Periodic representations for quadratic irrationals in the field of p-adic numbers

Stefano Barbero, Umberto Cerruti, Nadir Murru

Department of Mathematics G. Peano, University of Torino
Via Carlo Alberto 10, 10123, Torino, ITALY
stefano,barbero@unito.it, umberto.cerruti@unito.it, nadir.murru@unito.it

Abstract

Continued fractions have been widely studied in the field of p-adic numbers $\mathbb{Q}_p$, but currently there is no algorithm replicating all the good properties that continued fractions have over the real numbers regarding, in particular, finiteness and periodicity. In this paper, first we propose a periodic representation, which we will call *standard*, for any quadratic irrational via p-adic continued fractions, even if it is not obtained by a specific algorithm. This periodic representation provides simultaneous rational approximations for a quadratic irrational both in $\mathbb{R}$ and $\mathbb{Q}_p$. Moreover given two primes p_1 and p_2, using the Binomial transform, we are also able to pass from approximations in $\mathbb{Q}_{p_1}$ to approximations in $\mathbb{Q}_{p_2}$ for a given quadratic irrational. Then, we focus on a specific p–adic continued fraction algorithm proving that it stops in a finite number of steps when processes rational numbers, solving a problem left open in [6]. Finally, we study the periodicity of this algorithm showing when it produces *standard* representations for quadratic irrationals.

Keywords: Browkin algorithm, continued fractions, p–adic numbers, quadratic irrationals
2010 Mathematics Subject Classification: 11J70, 12J25, 11J61.

1 Introduction

Continued fractions have been widely studied in the field of p-adic numbers $\mathbb{Q}_p$. Unlike the real case, in $\mathbb{Q}_p$ there is no standard algorithm for continued fractions. In fact, we can find the classical works of Schneider [15], Ruban [13] and Browkin [5], where there are different approaches. Schneider [15] defined an algorithm that produces p-adic continued fractions of the kind

$$a_0 + \cfrac{p^{b_1}}{a_1 + \cfrac{p^{b_2}}{a_2 + \ddots}}, \tag{1}$$

with $a_i \in \{0, 1, \ldots, p-1\}$. Ruban [13] modified this algorithm to obtain simple continued fractions, i.e., continued fractions like (1) where any b_i is zero. Browkin [5] proposed another algorithm for obtaining simple continued fractions where the partial quotients belong to the set $\mathbb{Z}\left[\frac{1}{p}\right]$. However, none of the proposed algorithms satisfies all the good properties of classical continued fractions in $\mathbb{R}$, regarding in particular finiteness and periodicity. In $\mathbb{R}$ it is well known that the continued fraction algorithm characterizes rational numbers and quadratic irrationals. Indeed, this algorithm stops in a finite number of steps if and only if it processes a rational number and it is periodic if and only if processes a quadratic irrational. Instead, Schneider and Ruban p-adic continued fractions provide a finite or periodic expansion for rational numbers and they are not always periodic for square roots of rational integers. Indeed, Ooto in his paper [10] has been the first to show that an analogous of Lagrange's theorem does not hold for Ruban's continued fractions (for more details see also [11] and [19]). In this case, recently Capuano, Veneziano and Zannier [7] have given an effective criterion for periodicity of the expansion of quadratic irrationals for Ruban continued fractions. Browkin definition is very interesting because it provides a finite representation for rational numbers. However, its algorithm does not give a periodic expansion for any quadratic irrational, see also [3]. Thus, the problem of finding a p-adic continued fraction algorithm that it is always periodic when it processes quadratic irrationals appears to be very interesting. For this reason, Browkin proposed in [6] a further study, defining some new algorithms. Recently, some authors generalized the study of continued fractions in $\mathbb{Q}_p$ to the multidimensional case, see, e.g., [8], [9], [14], [18].

The paper is structured as follows. In Section 2, we start presenting and studying a periodic representation, that we will call *standard*, via p-adic continued fractions for quadratic irrationals, even if it is not determined by an algorithm. Then, in Section 3, we recall the classical algorithm proposed by Browkin [5] and a variant proposed by the same author in [6], named Browkin algorithm II, that appears to produce periodic expansions for more square roots than the original algorithm. Here, we prove that Browkin algorithm II stops in a finite number of steps when processes rational numbers, which is a problem left open in [6]. Finally, we study the periodicity of this algorithm showing when it produces *standard* periodic representations extending a result given in [6].

2 A standard periodic representation of quadratic irrationals in $\mathbb{Q}_p$

In this section, we present a periodic representation for any quadratic irrational in $\mathbb{Q}_p$ as a p-adic continued fraction, thanks to a generalization of the Rédei rational functions. In what follows, we will always use $|\cdot|$ for the Euclidean norm, $|\cdot|_p$ for the p–adic norm and $\nu_p(\cdot)$ for the p–adic valuation, where p is an odd prime number.

The Rédei rational functions $C_n(d,z)$ are defined by

$$C_n(d,z) := \frac{A_n(d,z)}{B_n(d,z)}, \quad \text{where for } n = 1,2,\ldots \quad A_n(d,z) + B_n(d,z)\sqrt{d} = \left(z+\sqrt{d}\right)^n,$$

and $z, d \in \mathbb{Z}$, $z \neq 0$, $d > 0$ not square, see [12]. By an induction argument, one can prove that the following matricial identity holds:

$$\begin{pmatrix} z & d \\ 1 & z \end{pmatrix}^n = \begin{pmatrix} A_n(d,z) & dB_n(d,z) \\ B_n(d,z) & A_n(d,z) \end{pmatrix}.$$

We start from this last identity to give a generalization of the Rédei rational functions. In particular we define the sequences $(N_n(h,d,z))_{n\geq 0}$ and $(D_n(h,d,z))_{n\geq 0}$ by means of

$$\begin{pmatrix} z+h & d \\ 1 & z \end{pmatrix}^n = \begin{pmatrix} N_n(h,d,z) + hD_n(h,d,z) & dD_n(h,d,z) \\ D_n(h,d,z) & N_n(h,d,z) \end{pmatrix} \tag{2}$$

where $h, d, z \in \mathbb{Z}$ and the *generalized Rédei rational functions* are defined by $Q_n(h,d,z) = \dfrac{N_n(h,d,z)}{D_n(h,d,z)}$, for $n = 1, 2, \ldots$.

It is immediate to see that the sequence of Rédei rational functions converges to $\sqrt{d}$ in $\mathbb{R}$, for any integer value of $z \neq 0$. Moreover, they are the convergents of the following continued fraction

$$\left[z, \overline{\frac{2z}{d-z^2}, 2z}\right],$$

see [1].

In the next propositions, we show that the generalized Rédei rational functions are the convergents of a certain periodic continued fraction and they converge to a root of $x^2 + hx - d$ in $\mathbb{R}$.

Lemma 1. *Let* $\left[a_0, \frac{a_1}{b_1}, \ldots, \frac{a_i}{b_i}, \ldots\right]$ *be a continued fraction with* $a_i, b_i \in \mathbb{Z}$, $b_i \neq 0$, *and let* $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ *be the sequences of numerators and denominators of convergents; then, for every* $n \geq 2$, *we have*

$$\begin{cases} p_n = a_n p_{n-1} + b_n b_{n-1} p_{n-2} \\ q_n = a_n q_{n-1} + b_n b_{n-1} q_{n-2} \end{cases}, \quad \textit{where} \quad \begin{cases} p_0 = a_0, \quad p_1 = a_0 a_1 + b_1 \\ q_0 = b_0 = 1, \quad q_1 = a_1 \end{cases}.$$

Proof. The proof is straightforward using the linear recurrence relations for numerators and denominators of the convergents of an irregular continued fraction (see, e. g., Section 9.1 of [4]). $\square$

For seek of simplicity, in the following, we set $N_n = N_n(h, d, z)$, $D_n = D_n(h, d, z)$, $Q_n = Q_n(h, d, z)$.

Proposition 2. *Given $h, d \in \mathbb{Z}$, the generalized Rédei rational functions $Q_n(h, d, z)$ are the convergents of the continued fraction*

$$\left[z, \overline{-\frac{h+2z}{z^2+hz-d}, h+2z}\right] \tag{3}$$

for every $z \in \mathbb{Z}$ not zero.

Proof. First, by (2) we can observe that the sequences $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ recur with characteristic polynomial

$$x^2 - (h+2z)\,x + z^2 + hz - d, \tag{4}$$

with initial conditions $1, h+z$ and $0, 1$, respectively. By Lemma 1, we have that the purely periodic continued fraction $\left[\overline{h+2z, -\frac{h+2z}{z^2+hz-d}}\right]$ has convergents $\frac{u_{n+2}}{u_{n+1}}$, where $(u_n)_{n\geq 0}$ is a linear recurrence sequence with characteristic polynomial (4) and initial conditions $0, 1$, i.e., $(u_n)_{n\geq 0} = (D_n)_{n\geq 0}$. Hence, the convergents of the periodic continued fraction (3) are

$$\frac{u_{n+2} - (h+z)\,u_{n+1}}{u_{n+1}},$$

and we can see that $u_{n+2} - (h+z)\,u_{n+1} = N_{n+1}$, for every $n \geq 0$. $\square$

Proposition 3. *Let α_1, α_2 be the real roots of $x^2 + hx - d$, for $h, d \in \mathbb{Z}$, then*

$$\lim_{n\to+\infty} Q_n(h, d, z) = \alpha_1, \quad \textit{if} \quad |-\alpha_1 + z| < |-\alpha_2 + z|,$$

$$\lim_{n\to+\infty} Q_n(h, d, z) = \alpha_2, \quad \textit{if} \quad |-\alpha_1 + z| > |-\alpha_2 + z|,$$

where $|\cdot|$ stands for the Euclidean norm.

Proof. Since the sequences $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ recur with the characteristic polynomial (4) it is straightforward to see that $-\alpha_1 + z$ and $-\alpha_2 + z$ are roots of the characteristic polynomial (4), and consequently

$$\lim_{n\to+\infty} \frac{N_{n+1}}{N_n} = \lim_{n\to+\infty} \frac{D_{n+1}}{D_n} = -\alpha_1 + z,$$

if $|-\alpha_1 + z| > |-\alpha_2 + z|$. Using the equality $N_{n+1} = zN_n + dD_n$, we have

$$\alpha_1 = \lim_{n\to+\infty} \frac{zN_n - N_{n+1}}{N_n} = \lim_{n\to+\infty} -\frac{dD_n}{N_n},$$

and

$$\lim_{n\to+\infty} Q_n = \alpha_2,$$

since $\alpha_1\alpha_2 = -d$. Similarly, we get

$$\lim_{n\to+\infty} Q_n = \alpha_1,$$

when $|-\alpha_1 + z| < |-\alpha_2 + z|$. $\square$

We recall the usual notation for the Legendre symbol

$$\left(\frac{m}{n}\right) = \begin{cases} 1 & \text{if } m \text{ is a quadratic residue modulo } n \text{ and } m \not\equiv 0 \pmod{n}, \\ -1 & \text{if } m \text{ is a non-quadratic residue modulo } n, \\ 0 & \text{if } m \equiv 0 \pmod{n}. \end{cases}$$

that we will use in the following theorem, where we study the convergence of the Rédei rational functions in the field of p–adic numbers $\mathbb{Q}_p$.

Theorem 4. *Let p be a prime number and $h, d, z \in \mathbb{Z}$ such that $z^2+hz-d = p$ and $\left(\frac{h^2+4d}{p}\right) = 1$. Then*

$$\lim_{n\to+\infty} Q_n(h,d,z) = \alpha,$$

where α root of x^2+hx-d in $\mathbb{Q}_p$. Moreover,

$$\lim_{n\to+\infty} Q_n(h,d,z) = \alpha_1, \quad \text{if} \quad h+2z < p,$$

$$\lim_{n\to+\infty} Q_n(h,d,z) = \alpha_2, \quad \text{if} \quad h+2z > p,$$

where

$$\alpha_1 = -\frac{h}{2} + \frac{\sqrt{h^2+4d}}{2} \quad \text{and} \quad \alpha_2 = -\frac{h}{2} - \frac{\sqrt{h^2+4d}}{2}.$$

Proof. First, we recall that the sequences $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ recur with characteristic polynomial (4). Moreover by the hypothesis $z^2+hz-d = p$, we have

$$(h+2z)^2 - \left(h^2+4d\right) = 4p. \tag{5}$$

Since $\left(\frac{h^2+4d}{p}\right) = 1$ we have $p \nmid h+2z$, otherwise from (5) we must have $p \mid h^2+4d$ and so $\left(\frac{h^2+4d}{p}\right) = 0$ by definition of Legendre symbol. Therefore $p \nmid D_n$ for any $n \geq 1$, indeed for all $n \geq 2$ we have

$$D_n = (h+2z)D_{n-1} - (z^2-hz+d)D_{n-2} = (h+2z)D_{n-1} - pD_{n-2}$$

with $D_0 = 0$ and $D_1 = 1$, so for all $n \geq 1$ we obtain $D_n \equiv (h+2z)^{n-1} \pmod{p}$. From (2), we have

$$\left(z^2+hz-d\right)^n = N_n^2 + hD_nN_n - dD_n,$$

and

$$Q_n^2 + hQ_n - d \equiv 0 \pmod{p^n},$$

i.e., $(Q_n)_{n\geq 0}$ converges to a root of x^2+hx-d in $\mathbb{Q}_p$.
From the recurrence relations of the sequences $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$, we can easily see that $N_n = zD_n - pD_{n-1}$, i.e.,

$$Q_n \equiv z \pmod{p}.$$

Hence, $(Q_n)_{n\geq 0}$ converges to the root $\alpha = a_0 + a_1p + \ldots$ of x^2+hx-d such that $z \equiv a_0 \pmod{p}$. From (5), we note that the solutions of $x^2 \equiv \frac{h^2+4d}{4} \pmod{p}$ are $y_1 \equiv \frac{h+2z}{2} \pmod{p}$ and $p - y_1 \equiv -\frac{h+2z}{2} \pmod{p}$. Thus, if $y_1 < \frac{p}{2}$ then $\frac{h+2z}{2} - \frac{\sqrt{h^2+4d}}{2} \equiv 0 \pmod{p}$, while if $y_1 > \frac{p}{2}$ we have $\frac{h+2z}{2} + \frac{\sqrt{h^2+4d}}{2} \equiv 0 \pmod{p}$ and the thesis follows. $\square$

From the previous results, we have that given any quadratic irrational, the continued fraction (3) provides a periodic representation, which we will call *standard*, in $\mathbb{R}$ (for any non-zero integer value of z) and in $\mathbb{Q}_p$ (when z satisfies the hypotheses of Theorem 4). It is very interesting to notice that, in this way, we can also construct simultaneous rational approximations for a quadratic irrational both in $\mathbb{R}$ and $\mathbb{Q}_p$. For the real case, surely the expansion does not provide best approximations; for the p–adic case the quality of the approximations depends on the p–adic norms of μ_1 and μ_2 eigenvalues of the following matrix:

$$\begin{pmatrix} 1 - \frac{(h+2z)^2}{p} & -\frac{h+2z}{p} \\ h+2z & 1 \end{pmatrix} = \begin{pmatrix} -\frac{h+2z}{p} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h+2z & 1 \\ 1 & 0 \end{pmatrix}.$$

Indeed, the rate of convergence of the continued fraction (3) is faster, the smaller is $\left|\frac{\mu_1}{\mu_2}\right|_p$ (or $\left|\frac{\mu_2}{\mu_1}\right|_p$).

Remark 1. *The Binomial transform (and its iterations) is a well–known and very studied transform of sequences in rings, see, e.g., [17]. In general, given a ring R, the Binomial interpolated operator $L^{(y)}$ with parameter $y \in R$ transforms a sequence $(a_n)_{n\geq 0}$ into a sequence $(b_n)_{n\geq 0}$ by means of*

$$b_n = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} a_i.$$

In [2], the authors showed that $L^{(y)}$ maps a linear recurrence sequence with characteristic polynomial $f(x)$ into another linear recurrence sequence with characteristic polynomial $f(x-y)$. In the case of the sequences $(N_n(h,d,z))_{n\geq 0}$ and $(D_n(h,d,z))_{n\geq 0}$, we have that

$$L^{(y)}(N_n(h,d,z)) = N_n(h,d,z+y), \quad L^{(y)}(D_n(h,d,z)) = D_n(h,d,z+y).$$

This gives an interesting explicit formula to pass from a sequence that converges in $\mathbb{Q}_{p_1}$ to a quadratic irrational into a sequence that converges to the same quadratic irrational in $\mathbb{Q}_{p_2}$, for p_1 and p_2 certain prime numbers. We see this in the following examples.

Example 1 (p–adic approximations of the Golden ratio)**.** *Let us consider the polynomial x^2-x-1, obtained for $h=-1, d=1$ following the above notation. There are many integers z such that z^2-z-1 is a prime number and $\left(\frac{5}{p}\right) = 1$. We have listed them in Table 1.*

Table 1: Integers z such that $z^2-z-1=p$ is prime and $\left(\frac{5}{p}\right)=1$. The sequences of such z and p are listed in OEIS [16] in sequences A002328 and A002327, respectively.

z	4	5	6	7	9	10	11	12	14	16	...
p	11	19	29	41	71	89	109	131	181	239	...

Thus, observing that we always have $h+2z<p$, we have the following standard representations for the Golden ratio in several p–adic fields:

$$\left[4, \overline{-\frac{7}{11}, 7}\right] \quad in \quad \mathbb{Q}_{11}, \qquad \left[5, \overline{-\frac{9}{19}, 9}\right] \quad in \quad \mathbb{Q}_{19}, \qquad \left[6, \overline{-\frac{11}{29}, 11}\right] \quad in \quad \mathbb{Q}_{29}, \qquad \left[7, \overline{-\frac{13}{41}, 13}\right] \quad in \quad \mathbb{Q}_{41},$$

and so on. Some p–adic approximations of the Golden ratio are

$$\left(\frac{N_n(-1,1,4)}{D_n(-1,1,4)}\right)_{n\geq 1} = \left(4, \frac{17}{7}, \frac{75}{38}, \frac{338}{119}, \frac{1541}{905}, \dots\right) \quad in \quad \mathbb{Q}_{11},$$

$$\left(\frac{N_n(-1,1,5)}{D_n(-1,1,5)}\right)_{n\geq 1} = \left(5, \frac{26}{9}, \frac{139}{62}, \frac{757}{387}, \frac{4172}{2305}, \dots\right) \quad in \quad \mathbb{Q}_{19},$$

$$\left(\frac{N_n(-1,1,6)}{D_n(-1,1,6)}\right)_{n\geq 1} = \left(6, \frac{37}{11}, \frac{233}{92}, \frac{1490}{693}, \frac{9633}{4955}, \dots\right) \quad in \quad \mathbb{Q}_{29},$$

$$\left(\frac{N_n(-1,1,7)}{D_n(-1,1,7)}\right)_{n\geq 1} = \left(7, \frac{50}{13}, \frac{363}{128}, \frac{2669}{1131}, \frac{19814}{9455}, \dots\right) \quad in \quad \mathbb{Q}_{41}.$$

Moreover, it is interesting to notice that the approximations in different p–adic fileds are related by the Binomial transform. For instance, we have

$$\frac{N_n(-1,1,7)}{D_n(-1,1,7)} = \frac{L(N_n(-1,1,6))}{L(D_n(-1,1,6))} = \frac{L(N_n(-1,1,5))}{L(D_n(-1,1,5))} = \frac{L(N_n(-1,1,4))}{L(D_n(-1,1,4))}.$$

The standard representation provides periodic continued fractions for any quadratic irrational in $\mathbb{Q}_p$, however we are not able to show that there exists an algorithm that gives such expansions. In the following, we focus on a specific algorithm that, under some conditions, provides the standard representation.

3 Finiteness and periodicity of a Browkin algorithm

In [5], the author defined the following function

$$s : \mathbb{Q}_p \to \mathbb{Q}, \quad s(\alpha) = \sum_{i=r}^{0} x_i p^i \tag{6}$$

for any p-adic number $\alpha = \sum_{i=r}^{\infty} x_i p^i$, with $x_i \in \left(-\frac{p}{2}, \frac{p}{2}\right)$, which plays the same role of the floor function over the real numbers. In fact, the original p-adic continued fraction algorithm proposed by Browkin [5] replicates the classical one over the real numbers. More pecifically, given a p-adic number α_0, its p-adic continued fraction expansion $[a_0, a_1, ...]$ is obtained by

$$\begin{cases} a_i = s(\alpha_i) \\ \alpha_{i+1} = \dfrac{1}{\alpha_i - a_i} \end{cases}, \quad i = 0, 1, \dots . \tag{7}$$

In [6], the author defined a new algorithm that exploits also the following function:

$$t : \mathbb{Q}_p \to \mathbb{Q}, \quad t(\alpha) = \sum_{i=r}^{-1} x_i p^i \tag{8}$$

for any p-adic number $\alpha = \sum_{i=r}^{\infty} x_i p^i$, with $x_i \in \left(-\frac{p}{2}, \frac{p}{2}\right)$. Given a p-adic number α_0, Browkin Algorithm II works as follows:

$$\begin{cases} a_i = s(\alpha_i), & \text{if } i \text{ is even} \\ a_i = t(\alpha_i), & \text{if } i \text{ is odd and } \nu_p(\alpha - t(\alpha)) = 0 \\ a_i = t(\alpha_i) - sign(t(\alpha_i)), & \text{if } i \text{ is odd and } \nu_p(\alpha - t(\alpha)) \neq 0 \\ \alpha_{i+1} = \dfrac{1}{\alpha_i - a_i} \end{cases} \quad i = 0, 1, \dots, \tag{9}$$

where $\nu_p(\cdot)$ is the p-adic valuation and for a real number x

$$sign(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Given a p–adic continued fraction $[a_0, a_1, \dots]$, we can define as usual the sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ by

$$\begin{cases} p_0 = a_0, & p_1 = a_0 a_1 + 1, \\ q_0 = 1, & q_1 = a_1, \end{cases} \qquad \begin{cases} p_n = a_n p_{n-1} + p_{n-2}, \\ q_n = a_n q_{n-1} + q_{n-2}, \end{cases} \quad n \geq 2.$$

In [5], [6], the author proved that actually the sequence $(p_n/q_n)_{n\geq 0}$ converges in $\mathbb{Q}_p$ to the starting number α_0 when the partial quotients a_i are obtained by (7) or (9). The author introduced the second algorithm since some experimental results showed that periodic expansions provided by this one are usually shorter than those produced by (7), and more square roots have a periodic expansion using (9). Thus, Browkin algorithm II appears to be very interesting, however the author left open the problem to prove that it stops in a finite number of steps when it processes a rational input. We are going to show that this is the case. The partial quotients a_i and the complete quotients α_i provided by (9) satisfy the following conditions:

$$\begin{cases} \nu_p(a_{2i}) = \nu_p(\alpha_{2i}) = 0, & i = 1, 2, \dots, \\ \nu_p(a_{2i+1}) = \nu_p(\alpha_{2i+1}) < 0, & i = 0, 1, 2, \dots, \end{cases} \tag{10}$$

see Lemma 1 and Algorithm II in [6].

We provide two examples about the use of Browkin algorithm II compared with the original one. We will denote with a_i and α_i the partial and complete quotients provided by Browkin algorithm I (see [5]), whereas b_i and β_i will be the partial and complete quotients provided by Browkin algorithm II.

Example 2. *Let us consider* $\alpha_0 = \beta_0 = \frac{3}{5}$ *in* $\mathbb{Q}_{11}$*; then, we have:*

$$a_0 = b_0 = s\left(\frac{3}{5}\right) = 5, \quad \alpha_1 = \beta_1 = -\frac{5}{22} = 3 \cdot 11^{-1} + 5 + 5 \cdot 11 + 5 \cdot 11^2 + \ldots,$$

indeed the first step of the algorithms are equal. In the second step, Browkin algorithm I still uses the function s*, whereas Browkin algorithm II uses the function* t*:*

$$a_1 = s\left(-\frac{5}{22}\right) = \frac{58}{11}, \quad b_1 = t\left(-\frac{5}{22}\right) = \frac{3}{11}.$$

Consequently the second complete quotients are different:

$$\alpha_2 = -\frac{2}{11}, \quad \beta_2 = -2,$$

and both the algorithms stop with

$$a_2 = -\frac{2}{11}, \quad b_2 = -2.$$

Finally, we get the following p*–adic continued fractions for* $\frac{3}{5}$*:*

$$\left[5, \frac{58}{11}, -\frac{2}{11}\right], \quad \left[5, \frac{3}{11}, -2\right].$$

Example 3. *Let us consider* $\alpha_0 = \beta_0 = \sqrt{11}$ *in* $\mathbb{Q}_5$*; then we have*

$$a_0 = b_0 = s\left(\sqrt{11}\right) = 1, \quad \alpha_1 = \beta_1 = \frac{1+\sqrt{11}}{10} = 1 \cdot 5^{-1} - 2 - 1 \cdot 5 - 2 \cdot 5^2 + \ldots,$$

from which we get

$$a_1 = s\left(\frac{1+\sqrt{11}}{10}\right) = -\frac{9}{5}, \quad b_1 = t\left(\frac{1+\sqrt{11}}{10}\right) = \frac{1}{5},$$

and

$$\alpha_2 = \frac{19-\sqrt{11}}{35} = -1 \cdot 5^{-1} + 2 - 2 \cdot 5 + \ldots, \quad \beta_2 = 1 + \sqrt{11} = 2 + 1 \cdot 5 + \ldots.$$

Thus

$$a_2 = s(\alpha_2) = \frac{9}{5}, \quad b_2 = s(\beta_2) = 2,$$

and

$$\alpha_3 = \frac{-44+\sqrt{11}}{55}, \quad \beta_3 = \frac{1+\sqrt{11}}{10} = \beta_1.$$

Hence, using Browkin algorithm II we get a periodic expansion of $\sqrt{11}$*, namely*

$$\sqrt{11} = \left[1, \overline{\frac{1}{5}, 2}\right],$$

whereas the expansion of $\sqrt{11}$ *obtained using Browkin algorithm I the expansion of* $\sqrt{11}$ *appears not to be periodic:*

$$\sqrt{11} = \left[1, -\frac{9}{5}, \frac{9}{5}, -\frac{8}{5}, \frac{9}{5}, \frac{6}{5}, \frac{2}{5}, \frac{56}{25}, \ldots\right].$$

Lemma 5. *We have*

$$\sum_{j=1}^{l} r_j p^{-j} > 0 \Rightarrow \sum_{j=1}^{l} r_j p^{-j} \geq \frac{1}{p^l} \tag{11}$$

and

$$\sum_{j=1}^{l} r_j p^{-j} < 0 \Rightarrow \sum_{j=1}^{l} r_j p^{-j} \leq -\frac{1}{p^l} \tag{12}$$

given any integer $r_i \in \left(-\frac{p}{2}, \frac{p}{2}\right)$ *and* $l \geq 1$.

Proof. It is easy to see that (12) follows from (11), thus we just prove (11) by induction on l. If $l = 1$, then $\frac{r_1}{p} > 0$ implies $\frac{r_1}{p} \geq \frac{1}{p}$ as wanted. Now, let us suppose that (11) is true for all $l = 1, \ldots, n-1$ and we prove it for $l = n$. First, we can observe that if $\sum_{j=1}^{n} r_j p^{-j} > 0$, with $r_n \neq 0$, then we can not have $\sum_{j=1}^{n-1} r_j p^{-j} < 0$. Indeed, we have

$$\frac{p^n \sum_{j=1}^{n-1} r_j p^{-j} + r_n}{p^n} = \frac{p\left(\sum_{j=1}^{n-1} r_j p^{-j+n-1}\right) + r_n}{p^n} > 0 \tag{13}$$

with $\sum_{j=1}^{n-1} r_j p^{-j+n-1} \leq -1$. Since $k_n \leq \dfrac{p-1}{2}$, we have $p\left(\sum_{j=1}^{n-1} r_j p^{-j+n-1}\right) + r_n < 0$ that contradicts (13). Hence, we can not have $\sum_{j=1}^{n-1} r_j p^{-j} < 0$. In the case that $\sum_{j=1}^{n-1} r_j p^{-j} > 0$, by inductive hypothesis we have $\sum_{j=1}^{n-1} r_j p^{-j} \geq \frac{1}{p^{n-1}}$. Consequently, if $\sum_{j=1}^{n-1} r_j p^{-j} + \dfrac{r_n}{p^n} > 0$, then

$$\sum_{j=1}^{n-1} r_j p^{-j} + \frac{r_n}{p^n} \geq \frac{1}{p^{n-1}} + \frac{r_n}{p^n} > \frac{1}{p^n}.$$

Finally, in the case that $\sum_{j=1}^{n-1} r_j p^{-j} = 0$, we have

$$\frac{r_n}{p^n} > 0 \Rightarrow \frac{r_n}{p^n} \geq \frac{1}{p^n}.$$

□

Lemma 6. *Given the sequence of partial quotients $(a_i)_{i=0}^{\infty}$ obtained applying algorithm (9), we have*

$$|a_{2k+1}| \leq 1 - \frac{1}{p^l}$$

for every $k \geq 0$, where $l = -\nu_p(\alpha_{2k+1})$.

Proof. Since a_{2k+1} is obtained by applying equations (9), we have that $a_{2k+1} = t(\alpha_{2k+1})$ or $a_{2k+1} = t(\alpha_{2k+1}) - sign(t(\alpha_{2k+1}))$. In the first case, we have

$$|a_{2k+1}| \leq \sum_{i=-l}^{-1} |r_{i,2k+1}| p^i \leq \frac{p-1}{2}\left(\sum_{j=1}^{l} p^{-j}\right) = \frac{p-1}{2}\left(\frac{1-p^{-l-1}}{1-p^{-1}} - 1\right) = \frac{1}{2}\left(1 - \frac{1}{p^l}\right).$$

In the second case, by the previous Lemma, if $t(\alpha_{2k+1}) > 0$ then $t(\alpha_{2k+1}) \geq \frac{1}{p^l}$ and

$$\frac{1}{p^l} \leq t(\alpha_{2k+1}) < \frac{1}{2}\left(1 - \frac{1}{p^l}\right) \Rightarrow -\left(1 - \frac{1}{p^l}\right) \leq t(\alpha_{2k+1}) - 1 < -\left(\frac{1}{2} + \frac{1}{2p^l}\right).$$

Similarly, if $t(\alpha_{2k+1}) < 0$ then $t(\alpha_{2k+1}) \leq -\frac{1}{p^l}$ and

$$\frac{1}{2} + \frac{1}{2p^l} \leq t(\alpha_{2k+1}) + 1 \leq 1 - \frac{1}{p^l}.$$

Thus, we get

$$|a_{2k+1}| = |t(\alpha_{2k+1}) - sign(t(\alpha_{2k+1}))| \leq 1 - \frac{1}{p^l},$$

and $|a_{2k+1}| \leq 1 - \frac{1}{p^l}$. □

Theorem 7. *If $\alpha \in \mathbb{Q}$, then the p-adic continued fraction expansion provided by Browkin algorithm II (9) is finite.*

Proof. Given $\alpha \in \mathbb{Q}$, by equations (9) we have $\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$ with $a_n \in \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right)$, for every $n \geq 0$. We can observe that $\nu_p(a_n) = \nu_p(\alpha_n)$ for every $n \geq 1$, and we recall that also equalities (10) hold. Thus, considering $a_n = c_n p^{\nu_p(a_n)}$, we have $a_{2k} = c_{2k}$ with

$$|a_{2k}| = |c_{2k}| < \frac{p}{2}. \tag{14}$$

If we consider $a_{2k+1} = c_{2k+1}p^{\nu_p(a_{2k+1})}$, since $|a_{2k+1}| \leq 1 - \frac{1}{p^l}$, by Lemma 6 we have

$$|a_{2k+1}| = |c_{2k+1}| \leq p^l \left(1 - \frac{1}{p^l}\right). \tag{15}$$

Now, taking

$$\alpha_n = \frac{A_n}{p^{-\nu_p(\alpha_n)}B_n}, \quad A_n, B_n \in \mathbb{Z},\ (A_n, B_n) = 1,\ p \nmid A_n B_n,$$

since $\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$, we obtain

$$\frac{A_{2k+1}}{p^l B_{2k+1}} = \alpha_{2k+1} = \frac{1}{\frac{A_{2k}}{B_{2k}} - c_{2k}} \Rightarrow B_{2k}B_{2k+1}p^l = A_{2k+1}\left(A_{2k} - c_{2k}B_{2k}\right), \tag{16}$$

hence

$$A_{2k+1} = \pm B_{2k}, \quad B_{2k+1} = \pm p^{-l}\left(A_{2k} - c_{2k}B_{2k}\right). \tag{17}$$

Since $\nu_p\left(\alpha_{2k}\right) = 0$, and recalling that $\alpha_{2k+1} = \frac{A_{2k+1}}{p^l B_{2k+1}}$ and $a_{2k+1} = c_{2k+1}p^{-l}$ we have

$$\frac{A_{2k+2}}{B_{2k+2}} = \alpha_{2k+2} = \frac{1}{\frac{A_{2k+1}}{p^l B_{2k+1}} - \frac{c_{2k+1}}{p^l}} \Rightarrow B_{2k+2}B_{2k+1}p^l = A_{2k+2}\left(A_{2k+1} - c_{2k+1}B_{2k+1}\right), \tag{18}$$

from which we have

$$A_{2k+2} = \pm B_{2k+1}, \quad B_{2k+2} = \pm p^{-l}\left(A_{2k+1} - c_{2k+1}B_{2k+1}\right). \tag{19}$$

By (17) and (14), it follows that

$$|B_{2k+1}| \leq \frac{1}{p^l}\left(|A_{2k}| + |c_{2k}|\,|B_{2k}|\right) < \frac{1}{p^l}\left(|A_{2k}| + \frac{p}{2}\,|B_{2k}|\right). \tag{20}$$

Moreover, by (19) and (15), we have

$$|B_{2k+2}| \leq \frac{1}{p^l}\left(|A_{2k+1}| + |c_{2k+1}|\,|B_{2k+1}|\right) \leq \frac{1}{p^l}\,|A_{2k+1}| + \left(1 - \frac{1}{p^l}\right)|B_{2k+1}|. \tag{21}$$

Using equations (19), the last inequality can be written as

$$|B_{2k+2}| \leq \frac{1}{p^l}\,|A_{2k+1}| + |B_{2k+1}| - \frac{1}{p^l}\,|A_{2k+2}| \Rightarrow |A_{2k+2}| + p^l\,|B_{2k+2}| \leq |A_{2k+1}| + p^l\,|B_{2k+1}|$$

and using (20), we finally get

$$|A_{2k+2}| + p^l\,|B_{2k+2}| < |A_{2k+1}| + |A_{2k}| + \frac{p}{2}\,|B_{2k}|. \tag{22}$$

From (17) and (19), we can observe that $|B_{2k+2}| = |A_{2k+3}|$ and $|A_{2k+1}| = |B_{2k}|$. Thus, from $p^l > \frac{p}{2} + 1$ and (22), we have

$$\left(\frac{p}{2} + 1\right)|A_{2k+3}| + |A_{2k+2}| < |A_{2k+2}| + p^l\,|B_{2k+2}| < \left(\frac{p}{2} + 1\right)|A_{2k+1}| + |A_{2k}|. \tag{23}$$

Now, defining $N_k = A_{2k+1}$ and $M_k = A_{2k}$, (23) is equivalent to

$$(p+2)\,|N_{k+1}| + 2\,|M_{k+1}| < (p+2)\,|N_k| + 2\,|M_k|,$$

i.e., the sequence of natural numbers $(p+2)\,|N_k| + 2\,|M_k|$ is strictly decreasing and consequently is finite. Hence, the sequences $|N_k| = |A_{2k+1}|$ and $|M_k| = |A_{2k}|$ are finite, as well as $|A_n|$ and $|B_n|$, since by (17) and (19) we have $|A_{n+1}| = |B_n|$, concluding the proof. □

Finally, in the next theorem, we address the problem of periodicity for Browkin algorithm II. In [6], the author observed experimentally that the algorithm (9) provides more periodic representations for square roots than the algorithm (7), and usually the period also appears to be shorter in the first case. Moreover, the author also gave some conditions for which the algorithm (9) produces a continued fraction of period 2 or period 4 and pre-period of length 1 for square roots. Here, we characterize when the algorithm (9) provides the standard representation for quadratic irrationals.

Theorem 8. *Given $\alpha \notin \mathbb{Q}$ a root of the polynomial $x^2 + hx - d$, with $h, d \in \mathbb{Z}$, algorithm (9) produces the p–adic continued fraction*

$$\left[z, \overline{-\frac{h+2z}{p}, h+2z}\right]$$

if and only if

$$1 \leq |z| \leq \frac{p-1}{2}, \quad 1 \leq |h+2z| \leq \frac{p-1}{2}$$

for z such that $z^2 + hz - d = p$.

Proof. From $\alpha^2 + h\alpha - d = 0$ and $z^2 + hz - d = p$, we have

$$\alpha = z - \frac{p}{\alpha + z + h} = z - \frac{p}{(\alpha - z) + h + 2z}.$$

Now, without loss of generality, we suppose $\nu_p(\alpha + z + h) = 0$. Indeed, $\nu_p(\alpha + z + h)$ can not be greater than 1 because $(\alpha - z)(\alpha + z + h) = -p$ and, if $\nu_p(\alpha + z + h) = 1$, then it is sufficient to consider $\tilde{z} = -h - z$ instead of z (note that $\tilde{z}^2 + h\tilde{z} - d = p$). Thus, if z and $h + 2z$ belong to $\left(-\frac{p}{2}, \frac{p}{2}\right)$, then $a_0 = s(\alpha) = z$ and

$$\alpha_1 = \frac{1}{\alpha - a_0} = -\frac{h+2z}{p} - \frac{\alpha - z}{p}.$$

Since $\nu_p\left(-\frac{\alpha - z}{p}\right) = \nu_p\left(\frac{1}{\alpha + z + h}\right) = 0$, we have

$$a_1 = t(\alpha_1) = -\frac{h+2z}{p}.$$

Finally,

$$\alpha_2 = \frac{1}{\alpha_1 - a_1} = \alpha + z + h = (\alpha - z) + h + 2z$$

with $\nu_p(\alpha - z) = 1$ and $\nu_p\left(-\frac{p}{\alpha - z}\right) = 0$. Hence, $a_2 = h + 2z$ and $\alpha_3 = \alpha_1$. Clearly, if z or $h + 2z$ do not belong to $\left(-\frac{p}{2}, \frac{p}{2}\right)$, equations (9) can not provide the standard representation. □

Remark 2. *If a quadratic irrational α has standard representation $\left[z, \overline{-\frac{h+2z}{p}, h+2z}\right]$ using algorithm (9), then also $-\alpha$ has a standard representation that is*

$$\left[-z, \overline{-\frac{h+2z}{p}, -(h+2z)}\right],$$

as well as the conjugate $-h - \alpha = \left[-h - z, \overline{\frac{h+2z}{p}, -(h+2z)}\right]$, when $1 \leq |h+z| \leq \frac{p-1}{2}$.

Question 1. *Might it be possible to define an algorithm that always produces standard representations for quadratic irrationals? We think that some modifications of Browkin algorithm II could give a new algorithm of such kind. We experimentally observed that Browkin algorithm II provides more standard representations than the algorithm defined by equations (7).*

References

[1] Abrate, M., Barbero, S., Cerruti, U., Murru, N., Periodic representations and rational approximations of square roots, J. of Approx. Theory, 175 (2013), 83–90.

[2] Barbero, S., Cerruti, U., Murru, N., Transforming recurrent sequences by using the Binomial and Invert operators, J. Integer Seq., 13 (2010), Article 10.7.7.

[3] Bedocchi, E., A note on p–adic continued fractions, Ann. Mat. Pura Appl., 152 (1988), 197–207.

[4] Borwein, J., van der Poorten, A., Shallit, J., Zudilin, W. Neverending Fractions: An Introduction to Continued Fractions, Cambridge University Press (2014).

[5] Browkin, J., Continued fractions in local fields I, Demonstr. Math., 11 (1978), 67–82.

[6] Browkin, J., Continued fractions in local fields II, Math. Comp., 70 (2000), 1281–1292.

[7] Capuano, L., Veneziano, F., Zannier, U., An effective criterion for periodicity of l–adic continued fractions, Math. Comp. 88 (2019), 1851-1882.

[8] Murru, N., Terracini, L., On p–adic multidimensional continued fractions, Math. Comp. 88 (2019), 2913-2934.

[9] Murru, N., Terracini, L., On the finiteness and periodicity of the p–adic Jacobi–Perron algorithm, Math. Comp. 89 (2020), 2913-2930.

[10] Ooto, T., Trascendental p–adic continued fractions, Math. Z. 287 (2017), no. 3-4, 1053–1064.

[11] van der Poorten, A. J., Schneider's continued fractions, Number theory with an emphasis on the Markoff spectrum, Lecture Notes in Pure and Appl. Math., 147 (1993), 271–281, Dekker, New York.

[12] Rédei, L., Uber eindeuting umkehrbare Polynome in endlichen Korpen, Acta Sci. Math., Szeged, 11 (1946), 85–92.

[13] Ruban, A., Certain metric properties of the p–adic numbers, Sibirsk Math. Z., 11 (1970), 222–227.

[14] Saito, A., Tamura, J., Yasutomi, S., Multidimensional p–adic continued fraction algorithms, Math. Comp. 89 (2020), 351-372.

[15] Schneider, T., Uber p–adische Kettenbruche, Symposia Mathematica, 4 (1968/69), 181–189.

[16] Sloane, N. J. A., The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[17] Spivey, M. Z., Steil, L. L., The k-binomial transforms and the Henkel transform, J. Integer Sequences, 9 (2006), Article 06.1.1.

[18] Tamura, J., A p–adic phenomenon related to certain integer matrices, and p-adic values of a multidimensional continued fraction, Summer School on the Theory of Uniform Distribution, RIMS Kokyuroku Bessatsu, B29 (2012), 1–40.

[19] Tilborghs, F., Periodic p–adic continued fractions, Simon Stevin 64 (1990), no. 3-4, 383–390.