

Research Article

Bayesian Compressive Sensing as Applied to Directions-of-Arrival Estimation in Planar Arrays

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The Bayesian compressive sensing (BCS) is applied to estimate the directions of arrival (DoAs) of narrow-band electromagnetic signals impinging on planar antenna arrangements. Starting from the measurement of the voltages induced at the output of the array elements, the performance of the BCS-based approach is evaluated when data are acquired at a single time instant and at consecutive time instants, respectively. Different signal configurations, planar array geometries, and noise conditions are taken into account, as well.

1. Introduction

In the last few years, we assisted to an extraordinary and still growing development and use of compressive sensing (CS)-based methods [1] in a wide number of applicative contexts such as communications [2], biomedicine [3], radar [4], and microwave imaging [5, 6]. CS has proven to be a very effective resolution tool when the relationship between the problem data and the unknowns is linear, and these latter are sparse (or they can be sparsified) with respect to some representation bases.

In this paper, a probabilistic version of the CS, namely, the Bayesian compressive sensing (BCS) [7], is used for estimating the directions of arrival (DoAs) of electromagnetic signals impinging on an array of sensors in a planar arrangement. Since the DoAs of the incoming signals are few with respect to the whole set of angular directions, they can be modeled as a sparse vector. Accordingly, the estimation problem at hand can be reformulated as the retrieval of such a sparse signal vector whose nonnull entries are related to the unknown angular directions of the signals.

Compared to the state-of-the-art estimation methods (e.g., the multiple signal classification (MUSIC) [8], the signal parameters via rotational invariance technique (ESPRIT) [9], the maximum likelihood (ML) DoAs estimators [10], and the class of techniques based on learning-by-examples (LBE) strategies [11–13]), CS-based approaches have shown

several interesting advantages. Likewise LBE-based methods, the computationally expensive calculation of the covariance matrix is not necessary since the voltages measured at the output of the array elements can be directly processed. CS-based methods turn out to be fast and also work with single time-instant (snapshot) data acquisitions. Moreover, unlike MUSIC and ESPRIT that require the incoherence of the impinging signals and a set of measurements larger than the number of signals, careful DoA estimates can be yielded also when the number of arriving signals is greater than the array sensors as well as in the presence of highly correlated sources.

Within the class of CS-based approaches, deterministic strategies recover the signal vector by enforcing the sparsity constraints through the l_1 -norm, while the l_2 -norm is adopted to quantify the mismatch between measured and estimated data as shown in [14] for the localization of narrowband sources when using a circular array. Hybrid l_1 -norm and l_2 -norm formulations have been considered [15, 16], as well. Other CS-based methods have been proposed [17–19] also dealing with the DoAs estimation of correlated sources [20]. Unfortunately, common formulations of the CS (i.e., based on deterministic strategies) require a minimum number of measurements equal to twice the number of impinging signals to satisfy the necessary condition for the well posedness of the problem (i.e., the restricted isometry property of the sampling matrix). To overcome such an issue,

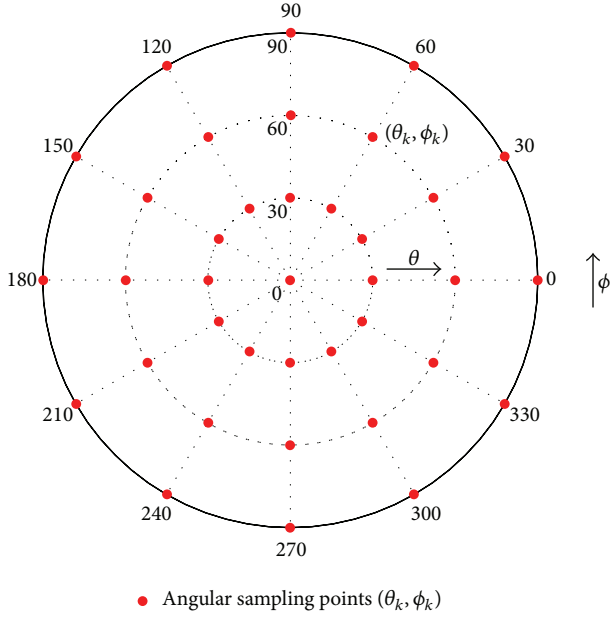


FIGURE 1: Sketch of the discretized observation domain for CS-based DoAs estimations.

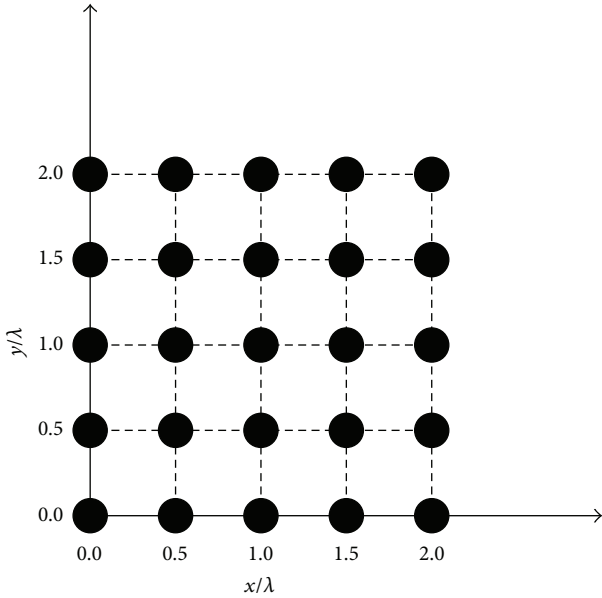


FIGURE 2: Geometry of the receiving fully populated array ($N = 25$).

probabilistic CS-based approaches have been taken into account [21–23] as the one considered in this work.

The outline of the paper is as follows. The DoAs estimation problem, its sparse reformulation, and the BCS-based DoAs estimation approach are presented in Section 2. A selected set of representative numerical results is reported in Section 3 to discuss, in a comparative fashion, the performance of the single and multiple snapshot implementations of the two-dimensional extension of the BCS method [24] for different array architectures. Eventually, some conclusions are drawn (Section 4).

TABLE 1: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $C = 50$)—actual DoAs of the impinging signals.

I	$\{\Psi_i = (\theta_i, \phi_i), i = 1, \dots, I\}$
2	$\{(25, 60); (60, 140)\}$
4	$\{(25, 60); (60, 140); (70, 210); (60, 300)\}$
8	$\{(25, 60); (60, 140); (70, 210); (60, 300); (40, 210); (80, 45); (15, 5); (30, 350)\}$

2. Mathematical Formulation

Let us consider a planar antenna array made of N isotropic sensors located on the $x - y$ plane. An unknown set of I signals $\mathbf{s}_i(\mathbf{r}, t) = \alpha_i(t)e^{j(2\pi f_0 t + \mathbf{k}_i \cdot \mathbf{r})}$, $i = 1, \dots, I$, is supposed to impinge on the array from the unknown directions $\Psi_i = (\theta_i, \phi_i)$, $i = 1, \dots, I$, being $0^\circ \leq \theta_i \leq 90^\circ$ and $0^\circ \leq \phi_i \leq 360^\circ$. Such signals are modeled as narrowband electromagnetic plane waves (i.e., $\alpha_i(t) \approx \alpha_i$, $i = 1, \dots, I$) at the carrier frequency f_0 , with $\mathbf{k}_i$ ($i = 1, \dots, I$) being the i th wave vector having amplitude $k = |\mathbf{k}_i| = 2\pi/\lambda$, for all $i = 1, \dots, I$, where λ is the free space wavelength.

By modelling the background noise as an additive Gaussian process with zero mean and variance σ^2 , the phasor voltage measured at the n th element is equal to

$$v_n(\tau) = \sum_{i=1}^I v_{i,n}(\tau) + \eta_n(\tau), \quad (1)$$

where τ is the measurement time-instant/snapshot and $\eta_n(\tau)$ is the noise sample at the same instant. Moreover,

$$v_{i,n}(\tau) = \alpha_i(\tau) e^{j(2\pi/\lambda)(x_n \sin \theta_i \cos \phi_i + y_n \sin \theta_i \sin \phi_i)} \quad (2)$$

is the open circuit voltage induced by the i th impinging wave at the n th planar array element located in the position $\mathbf{r}_n = (x_n, y_n)$.

The relationship between the measured data (i.e., $v_n(\tau)$, $n = 1, \dots, N$, $\tau = 1, \dots, T$) and the unknown DoAs [i.e., $\Psi_i = (\theta_i, \phi_i)$, $i = 1, \dots, I$] can be then represented in a compact matrix form as follows:

$$\mathbf{v}(\tau) = \mathbf{H}(\Psi) \mathbf{s}(\tau) + \boldsymbol{\eta}(\tau), \quad \tau = 1, \dots, T, \quad (3)$$

where $\mathbf{v}(\tau) = [v_1(\tau), v_2(\tau), \dots, v_N(\tau)]^*$ is the complex measurement vector, with $*$ denoting the transpose operation, and $\mathbf{H}(\Psi) = [\mathbf{h}(\Psi_1), \mathbf{h}(\Psi_2), \dots, \mathbf{h}(\Psi_I)]$ is the steering vector matrix where $\mathbf{h}(\Psi_i) = [h_{i,1}, h_{i,2}, \dots, h_{i,N}]^*$ being $h_{i,n} = e^{j(2\pi/\lambda)(x_n \sin \theta_i \cos \phi_i + y_n \sin \theta_i \sin \phi_i)}$. Moreover, $\mathbf{s}(\tau) = [\alpha_1(\tau), \alpha_2(\tau), \dots, \alpha_I(\tau)]^*$ is the signal vector, and $\boldsymbol{\eta}(\tau) = [\eta_1(\tau), \eta_2(\tau), \dots, \eta_N(\tau)]^*$ is the noise vector.

It is simple to observe that the solution of (3) is neither linear nor sparse with respect to the problem unknowns $\Psi_i = (\theta_i, \phi_i)$, $i = 1, \dots, I$, while it is linear versus $\mathbf{s}(\tau)$, for all τ . In order to apply the BCS to the DoAs estimation in planar arrays, the method in [24] for linear arrays has been exploited and here suitably customized to the dimensionality (2D) at hand.

To reformulate the original problem as a sparse one, the observation domain composed by all angular directions $0^\circ \leq$

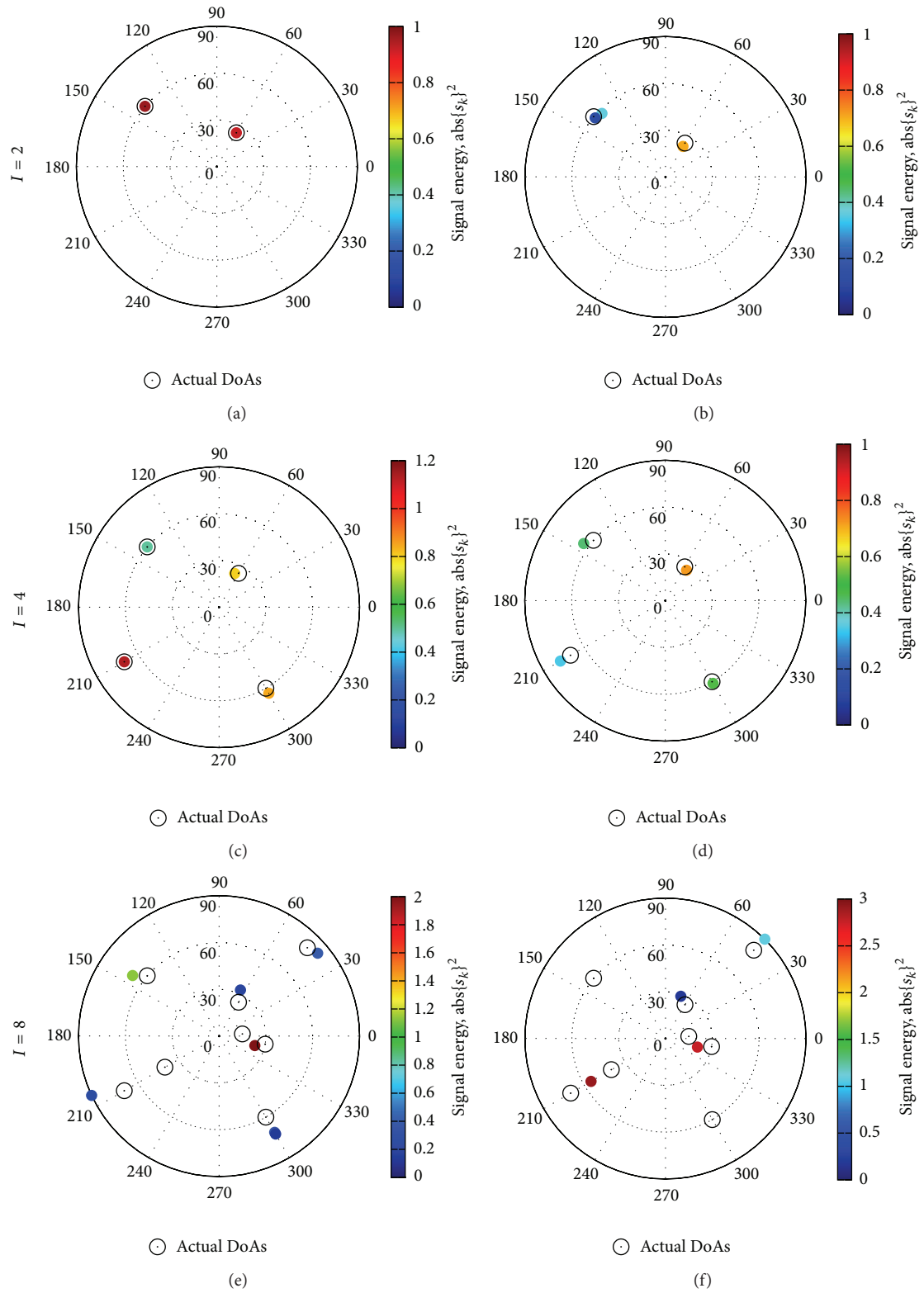


FIGURE 3: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = 1$; $C = 50$)—plot of the best (left column) and worst (right column) estimations obtained by means of the ST-BCS among the C different noisy scenarios when (a) (b) $I = 2$, (c) (d) $I = 4$, and (e) (f) $I = 8$.

TABLE 2: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = 1$; $C = 50$)—values of the DoAs for the best and worst estimation obtained by means of the ST-BCS among the C different noisy scenarios.

I	$\hat{I}^{(\text{bst})}$	$\{\hat{\Psi}_i^{(\text{bst})} = (\hat{\theta}_i^{(\text{bst})}, \hat{\phi}_i^{(\text{bst})}), i = 1, \dots, I\}$
2	2	$\{(25, 60); (60, 140)\}$
4	4	$\{(23.75, 65); (60, 140); (63.75, 300); (70, 210)\}$
8	7	$\{(23.75, 345); (32.5, 65); (67.5, 145); (71.25, 300); (72.5, 300); (82.5, 40); (90, 205)\}$
I	$\hat{I}^{(\text{wst})}$	$\{\hat{\Psi}_i^{(\text{wst})} = (\hat{\theta}_i^{(\text{wst})}, \hat{\phi}_i^{(\text{wst})}), i = 1, \dots, I\}$
2	3	$\{(22.5, 60); (57.5, 135); (58.75, 137.5)\}$
4	4	$\{(23.75, 55); (63.75, 145); (61.25, 300); (77.5, 210)\}$
8	4	$\{(21.25, 345); (28.75, 70); (55, 210); (90, 45)\}$

TABLE 3: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = \{1, 2\}$; $C = 50$)—statistics (minimum, maximum, average, and variance) of the location index ξ among C different noisy scenarios when using the ST-BCS ($T = 1$) and the MT-BCS ($T = 2$).

I	$\xi^{(\min)}$	$\xi^{(\max)}$	$\xi^{(\text{avg})}$	$\xi^{(\text{var})}$	$t^{(\text{avg})}$ [sec]
ST-BCS					
2	0.00	3.80	1.36	1.24	4.48×10^{-1}
4	1.34	3.70	2.07	6.02×10^{-1}	1.37
8	3.02×10^1	8.23×10^1	6.06×10^1	2.96×10^2	1.77
MT-BCS					
2	0.00	2.18	8.01×10^{-1}	4.06×10^{-1}	3.97
4	5.45×10^{-1}	1.91	1.37	1.19×10^{-1}	6.44
8	5.27	3.31×10^1	1.81×10^1	5.94×10^1	7.80

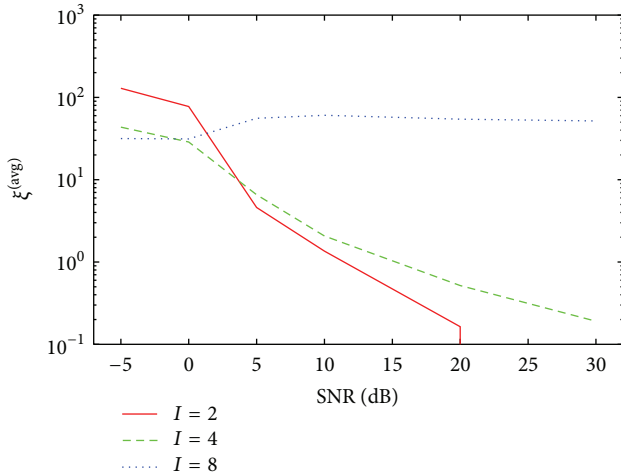


FIGURE 4: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR $\in [-5 : 30]$ dB; $T = 1$; $C = 50$)—behavior of the location index $\xi^{(\text{ave})}$ averaged among C different noisy scenarios versus the SNR when using the ST-BCS.

$\theta \leq 90^\circ$ and $0^\circ \leq \phi \leq 360^\circ$ is partitioned (Figure 1) in a fine grid of K samples satisfying the condition $K \gg I$. Therefore, the terms $\mathbf{H}(\Psi)$ and $\mathbf{s}(\tau)$ in (3) turn out being equal to

$$\begin{aligned} \check{\mathbf{H}}(\check{\Psi}) &= [\check{\mathbf{h}}(\check{\Psi}_1), \check{\mathbf{h}}(\check{\Psi}_2), \dots, \check{\mathbf{h}}(\check{\Psi}_k), \dots, \check{\mathbf{h}}(\check{\Psi}_K)], \\ \check{\mathbf{s}}(\tau) &= [\check{\alpha}_1(\tau), \check{\alpha}_2(\tau), \dots, \check{\alpha}_k(\tau), \dots, \check{\alpha}_K(\tau)]^*. \end{aligned} \quad (4)$$

By substituting (4) in (3), the problem is still linear with respect to also $\check{\mathbf{s}}(\tau)$, but $\check{\mathbf{s}}(\tau)$ [unlike $\mathbf{s}(\tau)$] is now sparse

since $K \gg I$. Accordingly, only few coefficients $\check{\alpha}_k(\tau)$, $k = 1, \dots, K$ are expected to differ from zero and exactly in correspondence with the steering vectors $\check{\mathbf{h}}(\check{\Psi}_k)$ at the angular direction $\check{\Psi}_k$ where the wave is estimated to impinge on the array. Accordingly, the original problem of determining the DoAs, $\Psi_i = (\theta_i, \phi_i)$, $i = 1, \dots, I$, is reformulated as the estimation of the (sparse) signal vector $\check{\mathbf{s}}(\tau)$. The signal DoAs are then retrieved as the directions $\check{\Psi}_k = (\hat{\theta}_k, \hat{\phi}_k)$ whose corresponding signal amplitudes $\check{\alpha}_k(\tau)$ are nonnull.

For single time-instant ($T = 1$) acquisitions, the single-task bayesian compressive sensing (ST-BCS) is used, and the sparsest vector $\check{\mathbf{s}}(\tau)$ is retrieved by maximizing the posterior probability [24, 25]:

$$\mathcal{P}(\check{\mathbf{s}}(\tau), \hat{\sigma}^2, \mathbf{a}(\tau) \mid \mathbf{v}(\tau)), \quad (5)$$

where $\hat{\sigma}^2$ is the estimate of the noise power, supposed to be not varying in time, and $\mathbf{a}(\tau)$ is the hyperparameter vector [26] enforcing the sparseness of the solution $\check{\mathbf{s}}(\tau)$ at the τ th snapshot. Accordingly, the analytic form of the solution turns out to be

$$\check{\mathbf{s}}(\tau) = \frac{1}{\hat{\sigma}^2} \left(\frac{\check{\mathbf{H}}(\check{\Psi})^* \check{\mathbf{H}}(\check{\Psi})}{\hat{\sigma}^2} + \text{diag}(\mathbf{a}(\tau)) \right)^{-1} \check{\mathbf{H}}(\check{\Psi})^* \check{\mathbf{v}}(\tau), \quad (6)$$

where all terms are real since the BCS works only with real numbers. The signal vector, $\check{\mathbf{s}}(\tau) = [\text{Re}\{\check{\mathbf{s}}(\tau)\}; \text{Im}\{\check{\mathbf{s}}(\tau)\}]^*$, has

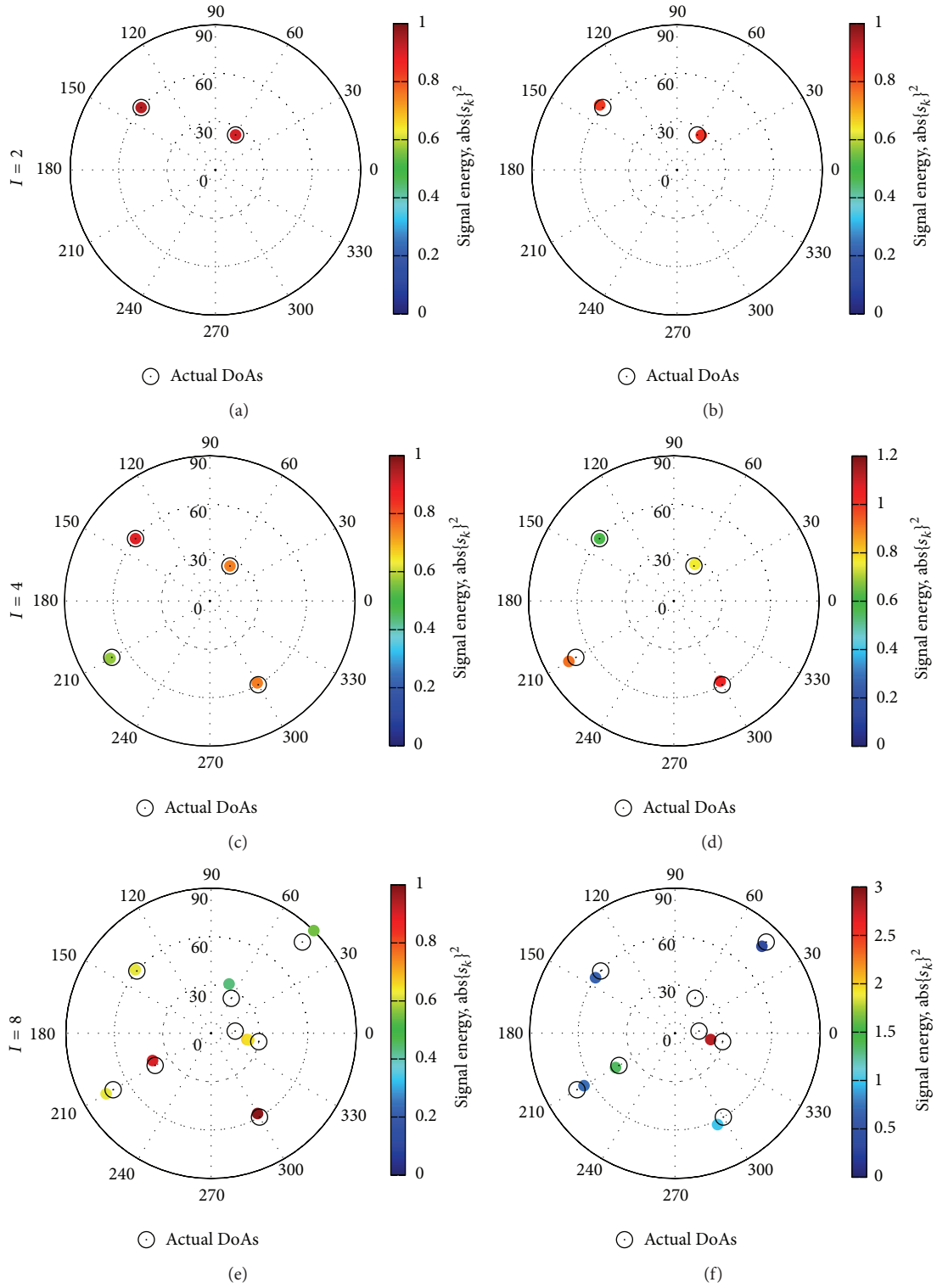


FIGURE 5: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = 2$; $C = 50$)—plot of the best (left column) and worst (right column) estimations obtained by means of the MT-BCS among the C different noisy scenarios when (a) (b) $I = 2$, (c) (d) $I = 4$, and (e) (f) $I = 8$.

TABLE 4: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = 2$; $C = 50$)—values of the DoAs for the best and worst estimation obtained by means of the MT-BCS among the C different noisy scenarios.

I	$\hat{I}^{(\text{bst})}$	$\{\hat{\Psi}_i^{(\text{bst})} = (\hat{\theta}_i^{(\text{bst})}, \hat{\phi}_i^{(\text{bst})}), i = 1, \dots, I\}$
2	2	$\{(25, 60); (60, 140)\}$
4	4	$\{(25, 60); (58.75, 300); (60, 140); (71.25, 210)\}$
8	8	$\{(22.5, 350); (23.75, 350); (32.5, 70); (40, 205); (57.5, 300); (61.25, 140); (75, 210); (90, 45)\}$
I	$\hat{I}^{(\text{wst})}$	$\{\hat{\Psi}_i^{(\text{wst})} = (\hat{\theta}_i^{(\text{wst})}, \hat{\phi}_i^{(\text{wst})}), i = 1, \dots, I\}$
2	2	$\{(26.25, 55); (62.5, 140)\}$
4	4	$\{(26.25, 60); (57.5, 300); (60, 140); (75, 210)\}$
8	6	$\{(22.5, 350); (42.5, 210); (60, 145); (62.5, 295); (65, 210); (76.25, 45)\}$

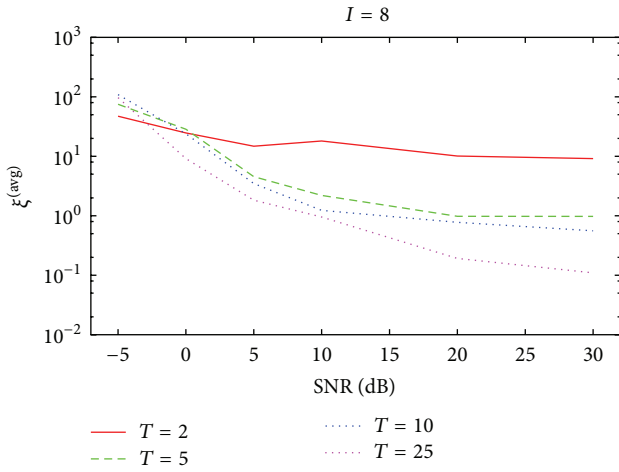


FIGURE 6: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I = 8$; SNR $\in [-5 : 30]$ dB; $T \in [2 : 25]$; $C = 50$)—behavior of the location index $\xi^{(\text{ave})}$ averaged among C different noisy scenarios versus the SNR when using the MT-BCS with different number of available snapshots T .

the dimension $2K \times 1$, and $\hat{\mathbf{v}}(\tau) = [\text{Re}\{\hat{\mathbf{v}}(\tau)\}; \text{Im}\{\hat{\mathbf{v}}(\tau)\}]^*$ is a $2N \times 1$ vector, while

$$\hat{\mathbf{H}}(\check{\Psi}) = \begin{bmatrix} \text{Re}\{\check{\mathbf{H}}(\check{\Psi})\} & -\text{Im}\{\check{\mathbf{H}}(\check{\Psi})\} \\ \text{Im}\{\check{\mathbf{H}}(\check{\Psi})\} & \text{Re}\{\check{\mathbf{H}}(\check{\Psi})\} \end{bmatrix} \quad (7)$$

is $2N \times 2K$ matrix, with $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ being the real and imaginary parts, respectively. The two control parameters in (6), $\mathbf{a}(\tau)$ and $\hat{\sigma}^2$, are obtained through the maximization of the function

$$\begin{aligned} \Pi_{\text{ST}}(\hat{\sigma}^2, \mathbf{a}(\tau)) \\ = K \log\left(\frac{1}{2\pi}\right) - \frac{\log|\mathbf{\Omega}(\tau)| + (\hat{\mathbf{v}}(\tau))^* (\mathbf{\Omega}(\tau))^{-1} \hat{\mathbf{v}}(\tau)}{2} \end{aligned} \quad (8)$$

by means of the relevance vector machine (RVM). In (6), $\mathbf{\Omega}(\tau) \triangleq \hat{\sigma}^2 \mathbf{I} + \hat{\mathbf{H}}(\check{\Psi}) \text{diag}(\mathbf{a}(\tau))^{-1} \hat{\mathbf{H}}(\check{\Psi})^*$ where $\mathbf{I}$ is the identity matrix.

When a set of consecutive snapshots is available, the multitask BCS (MT-BCS) implementation is used to statistically correlate the estimates derived for each snapshot by

setting a common hyperparameter vector: $\mathbf{a}(\tau) = \mathbf{a}$, for all $\tau = 1, \dots, T$. Hence, the final MT-BCS solution is given by [24, 27]:

$$\hat{\mathbf{s}} = \frac{1}{T} \sum_{\tau=1}^T \left\{ \left[\hat{\mathbf{H}}(\check{\Psi})^* \hat{\mathbf{H}}(\check{\Psi}) + \text{diag}(\mathbf{a}) \right]^{-1} \hat{\mathbf{H}}(\check{\Psi})^* \hat{\mathbf{v}}(\tau) \right\}, \quad (9)$$

where $\mathbf{a}$ is computed through the RVM maximization of the following function:

$$\begin{aligned} \Pi_{\text{MT}}(\mathbf{a}) \\ = -\frac{1}{2} \sum_{\tau=1}^T \{ \log(|\mathbf{\Omega}|) + (K + 2\beta_1) \log[(\hat{\mathbf{v}}(\tau))^* \mathbf{\Omega} \hat{\mathbf{v}}(\tau) + 2\beta_2] \}, \end{aligned} \quad (10)$$

where $\mathbf{\Omega} \triangleq \mathbf{I} + \hat{\mathbf{H}}(\check{\Psi}) \text{diag}(\mathbf{a})^{-1} \hat{\mathbf{H}}(\check{\Psi})^*$ and β_1 and β_2 are two user-defined parameters [28].

Although the condition $\hat{\alpha}_k(\tau) \approx 0$ or $\hat{\alpha}_k \approx 0$ usually holds true, the number of nonnull coefficients in either $\hat{\mathbf{s}}(\tau)$ (ST-BCS) or $\hat{\mathbf{s}}$ (MT-BCS) could be larger because of the presence of the noise. Hence, the energy thresholding techniques in [24] are exploited to firstly count the number of arriving signals, $\hat{I}$, and then to estimate the corresponding DoAs. More in detail, the coefficients $\hat{\alpha}_k(\tau)$ (or $\hat{\alpha}_k$) are firstly sorted according to their magnitude, and then only the first $\hat{I}$ coefficients whose cumulative power content is lower than a percentage χ of the totally received signal power, namely, $\|\hat{\mathbf{s}}(\tau)\| = \sum_{k=1}^K (\hat{\alpha}_k(\tau))^2$ (or $\|\hat{\mathbf{s}}\| = \sum_{k=1}^K (\hat{\alpha}_k)^2$), are preserved. Hence, $\hat{I}$ is selected such that $\sum_{i=1}^{\hat{I}} (\hat{\alpha}_i(\tau))^2 < \chi \|\hat{\mathbf{s}}(\tau)\|$ (or $\sum_{i=1}^{\hat{I}} (\hat{\alpha}_i)^2 < \chi \|\hat{\mathbf{s}}\|$).

3. Numerical Results

The planar array BCS-based estimation method is assessed by means of the following analysis devoted to evaluate (a) the performance of its different implementations in correspondence with single snapshot ($T = 1$) or multiple-snapshots ($T > 1$) acquisitions and (b) the impact of different array configurations. Throughout the numerical assessment, the array elements have been assumed uniformly spaced of $d_x = \lambda/2$ and $d_y = \lambda/2$ along the x -axis and y -axis, respectively, and all signals have been characterized with the same amplitude $\alpha_i(\tau) = \alpha_{i+1}(\tau)$, $i = 1, \dots, I - 1$. The

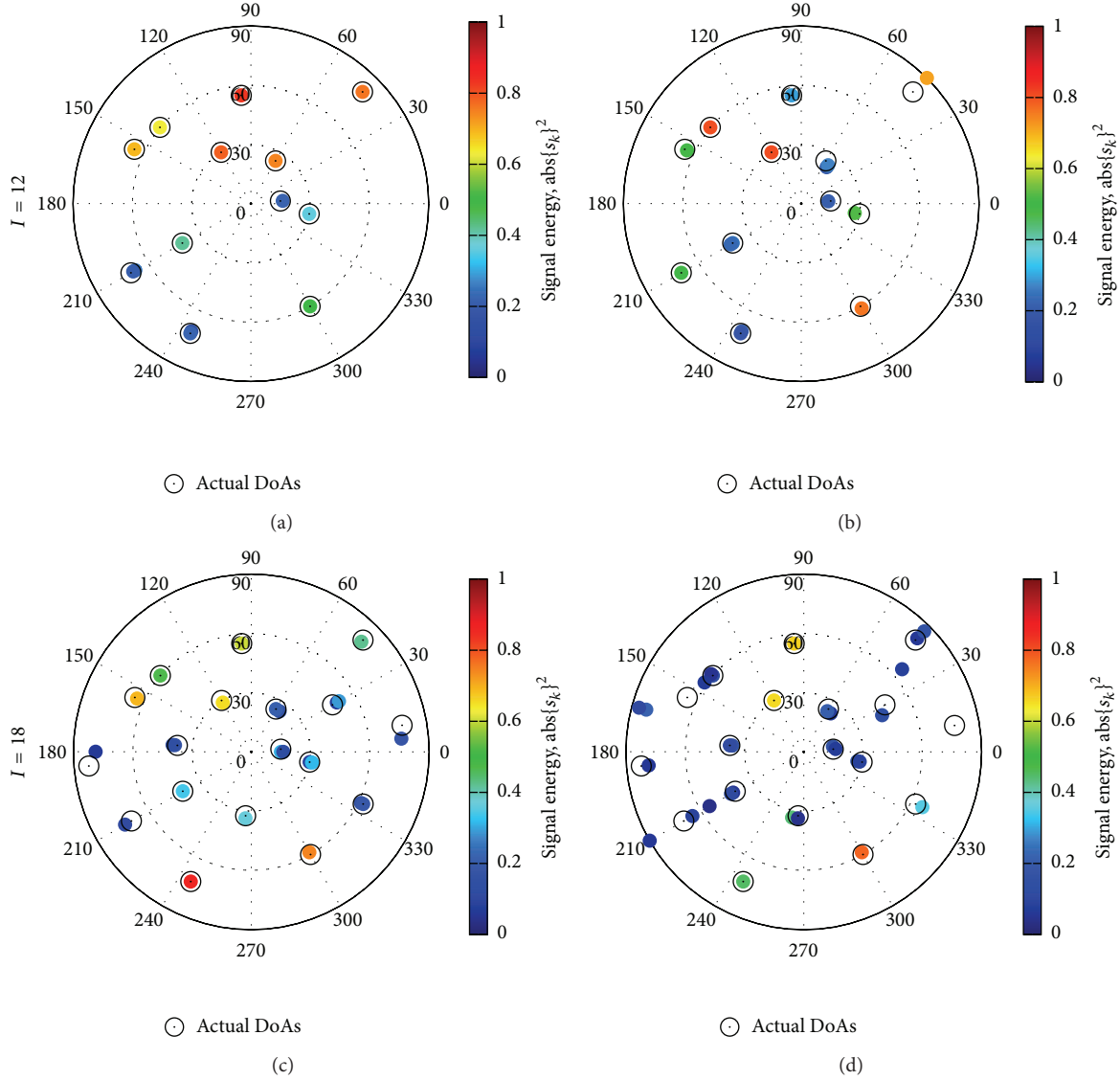


FIGURE 7: Fully populated array—($N = 25$; $d_x = d_y = 0.5\lambda$; $I = \{12, 18\}$; SNR = 10 dB; $T = 25$; $C = 50$)—plot of the best (left column) and worst (right column) estimations obtained by means of the MT-BCS among the C different noisy scenarios when (a) (b) $I = 12$ and (c) (d) $I = 18$.

measurements have been blurred with an additive Gaussian noise of variance σ^2 such that the resulting signal-to-noise ratio turns out to be

$$\text{SNR} = 10 \times \log \left[\frac{\sum_{n=1}^N |v_n^{\text{no-noise}}|^2}{N\sigma^2} \right], \quad (11)$$

with $v_n^{\text{no-noise}}$ ($n = 1, \dots, N$) being the voltage measured at the n th array element in the noiseless case. The angular observation domain (Figure 1) has been partitioned with a uniform grid characterized by a sampling step equal to $\Delta\theta = 1.25^\circ$ and $\Delta\phi = 1.25^\circ$ along the elevation and azimuthal direction, respectively. The energy threshold has been set to $\chi = 0.95$ according to [24].

In order to quantify the reliability and the effectiveness of the DoA estimation, the following indexes have been

computed. For each i th signal, the location index [13] is defined as

$$\xi_i = \xi(\Psi_i, \hat{\Psi}_i) \triangleq \frac{\Phi(\Psi_i, \hat{\Psi}_i)}{\Phi^{(\max)}} \times 100, \quad (12)$$

where

$$\begin{aligned} \Phi(\Psi_i, \hat{\Psi}_i) = & \left((\sin \theta_i \cos \phi_i - \sin \hat{\theta}_i \cos \hat{\phi}_i)^2 \right. \\ & + (\sin \theta_i \sin \phi_i - \sin \hat{\theta}_i \sin \hat{\phi}_i)^2 \\ & \left. + (\cos \theta_i - \cos \hat{\theta}_i)^2 \right)^{1/2} \end{aligned} \quad (13)$$

and $\Phi^{(\max)} = \max_{\Psi_i, \hat{\Psi}_i} \{\Phi(\Psi_i, \hat{\Psi}_i)\} = 2$ is the maximum admissible error in the DoA retrieval. Since the number of

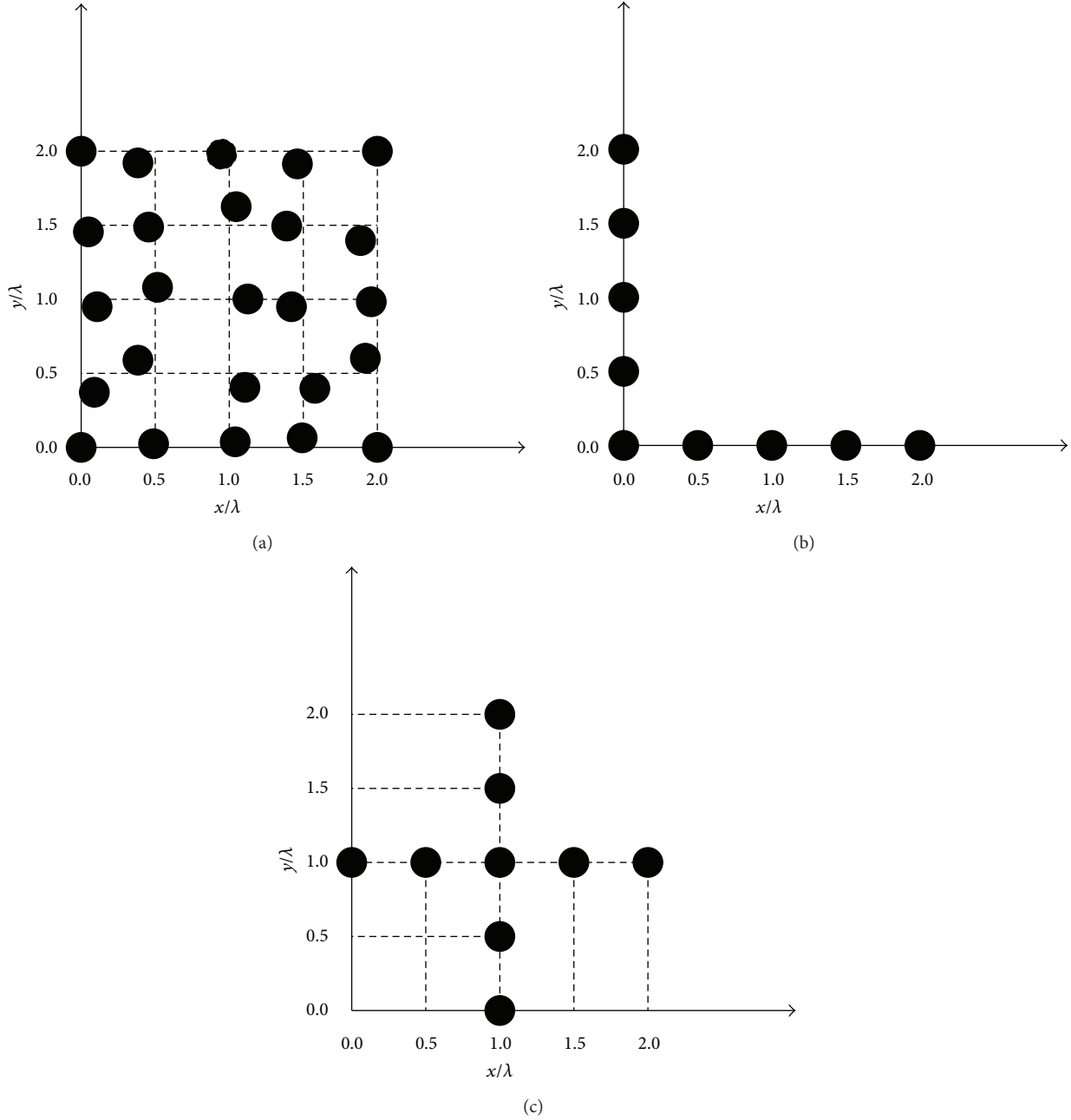


FIGURE 8: Array geometries comparison—geometries of the receiving (a) random ($N = 25$), (b) L-shaped ($N = 9$), and (c) cross-shaped ($N = 9$) arrays.

arriving signals $\hat{I}$ is unknown and it is derived from the BCS processing, the global location index has been also evaluated [24]:

$$\xi = \begin{cases} \frac{1}{\hat{I}} \left[\sum_{i=1}^{\hat{I}} \xi(\Psi_i, \hat{\Psi}_i) + (I - \hat{I}) \xi^{(\text{penalty})} \right] & \text{if } \hat{I} < I, \\ \frac{1}{I} \left[\sum_{i=1}^I \xi(\Psi_i, \hat{\Psi}_i) + \sum_{i=\hat{I}+1}^I \xi(\bar{\Psi}_i, \hat{\Psi}_i) \right] & \text{if } \hat{I} \geq I, \end{cases} \quad (14)$$

where $\xi^{(\text{penalty})} = \max_{\Psi_i, \hat{\Psi}_i} \{\xi_i\} = 100$ is the maximum of (12) and $\bar{\Psi}_i = \arg\{\min_{i=I+1} [\xi(\Psi_i, \hat{\Psi}_i)]\}$. Since it is preferred to detect all signals really present in the scenario, although overestimating their number then missing some of them, the penalty is considered only when $\hat{I} < I$.

3.1. Single and Multiple Snapshot BCS-Based DoAs Estimation Techniques. Let us consider the fully populated array of Figure 2 with $N = N_x \times N_y = 25$ elements, with $N_x = N_y = 5$ being the number of elements along the x and y axes, collecting the data $\mathbf{v}(\tau)$. Several different electromagnetic

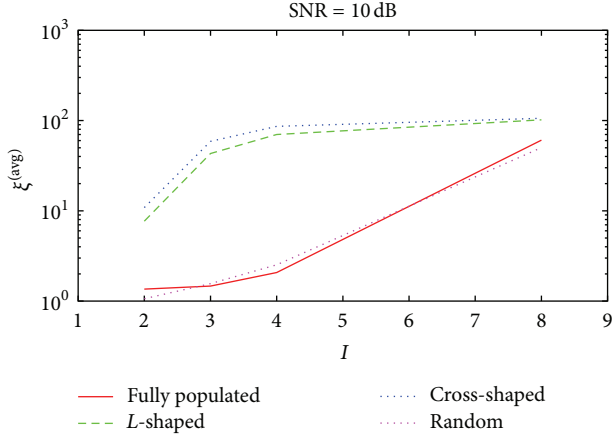


FIGURE 9: Array geometries comparison—($N = \{9, 25\}$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = 1$; $C = 50$)—behavior of the location index $\xi^{(\text{ave})}$ averaged among C different noisy scenarios versus the number of arriving signals I when using the ST-BCS.

scenarios have been considered in which $I = 2$, $I = 4$, and $I = 8$ signals are supposed to impinge on the planar array from the directions indicated in Table 1. (In the numerical results, the actual DoAs are chosen lying on the sampling grid of the observation domain. Whether this condition does not hold true, off-grid compensation methods [29, 30], already proposed in the state-of-the-art literature, can be profitably used).

The power of the background noise has been set to yield SNR = 10 dB. In order to test the behavior of the ST-BCS and the MT-BCS, the simulation for each signal configuration has been repeated $C = 50$ times, while varying the noise samples on the data. The DoAs estimation error has been therefore evaluated through the average location index defined as

$$\xi^{(\text{avg})} = \frac{1}{C} \sum_{c=1}^C \xi^{(c)}, \quad (15)$$

with $\xi^{(c)}$ being computed as in (14).

As for the ST-BCS, a single snapshot has been processed each time ($T = 1$). Figure 3 shows the best (Figure 3—left column) and the worst (Figure 3—right column) solutions in terms of minimum ($\xi^{(\min)} = \min_{c=1, \dots, C} \{\xi^{(c)}\}$) and maximum ($\xi^{(\max)} = \max_{c=1, \dots, C} \{\xi^{(c)}\}$) location errors, respectively, among the $C = 50$ DoAs estimations carried out when $I = 2$ (Figures 3(a) and 3(b)), $I = 4$ (Figures 3(c) and 3(d)), and $I = 8$ (Figures 3(e) and 3(f)). In Figure 3, the actual DoAs are denoted with a point at the center of a circle, while the color points indicate the estimated signal locations and amplitudes. For the sake of clarity, the retrieved DoAs are also reported in Table 2 where the number of estimated signals $\hat{I}$ is given, as well. As it can be observed, the strength of the estimated signals is different (Figure 3), even though they impinge on the antenna with the same energy because of the presence of the noise. On the other hand, the DoAs are predicted with a high degree of accuracy when $I = 2$ and $I = 4$ as confirmed by the values of the location error

(Table 3). As a matter of fact, the error values are low also for the worst solutions among the C trials (i.e., $\xi^{(\max)}|_{I=2} = 3.80\%$ and $\xi^{(\max)}|_{I=4} = 3.89\%$). It is worth also noting that for $I = 2$ the location error is small even though the numbers of detected signals are greater than the actual ones ($\hat{I}^{(\text{wst})}|_{I=2} = 3$) because two signals have very close DoAs (as compared to the sampling steps $\Delta\theta$ and $\Delta\phi$). However, if the ST-BCS shows being robust and accurate in such scenarios ($I = 2$ and $I = 4$), it is not able to correctly locate the actual DoAs when the number of signals increases to $I = 8$ (Figures 3(e) and 3(f)—Table 2). Indeed, the location error significantly increases as indicated by the indexes in Table 3.

As for the computational efficiency, the ST-BCS is able to perform the DoAs estimation in a limited CPU time ($t^{(\text{avg})} < 2.0$ [sec]—Table 3) (the simulations have been run using a standard processing unit (i.e., 2.4 GHz PC with 2 GB of RAM) with a nonoptimized code) also thanks to the single-snapshot processing. In order to investigate the effects of the SNR on the DoAs estimation capabilities of the ST-BCS, the SNR has been varied from -5 dB up to 30 dB with a step of 5 dB, while keeping the same DoAs of Table 1. In Figure 4, the values of the average location index are reported. As it can be noticed, the location index $\xi^{(\text{avg})}$ for $I = 2$ and $I = 4$ monotonically decreases, as one should expect, with the increment of the SNR. However, the ST-BCS estimates when $I = 8$ turn out to be still nonreliable also for higher SNR confirming the difficulty of dealing with such a complex scenario just processing one snapshot.

Let us now analyze the MT-BCS behavior. Firstly, the same problems addressed by means of the ST-BCS in Figure 3 are considered by taking into account only $T = 2$ snapshots. The best and worst MT-BCS results are reported in Figure 5, and the corresponding DoAs are given in Table 4. Unlike the ST-BCS (Table 2) the number of impinging signals is always correctly identified in the best case (Figure 5—left column), while in the worst case (Figure 5—right column), $\hat{I} = I$ only when $I = 2$ and $I = 4$ signals. As a matter of fact, the average location error when $I = 8$ is still high ($\xi^{(\text{avg})}|_{I=8} = 18.1\%$). The use of only $T = 2$ snapshots does not guarantee reliable performance also with the MT-BCS, even though the advantages in terms of accuracy of the MT-BCS over the ST-BCS are nonnegligible as pointed out by the values in Table 3. On the opposite, the computational cost of the MT-BCS is higher than that of the ST-BCS (Table 3).

More reliable MT-BCS estimations can be yielded when processing a larger number of snapshots. Figure 6 shows that, also for complex electromagnetic scenarios (i.e., $I = 8$ —Table 1), the average location error gets lower when T increases. By considering SNR = 10 dB as a representative example, one can observe that $\xi^{(\text{avg})}$ reduces almost one order of magnitude from $\xi^{(\text{avg})}|_{I=8} = 18.1\%$ ($T = 2$) to $\xi^{(\text{avg})}|_{I=8} = 2.20\%$ ($T = 5$). As expected, more accurate estimations arise with even more data (i.e., $\xi^{(\text{avg})}|_{I=8} = 1.23\%$ when $I = 10$, and $\xi^{(\text{avg})}|_{I=8} = 0.95\%$ when $I = 25$ —Figure 6). The benefits from the correlation of the information coming from different time instants thanks to the MT-BCS are also highlighted by the behavior of the plots

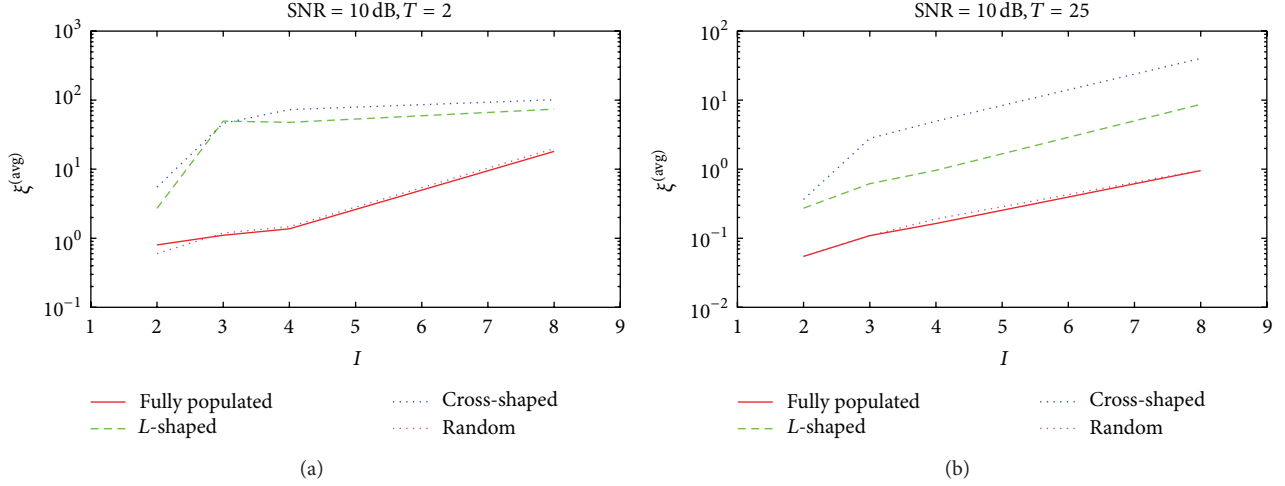


FIGURE 10: Array geometries comparison—($N = \{9, 25\}$; $d_x = d_y = 0.5\lambda$; $I \in [2 : 8]$; SNR = 10 dB; $T = \{2, 25\}$; $C = 50$)—behavior of the location index $\xi^{(\text{ave})}$ averaged among C different noisy scenarios versus the number of arriving signals I when using the MT-BCS with (a) $T = 2$ snapshots and (b) $T = 25$ snapshots.

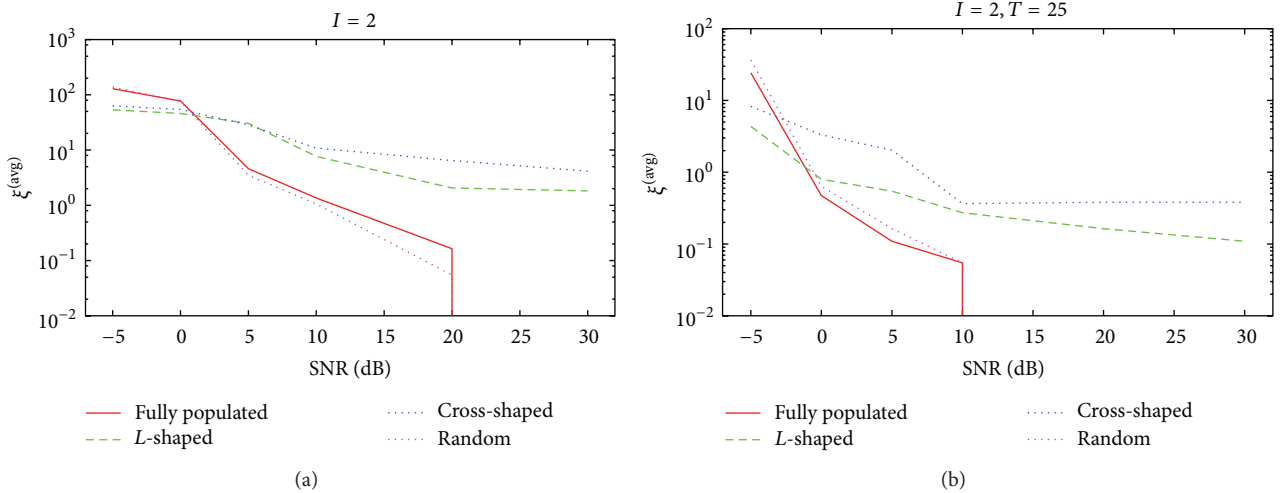


FIGURE 11: Array geometries comparison—($N = \{9, 25\}$; $d_x = d_y = 0.5\lambda$; $I = 2$; SNR = 10 dB; $T = \{1, 25\}$; $C = 50$)—behavior of the location index $\xi^{(\text{ave})}$ averaged among C different noisy scenarios versus the SNR when using (a) the ST-BCS with $T = 1$ snapshot and (b) the MT-BCS with $T = 25$ snapshots.

in Figure 6: $\xi^{(\text{avg})}$ more rapidly decreases for higher values of T when the quality of the data improves (i.e., higher SNR).

As long as the applications at hand do not require the fast or real-time identification of the DoAs, and there is the possibility to collect the data at consecutive time instants, the robust estimation of a larger number of impinging signals is allowed. In this context, Figure 7 shows the results obtained with the MT-BCS when $I = 12$ (Figures 7(a) and 7(b)) and $I = 18$ (Figures 7(c) and 7(d)) (SNR = 10 dB). As for the case $I = 12$, the DoAs are estimated with a good degree of accuracy also in the worst case within the C experiments (Figure 7(b)— $\xi^{(\text{max})}|_{I=12} = 1.77\%$), while the average location error amounts to $\xi^{(\text{avg})}|_{I=12} = 1.04\%$. Differently, the average error is $\xi^{(\text{avg})}|_{I=18} = 4.70\%$ and in the worst case (Figure 7(d)) is $\xi^{(\text{max})}|_{I=18} = 7.85\%$ when

$I = 18$. For the sake of completeness, the best solutions are reported in Figures 7(a) and 7(c) when $I = 12$ and $I = 18$, respectively.

3.2. DoAs Estimation Performance for Different Array Geometries. In this section, the behavior of the BCS-based single-snapshot and multiple-snapshots DoAs estimators is analyzed for different array architectures. The three array geometries in Figure 8 are taken into account. As it can be noticed, the first array (Figure 8(a)) has the same number of elements of the fully populated one, but the sensors are randomly located on the antenna aperture. The other two arrays (Figures 8(b) and 8(c)) have less elements (i.e., $N = 9$) but same aperture length of the fully populated array along the two coordinate axes.

In the first example, the performance of the ST-BCS is assessed when changing the number of impinging signals from $I = 2$ up to $I = 8$, while keeping the noise level to $\text{SNR} = 10$ dB. Figure 9 shows the average location error ($C = 50$) obtained in correspondence with the three arrays. Unlike the fully populated arrangement enabling good estimation features especially until $I = 4$ ($\xi^{(\text{avg})}|_{I=2,3,4} < 2.00\%$), both the L -shaped array and the cross-shaped one do not allow reliable estimations also for the simplest scenario (i.e., $\xi^{(\text{avg})}|_{I=2}^{L\text{-Shaped}} = 7.69\%$ and $\xi^{(\text{avg})}|_{I=2}^{\text{Cross-Shaped}} = 10.87\%$). This is due, on the one hand, to the limited information collected from a single snapshot acquisition and, on other hand, to the fact that the number of sensors is one third the elements of the fully-populated configuration (i.e., $N^{\text{Fully-Populated}}/N^{L/\text{Cross-Shaped}} = 2.78$). As for the random array, the achieved performances are almost equal to those of the fully populated solution thus confirming the higher reliability when having at disposal a larger number of sensors. When using the MT-BCS, no significant improvements occur in comparison with the ST-BCS when $T = 2$, since average errors higher than $\xi^{(\text{avg})} = 2.00\%$ (Figure 10(a)) are obtained with both the L -shaped or cross-shaped array. Whether $T = 25$ snapshots are at disposal (Figure 10(b)), it turns out that the estimates from the L -shaped array present average location errors below $\xi^{(\text{avg})} = 2.00\%$ until $I = 5$. Differently, always worse performance is achieved with the cross-shaped array (Figure 10(b)).

In order to give some insight on the effects of the SNR, let us consider the case $I = 2$ as a representative example. The results from the ST-BCS and the MT-BCS are reported in Figures 11(a) and 11(b), respectively. The location error tends to reduce as the SNR increases for all array structures, even though the L -shaped array outperforms the cross-shaped one, and the random array behavior is always very close to that of the fully populated configuration.

4. Conclusions

The BCS method has been customized for the DoAs estimation of multiple signals impinging on planar arrays. Two different implementations, one requiring the data measured at a single snapshot and the other using the data collected at multiple snapshots, have been tested on a wide number of different scenarios as well as using different array arrangements. Likewise in the linear array case, the reported results have shown that:

- (i) the two BCS-based implementations provide effective DoAs estimates just using as data the sensors output voltages without requiring the covariance matrix;
- (ii) the joint estimation of the signals number and DoAs is enabled;
- (iii) the correlation capability of the MT-BCS allows one to yield better results than the ST-BCS at the expenses of an increased computational burden.

As for the behavior of the two approaches versus the planar array geometry, it is possible to conclude that

- (i) the fully populated and the random arrays give the best performance as compared to both the L -shaped and the cross-shaped arrays, but using a larger number of sensors;
- (ii) under the assumption of the same number of elements, the L -shaped configuration always outperforms the precision from the cross-shaped arrangement.

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