S-Mod is essentially a bireflective subcategory of R-Mod.*

We need the following notion of equivalence of ring epimorphisms:

Definition 1.2.8. Let R be a ring. $f_1 : R \to S_1$, $f_2 : R \to S_2$ be ring epimorphisms. Then f_1 is *equivalent* to f_2 if there exists an isomorphism $g : S_1 \to S_2$ such that $f_2 = g \circ f_1$.

Theorem 1.2.9 ([37, Thm 1.2])**.** *Let R be a ring. Then there is a bijective correspondence between equivalence classes of ring epimorphisms starting at R and full replete abelian subcategories of R-Mod closed under limits and colimits. This correspondence assigns to each equivalence class of epimorphisms the essential image of the restriction of scalars functor induced by a chosen representative.*

In particular, a full abelian subcategory of R-Mod is bireflective if and only if it is closed under limits and colimits.

1.2.1 Universal localisation

There are special ring epimorphisms with further homological properties:

Definition 1.2.10. Let $f : R \to S$ be a ring epimorphism. We say that f is:

(1) *pseudo-flat* if for all $M, N \in S$-Mod, $\mathrm{Ext}^1_S(M, N) = \mathrm{Ext}^1_R(f^*(M), f^*(N))$. Equivalently $f^*(S\text{-Mod})$ is an extension-closed bireflective subcategory of R-Mod.

(2) *homological* if for all $M, N \in S$-Mod, for all $i \in \mathbb{N}$, $\mathrm{Ext}^i_S(M, N) = \mathrm{Ext}^i_R(f^*(M), f^*(N))$. This is equivalent to requiring that the restriction of scalars functors f^* induces a fully-faithful embedding at the level of derived categories $f^* : \mathrm{D}(S) \to \mathrm{D}(R)$.

Example 1.2.11. *Let R be a ring, $I = I^2$ an idempotent ideal of R. The canonical map $p : R \to R/I$ is a pseudo-flat ring epimorphism.*

Many interesting pseudo-flat ring epimorphisms are obtained as universal localisations:

Theorem 1.2.12 ([58, Theorem 4.1, Theorem 4.7])**.** *Let R be a ring, Σ a set of maps between finitely generated projective (left) modules over R. Then there exists a ring R_Σ, with a ring homomorphism $p_\Sigma : R \to R_\Sigma$, such that $R_\Sigma \otimes_R \alpha$ is an isomorphism for all $\alpha \in \Sigma$.*

For any other homomorphism $f : R \to S$ such that for all $\alpha \in \Sigma$, $S \otimes_R \alpha$ is invertible, there exists a unique factorisation of f through p_Σ.

Moreover, p_Σ is a pseudo-flat ring epimorphism.

Such a ring R_Σ is called the universal localisation *of R at Σ.*

Remark 1.2.13. *One can also define universal localisations with respect to a set of finitely presented modules $\mathcal{M}$. In this case, we localise the ring at the set of projective presentations of the modules in $\mathcal{M}$.*

Let $\mathcal{X}_{R_\Sigma}$ be the essential image of R_Σ - Mod in R - Mod under the restriction of scalars functor.

Proposition 1.2.14 ([27, Proposition 3.2]). *Let $R \to R_\Sigma$ be the universal localisation of R at Σ, let $X \in R$ - Mod. Then $X \in \mathcal{X}_{R_\Sigma}$ if and only if for all $\sigma \in \Sigma$ the map* $\mathrm{Hom}_R(\sigma, X)$ *is invertible.*

Remark 1.2.15. *If the ring R is hereditary, then for any module $X \in R$ - Mod and for any map between finitely generated projectives σ, the morphism* $\mathrm{Hom}(\sigma, X)$ *is invertible if and only if $X \in \{\ker\sigma, \operatorname{coker}\sigma\}^{\perp_{0,1}}$.*

In particular, the category of modules over the universal localisation at some set of modules $\mathcal{M}$ is equivalent to $\mathcal{M}^{\perp_{0,1}}$, as the projective presentations of the modules in $\mathcal{M}$ can be chosen to be injective.

We conclude with a result showing that over an hereditary ring universal localisations play a prominent role:

Theorem 1.2.16 ([46, Theorem 6.1]). *Let R be an hereditary ring and let $f : R \to S$ be a ring homomorphism. Then f is a homological (or equivalently psuedo-flat) ring epimorphism if and only if it is equivalent to a universal localisation.*

1.2.2 Covering and enveloping subcategories

We recall some terminology which will be useful later on.

Definition 1.2.17. Let $\mathcal{A}$ be a category, $\mathcal{S} \subseteq \mathcal{A}$ a subcategory. Let $M \in \mathcal{A}$. A morphism $f : M \to S$ is a $\mathcal{S}-$*preenvelope* if $S \in \mathcal{S}$ and for every $S' \in \mathcal{S}$ and every morphism $g : M \to S'$ there exists a map $h : S \to S'$ such that $g = h \circ f$. The map f is a $\mathcal{S}-$*envelope* if in addition it is left minimal.

The subcategory $\mathcal{S}$ is called *(pre)enveloping* if every object in $\mathcal{A}$ admits an $\mathcal{S}-$(pre)envelope.

Dually, we define $\mathcal{S}-$*(pre)covers* and *(pre)covering* subcategories.

Remark 1.2.18. *Preenvelopes and precovers are also known as* left approximations *and* right approximations *respectively.*

Moreover, in case the ambient category is essentially small and additive, preenveloping classes are often called covariantly finite, *precovering classes* contravariantly finite *and classes which are both preenveloping and precovering are said to be* functorially finite.

Example 1.2.19. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in some abelian category. Then, by definition, every object admits both a $\mathcal{T}-$cover and an $\mathcal{F}-$envelope.*

Lemma 1.2.20. *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$. Then $\mathcal{C}$ is an enveloping subcategory with envelopes given by the unit of the adjoint pair.*

Under some additional hypotheses we can determine when a preenveloping subcategory is reflective by means of the following result:

Proposition 1.2.21 ([30, Corollary 3.2]). *Let $\mathcal{A}$ be a complete well-powered abelian category. Then the following statements are equivalent for a full (replete) subcategory $\mathcal{F} \subseteq \mathcal{A}$:*

(i) $\mathcal{F}$ is a reflective subcategory.

(ii) $\mathcal{F}$ is preenveloping and closed under kernels.

There is a dual result for precovering classes and coreflective subcategorties.

1.3 Definable classes and purity

We collect some well known facts about definable classes and purity. A comprehensive reference can be found in [55].

Definition 1.3.1. A short exact sequence $0 \to L \to M \to N \to 0$ in $R\text{-}\mathrm{Mod}$ is *pure* if for every $U \in R\text{-}\mathrm{mod}$ the sequence

$$0 \longrightarrow \mathrm{Hom}(U, L) \longrightarrow \mathrm{Hom}(U, M) \longrightarrow \mathrm{Hom}(U, N) \longrightarrow 0$$

is an exact sequence of abelian groups. In this case, we say that L is a *pure submodule* of M or that the map $L \to M$ is a *pure monomorphism*.

Lemma 1.3.2 ([55, Lemma 2.1.2]). *Let R be a ring, then in $R\text{-}\mathrm{Mod}$:*

(i) If A is a direct summand of B, then the canonical embedding of A into B is a pure monomorphism.

(ii) The composition of two pure monomorphisms is a pure monomorphism.

(iv) Any direct limit of pure monomorphisms is a pure monomorphism.

(iii) A (set-indexed) product of pure monomorphisms is a pure monomorphism.

Definition 1.3.3. Let $\mathcal{D} \subseteq R\text{-}\mathrm{Mod}$. We say that $\mathcal{D}$ is *definable* if it is closed under products, pure submodules and direct limits.

Let $\mathcal{C} \subseteq R\text{-}\mathrm{Mod}$. We denote by $\langle \mathcal{C} \rangle$ the *definable subcategory generated* by $\mathcal{C}$, that is the smallest definable subcategory of $R\text{-}\mathrm{Mod}$ containing $\mathcal{C}$.

Remark 1.3.4. *Notice that a torsionfree class in $R\text{-}\mathrm{Mod}$ is definable if and only if it is closed under direct limits, as closure under products and (pure) submodules is granted.*

For artin algebras, we have the following:

Proposition 1.3.5 ([57, Section 1.F]). *Let Λ be an artin algebra. A short exact sequence $0 \to L \to M \to N \to 0$ in $\Lambda\text{-}\mathrm{Mod}$ is* pure *if for every $U \in \Lambda\text{-}\mathrm{mod}$ the sequence*

$$0 \longrightarrow \mathrm{Hom}(N, U) \longrightarrow \mathrm{Hom}(M, U) \longrightarrow \mathrm{Hom}(L, U) \longrightarrow 0$$

is an exact sequence of abelian groups.

Examples 1.3.6. *(1) Let $M \in R\text{-}\mathrm{mod}$. Then the torsionfree class $M^{\perp_0}$ is a definable subcategory of $R\text{-}\mathrm{Mod}$.*

(1') Let $M \in \Lambda\text{-}\mathrm{mod}$. Then, as shown in [29, Example 2.3], the torsion class ${}^{\perp_0}M$ is a definable subcategory of $\Lambda\text{-}\mathrm{Mod}$.

(2) Given a set-indexed family of definable classes $\{\mathcal{D}_i\}_{i\in I}$, the intersection $\mathcal{D} = \bigcap_{i\in I} \mathcal{D}_i$ is a definable class.

1.3.1 Pure-injective modules

Definition 1.3.7. A module E is *pure-injective* if every pure exact sequence starting at E is split exact. Dually, we have a notion of *pure-projective* module.

It turns out that there are enough pure-injective modules and in fact, that every module has a pure-injective hull:

Definition 1.3.8. Let M be a module. Then a pure monomorphism $f : M \to N$ is a *pure-injective hull* if N is pure-injective and f is left minimal.

Theorem 1.3.9 ([55, Theorem 4.3.18])**.** *Every module M has a pure-injective hull, which is unique up to isomorphism: if $f : M \to N$ and $f' : M \to N'$ are both pure-injective hulls of M then there exists an isomorphism $j : N \to N'$ such that $f' = j \circ f$.*

We denote by $\mathrm{PE}(M)$ the pure-injective hull of the module M.

Moreover indecomposable pure-injective modules have a nice endomorphism ring:

Theorem 1.3.10 ([55, Theorem 4.3.43])**.** *Every pure-injective indecomposable module has local endomorphism ring.*

We conclude with a result giving a decomposition of every pure-injective module

Definition 1.3.11. Let $E \in R\text{-}\mathrm{Mod}$ be a pure-injective module. We say that E is *superdecomposable* if it has no (non-zero) indecomposable summands.

We say that a pure-injective module E *has no superdecomposable part* if every non-zero direct summand of E has a non-zero indecomposable direct summand.

Theorem 1.3.12 ([55, Theorem 4.4.2])**.** *Let N be a pure-injective module. Then*

$$N \simeq \mathrm{PE}(\coprod_{\alpha} N_\alpha) \oplus N_c$$

where each N_α is indecomposable pure-injective and where N_c is superdecomposable. Both the N_α with their multiplicities and N_c are determined up to isomorphism by N.

1.3.2 The Ziegler spectrum

The collection of all the isoclasses of indecomposable pure-injective modules over a ring can be equipped with an interesting topology. The resulting topological space is known as the Ziegler spectrum.

We denote the collection of all the isomorphism classes of indecomposable pure-injective left modules over some ring R by $R\text{-}\mathrm{pinj}$.

We need the following result to ensure that our definition makes sense:

Theorem 1.3.13 ([55, Corollary 4.3.38]). *Let R be a ring. Then $R\text{-}\mathrm{pinj}$ is a set whose cardinality is bounded by $2^{\mathrm{card}(R)+\aleph_0}$.*

Now we can define the Ziegler spectrum:

Theorem 1.3.14 ([55, Theorem 5.1.1]). *There exists a topology on $R\text{-}\mathrm{pinj}$ whose closed sets are obtained as the "intersections" $R\text{-}\mathrm{pinj} \cap \mathcal{D}$, with $\mathcal{D}$ a definable subcategory of $R\text{-}\mathrm{Mod}$.*

The resulting topological space is called the (left) Ziegler spectrum *of R, denoted by ${}_R\mathrm{Zg}$,*

It turns out that a definable subcategory is completely determined by its indecomposable pure-injective modules, thus we have:

Theorem 1.3.15 ([55, Corollary 5.1.6]). *There is a bijection between closed subsets $\mathcal{C}$ of the Ziegler spectrum and definable subcategories $\mathcal{D}$ of $R\text{-}\mathrm{Mod}$, given by the assignments:*

$$\mathcal{D} \mapsto \mathcal{D} \cap R\text{-}\mathrm{pinj}$$
$$\mathcal{C} \mapsto \langle \mathcal{C} \rangle$$

We conclude this brief overview by identifying some of the closed points of the Ziegler spectrum.

Theorem 1.3.16 ([55, Theorem 5.1.12]). *Let $N \in R\text{-}\mathrm{Mod}$ be an indecomposable module of finite endolength (the length over its endomorphism ring), then N is a closed point of ${}_R\mathrm{Zg}$.*

1.3.3 Pure-injective modules and localisation

Pure-injective modules and definable subcategories can also be described through functor categories.

Definition 1.3.17. A torsion pair $(\mathcal{T}, \mathcal{F})$ in a Grothendieck category $\mathcal{A}$ is *hereditary* if $\mathcal{T}$ is closed under submodules. It is of *finite type* if $\mathcal{F}$ is closed under direct limits.

Remark 1.3.18. *Notice that in every hereditary torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ the torsion class is a Serre subcategory of $\mathcal{A}$, i.e. a subcategory closed under subobjects, quotients and extensions. Thus we can consider the abelian quotient $Q : \mathcal{A} \to \mathcal{A}_{\mathfrak{t}} := \mathcal{A}/\mathcal{T}$.*

Proposition 1.3.19 ([55, Proposition 11.1.15]). *Let $\mathcal{G}$ be a locally finitely generated Grothendieck category and let $\mathfrak{t}$ be an hereditary torsion pair of finite type in $\mathcal{G}$.*

Then $\mathcal{G}_{\mathfrak{t}}$ is a locally finitely generated Grothendieck category. Moreover, if $\mathcal{C}$ is a generating set of finitely generated objects in $\mathcal{G}$, then $Q(\mathcal{C})$ is a generating set of finitely generated objects in $\mathcal{G}_{\mathfrak{t}}$.

Hereditary torsion pairs of finite type are completely determined by some set of indecomposable injectives.

Proposition 1.3.20 ([55, Proposition 11.1.29]). *Let $\mathcal{A}$ be a locally finitely presented abelian category and let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair of finite type. Then $\mathcal{F}$ is cogenerated by the set of indecomposable torsionfree injective objects.*

Now we recall a fundamental embedding of the module category into the category of additive functors.

Definition 1.3.21. Let R be a ring, we denote by $(\text{mod-}R, \mathbf{Ab})$ the Grothendieck category of additive functors from $\text{mod-}R$ to the category of abelian groups, with their natural transformations.

Theorem 1.3.22 ([55, Theorem 12.1.3, Theorem 12.1.6]). *Let R be a ring, the functor $T : R\text{-Mod} \to (\text{mod-}R, \mathbf{Ab})$ given on objects by the assignment $M \mapsto - \otimes M$ is fully faithful and preserves direct limits and products.*

Moreover, a sequence of $R-$modules $0 \to L \to M \to N \to 0$ is pure-exact if and only if the sequence $0 \to TL \to TM \to TN \to 0$ is exact. A module M is pure-injective if and only if the functor $- \otimes M$ is injective.

In fact all the injective functors can be obtained from a pure-injective module:

Theorem 1.3.23 ([55, Corollary 12.1.9]). *There is a bijection between the set of isomorphism classes of indecomposable pure-injective modules in $R\text{-Mod}$ and the set of isomorphism classes of indecomposable injective functors in $(\text{mod-}R, \mathbf{Ab})$, given by the assignment $E \mapsto (- \otimes E)$.*

As we have seen hereditary torsion pairs of finite type in locally finitely presented abelian categories are uniquely determined by their injective objects, in particular every definable subcategory $\mathcal{D}$ of $R\text{-Mod}$ corresponds to a hereditary torsion pair of finite type in $(\text{mod-}R, \mathbf{Ab})$, this is made precise in the following:

Proposition 1.3.24 (Consequence of [55, Proposition 12.3.2]). *Let $\mathcal{D}$ be a definable subcategory of $R\text{-Mod}$. Then $({}^{\perp_0}T\mathcal{D}, \text{Cogen}(T\mathcal{D}))$ is a hereditary torsion pair of finite type in $(\text{mod-}R, \mathbf{Ab})$.*

This ultimately means that we can localise the functor category at a definable subcategory of $R\text{-Mod}$.

Theorem 1.3.25 ([55, Corollary 12.3.3]). *Suppose that $\mathcal{D}$ is a definable subcategory of $R\text{-Mod}$, let $\mathcal{C} \subseteq {}_R\text{Zg}$ be the corresponding 1.3.15 closed set and let $(\text{mod-}R, \mathbf{Ab})_{\mathcal{D}}$ be the corresponding localisation.*

Then the assignment $N \mapsto (- \otimes N)_{\mathcal{D}} \simeq (- \otimes N)$ is a bijection between the points of $\mathcal{C}$ and the isomorphism classes of indecomposable injective objects of $(\text{mod-}R, \mathbf{Ab})_{\mathcal{D}}$.

1.3.4 Definable torsion and torsionfree classes

Definable torsion classes and definable torsionfree classes will be fundamental in the rest of this work. Here we collect some useful results.

Since we will have a continuous interplay between torsion pairs in the small and in the large module category, we fix the following terminology, for a left noetherian ring R:

Definition 1.3.26. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$ and $(\mathbf{t}, \mathbf{f})$ a torsion pair in $R\text{-mod}$. Then $(\mathcal{T}, \mathcal{F})$ *extends* $(\mathbf{t}, \mathbf{f})$ if $\mathbf{t} = \mathcal{T} \cap R\text{-mod}$ and $\mathbf{f} = \mathcal{F} \cap R\text{-mod}$.

In the setting above, we will also say that $(\mathcal{T}, \mathcal{F})$ *restricts* to $(\mathbf{t}, \mathbf{f})$.

Example 1.3.27. *For any torsion pair* $(\mathbf{t}, \mathbf{f})$ *in* $R\text{-mod}$ *we have the following two, not necessarily distinct, extensions to* $R\text{-Mod}$*:*

(i) The extension with the largest torsion class $({}^{\perp_0}\mathbf{f}, \mathbf{F}(\mathbf{f}))$

(ii) The extension with the largest torsionfree class $(\mathbf{T}(\mathbf{t}), \mathbf{t}^{\perp_0})$

For definable torsionfree classes, we have a result valid for any noetherian ring, using [26, Section 4.4]:

Theorem 1.3.28. *Let R be a left noetherian ring. There is a bijection between torsion pairs in* $R\text{-mod}$ *and torsion pairs in* $R\text{-Mod}$ *with definable torsionfree class.*

This bijection associates to a torsion pair $(\mathbf{t}, \mathbf{f})$ *in* $R\text{-mod}$ *the limit closure* $(\mathcal{T} := \varinjlim \mathbf{t}, \mathcal{F} := \varinjlim \mathbf{f})$*. In this setting, $\mathcal{T}$ can also be described as* $\mathrm{Gen}(\mathbf{t})$ *and $\mathcal{F}$ as the orthogonal class* $\mathbf{t}^{\perp_0}$*.*

The inverse of this map sends a torsion pair $(\mathcal{T}, \mathcal{F})$ *to its restriction* $(\mathcal{T} \cap R\text{-mod}, \mathcal{F} \cap R\text{-mod})$*.*

A dual version of this bijection holds true in the setting of artin algebras as a direct consequence of the following observation [28, Section 2.2]:

Lemma 1.3.29. *Let* $M \in \Lambda\text{-Mod}$*. Then M is a pure submodule of the product of its finitely generated quotients.*

Proof. Let $\mathcal{S}$ be a chosen set of representatives of all the isomorphism classes of finitely generated Λ−modules. Let $\overline{M} := \prod_{S \in \mathcal{S}} \prod_{f \in \mathrm{Hom}(M,S)} \mathrm{Im}(f)$ and consider the map $f : M \to \overline{M}$ obtained through the universal property of products. Then by construction, every map from M to a finitely generated module must factor through f.

In particular the map f is injective: we have some set I such that there exists an embedding $g : M \to (D\Lambda)^I$, therefore, if $f(m) = 0$ for some $m \in M$, then $g_i(m) = 0$ for all $i \in I$, which means $m = 0$.

Using Proposition 1.3.5 we can conclude that f is a pure monomorphism. □

Theorem 1.3.30. *Let Λ be an artin algebra. There is a bijection between torsion pairs in* $\Lambda\text{-mod}$ *and torsion pairs in* $\Lambda\text{-Mod}$ *with definable torsion class.*

This bijection associates to a torsion pair $(\mathbf{t}, \mathbf{f})$ *in* Λ−mod *the torsion pair cogenerated by* $\mathbf{f}$*,* $(\mathcal{T} := {}^{\perp_0}\mathbf{f}, \mathcal{F} := ({}^{\perp_0}\mathbf{f})^{\perp_0})$*. In this setting, $\mathcal{T}$ can also be described as* $\mathrm{Cogen}_*(\mathbf{t})$*,* *the class of modules obtained as pure submodules of arbitrary direct products of modules in* $\mathbf{t}$.

The inverse of this map sends a torsion pair $(\mathcal{T}, \mathcal{F})$ *to its restriction* $(\mathcal{T} \cap \Lambda\text{-mod}, \mathcal{F} \cap \Lambda\text{-mod})$.

Proof. We prove that the restriction map is bijective, with inverse as above.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in Λ-Mod with definable torsion class. Let $(\mathbf{t}, \mathbf{f})$ be the corresponding restriction.

We want to show that $\mathcal{T} = {}^{\perp_0}\mathbf{f}$. The inclusion $\mathcal{T} \subseteq {}^{\perp_0}\mathbf{f}$ is immediate.

For the opposite one, let $M \in {}^{\perp_0}\mathbf{f}$. By Lemma 1.3.29, M can be obtained as a pure subobject of a direct product of finitely generated objects in ${}^{\perp_0}\mathbf{f}$ which are exactly the modules in $\mathbf{t}$.

Whence M is a pure submodule of a product of objects in $\mathcal{T}$, so that by definability it follows that $M \in \mathcal{T}$.

This proves the injectivity of the restriction map. Surjectivity follows form the fact that for any torsion pair $(\mathbf{t}, \mathbf{f})$ in Λ-mod, the torsion pair $({}^{\perp_0}\mathbf{f}, \mathbf{F}(\mathbf{f}))$ is an extension to Λ-Mod with definable torsion class. □

1.4 Silting and cosilting modules

We recall some fundamental results about silting and cosilting modules following the presentation given in the survey [2]. The original results can be found in [4], [8], [24], [25], [64].

Fix a ring R. The definition of silting and cosilting modules relies on some special classes. Let σ be a left R−module homomorphism. We define

$$\mathcal{D}_\sigma := \left\{ X \in R\text{-Mod} \;\middle|\; \operatorname{Hom}_R(\sigma, X) \text{ is surjective} \right\}$$

$$\mathcal{C}_\sigma := \left\{ X \in R\text{-Mod} \;\middle|\; \operatorname{Hom}_R(X, \sigma) \text{ is surjective} \right\}$$

$$\mathcal{F}_\sigma := \left\{ X \in \text{Mod-}R \;\middle|\; X \otimes \sigma \text{ is injective} \right\}$$

We give a lemma with some straightforward properties of these classes:

Lemma 1.4.1. *Let* $\mathcal{R} \in \{\mathcal{D}, \mathcal{C}\}$. *Let* σ *be a left* R−*module morphism.*

(i) If σ *is an* R−*module isomorphism, then* $\mathcal{R}_\sigma = R\text{-Mod}$.

(ii) If $\sigma = \sigma_1 \oplus \sigma_2$, *then* $\mathcal{R}_\sigma = \mathcal{R}_{\sigma_1} \cap \mathcal{R}_{\sigma_2}$.

(iii) If $\sigma = M \to 0$, *then* $\mathcal{D}_\sigma = M^{\perp_0}$.

(iv) If $\sigma = 0 \to N$, *then* $\mathcal{C}_\sigma = {}^{\perp_0}N$.

(v) If σ *is a map between projective modules, then the class* $\mathcal{D}_\sigma$ *is closed under products, quotients and extensions.*

(vi) *If σ is a map between injective modules, then the class $\mathcal{C}_\sigma$ is closed under coproducts, submodules and extensions.*

(vii) *If σ is a map between projective modules, $\mathcal{D}_\sigma \subseteq (\operatorname{coker}\sigma)^{\perp_1}$*

(viii) *If σ is a map between injective modules, $\mathcal{C}_\sigma \subseteq {}^{\perp_1}(\ker\sigma)$*

1.4.1 Cosilting modules

Definition 1.4.2. Let C be an $R-$module. We say that C is *cosilting* if there exists an injective copresentation $\omega : I_0 \to I_1$ such that:

$$\operatorname{Cogen}(C) = \mathcal{C}_\omega$$

Two cosilting modules C_1, C_2 are *equivalent* if $\operatorname{Prod}(C_1) = \operatorname{Prod}(C_2)$.

Remark 1.4.3. *Recall that a module C is* (1-)cotilting *if* $\operatorname{Cogen}(C) = {}^{\perp_1}C$. *Thus, a module is cotilting if and only if it is cosilting with respect to an injective copresentation which is surjective.*

Definition 1.4.4. Let $E \in R\text{-}\mathrm{Mod}$ and $\mathcal{C} \subseteq R\text{-}\mathrm{Mod}$. Then $E \in \mathcal{C}$ is *Ext-injective* in $\mathcal{C}$ if for every $C \in \mathcal{C}$, $\operatorname{Ext}^1_R(C, E) = 0$.

E is *split-injective* in $\mathcal{C}$ if every monomorphism $E \to C$ with $C \in \mathcal{C}$ splits.

Theorem 1.4.5. *The following statements are equivalent for an $R-$module C:*

(i) *C is a cosilting module.*

(ii) $\operatorname{Cogen}(C)$ *is a torsionfree class and C is a cotilting $R/\operatorname{Ann}(C)-$module.*

(iii) *C is Ext-injective in* $\operatorname{Cogen}(C)$ *and there is an injective cogenerator $E(R)$ of $R-$Mod and an exact sequence:*

$$0 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\;g\;} E(R)$$

such that $C_1, C_0 \in \operatorname{Prod}(C)$ and g is a $\operatorname{Cogen}(C)-$*precover.*

We can say something more about the approximation sequence for a cosilting module:

Proposition 1.4.6. *Let C be a cosilting module. Then every module admits a* $\operatorname{Cogen}(C)-$*cover with kernel in* $\operatorname{Prod}(C)$*. Moreover in the sequence obtained from the* $\operatorname{Cogen}(C)-$*cover of an injective cogenerator $E(R)$:*

$$0 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\;g\;} E(R)$$

C_0 is split-injective for $\operatorname{Cogen}(C)$ *and $C_0 \oplus C_1$ is a cosilting module equivalent to C.*

Lemma 1.4.7. *Let C be a cosilting $R-$module, with approximation sequence*

$$0 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{g} E(R)$$

Then

$$\operatorname{Im}(g) = \{x \in E(R) \mid \operatorname{Ann}(C)x = 0\}$$

is an injective cogenerator of $R/\operatorname{Ann}(C)$.

Moreover, $\operatorname{Cogen}(C) = {}^{\perp_1}C_1 \cap R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$.

Proof. Since C is a (1-)cotilting $R/\operatorname{Ann}(C)-$module, every $R/\operatorname{Ann}(C)-$ module admits a surjective $\operatorname{Cogen}(C)-$cover and clearly all modules with a surjective cover are in $R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$ (being annihilated by $\operatorname{Ann}(C)$).

Let $E = \{x \in E(R) \mid \operatorname{Ann}(C)x = 0\}$. This is the largest submodule of $E(R)$ belonging to $R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$. Thus $\operatorname{Im}(g) \subseteq E$.

On the other hand, as recalled above, E admits a surjective $\operatorname{Cogen} C-$cover $C' \to E$. The induced map $C' \to E \to E(R)$ must factor through $g : C_0 \to E(R)$ showing that $E \subseteq \operatorname{Im}(g)$.

Now, for any $M \in R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$ there is a set I such that M embeds in $E(R)^I$, but since M is in $R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$ this embedding must factor through E^I. This shows that E is a cogenerator.

Injectivity over $R/\operatorname{Ann}(C)$ is also immediate, using that E is a submodule of the injective $E(R)$ and that all maps from a module in $R/\operatorname{Ann}(C)\text{-}\operatorname{Mod}$ to $E(R)$ must factor through E.

To prove the last statement, notice that the sequence

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \operatorname{Im}(g) \longrightarrow 0$$

is an approximation sequence as in Theorem 1.4.5(iii). In particular, a module M is cogenerated by C precisely if it is annihilated by $\operatorname{Ann}(C)$ and $\operatorname{Ext}^1_{R/\operatorname{Ann}(C)}(M, C_1) = 0$. However $\operatorname{Cogen}(C) \subseteq \ker \operatorname{Ext}^1_R(-, C_1) \subseteq \ker \operatorname{Ext}^1_{R/\operatorname{Ann}(C)}(-, C_1)$ thus we obtain the desired identity. □

We also have the following generalisation of Bazzoni's result asserting that every cotilting module is pure-injective [20]:

Theorem 1.4.8 ([24, Theorem 4.7])**.** *Every cosilting module is pure-injective.*

As a consequence of Proposition 1.4.6, the precover g in Theorem 1.4.5(iii) can be chosen minimal and we obtain:

Theorem 1.4.9. *The following statements are equivalent for a torsionfree class* $\mathcal{F} \subseteq R-\operatorname{Mod}$*:*

(i) $\mathcal{F} = \operatorname{Cogen}(C)$ *for some cosilting module* C.

(ii) $\mathcal{F}$ *is covering.*

(iii) $\mathcal{F}$ is definable.

In light of the theorem above, we will denote by $\mathbf{Cosilt}(R)$ the set of definable torsionfree classes of R-Mod (or equivalently the set of torsion pairs with definable torsionfree class) and refer to such classes as cosilting classes (or cosilting torsion pairs). Using Theorem 1.3.28 we obtain:

Corollary 1.4.10. *Let R be a left noetherian ring, then we have a bijection:*

$$\mathbf{tpair}(R) \xrightarrow{\sim} \mathbf{Cosilt}(R)$$

1.4.2 Silting modules and duality

Definition 1.4.11. Let T be an R-module. We say that T is *silting* if there exists a projective presentation $\sigma : P_1 \to P_0$ such that:

$$\mathrm{Gen}(T) = \mathcal{D}_\sigma$$

Two silting modules T_1, T_2 are *equivalent* if $\mathrm{Add}(T_1) = \mathrm{Add}(T_2)$.

Remark 1.4.12. *Recall that a module is* (1-)tilting *if* $\mathrm{Gen}(T) = T^{\perp_1}$. *Thus, a module is tilting if and only if it is silting with respect to a monic projective presentation.*

Every silting class is a definable torsion class, as $\mathcal{D}_\sigma$ is definable, however, unlike in the cosilting case, over an arbitrary ring there might be definable torsion classes which are not of this shape.

However, in the noetherian case, it is possible to parametrize definable torsion classes by means of silting modules.

In general, for any definable subcategory $\mathcal{D}$ of Mod-R it is possible to define a *dual definable category* $(\mathcal{D})^\vee$ of R-Mod. As shown for instance in [55, Section 3.4.2], this assignment gives a bijection between definable subcategories of right and left modules.

When specialised to silting classes, this assignment enjoys the following properties, where we denote by $M^+ := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ the character dual of M:

Theorem 1.4.13. *Let $\sigma : P_1 \to P_0$ be a map between projective right modules. Then:*

(i) σ^+ is a map between injective left modules and $\mathcal{C}_{\sigma^+} = \mathcal{F}_\sigma$.

(ii) If $\mathcal{D}_\sigma$ is a silting class, then ${\mathcal{D}_\sigma}^\vee = \mathcal{F}_\sigma$.

(iii) If T is a silting module with respect to σ then T^+ is a cosilting module with respect to σ^+.

Definition 1.4.14. A cosilting module C is of *cofinite type* if the associated cosilting class $\mathrm{Cogen}(C)$ is the dual of a silting class.

The following lemma characterises cosilting classes of cofinite type:

Lemma 1.4.15. *A cosilting module C is of cofinite type if and only if the torsion class $^{\perp_0}C$ is generated by a set of finitely presented modules.*

Over a noetherian ring, Theorem 1.3.28 shows that every torsion pair with definable torsionfree class is generated by a set of finitely presented modules, thus we have:

Theorem 1.4.16. *Let R be a left noetherian ring. Then the assignment $\mathcal{D} \to \mathcal{D}^\vee$ yields a bijection between silting classes in* $\text{Mod-}R$ *and cosilting classes in* $R\text{-Mod}$. *In particular there are bijections between:*

(i) Equivalence classes of right silting modules.

(ii) Equivalence classes of left cosilting modules.

(iii) Definable torsionfree classes in $R\text{-Mod}$.

(iv) Definable torsion classes in $\text{Mod-}R$.

We will denote by $\mathbf{Silt}(R)$ the set of torsion classes of $R\text{-Mod}$ generated by a silting module (or equivalently the set of the corresponding torsion pairs). Over an artin algebra, we can use Theorem 1.3.30 and Theorem 1.4.16 to obtain:

Corollary 1.4.17. *Let Λ be an artin algebra, then we have a bijection:*

$$\mathbf{tpair}(\Lambda) \to \mathbf{Silt}(\Lambda)$$

1.4.3 Coherent torsion pairs

We collect in this section some results on torsion pairs whose torsion and torsionfree classes are definable. Here R is an arbitrary ring.

Definition 1.4.18 ([22]). A torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-}R$ is said to be *coherent* if both $\mathcal{T}$ and $\mathcal{F}$ are definable subcategories.

Remark 1.4.19. *Notice that a torsion pair is coherent if and only if the associated torsion radical (when composed with the forgetful functor to the category of abelian groups) is a coherent functor.*

Corollary 1.4.20 ([22, Corollary 1.7]). *If $(\mathcal{T}, \mathcal{F})$ is a coherent torsion pair in* $\text{Mod-}R$ *then all the Ext-projective objects in $\mathcal{T}$ are pure-projective.*

Notice that, as every module is a pure quotient of a direct sum of finitely presented modules, being pure-projective is equivalent to being in $\text{Add}(\text{mod-}R)$.

Theorem 1.4.21 ([22, Theorem 2.4]). *Let R be a right noetherian ring. Then the following statements are equivalent for a torsion pair $(\mathcal{T}, \mathcal{F})$ in* $\text{Mod-}R$, *with* $\mathcal{T} = \text{Gen}(S)$ *for some silting module S:*

(1) $(\mathcal{T}, \mathcal{F})$ is a coherent torsion pair, that is $\mathcal{F}$ is definable.

(2) The silting module S is pure-projective.

(3) There exists an exact sequence:

$$R_R \xrightarrow{f} S_0 \to S_1 \to 0$$

with f a $\mathcal{T}-$preenvelope, $S_0, S_1 \in \mathrm{Add}(S)$ and S_0 pure-projective.

(4) $\mathcal{T} = \varinjlim(\mathcal{T} \cap \text{mod-}R)$ and $\mathcal{T} \cap \text{mod-}R$ is a covariantly finite subcategory of $\text{mod-}R$.

Proof. The theorem in [22] is proved for (1-)tilting modules. We can extend it to silting modules using the fact that every silting module S is tilting over the factor algebra $R/\mathrm{Ann}(S)$. Let $\mathcal{T} = \mathrm{Gen}(S)$ a torsion class in $\text{Mod-}R$.

As a preliminary step, recall that, since R is noetherian, a module M annihilated by some ideal I is finitely presented in $\text{Mod-}R$ if and only if it is finitely presented in $\text{Mod-}R/I$. Thus an $R/I-$module is pure-projective in $\text{Mod-}R/I$ if and only if it is pure-projective in $\text{Mod-}R$.

"(1) $\implies$ (2)" : If $(\mathcal{T}, \mathcal{F})$ is coherent in $\text{Mod-}R$, then, since by [55, Theorem 5.5.3] the category $\text{Mod-}R/\mathrm{Ann}(S)$ can be seen as a definable subcategory of $\text{Mod-}R$, the torsion pair $(\mathcal{T}, \mathcal{F} \cap \text{Mod-}R/\mathrm{Ann}(S))$ is a coherent torsion pair in $\text{Mod-}R/\mathrm{Ann}(S)$ generated by a tilting module, thus we can apply [22, Theorem 2.4] to obtain that S is a pure-projective $R/\mathrm{Ann}(S)-$module and consequently, a pure-projective $R-$module.

"(2) $\implies$ (3)" : In this situation, as $\mathcal{T}$ is generated by a silting module, we have an exact sequence:

$$R \xrightarrow{f} S_0 \to S_1 \to 0$$

with f a $\mathcal{T}-$preenvelope, $S_0, S_1 \in \mathrm{Add}(S)$ (see [8, Proposition 3.14]). As S is assumed to be pure-projective, so is the module S_0.

"(3) $\implies$ (1)" : The pure-projective $\mathcal{T}-$preenvelope $R \to S_0$ in $\text{Mod-}R$ induces a special $\mathcal{T}-$preenvelope $0 \to R/\mathrm{Ann}(S) \to S_0$, see [8, proof of Proposition 3.2]. Moreover, S_0 is still pure-projective when seen as an $R/\mathrm{Ann}(S)-$module (for instance, if it was a direct summand of $\coprod_i M_i$, with all M_i finitely presented, it is also a summand of $\coprod_i (R/\mathrm{Ann}(S) \otimes_R M_i)$).

This means that S is pure-projective in $\text{Mod-}R/\mathrm{Ann}(S)$, using again [22, Theorem 2.4], and thus in $\text{Mod-}R$, and consequently $S^{\perp_0}$ is closed under direct limits. In conclusion, $S^{\perp_0}$ is a definable torsionfree class in $\text{Mod-}R$.

"(4) $\iff$ (1)" : is obtained combining Theorem 1.3.28 and a result of Lenzing, which can be found as [55, Corollary 3.4.37]. □

Remark 1.4.22. *The result above tells us that not every silting module is pure-projective, indeed over an artin algebra (and in general over a Krull-Schmidt ring) a pure-projective silting module is equivalent to a finitely presented one.*

This is a second striking difference with the well-behaved "dual" case: recall that every cosilting module is pure-injective.

1.5 t-structures

t-structures are the natural triangulated counterpart of torsion pairs in abelian categories. We recall some basic properties and a construction, due to Happel-Reiten-Smalø, relating torsion pairs in a Grothendieck category and t-structures in the corresponding derived category.

The definition of a triangulated category and the proofs of the properties of a t-structure presented in this section can be found in [49].

Definition 1.5.1. Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. A pair of subcategories $(\mathcal{U}, \mathcal{V})$ is a *torsion pair* if:

(i) $\mathcal{U}, \mathcal{V}$ are closed under direct summands

(ii) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$

(iii) For all objects $T \in \mathcal{T}$ we can find a triangle

$$U_T \longrightarrow T \longrightarrow V_T \longrightarrow \Sigma U_T$$

with $U_T \in \mathcal{U}$ and $V_T \in \mathcal{V}$.

A torsion pair $(\mathcal{U}, \mathcal{V})$ is a *t-structure* if in addition $\Sigma\mathcal{U} \subseteq \mathcal{U}$ or equivalently $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$. In a t-structure the class $\mathcal{U}$ is called the *aisle* and the class $\mathcal{V}$ is called the *co-aisle*.

Definition 1.5.2. A t-structure $(\mathcal{U}, \mathcal{V})$ is *non-degenerate* if

$$\bigcap_{i \in \mathbb{Z}} \Sigma^i \mathcal{U} = \bigcap_{i \in \mathbb{Z}} \Sigma^i \mathcal{V} = 0$$

The additional condition, requiring the closure of the aisle of a t-structure under the action of the suspension functor, is introduced to obtain the functoriality of the approximation triangle:

Proposition 1.5.3. *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure. The two assignments $T \mapsto U_T$ and $T \mapsto V_T$ can be extended to endofunctors $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$ of $\mathcal{T}$, called the truncation functors.*

We can associate an abelian category with every t-structure:

Proposition 1.5.4. *Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a t-structure. The intersection $\mathcal{H}_{\mathbb{T}} := \Sigma^{-1}\mathcal{U} \cap \mathcal{V}$ is an abelian category, called the* heart *of the t-structure.*

Moreover, there is a cohomological functor $H^0_{\mathbb{T}} := \tau_{\Sigma^{-1}\mathcal{U}} \circ \tau_{\mathcal{V}} : \mathcal{T} \to \mathcal{H}_{\mathbb{T}}$.

1.5.1 t-structures in derived categories and their hearts

In this section we will work in the derived category $\mathrm{D}(R)$ of some ring R. We follow the presentation of the subject given in [54].

Definition 1.5.5. Let $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathrm{D}(R)$. Then we say that $\mathbb{T}$ is:

(i) *smashing* if $\mathcal{V}$ is closed under coproducts.

(ii) *compactly generated* if there exists a set of compact objects $\mathcal{C}$ such that $\mathcal{V} = \mathcal{C}^{\perp_0}$. (Recall that an object C is *compact* if $\mathrm{Hom}_{\mathrm{D}(R)}(C, \coprod X_i) \simeq \coprod \mathrm{Hom}_{\mathrm{D}(R)}(C, X_i)$)

(iii) *cosilting* if there exists an object $C \in \mathrm{D}(R)$, called a *cosilting complex*, such that $\mathbb{T} = ({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$. Two cosilting complexes are *equivalent* if they determine the same t-structure in $\mathrm{D}(R)$.

Theorem 1.5.6 ([64, Theorem 1.3]). *There is a bijective correspondence between equivalence classes of 2-term cosilting complexes in* $\mathrm{D}(R)$ *and equivalence classes of cosilting modules in* R-Mod*, more specifically, given any 2-term cosilting complex* σ*, the homology* $H^0(\sigma)$ *is a cosilting module and conversely, given a module* C*, cosilting with respect to an injective copresentation* σ*, the map* σ *is 2-term cosilting when seen as an object of* $\mathrm{D}(R)$*.*

We denote by $\mathrm{D}(R)$-Mod the category of additive functors from the full subcategory of compact objects of $\mathrm{D}(R)$ to the category of abelian groups. This is clearly a Grothendieck category.

We have a concept of *pure-injective* object in $\mathrm{D}(R)$: namely an object is pure-injective if its image under the Yoneda functor $Y : \mathrm{D}(R) \to \mathrm{D}(R)$-Mod is injective.

Theorem 1.5.7 ([47, Theorem 4.6]). *The following are equivalent for a non-degenerate t-structure* $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ *in* $\mathrm{D}(R)$*:*

(i) $\mathbb{T}$ *is smashing and its heart is a Grothendieck category.*

(ii) $\mathbb{T}$ *is cosilting with respect to a pure-injective cosilting object.*

Moreover, every compactly generated t-structure has the properties above.

The following theorem, describing the procedure known as HRS-tilt is of paramount importance:

Theorem 1.5.8 ([40]). *Let* $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ *be a t-structure in* $\mathcal{T}$ *with heart* $\mathcal{H} = \mathcal{V} \cap \Sigma^{-1}\mathcal{U}$*. Let* $\mathsf{t} = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\mathcal{H}$*. Then the pair:*

$$\mathbb{T}_{\mathsf{t}}^{<0} := \mathcal{T} \star \mathcal{U}$$
$$\mathbb{T}_{\mathsf{t}}^{\geq 0} := \Sigma^{-1}\mathcal{V} \star \mathcal{F}$$

is a t-structure in $\mathcal{T}$ *with heart*

$$\mathcal{H}_{\mathsf{t}} := \left\{ X \in \mathcal{T} \;\middle|\; \mathrm{H}^0_{\mathbb{T}}(X) \in \mathcal{F} \text{ and } \mathrm{H}^0_{\mathbb{T}}(\Sigma X) \in \mathcal{T}, \mathrm{H}^0_{\mathbb{T}}(\Sigma^k X) = 0 \text{ for all } k \neq 0, 1 \right\}$$

Example 1.5.9. *We will apply the construction above in the special case of the standard t-structure* $\mathbb{D} := (\mathcal{D}^{<0}, \mathcal{D}^{\geq 0})$ *in the category* $\mathrm{D}(R)$ *whose heart is the module category.*

Before stating the main result, we recall the following:

Definition 1.5.10. Let $\mathcal{C}$ be a cocomplete category. An object $C \in \mathcal{C}$ is *finitely presented* if for every direct limit $\varinjlim X_i \in \mathcal{C}$, the canonical map

$$\varinjlim \operatorname{Hom}_{\mathcal{C}}(C, X_i) \to \operatorname{Hom}_{\mathcal{C}}(C, \varinjlim X_i)$$

is an isomorphism.

The following theorem ensures that the heart obtained from a cosilting torsion pair is a locally coherent category. See [54, Theorem 2.16 (iii)] for this formulation.

Theorem 1.5.11 ([51], [52, Theorem 1.2], [53, Theorem 7.1])**.** *Let R be a left noetherian ring. Let $\mathfrak{t}$ be a torsion pair in R-Mod. Let $\mathbb{D}_{\mathfrak{t}}$ be the HRS-tilt of the standard t-structure at the torsion pair $\mathfrak{t}$. Then the following are equivalent:*

(i) $\mathfrak{t} \in \mathbf{Cosilt}(R)$

(ii) $\mathcal{H}_{\mathfrak{t}}$, the heart of $\mathbb{D}_{\mathfrak{t}}$, is a locally coherent Grothendieck category whose subcategory of finitely presented objects is $\mathcal{H}_{\mathfrak{t}} \cap \mathrm{D}^b(R\text{-}\mathrm{mod})$.

For a general ring, we can still say that the heart is locally finitely presented:

Theorem 1.5.12 ([53, Theorem 6.1])**.** *Let R be a ring. Let $\mathfrak{t}$ be a torsion pair in R-Mod. Let $\mathbb{D}_{\mathfrak{t}}$ be the HRS-tilt of the standard t-structure at the torsion pair $\mathfrak{t}$. Then the following are equivalent:*

(i) $\mathfrak{t} \in \mathbf{Cosilt}(R)$

(ii) $\mathcal{H}_{\mathfrak{t}}$, the heart of $\mathbb{D}_{\mathfrak{t}}$, is a locally finitely presented Grothendieck category.

1.6 τ-tilting theory

Let Λ be an artin algebra. Denote by τ the Auslander-Reiten translation on Λ-mod (see [15, Section IV.1]). For any $M \in \Lambda$-mod denote by $|M|$ the number of non-isomorphic indecomposable summands of M.

1.6.1 Functorially finite torsion pairs over artin algebras

For the definition of a functorially finite subcategory see Section 1.2.2. Over an artin algebra, the following classical result of Smalø shows a special symmetry for torsion pairs:

Theorem 1.6.1 ([61])**.** *Let Λ be an artin algebra. Let $(\mathbf{t}, \mathbf{f})$ be a torsion pair in Λ-mod. Then $\mathbf{t}$ is functorially finite if and only if $\mathbf{f}$ is functorially finite.*

This result is crucial in all the theorems dealing with the notion of mutation of a $\tau-$tilting module, in particular the assumption that the base ring is an artin algebra will be necessary for many of the results in this thesis.

Example 1.6.2. *We give an example of a torsion pair in the category of finitely presented modules over a commutative noetherian ring whose torsionfree class is functorially finite, but whose torsion class is not.*

Consider the torsion pair (tors, add($\mathbb{Z}$)) *in* $\mathbb{Z}$ - mod *consisting of the finitely generated torsion groups and the finitely generated torsionfree groups.*

Then add($\mathbb{Z}$) *is functorially finite, but* tors *is not. In fact, assume we had a left approximation* $f : \mathbb{Z} \to T$. *Then* f *is not a monomorphism, as any subgroup of a torsion group is again torsion, so it has a non-zero kernel of the form* $n\mathbb{Z}$. *Let* p *be a prime number, such that* p *does not divide* n. *Then the map* $\mathbb{Z} \to \mathbb{Z}/(p\mathbb{Z})$ *can not factor through* f, *a contradiction.*

1.6.2 $\tau-$tilting modules

Note that, in the literature, $\tau-$tilting theory was developed for finite dimensional algebras. Here we only assume that our base ring is an artin algebra. The results we need still hold, with the same proofs, in this slightly more general context.

Definition 1.6.3 ([1, Definition 0.1])**.** A module $M \in \Lambda$ - mod is $\tau-$*tilting* if:

(i) $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ (i.e. M is $\tau-$*rigid*)

(ii) $|M| = |\Lambda|$.

Moreover, M is *support* $\tau-$*tilting* if there exists some idempotent $e \in \Lambda$ such that M is a $\tau-$tilting module over $\Lambda/\langle e\rangle$.

The results in this section have dual versions involving the concept of $\tau^{-1}-$tilting modules. We won't give the dual statements.

Definition 1.6.4. A module $M \in \Lambda$ - mod is τ^{-1}-*tilting* if:

(i) $\mathrm{Hom}_\Lambda(\tau^{-1}M, M) = 0$ (i.e. M is $\tau^{-1}-$*rigid*)

(ii) $|M| = |\Lambda|$.

Moreover, M is *support* $\tau^{-1}-$*tilting* if there exists some idempotent $e \in \Lambda$ such that M is a $\tau^{-1}-$tilting module over $\Lambda/\langle e\rangle$.

The following is the first fundamental result, showing that support $\tau-$tilting modules parametrise functorially finite torsion classes (thus in view of 1.6.1 functorially finite torsionfree classes).

Theorem 1.6.5 ([1])**.** *The map assigning to a module* M *the smallest torsion class containing it gives a bijection between:*

(i) The set $\mathbf{s}\tau - \mathbf{tilt}(\Lambda)$ *of isomorphism classes of basic support* $\tau-$*tilting modules over* Λ.

(ii) The set $\mathbf{ftors}(\Lambda)$ *of functorially finite torsion classes in* Λ - mod.

As a consequence of Theorem 1.4.16, (arbitrary) torsion pairs in Λ - mod are parametrised by silting modules. It turns out that support $\tau-$tilting modules are precisely the finitely generated silting modules:

Proposition 1.6.6 ([8]). *A module $M \in \Lambda$ - mod is a support $\tau-$tilting module if and only if it is a silting module.*

We can set a partial order on the set of isomorphism classes of support $\tau-$tilting modules:

Definition 1.6.7. Let $M, N \in \mathbf{s}\tau - \mathbf{tilt}(\Lambda)$, then we set $M \geq N$ iff $\mathrm{gen}(M) \supseteq \mathrm{gen}(N)$.

Support $\tau-$tilting modules are well behaved with respect to mutation. However, in order to describe this procedure, we need to keep track of some additional data.

Definition 1.6.8. Let (T, P) be a pair of finitely generated $\Lambda-$modules. Then (T, P) is a *(almost complete) support $\tau-$tilting pair* if:

(i) T is a $\tau-$rigid module.

(ii) P is projective and $\mathrm{Hom}(P, T) = 0$.

(iii) $|T| + |P| = |\Lambda|$ (resp. $|\Lambda| - 1$) .

Definition 1.6.9. Two basic support $\tau-$tilting pairs (T, P) and (T', P') for Λ are *mutations* of each other if there exists an almost complete support $\tau-$tilting pair (U, Q) which is a common direct summand of both.

In this case there is precisely one indecomposable direct summand X of (T, P) which is not a summand of (T', P') and we write $T' = \mu_X(T)$.

Remark 1.6.10. *We will introduce in the following chapters a concept of mutation for infinitely generated cosilting modules developed in [6].*

The conditions to obtain this generalized mutation are weaker and related to some properties of intervals of torsion pairs.

The mutations defined in 1.6.9 will be called irreducible *in the new terminology.*

Theorem 1.6.11 ([1, Prop. 2.28]). *Let $T = X \oplus U$ and T' be support $\tau-$tilting Λ-modules such that $T' = \mu_X(T)$ for some indecomposable Λ-module X. Then either $T > T'$ or $T < T'$ holds. We say that T' is a* left mutation *(respectively,* right mutation *) of T and we write $T' = \mu_X^+(T)$ (respectively, $T = \mu_X^-(T)$) if $X \in \mathrm{gen}(U)$ (resp. $X \notin \mathrm{gen}(U)$).*

Mutation describes the covering relation in the poset of functorially finite torsion classes.

Theorem 1.6.12 ([1, Thm. 2.33]). *The following are equivalent for $T, U \in \mathbf{s}\tau - \mathbf{tilt}(\Lambda)$:*

(i) U is a left mutation of T.

(ii) T is a right mutation of U.

(iii) $T > U$ *and there is no* $V \in \mathbf{s}\tau - \mathbf{tilt}(\Lambda)$ *such that* $T > V > U$.

This result was strengthened in [31]:

Theorem 1.6.13 ([31])**.** *Let* $\mathbf{t} \in \mathbf{ftors}(\Lambda)$. *If* $\mathbf{s}$ *is a torsion class with* $\mathbf{t} \supsetneq \mathbf{s}$ *then there exists a maximal torsion class* $\mathbf{t}'$ *with respect to the property* $\mathbf{t} \supsetneq \mathbf{t}' \supseteq \mathbf{s}$.

Such a class is functorially finite and the corresponding support $\tau-$*tilting module is therefore a mutation of the support* $\tau-$*tilting module corresponding to* $\mathbf{t}$.

Remark 1.6.14. *As an immediate consequence of the results above any class covered in the poset* $\mathbf{tors}(\Lambda)$ *by a functorially finite class must be functorially finite.*

Moreover, if there exists any non-functorially finite torsion class $\mathbf{t}$ *in* $\Lambda\text{-}\mathrm{mod}$, *an iterated application of Theorem 1.6.13 starting from the obvious inclusion* $\Lambda\text{-}\mathrm{mod} \supseteq \mathbf{t}$ *yields a countably infinite descending chain* $\{\mathbf{u}_i\}$ *of functorially finite torsion classes*

$$\Lambda\text{-}\mathrm{mod} = \mathbf{u}_0 \supsetneq \mathbf{u}_1 \supsetneq \mathbf{u}_2 \cdots \supsetneq \mathbf{u}_n \supsetneq \cdots \supseteq \mathbf{t}$$

1.6.3 τ-tilting finite algebras

The concept of a τ-tilting finite algebra was introduced in [31]. In the same paper the authors discovered the connection between τ-rigid modules and bricks. More recently, τ-tilting finite algebras were characterized in terms of the existence of infinitely generated silting modules [10].

Definition 1.6.15. Let Λ be an artin algebra. We say that Λ is *τ-tilting finite* if there are only finitely many isoclasses of basic τ-tilting modules.

Definition 1.6.16. A module $B \in \Lambda\text{-}\mathrm{Mod}$ is a *brick* if its endomorphism ring is a skew-field. A set $\mathcal{S}$ of bricks is a *semibrick* if for all $B, B' \in \mathcal{S}$ with $B \neq B'$ we have $\mathrm{Hom}_\Lambda(B, B') = 0$.

Theorem 1.6.17 ([31])**.** *The following statements are equivalent for an artin algebra* Λ*:*

(i) Λ *is* τ*-tilting finite.*

(ii) There exists only a finite number of bricks, up to isomorphism, in $\Lambda\text{-}\mathrm{mod}$.

(iii) Every torsion, or equivalently torsionfree class, in $\Lambda\text{-}\mathrm{mod}$ *is functorially finite.*

The third item in the theorem above was given a large analogue in the following:

Theorem 1.6.18 ([10])**.** *An artin algebra* Λ *is* τ*-tilting finite if and only if every torsion class in* $\Lambda\text{-}\mathrm{Mod}$ *is generated by some finitely generated silting module or, in other words, if and only if every silting module is equivalent to a finitely generated one.*

Examples 1.6.19. *(1) Every representation-finite algebra is in particular* $\tau-$*tilting finite.*

(2) *Nonetheless, there are many easy examples of $\tau-$tilting finite algebras which are of infinite representation type.*

Consider the local algebra $\Lambda = k[X,Y]/(X,Y)^3$, over some field k, which was shown to be wild by Gelfand and Ponomarev [38].

By [1, Example 6.1], every local algebra is $\tau-$tilting finite. So Λ is a representation infinite, but $\tau-$tilting finite algebra.

Chapter 2

A brick version of a theorem of Auslander

In this chapter we discuss the role of large bricks in the study of torsion pairs in the category of finitely presented modules over an artin algebra Λ. In particular, we will show that the existence of a large brick is equivalent to the existence of a non-functorially finite torsion class.

The main result of this chapter (Theorem 2.4.4) can be seen as a brick analogue of the following result of Auslander:

Theorem 2.0.1 ([13])**.** *An artin algebra Λ is representation-finite if and only if every indecomposable module is finitely generated.*

In the following, uniqueness or finiteness of some set of objects are always to be intended up to isomorphism.

2.1 Characteristic bricks

The concept of torsionfree, almost torsion modules was already being used by Herzog in 2009, as a tool for studying critical summands of cotilting modules.

More recently, Barnard, Carroll and Zhu introduced the related concept of minimal extending modules in their work on the lattice of torsion classes of the category Λ - mod.

In this section R will be a left noetherian ring.

Definition 2.1.1 ([42],[17])**.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in R - Mod (resp. in R - mod). Let $B \neq 0$ be a (finitely generated) R module. We say that B is *torsionfree, almost torsion* (resp. *minimal extending*) with respect to $(\mathcal{T}, \mathcal{F})$, if:

(1) $B \in \mathcal{F}$

(2) Every proper quotient of B is contained in $\mathcal{T}$.

(3) For every short exact sequence $0 \to B \to F \to M \to 0$, if $F \in \mathcal{F}$, then $M \in \mathcal{F}$.

Condition (2) is equivalent to:

(2') For any $F \in \mathcal{F}$ any nonzero morphism $B \to F$ is a monomorphism

Moreover, assuming conditions (1)-(2), we have the following reformulation of (3):

(3') For every non-split short exact sequence $0 \to B \to M \to T \to 0$, if $T \in \mathcal{T}$ then $M \in \mathcal{T}$.

Dually, we can define *torsion, almost torsionfree* (and *minimal co-extending*) modules.

Remark 2.1.2. *A note on the terminology: we will also use the expression torsionfree, almost torsion in or for $\mathcal{F}$ in place of torsionfree, almost torsion with respect to $(\mathcal{T}, \mathcal{F})$ when referring only to the torsionfree class.*

Moreover, if the torsion pair we are considering is easily identifiable, we will omit an explicit reference to it.

We will use the expression characteristic brick*, when referring to a torsion, almost torsionfree or torsionfree, almost torsion module. This name is justified, as all such modules are bricks.*

Lemma 2.1.3. *For a module satisfying (1) and (2), condition (3) is equivalent to condition (3').*

Proof. Fix a module $B \neq 0$ satisfying (1) and (2) with respect to some torsion pair $(\mathcal{T}, \mathcal{F})$.
"(3) $\implies$ (3')" : Assume B satisfies condition (3). Consider a short exact sequence $0 \to B \to M \to T \to 0$, with T torsion.

Take the short exact sequence given by the torsion pair

$$0 \to tM \to M \to M/tM \to 0$$

with tM torsion, M/tM torsionfree.

Since B satisfies (2) the map $g : B \to M \to M/tM$ is zero or injective. Assume it is zero. Then we have the following commutative diagram, obtained by the snake lemma and the universal property of the kernel:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & tM & \longrightarrow & tM/B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & M/tM & \xrightarrow{\sim} & M' & \longrightarrow & 0
\end{array}$$

Since T is torsion, the map $T \to M'$ must be zero, therefore $M/tM = 0$ and M is torsion.

Assume now that g is injective. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & tM & \xrightarrow{\sim} & T' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M/tM & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

By (3) the module M' is both torsion and torsion-free, therefore $M' = 0$ and the sequence is split. In conclusion, B satisfies (3').
"$(3') \implies (3)$" : Assume now B satisfies (3'). Take a short exact sequence $0 \to B \to F \to M \to 0$, with F torsionfree. Consider the torsion, torsionfree sequence for M as above and take the pullback to obtain the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & M' & \longrightarrow & tM & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Since M' is a submodule of F it is torsionfree. Now applying the contrapositive of (3') it follows that the sequence above splits, whence $tM = 0$ and M is torsionfree. □

2.1.1 A general notion

In [35] Enomoto studies torsion classes, torsionfree classes and wide subcategories by means of relative simple objects.

These simple objects in general don't satisfy Schur's lemma in the module category, one can thus consider the following notions of coherent-cocoherent (see [44, Exercise 8.23]) and simple objects:

Definition 2.1.4. Let $\mathcal{C} \subseteq \Lambda\text{-}\mathrm{Mod}$ be an additive subcategory. Let $M \in \mathcal{C}$.
Then M is *coherent* in $\mathcal{C}$ if for all $f : C \to M$ in $\mathcal{C}$, $\ker(f) \in \operatorname{gen}\mathcal{C}$.
Dually, M is *cocoherent* in $\mathcal{C}$ if for all $f : M \to C$ in $\mathcal{C}$, $\operatorname{coker}(f) \in \operatorname{cogen}\mathcal{C}$.
A module $0 \neq S \in \mathcal{C}$ is *$\mathcal{C}$-simple* if:

(S1) For all $0 \neq f : S \to C$ in $\mathcal{C}$, we have $\ker(f) \in \mathcal{C}^{\perp_0}$.

(S2) For all $0 \neq f : C \to S$ in $\mathcal{C}$ we have $\operatorname{coker}(f) \in {}^{\perp_0}\mathcal{C}$.

The following lemma is an immediate consequence of the definition:

Lemma 2.1.5. *Let $\mathcal{C} \subseteq \Lambda\text{-}\mathrm{Mod}$ be an additive subcategory. Then the collection of all coherent-cocoherent $\mathcal{C}$-simple objects is a semibrick in $\Lambda\text{-}\mathrm{Mod}$.*

Examples 2.1.6. *(1) If the category $\mathcal{C}$ is wide then every object is coherent and cocoherent.*

(2) *If $\mathcal{C}$ is a torsion class, then $0 \neq M$ is $\mathcal{C}$-simple if and only if every proper submodule of M is torsionfree.*

(2') *Dually, if $\mathcal{C}$ is a torsionfree class, then $0 \neq M$ is $\mathcal{C}$-simple if and only if every proper quotient of M is torsion.*

(3) *If $\mathcal{C}$ is a torsion class, then every object is cocoherent, as torsion classes are closed under quotients, and a coherent $\mathcal{C}$-simple is precisely a torsion, almost torsionfree module.*

(3') *Dually, for a torsionfree class the cocoherent simples are the torsionfree, almost torsion modules.*

2.1.2 The small and the large ones

Since the definition of minimal extending modules for a torsion pair in R-mod is identical to the definition of torsionfree, almost torsion modules for torsion pairs in R-Mod, we rightfully expect to have a relation between the two notions.

This is indeed the case, if we choose torsion pairs related by the bijection in Theorem 1.3.28 :

Proposition 2.1.7. *Let R be a left noetherian ring. Let $(\mathbf{t}, \mathbf{f})$ be a torsion pair in R-mod and $(\mathcal{T} = \varinjlim \mathbf{t}, \mathcal{F} = \varinjlim \mathbf{f})$ the corresponding torsion pair in R-Mod with definable torsionfree class.*

Then the minimal extending modules with respect to $(\mathbf{t}, \mathbf{f})$ are precisely the finitely generated torsionfree, almost torsion modules with respect to $(\mathcal{T}, \mathcal{F})$.

Moreover, all torsion, almost torsionfree modules with respect to $(\mathcal{T}, \mathcal{F})$ are finitely generated and coincide with the minimal co-extending modules with respect to $(\mathbf{t}, \mathbf{f})$.

We will need the following lemma in the proof:

Lemma 2.1.8. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair with $\mathcal{F} \in$ **Cosilt**(R). Then all the torsion, almost torsionfree modules are finitely generated.*

Over an artin algebra Λ, we can say dually that for a torsion pair in **Silt**(Λ) *all torsionfree, almost torsion modules are finitely generated.*

Proof. By Theorem 1.3.28 the torsion pair can be written as $(\mathrm{Gen}(\mathbf{t}), \mathbf{t}^{\perp_0})$ with $\mathbf{t} := \mathcal{T} \cap R$-mod a torsion class in R-mod.

Assume T is torsion, almost torsionfree and $T \notin R$-mod. By assumption, all proper submodules of T, in particular all possible images of morphisms from a finitely generated module, must be torsionfree, whence $T \in \mathbf{t}^{\perp_0}$. This yields a contradiction.

For the second case proceed dually using Theorem 1.3.30. □

Now we can prove the proposition:

Proof. It is clear that the finitely generated torsionfree, almost torsion modules with respect to $(\mathcal{T},\mathcal{F})$ are minimal extending with respect to $(\mathbf{t},\mathbf{f})$.

So suppose that S is minimal extending in $\mathbf{f}$. We have immediately $S \in \mathcal{F}$ and that all its proper quotients are in $\mathbf{t}$, so in particular in $\mathcal{T}$.

The only condition that we have to check is the last one: consider a short exact sequence

$$0 \longrightarrow S \xrightarrow{f} F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{F}$. Recall that $\mathcal{F} = \mathbf{t}^{\perp_0}$. Hence suppose we have a map $T \to M$ with $T \in \mathbf{t}$ which we may assume to be injective; taking the pullback along $F \to M$ we obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \xrightarrow{f} & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since the upper sequence is contained in R-mod, by the minimal extending property (3') of S, P must be torsion or the upper sequence must split.

Since P is a submodule of F it must be torsionfree, so the sequence splits, and $T = 0$, proving that $M \in \mathbf{t}^{\perp_0}$.

The proof of the second statement is more involved, as the available description of $\mathcal{T}$ is less practical to work with.

We need to check that every minimal co-extending module in $\mathbf{t}$ is torsion, almost torsionfree in $\mathcal{T}$.

Let S be minimal co-extending. By definition $S \in \mathbf{t} \subseteq \mathcal{T}$.

The second property is immediately verified, as every proper submodule of S is an element of $\mathbf{f} \subseteq \mathcal{F}$.

For the third property, consider a short exact sequence:

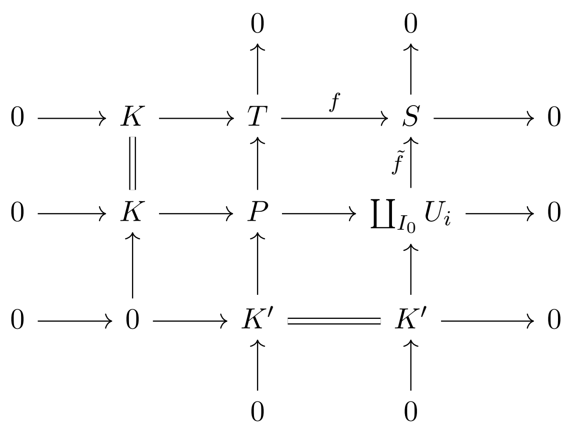
$$0 \longrightarrow K \longrightarrow T \xrightarrow{f} S \longrightarrow 0$$

with $T \in \mathcal{T}$. By Theorem 1.3.28 we can find a family $\{U_i\}_{i \in I}$ of objects of $\mathbf{t}$, with an epimorphism $h : \coprod_I U_i \to T$.

Since S is finitely generated, we can find a finite subset $I_0 \subseteq I$ such that $\tilde{f} = f \circ h \circ \iota_{I_0}$ is surjective (where $\iota_{I_0} : \coprod_{I_0} U_i \to \coprod_I U_i$ is the canonical inclusion).

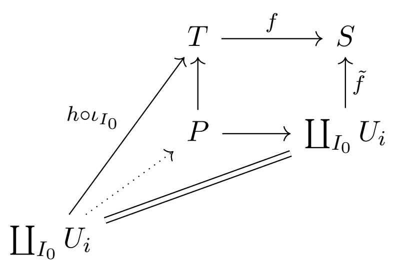
By definition, $\coprod_{I_0} U_i \in \mathbf{t}$, whence $K' = \ker \tilde{f}$ is a torsion module, as S is minimal co-extending.

Consider the following pullback diagram:



Notice that $P \in \mathcal{T}$ since T and K' are torsion modules.

Then consider the map $h \circ \iota_{I_0} : \coprod_{I_0} U_i \to T$. Using the universal property of the pullback, we obtain the following commutative diagram:



Thus the middle horizontal short exact sequence in the first diagram splits, hence $K \in \mathcal{T}$. □

Given any torsion pair $(\mathcal{T}, \mathcal{F})$ in R-Mod, the corresponding torsion, almost torsionfree and torsionfree, almost torsion modules have several orthogonality properties.

These follow immediately, once we have noticed that these modules are the simple objects of a suitable abelian category.

Proposition 2.1.9 ([3]). *Let $(\mathcal{T}, \mathcal{F})$ in R-Mod be a torsion pair, $\mathcal{H}$ the heart of the associated HRS t-structure in $\mathrm{D}(R)$.*

Then an object $M \in \mathcal{H}$ is simple if and only if $M = T[-1]$ for some torsion, almost torsionfree module or $M = F$ for some torsionfree, almost torsion module.

Corollary 2.1.10. *Let $(\mathcal{T}, \mathcal{F})$ in R-Mod be a torsion pair. Let T be torsion, almost torsionfree in $\mathcal{T}$ and F torsionfree, almost torsion in $\mathcal{F}$.*

Then $\mathrm{Ext}^1_R(T, F) = 0$.

Proof. In the HRS-heart $\mathcal{H}$ we have two distinct simple objects $T[-1]$ and F. It follows that $0 = \mathrm{Hom}_{\mathcal{H}}(T[-1], F) = \mathrm{Hom}_{\mathrm{D}(R)}(T[-1], F) = \mathrm{Ext}^1_R(T, F)$. □

2.2 Torsionfree, almost torsion in cosilting classes

R will be a left noetherian ring. In this section we prove directly the existence of some torsionfree, almost torsion module for any non-zero cosilting class.

We will obtain this as an application of Theorem 1.5.11:

Proposition 2.2.1. *Let $C \neq 0$ be a cosilting $R-$module. Then the class* $\operatorname{Cogen} C$ *admits some torsionfree, almost torsion module.*

Proof. Consider the HRS-t-structure constructed from $\operatorname{Cogen} C$. By Theorem 1.5.11 its heart is a locally coherent Grothendieck category and every finitely presented module in $\operatorname{Cogen} C$ gives a corresponding finitely presented object in the torsion class $\operatorname{Cogen} C$ in the heart.

Since $\operatorname{Cogen} C \neq 0$ by assumption, Theorem 1.3.28 implies the existence of some non-zero module $F \in \operatorname{Cogen} C \cap R\text{-}\operatorname{mod}$.

When seen as an object in the heart F is still finitely presented, in particular finitely generated. Thus, we can find a maximal subobject of F and a corresponding simple quotient S. As $\operatorname{Cogen} C$ is a torsion class in the heart, $S \in \operatorname{Cogen} C$ and, by Proposition 2.1.9, the corresponding module is torsionfree, almost torsion for $\operatorname{Cogen} C$. □

From the proof we can also deduce the following corollary:

Corollary 2.2.2. *Let C be a cosilting $R-$module. Then for every finitely presented module $M \in$* $\operatorname{Cogen} C$ *there exists a non-zero map $f : M \to F$ with F torsionfree, almost torsion.*

Over a left artin ring a cosilting module can be recovered from its torsionfree, almost torsion modules. This is a direct consequence of the following observation due to Enomoto (we include a proof for completeness):

Proposition 2.2.3 ([35])**.** *Let $\mathcal{L}$ be an abelian length category. Let $\mathbf{f} \subseteq \mathcal{L}$ be a torsionfree class. Then $\mathbf{f} = \operatorname{filt}(\mathbf{simp}(\mathbf{f}))$, where $\mathbf{simp}(\mathbf{f})$ is the set of $\mathbf{f}$-simples (in this case the objects of $\mathbf{f}$ whose proper quotients are torsion).*

Proof. Since $\mathbf{f}$ is closed under extensions, $\operatorname{filt}(\mathbf{simp}(\mathbf{f})) \subseteq \mathbf{f}$ is trivial.

For the other inclusion, let $F \in \mathbf{f}$. Since every object of $\mathcal{L}$ has finite length by definition, we can proceed by induction on n the length of F.

For $n = 1$, the object is simple in $\mathcal{L}$ thus it is also $\mathbf{f}$-simple and therefore it is an element of $\operatorname{filt}(\mathbf{simp}(\mathbf{f}))$.

Assume every object of length less than some fixed $n > 1$ has a filtration by $\mathbf{f}$-simples, and let F be an object of length n. Either F is $\mathbf{f}$-simple, and in that case we are done, or we can find a short exact sequence $0 \to K \to F \to Q \to 0$ with all terms in $\mathbf{f}$ and $K, Q \neq 0$.

In particular, the lengths of K and Q will be strictly less than n, so, by induction hypothesis, both K and Q are in $\operatorname{filt}(\mathbf{simp}(\mathbf{f}))$. Since this class is closed under extensions, $F \in \operatorname{filt}(\mathbf{simp}(\mathbf{f}))$. □

Proposition 2.2.4. *Let A be a left artinian ring. $\mathcal{F}, \mathcal{V} \in$ **Cosilt**(A). Then $\mathcal{F} = \mathcal{V}$ if and only if the torsionfree, almost torsion modules for $\mathcal{F}$ and $\mathcal{V}$ coincide.*

Proof. One of the implications is trivial. So assume the second condition holds. By Theorem 1.3.28 it is enough to show that $\mathbf{f} = \mathcal{F} \cap A\text{-}\mathrm{mod} = \mathbf{v} = \mathcal{V} \cap A\text{-}\mathrm{mod}$.

Since A is left artinian, $A\text{-}\mathrm{mod}$ is a length category thus Proposition 2.2.3 yields that it is sufficient to show that $\mathbf{simp}(\mathbf{f}) = \mathbf{simp}(\mathbf{v})$.

So let S be an $\mathbf{f}$-simple object. By Corollary 2.2.2 we have a non-zero map $f : S \to F$ with F a torsionfree, almost torsion module for $\mathcal{F}$. Since every proper quotient of S is torsion, this map must be injective.

Now F is also torsionfree, almost torsion for $\mathcal{V}$, thus since this class is closed under submodules, $S \in \mathbf{v}$. Therefore $\mathbf{simp}(\mathbf{f}) \subseteq \mathbf{v}$ and thus $\mathbf{f} \subseteq \mathbf{v}$. Inverting the roles of the two torsionfree classes we obtain the reverse inclusion. □

Remark 2.2.5. *Minimal extending modules (i.e. finitely generated torsionfree, almost torsion modules) are not enough to determine cosilting torsionfree classes: in fact there are non-zero cosilting classes without minimal extending modules, the easiest example being given by the torsion pair generated by the regular modules over the path algebra of a representation-infinite quiver (e.g. the Kronecker quiver).*

2.2.1 Extensions of functorially finite pairs

See Sections 1.3.4 and 1.4.3 for the terminology used in this section.

Let Λ be an artin algebra. Functorially finite torsion pairs in $\Lambda\text{-}\mathrm{mod}$ can be characterized as those pairs admitting a unique extension to $\Lambda\text{-}\mathrm{Mod}$, as observed in [63]. We include a proof for completeness.

Proposition 2.2.6 ([63, Proposition 5.3]). *Let $(\mathbf{t}, \mathbf{f})$ be a torsion pair in $\Lambda\text{-}\mathrm{mod}$. Then the following statements are equivalent:*

(1) $\mathbf{t}$ is functorially finite.

(2) There exists a coherent torsion pair $(\mathcal{T}, \mathcal{F})$ in $\Lambda\text{-}\mathrm{Mod}$ extending $(\mathbf{t}, \mathbf{f})$.

(3) There is a unique torsion pair extending $(\mathbf{t}, \mathbf{f})$ to $\Lambda\text{-}\mathrm{Mod}$.

(4) $\mathbf{t}^{\perp_0} \cap {}^{\perp_0}\mathbf{f} = 0$.

(5) For any silting module T generating the definable torsion class $\mathrm{Cogen}_*(\mathbf{t})$ and any cosilting module C cogenerating the definable torsionfree class $\varinjlim \mathbf{f}$, we have $\mathrm{Hom}(T, C) = 0$.

Proof. "(1) $\implies$ (2)" : By Theorem 1.6.5 we find a support $\tau-$tilting module T generating $\mathbf{t}$. The torsion pair $(\mathbf{T}(T), T^{\perp_0})$ in $\Lambda - \mathrm{Mod}$ extends the original torsion pair and it is definable on both sides: $\mathbf{T}(T) = \mathrm{Gen}(\mathrm{gen}(T)) = \mathrm{Gen}(T) = \varinjlim(\mathrm{gen}(T))$ by Theorem 1.3.28. To obtain definability of the last torsion class we can apply a result of Lenzing, see [55, Corollary 3.4.37], since $\mathrm{gen}(T)$ is functorially finite.

"(2) $\implies$ (1)" : Specialising Theorem 1.4.21 to the artin algebra case, we obtain that the torsion class is generated by a finitely presented silting module, that is a support $\tau-$tilting module. Therefore the restriction is functorially finite.

"(2) $\implies$ (3)" : Using Theorems 1.3.28 and 1.3.30 it follows that $\mathcal{T} = {}^{\perp_0}\mathbf{f}$ and $\mathcal{F} = \mathbf{t}^{\perp_0}$. Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\Lambda-$Mod extending $(\mathbf{t}, \mathbf{f})$. Then, we have $\mathcal{V} = \mathcal{U}^{\perp_0} \subseteq \mathbf{t}^{\perp_0}$ and $\mathcal{U} = {}^{\perp_0}\mathcal{V} \subseteq {}^{\perp_0}\mathbf{f}$. From this we deduce that $\mathcal{V} = \mathcal{U}^{\perp_0} = ({}^{\perp_0}\mathcal{V})^{\perp_0} \supseteq ({}^{\perp_0}\mathbf{f})^{\perp_0} = \mathcal{F}$. In conclusion $\mathcal{V} = \mathcal{F}$, proving that there is a unique extension.

"(3) $\implies$ (2)" : Immediate, using the fact that the extension with definable torsion class and the one with definable torsionfree class must coincide.

"(3) $\implies$ (4)" : By (3), $\mathrm{Gen}(\mathbf{t}) = {}^{\perp_0}\mathbf{f}$, thus $\mathbf{t}^{\perp_0} \cap {}^{\perp_0}\mathbf{f} = 0$.

"(4) $\implies$ (5)" : The image of any morphism $f : T \to C$ lies in $\mathbf{t}^{\perp_0} \cap {}^{\perp_0}\mathbf{f}$ which is 0 by (4).

"(5) $\implies$ (3)" : By (5) we have $\varinjlim \mathbf{t} \subseteq \mathrm{Cogen}_*(\mathbf{t}) = \mathrm{Gen}(T) \subseteq {}^{\perp_0}C = {}^{\perp_0}(\varinjlim \mathbf{f}) = \varinjlim \mathbf{t}$. Thus the largest and the smallest extensions of $\mathbf{t}$ coincide. □

2.3 The lattice of torsion classes

Let Λ be an artin algebra. There is a natural partial order on the collection of torsion classes $\mathbf{tors}(\Lambda)$ of Λ - mod given by inclusion.

As shown in [32] the resulting poset has the structure of a complete lattice and enjoys several nice lattice-theoretic properties.

More explicitely, we have the following description of the meet and join of a set indexed family $\{\mathbf{t}_i\}_{i\in I}$ of torsion classes:

$$\bigwedge_I \mathbf{t}_i := \bigcap_I \mathbf{t}_i \quad , \quad \bigvee \mathbf{t}_i := \widetilde{\mathbf{T}}\Big(\bigcup_I \mathbf{t}_i\Big)$$

We recall also some basic lattice theoretic terminology:

Definition 2.3.1. Let $(L, \leq)$ be a poset, $x, y \in L$:

(1) The *interval* $[x, y]$ is the poset supported by those $z \in L$ with $x \leq z \leq y$. Notice that if L is a (complete) lattice, any non-empty interval in L is a (complete) sublattice of L.

(2) We say that y *covers* x if $x < y$ and for any $z \in L$ such that $x \leq z \leq y$, either $z = x$ or $z = y$.

(3) Let L be a lattice. An element x is *meet irreducible* if whenever $x = y \wedge z$ we must have $x = y$ or $x = z$. If L is complete, an element x is *completely meet irreducible* if whenever $x = \bigwedge_I y_i$, with $y_i \in L$, we must have $x = y_j$ for some $j \in I$.

This condition can be restated as follows: there is a unique element $x^\star$ covering x, and for every $y > x$ we have $y \geq x^\star$.

(3') Let L be a lattice. An element x is *join irreducible* if whenever $x = y \vee z$ we must have $x = y$ or $x = z$. If L is complete, an element x is *completely join irreducible* if whenever $x = \bigvee_I y_i$, with $y_i \in L$, we must have $x = y_j$ for some $j \in I$.

This condition can be restated as follows: there is a unique element $x^\star$ covered by x, and for every $y < x$ we have $y \leq x^\star$.

(4) A poset L has *finite length* if every chain in L stabilizes.

We recall a result proven in [17], relating minimal extending modules with the covering relation in $\mathbf{tors}(\Lambda)$.

Theorem 2.3.2 ([17, Theorem 1.0.2])**.** *Let $\mathbf{t} \in \mathbf{tors}(\Lambda)$, $\mathcal{S}$ be a collection of representatives of the isoclasses of minimal extending modules with respect to $(\mathbf{t}, \mathbf{f})$.*

Then the elements of $\mathcal{S}$ are in bijection with torsion classes covering $\mathbf{t}$.

A typical phenomenon for the lattice of torsion pairs in the $\tau-$tilting infinite case is the presence of non-trivial locally maximal elements.

Definition 2.3.3. Let $\mathbf{t} \in \mathbf{tors}(\Lambda)$. We say that $\mathbf{t}$ is *locally maximal* if there are no elements of $\mathbf{tors}(\Lambda)$ covering $\mathbf{t}$.

Remark 2.3.4. *Any locally maximal torsion class is obtained as the meet of all the strictly larger torsion classes. In particular, such classes are never completely meet irreducible.*

Also, notice that there is a unique functorially finite locally maximal element, namely the torsion class $\Lambda\text{-}\mathrm{mod}$, which is by definition the meet of the empty set.

In fact, for any functorially finite torsion class $\mathbf{t}$ properly contained in some other class $\mathbf{u}$, it is possible, by means of mutation, to find a class $\mathbf{t}^\star$ covering $\mathbf{t}$ such that $\mathbf{t}^\star \leq \mathbf{u}$. See [31].

Lemma 2.3.5. *Let $\mathbf{t} \in \mathbf{tors}(\Lambda)$ be a meet irreducible, but not completely meet irreducible element, then $\mathbf{t}$ is locally maximal.*

Proof. Assume by contradiction $\mathbf{t}$ has some covering class. By meet irreducibility it has precisely one, say $\mathbf{t}^\star$.

As $\mathbf{t}$ is not completely meet-irreducible, but it has just one covering class, there must be some torsion class $\mathbf{u} \supsetneq \mathbf{t}$ such that $\mathbf{u} \not\supseteq \mathbf{t}^\star$.

But this is absurd, since $\mathbf{u} \wedge \mathbf{t}^\star = \mathbf{t}$. So $\mathbf{t}$ is locally maximal. □

Example 2.3.6. *We discuss the most common example of a torsion pair without minimal extending modules. Let k be an algebraically closed field.*

Let $\Lambda = kK_2$ be the Kronecker algebra, obtained as the path algebra of the quiver $0 \rightrightarrows 1$.

This is a finite-dimensional tame hereditary algebra, as such, we have that any indecomposable in $\Lambda\text{-}\mathrm{mod}$ is contained in the preprojective $\mathbf{p}$, regular $\mathbf{r}$ or preinjective $\mathbf{q}$ component of the AR-quiver.

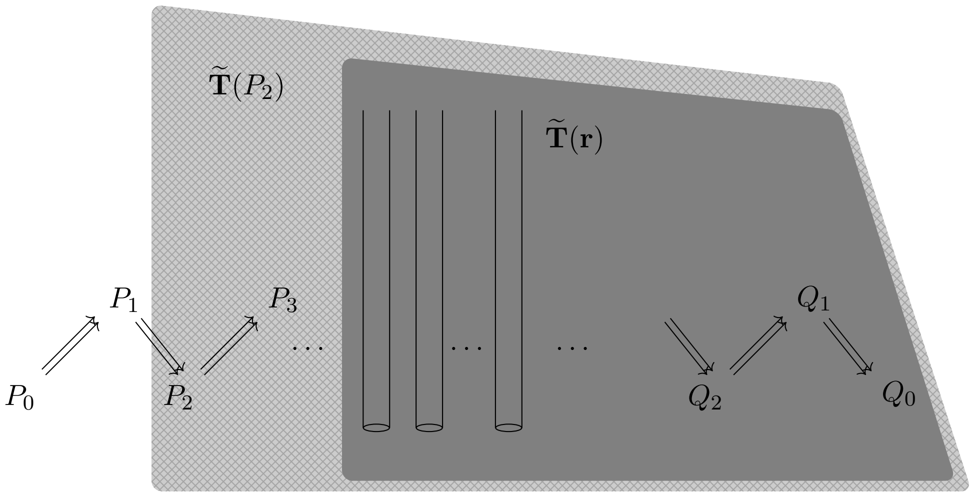


Figure 2.1: Torsion pairs in $kK_2 - \text{mod}$

Recall that the additive closure of the regular component $\mathbf{r}$ *is a wide subcategory of* Λ - mod *whose simple objects are called simple regular modules.*

The torsion class generated by the modules in the regular component $\widetilde{\mathbf{T}}(\mathbf{r})$ *contains all the regular and preinjective modules, but no preprojective module.*

The preprojective component contains a countable collection of bricks P_i*, such that* $\widetilde{\mathbf{T}}(P_i) \supset \widetilde{\mathbf{T}}(P_{i+1})$ *for* $i \geq 1$*. Moreover,* $\bigcap_{i\geq 1} \widetilde{\mathbf{T}}(P_i) = \widetilde{\mathbf{T}}(\mathbf{r})$*.*

Any torsion class larger than $\widetilde{\mathbf{T}}(\mathbf{r})$ *is of the form* $\widetilde{\mathbf{T}}(P_i)$ *for some* P_i*, so it follows that* $\widetilde{\mathbf{T}}(\mathbf{r})$ *is locally maximal.*

Consider now the corresponding cosilting torsion pair in Λ - Mod*. By Proposition 2.2.1, we know that there must be some torsionfree, almost torsion module for this torsion pair, which is necessarily infinitely generated as the corresponding torsion pair in* Λ - mod *has no minimal extending modules.*

It is easy to compute the torsion, almost torsionfree modules for this torsion pair: they are precisely the simple regular modules $\{S_l\}_L$*.*

By Corollary 2.1.10, it follows that the torsionfree, almost torsion modules must lie in the orthogonal category $(\{S_l\}_L)^{\perp_{0,1}}$ *which is known to be equivalent to the module category* $k(X)$ - Mod*.*

Such a subcategory contains a unique brick, up to isomorphism, since $k(X)$ *is a field, and this brick is the generic module* G *described by Ringel [57, Theorem 5.3, Section 5.7].*

Since a torsionfree, almost torsion module must exist, we conclude that G *is the unique torsionfree, almost torsion module for the extended torsion pair.*

2.3.1 Locally maximal torsion classes for $\tau-$tilting infinite algebras

We need some preparations to show the existence of non functorially finite locally maximal torsion classes:

Definition 2.3.7. Let L be a complete lattice. An element $x \in L$ is *compact* if for every set-indexed family $\{y_i\}_{i\in I}$ such that $x \leq \bigvee_{i\in I} y_i$ there exists a finite subset $J \subseteq I$ such that $x \leq \bigvee_{i\in J} y_i$. Dually we have the notion of a *co-compact* element.

We will use the following observation (the contrapositive of [31, Lemma 3.10]).

Lemma 2.3.8. *Let $\mathbf{t} \in \mathbf{tors}(\Lambda)$ be a functorially finite torsion class. Let $\{\mathbf{t}_i\}$ be a chain of torsion classes indexed by some ordinal.*

If $\mathbf{t} = \bigvee_i \mathbf{t}_i$, then there exists some j such that $\mathbf{t}_j = \mathbf{t}$.

Proof. Any functorially finite torsion class is both compact and co-compact [32, Proposition 3.2], in particular, there exists a finite subchain $\mathbf{t}_{i_n}$ such that $\mathbf{t} = \bigvee_n \mathbf{t}_{i_n}$. Whence, $\mathbf{t}_j = \mathbf{t}$ for some j. □

Lemma 2.3.9. *Let $\mathbf{u}_1, \mathbf{u}_2$ be functorially finite torsion classes in Λ-mod and $I = [\mathbf{u}_1, \mathbf{u}_2] \subseteq \mathbf{tors}(\Lambda)$ be the corresponding interval.*

Then if I has not finite length, it contains a maximal and a minimal non functorially finite torsion class $\mathbf{t}_{max}$ and $\mathbf{t}_{min}$.

Moreover, $\mathbf{t}_{max}$ is meet irreducible in I but not completely meet irreducible, while $\mathbf{t}_{min}$ is join irreducible in I but not completely join irreducible.

Proof. We denote by $\mathbf{nftors}(\Lambda) \subset \mathbf{tors}(\Lambda)$ the poset of non functorially finite torsion classes.

By assumption, I must contain either an infinite strictly ascending chain or an infinite strictly descending chain.

Since I is a complete sublattice, the join of the first chain, or the meet of the second one yields a non functorially finite class lying in I (using compactness, or co-compactness of functorially finite torsion classes), proving that the poset $nI = I \cap \mathbf{nftors}(\Lambda)$ is not empty.

This poset and its dual satisfy the hypotheses of Zorn's lemma, in fact for any chain in nI the join, or meet, of the chain in I is again a non functorially finite torsion class by Lemma 2.3.8 and its dual, giving the required upper, or lower, bound.

So we conclude that nI has a maximal and a minimal element.

Now, if $\mathbf{t}$ is such a maximal element, then starting with the obvious inclusion $\mathbf{t} \subsetneq \mathbf{u}_2$ and applying inductively [31, Theorem 3.1] it is possible to construct an infinite descending chain of functorially finite torsion classes $\mathbf{t} \subseteq ... \subsetneq \mathbf{t}_n \subsetneq \cdots \subsetneq \mathbf{t}_1 \subsetneq \mathbf{t}_0 = \mathbf{u}_2$.

Now, by co-compactness the meet of an infinite strictly descending chain is not functorially finite, whence we can conclude by maximality that $\mathbf{t} = \bigwedge_{i\in\mathbb{N}} \mathbf{t}_i$, proving that it is not completely meet irreducible.

Assume now that $\mathbf{t} = \mathbf{s}_1 \wedge \mathbf{s}_2$, for some $\mathbf{s}_i \in I$.

By the definition of meet, $\mathbf{t} \leq \mathbf{s}_i$, so if any of the two is not functorially finite, we must have equality, by maximality in nI.

So assume they are both functorially finite. By the argument above, $\mathbf{t} = \bigwedge_{i\in\mathbb{N}} \mathbf{t}_i$, but by co-compactness of $\mathbf{s}_i$ there is some index j, such that $\mathbf{t}_j \leq \mathbf{s}_1, \mathbf{s}_2$, but this is a contradiction, since $\mathbf{t}_j > \mathbf{t}$. So maximal non functorially finite torsion classes are meet irreducible in I.

Dual arguments yield the dual results. □

Corollary 2.3.10. *Let Λ be a $\tau-$tilting infinite algebra, then there exists a maximal non functorially finite torsion class. Such torsion class is meet irreducible, but not completely meet irreducible, hence locally maximal.*

Proof. Apply the lemma above to the interval $[0, \Lambda\text{-}\mathrm{mod}]$ which has infinite length, see [31, Proposition 3.9], to obtain a maximal element $\mathbf{t}$ in $\mathbf{nftors}(\Lambda)$ with the required properties. □

2.4 Large bricks

We need a last lemma before proceeding into the proof of the main theorem of this chapter. This construction is already present in the literature, see [17] and [18], we give a proof for the convenience of the reader:

Lemma 2.4.1. *Let $B \in \Lambda\text{-}\mathrm{Mod}$ be a brick. Then B is the unique torsion, almost torsionfree module for the torsion pair $(\mathbf{T}(B), B^{\perp_0})$.*

Proof. We check the three conditions dual to those in Definition 2.1.1:

(1) $B \in \mathbf{T}(B)$ by definition.

(2) Since B is a brick, for every proper submodule M of B we must have $M \in B^{\perp_0}$, that is, M is torsionfree.

(3') Consider a short exact sequence:

$$0 \longrightarrow F \longrightarrow M \xrightarrow{f} B \longrightarrow 0$$

with $F \in B^{\perp_0}$. If $M \notin B^{\perp_0}$, let $0 \neq g : B \to M$. Since F is torsionfree, g can not factor through F, in particular $f \circ g \neq 0$.

Since B is a brick this endomorphism must be invertible, which means that the sequence splits. This proves that B is torsion, almost torsionfree.

Any other torsion, almost torsionfree module S, if not isomorphic to B, would be orthogonal to it, in particular torsionfree. This is a contradiction, yielding uniqueness (up to isomorphism). □

Lemma 2.4.2. *Let Λ be a $\tau-$tilting finite algebra. Then every brick in $\Lambda\text{-}\mathrm{Mod}$ is finitely generated.*

Proof. Let B be a brick. By Lemma 2.4.1, the module B is torsion, almost torsionfree with respect to $(\mathbf{T}(B), B^{\perp_0})$.

The restriction of $(\mathbf{T}(B), B^{\perp_0})$ to $\Lambda\text{-}\mathrm{mod}$ is necessarily functorially finite, as all torsion classes in $\Lambda\text{-}\mathrm{mod}$ are functorially finite by hypothesis.

By Proposition 2.2.6, $(\mathbf{T}(B), B^{\perp_0})$ is the unique extension of the functorially finite torsion pair obtained above, thus it is a cosilting torsion pair.

Whence, by Lemma 2.1.8, all the torsion, almost torsionfree modules for $(\mathbf{T}(B), B^{\perp_0})$ are finitely generated. This means that the brick B is finitely generated. □

Lemma 2.4.3. *Let Λ be a $\tau-$tilting infinite algebra. Then there exists some infinitely generated brick in* Λ - Mod.

Proof. Apply Corollary 2.3.10 to obtain a locally maximal non functorially finite torsion class $\mathbf{t}$ in Λ - mod.

By Theorem 2.3.2, the torsion pair $(\mathbf{t}, \mathbf{f})$ has no minimal extending modules.

Consider now the corresponding torsion pair under the bijection in Theorem 1.3.28,

$$(\mathcal{T} = \varinjlim \mathbf{t}, \mathcal{F} = \varinjlim \mathbf{f}).$$

By Proposition 2.2.1, there is some torsionfree, almost torsion module B for this torsion pair.

If B were finitely generated, by Proposition 2.1.8, it would be minimal extending for the original torsion pair $(\mathbf{t}, \mathbf{f})$ which gives a contradiction. □

Combining the two lemmas we can finally obtain:

Theorem 2.4.4. *An artin algebra Λ is $\tau-$tilting finite if and only if every brick over Λ is finitely generated.*

Several questions remain open at this point: can we characterise the bricks which can be obtained as torsionfree, almost torsion modules with respect to a cosilting torsion pair? We know that all such modules can be embedded in some indecomposable pure-injective module, thus there is a bound on their cardinality.

We know that finitely generated bricks parametrize all the completely meet-irreducible torsion classes. Is there any connection between large bricks and meet-irreducible, but not completely meet-irreducible torsion classes?

Moreover, classically we have a connection between ring epimorphisms from Λ to a divison ring and endofinite bricks. If it were possible to show the existence of some endofinite infinitely generated torsionfree, almost torsion module in the $\tau-$tilting infinite case, this could give some further insight on the following conjecture proposed in [10]:

Conjecture 2.4.5. *An artin algebra Λ is $\tau-$tilting infinite if and only if there exists a pseudo-flat ring epimorphism $\Lambda \to \Gamma$ such that Γ is not artinian.*

Chapter 3

Wide subcategories and extending modules

The content of this chapter is joint work with Lidia Angeleri-Hügel.

In the previous chapter we studied characteristic bricks as simple objects in some heart in the derived category.

In this chapter we consider them as simple objects in some wide subcategory of the module category. This approach is not new: for every torsion class in the category of finitely generated modules over an artin algebra, brick labels, in the sense of [32], were shown to be simple objects in some special wide subcategory in [11], [12].

More closely to our terminology, minimal co-extending modules are shown to be the simple objects in the wide subcategory of coherent objects of a torsion class in [18].

The wide subcategories we are interested in were firstly introduced and studied for arbitrary artin algebras in [48] to extend the Ingalls-Thomas bijections [43] holding for hereditary algebras. Such subcategories were further investigated for "large" torsion classes in [10].

In general, these categories consist of coherent/co-coherent objects in the sense of Definition 2.1.4.

Focusing once again on the interplay between torsion pairs in the small and the large module category we will give a compatibility result (Proposition 3.1.12) for cosilting torsion pairs, extending Proposition 2.1.7 from the simple objects to the whole wide subcategories.

We then use the approximation sequence provided by every cosilting module, see Proposition 1.4.6, to give a description of these Ingalls-Thomas subcategories as some orthogonal categories in Theorems 3.2.1, 3.2.13. From these two theorems we can immediately deduce that all the wide subcategories obtained from a cosilting torsion pair must be closed under coproducts.

In the last sections of this chapter we give applications of our results to $\tau-$tilting theory, Propositions 3.2.32 and 3.3.8, and Ext-orthogonal pairs, see Proposition 3.3.18.

Before starting we recall the definition of a wide subcategory:

Definition 3.0.1. Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{W} \subseteq \mathcal{A}$ is *wide* if it is closed under kernels, cokernels and extensions.

3.1 Coherent and co-coherent objects

In this section we adapt to our terminology and dualize some already known results.

Definition 3.1.1. Let $(\mathcal{T}, \mathcal{F})$ a torsion pair in some abelian category $\mathcal{A}$. We define:

$$\begin{aligned}
\mathrm{A}(\mathcal{T}) &= \{X \in \mathcal{A} \mid \text{ for all } T \in \mathcal{T}, f : T \to X, \ker(f) \in \mathcal{T}\} \\
\mathrm{B}(\mathcal{F}) &= \{X \in \mathcal{A} \mid \text{ for all } F \in \mathcal{F}, f : X \to F, \operatorname{coker}(f) \in \mathcal{F}\} \\
\alpha(\mathcal{T}) &= \mathcal{T} \cap \mathrm{A}(\mathcal{T}) \\
\beta(\mathcal{F}) &= \mathcal{F} \cap \mathrm{B}(\mathcal{F})
\end{aligned}$$

If $\mathcal{A}$ is a small category in some sense, e.g. if it is the category of finitely presented objects in some locally noetherian Grothendieck category $\mathcal{G}$, then we will use the symbols $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{B}}$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ for these operators in $\mathcal{A}$.

Remark 3.1.2. *According to Definition 2.1.4, the category $\alpha(\mathcal{T})$ consists of the coherent objects of the torsion class. Dually, the category $\beta(\mathcal{F})$ consists of the co-coherent objects of the torsionfree class.*

Lemma 3.1.3. *Let $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$. Then:*

(i) *The subcategory $\mathrm{A}(\mathcal{T})$ is closed under subobjects and extensions. Moreover, $\mathcal{F} \subseteq \mathrm{A}(\mathcal{T})$.*

(ii) *The subcategory $\mathrm{B}(\mathcal{F})$ is closed under quotients and extensions. Moreover, $\mathcal{T} \subseteq \mathrm{B}(\mathcal{F})$.*

(iii) *$\alpha(\mathcal{T})$ is a wide subcategory of $\mathcal{A}$. It is closed under torsion subobjects.*

(iv) *$\beta(\mathcal{F})$ is a wide subcategory of $\mathcal{A}$. It is closed under torsionfree quotients.*

(v) $\mathrm{A}(\mathcal{T}) = \alpha(\mathcal{T}) \star \mathcal{F}$

(vi) $\mathrm{B}(\mathcal{F}) = \mathcal{T} \star \beta(\mathcal{F})$

(vii) $\alpha(\mathcal{T}) = 0$ *if and only if* $\mathrm{A}(\mathcal{T}) = \mathcal{F}$.

(viii) $\beta(\mathcal{F}) = 0$ *if and only if* $\mathrm{B}(\mathcal{F}) = \mathcal{T}$.

Proof. We give a proof of (i), (iii), (v) and (vii), the other points having similar proofs.

(i) First, notice that if $F \in \mathcal{F}$ then for every $T \in \mathcal{T}$, $\mathrm{Hom}_R(T, F) = 0$. Thus, every such object F is trivially an element of $\mathrm{A}(\mathcal{T})$.

Let $X \in \mathrm{A}(\mathcal{T})$, consider $Y \leq X$. Then for every torsion object T and every map $f : T \to Y$, the kernel of f is equal to the kernel of the composition of f with an embedding of Y into X. Thus, Y is in $\mathrm{A}(\mathcal{T})$.

Let $0 \longrightarrow X' \longrightarrow Y \xrightarrow{g} X'' \longrightarrow 0$ be a short exact sequence with $X', X'' \in \mathrm{A}(\mathcal{T})$. Let $f : T \to Y$ be some map. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(g \circ f) & \longrightarrow & T & \longrightarrow & \mathrm{Im}(g \circ f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y & \xrightarrow{g} & X'' & \longrightarrow & 0 \end{array}$$

By the previous point, $\mathrm{Im}(g \circ f) \in \mathrm{A}(\mathcal{T})$, thus $\ker(g \circ f) \in \mathcal{T}$. An application of the snake lemma yields that $\ker(f) = \ker(h)$ and this is a torsion module since $X' \in \mathrm{A}(\mathcal{T})$. Thus, $\mathrm{A}(\mathcal{T})$ is closed under extensions.

(iii) First, $\mathrm{A}(\mathcal{T})$ is closed under subobjects. So every subobject of an object in $\alpha(\mathcal{T})$ which is in $\mathcal{T}$ is an element of $\alpha(\mathcal{T})$.

To see that $\alpha(\mathcal{T})$ is wide we need to check that it is closed under extensions, kernels and cokernels. Closure under extensions is immediate as both $\mathcal{T}$ and $\mathrm{A}(\mathcal{T})$ are extension-closed. For the same reason, $\alpha(\mathcal{T})$ is also closed under images, thus it is enough to check that every monomorphism has a cokernel and that every epimorphism has a kernel.

For kernels, consider a short exact sequence $0 \to K \to T_1 \to T_2 \to 0$, with $T_1, T_2 \in \alpha(\mathcal{T})$. Then $K \in \mathcal{T}$ and it is a subobject of an object in $\mathrm{A}(\mathcal{T})$, thus $K \in \alpha(\mathcal{T})$.

For cokernels, consider the pull-back diagram:

$$\begin{array}{ccccccccc} & & & & \ker(h) & = & \ker(h) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & T_1 & \xrightarrow{f} & T_2 & \longrightarrow & \mathrm{coker}(f) & \longrightarrow & 0 \end{array}$$

were $T_1, T_2 \in \alpha(\mathcal{T})$ and $T \in \mathcal{T}$. $P \in \mathcal{T}$ as it is an extension of two torsion objects, and thus $\ker(h) \in \mathcal{T}$ since $T_2 \in \mathrm{A}(\mathcal{T})$. So $\mathrm{coker}(f) \in \mathrm{A}(\mathcal{T})$. It is also in $\mathcal{T}$ as it is a quotient of a torsion object.

(v) $\mathrm{A}(\mathcal{T}) \supseteq \alpha(\mathcal{T}) \star \mathcal{F}$ is immediate as by the previous points we have that $\mathrm{A}(\mathcal{T}) \supseteq \mathcal{F}$ and that it is closed under extensions.

For the reverse inclusion, given $X \in \mathrm{A}(\mathcal{T})$, the approximation sequence $0 \to tX \to X \to X/tX \to 0$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ has the required shape.

(vii) Immediate by point (v). □

Remark 3.1.4. *Let $\mathcal{L}$ be a length category, $(\mathbf{t}, \mathbf{f})$ a torsion pair in $\mathcal{L}$. In this situation, by Corollary 1.1.10, closure under subobjects (resp. quotients) and extensions is enough to ensure that $\widetilde{\mathrm{A}}(\mathbf{t})$ is a torsionfree class (resp. $\widetilde{\mathrm{B}}(\mathbf{f})$ is a torsion class) in $\mathcal{L}$.*

The following result is due to Enomoto and Sakai, again we include a proof for completeness. Following their terminology we call a subcategory *ICE-closed* (resp. *IKE-closed*) if it is closed under images, cokernels (resp. kernels) and extensions.

Proposition 3.1.5 ([36])**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then:*

(i) Let $(\mathcal{T}', \mathcal{F}')$ be a torsion pair, with $\mathcal{F} \subseteq \mathcal{F}'$ and such that $\mathcal{F}' \cap \mathcal{T}$ is closed under kernels (thus IKE-closed). Then $\mathcal{F}' \subseteq \mathrm{A}(\mathcal{T})$.

(ii) Let $(\mathcal{T}', \mathcal{F}')$ be a torsion pair, with $\mathcal{T} \subseteq \mathcal{T}'$ and such that $\mathcal{T}' \cap \mathcal{F}$ is closed under cokernels (thus ICE-closed). Then $\mathcal{T}' \subseteq \mathrm{B}(\mathcal{F})$.

Proof. We prove (i). By Lemma 1.1.13, $\mathcal{F}'$ can be written as $(\mathcal{F}' \cap \mathcal{T}) \star \mathcal{F}$. Hence to show that $\mathcal{F}' \subseteq \mathrm{A}(\mathcal{T})$ is enough to show $(\mathcal{F}' \cap \mathcal{T}) \subseteq \mathrm{A}(\mathcal{T})$.

Let $M \in \mathcal{F}' \cap \mathcal{T}$. Let $T \in \mathcal{T}$, $f : T \to M$. Without loss of generality we might assume f surjective (in fact $\mathcal{F}' \cap \mathcal{T}$ is closed under submodules which belong to $\mathcal{T}$).

Notice that the map f factors through the torsionfree part of T with respect to $(\mathcal{T}', \mathcal{F}')$ so that we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{t}_{\mathcal{T}'}(T) & \longrightarrow & \ker(f) & \longrightarrow & \ker(h) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}_{\mathcal{T}'}(T) & \longrightarrow & T & \longrightarrow & T/\mathrm{t}_{\mathcal{T}'}(T) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\
 & & & & M & = & M & &
\end{array}
$$

Now, h is a morphism in $\mathcal{F}' \cap \mathcal{T}$, which by hypothesis is closed under kernels, thus $\ker(h) \in \mathcal{T}$. Hence, $\ker(f) \in \mathcal{T}$ as an extension of two objects in $\mathcal{T}$. (Recall that by hypothesis $\mathcal{T}' \subseteq \mathcal{T}$). Thus $\mathcal{F}' \cap \mathcal{T} \subseteq \mathrm{A}(\mathcal{T})$ as desired. □

Remark 3.1.6. *From now on, we will work with module categories: notice that in this setting not every result can be dualised.*

The following result from [12] and [48] tells that for some special torsion pairs the computation of coherent and co-coherent objects is particularly easy:

Theorem 3.1.7 ([12], [48])**.** *Let R be a left artinian ring. Let $\mathcal{W}$ be a subcategory of R-mod. Then:*

(i) $\widetilde{\alpha}(\widetilde{\mathbf{T}}(\mathcal{W})) = \mathcal{W}$ if and only if $\mathcal{W}$ is a wide subcategory.

(ii) $\widetilde{\beta}(\widetilde{\mathbf{F}}(\mathcal{W})) = \mathcal{W}$ *if and only if* $\mathcal{W}$ *is a wide subcategory.*

Remark 3.1.8. *For part (i) of the theorem above it is enough to assume that the ring is left noetherian. In fact, the proof of the result uses the description of* $\widetilde{\mathbf{T}}$ *as* filt gen .

Such a result is valid in any abelian category in which a torsion class is precisely a subcategory closed under extensions and quotients. This holds for the category of finitely generated left modules over a left noetherian ring (see Corollary 1.1.8).

A "large" partial version of this result was obtained for torsion classes:

Proposition 3.1.9 ([10])**.** *Let* R *be a ring. Let* $\mathcal{W}$ *be a subcategory of* R-Mod*. Then*

$$\alpha(\mathbf{T}(\mathcal{W})) = \mathcal{W} \text{ if } \mathcal{W} \text{ is a wide subcategory closed under coproducts.}$$

We will prove the equivalence of the two conditions above for cosilting torsion pairs in Proposition 3.2.18.

We conclude this section giving a name to such torsion pairs

Definition 3.1.10 ([12])**.** Let R be a (left noetherian) ring. A torsion pair $(\mathcal{T}, \mathcal{F})$ in R-Mod (resp. $(\mathbf{t}, \mathbf{f})$ in R-mod) is said to be *widely generated* if there exists a wide subcategory $\mathcal{W}$ of the ambient category with $\mathbf{T}(\mathcal{W}) = \mathcal{T}$ (resp. $\widetilde{\mathbf{T}}(\mathcal{W}) = \mathbf{t}$).

3.1.1 The connection with characteristic bricks

We now give the announced connection with torsionfree, almost torsion and torsion, almost torsionfree modules.

Proposition 3.1.11. *Let* R *be a ring. Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* R-Mod*. Then:*

(i) The simple objects of $\alpha(\mathcal{T})$ *are precisely the torsion, almost torsionfree modules in* $\mathcal{T}$*.*

(ii) The simple objects of $\beta(\mathcal{F})$ *are precisely the torsionfree, almost torsion modules in* $\mathcal{F}$*.*

Proof. We give a proof of (i). Notice that an object S is torsion, almost torsionfree in $\mathcal{T}$ if and only if it is a coherent $\mathcal{T}-$simple object (see Definition 2.1.4).

Now $\alpha(\mathcal{T})$ consists precisely of the coherent objects in $\mathcal{T}$. Thus it is enough to show that S is $\mathcal{T}-$simple if and only if it is simple in $\alpha(\mathcal{T})$.

Since $\alpha(\mathcal{T}) \subseteq \mathcal{T}$, we have that a $\mathcal{T}-$simple is necessarily simple in $\alpha(\mathcal{T})$. So, it remains to show that every simple object S of $\alpha(\mathcal{T})$ is $\mathcal{T}-$simple.

Since $\mathcal{T}$ is a torsion class, it is enough to show that every proper submodule of S is torsionfree. But this is immediate, as $\alpha(\mathcal{T})$ is closed under submodules in $\mathcal{T}$ and S is simple in $\alpha(\mathcal{T})$. □

In the case of a cosilting torsion pair over a left noetherian ring we have the following compatibility result:

Proposition 3.1.12. *Let R be a left noetherian ring. Let $(\mathcal{T},\mathcal{F})$ be a cosilting torsion pair in $R\text{-}\mathrm{Mod}$ with restriction $(\mathbf{t},\mathbf{f})$ to $R\text{-}\mathrm{mod}$. Then:*

(i) $\mathrm{A}(\mathcal{T})\cap R\text{-}\mathrm{mod}=\widetilde{\mathrm{A}}(\mathbf{t})$ *and thus* $\alpha(\mathcal{T})\cap R\text{-}\mathrm{mod}=\widetilde{\alpha}(\mathbf{t})$

(ii) $\mathrm{B}(\mathcal{F})\cap R\text{-}\mathrm{mod}=\widetilde{\mathrm{B}}(\mathbf{f})$ *and thus* $\beta(\mathcal{F})\cap R\text{-}\mathrm{mod}=\widetilde{\beta}(\mathbf{f})$

Proof. We can adapt the proof of 2.1.7 to this situation. We show how to proceed for case (ii).

First, notice that $\mathrm{B}(\mathcal{F})\cap R\text{-}\mathrm{mod}\subseteq\widetilde{\mathrm{B}}(\mathbf{f})$ by definition. So assume $X\in\widetilde{\mathrm{B}}(\mathbf{f})$, let $F\in\mathcal{F}$ and $f:X\to F$ with cokernel C. To show that $X\in\mathrm{B}(\mathcal{F})$ we need to prove that C is torsionfree. As $\mathcal{F}=\mathbf{t}^{\perp_0}$, assume we have an injection $T\to C$, with $T\in\mathbf{t}$, and consider the following pull-back diagram:

$$\begin{array}{ccccccc}
X & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0\\
\| & & \downarrow & & \downarrow & & \\
X & \xrightarrow{f} & F & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

By construction, P is a finitely generated torsionfree module, thus, using that $X\in\widetilde{\mathrm{B}}(\mathbf{f})$ we must have that $T\in\mathbf{f}$. Thus $T=0$ being both torsion and torsionfree. □

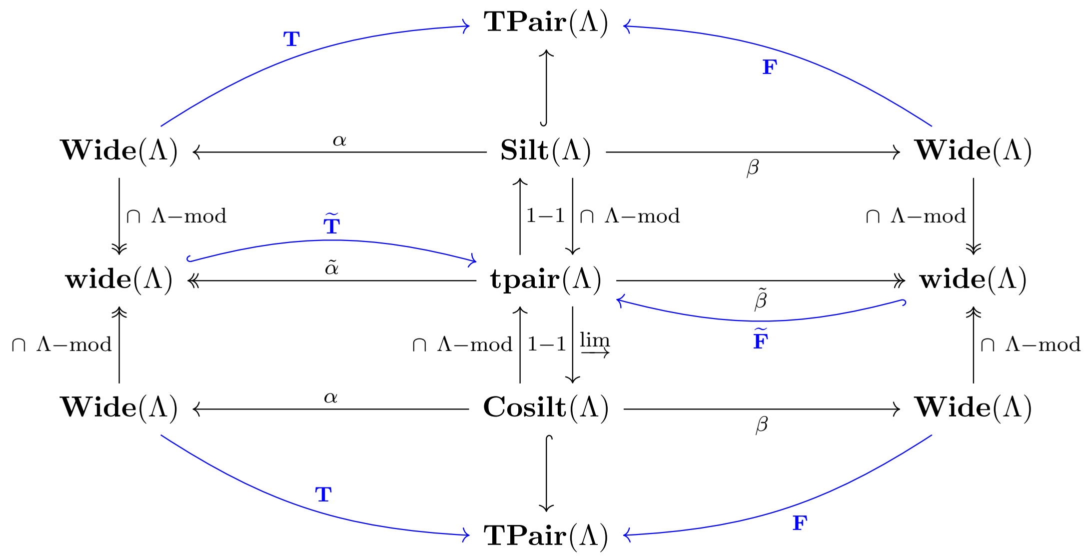
Over an artin algebra, we can prove the analogous result for the silting case by dual arguments:

Proposition 3.1.13. *Let Λ be an artin algebra. Let $(\mathcal{T},\mathcal{F})$ be a silting torsion pair in $\Lambda\text{-}\mathrm{Mod}$ with restriction $(\mathbf{t},\mathbf{f})$ to $\Lambda\text{-}\mathrm{mod}$. Then:*

(i) $\mathrm{A}(\mathcal{T})\cap\Lambda\text{-}\mathrm{mod}=\widetilde{\mathrm{A}}(\mathbf{t})$ *and thus* $\alpha(\mathcal{T})\cap\Lambda\text{-}\mathrm{mod}=\widetilde{\alpha}(\mathbf{t})$

(ii) $\mathrm{B}(\mathcal{F})\cap\Lambda\text{-}\mathrm{mod}=\widetilde{\mathrm{B}}(\mathbf{f})$ *and thus* $\beta(\mathcal{F})\cap\Lambda\text{-}\mathrm{mod}=\widetilde{\beta}(\mathbf{f})$

Summarising what we discussed, we can say that the black diagram in the picture below is commutative.



3.2 Wide subcategories from cosilting torsion pairs

For cosilting torsion pairs we can obtain an explicit description of the classes defined in the previous sections. For the necessary preliminaries about cosilting modules we refer back to section 1.4.

3.2.1 Wide subcategories from torsionfree classes

The following result is the basis for the following observations:

Theorem 3.2.1. *Let R be a ring. Let C be a cosilting $R-$module,* $\mathrm{Cogen}(C)$ *the corresponding definable torsionfree class. Let*

$$0 \to C_1 \to C_0 \to E(R)$$

be the $\mathrm{Cogen}(C)-$*cover of an injective cogenerator of* $R\text{-}\mathrm{Mod}$*. Then:*

$$\mathrm{B}(\mathrm{Cogen}(C)) = {}^{\perp_0}C_1$$

In particular, $\mathrm{B}(\mathrm{Cogen}(C))$ *is a torsion class in* $R\text{-}\mathrm{Mod}$*.*

Proof. "$\subseteq$" : As $\mathrm{B}(\mathrm{Cogen}(C))$ is closed under quotients (by Lemma 3.1.3), it is enough to show that we can't have a non-zero monomorphism from some object $B \in \mathrm{B}(\mathrm{Cogen}(C))$ to C_1.

Assume that we can find a monomorphism $i : B \to C_1$ and consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & B & = & B & & \\
 & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_0 & \xrightarrow{f} & E(R) \\
 & & \downarrow & & {\scriptstyle r}\uparrow\downarrow{\scriptstyle l} & & \| \\
0 & \longrightarrow & \operatorname{coker}(i) & \longrightarrow & P & \xrightarrow{m} & E(R)
\end{array}$$

Since $B \in \mathrm{B}(\operatorname{Cogen}(C))$, we have that $P \in \operatorname{Cogen}(C)$. Thus the map $m : P \to E(R)$ must factor through f via some $r : P \to C_0$. Whence we obtain that $f = m \circ l = (f \circ r) \circ l$. By right minimality of f, the map $r \circ l$ is an isomorphism. Thus l is a monomorphism. This implies that $B = 0$.

"$\supseteq$" : Let $X \in {}^{\perp_0}C_1$. We must show that every map $X \to F$ for $F \in \operatorname{Cogen}(C)$ has torsionfree cokernel. Since ${}^{\perp_0}C_1$ is closed under quotients, without loss of generality we consider only injective maps.

So let $0 \to X \to F \to M \to 0$ be a short exact sequence. The long exact sequence obtained applying the functor $\operatorname{Hom}_R(-, C_1)$ shows that $\operatorname{Ext}^1_R(M, C_1) = 0$, as $\operatorname{Ext}^1_R(F, C_1) = 0$ by Lemma 1.4.7.

Moreover, as M is a quotient of F, we have $M \in R/\operatorname{Ann}(C)\text{-Mod}$. It follows that $M \in \operatorname{Cogen}(C)$ (recall that $\operatorname{Cogen}(C) = {}^{\perp_1}C_1 \cap R/\operatorname{Ann}(C)\text{-Mod}$ again by Lemma 1.4.7). □

Corollary 3.2.2. *Let R be a ring, C a cotilting $R-$module. Then, using the same notation as in Theorem 3.2.1*

$$\beta(\operatorname{Cogen}(C)) = {}^{\perp_{0,1}}C_1$$

Proof. It is enough to notice that in this case $\operatorname{Cogen}(C) = {}^{\perp_1}C_1$. □

Corollary 3.2.3. *Let R be a ring, C a cosilting $R-$module, then $\beta(\operatorname{Cogen}(C))$ is a wide subcategory closed under coproducts.*

Proof. $\beta(\operatorname{Cogen}(C))$ is always wide by Lemma 3.1.3, closure under coproducts comes immediately noticing that $\beta(\operatorname{Cogen}(C)) = {}^{\perp_0}C_1 \cap \operatorname{Cogen}(C)$. □

In fact, using the following theorem, we obtain that these subcategories are always coreflective:

Theorem 3.2.4 ([21, Theorem 2.5]). *A class of modules is precovering, if it is closed under coproducts and pure quotients.*

Proposition 3.2.5. *Let R be a ring, C a cosilting $R-$module, then $\beta(\operatorname{Cogen}(C))$ is a coreflective subcategory.*

Proof. The subcategory $\beta(\mathrm{Cogen}(C))$ is a wide subcategory closed under coproducts, moreover it is also closed under pure quotients, as it is the intersection of two classes closed under such quotients. Thus Theorem 3.2.4 gives that $\beta(\mathrm{Cogen}(C))$ is precovering. We obtain the result using the dual of Proposition 1.2.21. □

Examples 3.2.6. *(i) An example of a wide subcategory of the form $\beta(\mathcal{F})$ for a cosilting class $\mathcal{F}$ which is not closed under products is discussed in [5, Example 4.10]. This is the perpendicular class ${}^{\perp_{0,1}}G$ to the generic module G over the Kronecker algebra kK_2.*

(ii) For a torsion pair $(\mathcal{T}, \mathcal{F}) \in \mathbf{Cosilt}(A)$ over some hereditary algebra Λ, we have that $\beta(\mathcal{F})$ is closed under products, if and only if the torsion class is generated by a wide subcategory of $\Lambda\text{-}\mathrm{mod}$. Further details are given in Section 3.3.3.

Proposition 3.2.7. *Let A be a left artinian ring. Then the map*

$$\begin{aligned} \beta\colon \mathbf{TPair}(A) &\longrightarrow \mathbf{Wide}(A) \\ (\mathcal{T}, \mathcal{F}) &\mapsto \beta(\mathcal{F}) \end{aligned}$$

restricts to an injection

$$\beta : \mathbf{Cosilt}(A) \longrightarrow \mathbf{Wide}_{\coprod}(A)$$

between torsion pairs with definable torsionfree class and the class $\mathbf{Wide}_{\coprod}(A)$ of wide subcategories closed under coproducts.

Proof. By Corollary 3.2.3 the map is well-defined. Moreover, by Proposition 3.1.11, if $\beta(\mathcal{F}) = \beta(\mathcal{F}')$, then $\mathcal{F}$ and $\mathcal{F}'$ have the same torsionfree, almost torsion modules. At this point, Proposition 2.2.4 yields $\mathcal{F} = \mathcal{F}'$. □

Remark 3.2.8. *By Proposition 3.2.7 the map $\beta : \mathbf{Cosilt}(A) \longrightarrow \mathbf{Wide}_{\coprod}(A)$ is an injection, however in general there is no hope of obtaining all the wide subcategories closed under coproducts as images of some cosilting class under β.*

As an example, in case $A = k\mathcal{K}_2$ is the path-algebra of the Kronecker quiver we have, for each non-empty collection P of tubes in the AR-quiver, a wide subcategory $\mathcal{W}_P$ of $A\text{-}\mathrm{mod}$. The direct limit closure of any such subcategory is a wide subcategory of $A\text{-}\mathrm{Mod}$ closed under coproducts.

However, if P is a proper subset of the set of all tubes, none of these categories is in the image of β.

Assume there exists a cosilting torsionfree class $\mathcal{F}$ such that $\beta(\mathcal{F}) = \varinjlim \mathcal{W}_P$. Then $\widetilde{\beta}(\mathcal{F} \cap k\mathcal{K}_2\text{-}\mathrm{mod}) = \mathcal{W}_P$. The unique torsionfree class in $k\mathcal{K}_2\text{-}\mathrm{mod}$ with such $\widetilde{\beta}$ is $\widetilde{\mathbf{F}}(\mathcal{W}_P)$.

Therefore we have that the cosilting class $\mathcal{F} = \varinjlim \widetilde{\mathbf{F}}(\mathcal{W}_P) = \mathcal{W}_Q^{\perp_0}$, where Q is the complement of P. For such a torsionfree class in the hereditary setting, the subcategory $\beta(\mathcal{W}_Q^{\perp_0}) = \mathcal{W}_Q^{\perp_{0,1}}$ by Lemma 3.3.10.

This is a bireflective subcategory, equivalent to the category of modules over a localisation of the polynomial ring $k[X]$. As such $\beta(\mathcal{F}) \neq \varinjlim \mathcal{W}_P$, giving a contradiction.

3.2.2 Wide subcategories from torsion classes

For the description of $\mathrm{A}({}^{\perp_0}C)$ we give the following definition:

Definition 3.2.9. Let $M \in R\text{-}\mathrm{Mod}$. Then we define:

$${}^{\perp_{1h}}M := \{X \in R\text{-}\mathrm{Mod} \mid \text{ for all } Y \leq X, \mathrm{Ext}^1(Y, M) = 0\}$$

Remark 3.2.10. *In case the injective dimension of M is at most one, then ${}^{\perp_{1h}}M = {}^{\perp_1}M$.*

The following property is well-known, the proof is short so we include it:

Lemma 3.2.11. *Let $M \in R\text{-}\mathrm{Mod}$ be a pure-injective module, then ${}^{\perp_1}M$ is closed under direct limits.*

Proof. Let $\{(X_i, f_i)\}_{i \in I}$ be a directed system in ${}^{\perp_1}M$. Then we have a pure-exact sequence

$$0 \to K \to \coprod_I X_i \to \varinjlim_I X_i \to 0$$

As M is pure-injective, the sequence

$$0 \to \mathrm{Hom}_R(\varinjlim_I X_i, M) \to \mathrm{Hom}_R(\coprod_I X_i, M) \to \mathrm{Hom}_R(K, M) \to 0$$

is exact, thus $\mathrm{Ext}^1_R(\varinjlim_I X_i, M)$ is a subgroup of $\prod_I \mathrm{Ext}^1_R(X_i, M) = 0$. □

Lemma 3.2.12. *Let M be an $R-$module. Then ${}^{\perp_{1h}}M$ is closed under extensions and submodules. Morevoer, if M is pure-injective, the class ${}^{\perp_{1h}}M$ is also closed under direct unions (and thus direct sums).*

Proof. Closure under submodules is immediate, by the definition. For the extensions, notice that ${}^{\perp_1}M$ is closed under extensions and that every submodule of a module Y obtained as an extension $0 \to X' \to Y \to X'' \to 0$ can be obtained as an extension of submodules of X' and X''.

In case M is pure-injective ${}^{\perp_1}M$ is closed under direct limits as seen in Lemma 3.2.11. If $\{X_i, (\alpha_{ij})\}$ is a directed sequence of monomorphisms in ${}^{\perp_{1h}}M$, then any submodule Y of $\bigcup_i X_i$ can be written as the union of the submodules $Y_i := Y \cap X_i$ of the X_i. Thus, it is in ${}^{\perp_1}M$. □

Theorem 3.2.13. *Let R be a ring. Let C be a cosilting $R-$module and ${}^{\perp_0}C$ the corresponding torsion class. Let*

$$0 \to C_1 \to C_0 \to E(R)$$

be the $\mathrm{Cogen}(C)-$cover of an injective cogenerator of $R\text{-}\mathrm{Mod}$. Then:

$$\mathrm{A}({}^{\perp_0}C) = {}^{\perp_{1h}}C_0$$

Proof. " $\supseteq$ " : Let $X \in {}^{\perp_{1h}}C_0$. We have to show that for every $T \in {}^{\perp_0}C$, and every map $f : T \to X$, $\ker(f) \in {}^{\perp_0}C$.

Since ${}^{\perp_{1h}}C_0$ is closed under submodules, we may assume, without loss of generality, that f is an epimorphism.

Consider the short exact sequence $0 \to \ker(f) \to T \to X \to 0$. Applying $\mathrm{Hom}_R(-, C_0)$ to the sequence, we obtain that $\mathrm{Hom}(\ker(f), C_0) = 0$. However, since C_0 cogenerates $\mathrm{Cogen}(C)$, by Proposition 1.4.6, it follows that $\mathrm{Hom}(\ker(f), C) = 0$ as desired.

" $\subseteq$ " : Let $X \in \mathrm{A}({}^{\perp_0}C)$. This class is closed under submodules, by Lemma 3.1.3, so it is enough to show that $\mathrm{Ext}^1(X, C_0) = 0$.

Let $0 \to C_0 \to M \xrightarrow{f} X \to 0$ be a short exact sequence. Applying the snake lemma to the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \mathrm{t}\,M & \longrightarrow & f(\mathrm{t}\,M) = I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0 & \longrightarrow & M & \xrightarrow{f} & X & \longrightarrow & 0
\end{array}$$

we obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \mathrm{t}\,M & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0 & \longrightarrow & M & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow h & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M/\mathrm{t}\,M & \longrightarrow & \overline{X} & \longrightarrow & 0
\end{array}$$

Since I is a submodule of X it is in $\mathrm{A}({}^{\perp_0}C)$, thus $F \in {}^{\perp_0}C$. But then $F \in \mathrm{Cogen}(C) \cap {}^{\perp_0}C = 0$.

This forces $L = C_0$. Then, since C_0 is split-injective in $\mathrm{Cogen}(C)$ and $M/\mathrm{t}\,M$ is in $\mathrm{Cogen}(C)$, the third short exact sequence splits. Therefore we get a map $g : M/\mathrm{t}\,M \to L$. Then the map $g \circ h$ is a splitting epimorphism for the middle sequence. □

Corollary 3.2.14. *With the same notation as in Theorem 3.2.13, if the module C is cotilting, we have:*

$$\alpha({}^{\perp_0}C) = {}^{\perp_{0,1}}C_0$$

Proof. The injective dimension of a cotilting module is less than 1 by definition. □

Corollary 3.2.15. *Let R be a ring, C a cosilting $R-$module, then $\alpha({}^{\perp_0}C)$ is a wide subcategory closed under coproducts.*

Proof. This class is wide by Lemma 3.1.3. It is closed under coproducts since it is an intersection of coproduct-closed subcategories. □

Proposition 3.2.16. *Let R be a ring. Then the map*

$$\begin{aligned}\alpha\colon \mathbf{TPair}(R) &\longrightarrow \mathbf{Wide}(R)\\ (\mathcal{T},\mathcal{F}) &\mapsto \alpha(\mathcal{T})\end{aligned}$$

restricts to a map

$$\alpha : \mathbf{Cosilt}(R) \longrightarrow \mathbf{Wide}_{\coprod}(R)$$

Example 3.2.17. *In general, the subcategories $\alpha(\mathcal{T})$ are not closed under products. In fact, we will see that over a left artinian ring A this can only occur if the restriction $\alpha(\mathcal{T}) \cap A\text{-}\mathrm{mod}$ is functorially finite.*

We can now complete a large version of Theorem 3.1.7(i) for cosilting torsion pairs:

Proposition 3.2.18. *Let R be a ring, $\mathcal{W} \subseteq R\text{-}\mathrm{Mod}$ such that $\mathcal{W}^{\perp_0}$ is a definable torsionfree class, then:*

$$\alpha(\mathbf{T}(\mathcal{W})) = \mathcal{W} \text{ if and only if } \mathcal{W} \in \mathbf{Wide}_{\coprod}(R)$$

Proof. If $\mathcal{W} \in \mathbf{Wide}_{\coprod}(R)$ then $\alpha(\mathbf{T}(\mathcal{W})) = \mathcal{W}$ by Proposition 3.1.9; conversely, if the equality holds, $\mathcal{W}$ is wide and closed under coproducts by Corollary 3.2.15. □

3.2.3 Mutation of cosilting torsion pairs

There is a concept of mutation for cosilting objects and for cosilting torsion pairs, introduced in [6].

We start with the definition of mutation for cosilting complexes in the derived category:

Definition 3.2.19. Let $\sigma, \sigma' \in \mathrm{D}(R)$ be cosilting complexes. Let $\mathcal{E} = \mathrm{Prod}(\sigma) \cap \mathrm{Prod}(\sigma')$.

We say that σ' is a *right mutation of* σ (with respect to $\mathcal{E}$) if there is a triangle $\sigma[-1] \longrightarrow \gamma_0 \longrightarrow \gamma_1 \xrightarrow{\Phi} \sigma$ such that:

(i) Φ is a $\mathcal{E}-$precover of σ in $\mathrm{D}(R)$

(ii) $\gamma_0 \oplus \gamma_1$ is a cosilting object equivalent to σ'.

Let $\mathfrak{t}, \mathfrak{u}$ be torsion pairs in $\mathbf{Cosilt}(R)$. Then, recalling the correspondence given in Theorem 1.5.6, we can say that $\mathfrak{t}$ is a right mutation of $\mathfrak{u}$ if this holds for the corresponding cosilting complexes in $\mathrm{D}(R)$.

Here we use the following equivalent characterisation, valid over left noetherian rings:

Definition 3.2.20 ([6])**.** Let R be a left noetherian ring. Let $(\mathcal{T},\mathcal{F})$, $(\mathcal{U},\mathcal{V})$ be torsion pairs with definable torsionfree classes in $R\text{-}\mathrm{Mod}$. We say that $(\mathcal{T},\mathcal{F})$ is a *right mutation* of $(\mathcal{U},\mathcal{V})$ if $\mathcal{U} \subseteq \mathcal{T}$ and $\mathcal{T} \cap \mathcal{V}$ is a wide subcategory (closed under coproducts).

Remark 3.2.21. *Notice that if $(\mathcal{T},\mathcal{F})$ is a right mutation of $(\mathcal{U},\mathcal{V})$ then the subcategory $\mathcal{T} \cap \mathcal{V}$ is both ICE and IKE-closed, in particular in this setting $\mathcal{T} \subseteq \mathrm{B}(\mathcal{V})$ and $\mathcal{V} \subseteq \mathrm{A}(\mathcal{T})$, see Proposition 3.1.5.*

In the artinian case, we can always produce a maximal mutation (which might be trivial) of every cosilting torsion pair.

Proposition 3.2.22. *Let A be a left artinian ring. C a cosilting module in A-Mod. Then $\mathrm{A}({}^{\perp_0}C)$ is a definable torsionfree class.*

Proof. Set $\mathcal{V} = \mathrm{A}({}^{\perp_0}C)$. We know that $\mathcal{V}$ is closed under extensions and submodules, thus by Corollary 1.1.9, the restriction $\mathcal{V} \cap A$-mod is a torsionfree class in A-mod. Therefore, $\mathcal{F} = \varinjlim(\mathcal{V} \cap A\text{-mod})$ is a definable torsionfree class by Theorem 1.3.28.

To prove our claim, it is enough to show $\mathcal{V} = \mathcal{F}$. That $\mathcal{V} \subseteq \mathcal{F}$ is immediate, as $\mathcal{V}$ is closed under submodules, thus we concentrate on the opposite inclusion.

Let $F \in \mathcal{F}$, then F can be written as the union of its finitely generated submodules, which are elements of $\mathcal{V}$ by construction.

Now by Lemma 3.2.12 and Theorem 3.2.13, the subcategory $\mathcal{V}$ is closed under directed unions, thus $\mathcal{F} \subseteq \mathcal{V}$. □

Since $\alpha(\mathcal{T})$ is wide, combining Proposition 3.2.22 with Remark 3.2.21 we can obtain the following

Corollary 3.2.23 ([6, Corollary 9.9])**.** *Let A be a left artinian ring, $(\mathcal{T}, \mathcal{F})$ a cosilting torsion pair in A-Mod.*

Then $\mathcal{V} = \mathrm{A}(\mathcal{T})$ is the largest torsionfree class such that $(\mathcal{T}, \mathcal{F})$ is a right mutation of $(\mathcal{U}, \mathcal{V})$.

We conclude this discussion of mutation for cosilting pairs with a result showing that the alpha-subcategories are locally small:

Proposition 3.2.24. *Let R be a left noetherian ring. $(\mathcal{T}, \mathcal{F})$ a cosilting torsion pair in R-Mod with restriction $(\mathbf{t}, \mathbf{f})$. Then:*

$$\alpha(\mathcal{T}) = \varinjlim[\widetilde{\alpha}(\mathbf{t})]$$

Proof. " $\supseteq$ " : By Corollary 3.2.15, we have that $\alpha(\mathcal{T})$ is wide and closed under coproducts, whence it is closed under direct limits. By Proposition 3.1.12, $\widetilde{\alpha}(\mathbf{t}) \subseteq \alpha(\mathcal{T})$ so we obtain the first inclusion.

" $\subseteq$ " : Let $X \in \alpha(\mathcal{T})$. Then $X \in \mathcal{T} \cap \mathrm{A}(\mathcal{T})$. In particular, by Proposition 3.1.12, all its finitely generated submodules are in $\widetilde{\mathrm{A}}(\mathbf{t})$.

Hence, we can write $X = \varinjlim(X_i)$ with $X_i \in \widetilde{\mathrm{A}}(\mathbf{t})$. Since $\mathcal{F}$ is definable, the torsion radical of the torsion pair commutes with direct limits, in particular $X = t(X) = t(\varinjlim X_i) \cong \varinjlim t(X_i)$.

Now, each $t(X_i) \in (\mathcal{T} \cap \widetilde{\mathrm{A}}(\mathbf{t})) = \widetilde{\alpha}(\mathbf{t})$. This proves the second inclusion. □

As an immediate application of the Proposition, we can identify the very special case in which $\alpha(\mathcal{T})$ is also closed under products.

Corollary 3.2.25. *Let A be a left artinian ring. Let C be a cosilting module, $\mathcal{T} = {}^{\perp_0}C$ and $\mathbf{t} = \mathcal{T} \cap A\text{-mod}$.*

Then $\alpha(\mathcal{T})$ is closed under products (and therefore it is a bireflective subcategory) if and only if $\widetilde{\alpha}(\mathbf{t})$ is functorially finite in $A\text{-mod}$.

Proof. $\widetilde{\alpha}(\mathbf{t})$ is covariantly finite if and only if $\varinjlim \widetilde{\alpha}(\mathbf{t}) = \alpha(\mathcal{T})$ is a definable subcategory by [26, Section 4.2].

In this case, we can show that it is also contravariantly finite. In fact, $\widetilde{\alpha}(\mathbf{t}) = B\text{-mod}$ for some left artin ring B, finitely generated as $A-$module.

Indeed, assume $\alpha(\mathcal{T})$ is bireflective. Then, by Theorem 1.2.9, there exists a ring epimorphism $A \to B$ with $\alpha(\mathcal{T}) \cong B\text{-Mod}$. Consider a small progenerator of $\alpha(\mathcal{T})$, which we denote again by B. Then B can be written as a direct limit of objects B_i in $\widetilde{\alpha}(\mathbf{t})$, in particular, it is a quotient of $\coprod B_i$. Since B is projective in the subcategory, we have that B is actually a direct summand of $\coprod B_i$. But B is also compact in the category, thus it is a summand of a finite direct sum of finitely generated modules. In particular B is finitely generated.

This shows that $\widetilde{\alpha}(\mathbf{t}) = B\text{-mod}$. □

We can restate the previous results without explicitly mentioning cosilting modules:

Corollary 3.2.26. *Let A be a left artinian ring. For a wide subcategory $\mathcal{W} \in \mathbf{wide}(A)$, fix $\overline{\mathcal{W}} = \varinjlim \mathcal{W}$. Then:*

(i) $\overline{\mathcal{W}} \in \mathbf{Wide}_{\coprod}(A)$ and $\overline{\mathcal{W}} \cap A\text{-mod} = \mathcal{W}$.

(ii) $\mathcal{W}$ is functorially finite if and only if $\overline{\mathcal{W}}$ is bireflective.

Proof. For (i) notice that every $\mathcal{W} \in \mathbf{wide}(A)$ is obtained as $\widetilde{\alpha}(\widetilde{\mathbf{T}}(\mathcal{W}))$ by Theorem 3.1.7, then apply Propositions 3.1.12 and Proposition 3.2.24. For closure under coproducts use Corollary 3.2.15.

Point (ii) now follows immediately from Corollary 3.2.25. □

3.2.4 Coreflective subcategories from cosilting classes

We prove that over a left noetherian ring all the wide subcategories of the form $\alpha(\mathcal{T})$ for a cosilting torsion pair $(\mathcal{T}, \mathcal{F})$ are coreflective.

We need the following theorem of El Bashir:

Theorem 3.2.27 ([33])**.** *Let $\mathcal{C}$ be a full subcategory of some Grothendieck category. Then $\mathcal{C}$ is a covering class if and only if $\mathcal{C}$ is closed under coproducts and directed colimits and there is some set of objects $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{C} = \varinjlim \mathcal{S}$.*

Remark 3.2.28. *Under the assumption of some sufficiently powerful large cardinal axiom (Vopenka's principle) every subcategory closed under coproducts and directed colimits is covering, see [33] for details.*

Theorem 3.2.29. *Let R be a left noetherian ring, $(\mathcal{T}, \mathcal{F})$ a cosilting torsion pair in R-Mod.*

Then $\alpha(\mathcal{T})$ is a wide coreflective subcategory of R-Mod.

Proof. By Proposition 3.2.24, the wide subcategory $\alpha(\mathcal{T})$ is the limit closure of some set, in particular it satisfies the hypotheses of Theorem 3.2.27 and is therefore a covering class.

Therefore $\alpha(\mathcal{T})$ satisfies the dual of the second condition of Proposition 1.2.21 and is therefore coreflective in R-Mod. □

3.2.5 Wide subcategories with coproducts and τ-tilting finiteness

The wide subcategory closed under coproducts $\overline{\mathcal{W}}$ extending a given wide subcategory $\mathcal{W}$ of Λ-mod given in Corollary 3.2.26 is not unique in general.

Recall the following result:

Lemma 3.2.30 ([48, Corollary 3.11]). *Let Λ be an artin algebra and $\mathbf{t}$ be a functorially finite torsion class in Λ-mod, then $\widetilde{\alpha}(\mathbf{t})$ is a functorially finite wide subcategory. If Λ is $\tau-$tilting finite every wide subcategory of Λ-mod is functorially finite.*

Remark 3.2.31. *Given a functorially finite wide subcategory $\mathcal{W}$ of Λ-mod, with Λ an arbitrary artin algebra, it is not always true that $\widetilde{\mathbf{T}}(\mathcal{W})$ is functorially finite, see [11, Example 3.13].*

We present a further categorical characterisation of τ-tilting finiteness:

Proposition 3.2.32. *Let Λ be an artin algebra. The following statements are equivalent:*

(i) *Λ is τ-tilting finite.*

(ii) *For every wide subcategory $\mathcal{W}$ of Λ-mod there exists a unique wide subcategory $\overline{\mathcal{W}}$ of Λ-Mod closed under coproducts with $\mathcal{W} = \overline{\mathcal{W}} \cap \Lambda$-mod.*

(iii) *If $\mathcal{W} \in \mathbf{Wide}_{\coprod}(\Lambda)$, then $\mathcal{W} \cap \Lambda\text{-mod} = 0$ if and only if $\mathcal{W} = 0$.*

If any of the equivalent conditions above holds, then every $\mathcal{W} \in \mathbf{Wide}_{\coprod}(\Lambda)$ is of the form $\varinjlim(\mathcal{W} \cap \Lambda\text{-mod})$ and is a bireflective subcategory.

Proof. " (i) $\Longrightarrow$ (ii) " : For existence, let $\mathcal{W}$ be a wide subcategory of Λ-mod. Consider the cosilting torsion pair $(\mathbf{T}(\mathcal{W}), \mathcal{W}^{\perp_0})$. By Theorem 3.1.7 and Proposition 3.1.12 $\alpha(\mathbf{T}(\mathcal{W})) \cap \Lambda\text{-mod} = \mathcal{W}$ and by Corollary 3.2.15 $\alpha(\mathbf{T}(\mathcal{W}))$ is wide and closed under coproducts.

For uniqueness, let $\overline{\mathcal{W}}$ be a wide subcategory closed under coproducts which restricts to $\mathcal{W}$. Since Λ is τ-tilting finite, every torsionfree class is definable by Proposition 2.2.6, thus the torsion class $\mathbf{T}(\overline{\mathcal{W}})$ generated by $\overline{\mathcal{W}}$ is part of a cosilting torsion pair.

In particular, using Proposition 3.1.9 we obtain that $\overline{\mathcal{W}} = \alpha(\mathbf{T}(\overline{\mathcal{W}}))$, which by Proposition 3.2.24 is equal to the limit closure $\varinjlim \widetilde{\alpha}(\mathbf{T}(\overline{\mathcal{W}}) \cap \Lambda\text{-mod})$. Now by means of Proposition 3.1.12, we have $\varinjlim \widetilde{\alpha}(\mathbf{T}(\overline{\mathcal{W}}) \cap \Lambda\text{-mod}) = \varinjlim[\alpha(\mathbf{T}(\overline{\mathcal{W}})) \cap \Lambda\text{-mod}]$, so that using once again Proposition 3.1.9 we conclude that $\varinjlim[\alpha(\mathbf{T}(\overline{\mathcal{W}})) \cap \Lambda\text{-mod}] = \varinjlim[\overline{\mathcal{W}} \cap \Lambda\text{-mod}] = \varinjlim \mathcal{W}$.

" (ii) $\implies$ (iii) " : Since the zero subcategory has a unique extension to Λ-Mod, (iii) is immediate.

" (iii) $\implies$ (i) " : By (iii) for every cosilting module C, $\beta(\mathrm{Cogen}(C)) \cap \Lambda\text{-mod} = 0$ if and only if $\beta(\mathrm{Cogen}(C)) = 0$. Now recall that a torsion class is locally maximal if there are no finitely presented torsionfree, almost torsion modules with respect to the corresponding torsion pair. Since the torsionfree, almost torsion modules are precisely the simple objects of $\beta(\mathrm{Cogen}(C))$, there is no locally maximal non functorially-finite torsion class in Λ-mod. This means that Λ is τ-tilting finite by Corollary 2.3.10.

For the last part use condition (ii) with Lemma 3.2.30 and Corollary 3.2.26. □

The proposition above can be restated as follows:

Corollary 3.2.33. *Let Λ be an artin algebra. The following statements are equivalent:*

(i) Λ is τ-tilting finite.

(ii) The class of wide subcategories closed under coproducts is a finite set.

Proof. "$(i) \implies (ii)$" : notice that every wide subcategory of Λ-mod is obtained as $\widetilde{\alpha}(\mathbf{t})$ for some torsion class $\mathbf{t}$, by Theorem 3.1.7. Since Λ is $\tau-$tilting finite, this implies that the set of wide subcategories of Λ-mod is finite, thus the class of wide subcategories of Λ-Mod closed under coproducts is also finite by Proposition 3.2.32(ii).

"$(ii) \implies (i)$" : notice that the cardinality of the set of cosilting torsion pairs is bounded by the cardinality, if it exists, of the class of wide subcategories closed under coproducts (by Proposition 3.2.7). Thus finiteness of the second class implies finiteness of the first. Now use Theorem 1.3.28 to conclude. □

As observed earlier, for a $\tau-$tilting finite algebra all wide subcategories closed under coproducts must be bireflective.

Moreover, we have that from Corollary 3.2.26:

Corollary 3.2.34. *Let Λ be an artin algebra, assume every wide subcategory closed under coproducts of Λ-Mod is also closed under products. Then every wide subcategory of Λ-mod is functorially finite.*

However, it is not clear to the author if the following holds:

Conjecture 3.2.35. *An artin algebra Λ is $\tau-$tilting finite if and only if every wide subcategory of Λ-mod is functorially finite.*

This would also imply the following:

Conjecture 3.2.36. *An artin algebra Λ is $\tau-$tilting finite if and only if every wide subcategory closed under coproducts of Λ-Mod is bireflective.*

3.3 Minimal cosilting modules

Recall the definition of a minimal cosilting module given in [5, Definition 4.12, Remark 4.18]:

Definition 3.3.1. A cosilting module C over some ring R is *minimal* if the sequence

$$0 \to C_1 \to C_0 \xrightarrow{g} E(R)$$

with g the $\mathrm{Cogen}(C)$-cover, $C_1 \in \mathrm{Prod}(C)$ satisfies:

(i) $\beta(\mathrm{Cogen}(C)) = \mathrm{Cogen}(C) \cap {}^{\perp_0}C_1$ is a bireflective subcategory of R-Mod

(ii) $\mathrm{Hom}_R(C_0, C_1) = 0$

Minimal cosilting modules are connected with ring epimorphisms. Before we recall the result, we need a further piece of terminology:

Definition 3.3.2. Let R be a ring, $C \in R$-Mod. An injective copresentation $0 \to C \to I_0 \xrightarrow{\omega} I_1$ of the module C is called a *precosilting copresentation* if $\mathrm{Cogen}(C) \subseteq \mathcal{C}_\omega$.

For a module $M \in R$-Mod we denote by $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ its character dual.

Theorem 3.3.3 ([5, Theorem 4.17])**.** *The map assigning a ring epimorphism $\lambda : R \to S$ to the class $\mathrm{Cogen}(S^+)$ yields a bijection between:*

(i) Equivalence classes of ring epimorphisms $\lambda : R \to S$ such that S^+ has a precosilting copresentation.

(ii) Equivalence classes of minimal cosilting R−modules.

3.3.1 Torsion, almost torsionfree modules for minimal cosilting modules

We will prove that, over an artin algebra, all minimal cosilting modules (with the exception of injective cogenerators) admit some torsion, almost torsionfree module.

Lemma 3.3.4. *Let R be a ring. Let C be a minimal cosilting module with approximation sequence $0 \to C_1 \to C_0 \to E(R)$.*

Assume that the module C_0 is cosilting, then $C_1 = 0$.

Proof. By Proposition 1.4.6 we have that $\mathrm{Cogen}(C_0) = \mathrm{Cogen}(C)$, thus the two cosilting modules are equivalent and $\mathrm{Prod}(C) = \mathrm{Prod}(C_0)$. Therefore, $C_1 \in \mathrm{Prod}(C_0)$. However, by assumption, $\mathrm{Prod}(C_0) \subseteq \beta(\mathrm{Cogen}(C))$. But, by Theorem 3.2.1, we have $\beta(\mathrm{Cogen}(C)) \subseteq {}^{\perp_0}C_1$, thus $C_1 = 0$. □

Lemma 3.3.5. *Let R be a ring, let C be a cosilting module. If $\alpha({}^{\perp_0}C) = 0$ then C_0 is cosilting.*

Moreover if C is cotilting, then C_0 is cotilting if and only if $\alpha({}^{\perp_0}C) = 0$.

Proof. We must show that there is some injective copresentation ω of C_0, such that $\mathcal{C}_\omega = \mathrm{Cogen}(C_0) = \mathrm{Cogen}(C)$.

Notice that, for every copresentation ω, the class $\mathcal{C}_\omega$ is closed under submodules.

Moreover, $\mathcal{C}_\omega \subseteq {}^{\perp_1}C_0$: let $M \in \mathcal{C}_\omega$, then M is also in $\mathcal{C}_\pi$ where $\pi : I_0 \to \mathrm{Im}(\omega)$. In particular, applying the $\mathrm{Hom}(M, -)$ functor to the short exact sequence

$$0 \longrightarrow C_0 \longrightarrow I_0 \xrightarrow{\pi} \mathrm{Im}(\omega) \longrightarrow 0$$

we obtain that $\mathrm{Ext}^1(M, C_0) = 0$.

Thus, $\mathcal{C}_\omega \subseteq {}^{\perp_{1h}}C_0 = \mathrm{A}({}^{\perp_0}C) = \alpha({}^{\perp_0}C) \star \mathrm{Cogen}(C) = \mathrm{Cogen}(C_0)$, as $\alpha({}^{\perp_0}C) = 0$ by assumption.

Moreover, by Proposition 1.4.6 C_0 is a summand of a cosilting module equivalent to C, thus by [64, Lemma 4.13], $\mathrm{Cogen}(C) \subseteq \mathcal{C}_\omega$, with ω the minimal injective copresentation of C_0.

Then $\mathrm{Cogen}(C_0) \subseteq \mathcal{C}_\omega \subseteq \mathrm{Cogen}(C_0)$. Thus C_0 is cosilting with respect to ω.

For the cotilting case, notice that $\alpha({}^{\perp_0}C) = {}^{\perp_{0,1}}C_0$, thus $\alpha({}^{\perp_0}C) = 0$ if and only if $\mathrm{Cogen}(C_0) = {}^{\perp_1}C_0$, that is, if and only if C_0 is cotilting. □

Remark 3.3.6. *The dual case, with the condition* $\beta(\mathrm{Cogen}(C)) = 0$*, is not interesting: by the injectivity of* β *this assumption forces* $C_0 = C_1 = 0$ *and trivially they are both cosilting modules.*

Proposition 3.3.7. *Let* Λ *be an artin algebra. Let* C *be a minimal cosilting module, with* $\mathrm{Cogen}(C) \neq \Lambda\text{-}\mathrm{Mod}$.

Then $\alpha({}^{\perp_0}C) \neq 0$ *and the torsion pair has some torsion, almost torsionfree module.*

Proof. Let $0 \to C_1 \to C_0 \to D\Lambda$ be the approximation sequence.

If $\alpha({}^{\perp_0}C) = 0$, then C_0 is a cosilting module and thus $C_1 = 0$. This implies that C_0 is a finitely generated cosilting module, thus C_0 is support τ^{-1}−tilting.

Since $\alpha({}^{\perp_0}C) = 0$ the (functorially finite) torsionfree class cogenerated by C_0 in $\Lambda\text{-}\mathrm{mod}$ does not have any covering torsionfree class (by the dual of Theorem 2.3.2). Thus, we must have $\Lambda\text{-}\mathrm{mod} = \mathrm{cogen}(C_0)$, which contradicts our hypothesis (Theorem 1.3.28, Remark 2.3.4).

Hence $\alpha({}^{\perp_0}C) \neq 0$. But, by Proposition 3.2.24, this implies $\widetilde{\alpha}({}^{\perp_0}C \cap \Lambda\text{-}\mathrm{mod}) \neq 0$. Now, every wide subcategory of $\Lambda\text{-}\mathrm{mod}$ is completely determined by its simple objects, in particular, we can find some simple object $S \in \widetilde{\alpha}({}^{\perp_0}C \cap \Lambda\text{-}\mathrm{mod})$.

By Proposition 2.1.7, S is a torsion, almost torsionfree module for the torsion pair $({}^{\perp_0}C, \mathrm{Cogen}(C))$. □

3.3.2 An application to τ−tilting theory

Proposition 3.3.8. *Let* Λ *be an artin algebra. Then the following statements are equivalent:*

(i) Λ *is* τ*−tilting-finite.*

(ii) Every cosilting module which is not equivalent to a finitely generated one is minimal.

Proof. " $(i) \implies (ii)$ " : By Proposition 2.2.6, every torsion pair in Λ-Mod is the extension of a functorially finite one, this means that every cosilting module is equivalent to a support τ^{-1}−tilting module and all such modules are finitely generated.

" $(ii) \implies (i)$ " : Assume the algebra were τ−tilting infinite. By the dual of Corollary 2.3.10, we have some non functorially-finite locally minimal torsion class $\mathbf{t}$ in Λ-mod. Recall that a torsion class is locally minimal if and only if it does not have any torsion, almost torsionfree module. This is equivalent to the condition $\widetilde{\alpha}(\mathbf{t}) = 0$.

However, the cosilting extension of the corresponding torsion pair $(\mathbf{t}, \mathbf{f})$, satisfies the conditions of Proposition 3.3.7, since every infinitely generated cosilting module is assumed to be minimal and $\varinjlim \mathbf{f} \neq \Lambda$-Mod. Thus $\alpha(\varinjlim \mathbf{t}) = \varinjlim \widetilde{\alpha}(\mathbf{t}) \neq 0$, a contradiction. □

Remark 3.3.9. *From the proof of the Theorem above and of Theorem 2.4.4, we can see that the "pathological" behaviour of the lattice of torsion classes in* Λ-mod *in the* τ−*tilting infinite case is directly connected with pathological behaviour of the corresponding cosilting modules: in one case it ensures the existence of large torsionfree, almost torsion modules, in the other the non-minimality of the cosilting class.*

3.3.3 Minimal cosilting modules over hereditary algebras

In general, it is not easy to understand if a certain cosilting class is cogenerated by a minimal cosilting module, however over an hereditary algebra there is a handy criterion.

First we give a small lemma simplifying some computations:

Lemma 3.3.10. *Let A be a left artinian ring, $\mathcal{W}$ a wide subcategory of* A-mod *whose simple objects have projective dimension less than one. Then:*

$$\beta(\mathcal{W}^{\perp_0}) = \mathcal{W}^{\perp_{0,1}}$$

Proof. Notice that $\mathbf{T}(\mathcal{W}) = \varinjlim \widetilde{\mathbf{T}}(\mathcal{W})$, thus $\mathcal{W} \subseteq \alpha(\mathbf{T}(\mathcal{W}))$ by Proposition 3.1.12.

We can immediately verify that a module M in $\mathcal{W}^{\perp_1}$ is in $\mathrm{B}(\mathcal{W}^{\perp_0})$ once we have noticed that it is enough to check the condition for injective maps $0 \to M \to F$ with $F \in \mathcal{W}^{\perp_0}$ (here we need that $\mathcal{W}^{\perp_1}$ is closed under quotients).

For the other inclusion, take $M \in \beta(\mathcal{W}^{\perp_0}) = \mathcal{W}^{\perp_0} \cap \mathrm{B}(\mathcal{W}^{\perp_0})$. Consider a short exact sequence:

$$0 \to M \to N \to W \to 0$$

with $W \in \mathcal{W}$. Then, taking the torsion part of N we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{t}N & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \overline{N} & \longrightarrow & \widetilde{W} & \longrightarrow & 0
\end{array}$$

then since $M \in \mathrm{B}(\mathcal{W}^{\perp_0})$ and this class is closed under quotients, $L \in \mathrm{B}(\mathcal{W}^{\perp_0})$. Therefore, $\widetilde{W}$ is in $\mathcal{W}^{\perp_0}$ as $\overline{N} \in \mathcal{W}^{\perp_0}$, but $\widetilde{W}$ is a quotient of W, thus it must be zero.

Thus $I = W$, and since $\mathcal{W} \subseteq \alpha(\mathbf{T}(\mathcal{W}))$ we have that $K \in \mathbf{T}(\mathcal{W})$. However, K is also a submodule of M which is in $\mathcal{W}^{\perp_0}$, therefore, $K = 0$ as it is both torsion and torsionfree. This shows that the middle sequence splits. □

Proposition 3.3.11. *Let Λ be an hereditary artin algebra. Then a cosilting module C is equivalent to a minimal one if and only if there exists a wide subcategory $\mathcal{W} \subseteq \Lambda$-mod such that* $\mathrm{Cogen}(C) = \mathcal{W}^{\perp_0}$.

Proof. Assume C is a minimal cosilting module. Then, it arises from a pseudo-flat ring epimorphism, which, since Λ is hereditary, must be homological. By [46, Theorem 6.1] every such epimorphism is a universal localisation. The corresponding bireflective subcategory is $\beta(\mathrm{Cogen}(C))$.

By [60, Theorem 2.3], every universal localisation is obtained as the perpendicular category of a unique wide subcategory $\mathcal{W}$ of Λ-mod and we claim that ${}^{\perp_0}C = \mathbf{T}(\mathcal{W})$.

This follows immediately from the injectivity of β, Proposition 3.2.7, as $\beta(\mathcal{W}^{\perp_0}) = \mathcal{W}^{\perp_{0,1}} = \beta(\mathrm{Cogen}(C))$, by Lemma 3.3.10.

On the other hand, assume we have a torsion pair $(\mathbf{T}(\mathcal{W}), \mathcal{W}^{\perp_0})$. The torsionfree class is definable, therefore, this is a cosilting torsion pair.

Moreover, $\beta(\mathcal{W}^{\perp_0}) = \mathcal{W}^{\perp_{0,1}}$ is a bireflective subcategory. Since we are in the hereditary case, we can obtain a corresponding minimal cosilting module, by [5, Proposition 4.6], with torsionfree class $\mathbf{F}(\beta(\mathrm{Cogen}(C))) = \mathrm{Cogen}(\beta(\mathrm{Cogen}(C)))$.

Applying once again Proposition 3.2.7, we conclude that this new torsion pair is the original one, and thus it is cogenerated by a minimal cosilting module as required. □

Example 3.3.12. *Let $\Lambda = kQ$ be a representation-infinite (hereditary) algebra. Then we always have an easy example of a non-minimal cosilting torsion pair: let $\mathbf{q}$ be a collection of iso-classes of all the preinjective modules in Λ-mod, then $(\mathbf{T}(\mathbf{q}), \mathbf{q}^{\perp_0})$ is a cosilting torsion pair in Λ-Mod which is not widely generated, thus not minimal.*

3.3.4 Torsion pairs and Ext-orthogonal pairs

We give some applications to Ext-orthogonal pairs over an hereditary algebra, following [46].

Definition 3.3.13 ([46, Def. 2.1]). Let R be a ring, then $(\mathcal{X}, \mathcal{Y})$ a pair of full subcategories of R-Mod is said to be an *Ext-orthogonal pair* if:

$$X \in \mathcal{X} \iff \forall n \in \mathbb{Z}\ \mathrm{Ext}^n(X, \mathcal{Y}) = 0$$
$$Y \in \mathcal{Y} \iff \forall n \in \mathbb{Z}\ \mathrm{Ext}^n(\mathcal{X}, Y) = 0$$

An Ext-orthogonal pair is *complete* if for all $M \in R$-Mod we have an exact sequence:

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

The following two results will be used several times in the rest of the section:

Proposition 3.3.14 ([46, Proposition 3.1]). *Let R be an hereditary ring, $f : R \to S$ a homological ring epimorphism. Let $\mathcal{Y} = f^*(S\text{-}\mathrm{Mod})$ and set $\mathcal{X} = {}^{\perp_{0,1}}\mathcal{Y}$, $\mathcal{Z} = \mathcal{Y}^{\perp_{0,1}}$. Then $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are complete Ext-orthogonal pairs in $R\text{-}\mathrm{Mod}$ with $\mathcal{Y} = (\ker f \oplus \operatorname{coker} f)^{\perp_{0,1}}$ and $\mathcal{Z} = S^{\perp_{0,1}}$.*

Proposition 3.3.15 ([46, Theorem 5.1]). *Let R be an hereditary ring and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair in $R\text{-}\mathrm{Mod}$. The following statements are equivalent:*

- *(i)* $\mathcal{Y}$ *is closed under coproducts.*
- *(ii)* $\mathcal{X} = \varinjlim(\mathcal{X} \cap R\text{-}\mathrm{mod})$
- *(iii) There exists a subcategory $\mathcal{C} \subseteq R\text{-}\mathrm{mod}$ such that $\mathcal{C}^{\perp_{0,1}} = \mathcal{Y}$.*

Minimal cosilting modules and Ext-orthogonal pairs

One can show that, over an hereditary ring every complete Ext-orthogonal pair is obtained from a torsion pair:

Proposition 3.3.16. *Let R be an hereditary ring, $(\mathcal{X}, \mathcal{Y})$ a complete Ext-orthogonal pair. Then there is a (uniquely determined) torsion pair $(\mathcal{T}, \mathcal{F})$ in $R\text{-}\mathrm{Mod}$ such that $(\mathcal{X}, \mathcal{Y}) = (\alpha(\mathcal{T}), \beta(\mathcal{F}))$.*

Proof. As noticed in [46], $(\mathcal{X}, \mathcal{Y})$ gives rise to a torsion and a cotorsion pair, from which it can be recovered.

So, let $(\mathbf{T}(\mathcal{X}), \mathcal{X}^{\perp_0})$ be the torsion pair generated by $\mathcal{X}$.

We have that $\mathrm{Cogen}(\mathcal{Y}) \subseteq \mathcal{X}^{\perp_0}$, since the second is a torsionfree class containing $\mathcal{Y}$.

Moreover, if $L \in \mathcal{X}^{\perp_0}$, the approximation sequence of the Ext-orthogonal pair, gives an embedding $L \to Y^L$ with $Y^L \in \mathcal{Y}$. Thus $\mathrm{Cogen}(\mathcal{Y}) = \mathcal{X}^{\perp_0}$. In a similar way we can obtain that $\mathbf{T}(\mathcal{X}) = \mathrm{Gen}(\mathcal{X})$.

Now, being the left part of an Ext-orthogonal pair, $\mathcal{X}$ is a wide subcategory closed under coproducts. Thus, Proposition 3.1.9 gives $\alpha(\mathbf{T}(\mathcal{X})) = \mathcal{X}$.

Moreover, $\mathcal{Y}$ is a wide subcategory closed under products. We prove that $\beta(\mathrm{Cogen}(\mathcal{Y})) = \mathcal{Y}$.

"$\subseteq$" : Let $B \in \beta(\mathrm{Cogen}(\mathcal{Y}))$, then there is some element $Y \in \mathcal{Y}$ and a short exact sequence $0 \to B \to Y \to F \to 0$, with $F \in \mathrm{Cogen}(\mathcal{Y})$. In particular, as F also can be embedded in some Y', B can be realized as the kernel of a map in $\mathcal{Y}$.

"$\supseteq$" : Let $Y \in \mathcal{Y}$ and $Y \to F \to M \to 0$ a short exact sequence with $F \in \mathrm{Cogen}(\mathcal{Y})$. Once again, we can embed F in some $Y' \in \mathcal{Y}$. The cokernel C of the composite $Y \to F \to Y'$ is then a module in $\mathcal{Y}$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} & & Y & \xrightarrow{f} & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(i \circ f) & \longrightarrow & Y' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

we can see that M embeds in C. Thus $M \in \mathrm{Cogen}(\mathcal{Y})$ and $Y \in \beta(\mathrm{Cogen}(\mathcal{Y}))$.

For uniqueness, let $(\mathcal{T}, \mathcal{F})$ be a torsion pair with $(\mathcal{X}, \mathcal{Y}) = (\alpha(\mathcal{T}), \beta(\mathcal{F}))$. Then, obviously $\mathbf{T}(\mathcal{X}) \subseteq \mathcal{T}$ and $\mathrm{Cogen}(\mathcal{Y}) \subseteq \mathcal{F}$. But for a couple of torsion pairs, the inclusion of the torsion classes is equivalent to the reverse containement for the torsionfree class, thus we can conclude. □

Corollary 3.3.17. *Let A be a hereditary left artin ring. $(\mathcal{X}, \mathcal{Y})$ a complete Ext-orthogonal pair. Then the corresponding torsion pair is cosilting if and only if $\mathcal{X} = \varinjlim(\mathcal{X} \cap A\text{-}\mathrm{mod})$.*

All the cosilting torsion pairs obtained in this way are minimal, therefore $\mathcal{Y}$ is a bireflective subcategory of $A\text{-}\mathrm{Mod}$.

Proof. If the torsion pair is cosilting then the wide subcategory $\mathcal{X}$ has the required form by Proposition 3.2.24.

For the other implication, $\mathcal{T} = \mathrm{Gen}(\mathcal{X}) = \mathrm{Gen}(\varinjlim(\mathcal{X} \cap A\text{-}\mathrm{mod})) = \mathrm{Gen}(\mathcal{X} \cap A\text{-}\mathrm{mod}) = \varinjlim \widetilde{\mathbf{T}}(\mathcal{X} \cap A\text{-}\mathrm{mod})$, which is a torsion class in a cosilting pair.

It is minimal, since the torsion class is widely generated (see Proposition 3.3.11). □

The Corollary above ultimately yields the following bijection by means of Theorem 3.3.15.

Proposition 3.3.18. *Let A be a hereditary left artin ring. Then there is a bijection between minimal cosilting torsion pairs and (complete) Ext-orthogonal pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{Y}$ bireflective.*

Proof. It is enough to show that for every minimal cosilting torsion pair $(\mathcal{T}, \mathcal{F})$ the pair $(\alpha(\mathcal{T}), \beta(\mathcal{F}))$ is complete Ext-orthogonal.

Since we are working over an hereditary ring, the minimal cosilting module corresponds to an homological epimorphism $\lambda : A \to B$ and $\beta(\mathcal{F}) = \lambda^*(B\text{-}\mathrm{Mod})$. Thus by Proposition 3.3.14 we have a complete Ext-orthogonal pair $(\mathcal{X}, \beta(\mathcal{F}))$, with $\mathcal{X} = \varinjlim(\mathcal{X} \cap A\text{-}\mathrm{mod})$ (Theorem 3.3.15).

By Corollary 3.3.17 there is some cosilting module C, with $\beta(\mathrm{Cogen}(C)) = \beta(\mathcal{F})$. Since β is injective $\mathrm{Cogen}(C) = \mathcal{F}$ and $\mathcal{X} = \alpha(\mathcal{T})$. □

Minimal silting modules and Ext-orthogonal pairs

Dually, there is a concept of minimal silting modules. Minimal silting modules are defined for general rings, but here we will use the following, more accessible, definition:

Definition 3.3.19 ([7, Definition 5.4])**.** Let R be an hereditary ring. Let T be a silting $R-$module.

Then T is *minimal silting* if R admits an $\mathrm{Add}(T)-$envelope.

For a minimal silting module T, the wide subcategory $\alpha(\mathrm{Gen}(T))$ is bireflective [7, Remark 5.7].

Moreover, we have the following lemma:

Lemma 3.3.20. *Let T be a minimal silting module over an hereditary ring R. Let $R \to T_0 \to T_1 \to 0$ be the exact sequence induced by the* $\operatorname{Add}(T)-$ *envelope.*

Then $\beta(T^{\perp_0}) = T_0^{\perp_{0,1}}$.

Proof. Dualise the arguments in the proof of Theorem 3.2.13. □

Proposition 3.3.21. *Let A be a hereditary ring. Then there is a bijection between minimal silting torsion pairs and (complete) Ext-orthogonal pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{X}$ bireflective.*

Proof. As a preliminary observation, notice that for any minimal silting module T we have a ring epimorphism $\lambda : A \to B$ such that $\lambda^*(B\text{-}\mathrm{Mod}) = \alpha(\mathrm{Gen}(T))$ and $\mathrm{Gen}(B) = \mathrm{Gen}(T)$.

In particular, the induced $A-$module map $A \to {}_AB$ is a $\mathrm{Gen}(B)-$envelope, thus we must have that ${}_AB \simeq T_0$, where T_0 is the $\mathrm{Gen}(T)-$envelope of A.

This shows that the map $T \mapsto (\alpha(\mathrm{Gen}(T)), \beta(T^{\perp_0})) = (\lambda^*(B\text{-}\mathrm{Mod}), B^{\perp_{0,1}})$ assigns a complete Ext-orthogonal pair to each minimal silting module (Lemma 3.3.20 and Proposition 3.3.14).

Moreover this map is surjective : If $(\mathcal{X}, \mathcal{Y})$ is any complete Ext-orthogonal pair with $\mathcal{X}$ bireflective, then $\mathcal{X} = \lambda^*(B\text{-}\mathrm{Mod})$ for some ring epimorphism.

Thus, once again by Proposition 3.3.14, this pair is obtained via the map defined above from the minimal silting torsion pair $(\mathrm{Gen}(B), B^{\perp_0})$.

The map is also injective, as α induces a bijection between equivalence classes of minimal silting modules and epiclasses of homological ring epimorphisms starting at A ([7, Theorem 5.8]). □

Almost complete Ext-orthogonal pairs

Weakening the completeness requirement as follows we can obtain a general bijection:

Definition 3.3.22. Let R be an hereditary ring. An Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ in $R\text{-}\mathrm{Mod}$ is *almost complete* if for all $M \in R\text{-}\mathrm{Mod}$ we have an exact sequence:

$$0 \longrightarrow F_M \longrightarrow X_M \longrightarrow M \longrightarrow Y^M \longrightarrow T^M \longrightarrow 0$$

with $F_M \in \mathcal{X}^{\perp_0}$, $X_M \in \mathcal{X}$, $Y^M \in \mathcal{Y}$ and $T^M \in {}^{\perp_0}\mathcal{Y}$.

Definition 3.3.23. A torsion pair $(\mathcal{T}, \mathcal{F})$ in $R\text{-}\mathrm{Mod}$ is *coherently determined* if $\mathcal{T} = \mathrm{Gen}(\alpha(\mathcal{T}))$ and $\mathcal{F} = \mathrm{Cogen}(\beta(\mathcal{F}))$.

Proposition 3.3.24. *Let R be an hereditary ring. Then there is a bijection between coherently determined torsion pairs and almost complete Ext-orthogonal pairs.*

Proof. From the proof of Proposition 3.3.16 we obtain that the map

$$(\mathcal{X}, \mathcal{Y}) \mapsto (\mathrm{Gen}(\mathcal{X}), \mathrm{Cogen}(\mathcal{Y}))$$

is well-defined and injective.

Moreover, given any coherently determined torsion pair $(\mathcal{T}, \mathcal{F})$ we can see that $(\mathcal{X}, \mathcal{Y}) = (\alpha(\mathcal{T}), \beta(\mathcal{F}))$ is an Ext-orthogonal pair: we obtain that $\mathcal{X}^{\perp_{0,1}} = \mathcal{Y}$ using that $\mathcal{F} = \mathcal{X}^{\perp_0}$ and adopting the strategy used in the proof of Lemma 3.3.10. We can dualise these arguments for $\mathcal{X} = {}^{\perp_{0,1}}\mathcal{Y}$.

Assume we have a torsionfree module M, then there is some $Y \in \mathcal{Y}$ (which is closed under products), such that M embeds in Y. This allows us to build the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & \mathrm{t}_{\mathcal{T}} C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/\mathrm{t}_{\mathcal{T}} C & = & C/\mathrm{t}_{\mathcal{T}} C & &
\end{array}$$

with F the pull-back along the embedding of the torsion part of C with respect to $(\mathcal{T}, \mathcal{F})$.

Now $Y \in \mathcal{Y} = \mathcal{X}^{\perp_{0,1}} \subseteq \mathcal{X}^{\perp_1}$, therefore $C/\mathrm{t}_{\mathcal{T}} C \in \mathcal{X}^{\perp_1}$ as this subcategory is closed under quotients, since the ring is hereditary. This gives $C/\mathrm{t}_{\mathcal{T}} C \in \mathcal{Y}$. Now $\mathcal{Y}$ is a wide subcategory, thus $F \in \mathcal{Y}$, being the kernel of a map in $\mathcal{Y}$.

In conclusion, for every torsionfree module M, we can find a short exact sequence starting at M with the middle term in $\mathcal{Y}$ and the third term in $\mathcal{T} = {}^{\perp_0}\mathcal{Y}$.

With dual arguments we can find, for every torsion module M, a short exact sequence ending at M with first term in $\mathcal{F} = \mathcal{X}^{\perp_0}$ and middle term in $\mathcal{X}$.

For a general module, we can construct the required 5-term exact sequence applying the two special cases to the torsion and torsionfree part of it.

This proves that every such pair $(\alpha(\mathcal{T}), \beta(\mathcal{F}))$ is an almost complete Ext-orthogonal pair.

It is immediate to verify that the assignment $(\mathcal{T}, \mathcal{F}) \mapsto (\alpha(\mathcal{T}), \beta(\mathcal{F}))$ is the inverse of the first map. □

Chapter 4

Computing mutations: approximations and cosilting pairs

The content of this chapter is joint work with Lidia Angeleri-Hügel and Rosanna Laking.

In this chapter we introduce several tools to understand minimal inclusions of torsion classes in Λ - mod in terms of mutation of the corresponding cosilting modules. We will see that this will amount to some operation on the Ziegler spectrum of Λ.

The rigid systems and the cosilting pairs we consider in this chapter were inspired by the theory developed in [1] and [19].

4.1 Rigid systems and cosilting pairs

Let R be a ring. We start recalling some preliminary results on cosilting modules and complexes. This first part of the section is intended as a reference and does not contain any new result.

Lemma 4.1.1 ([6, Proposition 2.5]). *Let σ be a cosilting complex in $\mathrm{D}(R)$. Then the homological functor $H^0_\sigma : \mathrm{D}(R) \to \mathcal{H}_\sigma$ to the heart of the cosilting t-structure induces an equivalence: $H^0_\sigma : \mathrm{Prod}(\sigma) \simeq \mathrm{Inj}(\mathcal{H}_\sigma)$ where $\mathrm{Inj}(\mathcal{H}_\sigma)$ is the category of injective objects in the heart. Moreover we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(R)}(-, \sigma) \simeq \mathrm{Hom}_{\mathcal{H}_\sigma}(H^0_\sigma(-), H^0_\sigma(\sigma))$$

Lemma 4.1.2. *Let M be a pure-injective module with injective copresentation $0 \to M \to I_0 \xrightarrow{\sigma} I_1$.*

Then $\mathcal{C}_\sigma$ is closed under $\varinjlim$.

Proof. Note that $\mathcal{C}_\sigma = {}^{\perp_1}M \cap \mathcal{C}_\varrho$, where $\varrho : \mathrm{Im}(\sigma) \to I_1$. Since M is pure-injective ${}^{\perp_1}M$ is closed under $\varinjlim$ by Lemma 3.2.11, thus it is enough to show that $\mathcal{C}_\varrho$ is closed under $\varinjlim$.

This is true for any monomorphism ϱ, in fact, let $(X_i, \{\varphi_{ij}\})$ be a directed system in $\mathcal{C}_\varrho$. Let $f : \varinjlim X_i = X \to I_1$.

Then for each $i \in I$ we obtain a commutative diagram:

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & X \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle f} \\ \mathrm{Im}(\sigma) & \xrightarrow{\varrho} & I_1 \end{array}$$

Moreover, we have :

$$(\varrho h_j)\varphi_{ij} = f\varphi_j\varphi_{ij} = f\varphi_i = \varrho h_i$$

and since ϱ is a monomorphism, $h_j\varphi_{ij} = h_i$, thus $\{h_i\}$ is compatible with the directed system and it induces, by the universal properties of colimits, a factorisation $h : X \to \mathrm{Im}(\sigma)$.

Now $\varrho h\varphi_i = \varrho h_i = f\varphi_i$, for all $i \in I$, thus, by uniqueness of factorisation $\varrho h = f$ and $X \in \mathcal{C}_\varrho$. □

Corollary 4.1.3. *If $\sigma : I_0 \to I_1$ is a pure-injective 2-term complex of injectives in* $\mathrm{D}(R)$, *then $\mathcal{C}_\sigma \subseteq R\text{-}\mathrm{Mod}$ is closed under* $\varinjlim$.

Moreover, if R is left artinian, then $\mathcal{C}_\sigma$ is a cosilting torsionfree class in $R\text{-}\mathrm{Mod}$.

Proof. By Theorem [55, 17.3.19], the zeroth homology of σ is a pure-injective $R-$module. Thus we can apply Lemma 4.1.2 to $H^0(\sigma)$ and obtain the closure under direct limits.

Assuming R left artinian, it remains to show that $\mathcal{C}_\sigma$ is a torsionfree class, but this is immediate as it coincides with the limit closure of $\mathcal{C}_\sigma \cap R\text{-}\mathrm{mod}$, which is a torsionfree class in $R\text{-}\mathrm{mod}$ as $\mathcal{C}_\sigma$ is closed under submodules and extensions. □

In the artin algebra case we can also choose a "nice" injective copresentation of any given cosilting module:

Proposition 4.1.4. *Let Λ be an artin algebra, $C \in \Lambda\text{-}\mathrm{Mod}$ a cosilting module.*

Then C is cosilting with respect to $\omega = \mu \oplus (0 \to I)$ where μ is the minimal injective copresentation of C and I is the direct sum of a collection of representatives of the isomorphism classes of indecomposable injective modules in $C^{\perp_0}$.

Proof. Let σ be an injective copresentation of C such that $\mathrm{Cogen}(C) = \mathcal{C}_\sigma$.

Then $\sigma \simeq \mu \oplus \nu \oplus (0 \to J)$, with ν an isomorphism and J an injective module. Since $\mathcal{C}_\nu = \Lambda\text{-}\mathrm{Mod}$ and $\mathcal{C}_\sigma = \mathcal{C}_\mu \cap \mathcal{C}_\nu \cap {}^{\perp_0}J$ by Lemma 1.4.1, we may assume that $\nu = 0$.

We claim that $J \in \mathrm{Prod}(I)$, with I as above. In fact, since $\mathrm{Cogen}(C) \subseteq {}^{\perp_0}J$, it must be the case that every indecomposable summand of J occurs as a summand of I.

Thus $\mathcal{C}_\omega \subseteq \mathrm{Cogen}(C)$ (as ${}^{\perp_0}I \subseteq {}^{\perp_0}J$). Moreover, $\mathrm{Cogen}(C) \subseteq {}^{\perp_0}I$ by assumption (since I is a finitely generated module over an artin algebra, ${}^{\perp_0}I$ is definable [29, Example 2.3]) and $\mathrm{Cogen}(C) = \mathcal{C}_\sigma \subseteq \mathcal{C}_\mu$.

We can thus conclude that $\mathcal{C}_\omega = \mathrm{Cogen}(C)$. □

We now recall a couple of results from the literature for the convenience of the reader:

Lemma 4.1.5 ([2, Lemma 3.3]). *Let C be an $R-$module with injective copresentation $0 \to C \to I_0 \xrightarrow{\omega} I_1$, then:*

(i) A module X belongs to $\mathcal{C}_\omega$ if and only if $\mathrm{Hom}_{\mathrm{D}(R)}(X, \omega[1]) = 0$, if and only if $\mathrm{Hom}_{\mathrm{D}(R)}(\sigma, \omega[1]) = 0$ for any injective copresentation $0 \to X \to E_0 \xrightarrow{\sigma} E_1$.

(ii) Assume that ω is a minimal injective copresentation of C. Then $\mathrm{Cogen}(C) \subseteq {}^{\perp_1}C$ if and only if $\mathrm{Hom}_{\mathrm{D}(R)}(\omega^I, \omega[1]) = 0$ for any set I, if and only if $\mathrm{Hom}_{\mathrm{D}(R)}(X, \omega[1]) = 0$ for all $X \in \mathrm{Cogen}(C)$.

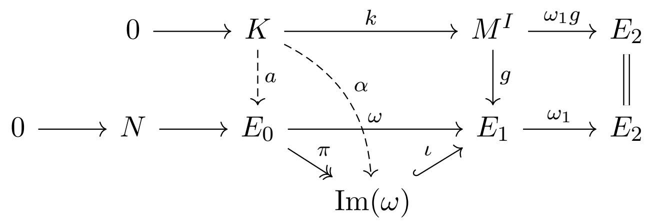
We need the following adaptation of Lemma 4.1.5, the proof is already found in [64, Lemma 4.13]:

Lemma 4.1.6. *Let $M, N \in R\text{-}\mathrm{Mod}$ and let $\omega : E_0 \to E_1$ be a minimal injective copresentation of N. Then: $\mathrm{Cogen}(M) \subseteq {}^{\perp_1}N$ if and only if $M^I \in \mathcal{C}_\omega$ for all sets I.*

Proof. " $\Leftarrow$ ": Recall that $\mathcal{C}_\omega \subseteq {}^{\perp_1}N$ and it is closed under submodules, thus if it contains all the direct products of copies of M, then it must also contain $\mathrm{Cogen}(M)$.

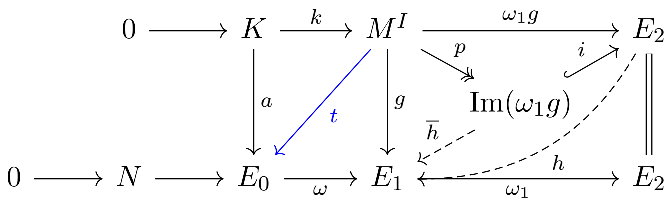
" $\Rightarrow$ ": If we assume that $\mathrm{Cogen}(M) \in {}^{\perp_1}N$, then we can proceed as follows: let $0 \to N \to E_0 \to E_1 \to E_2$ be an injective coresolution of N and assume we have a map $g : M^I \to E_1$. We must find a map $t : M^I \to E_0$, such that $\omega \circ t = f$.

We start with the following commutative diagram:



where K is the kernel of $\omega_1 g$ and the map α is obtained through the universal property of kernels. We obtain a using the hypothesis, in fact since $K \in \mathrm{Cogen}(M)$, we have that $K \in {}^{\perp_1}N$, thus the map $\mathrm{Hom}(K, E_0) \to \mathrm{Hom}(K, \mathrm{Im}(\omega))$ is surjective.

We can then produce the following diagram:



the map $t : M^I \to E_0$ is obtained extending a along k, using that the module E_0 is injective, and satisfies $a = tk$. The map $\overline{h}$ is obtained applying the universal property of the cokernel to the map $\beta = (g - \omega t) : M^I \to E_1$. Then h is the extension of $\overline{h}$ along the embedding $i : \mathrm{Im}(\omega_1 g) \to E_2$, using the injectivity of E_1.

Now, we have that $h\omega_1\beta = h\omega_1(g - \omega t) = h\omega_1 g = h(ip) = \overline{h}p = \beta$. However, $\mathrm{Im}(\iota) \cap \mathrm{Im}(\beta) = 0$, in fact if $e \in \mathrm{Im}(\iota) \cap \mathrm{Im}(\beta)$, then $e = \beta(y) = h\omega_1\beta(y) = h\omega_1\iota(x) = 0$ as $\iota(x)$ is in the kernel of ω_1.

By assumption, ω is a minimal injective copresentation, thus the inclusion $\iota : \mathrm{Im}(\omega) \to E_1$ is essential, therefore $\mathrm{Im}(\beta) = 0$ and $g = \omega t$. □

Definition 4.1.7. An $R-$module C is *precosilting* if there is an injective copresentation ω of C such that $C \in \mathcal{C}_\omega$ and $\mathcal{C}_\omega$ is a torsionfree class.

Proposition 4.1.8 ([64, Proposition 3.10]). *Every precosilting module M with respect to an injective copresentation σ is a direct summand of a cosilting module $\overline{M} = M \oplus N$ such that* $\mathrm{Cogen}(\overline{M}) = \mathcal{C}_\sigma$.

We conclude with an obvious observation:

Lemma 4.1.9. *Let C be a precosilting module. Assume C' is a cosilting module with* $\mathrm{Prod}(C) = \mathrm{Prod}(C')$*, then C is cosilting.*

Proof. In this setting, $\mathrm{Cogen}(C) = \mathrm{Cogen}(C')$ is a torsionfree class and C is cotilting over $R/\mathrm{Ann}(C) = R/\mathrm{Ann}(C')$. Thus it is cosilting by Theorem 1.4.5. □

4.1.1 Rigid systems and cosilting objects

We introduce the concept of a rigid system in the derived category of a ring and show that over a left artinian ring A such systems are in bijection with 2-term cosilting complexes, or equivalently, cosilting modules.

Definition 4.1.10. Let $\mathcal{N} = \{\sigma_i\}_{i\in I}$ be a set of 2-term complexes of injectives, such that each σ_i is an indecomposable pure-injective object in $\mathrm{D}(R)$.

Then $\mathcal{N}$ is called a *rigid system* if:

$$\mathrm{Hom}_{\mathrm{D}(R)}(\sigma_i, \sigma_j[1]) = 0, \text{ for all } i, j \in I$$

A rigid system $\mathcal{N}$ is *maximal* if for every rigid system $\mathcal{L}$ with $\mathrm{Prod}(\mathcal{N}) \subseteq \mathrm{Prod}(\mathcal{L})$ we have $\mathrm{Prod}(\mathcal{N}) = \mathrm{Prod}(\mathcal{L})$.

Two rigid systems $\mathcal{L}, \mathcal{N}$ are *equivalent* if $\mathrm{Prod}(\mathcal{L}) = \mathrm{Prod}(\mathcal{N})$.

We will use the following notation

Notation 4.1.11. *Let σ be a 2-term cosilting complex in* $\mathrm{D}(R)$*. We denote by $\mathcal{N}_\sigma$ the set of indecomposable objects in* $\mathrm{Prod}(\sigma)$*.*

Moreover, consider $\{S_i\}_{i\in I}$ the set of simple objects in $\mathcal{H}_\sigma$.

By Lemma 4.1.1 , each of their injective envelopes $E(S_i)$ in the heart, corresponds to an element of $\mathcal{N}_\sigma$ which we denote by σ_{S_i}. We fix $\widetilde{\sigma} = \prod_{i\in I} \sigma_{S_i}$.

We proceed with some small observations:

Lemma 4.1.12. *Let $\sigma : I_0 \to I_1$ be a 2-term cosilting complex in* $\mathrm{D}(R)$ *and* $\mathcal{H}$ *the heart of the corresponding t-structure* $({}^{\perp_{\leq 0}}\sigma, {}^{\perp_{>0}}\sigma)$. *Then:*

$$\mathrm{Prod}(\sigma) = \mathrm{Prod}(\widetilde{\sigma}) = \mathrm{Prod}(\mathcal{N}_\sigma)$$

Proof. Recall that the heart $\mathcal{H}$ is a locally finitely presented Grothendieck category, by Theorem 1.5.12. In particular, every object X has a (non-zero) finitely generated subobject Y. This can be used to show that $\prod_{i\in I} E(S_i)$ is an injective cogenerator of the heart: Y has some simple quotient S thus the induced map $Y \to E(S)$ can be extended to a non-zero map $f : X \to \prod_{i\in I} E(S_i)$.

Now we can obtain the equality using the equivalence in Lemma 4.1.1: we have $H^0_\sigma(\mathrm{Prod}(\sigma)) = \mathrm{Inj}(\mathcal{H}) = \mathrm{Prod}_{\mathcal{H}}(\prod_i E(S_i)) = \mathrm{Prod}_{\mathcal{H}}(\prod_i H^0_\sigma(\sigma_{S_i})) = H^0_\sigma(\mathrm{Prod}(\widetilde{\sigma}))$, thus:

$$\mathrm{Prod}(\sigma) = \mathrm{Prod}(\widetilde{\sigma}) \subseteq \mathrm{Prod}(\mathcal{N}_\sigma) \subseteq \mathrm{Prod}(\sigma)$$

□

Lemma 4.1.13. *Let A be a left artinian ring and let $\mathcal{N}$ be a rigid system in* $\mathrm{D}(A)$. *Then for all $\sigma_1, \sigma_2 \in \mathrm{Prod}(\mathcal{N})$ we have* $\mathrm{Hom}_{\mathrm{D}(A)}(\sigma_1, \sigma_2[1]) = 0$.

Proof. As $\mathrm{Hom}_{\mathrm{D}(A)}(X, \prod_i Y_i) \simeq \prod_i \mathrm{Hom}_{\mathrm{D}(A)}(X, Y_i)$, it is enough to show that for $\sigma \in \mathrm{Prod}(\mathcal{N})$ and $\omega \in \mathcal{N}$ $\mathrm{Hom}_{\mathrm{D}(A)}(\sigma, \omega[1]) = 0$.

As every element of $\mathrm{Prod}(\mathcal{N})$ is a 2-term complex of injectives, by Lemma 4.1.5 we have that $\mathrm{Hom}_{\mathrm{D}(A)}(\sigma, \omega[1]) = 0$ if and only if $H^0(\sigma) \in \mathcal{C}_\omega$.

Since ω is pure-injective, by Corollary 4.1.3 we have that $\mathcal{C}_\omega$ is closed under products.

Assume σ is a direct summand of $\prod_j \omega_j$ with $\omega_j \in \mathcal{N}$. Then $H^0(\omega_j) = \prod_j H^0(\omega_j) \in \mathcal{C}_\omega$, as $\mathrm{Hom}_{\mathrm{D}(A)}(\omega_j, \omega[1]) = 0$ by hypothesis and consequently $\sigma \in \mathcal{C}_\omega$. □

Proposition 4.1.14. *Let A be a left artinian ring and σ a 2-term cosilting complex in* $\mathrm{D}(A)$. *Then $\mathcal{N}_\sigma$ is a maximal rigid system in* $\mathrm{D}(A)$.

Proof. Since every 2-term cosilting complex in $\mathrm{D}(A)$ is pure-injective, for instance by Theorem 1.5.7, every element of $\mathcal{N}_\sigma$ is automatically pure-injective.

Since $\mathrm{Hom}_{\mathrm{D}(A)}(\sigma^I, \sigma[1]) = 0$ for all I, we have $\mathrm{Hom}_{\mathrm{D}(A)}(N_1, N_2[1]) = 0$ for all $N_1, N_2 \in \mathcal{N}_\sigma$ showing that $\mathcal{N}_\sigma$ is a rigid system.

It remains to show maximality. Let $\mathcal{L}$ be a rigid system, with $\mathrm{Prod}(\mathcal{N}_\sigma) \subseteq \mathrm{Prod}(\mathcal{L})$.

Let $L \in \mathcal{L}$. Since $\widetilde{\sigma} \in \mathrm{Prod}(\mathcal{N}_\sigma)$, we have that $\mathrm{Hom}_{\mathrm{D}(A)}(L, \widetilde{\sigma}[1]) = 0$ by Lemma 4.1.13, and since these are 2-term complexes of injectives, $L \in {}^{\perp_{>0}}\widetilde{\sigma} = {}^{\perp_{>0}}\sigma$.

Since $L \in {}^{\perp_{>0}}\sigma$, by means of [64, Lemma 2.12], we can find the two following triangles in $\mathrm{D}(A)$:

$$L = L_0 \longrightarrow T_0 \longrightarrow L_1 \longrightarrow L_0[1]$$

$$L_1 \longrightarrow T_1 \longrightarrow L_2 \longrightarrow L_1[1]$$

with $T_0, T_1 \in \mathrm{Prod}(\sigma) = \mathrm{Prod}(\mathcal{N}_\sigma)$ and $L_2 \in \mathcal{D}^{\geq 0}(A)$.

Applying $\mathrm{Hom}_{\mathrm{D}(A)}(-, L)$ to the second triangle we obtain the exact sequence

$$\mathrm{Hom}_{\mathrm{D}(A)}(T_1, L[1]) \to \mathrm{Hom}_{\mathrm{D}(A)}(L_1, L[1]) \to \mathrm{Hom}_{\mathrm{D}(A)}(L_2, L[2])$$

where $\mathrm{Hom}_{\mathrm{D}(A)}(T_1, L[1]) = 0$ since $T_1 \in \mathrm{Prod}(\mathcal{N}_\sigma) \subseteq \mathrm{Prod}(\mathcal{L})$ and $\mathrm{Hom}_{\mathrm{D}(A)}(L_2, L[2]) = \mathrm{Hom}_{\mathrm{K}^b(\mathrm{Inj}(A))}(L_2, L[2]) = 0$ since L is 2-term.

It follows that $\mathrm{Hom}_{\mathrm{D}(A)}(L_1, L[1]) = 0$ so that the first triangle must split and L is a summand of $T_0 \in \mathrm{Prod}(\mathcal{N}_\sigma)$, thus $\mathcal{L} \subseteq \mathrm{Prod}(\mathcal{N}_\sigma)$. □

Lemma 4.1.15. *Let $\mathcal{N}$ be a rigid system in $\mathrm{D}(A)$, $\sigma_\mathcal{N} = \prod_{\nu \in \mathcal{N}} \nu$, then $\sigma_\mathcal{N}$ is a direct summand of a 2-term cosilting complex γ.*

Moreover, if $\mathcal{N}$ is maximal the complex $\sigma_\mathcal{N}$ is cosilting.

Proof. By Corollary 4.1.3, the class $\mathcal{C}_{\sigma_\mathcal{N}}$ is a torsionfree class in A-Mod and $N := H^0(\sigma_\mathcal{N}) \in \mathcal{C}_{\sigma_\mathcal{N}}$ by Lemma 4.1.13. Thus N is a precosilting module with respect to $\sigma_\mathcal{N}$.

In particular, we can apply Proposition 4.1.8 to find a module M with injective copresentation ϱ, such that $N \oplus M$ is cosilting with respect to $\gamma = \sigma_\mathcal{N} \oplus \varrho$ and with cosilting class $\mathcal{C}_{\sigma_\mathcal{N}}$. In particular, γ is a cosilting complex.

Assuming $\mathcal{N}$ maximal, we have:

$$\mathrm{Prod}(\mathcal{N}) \subseteq \mathrm{Prod}(\mathcal{N}_\gamma) \implies \mathrm{Prod}(\mathcal{N}) = \mathrm{Prod}(\mathcal{N}_\gamma)$$

thus:

$$\mathrm{Prod}(\sigma_\mathcal{N}) = \mathrm{Prod}(\mathcal{N}) = \mathrm{Prod}(\mathcal{N}_\gamma) = \mathrm{Prod}(\gamma)$$

and $\sigma_\mathcal{N}$ is a cosilting complex equivalent to γ. □

Theorem 4.1.16. *Let A be a left artinian ring. The assignments*

$$\sigma \mapsto \mathcal{N}_\sigma$$
$$\mathcal{N} \mapsto \sigma_\mathcal{N} := \prod_{\omega \in \mathcal{N}} \omega$$

are mutually inverse bijections between:

- *Equivalence classes of 2-term cosilting complexes in $\mathrm{D}(A)$*
- *Equivalence classes of maximal rigid systems in $\mathrm{D}(A)$*

Proof. The maps are well-defined by Proposition 4.1.14 and Lemma 4.1.15.

Verifying that they are mutually inverse is an easy computation. □

4.1.2 Cosilting pairs in module categories

In the last section we considered rigid systems in the derived category. Here we propose an analogous notion at the level of the module category. Ideally, this will be a large version of the concept of support τ^{-1}–tilting pair from [1].

Throughout this section, Λ will be an artin algebra. We will use some properties of the Ziegler spectrum of Λ, see Section 1.3.2 for a brief overview.

Notation 4.1.17. *We denote by* μ_X *a minimal injective copresentation of the module* X*. Denote by* $\operatorname{inj}(\Lambda)$ *the collection of some chosen representatives of all the isoclasses of indecomposable injective modules in* $\Lambda\text{-}\mathrm{Mod}$.

To make the statements more readable, we will often consider, with an abuse of notation, points of the Ziegler spectrum as modules, instead of isomorphism classes.

Definition 4.1.18. Let $\mathcal{Z} \subseteq {}_\Lambda\mathrm{Zg}$ be a closed subset, $\mathcal{I} \subseteq \operatorname{inj}(\Lambda)$. Then we say that $(\mathcal{Z}, \mathcal{I})$ is a *rigid pair* if:

(i) For all $X, Y \in \mathcal{Z}$, we have $X \in \mathcal{C}_{\mu_Y}$.

(ii) For all $X \in \mathcal{Z}, I \in \mathcal{I}$, $\operatorname{Hom}_\Lambda(X, I) = 0$.

If moreover, for all the rigid pairs $(\mathcal{Z}', \mathcal{I}')$ with $\mathcal{Z} \subseteq \mathcal{Z}'$ and $\mathcal{I} \subseteq \mathcal{I}'$ we have $(\mathcal{Z}, \mathcal{I}) = (\mathcal{Z}', \mathcal{I}')$, then we say that $(\mathcal{Z}, \mathcal{I})$ is a *cosilting pair*.

Remark 4.1.19. *Every finitely generated indecomposable* $\Lambda-$*module is both open and closed in* ${}_\Lambda\mathrm{Zg}$*. For the first, see [55, 5.3.37]; for the second, notice that any such module is endofinite, [55, Remark 4.5.34], thus we can apply [55, Theorem 5.1.12].*

Moreover, for any $X, Y \in \Lambda\text{-}\mathrm{mod}$, $\operatorname{Ext}^1_\Lambda(\operatorname{cogen}(X), Y) = 0$ *if and only if* $\operatorname{Hom}_\Lambda(\tau^{-1}Y, X) = 0$ *by [16, Proposition 5.6]; thus, using Lemma 4.1.6 we have that* $\operatorname{Hom}_\Lambda(\tau^{-1}Y, X) = 0$ *if and only if* $X \in \mathcal{C}_{\mu_Y}$.

It follows that the cosilting pairs with $\prod_{Z \in \mathcal{Z}} Z$ *finitely generated are precisely the support* $\tau^{-1}-$*tilting pairs.*

Before stating the main result, we need some preliminary work:

Definition 4.1.20. A pure-injective module E is an *elementary cogenerator* if the subcategory $\mathrm{Cogen}_*(E)$ is definable or, equivalently, if $\langle E \rangle = \mathrm{Cogen}_*(E)$.

Remark 4.1.21. *Notice that if* E *is an elementary cogenerator in* $R\text{-}\mathrm{Mod}$*, then* $\langle E \rangle \cap R\text{-}\mathrm{pinj}$ *is precisely* $\mathrm{Prod}(E) \cap R\text{-}\mathrm{pinj}$.

In fact, since E *is an elementary cogenerator* $\mathrm{Cogen}_*(E) = \langle E \rangle$*. If* I *is any indecomposable pure-injective module in* $\mathrm{Cogen}_*(E)$ *it must be in* $\mathrm{Prod}(E)$*, as any pure monomorphism starting at* I *splits, thus* $\mathrm{Prod}(E) \cap R\text{-}\mathrm{pinj}$ *is a closed set and coincides with* $\langle E \rangle \cap R\text{-}\mathrm{pinj}$

Proposition 4.1.22 ([47, Theorem 5.12, Remark 5.13])**.** *Let* R *be a left noetherian ring,* $C \in R\text{-}\mathrm{Mod}$ *a cosilting module. Then* C *is an elementary cogenerator.*

Proof. Since for any ideal I the subcategory $R/I\text{-}\mathrm{Mod}$ is definable in $R\text{-}\mathrm{Mod}$ and every cosilting module is cotilting in a subcategory of this form, it is enough to show that every cotilting module is an elementary cogenerator.

By means of the results in [47] this is equivalent to require that the heart of the corresponding t-structure in $\mathrm{D}(R)$ is locally coherent.

So we conclude using Theorem 1.5.11. □

We also need the following lemma (see Theorem 1.3.12 for the context):

Lemma 4.1.23 ([19, Lemma 1.12]). *Let R be a left noetherian ring and C a cosilting $R-$module. Then there exists a direct summand D of C such that D has no superdecomposable part and is a cosilting module equivalent to C.*

In particular, $\operatorname{Prod}(C) = \operatorname{Prod}(\prod_{Z\in\mathcal{Z}} Z)$ *where* $\mathcal{Z} = \operatorname{Prod}(C) \cap R\text{-}\operatorname{pinj}$.

Proof. Let $\overline{R} = R/\operatorname{Ann}(C)$ and consider C as an $\overline{R}-$cotilting module. Then using [19, Lemma 1.12] we obtain an $\overline{R}-$cotilting module D equivalent to C without a superdecomposable part.

Now $\operatorname{Cogen}_R(D) = \operatorname{Cogen}_{\overline{R}}(D) = \operatorname{Cogen}_{\overline{R}}(C) = \operatorname{Cogen}_R(C)$ are all torsionfree classes in $R\text{-}\operatorname{Mod}$.

Thus using Theorem 1.4.5, D is a cosilting $R-$module equivalent to C without a superdecomposable part.

For the last statement, notice that $D \simeq \operatorname{PE}(\coprod D_i)$ with each D_i being an indecomposable pure-injective summand of D by Theorem 1.3.12. Thus we have $\operatorname{Prod}(C) = \operatorname{Prod}(D) \supseteq \operatorname{Prod}(\operatorname{Prod}(D) \cap R\text{-}\operatorname{pinj}) \supseteq \operatorname{Prod}(\prod_i D_i) \supseteq \operatorname{Prod}(\operatorname{PE}(\coprod D_i)) = \operatorname{Prod}(D)$ as $\operatorname{PE}(\coprod D_i)$ is a direct summand of $\prod_i D_i$. □

Lemma 4.1.24. *Let Λ be an artin algebra, $C \in \Lambda\text{-}\operatorname{Mod}$ a cosilting module. Let $\mathcal{I}_C := C^{\perp_0} \cap \operatorname{inj}(\Lambda)$ and $\mathcal{Z}_C := \operatorname{Prod}(C) \cap \Lambda\text{-}\operatorname{pinj}$.*

Then $(\mathcal{Z}_C, \mathcal{I}_C)$ is a cosilting pair in $\Lambda\text{-}\operatorname{Mod}$.

Proof. By Proposition 4.1.22, C is an elementary cogenerator, so $\mathcal{Z}_C$ is a closed subset of the Ziegler spectrum.

Moreover, for every $Z \in \mathcal{Z}_C$, we have $Z \in {}^{\perp_0}\mathcal{I}_C$ by definition and for every pair $X, Y \in \mathcal{Z}_C$, we have $X \in \mathcal{C}_{\mu_Y}$ as there is a cosilting module C^I such that X and Y are both summands of C^I and μ_Y is a summand of every injective copresentation of C^I.

This shows that $(\mathcal{Z}_C, \mathcal{I}_C)$ is a rigid pair.

Now we show maximality: let $(\mathcal{Z}', \mathcal{I}')$ be a rigid pair containing $(\mathcal{Z}_C, \mathcal{I}_C)$.

By Lemma 4.1.23 there is a cosilting module equivalent to C with no superdecomposable part, that is the class $\operatorname{Cogen}(C)$ is completely determined by the modules in $\mathcal{Z}_C$, namely we can chose a cosilting copresentation σ of the product of the modules in $\mathcal{Z}_C$ such that $\mathcal{C}_\sigma = \operatorname{Cogen}(C)$. Thus, any $Z \in \mathcal{Z}'$ must be an Ext-injective module in $\operatorname{Cogen}(C)$, so that $Z \in \mathcal{Z}_C$.

In the same way, using that all the indecomposable injective modules are finitely generated and that the class ${}^{\perp_0}M$ is closed under products for every such module over an artin algebra, we obtain that $\mathcal{I}_C = C^{\perp_0} \cap \operatorname{inj}(\Lambda) = (\mathcal{Z}_C)^{\perp_0} \cap \operatorname{inj}(\Lambda) \supseteq (\mathcal{Z}')^{\perp_0} \cap \operatorname{inj}(\Lambda) \supseteq \mathcal{I}'$, so that $\mathcal{I}_C = \mathcal{I}'$. □

Remark 4.1.25. *In Proposition 4.1.4 we have shown that over an artin algebra, for every cosilting module C, we have a nice injective copresentation $\mu_C \oplus (0 \to I_C)$ which we can use to describe the cosilting class. However, if $C = \prod C_i$, with the C_i indecomposable pure-injective modules, it might be more convenient to consider a different copresentation, namely the one of the form $(\prod \mu_{C_i}) \oplus (0 \to I_C)$.*

By minimality μ_C is a direct summand of $\sigma = \prod \mu_{C_i}$, thus $\mathcal{C}_{\mu_C} \supseteq \mathcal{C}_\sigma$. However, notice that each μ_{C_i} is a direct summand of μ_C, as we can write $C = C_i \oplus (\prod_{j \neq i} C_j)$, thus $\mathcal{C}_{\mu_{C_i}} \supseteq \mathcal{C}_{\mu_C}$. Therefore, $\mathcal{C}_{\mu_C} = \mathcal{C}_\sigma$.

Lemma 4.1.26. *Let Λ be an artin algebra, then for every cosilting pair $(\mathcal{Z}, \mathcal{I})$ in $\Lambda\text{-}\mathrm{Mod}$ the module*

$$C_{(\mathcal{Z},\mathcal{I})} := \prod_{Z \in \mathcal{Z}} Z$$

is cosilting with respect to the injective copresentation

$$\sigma_{(\mathcal{Z},\mathcal{I})} := (\prod_{Z \in \mathcal{Z}} \mu_Z) \oplus (0 \to \prod_{I \in \mathcal{I}} I)$$

More explicitely, $\mathrm{Cogen}(\mathcal{Z}) = \mathcal{C}_{\sigma_{(\mathcal{Z},\mathcal{I})}}$

Proof. Since $C_{(\mathcal{Z},\mathcal{I})}$ is pure-injective and Λ is an artin algebra we can apply Lemma 4.1.2 and Corollary 4.1.3 to obtain that $\mathcal{C}_{\sigma_{(\mathcal{Z},\mathcal{I})}} \subseteq \Lambda\text{-}\mathrm{Mod}$ is a cosilting class.

Moreover, $C_{(\mathcal{Z},\mathcal{I})} \in \mathcal{C}_{\sigma_{(\mathcal{Z},\mathcal{I})}}$ by the definition of rigid pair, thus $C_{(\mathcal{Z},\mathcal{I})}$ is precosilting.

Thus, by means of Proposition 4.1.8, we can find a module C', such that $C = C_{(\mathcal{Z},\mathcal{I})} \oplus C'$ is cosilting with cosilting class $\mathcal{C}_{\sigma_{(\mathcal{Z},\mathcal{I})}}$.

By Proposition 4.1.22, C is an elementary cogenerator, thus the indecomposable pure-injective modules in $\mathrm{Prod}(C)$ form a closed subset $\mathcal{Z}'$ of the Ziegler spectrum. Moreover, by construction $\mathrm{Hom}(C, \mathcal{I}) = 0$ thus $(\mathcal{Z}', \mathcal{I})$ is a rigid pair.

However since $\mathcal{Z} \subseteq \mathcal{Z}'$ and $(\mathcal{Z}, \mathcal{I})$ is a cosilting pair we have $\mathcal{Z} = \mathcal{Z}'$. However, by Lemma 4.1.23, $\mathrm{Prod}(C) = \mathrm{Prod}(\prod_{Z \in \mathcal{Z}'} Z)$, so that $\mathrm{Prod}(C_{(\mathcal{Z},\mathcal{I})}) = \mathrm{Prod}(C)$ and therefore $C_{(\mathcal{Z},\mathcal{I})}$ is a cosilting module equivalent to C by Lemma 4.1.9. □

Theorem 4.1.27. *Let Λ be an artin algebra. The assignments*

$$C \mapsto (\mathcal{Z}_C, \mathcal{I}_C)$$
$$(\mathcal{Z}, \mathcal{I}) \mapsto C_{(\mathcal{Z},\mathcal{I})}$$

are mutually inverse bijections between:

- *Equivalence classes of cosilting modules in* $\Lambda\text{-}\mathrm{Mod}$
- *Cosilting pairs in* $\Lambda\text{-}\mathrm{Mod}$

Proof. We proved in Lemmas 4.1.26 and 4.1.24 that the assignments are well defined.

Let $(\mathcal{Z}, \mathcal{I})$ be a cosilting pair, with associated cosilting module $C_{(\mathcal{Z},\mathcal{I})} = \prod_{Z \in \mathcal{Z}} Z$, then $\mathrm{Prod}(C_{(\mathcal{Z},\mathcal{I})}) \cap \Lambda\text{-}\mathrm{pinj}$ is equal to the closed set $\mathcal{Z}$. The non-trivial inclusion is $\mathrm{Prod}(C_{(\mathcal{Z},\mathcal{I})}) \cap \Lambda\text{-}\mathrm{pinj} \subseteq \mathcal{Z}$. As $\mathcal{Z}$ is a closed set, by Theorem 1.3.15 there is a unique definable subcategory $\mathcal{D}$ of $\Lambda\text{-}\mathrm{Mod}$ such that $\mathcal{Z} = \mathcal{D} \cap \Lambda\text{-}\mathrm{pinj}$.

However, let $M \in \mathrm{Prod}(C_{(\mathcal{Z},\mathcal{I})}) \cap \Lambda\text{-}\mathrm{pinj}$, then M is a direct summand of $(\prod_{Z \in \mathcal{Z}} Z)^I \in \mathcal{D}$, thus $M \in \mathcal{D} \cap \Lambda\text{-}\mathrm{pinj} = \mathcal{Z}$.

Conversely, any cosilting module C is determined, up to equivalence, by the pure-injective indecomposable modules in $\mathrm{Prod}(C)$ by means of Lemma 4.1.23, thus the cosilting class $\mathcal{C}_{\sigma_{(\mathcal{Z}_C, \mathcal{I}_C)}} = \mathrm{Cogen}(C)$ i.e. C is equivalent to $C_{(\mathcal{Z}_C, \mathcal{I}_C)}$. □

Summarising what we discussed in this first section we have:

Corollary 4.1.28. *Let Λ be an artin algebra, then we have bijections between:*

(i) The set $\mathbf{tors}(\Lambda)$ *of torsion classes in* $\Lambda\text{-}\mathrm{mod}$.

(ii) The set of equivalence classes of maximal rigid systems in $\mathrm{D}(\Lambda)$.

(iii) The set of cosilting pairs in $\Lambda\text{-}\mathrm{Mod}$.

4.2 Mutation of cosilting pairs

In this section we will discuss a notion of mutation for cosilting pairs. Recall the following terminology:

Definition 4.2.1. We say that B is a *characteristic brick* for the torsion pair $(\mathcal{T}, \mathcal{F})$ if B is torsion, almost torsionfree or torsionfree, almost torsion with respect to $(\mathcal{T}, \mathcal{F})$.

In the artin algebra setting, the mutation of cosilting modules developed in [6] is deeply connected with the operation of exchanging characteristic bricks between torsion pairs.

Thus, before discussing the idea of mutation for cosilting pairs, we need a better understanding of characteristic bricks for cosilting torsion pairs.

4.2.1 Characteristic bricks for cosilting torsion pairs

Let R be left noetherian.

For every cotilting module C, we know that $\mathrm{Gen}(\mathrm{Cogen}(C)) = R\text{-}\mathrm{Mod}$, as the projective generator R is always contained in ${}^{\perp_1}C = \mathrm{Cogen}(C)$. In particular, every module admits a surjective $\mathrm{Cogen}(C)-$cover. This needs not be the case for cosilting modules. We will discuss specifically the case of torsion, almost torsionfree modules without a surjective cover.

Fix $\overline{R} = R/\operatorname{Ann}(C)$ for a cosilting module C. We begin with a lemma that identifies when a R-module is contained within $\overline{R}\text{-}\mathrm{Mod}$.

Lemma 4.2.2. *Let M be a R-module. Then M has a surjective* $\mathrm{Cogen}(C) = \mathcal{F}$*-cover in* $R\text{-}\mathrm{Mod}$ *if and only if M is contained in* $\overline{R}\text{-}\mathrm{Mod}$.

Proof. First suppose that M has a surjective $\mathcal{F}$-cover. Then there exists a short exact sequence

$$0 \to K_M \to F_M \xrightarrow{f} M \to 0$$

such that f is an $\mathcal{F}$-cover of M. It follows that M is contained in $\overline{R}\text{-}\mathrm{Mod}$, as $F_M \in \overline{R}\text{-}\mathrm{Mod}$ and this category is closed under quotients.

Now assume that M is contained in $\overline{R}\text{-}\mathrm{Mod}$. Then, since $\mathcal{F}$ is a cotilting torsion-free class, it is generating in $\overline{R}\text{-}\mathrm{Mod}$ and hence every module in $\overline{R}\text{-}\mathrm{Mod}$ has a surjective $\mathcal{F}$-cover. □

Given any cosilting torsion pair $(\mathcal{T}, \mathcal{F})$ in R-Mod we have a corresponding cotilting torsion pair $(\mathcal{T} \cap \overline{R}\text{-Mod}, \mathcal{F})$ in $\overline{R}$-Mod. In the next proposition we discuss the relation between the corresponding characteristic bricks:

Proposition 4.2.3. *Let C be a cosilting module in R-Mod with torsion pair $(\mathcal{T}, \mathcal{F})$. Denote by $(\mathcal{T}', \mathcal{F}')$ the corresponding cotilting torsion pair in $\overline{R}$-Mod. Then the following statements hold for a module B :*

(1) B is torsionfree, almost torsion with respect to $(\mathcal{T}, \mathcal{F})$ if and only if it is torsionfree, almost torsion with respect to $(\mathcal{T}', \mathcal{F}')$

(2) B is torsion, almost torsionfree with respect to $(\mathcal{T}', \mathcal{F}')$ if and only if it is torsion, almost torsionfree with respect to $(\mathcal{T}, \mathcal{F})$ and it admits a surjective $\mathcal{F}-$cover.

Proof. The first statement is immediate, as all the conditions which must be checked can be verified in the torsionfree class, which is contained in $\overline{R}$-Mod.

For the second point, notice that by Lemma 4.2.2 B admits a surjective cover if and only if it is an element of $\overline{R}$-Mod.

It is clear that, in this situation, if B is torsion, almost torsionfree with respect to $(\mathcal{T}, \mathcal{F})$ it is also torsion, almost torsionfree with respect to $(\mathcal{T}', \mathcal{F}')$.

For the converse, notice that B is surely a torsion module and that all its proper submodules are contained in $\mathcal{F}' = \mathcal{F}$. Thus it remains to check only the last condition; let $0 \to K \to T \to B \to 0$ be an exact sequence in R-Mod with $T \in \mathcal{T}$. Assume that K is torsionfree, then using the surjective cover of B, $f : F \to B$, we can construct the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

By assumption K and F are in $\mathcal{F}$, thus $P \in \mathcal{F}$ and therefore T has a surjective cover. This forces T to be in $\overline{R}$-Mod by Lemma 4.2.2. Hence, we can use the torsion, almost torsionfree property of B to conclude that $K \in \mathcal{T}'$. So $K \in \mathcal{T}' \cap \mathcal{F} = 0$.

For a generic M, consider the following pushout diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & T & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M/\operatorname{t} M & \longrightarrow & T' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Apply the special case to the lower sequence to conclude that $M = \operatorname{t} M$ is a torsion module.

This shows that B is torsion, almost torsionfree with respect to $(\mathcal{T}, \mathcal{F})$. □

Therefore, it remains to consider the set of torsion, almost torsionfree modules which are not contained in $\overline{R}$-Mod. The following proposition deals with those:

Proposition 4.2.4. *Let C be a cosilting module, T a torsion, almost torsionfree module with respect to $(\mathcal{T}, \mathcal{F} = \mathrm{Cogen}(C))$ whose $\mathcal{F}-$cover $f : F \to T$ is not surjective.*

Then there exists a simple $R-$module $S \in \mathcal{F}^{\perp_0}$ and a short exact sequence

$$0 \longrightarrow F \xrightarrow{f} T \longrightarrow S \longrightarrow 0$$

Moreover, T has a unique maximal submodule.

Proof. First we show that T has a unique maximal submodule. By hypothesis, any maximal submodule is in $\mathcal{F}$, as T is torsion, almost torsionfree. Recall that T is finitely generated by Proposition 2.1.7, thus we have at least one maximal submodule. If T had two maximal submodules $F_1 \neq F_2$ then the summation map $F_1 \oplus F_2 \to T$ would be surjective, contradicting the assumption we made on the $\mathcal{F}-$cover.

This shows the existence of a simple module S fitting in the short exact sequence. If we had any non-zero morphism from a torsionfree module to S, then it would be an epimorphism, as S is simple, and we could obtain a surjective map from a torsionfree module to T via pullback. This would contradict the assumption, thus $S \in \mathcal{F}^{\perp_0}$. □

4.2.2 Neg-isolated points

Given a cosilting $\Lambda-$module C, the characteristic bricks in $\overline{\Lambda}$-Mod correspond to certain indecomposable modules in $\mathrm{Prod}(C)$. These modules will play a fundamental role in the final part of the chapter, to understand it we need the concept of neg-isolated point.

In this section, we recall the concept of neg-isolated point in some closed subset of the Ziegler spectrum. For an in depth discussion of this technical notion we refer to [55, Section 5.3]. The results concerning critical and special modules can be found in [3].

Definition 4.2.5. Let $\mathcal{C} \subseteq R$-Mod be an additive subcategory. A morphism $f : C \to C'$ in $\mathcal{C}$ is *left almost split in* $\mathcal{C}$ if it is not a split monomorphism and for every non-split mono $g : C \to C''$ there exists $h : C' \to C''$ such that $g = h \circ f$.

Definition 4.2.6. Let Z be an indecomposable pure-injective module in some definable subcategory $\mathcal{D} \subseteq R$-Mod.

Then Z is *neg-isolated in* $\mathcal{D}$ if there is a left almost split map $Z \to Z'$ in $\mathcal{D}$.

Remark 4.2.7. *There is a finer topology on the set of indecomposable pure-injective modules, known as the full-support topology, such that the neg-isolated points are precisely the open, or isolated, points, see [55, Proposition 5.3.67] for details.*

There are equivalent characterisations of the isolation condition: the original definition given in [55] is that a module N is neg-isolated in a definable subcategory $\mathcal{D}$ if and only if the tensor product functor $(- \otimes N)$ is the injective envelope of some simple object in the localised functor category $(\mathrm{mod}\text{-}R, \mathbf{Ab})_{\mathcal{D}}$, see Section 1.3.3 for the terminology and notation.

Proposition 4.2.8 ([3, Proposition 6.6])**.** *Let $\mathcal{D}$ be a definable subcategory of R-Mod. The following statements are equivalent for a module $N \in \mathcal{D}$:*

(i) N *is neg-isolated in* $\mathcal{D}$.

(ii) $(-\otimes N)$ *is the injective envelope of a simple object in* $(\text{mod-}R, \mathbf{Ab})_{\mathcal{D}}$.

The following result gives a relation between elementary cogenerators and neg-isolated points:

Theorem 4.2.9 ([55, Theorem 5.3.50]). *A pure-injective module* E *is an elementary cogenerator if and only if every neg-isolated point in* $\langle E\rangle \cap {}_R\mathrm{Zg}$ *occurs as a direct summand of* E.

Definition 4.2.10. Let Z be a neg-isolated module in a definable subcategory $\mathcal{D}$. Then Z is *critical* if there exists a non-injective left almost split map in $\mathcal{D}$ starting at Z.

Proposition 4.2.11 ([3, Proposition 6.13]). *Let* $\mathcal{D}$ *be a definable subcategory,* $\mathcal{C}_\bullet$ *the set of critical modules in* $\mathcal{D}$. *Then* $\mathcal{D} \subseteq \mathrm{Cogen}(\mathcal{C}_\bullet)$ *and every module in* $\mathcal{C}_\bullet$ *is split-injective in* $\mathrm{Cogen}(\mathcal{C}_\bullet)$.

Remark 4.2.12. *Given a cosilting module* C *we will be interested in critical modules in* $\mathrm{Cogen}(C)$ *as, by the proposition above,* $\mathrm{Cogen}(\mathcal{C}_\bullet) = \mathrm{Cogen}(C)$.

One can show that every critical module in $\mathrm{Cogen}(C)$ *is actually a direct summand of* C *(in fact the critical modules in* $\mathrm{Cogen}(C)$ *are precisely the critical modules in* $\langle C\rangle$*)*

Proposition 4.2.13 ([3, Lemma 4.3]). *Let* C *be a cosilting* $R-$*module and* $N \in \mathrm{Prod}(C)$ *a neg-isolated module in* $\mathrm{Cogen}(C)$. *Then:*

(i) N *is critical if and only if there exists a surjective left almost split map in* $\mathrm{Cogen}(C)$ *starting at* N.

(ii) N *is not critical if and only if there exists an injective left almost split map in* $\mathrm{Cogen}(C)$ *starting at* N.

Proof. In [3] this is proved when C is a cotilting module. However, since every cosilting module C is cotilting over $\overline{R} = R/\mathrm{Ann}(C)$ and all the concepts involved are relative to the subcategory $\mathrm{Cogen}_R(C) = \mathrm{Cogen}_{\overline{R}}(C) \subseteq \overline{R}\text{-Mod}$ the same statements hold in this case. □

We now consider the non critical case:

Definition 4.2.14 ([3]). Let C be a cosilting module. A module $N \in \mathrm{Prod}(C)$ is *special* (in $\mathrm{Cogen}(C)$) if there exists an injective left almost split map in $\mathrm{Cogen}(C)$ starting at N.

Critical and special summands of a cosilting module are connected with characteristic bricks:

Theorem 4.2.15 ([3, Propositions 6.11 and 6.18]). *Let* $(\mathcal{T}, \mathcal{F})$ *be a cosilting torsion pair in* $R\text{-Mod}$, *with cosilting module* C. *Then the following statements are equivalent for a module* N*:*

(1) N is special neg-isolated in $\mathcal{F}$

(2) There is a short exact sequence

$$0 \longrightarrow N \xrightarrow{b} \overline{N} \xrightarrow{f} T \longrightarrow 0$$

where T is torsion, almost torsionfree with respect to $(\mathcal{T}, \mathcal{F})$, the map b is a strong left almost split morphism in $\mathcal{F}$ and f is an $\mathcal{F}-$cover.

Moreover, the following statements are equivalent for a module M:

(1) M is critical neg-isolated in $\mathcal{F}$

(2) There is a short exact sequence

$$0 \longrightarrow F \xrightarrow{g} M \xrightarrow{a} \overline{M} \longrightarrow 0$$

where F is torsionfree, almost torsion with respect to $(\mathcal{T}, \mathcal{F})$, the map a is a strong left almost split morphism in $\mathcal{F}$ and g is a $\mathrm{Prod}(C)-$envelope.

Proof. The theorem is proved in [3] for cotilting modules. We can extend it using the compatibility results for characteristic bricks obtained in Section 4.2.1.

First notice that a module N is special neg-isolated in $\mathcal{F}$ as a $R-$module if and only if it is special neg-isolated in $\mathcal{F}$ as a $\overline{R}-$module: the conditions only involve the torsionfree class $\mathcal{F}$.

"(1) $\implies$ (2)" : If N is special neg-isolated we have a short exact sequence in $\overline{R}$-Mod

$$0 \longrightarrow N \xrightarrow{b} \overline{N} \xrightarrow{f} T \longrightarrow 0$$

with the required properties. As T is a torsion, almost torsionfree module with respect to the cotilting torsion pair in $\overline{R}$-Mod we can use Proposition 4.2.3 to obtain that T is also torsion, almost torsionfree with respect to $(\mathcal{T}, \mathcal{F})$.

"(2) $\implies$ (1)" : In this setting, as T has a surjective $\mathcal{F}-$cover, we can use once again Proposition 4.2.3 to infer that T is torsion, almost torsionfree with respect to the torsion, pair in $\overline{R}$-Mod. Thus we can apply the original theorem in $\overline{R}$-Mod to conclude that N is special neg-isolated in $\mathcal{F}$.

In the same way we can prove the equivalence of the two conditions for critical modules. □

For cotilting modules we have a cleaner connection between injective envelopes and neg-isolated modules

Theorem 4.2.16 ([3, Proposition 4.1, Theorem 4.2, Corollary 4.4])**.** *Let C be a cotilting module, with cotilting class $\mathcal{F}$. Then a module X is either critical or special neg-isolated in $\mathcal{F}$ if and only if, when seen as an object in the cosilting heart, it is the injective envelope of some simple object.*

In particular, there is a bijection between the set of special neg-isolated modules and the set of torsion, almost torsionfree modules. This bijection sends a torsion, almost torsionfree module T to the kernel N of the $\mathcal{F}-$cover of T. This N is the unique indecomposable module in $\mathrm{Prod}(C)$ *such that* $\mathrm{Ext}^1(T, N) = \mathrm{Hom}_{\mathrm{D}(R)}(T[-1], N) \neq 0$.

In the same way we have a bijection between the set of critical neg-isolated modules and the set of torsionfree, almost torsion modules. This bijection sends a torsionfree, almost torsion module F to the $\mathrm{Prod}(C)-$*envelope M of F. This M is the unique indecomposable module in* $\mathrm{Prod}(C)$ *such that* $\mathrm{Hom}(F, M) = \mathrm{Hom}_{\mathrm{D}(R)}(F, M) \neq 0$.

In the next three corollaries we adapt this theorem to the cosilting setting. This will yield several useful orthogonality properties which will be essential in the next sections.

Corollary 4.2.17. *Let C be a cosilting module, F a torsionfree, almost torsion module for* $\mathrm{Cogen}(C)$*. Then there exists a unique indecomposable module M in* $\mathrm{Prod}(C)$ *such that* $\mathrm{Hom}(F, M) \neq 0$*, this M is critical neg-isolated in* $\mathrm{Cogen}(C)$.

Proof. We can apply Theorem 4.2.16 to the cotilting module C over $\overline{\Lambda} = \Lambda / \mathrm{Ann}(C)$.

This will yield a unique critical neg-isolated M with the required property in $\overline{\Lambda}$-Mod, and as all these conditions are local to $\mathrm{Cogen}(C)$ we can obtain the same properties for M when seen as $\Lambda-$module. □

Corollary 4.2.18. *Let C be a cosilting module, T a torsion, almost torsionfree module in ${}^{\perp_0}C$ with surjective* $\mathrm{Cogen}(C)-$*cover.*

Then there exists a unique indecomposable module N in $\mathrm{Prod}(C)$ *such that $T \notin \mathcal{C}_{\mu_N}$, or equivalently such that* $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(T[-1], \mu_N) \neq 0$*. This N is special neg-isolated in* $\mathrm{Cogen}(C)$.

Moreover, for every indecomposable injective $I \in C^{\perp_0}$ we have $\mathrm{Hom}(T, I) = 0$.

Proof. Let $\mathcal{F} = \mathrm{Cogen}(C)$. By Proposition 4.2.3 the module T is still torsion, almost torsionfree for the cotilting torsion pair cogenerated by C in $\overline{\Lambda}$-Mod. Therefore, we can apply Theorem 4.2.16 to obtain the unique (special) neg-isolated point N such that $\mathrm{Ext}^1_{\overline{\Lambda}}(T, N) \neq 0$.

Once again, when seen as a $\Lambda-$module this module N is still special neg-isolated. Moreover, as $\mathrm{Ext}^1_{\Lambda}(T, N) \neq 0$ we have that $T \notin \mathcal{C}_{\mu_N}$.

For every other indecomposable module X in $\mathrm{Prod}(C)$, $T \in \mathcal{C}_{\mu_X}$ for the minimal injective copresentation μ_X of X. In fact, in every non-split extension $0 \to X \to L \to T \to 0$ in Λ-Mod, we have that $L \in \mathcal{F}$, as T is torsion, almost torsionfree. Thus $0 = \mathrm{Ext}^1_{\overline{\Lambda}}(T, X) = \mathrm{Ext}^1_{\Lambda}(T, X)$. Moreover all proper submodules F of T are torsionfree, and since $\mathcal{F}$ is an extension closed subcategory $\mathrm{Ext}^1_{\overline{\Lambda}}(F, X) = \mathrm{Ext}^1_{\Lambda}(F, X)$. However, in $\overline{\Lambda}$-Mod, X has injective dimension less than 1, being in $\mathrm{Prod}(C)$, thus for all $F \leq T$ we have $\mathrm{Ext}^1_{\Lambda}(F, X) = 0$ which implies that $T \in \mathcal{C}_{\mu_X}$, as in the proof of Lemma 4.1.6.

For the last statement, notice that if we had any morphism from T to any such indecomposable injective I, the surjective $\mathcal{F}-$cover $F \twoheadrightarrow T$ would induce a non-zero map $C \to I$. □

Corollary 4.2.19. *Let C be a cosilting module, T a torsion, almost torsionfree module in ${}^{\perp_0}C$ with injective* $\mathrm{Cogen}(C)$*−cover.*

Then for every indecomposable module N in $\mathrm{Prod}(C)$ *we have that $T \in \mathcal{C}_{\mu_N}$, or equivalently* $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(T[-1], \mu_N) = 0$.

Moreover, there exists a unique indecomposable injective module I in $C^{\perp_0}$ such that $\mathrm{Hom}(T, I) \neq 0$.

Proof. Let $\mathcal{F} = \mathrm{Cogen}(C)$. We start from the second statement. Let T be a torsion, almost torsionfree module with injective $\mathcal{F}-$cover, then by Proposition 4.2.4 the cokernel of the cover is a simple module S with $E(S) \in C^{\perp_0}$. Clearly for such injective $\mathrm{Hom}(T, E(S)) \neq 0$.

Moreover, assume we had another indecomposable injective $I \in C^{\perp_0}$ with some non-zero $f : T \to I$. Then we can consider two settings, either f is monic or it has non-zero kernel. If f is monic then in case T is not simple we have a non-zero map from a non-zero proper submodule of T, which is in $\mathrm{Cogen}(C)$ to I, thus $I \notin C^{\perp_0}$ a contradiction. If $T = S$ is simple, then $I = E(S)$, again a contradiction.

Thus we might assume that the map f has some non-trivial kernel K. Let S' be the simple socle of the indecomposable injective I. We can produce the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & S' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0 \end{array}$$

But as T is torsion, almost torsionfree, P must be torsionfree or isomorphic to T, in the first case we would get a non-zero map from $\mathrm{Cogen}(C)$ to I, in the second Proposition 4.2.4 implies that $S' \simeq S$ and $I = E(S)$.

For the first statement, let $N \in \mathrm{Prod}(C)$ indecomposable, then to show that $T \in \mathcal{C}_{\mu_N}$ it is equivalent to prove that every submodule of T is in ${}^{\perp_1}(N)$. However, every proper submodule of T is torsionfree, thus Ext-orthogonal to N as all the modules in $\mathrm{Prod}(C)$ are Ext-injective in $\mathcal{F}$. In conclusion it is enough that $\mathrm{Ext}^1(T, N) = 0$.

Assume we had a non-split short exact sequence $0 \to N \to L \to T \to 0$. As T is torsion, almost torsionfree, the module L must be torsionfree, however if this were the case we would have a surjective map from a module in $\mathcal{F}$ to T, which contradicts our assumption on the injectivity of the $\mathcal{F}-$cover. □

We close this section showing some additional properties of critical and special modules. First, critical and special modules occur as summands of the approximation sequence induced by a cosilting module:

Theorem 4.2.20 ([3]). *Let $E(R)$ be an injective cogenerator of R-*Mod*, let C be a cosilting module with approximation sequence:*

$$0 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{f} E(R)$$

Then every special module is a summand of C_1 and every critical module is a summand of C_0.

Proof. Once again in [3] this result is proved only for cotilting modules. However, since we know that:

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathrm{Im}(f) \longrightarrow 0$$

is an approximation sequence for the injective cogenerator $\mathrm{Im}(f)$ of $\overline{R}$-Mod, by Lemma 1.4.7, and that being critical, or being special is a condition local to $\mathcal{F}$, we can immediately extend the result to the cosilting case. □

For the last result we need the following generalisation of the radical:

Definition 4.2.21. Let X, Y be modules in R-Mod. Then we define

$$\mathrm{Rad}(X,Y) := \left\{ f \in \mathrm{Hom}_R(X,Y) \;\middle|\; \begin{array}{l} \forall Z \text{ such that } \mathrm{End}(Z) \text{ is local }, \\ \forall g \in \mathrm{Hom}_R(Z,X),\ \forall h \in \mathrm{Hom}_R(Y,Z), \\ h \circ f \circ g \text{ is not invertible} \end{array} \right\}$$

Recall that by Theorem 1.3.10 every indecomposable pure-injective module has local endomorphism ring.

Lemma 4.2.22. *Let R be a ring, then* Rad *is an ideal in* R-Mod.

Proof. Same proof as [15, Proposition 7.1]. □

We conclude this brief recap with the following Lemma.

Lemma 4.2.23 (cf. [1, Lemma 2.20])**.** *Let C be a cosilting module. Then for the minimal approximation:*

$$0 \longrightarrow C_1 \xrightarrow{\ f\ } C_0 \xrightarrow{\ g\ } E(R)$$

we have that C_1 and C_0 do not have a common, up to isomorphism, indecomposable summand; in particular C_1 has no critical summands and C_0 has no special summands.

Proof. The proof is the dual of the proof of [1, Lemma 2.20], using the modified definition of the radical.

We show that $\mathrm{Hom}_R(C_1, C_0) = \mathrm{Rad}(C_1, C_0)$. First, notice that $f \in \mathrm{Rad}(C_1, C_0)$: assume by contradiction that we have a (non-zero) module Z with local endomorphism ring and two maps $j : Z \to C_1$, $h : C_0 \to Z$ such that $h \circ f \circ j$ is invertible. Then Z is isomorphic to a direct summand of C_0 which is contained in the kernel of g. However, the map g is right minimal and the existence of such a Z leads to a contradiction.

Applying $\mathrm{Hom}(-, E(R))$ to the short exact sequence:

$$0 \longrightarrow C_1 \xrightarrow{\ f\ } C_0 \xrightarrow{\ \bar{g}\ } \mathrm{Im}(g) \longrightarrow 0$$

we observe that the map $\mathrm{Hom}(f, E(R)) : \mathrm{Hom}(C_0, E(R)) \to \mathrm{Hom}(C_1, E(R))$ is surjective.

So let $s : C_1 \to C_0$ be an arbitrary morphism.

By the observation above, there is some map $b : C_0 \to E(R)$ such that $g \circ s = b \circ f$. However, since g is a $\mathrm{Cogen}(C)-$cover, the map b must also factor through g, so we can write $b = g \circ l$ for some endomorphism $l : C_0 \to C_0$.

In this way we can conclude that $g \circ s = (g \circ l) \circ f$ and therefore $g \circ (s - l \circ f) = 0$.

Thus $(s - l \circ f) = f \circ v$ since f is the kernel of g and ultimately we can express $s = f \circ v + l \circ f$.

Since the radical is an ideal in R-Mod, we conclude that $s \in \mathrm{Rad}(C_1, C_0)$.

It follows that C_1 and C_0 can not have any common indecomposable (pure-injective) summand. □

Remark 4.2.24. *For Lemma 4.2.23 pure-injectivity of the modules C_1, C_0 is a fundamental point. In particular, a large version of [1, Lemma 2.20] for silting modules is unlikely to exist.*

4.2.3 Identifying well-behaved points of cosilting pairs

Let Λ be an artin algebra. In this section we will identify the classes of points at which it is possible to perform some kind of mutation. We will see that these points correspond to some neg-isolated modules.

Whenever we work with a subset $\mathcal{S}$ of some topological space $\mathcal{T}$ we will always consider $\mathcal{S}$ as a topological space with respect to the subspace topology.

Definition 4.2.25. Let $(\mathcal{Z}, \mathcal{I})$ be a cosilting pair in Λ-Mod. Then we say that:

(i) a point $N \in \mathcal{Z}$ is *special* if there exists a monomorphism $N \to \overline{N}$ that is a left almost split map in $\mathrm{Cogen}(\mathcal{Z})$

(ii) a point $N \in \mathcal{Z}$ is *(very) critical* if there exists an epimorphism $f : N \to \overline{N}$ that is a left almost split map in $\mathrm{Cogen}(\mathcal{Z})$ (and $\ker f$ is finitely generated).

(iii) An injective $I \in \mathcal{I}$ is *special*, if $I = E(S)$ for some simple module S and there exists a torsion, almost torsionfree module T with respect to $({}^{\perp_0}\mathcal{Z}, \mathrm{Cogen}(\mathcal{Z}))$ with a short exact sequence:

$$0 \longrightarrow F \xrightarrow{c} T \longrightarrow S \longrightarrow 0$$

where $c : F \to T$ is a $\mathrm{Cogen}(\mathcal{Z})-$cover.

We prove in the next Lemma that all the injective modules in a cosilting pair are well-behaved, i.e. special. This is in accordance with intuition as the injective part of a rigid pair does not involve topological notions.

The lemma also shows that a non-sincere definable torsionfree class must admit some torsion, almost torsionfree module.

Lemma 4.2.26. *Let $\mathcal{F}$ be a definable torsionfree class in* Λ-Mod*, assume there is a simple module S with $E(S) \in \mathcal{F}^{\perp_0}$. Then there exists a module T, torsion, almost torsionfree with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$, with a short exact sequence:*

$$0 \longrightarrow F \xrightarrow{c} T \longrightarrow S \longrightarrow 0$$

where $c : F \to T$ is a $\mathcal{F}-$cover.

Proof. Since we are dealing with a question about torsion, almost torsionfree modules, we can work with the restricted torsion pair $(\mathbf{t}, \mathbf{f})$ in Λ-mod and apply the (dual of the) techniques in [35]. We will still use the torsion, almost torsionfree terminology even if we actually work with minimal co-extending modules.

First, notice that ${}^{\perp_0}E(S) \cap \Lambda$-mod is a Serre subcategory of Λ-mod, which we will consider now as a torsionfree class. It is functorially finite and it is distinct from the whole category, thus we can find a torsion, almost torsionfree module U with respect to $(\mathbf{u}, \mathbf{v} = {}^{\perp_0}E(S) \cap \Lambda\text{-mod})$ by means of Theorem 1.6.13 and by [17, Theorem 2.3.2].

Such module admits a $\mathbf{v}-$cover, which must be a monomorphism as $\mathbf{v}$ is closed under quotients and $0 \neq U \in \mathbf{u}$. Being torsion, almost torsionfree, by Proposition 4.2.4, U must have a unique maximal submodule and we then have a sequence:

$$0 \to V \to U \to S' \to 0$$

moreover, we must have $S' \simeq S$ as S is a composition factor of U, but it is not a factor of any of its proper submodules.

Now U is a module in $\mathbf{u} \subseteq \mathbf{t}$, however it might not be $\mathbf{t}-$simple. Thus, consider the set $\mathcal{S}$ consisting of $\mathbf{t}-$simple modules occurring as quotients of U and having a non-zero morphism to S. This set is clearly non-empty, as the simple module S itself satisfy these conditions.

Let T be a module in $\mathcal{S}$ of maximal dimension. We claim that T is torsion, almost torsionfree with respect to $(\mathbf{t}, \mathbf{f})$. It is clearly a $\mathbf{t}-$simple by construction, thus by the dual of [35, Lemma 4.4], it remains to show that it is maximal in the epi-brick (collection of bricks with the only non-zero maps between them being epimorphisms) of $\mathbf{t}-$simple modules.

Let T' be a $\mathbf{t}-$simple module, assume we have an epimorphism $T' \to T$.

Consider the following pull-back diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

By $\mathbf{t}-$simplicity of T', the kernel $K \in \mathbf{f} \subseteq \mathbf{v}$. Since U is torsion, almost torsionfree with respect to $(\mathbf{u}, \mathbf{v})$ we have two possibilities: either $P \in \mathbf{v}$ or the upper sequence splits. In the first case, we reach a contradiction, as we would obtain a non-zero map $P \to T' \to T \to S \to E(S)$. In the second case, writing $P = K \oplus U$, we notice that the map $U \to T'$ must be an epimorphism, as every proper submodule of T' is an element of $\mathbf{f} \subseteq \mathbf{v}$. This means that T' is an element of the set $\mathcal{S}$, thus, by maximality of T, we must have $T' \simeq T$.

Now using that T is torsion, almost torsionfree, and that S is one of its composition factors we obtain the required short exact sequence. □

Now we show that the special and very critical points we defined have a topological meaning: they are open in the relative topology of the closed set $\mathcal{Z}$. This result is essentially an adaptation of [55, Corollary 5.3.3].

Proposition 4.2.27. *Let* $(\mathcal{Z}, \mathcal{I})$ *be a cosilting pair in* Λ - Mod. *Every special or very critical point* X *in* $\mathcal{Z}$ *is open in* $\mathcal{Z}$. *In particular, for any such* X *the pair* $(\mathcal{Z} \setminus \{X\}, \mathcal{I})$ *is a rigid pair.*

Proof. We sketch the proof. The argument involves several technical statements and definitions, which will not be explained. The relevant notions can be found in [41] and [55].

Note that each bireflective subcategory of Λ - Mod is definable by [55, Theorem 5.5.3]. In particular, if $(\mathcal{Z}, \mathcal{I})$ is a cosilting pair with corresponding cosilting module C and $\overline{\Lambda} = \Lambda / \operatorname{Ann}(C)$, then $\mathcal{Z} \subseteq {}_{\overline{\Lambda}}\mathrm{Zg}$ and ${}_{\overline{\Lambda}}\mathrm{Zg}$ is a closed subspace of ${}_{\Lambda}\mathrm{Zg}$, therefore the subspace topology of $\mathcal{Z} \subseteq {}_{\overline{\Lambda}}\mathrm{Zg}$ is the same as the subspace topology in ${}_{\Lambda}\mathrm{Zg}$ and a set is closed in ${}_{\overline{\Lambda}}\mathrm{Zg}$ if and only if it is closed in ${}_{\Lambda}\mathrm{Zg}$.

Therefore we can reduce to the case where the set $\mathcal{I}$ is empty and $\mathcal{Z}$ corresponds to a cotilting module.

Recall that by Lemma 4.1.24 the definable category corresponding to the closed set $\mathcal{Z}$ is $\langle C \rangle$, the definable subcategory generated by C, and, since we are in the left noetherian setting $\langle C \rangle = \operatorname{Cogen}_*(C)$, see Proposition 4.1.22, thus the pure-injective modules in $\langle C \rangle$ are precisely the elements of $\operatorname{Prod}(C)$.

Let ${}_{\Lambda}\mathcal{C} := (\text{mod-}\Lambda, \mathbf{Ab})$ be the category of additive functors from mod - Λ to the category of abelian groups.

Then ${}_{\Lambda}\mathcal{C}$ is a locally coherent Grothendieck category and thus it is possible to define a topology on the set of indecomposable injective objects in ${}_{\Lambda}\mathcal{C}$. The resulting topological space is denoted as $\mathrm{Zg}({}_{\Lambda}\mathcal{C})$.

The fully faithful functor $T : \Lambda\text{-Mod} \to {}_{\Lambda}\mathcal{C}, M \mapsto - \otimes_{\Lambda} M$ induces a homeomorphism $\overline{T} : {}_{\Lambda}\mathrm{Zg} \to \mathrm{Zg}({}_{\Lambda}\mathcal{C})$, see Theorem 1.3.22.

Consider the hereditary torsion pair cogenerated by the set of injectives $\overline{T}(\mathcal{Z})$. This gives a localization of the functor category ${}_{\Lambda}\mathcal{C}_{\langle C \rangle}$, which will be once again a locally coherent Grothendieck category by [41, Theorem 2.16] and its Ziegler spectrum $\mathrm{Zg}({}_{\Lambda}\mathcal{C}_{\langle C \rangle})$ is homeomorphic to the closed subset $\overline{T}(\mathcal{Z})$ of $\mathrm{Zg}({}_{\Lambda}\mathcal{C})$ equipped with the subspace topology by [41, Proposition 3.6]. So in conclusion $\mathcal{Z}$ is homeomorphic to $\mathrm{Zg}({}_{\Lambda}\mathcal{C}_{\langle C \rangle})$.

Now, it turns out that ${}_{\Lambda}\mathcal{C}_{\langle C \rangle}$ is equivalent to the heart $\mathcal{H}$ of the cotilting HRS-t-structure, as shown in [62, Section 6.1]. This result is obtained exploiting the equivalence between the categories of injective objects in $\mathcal{H}$ and ${}_{\Lambda}\mathcal{C}_{\langle C \rangle}$: recall that two abelian categories with enough injectives are equivalent if and only if the corresponding full subcategories of injective objects are equivalent, for a proof of the dual case see [15, Proposition IV.1.2].

In fact, in the cotilting case Lemma 4.1.1 tells us that $\mathrm{Inj}(\mathcal{H})$ is equivalent to $\mathrm{Prod}(C)$ and by Theorem 1.3.25 the injective objects of ${}_{\Lambda}\mathcal{C}_{\langle C\rangle}$ are precisely the pure-injectives in $\langle C\rangle$, thus the elements of $\mathrm{Prod}(C)$.

In conclusion we have that the spaces $\mathcal{Z}, \mathrm{Zg}(\mathcal{H})$ and $\mathrm{Zg}({}_{\Lambda}\mathcal{C}_{\langle C\rangle})$ are homeomorphic.

Thus to show that every very critical or special point M is open in $\mathcal{Z}$ it is enough to show that the corresponding object M in $\mathcal{H}$ is open in $\mathrm{Zg}(\mathcal{H})$. To do this, using [41, Theorem 3.8], it suffices to find a coherent simple object S in the heart, such that $\{M\} = \mathcal{O}(S) := \{E \in \mathrm{Zg}(\mathcal{H}) \mid \mathrm{Hom}_{\mathcal{H}}(S, E) \neq 0\}$.

We can obtain this simple S by Theorem 4.2.16: a module N is critical or special neg-isolated in $\mathrm{Cogen}(C)$ if and only if it is the injective envelope of a simple in $\mathcal{H}$. The resulting S is finitely presented by hypothesis: recall that a stalk complex in the heart is finitely presented if the corresponding module M is finitely presented by Theorem 1.5.11. □

In light of this result we give the following definition:

Definition 4.2.28. We say that a rigid pair $(\mathcal{U}, \mathcal{I})$ is *almost complete*, if it is not a cosilting pair and there exists a point X of the Ziegler spectrum, resp. an indecomposable injective J, such that $(\mathcal{U} \cup \{X\}, \mathcal{I})$, resp. $(\mathcal{U}, \mathcal{I} \cup \{J\})$, is a cosilting pair and X is either very critical or special in $\mathcal{U} \cup \{X\}$.

We say that a cosilting pair $(\mathcal{Z}, \mathcal{I})$ *completes* a rigid pair $(\mathcal{U}, \mathcal{J})$ if $\mathcal{U} \subseteq \mathcal{Z}$ and $\mathcal{J} \subseteq \mathcal{I}$.

Remark 4.2.29. *Let $\mathcal{C}$ be a closed subspace of a topological space $\mathcal{T}$. Then, given a point $P \in \mathcal{C}$ the set $\mathcal{C} \setminus \{P\}$ is closed if and only if the point P is open in $\mathcal{C}$.*

If $\mathcal{T} = {}_{\Lambda}\mathrm{Zg}$ then all the open points in a closed subset are in particular neg-isolated in the corresponding definable subcategory: recall that the neg-isolated points are precisely the open points with respect to a finer topology, see Remark 4.2.7.

The next proposition in conjunction with our previous remark shows that the conditions imposed in Definition 4.2.28 are reasonable:

Proposition 4.2.30. *Let R be a left noetherian ring, $C \in R\text{-}\mathrm{Mod}$ a cosilting module. Then every neg-isolated module in $\langle C\rangle$ is either a special or critical neg-isolated module in* $\mathrm{Cogen}(C)$.

Proof. First, notice that we can once again assume, without loss of generality, that C is cotilting in fact the definable closure of C in $\Lambda\text{-}\mathrm{Mod}$ is equal to the definable closure in $\overline{\Lambda}\text{-}\mathrm{Mod}$ as $\overline{\Lambda}\text{-}\mathrm{Mod}$ is definable.

Recall that, by Proposition 4.2.8, a module is neg-isolated in a definable subcategory $\mathcal{X}$ if and only if its image under the tensor embedding T to the localised functor category $(\mathrm{mod}\text{-}R, \mathbf{Ab})_{\mathcal{X}}$ is the injective envelope of a simple object.

Now, if the definable subcategory is $\langle C\rangle$, then we have an equivalence of categories

$$\mathcal{H} \simeq (\mathrm{mod}\text{-}R, \mathbf{Ab})_{\langle C\rangle}$$

as discussed in the proof of Proposition 4.2.27.

In conclusion, the neg-isolated modules correspond to injective envelopes of the simple objects in the heart $\mathcal{H}$.

As we know by Proposition 2.1.9, the simple objects correspond to the characteristic bricks for the cotilting torsion pair and their injective envelopes are precisely the special and critical modules in $\mathrm{Cogen}(C)$, see Theorem 4.2.16. □

Combining the previous proposition with the Corollaries 4.2.17, 4.2.18 and 4.2.19 we obtain:

Remark 4.2.31. *Given a cosilting $\Lambda-$module C with cosilting pair $(\mathcal{Z},\mathcal{I})$ we have a bijection:*

$$\{ \text{ characteristic bricks } \} \overset{1-1}{\longleftrightarrow} \{ \text{ neg-isolated modules in } \langle C\rangle\} \cup \mathcal{I}$$

realised as follows:
F torsionfree, almost torsion $\mapsto M$ in $\mathcal{Z}$ with $\mathrm{Hom}(F, \mu_M) \neq 0$
T torsion, almost torsionfree in $\overline{\Lambda}\text{-}\mathrm{Mod} \mapsto N$ in $\mathcal{Z}$ with $T \notin \mathcal{C}_{\mu_N}$
T torsion, almost torsionfree not in $\overline{\Lambda}\text{-}\mathrm{Mod} \mapsto I$ indecomposable injective in $\mathcal{I}$ with $\mathrm{Hom}(T, I) \neq 0$.

In the other direction, knowing that a module is neg-isolated in $\langle C\rangle$ if and only if it is special or critical neg-isolated in $\mathcal{F}$ we associate to every critical (resp. special), the kernel (resp. cokernel) of the corresponding left almost split map. To an injective $I \in \mathcal{I}$ we associate the torsion, almost torsionfree module T, such that $T[-1]$ is the socle of the indecomposable injective $H^0_\sigma(0 \to I)$ in the cosilting heart $\mathcal{H}$.

In conclusion, we have that to obtain a possible mutation we must work with open points of cosilting pairs. Every open point corresponds to some critical or special neg-isolated module in the cosilting class and we know that all very critical or special points are open.

Example 4.2.32. *Let $\Lambda = k\mathcal{K}_2$ be the Kronecker algebra over some algebraically closed field k. See Example 2.3.6 for a description of the category $\Lambda\text{-}\mathrm{mod}$. The tubes in the regular component of the AR-quiver are parametrised by some set L. Let $P \subsetneq L$ be a non-empty subset and fix $Q = L \setminus P$, then we have a cosilting pair, corresponding to a cotilting module C, whose closed set $\mathrm{Prod}(C) \cap {}_\Lambda\mathrm{Zg}$ is*

$$\{S_q[-\infty]\}_{q\in Q} \cup \{S_p[\infty]\}_{p\in P} \cup \{G\}$$

where $S_q[-\infty]$ is the adic module obtained as the inverse limit of the sequence of irreducible epimorphisms in the tube indexed by q, $S_p[\infty]$ is the Prüfer module obtained as the direct limit of the irreducible monomorphisms in the tube indexed by p and G is the generic module.

The torsionfree, almost torsion modules for the corresponding torsion pair are the simple regulars $S_p[1]$ for $p \in P$, while the torsion, almost torsionfree modules are the remaining simple regulars $S_q[1]$ for $q \in Q$. Knowing this, we can identify the open points *of this pair: they are the very critical points of the form* $S_p[\infty]$ *and the special points* $S_q[-\infty]$.

The point G *is not the injective envelope of any simple in the cotilting heart, in particular it is not open in the closed set we considered.*

As in the example above, all infinite cosilting pairs will contain non-open points, in fact:

Lemma 4.2.33. *Let* $\mathcal{Z}$ *be a closed subspace of* ${}_\Lambda\mathrm{Zg}$*. Then if every point of* $\mathcal{Z}$ *is open,* $\mathcal{Z}$ *is a finite set.*

Proof. The Ziegler spectrum is a compact topological space [55, Corollary 5.1.23], therefore the closed subspace $\mathcal{Z}$ is also compact. If every point $z \in \mathcal{Z}$ is open, then we have an open cover given by the collection of all the singletons $\{z\}$. By compactness, this collection must be finite. □

Using the results we obtained, we can now get a more practical criterion to determine if a given rigid pair is a cosilting pair, this is the analogue of [1, Corollary 2.13]:

Corollary 4.2.34. *A rigid pair* $(\mathcal{Z}, \mathcal{I})$ *is cosilting if and only if* $\mathrm{Cogen}(\mathcal{Z}) = \mathcal{C}_{\sigma(\mathcal{Z},\mathcal{I})}$.

Proof. If $(\mathcal{Z}, \mathcal{I})$ is a cosilting pair then by Lemma 4.1.26 $\mathrm{Cogen}(\mathcal{Z}) = \mathcal{C}_{\sigma(\mathcal{Z},\mathcal{I})}$.

Conversely, if we assume the second condition, we obtain that $C = \prod_{Z\in\mathcal{Z}} Z$ is a cosilting module with respect to the presentation $\sigma_{(\mathcal{Z},\mathcal{I})}$, therefore applying Lemma 4.1.24 to C we obtain a cosilting pair $(\mathcal{Z}_C, \mathcal{I}_C)$. Using the fact that $\mathcal{Z}$ is closed, as in the proof of Theorem 4.1.27, we can see that $\mathrm{Prod}(C) \cap \Lambda\text{-}\mathrm{pinj} = \mathcal{Z}_C = \mathcal{Z}$. Moreover, $\mathcal{I}_C \supseteq \mathcal{I}$ as all the elements of $\mathcal{I}$ are necessarily in $C^{\perp_0}$.

Assume now that $\mathcal{I}_C$ contains an injective $I \notin \mathcal{I}$. Then by Remark 4.2.31 we would have a torsion, almost torsionfree module T corresponding with it, and by Corollary 4.2.19 such that $T \in \mathcal{C}_{\sigma(\mathcal{Z}_C, \mathcal{I}_C \setminus \{I\})}$.

However, $\mathcal{I}_C \setminus \{I\} \supseteq \mathcal{I}$, therefore $\mathcal{C}_{\sigma(\mathcal{Z}_C,\mathcal{I}_C\setminus\{I\})} = \mathcal{C}_{\sigma(\mathcal{Z},\mathcal{I})}$ and we have a contradiction. □

4.2.4 Exchanging points of cosilting pairs

Before starting with the proper discussion of mutation, we recall a lemma giving a procedure for exchanging characteristic bricks. This gives a mutation operation on torsion pairs in $\Lambda\text{-}\mathrm{mod}$ on which we will build our mutation of cosilting pairs.

The next lemma is a summary of some results from the literature, for instance [17, Theorem 2.2.6, Proposition 2.3.3]. It could also be recovered in terms of brick labelling from [11] and [32].

Lemma 4.2.35. *Let* $(\mathbf{t}, \mathbf{f})$ *be a torsion pair in* $\Lambda\text{-}\mathrm{mod}$*. Let* F *be a torsionfree, almost torsion module with respect to* $(\mathbf{t}, \mathbf{f})$*. Then* F *is torsion, almost torsionfree with respect to* $(\mathbf{t}', \mathbf{f} \cap F^{\perp_0})$*. Moreover* $\mathbf{t}'$ *covers* $\mathbf{t}$ *in the lattice* $\mathbf{tors}(\Lambda)$.

Dually, a torsion, almost torsionfree module T *with respect to* $(\mathbf{t}, \mathbf{f})$ *is torsionfree, almost torsion with respect to* $(\mathbf{t} \cap {}^{\perp_0}T, \mathbf{f}')$ *and* $\mathbf{t}$ *covers* $(\mathbf{t} \cap {}^{\perp_0}T)$.

Remark 4.2.36. *We can apply Lemma 4.2.35 to a cosilting torsion pair* $(\mathcal{T}, \mathcal{F})$ *with restriction* $(\mathbf{t}, \mathbf{f})$ *as follows: let* F *be a finitely generated torsionfree, almost torsion module with respect to* $(\mathcal{T}, \mathcal{F})$*, then by Proposition 2.1.7* F *is also torsionfree, almost torsion with respect to* $(\mathbf{t}, \mathbf{f})$*. Applying Lemma 4.2.35 we obtain a torsion pair* $(\mathbf{u}, \mathbf{v})$ *such that* F *is torsion, almost torsionfree with respect to it and* $\mathbf{u}$ *covers* $\mathbf{t}$*.*

Consider now the cosilting pair $(\mathcal{U}, \mathcal{V})$ *extending* $(\mathbf{u}, \mathbf{v})$ *and use again Proposition 2.1.7 to obtain that* F *is torsion, almost torsionfree with respect to* $(\mathcal{U}, \mathcal{V})$*.*

Moreover, using Theorem 1.3.28 it follows that $\mathcal{U}$ *covers* $\mathcal{T}$ *in the lattice of torsion classes which are part of a cosilting torsion pair as this lattice is isomorphic to* $\mathbf{tors}(\Lambda)$*.*

We can now start with the main discussion: the following lemma will describe the mutation of a cosilting pair at a point corresponding to a finitely generated torsionfree, almost torsion module with respect to the corresponding cosilting torsion pair.

Lemma 4.2.37. *Let* $(\mathcal{Z}, \mathcal{I})$ *be a cosilting pair in* $\Lambda\text{-}\mathrm{Mod}$ *with* $X \in \mathcal{Z}$ *a very critical point. Let* F *be the finitely generated torsionfree, almost torsion module in* $\mathrm{Cogen}(C_{(\mathcal{Z},\mathcal{I})})$ *obtained as the kernel of the left almost split epimorphism corresponding to* X*. Then:*

$$\mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})}) = \mathrm{Cogen}(C_{(\mathcal{Z},\mathcal{I})}) \cap F^{\perp_0}$$

In particular, $\mathrm{Cogen}(C_{(\mathcal{Z},\mathcal{I})})$ *covers* $\mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})})$ *in* $\mathbf{Cosilt}(\Lambda)$*.*

Proof. First notice that $\mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})})$ is a torsionfree class, as all the modules in $\mathcal{Z} \setminus \{X\}$ are Ext-injective in the larger class $\mathrm{Cogen}(C_{(\mathcal{Z},\mathcal{I})})$ (see [2, Remark 3.4]). It is definable, as it is obtained as the class of subobjects of some definable subcategory, see [55, Proposition 3.4.15], namely the definable subcategory corresponding to the closed set $\mathcal{Z} \setminus \{X\}$. Now let's fix $(\mathcal{T}, \mathcal{F}) = ({}^{\perp_0}C_{(\mathcal{Z},\mathcal{I})}, \mathrm{Cogen}(C_{(\mathcal{Z},\mathcal{I})}))$ and $(\mathcal{T}', \mathcal{F}') = (\mathbf{T}(\mathcal{T} \cup \{F\}), \mathcal{F} \cap F^{\perp_0})$.

As F is $\mathcal{F}$-simple, we have that $\mathcal{F}'$ is the largest torsionfree class contained in $\mathcal{F}$ which does not contain F. Thus, to show that $\mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})}) \subseteq \mathcal{F}'$, it is enough to show that $F \notin \mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})})$. This is obtained from Corollary 4.2.17: the only element Z of $\mathcal{Z}$ such that $\mathrm{Hom}(F, Z) \neq 0$ is precisely the very critical module X.

For the reverse inclusion, we will prove that every critical neg-isolated module M in $\mathcal{F}'$ is actually contained in the set $\mathcal{Z} \setminus \{X\}$ (it is sufficient as critical modules cogenerate the torsionfree class by Proposition 4.2.11). This is equivalent to require that every such critical module is Ext-injective in $\mathcal{F}$, as $\mathcal{Z} = \mathrm{Prod}(C) \cap \Lambda\text{-}\mathrm{pinj}$ is precisely the set of indecomposable Ext-injectives in $\mathcal{F}$ and $X \notin \mathcal{F}'$ once again by Corollary 4.2.17.

The class $\mathcal{F}$ can be expressed as the class of extensions $(\mathcal{F} \cap \mathcal{T}') \star \mathcal{F}'$ by Lemma 1.1.13. As M is split-injective in $\mathcal{F}'$ by Proposition 4.2.11, it will be enough to show that M is Ext-injective in $(\mathcal{F} \cap \mathcal{T}')$.

However $(\mathcal{T}', \mathcal{F}')$ is a mutation of the torsion pair $(\mathcal{T}, \mathcal{F})$ at the torsionfree, almost torsion module F as per Lemma 4.2.35, so that we have a minimal inclusion between the corresponding torsion classes $\mathbf{t} \subseteq \mathbf{t}'$ in $\Lambda\text{-}\mathrm{mod}$ thus by [6, Corollary 9.14], we have $\mathcal{F} \cap \mathcal{T}' = \varinjlim \mathrm{filt}(F)$. Notice that all the split-injectives M in $\mathcal{F}'$ are pure-injective modules, thus ${}^{\perp_1}M$ is closed under direct limits by Lemma 3.2.11.

In conclusion, it is enough to prove that $\mathrm{Ext}^1(F, M) = 0$. Let

$$0 \longrightarrow M \xrightarrow{k} L \xrightarrow{j} F \longrightarrow 0$$

be an arbitrary extension. As L is an extension of modules in $\mathcal{F}$, we have $L \in \mathcal{F}$. Moreover, if $L \in F^{\perp_0}$ then the sequence would split, as M is split-injective in $\mathcal{F}'$.

Thus we can focus on the case in which we have a non-zero map $f : F \to L$. Any such f must be a monomorphism, as F is torsionfree, almost torsion in $\mathcal{F}$. If $j \circ f = 0$, then f would factor through the kernel $k : M \to L$. But $M \in F^{\perp_0}$, so this can not occur. Thus $j \circ f \neq 0$ and since F is a brick, it must be an automorphism. Thus the sequence splits. □

Remark 4.2.38. *Notice, in the last part of the proof of the Proposition, that the fact that F is Ext-orthogonal to the criticals in $\mathcal{F}'$ could be immediately deduced using that F is torsion, almost torsionfree in the new torsion pair. This means that its injective envelope in the new heart corresponds to a special point.*

Remark 4.2.39. *It is not clear if a critical, but not very-critical point X in a cosilting pair $(\mathcal{Z}, \mathcal{I})$ could be open. If any such point existed, the description of the class* $\mathrm{Cogen}(C_{(\mathcal{Z}\setminus\{X\},\mathcal{I})})$ *would not be the same as the one given in Lemma 4.2.37.*

In fact, if F is an infinitely generated torsionfree, almost torsion module for a definable class $\mathcal{F}$, then $\mathcal{V} := \mathcal{F} \cap F^{\perp_0}$ is not definable: all the finitely generated quotients of F must be torsion modules, thus $\mathcal{V} \cap \Lambda\text{-}\mathrm{mod} = \mathcal{F} \cap \Lambda\text{-}\mathrm{mod}$. *Hence, if $\mathcal{V}$ were definable, it would be equal to $\mathcal{F}$, but this is not the case as $\mathcal{V}$ doesn't contain F.*

The next lemma describes the mutation of a cosilting pair at a point corresponding to a (finitely generated) torsion, almost torsionfree module with respect to the corresponding cosilting torsion pair.

Lemma 4.2.40. *Let $(\mathcal{Z}, \mathcal{I})$ be a cosilting pair in* $\Lambda\text{-}\mathrm{Mod}$ *with $X \in \mathcal{Z}$ a special point. Let T be the torsion, almost torsionfree module obtained as the cokernel of the corresponding left almost split monomorphism. Let* $\mathbf{f} = \mathrm{Cogen}(\mathcal{Z}) \cap \Lambda\text{-}\mathrm{mod}$. *Then:*

$$\mathcal{C}_{\sigma(\mathcal{Z}\setminus\{X\},\mathcal{I})} = \varinjlim \widetilde{\mathbf{F}}(\mathbf{f} \cup \{T\})$$

In particular, $\mathcal{C}_{\sigma(\mathcal{Z}\setminus\{X\},\mathcal{I})}$ covers $\mathcal{C}_{\sigma(\mathcal{Z},\mathcal{I})} = \mathrm{Cogen}(\mathcal{Z})$ *in* $\mathbf{Cosilt}(\Lambda)$.

The same result applies for any injective $I \in \mathcal{I}$, choosing as T the torsion, almost torsionfree module making I special.

Proof. Both the involved classes are definable torsionfree classes by construction. Thus, it is enough to show that the corresponding restrictions $\mathbf{f}'' = \mathcal{C}_{\sigma(\mathcal{Z}\setminus\{X\},\mathcal{I})} \cap \Lambda\text{-}\mathrm{mod}$ and $\mathbf{f}' = \widetilde{\mathbf{F}}(\mathbf{f} \cup \{T\})$ coincide by Theorem 1.3.28.

By Lemma 4.2.35, the torsionfree class $\mathbf{f}'$ covers $\mathbf{f}$ in the lattice of torsionfree classes in $\Lambda\text{-}\mathrm{mod}$.

Moreover, the brick T is also contained in $\mathbf{f}''$. In fact, $\sigma_{(\mathcal{Z}\setminus\{X\},\mathcal{I})} = \prod_{N\in\{Z\setminus\{X\}\}} \mu_N \oplus \prod_{I\in\mathcal{I}}(0 \to I)$ and $T \in \mathcal{C}_{\mu_N}$ for all such N by Corollary 4.2.18. Finally, $T \in \mathcal{C}_{0\to I} = {}^{\perp_0}I$ for all the injectives I again using Corollary 4.2.18.

Now, since $\mathbf{f}'$ is the smallest torsionfree class in Λ-mod containing $\mathbf{f}$ and T, and $\mathbf{f} = \mathcal{C}_{\sigma_{(\mathcal{Z},\mathcal{I})}} \cap \Lambda\text{-mod} \subseteq \mathbf{f}''$, this shows that $\mathbf{f}' \subseteq \mathbf{f}''$. For the converse, it will be enough to show that $\mathbf{w} := {}^{\perp_0}\mathbf{f}' \cap \mathbf{f}'' = 0$. Now ${}^{\perp_0}\mathbf{f}' = \mathbf{t} \cap {}^{\perp_0}T$, where $\mathbf{t} = {}^{\perp_0}\mathbf{f}$.

So let $M \in \mathbf{w}$ and consider the object $M[-1]$ in the cosilting heart $\mathcal{H}$ corresponding to the cosilting module $C_{(\mathcal{Z},\mathcal{I})}$. We know by Theorem 1.5.11 that $\mathcal{H}$ is a locally coherent Grothendieck category whose finitely presented objects are the ones in $\mathcal{H} \cap \mathrm{D}^b(\Lambda\text{-mod})$. In particular, $M[-1]$ is a finitely presented object, thus if $M[-1] \neq 0$ it must have some simple quotient S in the heart.

Recall that the simple objects in $\mathcal{H}$ are in bijection with characteristic bricks for the cosilting torsion pair and in particular, as seen in Remark 4.2.31, a simple S corresponds to the unique neg-isolated module Z in $\mathcal{Z}$ such that $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(S, \mu_Z) \neq 0$ or to the unique indecomposable injective $I \in \mathcal{I}$ with $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(S, 0 \to I) \neq 0$.

But the fact that $M \in \mathcal{C}_{\sigma(\mathcal{Z}\setminus\{X\},\mathcal{I})}$ implies by Lemma 4.1.5 that, for every $Z \in \mathcal{Z} \setminus \{X\}$, $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(M[-1], \mu_Z) = 0$ and $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(M[-1], 0 \to I) = 0$ for all $I \in \mathcal{I}$.

This means that the simple quotient S must be the one corresponding to X, that is $T[-1]$. However, this would show the existence of a non-zero map $M \to T$. This contradicts our hypothesis on M, thus $\mathbf{w} = 0$ and this shows that $\mathbf{f}' = \mathbf{f}''$.

The proof in the case of the indecomposable injective is identical, using Corollary 4.2.19. □

The next lemma fixes the asymmetry in the definition of an almost complete pair:

Lemma 4.2.41. *Let $(\mathcal{U},\mathcal{I})$ be a rigid pair. Then $(\mathcal{U},\mathcal{I})$ is almost complete with respect to a very critical point if and only if it is almost complete with respect to a special point or indecomposable injective.*

Proof. We start the proof in the case of a rigid pair $(\mathcal{U},\mathcal{I})$ which is almost complete with respect to a very critical point X. Let F be the torsionfree, almost torsion module corresponding to X.

Consider the cosilting pair $(\mathcal{Z},\mathcal{I}) = (\mathcal{U} \cup \{X\},\mathcal{I})$, and fix $\mathcal{F} = \mathrm{Cogen}(\mathcal{Z}) = \varinjlim \mathbf{f}$, where $\mathbf{f} = \mathcal{F} \cap \Lambda\text{-mod}$.

By Lemma 4.2.37 we have $\mathcal{F}' := \mathrm{Cogen}(\mathcal{U}) = \mathcal{F} \cap F^{\perp_0}$. This is a cosilting torsionfree class, in particular, we have a corresponding cosilting pair $(\mathcal{U}',\mathcal{I}')$ extending the rigid pair $(\mathcal{U},\mathcal{I})$: every element of $\mathcal{U}$ is Ext-injective in $\mathrm{Cogen}(\mathcal{Z}) \supseteq \mathrm{Cogen}(\mathcal{U})$, thus it is also an element of $\mathcal{U}'$ being in $\mathrm{Prod}(C_{(\mathcal{U}',\mathcal{I}')})$ and $\mathcal{I}' = \mathrm{Cogen}(\mathcal{U}')^{\perp_0} \cap \mathrm{inj}(\Lambda) = \mathrm{Cogen}(\mathcal{U})^{\perp_0} \cap \mathrm{inj}(\Lambda) \supseteq \mathrm{Cogen}(\mathcal{Z})^{\perp_0} \cap \mathrm{inj}(\Lambda) = \mathcal{I}$.

Moreover, the module F is torsion, almost torsionfree with respect to the torsion pair $({}^{\perp_0}\mathcal{F}', \mathcal{F}')$, in particular we have either a special point $X' \in \mathcal{U}'$ or an injective $I' \in \mathcal{I}'$ corresponding to it.

Assume we are in the special point case, we want to prove that $\mathcal{U}' = \mathcal{U} \cup \{X'\}$.

Applying Lemma 4.2.40 we obtain that $\mathcal{C}_{\sigma(\mathcal{U}'\setminus\{X'\},\mathcal{I}')} = \varinjlim \widetilde{\mathbf{F}}(\mathbf{f}' \cup \{F\})$. However, this is a definable torsionfree class strictly containing $\mathcal{F}'$ and contained in $\mathcal{F}$, therefore since these two classes cover each other, we must have $\mathcal{F} = \mathcal{C}_{\sigma(\mathcal{U}'\setminus\{X'\},\mathcal{I}')}$.

In particular, we have that $(\mathcal{Z},\mathcal{I})$ is a cosilting pair completing $(\mathcal{U}'\setminus\{X'\},\mathcal{I}')$, so that $\mathcal{I}'=\mathcal{I}$ and $\mathcal{U}'\setminus\{X'\}\subseteq\mathcal{Z}=\mathcal{U}\cup\{X\}$. However, X can not be an element of $\mathcal{U}'$, otherwise $F\in\mathcal{F}'$, thus $\mathcal{U}'\setminus\{X'\}=\mathcal{U}$.

In the injective case, we want to show that $\mathcal{I}'=\mathcal{I}\cup\{I'\}$. Again, we can apply Lemma 4.2.40 to obtain that $\mathcal{C}_{\sigma(\mathcal{U}',\mathcal{I}'\setminus\{I'\})}=\varinjlim\widetilde{\mathbf{F}}(\mathbf{f}'\cup\{F\})=\mathcal{F}$ and conclude using that $\mathcal{I}\subseteq\mathcal{I}'$, but $\mathcal{I}'\setminus\{I'\}\subseteq\mathcal{I}$.

Assume now that $(\mathcal{U},\mathcal{I})$ is almost complete with respect to an indecomposable injective I. Let T be the corresponding torsion, almost torsionfree module. Let $\mathcal{I}'=\mathcal{I}\cup\{I\}$ and $\mathcal{F}=\varinjlim\mathbf{f}=\mathcal{C}_{\sigma(\mathcal{U},\mathcal{I}')}$.

Using Lemma 4.2.40 we obtain that $\mathcal{F}':=\mathcal{C}_{\sigma(\mathcal{U},\mathcal{I})}=\varinjlim\widetilde{\mathbf{F}}(\mathbf{f}\cup\{T\})$ covers $\mathcal{F}$ in the lattice of definable torsionfree classes.

As the class $\mathcal{F}'$ is cosilting, we have a cosilting pair $(\mathcal{Z},\mathcal{J})$ extending $(\mathcal{U},\mathcal{I})$ with $\mathrm{Cogen}(\mathcal{Z})=\mathcal{C}_{\sigma(\mathcal{Z},\mathcal{J})}=\mathcal{F}'$. Then T is a finitely generated torsionfree, almost torsion module in $\mathcal{F}'$, thus it corresponds to a very critical point $X'\in\mathcal{Z}$.

Using Lemma 4.2.37 we obtain that the class $\mathrm{Cogen}(\mathcal{Z}\setminus\{X'\})=\mathcal{F}'\cap T^{\perp_0}$, therefore it must coincide with $\mathcal{F}$ and thus $(\mathcal{U},\mathcal{I}')$ is a completion of $(\mathcal{Z}\setminus\{X'\},\mathcal{J})$ so that $\mathcal{Z}\setminus\{X'\}\subseteq\mathcal{U}$, but knowing that $\mathcal{U}\subseteq\mathcal{Z}$ we must have $\mathcal{Z}\setminus\{X'\}=\mathcal{U}$. Moreover $\mathcal{I}\subseteq\mathcal{J}\subseteq\mathcal{I}'=\mathcal{I}\cup\{I\}$. However, $\mathcal{J}$ does not contain I as $T\in\mathcal{F}'$ and thus $\mathcal{J}=\mathcal{I}$.

The case in which the rigid pair is almost complete with respect to a special point can be treated in the same way. □

From the proof of the previous lemma we can extract the following proposition, ensuring that every almost complete rigid pair admits exactly two complements, this result is the cosilting analogue of [1, Theorem 2.18]:

Proposition 4.2.42. *Given an almost complete rigid pair $(\mathcal{U},\mathcal{I})$, there exists exactly two cosilting pairs completing it: one is obtained by adding a very critical point, the corresponding cosilting class is $\mathcal{C}_{\sigma(\mathcal{U},\mathcal{I})}$, the other by adding a special point or an indecomposable injective, the corresponding class is* $\mathrm{Cogen}(\mathcal{U})$.

Proof. In the proof of Lemma 4.2.41 we see that there are two completions $(\mathcal{U}',\mathcal{I}')$ and $(\mathcal{U}'',\mathcal{I}'')$ with the required properties.

Let $(\mathcal{V},\mathcal{J})$ be a cosilting pair completing $(\mathcal{U},\mathcal{I})$, then $\mathcal{F}=\mathrm{Cogen}(\mathcal{U})\subseteq\mathrm{Cogen}(\mathcal{V})=\mathcal{C}_{\sigma(\mathcal{V},\mathcal{J})}\subseteq\mathcal{C}_{\sigma(\mathcal{U},\mathcal{I})}=\mathcal{F}'$.

As $\mathcal{F}'$ covers $\mathcal{F}$, we have that $(\mathcal{V},\mathcal{J})$ is either $(\mathcal{U}',\mathcal{I}')$ or $(\mathcal{U}'',\mathcal{I}'')$. □

In conclusion, we have the following theorem describing the process of mutation:

Theorem 4.2.43. *Let $(\mathcal{Z},\mathcal{I})$ be a cosilting pair. The following procedure yields the cosilting pairs corresponding to the minimal extensions and coextensions of $\mathcal{F}=\mathrm{Cogen}(\mathcal{Z})$:*

- *Given F, a finitely generated torsionfree, almost torsion module and X the corresponding very critical point in $\mathcal{Z}$, then the cosilting pair corresponding to the coextension $\mathcal{V}=\mathrm{Cogen}(\mathcal{Z})\cap F^{\perp_0}$ is:*

  (i) $((\mathcal{Z} \setminus \{X\}) \cup \{X'\}, \mathcal{I})$ *if* F *has a surjective* $\mathcal{V}-$*cover and* X' *is the kernel of this cover.*

  (ii) $(\mathcal{Z} \setminus \{X\}, \mathcal{I} \cup \{E(S)\})$ *if* F *has an injective* $\mathcal{V}-$*cover and* S *is the simple module obtained as the cokernel of this cover.*

- *If* T *is torsion, almost torsionfree module, then the cosilting pair corresponding to* $\mathrm{Cogen}(C') = \varinjlim \widetilde{F}(\mathbf{f} \cup \{T\})$ *is:*

  (i) $((\mathcal{Z} \setminus \{X\}) \cup \{X'\}, \mathcal{I})$, *if* T *has a surjective* $\mathcal{F}-$*cover and* X *is the corresponding special point in* $\mathcal{Z}$

  (i) $(\mathcal{Z} \cup \{X'\}, \mathcal{I} \setminus \{E(S)\})$, *if* T *has an injective* $\mathcal{F}-$*cover and* S *is the simple cokernel of such a cover.*

  where X' *is a* $\mathrm{Prod}(C')-$*envelope of* T*.*

Proof. The proof is just a collection of the previous lemmas. I will give some details for the torsion, almost torsionfree case: by Lemma 4.2.41 we know that we must substitute precisely one object from the cosilting pair $(\mathcal{Z}, \mathcal{I})$. As the new class $\mathrm{Cogen}(C')$ must contain the module T we have to remove the obstructions to its presence. If T has an injective $\mathcal{F}-$cover then we have an indecomposable injective I, with $\mathcal{F} \subseteq {}^{\perp_0} I$, such that $\mathrm{Hom}(T, I) \neq 0$. So we must remove I from the set $\mathcal{I}$.

If T has a surjective cover, then we have a special point $X \in \mathcal{Z}$ such that $\mathrm{Ext}^1(T, X) \neq 0$, so in particular $T \notin \mathcal{C}_{\mu_X}$. Therefore in this case we must remove the point X.

To complete the modified rigid pair, recall that T becomes torsionfree, almost torsion in $\mathrm{Cogen}(C')$, therefore by Theorem 4.2.15 and Corollary 4.2.17 we must have a sequence:

$$0 \to T \to X' \to \overline{X} \to 0$$

with X' a $\mathrm{Prod}(C')-$envelope which is critical in $\mathrm{Cogen}(C')$. The module X' was not an element of $\mathcal{Z}$, as $T \notin \mathcal{F}$, thus it is the required complement. □

Appendix A

Torsion classes over the wild 3-Kronecker quiver

We discuss the structure of the lattice of torsion classes over the three Kronecker algebra $\Lambda = k(\ 1 \Rrightarrow 2\ )$, for some algebraically closed field k.

The fundamental results about wild hereditary algebras which we use can be found in the extensive survey [45].

First, recall that the indecomposable modules in Λ - mod can be divided in three sets: preprojective modules $\mathfrak{p}$, that is modules which can be obtained as inverse-AR-translations of some indecomposable projective, preinjective modules $\mathfrak{q}$ obtained as AR-translations of some indecomposable injective, and regular modules $\mathfrak{r}$, that is indecomposable modules R such that $\tau^{-n}(\tau^n R) \simeq R \simeq \tau^n(\tau^{-n} R)$ for all $n \in \mathbb{N}$.

We have the usual orthogonality properties: $\mathrm{Hom}(\mathfrak{q}, \mathfrak{r}) = \mathrm{Hom}(\mathfrak{q}, \mathfrak{p}) = \mathrm{Hom}(\mathfrak{r}, \mathfrak{p}) = 0$.

Every preprojective and preinjective module is a *stone*, that is it has no self-extensions, and thus, since the algebra is hereditary, it is also a brick. As such, we have a torsion class $\mathrm{gen}(M)$ in Λ - mod for each isoclass of preinjectives and preprojective modules, such that M is the unique torsion, almost torsionfree module.

There are countably many isomorphism classes of preprojective and preinjective modules, which we can denote in the following way:

$$
\begin{aligned}
P^0 &= \Lambda e_2 & Q^0 &= D(e_1\Lambda)\\
P^1 &= \Lambda e_1 & Q^1 &= D(e_2\Lambda)\\
P^{2n} &= \tau^{-n}(P^0) & Q^{2n} &= \tau^n(Q^0)\\
P^{2n+1} &= \tau^{-n}(P^1) & Q^{2n+1} &= \tau^n(Q^1)
\end{aligned}
$$

Notice that, for every natural $n \neq 0$ and for every $m \in \mathbb{N}$, $\mathrm{gen}(P^n) \supseteq \mathrm{gen}(P^{n+1}) \supseteq \mathrm{gen}(Q^{m+1}) \supseteq \mathrm{gen}(Q^m)$. On the other hand $\mathrm{gen}(P^0) = \mathrm{add}(S_2)$ does not properly contain any non-zero torsion class, and it is only contained in the trivial class $\mathrm{gen}(\Lambda) = \Lambda$ - mod.

What about regular modules? For any such module R we have that $R \in \mathrm{Gen}(P^n)$, for every $n \neq 0$ and $\widetilde{\mathbf{T}}(R) \supseteq \mathrm{Gen}(Q^m)$, for every m.

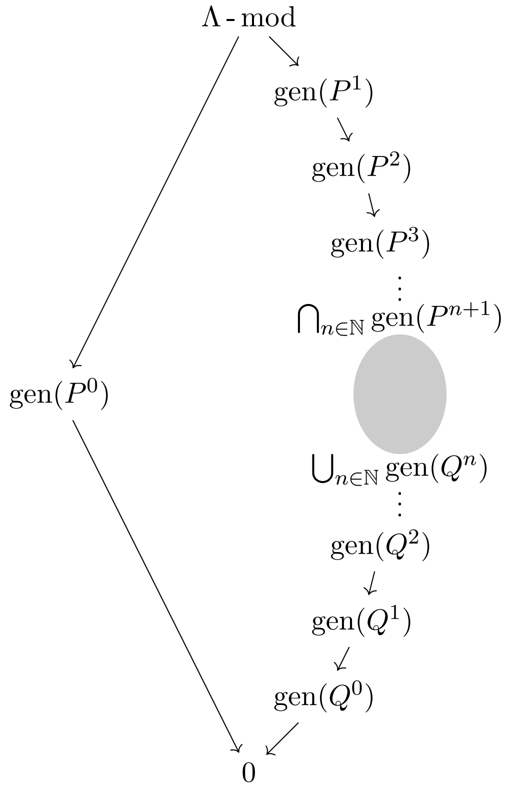


Figure A.1: Global structure of torsion classes over $k\mathcal{K}_3$

As such, the global structure of the lattice of torsion classes is close to the tame case, see Figure A.

A first important difference is that the interval

$$\left[\operatorname{gen}(\mathfrak{q}) = \bigcup_{n\in\mathbb{N}} \operatorname{gen}(Q^n),\ \operatorname{gen}(\mathfrak{r}) = \bigcap_{n\in\mathbb{N}} \operatorname{gen}(P^{n+1})\right]$$

is not wide. In fact the heart of these two classes is $\operatorname{reg}-\Lambda$, the additive closure of $\mathfrak{r}$, which is not closed under kernels and cokernels.

As in the tame case, $\operatorname{gen}(\mathfrak{q})$ does not admit any minimal co-extending module and $\operatorname{gen}(\mathfrak{r})$ does not admit any minimal extending module.

However, we will see that these classes are actually "isolated". To this aim, it will be interesting to consider the notion of elementary module:

Definition A.0.1. Let Λ be an hereditary algebra, a module $E \neq 0 \in \operatorname{reg}-\Lambda$ is said to be *elementary* if for every short exact sequence:

$$0 \to M \to E \to N \to 0$$

with $M \neq 0, N \neq 0$, either $M \notin \operatorname{reg}-\Lambda$ or $N \notin \operatorname{reg}-\Lambda$.

Since reg $-\,\Lambda$ is an additive subcategory, closed under extensions (and images), it inherits a natural exact structure from Λ - mod. It is immediate to check that elementary modules are precisely the simple objects in reg $-\,\Lambda$ relatively to this exact structure.

Notice that every regular module admits a finite filtration by elementary modules.

We have the following equivalent characterisations of elementary modules:

Proposition A.0.2. *Let Λ be a wild hereditary algebra, E an indecomposable module. The following statements are equivalent:*

(i) E is elementary.

(ii) There exists an integer N such that $\tau^l E$ has no non-trivial regular factors for all $l \geq N$.

(iii) If $Y \neq 0$ is a regular submodule of E, then E/Y is preinjective.

From which we can obtain:

Corollary A.0.3. *Every elementary module is a brick.*

Now, we need a further general result about regular modules:

Proposition A.0.4. *Let Λ be a wild hereditary algebra, X, Y non-zero regular modules, then there exists some positive integer m such that X is generated by $\tau^{-m}Y$ and cogenerated by $\tau^m Y$.*

Finally, we can show that:

Proposition A.0.5. *Let $\Lambda = k\mathcal{K}_3$. The torsion class* $\mathrm{gen}(\mathfrak{q})$ *does not admit any minimal co-extending module, nor any minimal extending module.*

Proof. We already noticed that $\mathrm{gen}(\mathfrak{q})$ does not cover any torsion class, in particular, it does not admit any minimal co-extending module.

Assume there is a torsion class $\mathbf{t}$ covering $\mathrm{gen}(\mathfrak{q})$. Then there is some regular module R, which we might assume elementary, in $\mathbf{t}$. Moreover, since this class is covering, it must be the case that $\mathbf{t} = \widetilde{\mathbf{T}}(R)$.

However, Proposition A.0.4 tells us that R generates the elementary brick $\tau^m R$, for some $m > 1$, therefore $\mathbf{t} = \widetilde{\mathbf{T}}(R) \supsetneq \widetilde{\mathbf{T}}(\tau^m R) \supsetneq \mathrm{gen}(\mathfrak{q})$, a contradiction. $\mathrm{gen}(\mathfrak{q})$ does not have any minimal extending module. $\square$

With dual arguments for the corresponding torsionfree class we obtain:

Proposition A.0.6. *Let $\Lambda = k\mathcal{K}_3$. The torsion class* $\mathrm{gen}(\mathfrak{r})$ *does not admit any minimal co-extending module, nor any minimal extending module.*

We give some further examples of pathological torsion pairs in Λ - mod:

Proposition A.0.7. *Let $\Lambda = k\mathcal{K}_3$. Let B_1 be the brick obtained as the cokernel of the map $\alpha : P_2 \to P_1$ induced by the first arrow of the quiver.*

Then the minimal extending modules with respect to $(\widetilde{\mathbf{T}}(B_1), B_1^{\perp_0} \cap \Lambda\text{-}\mathrm{mod})$ are in bijection with the finite-dimensional simple modules over the free (associative) algebra in two variables $k\langle X, Y\rangle$.

Proof. This is an immediate consequence of Lemma 3.3.10: $\beta(B_1^{\perp_0}) = B_1^{\perp_{0,1}}$, which by Remark 1.2.15 is equivalent to the category of modules over the universal localisation at the map α.

It is well-known that such universal localisation is Morita-equivalent to the ring $k\langle X, Y\rangle$.

By Proposition 3.1.12 the minimal extending modules with respect to $(\widetilde{\mathbf{T}}(B_1), B_1^{\perp_0} \cap \Lambda\text{-}\mathrm{mod})$ are precisely the finite-dimensional simples of $\beta(B_1^{\perp_0})$. □

Universal localisations are also useful in finding locally maximal torsion classes:

Proposition A.0.8. *Let $\Lambda = k\mathcal{K}_3$. Then there are orthogonal bricks B_1, B_2 such that the torsion class $(\widetilde{\mathbf{T}}(B_1, B_2), \{B_1, B_2\}^{\perp_0} \cap \Lambda\text{-}\mathrm{mod})$ has no minimal extending modules.*

As an example, it is enough to chose orthogonal bricks with dimension vector $(1, 1)$ and $(1, 2)$: in fact $\beta(\{B_1, B_2\}^{\perp_0}) = \{B_1, B_2\}^{\perp_{0,1}}$, but if a finite-dimensional module M is perpendicular to both B_1 and B_2 then its dimension vector must be orthogonal to both $(1, 1)$ and $(1, 2)$ with respect to the Euler form associated to $k\mathcal{K}_3$. This means that $\widetilde{\beta}(\{B_1, B_2\}^{\perp_0}) = 0$.

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