

Fast Procedures for Accurate Parameter Estimation of Sine-Waves Affected by Noise and Harmonic Distortion

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Abstract — *In this paper two procedures are proposed for the estimation of the frequency, amplitude, and phase of sine-waves affected by either wideband noise or by both noise and harmonic distortion, respectively. The procedures are very simple and suitable for real-time applications. According to them, firstly the signal parameters are estimated by means of the new Corrected Interpolated Discrete Fourier transform (IpDFTc) algorithm, which compensates the contribution of the spectral interference from the fundamental image component and harmonics on the parameter estimates returned by the classical IpDFT algorithm. Then, a linear sine-fit algorithm is applied to optimize the noise robustness of the estimators, which is worsened by signal windowing employed to reduce the effect of spectral leakage on the IpDFTc estimates. In the paper, the Hann window is applied since it exhibit both the highest noise robustness among the Maximum Sidelobe Decay (MSD) windows and a good reduction of long-range spectral leakage. It has been shown that both proposed procedures almost attain the Cramér-Rao Lower Bounds (CRLBs) for unbiased estimators when at least about 1.5 sine-wave cycles are observed so that the effect of interfering tones on the IpDFTc estimates can be effectively compensated by windowing. The performances of both procedures are analysed through computer simulations and experimental results.*

Keywords: Harmonics, interpolated DFT algorithms, linear sine-fit algorithm, parameter estimation, sine-waves.

1. Introduction

High spectral purity sine-waves are used in ICs or electronic components testing [1, 2]. Conversely, sine-waves encountered in common application fields, such as communications, power systems, and vibration analysis, are often corrupted by disturbances like harmonics and spurious tones. In both situations, sine-waves are affected by wideband noise. Thus, noise immunity is a relevant characteristic of all sine-wave parameter estimation algorithms. The classical Pisarenko harmonic decomposition method, MUSIC and ESPRIT algorithms can provide optimal wideband noise rejection capability, but they strongly rely on the accuracy of the adopted signal model. Moreover the required processing effort is quite high [3, 4]. Other state-of-the-art time-domain algorithms, such as estimators based on linear prediction [5, 6], singular value decomposition [7], or autocorrelation [8] are affected by the same drawbacks. Time-domain methods widely adopted in Analog-to-Digital (ADC) testing are the three-parameter sine-fit (3PSF) and the four-parameter sine-fit (4PSF) algorithms [1, 2, 9-12], which are named according to the number of parameters to be estimated. Specifically, the 3PSF algorithm assumes that the sine-wave frequency is *a-priori* known and it returns estimates for the waveform amplitude, phase, and offset by solving a simple linear least-squares problem that exploits closed-form formulas requiring matrix inversion. Conversely, the 4PSF algorithm assumes that all the sine-wave parameters are unknown and the resulting nonlinear estimation problem is solved through an iterative and computationally expensive procedure. Both the 3PSF and the 4PSF algorithms provide statistically optimal estimators if the noise superimposed to the sine-wave is Gaussian and sufficiently accurate initial frequency estimates are available [12, 13]. Unfortunately, both these algorithms exhibit low immunity to harmonics and spurious tones [1, 2, 14]. To overcome that issue the so called multi-harmonic sine-fit (MHSF) algorithms have been proposed in the case of either known or unknown signal frequency. However their computational burden is quite high [15].

Frequency-domain algorithms based on the Discrete Fourier Transform (DFT) are often used when fast and accurate sine-wave parameter estimates are required. Unfortunately, since in practice non-coherent sampling usually occurs, frequency estimates returned by these algorithms are affected by the so called picket-fence effect [16, 17]. A variety of Interpolated DFT (IpDFT) algorithms has been proposed to

compensate that detrimental effect in either real-valued [16-24] or complex-valued [25-29] sine-waves. These algorithms evaluate the inter-bin frequency location using two or more DFT samples as interpolation points. Then, the related sine-wave amplitude and phase are estimated exploiting the knowledge of the obtained frequency value. However, when multi-tone signals are observed, estimation accuracy is limited by the mutual interference between tones induced by the spectral leakage that arises because of the finite length of the observation interval [16]. To reduce that phenomenon, the acquired signal is weighted by a suitable window sequence before applying the IpDFT algorithm [30]. Cosine windows are often used, since they are simple to be implemented and ensure a good leakage reduction capability [30], [31]. In particular, the Maximum Sidelobe Decay (MSD) cosine windows [31] are often adopted because of two main reasons. Firstly, they exhibit the highest sidelobe decay rate among the cosine windows with a given number of terms, thus ensuring an optimal long-range spectral leakage reduction. Secondly, the related IpDFT estimators can be expressed with simple formulas [16, 18].

Unfortunately, any IpDFT algorithms cannot provide optimal robustness to wideband noise and the uncertainties of the returned estimates do not attain the corresponding Cramér-Rao Lower Bounds (CRLBs) [13]. This is mainly due to both windowing and the use of only two or a few DFT samples in the interpolation formulas [24]. When real-valued sine-waves are analyzed, instead of reducing the spectral interference from the fundamental image component by windowing, a three-step iterative algorithm has been proposed in [32]. In the first step a coarse frequency estimate is obtained. In the second step the fundamental image component is removed from the signal. Finally, the third step estimates the sine-wave frequency by applying the iterative IpDFT algorithm proposed in [25] to the residual signal. In such a way, the impact of wideband noise on the estimated frequency is minimized and an estimation variance very close to the CRLB is obtained. Leveraging on the same idea, an iterative algorithm that compensates the contribution of the fundamental image component on the frequency estimator presented in [25] has been developed in [33]. The achieved estimators attain the CRLB and, in particular, the frequency estimator outperforms the one proposed in [32] when few signal samples are acquired. However, even if both these estimators attain the CRLB, they require the application of an iterative procedure and they assume that a pure sine-wave is

acquired. A new iterative estimator for the parameters of harmonically distorted noisy sine-waves that is based on the results reported in [25] has been proposed in [33]. It can be applied to both complex-valued and real-valued signals and it is capable to attain the asymptotic (that is long data record) CRLB, which coincides with the CRLB of complex-valued single-tone estimators.

As known, the 4PSF and the MHSF algorithms with initial frequency estimated by the IpDFT algorithm provides almost optimal sine-wave parameter estimates [12, 15], but - as already specified above - they require a significant processing effort. The goal of this paper is to overcome this drawback by using a two-step approach. The first step provides a preliminary estimation of the sine-wave parameters using a new modified version of the classical two-point IpDFT algorithm. Then, instead of using the 4PSF or the MHSF algorithms, the obtained estimates are refined by applying a linear sine-fit algorithm based on the Gauss-Newton method [13]. Two sine-wave parameter estimation procedures are proposed: in the first one the available data are modeled as a distortion free noisy sine-wave, while the second procedure takes into account also harmonic distortion.

The modified version of the classical two-point IpDFT algorithm based on the two-terms MSD (or Hann) window [31] – which is called corrected IpDFT (IpDFTc) algorithm - compensates the contribution on the estimated parameters of the spectral interference from the fundamental image component and harmonics by leveraging on the analytical expressions presented in [35] and [36]. It is worth noticing that an IpDFT algorithm based on the same idea has been proposed in [19]. However, that algorithm estimates the signal parameters through an iterative procedure, while the proposed IpDFTc algorithm exploits analytical expressions, thus ensuring fast and easy application.

The proposed estimation procedures exploits the combined application of two simple algorithms: the IpDFTc algorithm and a linear sine-fit method [13]. The IpDFTc algorithm returns a first estimate of the signal parameters, which is then refined by using a linear sine-fit algorithm based on one iteration of the Gauss-Newton method [13]. Both proposed procedures are very fast, simple to be implemented, and exhibit almost optimal accuracy when at least 1.5 signal cycles are observed. The procedure aimed at distortion-free noisy sine-waves is called Parameter estimation by linear Sine-Fit and IpDFTc (PSF-IpDFTc)

algorithm, while the one proposed for harmonically distorted noisy sine-waves is called Harmonics PSF-IpDFTc (HPSF-IpDFTc) algorithm. These procedures and the IpDFTc algorithm represent the main contributions of the paper.

The performance of the proposed sine-wave parameter estimators are analyzed and compared with those of several state-of-the-art algorithms using both computer simulations and experimental results.

2. The linear sine-fit algorithm for parameter estimation

Let's assume that a continuous-time harmonically distorted noisy sine-wave is sampled with a constant sampling rate $f_s = 1/T_s$. Thus, the record of M samples acquired at the sampling instants $t_m, m = 0, 1, \dots, M-1$ can be modeled as:

$$x(t_m) = \sum_{k=1}^H A_k \cos(2\pi k f_1 t_m + \phi_k) + e(t_m), \quad m = 0, 1, 2, \dots, M-1 \quad (1)$$

where A_1, f_1 , and ϕ_1 are the amplitude, input frequency, and the offset of the fundamental component (which corresponds to $k = 1$), $A_k, k f_1$, and ϕ_k are the amplitude, frequency, and phase of the k th harmonic ($k = 2, 3, \dots, H$), and $e(\cdot)$ is a wideband noise with Gaussian distribution of zero mean and variance σ^2 . The Signal-to-Noise Ratio (SNR) of (1) is defined as $SNR \stackrel{\text{def}}{=} A_1^2 / (2\sigma^2)$.

The ratio between the sine-wave frequency f_1 and the sampling frequency f_s represents the normalized sine-wave frequency, which can be expressed as:

$$f \stackrel{\text{def}}{=} \frac{f_1}{f_s} = \frac{\nu}{M} = \frac{l + \delta}{M}, \quad (2)$$

where $\nu = l + \delta$ is the number of acquired signal cycles, in which l is its integer part and δ is the inter-bin frequency location ($-0.5 \leq \delta < 0.5$). When sampling is coherent, i.e., $\delta = 0$, the picket-fence effect does not affect the estimated normalized frequency. Unfortunately, non-coherent sampling usually occurs in practice and the inter-bin frequency location δ has to be estimated by suitable processing.

From (2) it follows that the number of acquired signal cycles ν , or normalized frequency expressed in bins, is equal to $\nu = M \cdot f$. That variable will be widely used in the following analysis.

When frequency SNR , equal to $M \cdot SNR$ [17], is higher than a threshold of about 18 dB, the integer part l of the number of acquired signal cycles coincides, with very high probability, with the location of the largest sample of the DFT spectrum of the signal (1) [17, 37]. Thus, the estimated number of acquired cycles $\hat{\nu}$ can be written as $\hat{\nu} = l + \hat{\delta}$, and from (2) it follows that the corresponding correction term for the normalized frequency is:

$$\Delta f = \frac{\Delta f_1}{f_s} = \frac{\Delta \nu}{M} = \frac{\Delta \delta}{M}, \quad (3)$$

where $\Delta \nu = \hat{\nu} - \nu = \Delta \delta = \hat{\delta} - \delta$ are the number of acquired signal cycles and inter-bin frequency location correction terms, respectively.

Let $\hat{f}_1$, $\hat{A}_1$, and $\hat{\phi}_1$ be the estimators of the unknown frequency f_1 , amplitude A_1 , and phase ϕ_1 so that the differences $\Delta f_1 = \hat{f}_1 - f_1$, $\Delta A_1 = A_1 - \hat{A}_1$, and $\Delta \phi_1 = \phi_1 - \hat{\phi}_1$ represents the corresponding error correction terms. Since the IpDFTc algorithm returns accurate estimates when a sufficient number of sine-wave cycles is observed [17, 18], in (1) we can assume $\hat{f}_1 \cong f_1$, $\hat{A}_1 \cong A_1$, and $\hat{\phi}_1 \cong \phi_1$ and refine the initial estimates by applying the Gauss-Newton method [13].

By using the Taylor's series expansion of the cosine function around $\hat{f}_1$ truncated to the first order term we obtain:

$$x(t_m) \cong \sum_{k=1}^H A_k \cos(2\pi k \hat{f}_1 t_m + \phi_k) - 2\pi \left(\sum_{k=1}^H k A_k t_m \sin(2\pi k \hat{f}_1 t_m + \phi_k) \right) \Delta f_1 + e(t_m), \quad (4)$$

which, using (3) and the above approximations, becomes:

$$x(t_m) \cong \sum_{k=1}^H \hat{A}_k \cos(2\pi k \hat{f}_1 t_m + \hat{\phi}_k) - \frac{2\pi f_s}{M} \left(\sum_{k=1}^H k \hat{A}_k t_m \sin(2\pi k \hat{f}_1 t_m + \hat{\phi}_k) \right) \Delta \nu + e(t_m). \quad (5)$$

Expression (5) can be written as follows:

$$\varepsilon(t_m) \cong -\frac{2\pi f_s}{M} \left(\sum_{k=1}^H k \hat{A}_k t_m \sin(2\pi k \hat{f}_1 t_m + \hat{\phi}_k) \right) \Delta \nu + e(t_m), \quad (6)$$

where $\varepsilon(t_m) = x(t_m) - \sum_{k=1}^H \hat{A}_k \cos(2\pi k \hat{f}_1 t_m + \hat{\phi}_k)$.

By applying the least-squares approach to (6), the following expression for Δv is finally achieved:

$$\Delta v = (D_v^T D_v)^{-1} D_v^T \varepsilon(t_m) = \frac{M}{2\pi f_s} \frac{\sum_{m=0}^{M-1} \hat{s}_v(t_m) \varepsilon(t_m)}{\sum_{m=0}^{M-1} \hat{s}_v^2(t_m)}, \quad (7)$$

where $D_v = [\hat{s}_v(t_0) \ \hat{s}_v(t_1) \ \dots \ \hat{s}_v(t_{M-1})]^T$, in which $\hat{s}_v(t_m) = -\sum_{k=1}^H k \hat{A}_k t_m \sin(2\pi k \hat{f}_1 t_m + \hat{\phi}_k)$, and

$(\cdot)^T$ is the transpose operator,

Thus, the new estimate for the number of acquired sine-wave cycles is $\tilde{v} = \hat{v} + \Delta v$.

In addition, by replacing $A_1 = \hat{A}_1 + \Delta A_1$ in (1) and since $\hat{f}_1 \cong f_1$ and $\hat{\phi}_k \cong \phi_k$, we have:

$$x(t_m) \cong \sum_{k=1}^H \hat{A}_k \cos(2\pi k \hat{f}_1 t_m + \hat{\phi}_k) + \cos(2\pi \hat{f}_1 t_m + \hat{\phi}_1) \Delta A_1 + e(t_m), \quad (8)$$

from which:

$$\varepsilon(t_m) \cong \cos(2\pi \hat{f}_1 t_m + \hat{\phi}_1) \Delta A_1 + e(t_m). \quad (9)$$

By applying the least-squares approach to (9), we obtain:

$$\Delta A_1 = (D_a^T D_a)^{-1} D_a^T \varepsilon(t_m) = \frac{\sum_{m=0}^{M-1} \hat{s}_a(t_m) \varepsilon(t_m)}{\sum_{m=0}^{M-1} \hat{s}_a^2(t_m)}, \quad (10)$$

where $D_a = [\hat{s}_a(t_0) \ \hat{s}_a(t_1) \ \dots \ \hat{s}_a(t_{M-1})]^T$, in which $\hat{s}_a(t_m) = \cos(2\pi \hat{f}_1 t_m + \hat{\phi}_1)$,

Hence, the new estimate of the fundamental amplitude is $\tilde{A}_1 = \hat{A}_1 + \Delta A_1$.

Finally, by expanding the cosine function in (1) using the Taylor's series around $\hat{\phi}_1$ truncated to the first order term and since $\hat{f}_1 \cong f_1$ and $\hat{A}_k \cong A_k$, we obtain:

$$x(t_m) \cong \sum_{k=1}^H \hat{A}_k \cos(2\pi k \hat{f}_1 t_m + \hat{\phi}_k) - A_1 \sin(2\pi \hat{f}_1 t_m + \hat{\phi}_1) \Delta \phi_1 + e(t_m), \quad (11)$$

which can be written as:

$$\varepsilon(t_m) \cong -A_1 \sin(2\pi k \hat{f}_1 t_m + \hat{\phi}_1) \Delta \phi_1 + e(t_m). \quad (12)$$

Thus, by applying the least-squares approach to (12), we have:

$$\Delta\phi_1 = (D_\phi^T D_\phi)^{-1} D_\phi^T \varepsilon(t_m) = \frac{\sum_{m=0}^{M-1} \hat{s}_\phi(t_m) \varepsilon(t_m)}{\sum_{m=0}^{M-1} \hat{s}_\phi^2(t_m)}, \quad (13)$$

where $D_\phi = [\hat{s}_\phi(t_0) \ \hat{s}_\phi(t_1) \ \dots \ \hat{s}_\phi(t_{M-1})]^T$, in which $\hat{s}_\phi(t_m) = -\hat{A}_1 \sin(2\pi \hat{f}_1 t_m + \hat{\phi}_1)$, and the new estimate for the phase of the fundamental phase is $\tilde{\phi}_1 = \hat{\phi}_1 + \Delta\phi_1$.

Simulations showed that, when the uncertainties of the estimators $\hat{A}_k$, and $\hat{\phi}_k$ are enough small, as may happen when they are provided by the IpDFTc algorithm, the variances of the estimator $\tilde{v}$, $\tilde{A}_1$, and $\tilde{\phi}_1$ can attain the related CRLBs for unbiased estimators, i.e., $(\sigma_v^2)_{CRLB} \cong 3/(\pi^2 \cdot M \cdot SNR)$, $(\sigma_A^2)_{CRLB} \cong \frac{2\sigma^2}{M}$, and $(\sigma_\phi^2)_{CRLB} \cong 1/(M \cdot SNR)$, respectively [13, 17].

3. The PSF-based and HPSF-based procedures

In the following the proposed PSF-IpDFTc and HPSF-IpDFTc parameter estimators for a noisy sine-wave and a harmonically distorted noisy sine-wave are presented. Both of them use the linear sine-fit algorithm presented in the previous Section. The parameters of the fundamental component of the signal (1) are estimated by means of the IpDFTc algorithm based on the Hann window, which is defined as [30, 31]:

$$w(m) \stackrel{\text{def}}{=} 0.5 \cos\left(\frac{2\pi}{M} \left(m - \frac{M-1}{2}\right)\right) + 0.5 \cos\left(\frac{4\pi}{M} \left(m - \frac{M-1}{2}\right)\right), m = 0, 1, 2, \dots, M-1 \quad (14)$$

That window has been chosen since it ensures the smallest variances of the IpDFT estimators due to wideband noise in the class of MSD windows [18]. In addition, the Hann window ensures an efficient reduction of spectral interferences from the fundamental image component and harmonics thanks to its good decay rate, equal to 18 dB/octave [31].

To ensure that the phase estimators $\hat{\phi}_k$ are independent of the frequency estimator $\hat{v}$, the time instants are chosen equal to: $t_m = \left(m - \frac{M-1}{2}\right) T_s$, $m = 0, 1, \dots, M-1$ [17, 38]. Thus, the analyzed discrete-time signal is:

$$x(m) = \sum_{k=1}^H A_k \cos\left(2\pi k \frac{\nu}{M} \left(m - \frac{M-1}{2}\right) + \phi_k\right) + e(m), \quad m = 0, 1, \dots, M-1 \quad (15)$$

The Discrete-Time Fourier Transform (DTFT) of the windowed signal $x_w(m) = x(m) \cdot w(m)$ is given by:

$$\begin{aligned} X_w(\lambda) &\stackrel{\text{def}}{=} \sum_{m=0}^{M-1} x_w(m) e^{-j2\pi \frac{\nu}{M} \left(m - \frac{M-1}{2}\right)} \\ &= \sum_{k=1}^H \frac{A_k}{2} [W(\lambda - k\nu) e^{j\phi_k} + W(\lambda + k\nu) e^{-j\phi_k}] + E_w(\lambda), \quad \lambda \in [0, M) \end{aligned} \quad (16)$$

where $E_w(\cdot)$ is the DTFT of the windowed noise $e_w(m) = e(m) \cdot w(m)$, $w(\cdot)$ is the Hann window, and $W(\cdot)$ is its DTFT:

$$W(\lambda) \stackrel{\text{def}}{=} \sum_{m=0}^{M-1} w(m) e^{-j2\pi \frac{\lambda}{M} \left(m - \frac{M-1}{2}\right)} = \frac{M \sin(\pi \lambda)}{2\pi \lambda (1 - \lambda^2)}, \quad \lambda \in [0, M) \quad (17)$$

which is a real-valued even function.

The estimators for the parameters ν , A_1 , and ϕ_1 provided by the IpDFTc algorithm based on the Hann window are:

- estimator $\hat{\nu}$

$$\hat{\nu} = \hat{\nu}_0 + \Delta_c \nu, \quad (18)$$

in which $\hat{\nu}_0 = l + \hat{\delta}_0$ where the estimator $\hat{\delta}_0$ is returned by the IpDFT algorithm based on the Hann

window, that is [18] $\hat{\delta}_0 = \frac{(i+1)\alpha+i-2}{\alpha+1}$ where $\alpha = \frac{|X_w(l+i)|}{|X_w(l+i-1)|}$ with $i = 0$ if $|X_w(l-1)| > |X_w(l+1)|$ and $i = 1$

if $|X_w(l-1)| < |X_w(l+1)|$. In (18) the correction term $\Delta_c \nu$ is given by [35]:

$$\begin{aligned} \Delta_c \nu = \Delta_c \delta = & -(2 + (-1)^i \hat{\delta}_0) \left[\frac{(-1)^i 2 \hat{\nu}_0}{2 \hat{\nu}_0 - \hat{\delta}_0 + (-1)^{i+1} 2} \frac{W(2 \hat{\nu}_0 - \hat{\delta}_0)}{W(\hat{\delta}_0)} \cos(2 \hat{\phi}_{10}) \right. \\ & \left. + \sum_{k=2}^H \frac{(-1)^i (k-1)}{(k-1) \hat{\nu}_0 + \hat{\delta}_0 + (-1)^i 2} \frac{\hat{A}_{k0}}{\hat{A}_{10}} \frac{W((k-1) \hat{\nu}_0 + \hat{\delta}_0)}{W(\hat{\delta}_0)} \cos(\hat{\phi}_{k0} - \hat{\phi}_{10}) \right], \end{aligned}$$

in which $W(\cdot)$ is given by (17) and $\hat{A}_{k0}$ and $\hat{\phi}_{k0}$ are the amplitude and phase estimators of the fundamental ($k = 1$) and harmonic ($k \geq 2$) components returned by the IpDFT algorithm based on the Hann window, i.e.:

$$\hat{A}_{10} = \frac{2|X_w(l)|}{W(\hat{\delta}_0)}, \quad \hat{\phi}_{10} = \text{angle}\{X_w(l)\}$$

and:

$$\hat{A}_{k0} = \frac{2|X_{rw}(l_k)|}{W(\hat{\delta}_{k0})}, \quad \hat{\phi}_{k0} = \text{angle}\{X_{rw}(l_k)\}, \quad k = 2, 3, \dots, H$$

in which X_{rw} is the DTFT of the residual windowed signal $x_{rw}(m) = x_r(m) \cdot w(m)$, in which $x_r(\cdot)$ is obtained from (15) after the removal of the fundamental component, that is: $x_r(m) = x(m) - \hat{A}_{10} \cos\left(2\pi \frac{\hat{\nu}_0}{M} \left(m - \frac{M-1}{2}\right) + \hat{\phi}_{10}\right)$, $m = 0, 1, \dots, M-1$, $l_k = \text{round}(k\hat{\nu}_0)$, where $\text{round}(z)$ returns the integer closest to z , and $\hat{\delta}_{k0} = k\hat{\nu}_0 - l_k$.

• *estimator $\hat{A}_1$*

$$\hat{A}_1 = \hat{A}_{10c} + \Delta_c A_1, \quad (19)$$

where $\hat{A}_{10c} = \frac{2|X_w(l)|}{W(\hat{\delta}_0 + \Delta_c \delta)}$ and $\Delta_c A_1$ is the related correction term, given by [36]:

$$\Delta_c A_1 = -\hat{A}_{10} \frac{W(2\hat{\nu}_0 - \hat{\delta}_0)}{W(\hat{\delta}_0)} \cos(2\hat{\phi}_{10}) - \sum_{k=2}^H \hat{A}_{k0} \frac{W((k-1)\hat{\nu}_0 + \hat{\delta}_0)}{W(\hat{\delta}_0)} \cos(\hat{\phi}_{k0} - \hat{\phi}_{10}).$$

• *estimator $\hat{\phi}_1$*

$$\hat{\phi}_1 = \hat{\phi}_{10} + \Delta_c \phi_1, \quad (20)$$

where $\Delta_c \phi_1$ is the ϕ_1 correction term, given by [36]:

$$\Delta_c \phi_1 = \frac{W(2\hat{\nu}_0 - \hat{\delta}_0)}{W(\hat{\delta}_0)} \sin(2\hat{\phi}_{10}) - \sum_{k=2}^H \frac{\hat{A}_{k0}}{\hat{A}_{10}} \frac{W((k-1)\hat{\nu}_0 + \hat{\delta}_0)}{W(\hat{\delta}_0)} \sin(\hat{\phi}_{k0} - \hat{\phi}_{10}).$$

Of course, for noisy sine-waves, the harmonics contributions are null in the expressions above.

The steps required to perform the PSF-IpDFTc and the HPSF-IpDFTc procedures are presented in Table 1 and Table 2, respectively.

Table 1. Steps of the PSF-IpDFTc procedure.

<i>Step 1: Apply the IpDFT algorithm based on the Hann window to the analyzed signal and determine the estimators $\hat{v}_0$, $\hat{A}_{10}$, and $\hat{\phi}_{10}$.</i>
<i>Step 2: Compute the estimators $\hat{v}$, $\hat{A}_1$, and $\hat{\phi}_1$ by using (18), (19), and (20) neglecting the harmonics contribution.</i>
<i>Step 3: Compute the correction terms Δv, ΔA_1, and $\Delta \phi_1$ by using (7), (10), and (13) in which</i> $\varepsilon(m) = x(m) - \hat{A}_1 \cos\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right),$ $\hat{s}_v(m) = -\hat{A}_1\left(m - \frac{M-1}{2}\right) T_s \sin\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right),$ $\hat{s}_a(m) = \cos\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right), \text{ and}$ $\hat{s}_\phi(m) = -\hat{A}_1 \sin\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right), \quad m = 0, 1, \dots, M-1$
<i>Step 4: Return the estimates for the number of acquired sine-wave cycle, amplitude, and phase $\tilde{v} = \hat{v} + \Delta v$, $\tilde{A}_1 = \hat{A}_1 + \Delta A_1$, and $\tilde{\phi}_1 = \hat{\phi}_1 + \Delta \phi_1$.</i>

Table 2. Steps of the HPSF-IpDFTc procedure.

<i>Step 1: Apply the IpDFT algorithm based on the Hann window to the analyzed signal and determine the estimators $\hat{v}_0$, $\hat{A}_{10}$, and $\hat{\phi}_{10}$</i>
<i>Step 2: Determine the residual signal removing the fundamental component:</i> $x_r(m) = x(m) - \hat{A}_{10} \cos\left(2\pi \frac{\hat{v}_0}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_{10}\right).$
<i>Step 3: Apply the IpDFT algorithm based on the Hann window to $x_r(\cdot)$ and determine the estimators $\hat{A}_{k0}$ and $\hat{\phi}_{k0}$.</i>
<i>Step 4: Compute the estimators $\hat{v}$, $\hat{A}_1$, and $\hat{\phi}_1$ by using (18), (19), and (20).</i>
<i>Step 5: Compute the correction terms Δv, ΔA_1, and $\Delta \phi_1$ by using (7), (10), and (13) in which</i> $\varepsilon(m) = x(m) -$ $\hat{A}_1 \cos\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right) - \sum_{k=2}^H \hat{A}_{k0} \cos\left(2\pi k \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_{k0}\right),$ $\hat{s}_v(m) = -\hat{A}_1\left(m - \frac{M-1}{2}\right) T_s \sin\left(2\pi k \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right) -$ $\sum_{k=2}^H k \hat{A}_{k0}\left(m - \frac{M-1}{2}\right) T_s \sin\left(2\pi k \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_{k0}\right),$ $\hat{s}_a(m) = \cos\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right), \text{ and}$ $\hat{s}_\phi(m) = -\hat{A}_1 \sin\left(2\pi \frac{\hat{v}}{M}\left(m - \frac{M-1}{2}\right) + \hat{\phi}_1\right), \quad m = 0, 1, \dots, M-1$

Step 6: Return the estimates for the number of acquired sine-wave cycle, amplitude, and phase $\tilde{v} = \hat{v} + \Delta v$, $\tilde{A}_1 = \hat{A}_1 + \Delta A_1$, and $\tilde{\phi}_1 = \hat{\phi}_1 + \Delta \phi_1$.

When the number of acquired sine-wave cycles is quite high (simulations showed that four cycles are enough) the above described PSF-IpDFTc and HPSF-IpDFTc estimators can be replaced by more simple PSF-IpDFT and HPSF-IpDFT procedures, which are based on the IpDFT algorithm instead of the IpDFTc algorithm (i.e. only the estimators $\hat{v}_0, \hat{A}_{10}$, and $\hat{\phi}_{10}$ can be used instead of $\hat{v}, \hat{A}_1$, and $\hat{\phi}_1$).

In the case of a noisy sine-wave the variances of the IpDFTc estimators based on the Hann window are given by [18, 38]:

$$\sigma_{\tilde{v}}^2 \cong \frac{(2 - |\delta|)^2(10\delta^2 - 10|\delta| + 7)}{27} \cdot \frac{ENBW}{M \cdot SNR} \cdot \frac{1}{SL^2(\delta)}, \quad (21)$$

$$\sigma_{\tilde{A}_1}^2 \cong 2 \frac{ENBW}{SL^2(\delta)} \cdot \frac{\sigma^2}{M}, \quad (22)$$

$$\sigma_{\tilde{\phi}_1}^2 \cong \frac{ENBW}{M \cdot SNR} \cdot \frac{1}{SL^2(\delta)}, \quad (23)$$

where $ENBW$ and $SL(\cdot)$ represent the window Equivalent Noise BandWidth and Scalping Loss, respectively, which for the Hann window are equal to $ENBW = 1.5$ and $SL(\delta) = \frac{\sin(\pi\delta)}{(1-\delta^2)\pi\delta}$ [18, 30].

It is worth noticing that, similarly to the IpDFT algorithm, the parameter estimates returned by the IpDFTc algorithm follow a normal distribution [39]. Thus, with 99.73% probability, we have:

$$|\Delta v| \leq 3 \cdot (\sigma_{\tilde{v}})_{max} = 3 \cdot (\sigma_{\tilde{v}})_{|\delta=0}| \text{ so that } |\Delta v| < 6.79 \cdot (\sigma_v)_{CR} \quad (24)$$

$$|\Delta A_1| \leq 3 \cdot (\sigma_{\tilde{A}_1})_{max} = 3 \cdot (\sigma_{\tilde{A}_1})_{|\delta=-0.5}| \text{ so that } |\Delta A_1| < 4.33 \cdot (\sigma_A)_{CR} \quad (25)$$

$$|\Delta \phi_1| \leq 3 \cdot (\sigma_{\tilde{\phi}_1})_{max} = 3 \cdot (\sigma_{\tilde{\phi}_1})_{|\delta=-0.5}| \text{ so that } |\Delta \phi_1| < 4.33 \cdot (\sigma_\phi)_{CR} \quad (26)$$

where $(\sigma_v)_{CR}$, $(\sigma_A)_{CR}$ and $(\sigma_\phi)_{CR}$ are the standard deviations corresponding to the CRLBs [13, 17]. Thus, considering the number of samples M used in practical engineering applications, (24) – (26) show that the IpDFTc algorithm returns estimates with small errors and suitable to be used as initial parameter values of the proposed linear sine-fit algorithm.

4. Computer simulations and experimental results

In this Section, the accuracies the PSF-based procedures are compared with those of the IpDFTc algorithm, the 4PSF algorithm, the corrected Aboutanios and Mulgrew (AMc) algorithm [33], the Harmonic Aboutanios and Mulgrew (HAM) algorithm [34], and the Weighted Least Squares with the monic constraint (WLS1-monic) algorithm for single tone [6]. Moreover, the accuracies of the HPSF-based procedures are compared with those of the IpDFTc algorithm, the nonlinear MHSF algorithms [15], and the WLS-monic for multi-tone signals (WLS2-monic) algorithm [6]. The comparison is performed through both computer simulations and experimental results and the *RMSE* or the variance are used as accuracy parameters, respectively. In the case of real-valued single-tone signals the obtained *RMSE* values are referred to the related CRLB, which is equal to $(\sigma_v^2)_{CR} \cong \frac{3}{\pi^2 \cdot M \cdot SNR}$ for unbiased estimators of the number of acquired sine-wave cycles, $(\sigma_A^2)_{CR} \cong \frac{2\sigma^2}{M}$ for unbiased estimators of the amplitude, $(\sigma_\phi^2)_{CR} \cong \frac{1}{M \cdot SNR}$ for unbiased estimators of the sine-wave phase that do not depend upon ν , and $(\sigma_\phi^2)_{CR} \cong \frac{4}{M \cdot SNR}$ for unbiased estimators of the sine-wave phase that depend upon ν [13]. The latter expression for the CRLB on phase estimators is used for the AMc and WLS1-monic algorithms.

The initial values used of the parameters used in the 4PSF and the MHSF algorithms are determined by applying the IpDFT estimator based on the Hann window to the analyzed signal and performing three iterations. All analyzed testing conditions assume $M = 512$ processed samples, except when the HAM and WLS-monic algorithms are considered, for which only $M = 128$ samples are processed due to the relevant processing effort required. The *frequency SNR* is set greater than 18 dB to avoid spurious spectrum peaks

(the corresponding *time-domain SNR* is greater than -9 dB or -3 dB when $M = 512$ or $M = 128$ samples are considered, respectively) [37]. Thus, the integer part l of the number of acquired signal cycles can be assumed to be known *a priori*. Indeed it can be easily determined with very high probability as the location of the largest DFT spectrum sample [37].

4.1. Computer simulations

The amplitude of the simulated sine-wave is $A_1 = 1$ p.u. while the phase ϕ_1 is chosen at random in the range $[0, 2\pi)$ rad. For harmonically distorted sine-waves the first 7 harmonics with phases chosen at random are considered in the signal model. The number of acquired sine-wave cycles ν varies in the range (1.5, 8) cycles. For each value of ν , 1000 runs of M samples each are considered. It is worth noticing that very similar results have been obtained using acquisition lengths between 128 and 4096. In the 4PSF and MHSF algorithms 3 iterations are performed.

Fig. 1 shows the *RMSEs* normalized to the $\sqrt{CRLB}$ for the estimated number of cycles $\tilde{\nu}$ (Fig. 1(a)), amplitude $\tilde{A}_1$ (Fig. 1(b)), and phase $\tilde{\phi}_1$ (Fig. 1(c)) as a function of ν . The IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, and the 4PSF algorithms are adopted, $SNR = 20$ dB. However, very similar estimator *RMSEs* behaviors occur for $SNR \geq -5$ dB. Insets are reported in Fig. 1 for a better visual comparison between the performance of the PSF-IpDFTc and the 4PSF algorithms.

From Fig. 1 it follows that the PSF-IpDFTc procedure outperforms the IpDFTc algorithm for all the considered values of ν . Conversely, the PSF-IpDFTc procedure outperforms the more simple PSF-IpDFT procedure only when the number of acquired sine-wave cycles ν is small, that is when the contribution of the spectral interference from the fundamental image component on the estimated parameters overcomes the effect of noise. Of course, the value of ν above which that occurs increases as SNR increases. Both the PSF-IpDFTc and the 4PSF algorithms almost attain the $\sqrt{CRLB}$ when at least 1.5-2 cycles are acquired. Indeed, below such a threshold the results returned by the PSF-IpDFTc procedure are affected by significant spectral interference from the fundamental image component.

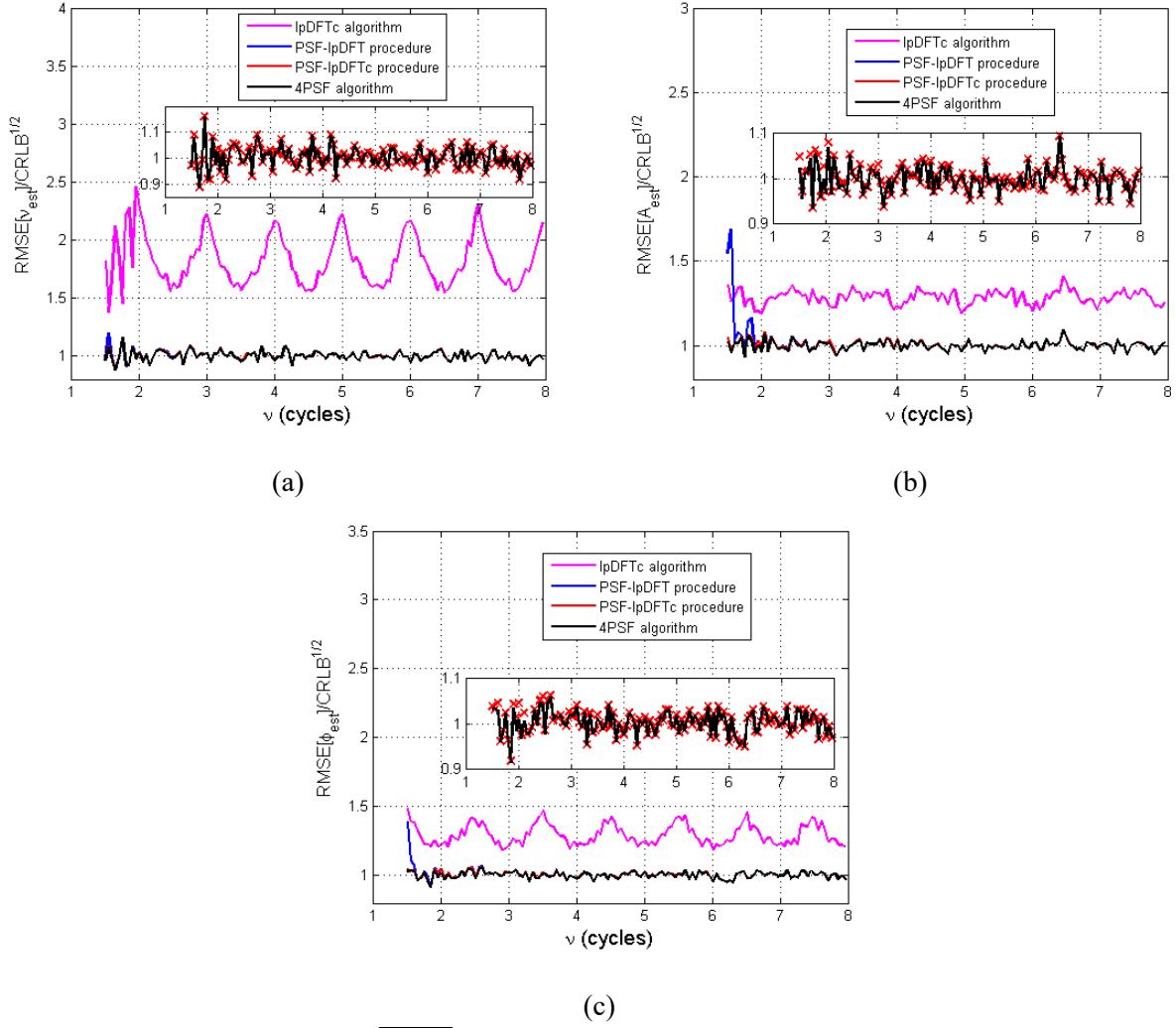


Fig. 1. $RMSE$ s normalized to the $\sqrt{CRLB}$ for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, and the 4PSF algorithms when $SNR = 20$ dB. Insets showing the results achieved by the PSF-IpDFTc and the 4PSF algorithms are also reported; 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

Fig. 2 shows the $RMSE$ s normalized to the $\sqrt{CRLB}$ for the estimated number of cycles $\tilde{\nu}$ (Fig. 2(a)), amplitude $\tilde{A}_1$ (Fig. 2(b)), and phase $\tilde{\phi}_1$ (Fig. 2(c)) returned by the PSF-IpDFTc procedure and the AMc algorithm when $SNR = 20$ dB and $M = 512$ samples. In the AMc algorithm 4 iterations are performed; further iterations do not significantly improved the estimation accuracy.

In Fig. 2 it can be observed that the $RMSE$ s ensured by both the PSF-IpDFTc and the AMc algorithms are very close each other, except for phase estimates where the AMc algorithm results exhibit higher variation. It is worth noticing that the computational efforts of both algorithms are of order $O(M \log_2 M)$ because the integer number

of acquired cycles l is determined by means of the FFT algorithm. However, the AMc algorithm requires an iterative procedure and the number of iterations depends on the estimation accuracy and the number of analysed samples M . Thus, the application of PSF-IpDFTc results to be more straightforward.

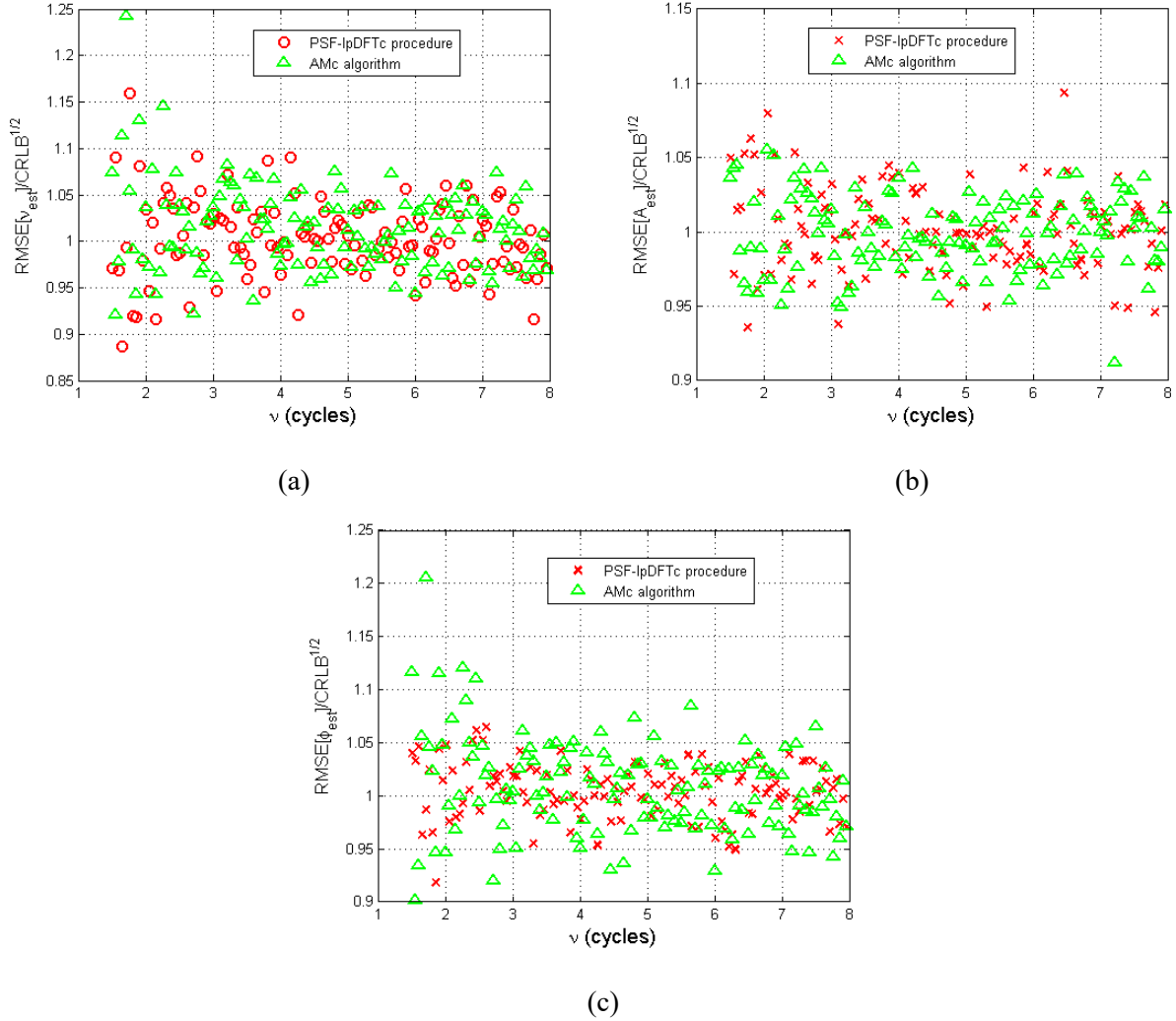


Fig. 2. $RMSEs$ normalized to the $\sqrt{CRLB}$ for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the PSF-IpDFT and the AMc algorithms when $SNR = 20$ dB. 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

Fig. 3 shows the $RMSEs$ normalized to the $\sqrt{CRLB}$ for the estimated number of cycles $\tilde{\nu}$ returned by the PSF-IpDFTc procedure and the WLS1-monic algorithm when $SNR = 20$ dB and $M = 128$ samples. In the WLS1-monic algorithm 3 iterations are performed [6]. As it can be seen, the $RMSEs$ returned by the proposed PSF-IpDFTc and the WLS1-monic algorithms are very close to each other. However, the

computational effort required by the WLS1-monic algorithm is of the order $O(M^3)$, while that required by the PSF-IpDFTc procedure is of order $O(M \log_2 M)$.

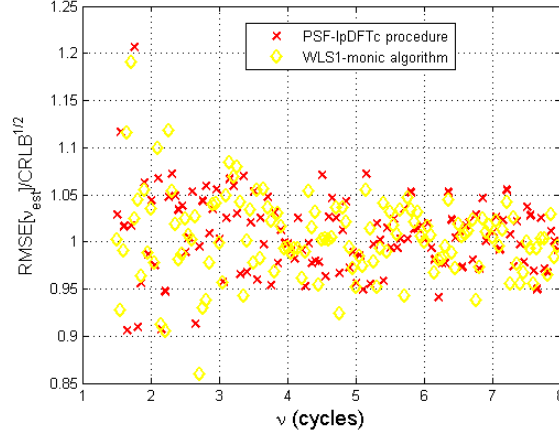


Fig. 3. *RMSEs* normalized to the $\sqrt{CRLB}$ for the estimators of the acquired number of cycles $\tilde{v}$ returned by the PSF-IpDFTc and the WLS1-monic algorithms when $SNR = 20$ dB. 1000 runs of $M = 128$ samples for each value of v and sine-wave phase selected at random; Hann window is used.

Fig. 4 shows the *RMSEs* normalized to the $\sqrt{CRLB}$ for the estimated acquired number of cycles $\tilde{v}$ (Fig. 4(a)), amplitude $\tilde{A}_1$ (Fig. 4(b)), and phase $\tilde{\phi}_1$ (Fig. 4(c)) versus SNR . The IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, the AMc, and the 4PSF algorithms are applied and $v = 2.1$ cycles. For each value of SNR , 1000 runs of $M = 512$ samples each with randomly selected sine-wave phase are considered. Four iterations are performed in the AMc algorithm.

As it can be observed, the PSF-IpDFTc and the 4PSF algorithms almost attain the corresponding $\sqrt{CRLB}$ when $SNR > -5$ dB (that is when *frequency SNR* is at least 22 dB). Also, these methods outperform the IpDFTc and the PSF-IpDFT algorithms. Moreover, when $SNR > 40$ dB the estimates returned by the PSF-IpDFT algorithm are strongly affected by the spectral interference from the image component. However, when at least 4 cycles are observed, the contribution of spectral interference becomes negligible and the estimates returned by the PSF-IpDFT algorithm are almost efficient for all the considered SNR values. Observe also that the *RMSEs* of amplitude and phase estimates returned by the IpDFTc algorithm are quite close to the corresponding $\sqrt{CRLB}$. Moreover, it can be seen that the PSF-IpDFTc procedure provide almost the same *RMSEs* as the AMc algorithm, except when the number of acquired signal cycles

and the phase are estimated and $SNR > 45$ dB, where the proposed PSF-IpDFTc procedure has better accuracy. In the case of small ν and moderate or high $SNRs$ the spectral interference from the image component on the number of acquired signal cycles and phase AMc estimators is important and to avoid this contribution a higher number of iterations should be used. For the considered ν value 6 iterations are needed in the AMc algorithm to achieve $RMSEs$ very close to those achieved by the proposed PSF-IpDFTc procedure and the related CRLBs.

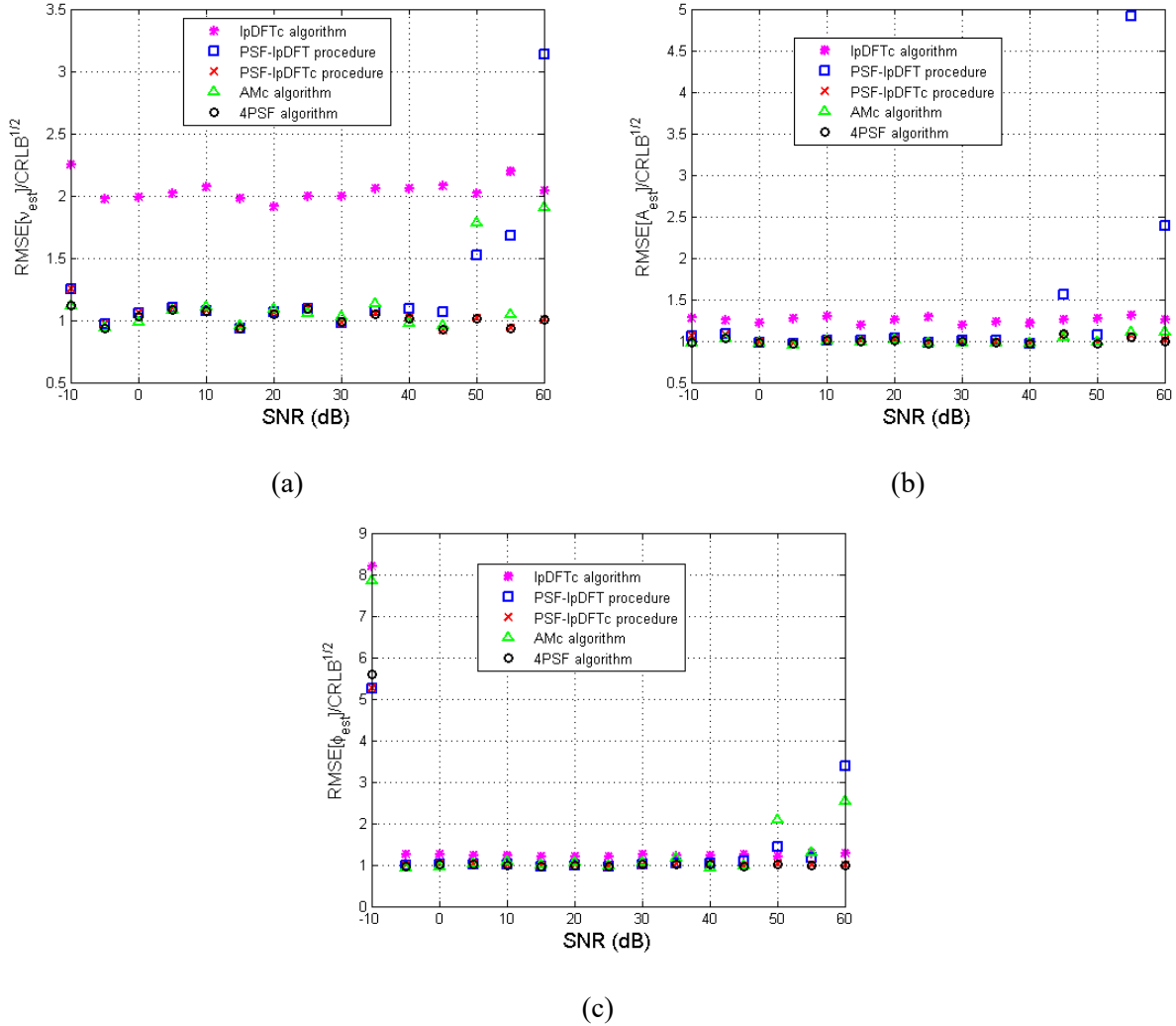


Fig. 4. $RMSEs$ normalized to the $\sqrt{CRLB}$ for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, the AMc, and the 4PSF algorithms versus SNR ; 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

Fig. 5 shows the *RMSEs* of the number of cycles $\tilde{\nu}$ (Fig. 5(a)), amplitude $\tilde{A}_1$ (Fig. 5(b)), and phase $\tilde{\phi}_1$ (Fig. 5(c)) estimated by the IpDFTc, the HPSF-IpDFT, the HPSF-IpDFTc, and the MHSF algorithms as a function of ν when the sine-wave is affected by the first 7 harmonics and $SNR = 20$ dB with respect to the fundamental component. The amplitude of all harmonics is equal to 0.1 p.u.

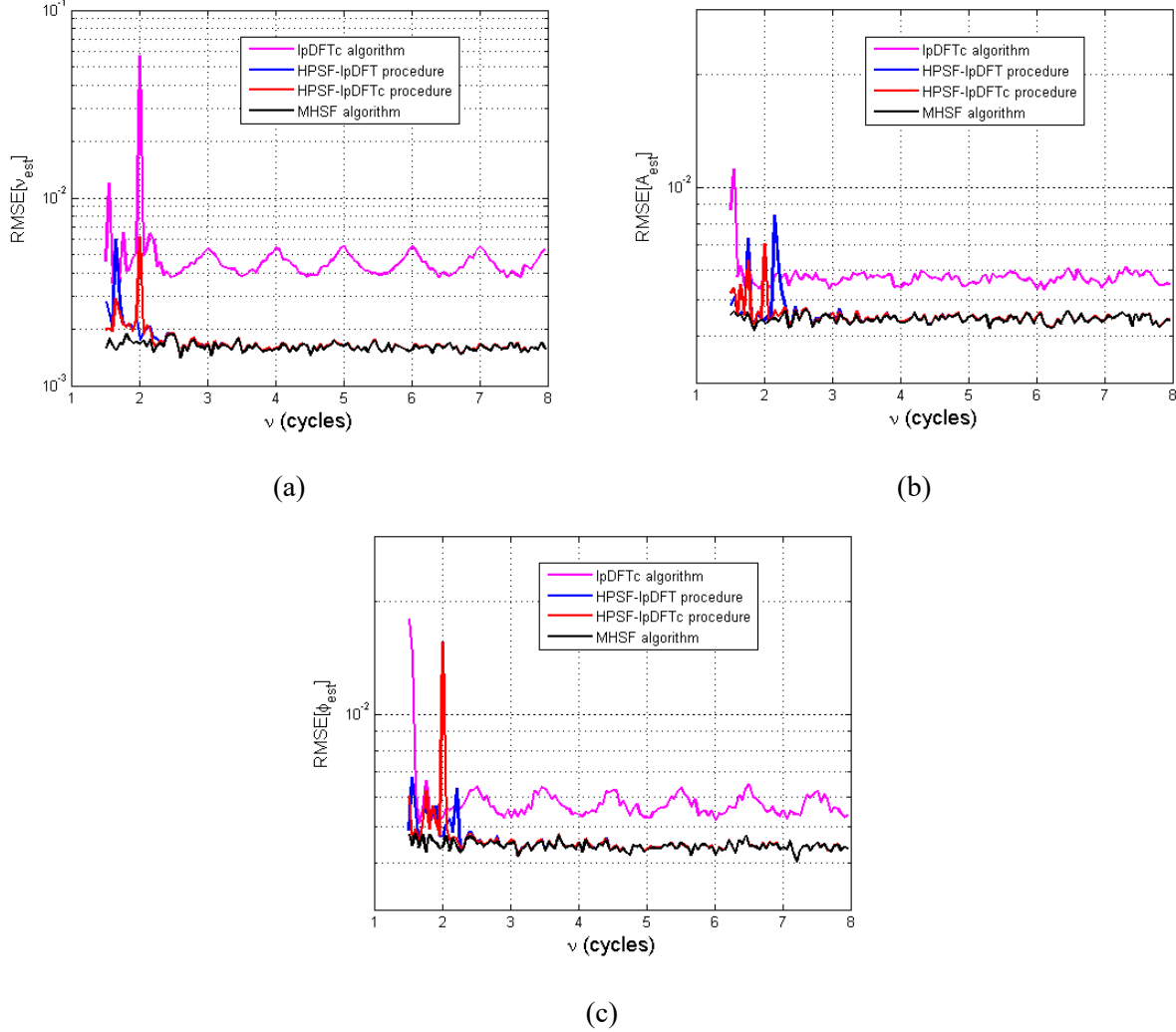


Fig. 5. *RMSEs* for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the IpDFTc, the HPSF-IpDFT, the HPSF-IpDFTc, and the MHSF algorithms versus ν . Sine-wave affected by the first 7 harmonics of amplitude 0.1 p.u. and $SNR = 20$ dB; 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

As shown, the *RMSEs* of the HPSF-IpDFTc and the MHSF algorithms are very close each other when $\nu > 2.5$ cycles so that the effect of the spectral interference on the estimated parameters from both the fundamental image component and harmonics becomes negligible. The threshold on ν above which that behavior occurs increases as the SNR and the number of harmonics increase. However, the effect on that

threshold of harmonics of order higher than 4 is negligible. The HPSF-IpDFTc, the HPSF-IpDFT, and the MHSF algorithms clearly outperform the IpDFTc algorithm. It is worth noticing that the MHSF algorithm requires to consider all harmonics contained in the analyzed signal since significant errors can arise if some harmonics are neglected. This implies a relevant computational burden [35].

The accuracy of the HPSF-IpDFTc procedure as a function of the harmonics considered by the sine-fit algorithm is analyzed when the signal is affected by the first 7 harmonics of amplitudes 1/4, 1/8, 1/128, 1/16, 1/256, and 1/32 p.u. respectively and $SNR = 20$ dB. For this value of SNR and $M = 512$, the 4th and 6th harmonics are below the noise floor, as shown in Fig. 6 in which the spectrum of a generic signal record related to $\nu = 6.41$ cycles is plotted.

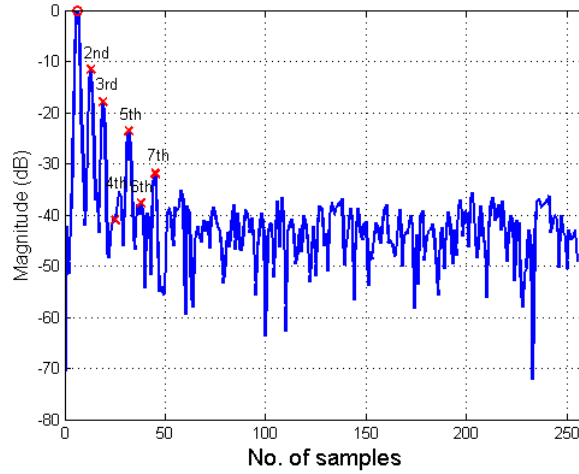


Fig. 6. Spectrum of a generic signal record related to $\nu = 6.41$ cycles and weighted by the Hann window. 7 harmonics of amplitudes 1/4, 1/8, 1/128, 1/16, 1/256, and 1/32 p.u. respectively and $SNR = 20$ dB.

Fig. 7 shows the $RMSE$ s for the number of cycles $\tilde{\nu}$ (Fig. 7(a)), amplitude $\tilde{A}_1$ (Fig. 7(b)), and phase $\tilde{\phi}_1$ (Fig. 7(c)) estimated by the HPSF-IpDFTc procedure when all harmonics or only part of them are taken into account in the sine-fit algorithm. Fig. 7 confirms that only the most significant harmonics have a significant impact on the algorithm accuracy.

Moreover, the accuracy of the HPSF-IpDFTc procedure has been analyzed when all harmonics or only the 2nd harmonic are considered by the IpDFTc algorithm. The results obtained using the same signal considered in Fig. 7 are shown in Fig. 8.

The obtained results show that the IpDFTc algorithm returns accurate estimates of the fundamental component parameters even if only the 2nd harmonic contribution is taken into account.

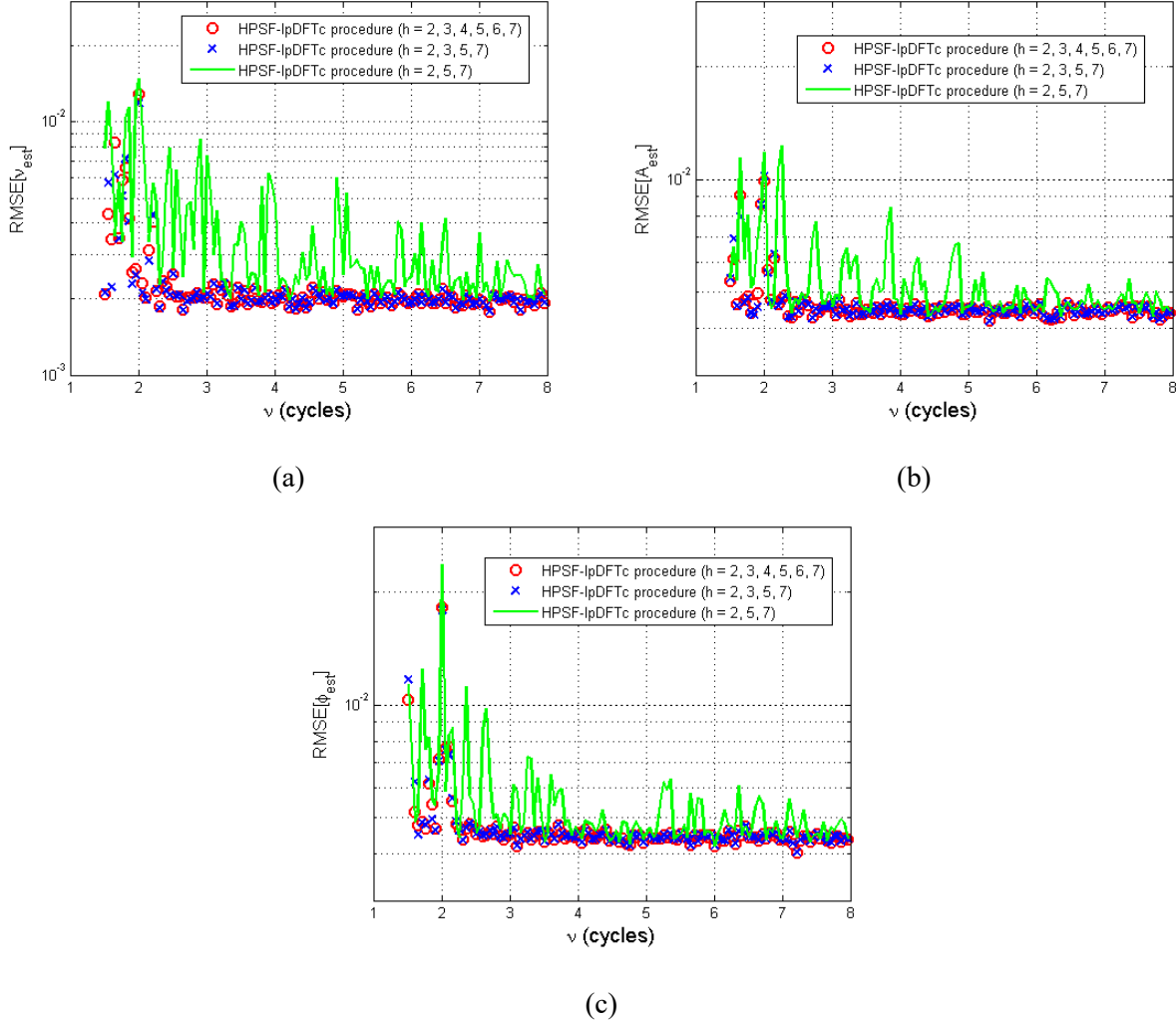


Fig. 7. $RMSE$ s for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the HPSF-IpDFTc procedure versus ν when all the first 7 harmonics or only part of them are considered. Sine-wave affected by the first 7 harmonics of amplitudes 1/4, 1/8, 1/128, 1/16, 1/256, and 1/32 p.u. respectively and $SNR = 20$ dB; 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

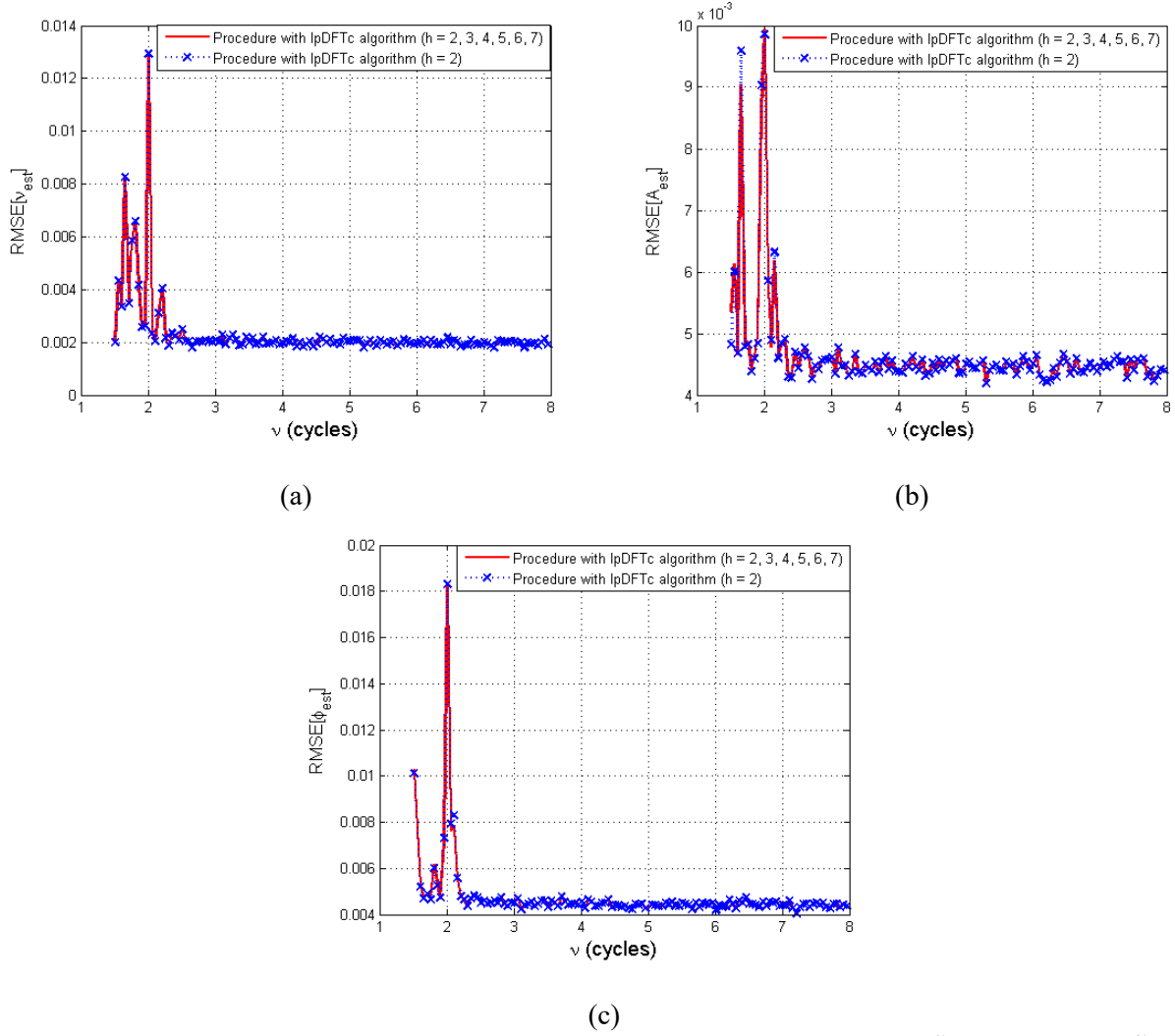


Fig. 8. RMSEs for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the HPSF-IpDFTc procedure versus ν when all harmonics or only the 2nd harmonic are considered in the IpDFTc algorithm. Sine-wave affected by the first 7 harmonics of amplitudes 1/4, 1/8, 1/128, 1/16, 1/256, and 1/32 p.u. respectively and $\text{SNR} = 20$ dB; 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase selected at random; Hann window is used.

The accuracies of the proposed HPSF-IpDFTc procedure, the HAM algorithm and the WLS2-monic algorithm have been compared each other. The analyzed signal exhibits a fundamental component of amplitude $A_1 = 1$ p.u. and a 3rd harmonic of amplitude $A_3 = 0.3$ p.u. The CRLB for unbiased estimators of the parameters of sine-waves distorted by one harmonic is expressed by $(\sigma_{\nu}^2)_{CR} \cong \frac{6\sigma^2}{4\pi^2 \cdot M \cdot (A_2^2 + 9A_3^2)}$, for the number of acquired sine-wave cycles, $(\sigma_A^2)_{CR} \cong \frac{2\sigma^2}{M}$, for the amplitude, $(\sigma_{\phi}^2)_{CR} \cong \frac{2\sigma^2}{M \cdot (A_2^2 + 9A_3^2)} \left[1 + \left(\frac{3A_3}{A_1} \right)^2 \right]$,

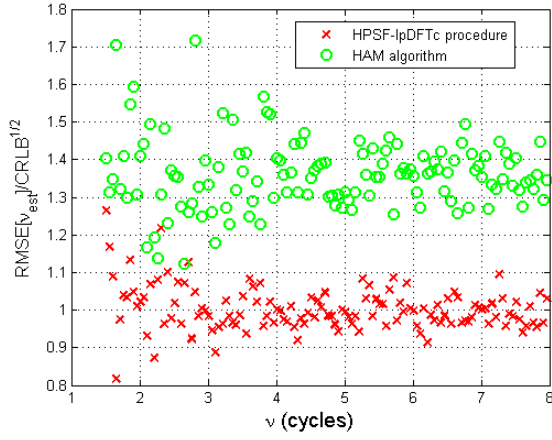
when the phase does not depend upon ν , and $(\sigma_\phi^2)_{CR} \cong \frac{2\sigma^2}{M \cdot (A_2^2 + 9A_3^2)} \left[4 + \left(\frac{3A_3}{A_1} \right)^2 \right]$ when the phase depends upon ν (see the Appendix). The latter expression for the CRLB on phase estimators is used for the HAM and the WLS2-monic algorithms.

Fig. 9 shows the *RMSEs* normalized to the $\sqrt{CRLB}$ for the number of cycles $\tilde{\nu}$ (Fig. 9(a)), amplitude $\tilde{A}_1$ (Fig. 9(b)), and phase $\tilde{\phi}_1$ (Fig. 9(c)) estimated by the HPSF-IpDFT and HAM algorithms when $SNR = 20$ dB. For each value of ν , 1000 runs of $M = 128$ samples each are considered and the phases of the both fundamental and 3rd harmonic are chosen at random in the range $[0, 2\pi)$ rad. The HAM algorithm is implemented by performing four iterations [34].

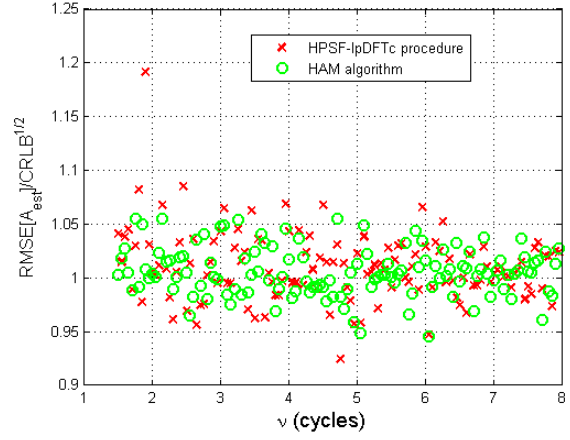
In Fig. 9 it can be observed that the HPSF-IpDFTc procedure outperforms the HAM algorithm when the frequency or the phase are estimated, while the two algorithms have almost the same accuracy with amplitude estimates. Also, the *RMSEs* of the frequency or the phase estimates returned by the HPSF-IpDFTc procedure exhibit smaller variations than the HAM algorithm. Moreover, the proposed HPSF-IpDFTc procedure returns *RMSEs* very close to the related CRLBs.

Fig. 10 shows the *RMSEs* normalized to the $\sqrt{CRLB}$ for the number of cycles $\tilde{\nu}$ estimated by the HPSF-IpDFTc and the WLS1-monic algorithms when $SNR = 40$ dB. For each value of ν , 1000 runs of $M = 128$ samples each are considered and the phases of the fundamental and the 3rd harmonic are selected at random in the range $[0, 2\pi)$ rad. The WLS2-monic algorithm is implemented by performing three iterations [6].

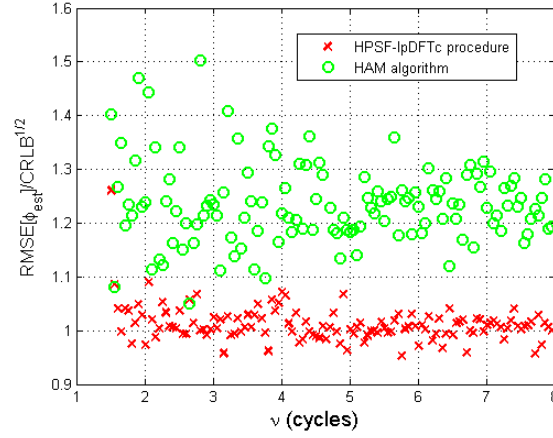
From Fig. 10 it follows that the proposed HPSF-IpDFTc algorithm outperforms the WLS2-monic algorithm, except when less than about 1.5 cycles are observed. Indeed, in such conditions the effect on frequency estimates of the spectral interference from both the fundamental image component and harmonics is quite high and the proposed algorithm is not capable to effectively compensate it. In addition, when at least 1.5 cycles are observed, the proposed HPSF-IpDFTc procedure ensures a frequency *RMSEs* very close to the CRLB. Moreover, both algorithms ensures *RMSEs* that are almost independent of the observed number of cycles.



(a)



(b)



(c)

Fig. 9. $RMSEs$ normalized to the $\sqrt{CRLB}$ for the estimators of the acquired number of cycles $\tilde{\nu}$ (a), amplitude $\tilde{A}_1$ (b), and phase $\tilde{\phi}_1$ (c) returned by the HPSF-IpDFTc and the HAM algorithms when $SNR = 20$ dB. Sine-wave affected by a 3rd harmonic of amplitude 0.3 p.u. and $SNR = 20$ dB; 1000 runs of $M = 128$ samples for each value of ν ; fundamental and 3rd harmonic phases selected at random; Hann window is used.

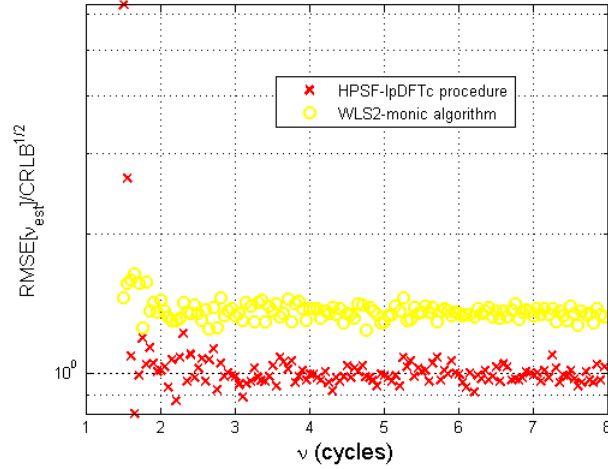


Fig. 10. *RMSEs* normalized to the $\sqrt{CRLB}$ for the acquired number of cycles $\tilde{\nu}$ estimated by the HPSF-IpDFTc and the WLS2-monic algorithms. Sine-wave affected by a 3rd harmonic of amplitude 0.3 p.u. and $SNR = 40$ dB; 1000 runs of $M = 128$ samples for each value of ν ; fundamental and 3rd harmonic phases selected at random; Hann window is used.

4.2. Experimental results

The signals analyzed in the experiments were provided by an Agilent 33220A signal generator and acquired by means of an NI6023E acquisition board with sampling rate $f_s = 100$ kHz, full scale range 10 V and resolution 12-bit. The *SNRs* of the generated signals were estimated by the MHSF algorithm and resulted close to 60 dB. The signal frequencies varied in the range (500, 680) Hz with steps of 20 Hz. The sine-wave amplitude was $A_1 = 2$ V and 1000 acquisitions of $M = 512$ samples each were performed at each considered frequency.

Fig. 11 shows the standard deviations of the frequency and amplitude estimators provided by the IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, the AMc, and the 4PSF algorithms as a function of ν when noisy sine-waves are considered. In Fig. 11(a) the standard deviation of the frequency estimator returned by the WLS1-monic algorithm is also reported. The values of ν reported in the abscissa coincides with the average of the estimates returned by the 4PSF algorithm. In the AMc algorithm 4 iterations are performed; further iterations do not significantly improved the estimation accuracy. Conversely, the WLS1-monic algorithm requires 3 iterations to ensure accurate results [6].

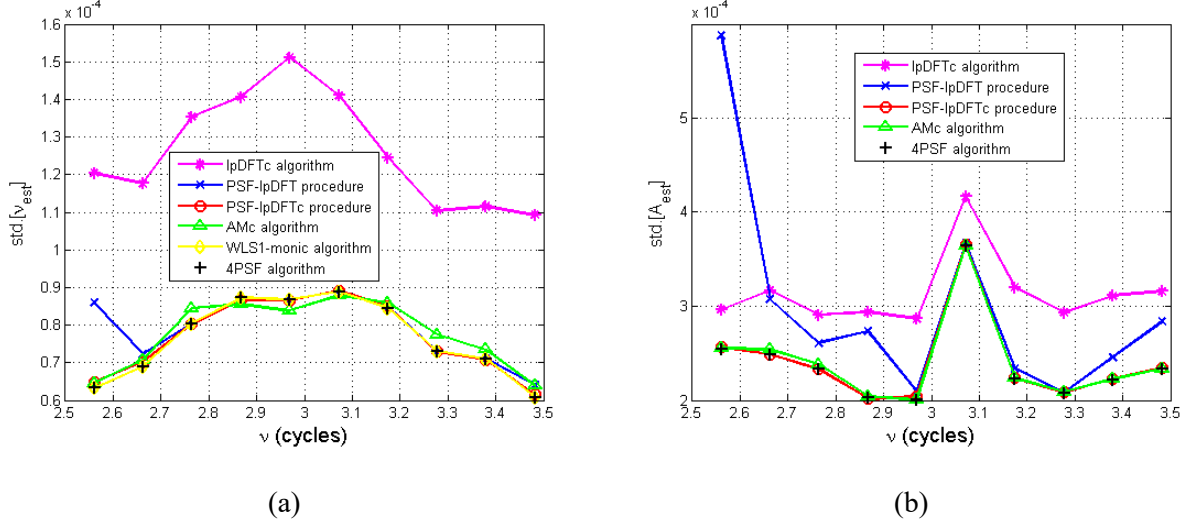


Fig. 11. Real noisy sine-waves: standard deviations of the estimated number of cycles ν (a) and amplitude (b) returned by the IpDFTc, the PSF-IpDFT, the PSF-IpDFTc, the AMc, and the 4PSF algorithms versus ν . The standard deviation of the estimated number of cycles ν returned by the WLS1-monic algorithm is also shown in Fig. 11(a). 1000 runs of $M = 512$ samples for each value of ν and sine-wave phase at random; Hann window is used.

Fig. 11 shows that the standard deviations of the frequency estimates returned by the PSF-IpDFTc, the PSF-IpDFT, the AMc, the WLS1-monic, and the 4PSF algorithms almost coincides, except at the lowest considered frequency, where the IpDFTc algorithm provides more accurate estimates than the IpDFT algorithm. Conversely, the amplitude estimates returned by the PSF-IpDFTc, the AMc, and the 4PSF algorithms are almost equal and they are more accurate than those obtained by applying the PSF-IpDFT procedure and the IpDFTc algorithm, except for small values of the inter-bin frequency location (i.e., in quasi-coherent sampling conditions) where the PSF-IpDFT procedure almost achieves the accuracy of the 4PSF algorithm.

The algorithms were applied also to real-life harmonically distorted signals provided by the power grid. To that aim, the voltage grid was applied to a transformer and then acquired by using the NI6023E acquisition board with sampling rate $f_s = 5$ kHz and full scale-range 10 V. 1000 acquisitions of $M = 512$ samples each were performed. Since the grid frequency is close to 50 Hz, it follows from (2) that about $\nu \cong 5.12$ cycles are observed in each acquired record. The signal corresponding to the 456th acquisition and its DFT spectrum obtained using the Hann window are shown in Fig. 12.

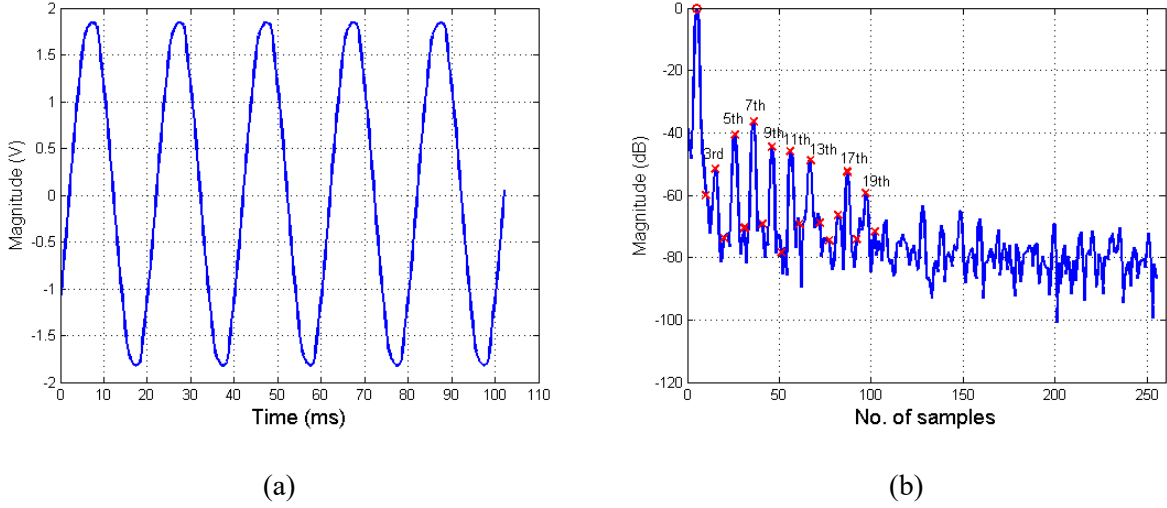


Fig. 12. Signal acquired from the power grid (a) and its DFT spectrum obtained using the Hann window (b).

Fig. 12(b) shows that odd harmonics up to 19th order are significant. Thus, they have been taken into account in the HPSF-IpDFT, the HPSF-IpDFTc and the MHSF algorithms. The standard deviations of the estimates for the number of acquired signal cycles and the fundamental amplitude are given in Table 1.

Table 1. Signal acquired from the power grid: standard deviations of the estimates returned by the HPSF-IpDFT, the HPSF-IpDFTc, and the MHSF algorithms for the number of acquired signal cycles ν and the fundamental amplitude A_1 .

Parameter	HPSF-IpDFT procedure	HPSF-IpDFTc procedure	MHSF algorithm
ν	$1.31 \cdot 10^{-3}$	$1.31 \cdot 10^{-3}$	$1.32 \cdot 10^{-3}$
A_1	$2.60 \cdot 10^{-3}$	$2.60 \cdot 10^{-6}$	$2.56 \cdot 10^{-6}$

As expected from theory since $\nu > 5$ cycles, the results reported in Table 1 show that the HPSF-IpDFT, the HPSF-IpDFTc and the MHSF algorithms exhibit very close standard deviations.

5. Conclusions

In this paper the PSF-IpDFTc and the HPSF-IpDFTc procedures have been proposed for frequency, amplitude, and phase estimation of distortion-free and harmonically-distorted noisy sine-waves, respectively. Both procedures obtain an initial estimate of the unknown parameters by applying the IpDFTc

algorithm which compensates of the contribution of the spectral interference from the fundamental image component and harmonics on the selected DFT spectrum samples. A linear sine-fit algorithm is then applied to optimized estimator noise robustness which is worsened by signal windowing employed to reduce the effect of spectral leakage on the IpDFTc estimates.

Also, the accuracies of the PSF-IpDFT and the HPSF-IpDFT procedures, which are based on the simpler IpDFT algorithm instead of the IpDFTc algorithm, have been analyzed.

Both computer simulations and experiments showed that the PSF-IpDFTc and the HPSF-IpDFTc procedures provide more accurate results than the corresponding PSF-IpDFT and HPSF-IpDFT procedures when less than about 3-4 sine-wave cycles are acquired, even if at the cost of a slightly higher computational effort.

For distortion-free noisy sine-waves the PSF-IpDFTc procedure is as accurate as the 4PSF algorithm, thus it almost attains the CRLB for all estimated parameters, except when less than about 1.5 sine-wave cycles are observed. Conversely, the statistical efficiency of the simpler PSF-IpDFT procedure is assured when at least about 4 sine-wave cycles are acquired. Also, when at least 1.5 cycles are observed, the PSF-IpDFTc procedure is as accurate as the WLS1-monic algorithm [6], despite its significantly smaller computational complexity. In addition, the PSF-IpDFTc and the AMc algorithms [33] provide almost the same accuracy and require almost the same processing effort. However, the application of the former one is more straightforward since the AMc algorithm requires to implement an iterative procedure.

For harmonically distorted noisy sine-waves the HPSF-IpDFTc and the HPSF-IpDFT procedures provide close estimates to the more computational demanding MHSF algorithm if at least about 4.5 sine-wave cycles are acquired even though only the second harmonic is considered in the IpDFTc algorithm, while the effect of higher order harmonics is neglected. Also, the HPSF-IpDFTc procedure outperforms the HAM algorithm [34] in the case of frequency and phase estimation, while the two algorithms ensure almost the same accuracy in the case of amplitude estimation. In addition, the HPSF-IpDFTc procedure outperforms the WLS2-monic algorithm [6] for all estimated parameters. The superior accuracy of the

HPSF-IpDFTc procedure is obtained despite it requires a much smaller effort than the iterative HAM and WLS2-monic algorithms.

Concluding, the authors believe that the PSF-based and HPSF-based procedures can be advantageously used in real-time applications since they are simple to be implemented, require a low processing effort and ensure optimal sine-wave parameter estimation accuracy.

Appendix

Closed-form expressions for the CRLBs of unbiased estimators on the parameters of a sine-wave affected by one harmonic

Let's assume that the analyzed discrete-time signal is composed by a fundamental component of amplitude A_1 , normalized frequency f , and phase ϕ_1 , and a k th harmonic of amplitude A_k , and phase ϕ_k . i.e.:

$$s(m) = A_1 \cos(2\pi f m + \phi_1) + A_k \cos(2\pi k f m + \phi_k), \quad m = 0, 1, \dots, M-1 \quad (\text{A1})$$

Assume also that the number of acquired samples M is high and that all the parameters in (A.1) are unknown. Thus, the unknown parameter vector is $\theta = [A_1 \ A_2 \ \phi_1 \ \phi_2 \ f]^T$, where $(\cdot)^T$ denotes the transpose operator.

Reminding that [13]:

$$\frac{1}{M^{i+1}} \sum_{m=0}^{M-1} m^i \sin(2\pi r f m + \varphi) \cong 0,$$

$$\frac{1}{M^{i+1}} \sum_{m=0}^{M-1} m^i \cos(2\pi r f m + \varphi) \cong 0,$$

for $i = 0, 1, 2$, where r is a positive integer and $\varphi \in [0, 2\pi)$ rad, after some algebra the following Fisher information matrix for θ is obtained:

$$I(\theta) \cong \frac{1}{2\sigma^2} \begin{pmatrix} M & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 \\ 0 & 0 & A_1^2 M & 0 & 2\pi A_1^2 \sum_{m=0}^{M-1} m \\ 0 & 0 & 0 & A_k^2 M & 2\pi k A_k^2 \sum_{m=0}^{M-1} m \\ 0 & 0 & 2\pi A_1^2 \sum_{m=0}^{M-1} m & 2\pi k A_k^2 \sum_{m=0}^{M-1} m & 4(A_1^2 + k^2 A_k^2) \pi^2 \sum_{m=0}^{M-1} m^2 \end{pmatrix} \quad (\text{A2})$$

Since [13]:

$$\sum_{m=0}^{M-1} m = \frac{M(M-1)}{2},$$

$$\sum_{m=0}^{M-1} m^2 = \frac{M(M-1)(2M-1)}{6},$$

after some calculations the determinant of the matrix $I(\theta)$ results:

$$\Delta = \frac{M^5(M^2-1)\pi^2 A_1^2 A_k^2 (A_1^2 + k^2 A_k^2)}{3} \cong \frac{M^7 \pi^2 A_1^2 A_k^2 (A_1^2 + k^2 A_k^2)}{3} \quad (\text{A3})$$

Using (A.3), after some algebra, the entries of the inverse Fisher matrix (which are the CRLBs for unbiased estimators of the unknown parameters) are obtained:

$$(\sigma_{A_1}^2)_{CR} = [I(\theta)]_{11}^{-1} \cong \frac{2\sigma^2}{M}, \quad (\text{A4})$$

$$(\sigma_{A_k}^2)_{CR} = [I(\theta)]_{22}^{-1} \cong \frac{2\sigma^2}{M}, \quad (\text{A5})$$

$$(\sigma_{\phi_1}^2)_{CR} = [I(\theta)]_{33}^{-1} \cong \frac{2\sigma^2}{M \cdot (A_1^2 + k^2 A_k^2)} \left[4 + \left(\frac{k A_k}{A_1} \right)^2 \right], \quad (\text{A6})$$

$$(\sigma_{\phi_k}^2)_{CR} = [I(\theta)]_{44}^{-1} \cong \frac{2\sigma^2}{M \cdot (A_1^2 + k^2 A_k^2)} \left[4k^2 + \left(\frac{A_1}{A_k} \right)^2 \right], \quad (\text{A7})$$

$$(\sigma_f^2)_{CR} = [I(\theta)]_{55}^{-1} \cong \frac{6\sigma^2}{\pi^2 \cdot M^3 \cdot (A_1^2 + k^2 A_k^2)}. \quad (\text{A8})$$

It is worth noticing that (A.8) is a particular case of the CRLB for the normalized frequency of a harmonically-distorted sine-wave, which was derived in [40].

Observe also that, when the signal frequency is known, it can be shown that expressions (A.6) and (A.7) for the CRLBs on unbiased phase estimators become [17]:

$$(\sigma_{\phi_1}^2)_{CR} \cong \frac{2\sigma^2}{M \cdot (A_1^2 + k^2 A_k^2)} \left[1 + \left(\frac{k A_k}{A_1} \right)^2 \right], \quad (\text{A9})$$

$$(\sigma_{\phi_k}^2)_{CR} \cong \frac{2\sigma^2}{M \cdot (A_1^2 + k^2 A_k^2)} \left[k^2 + \left(\frac{A_1}{A_k} \right)^2 \right]. \quad (\text{A10})$$

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