

Nonlinear Stochastic Dynamic Analysis by Evolutionary Tail-Equivalent Linearization Method

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ABSTRACT: This study introduces the evolutionary tail-equivalent linearization method (ETELM) for nonlinear stochastic dynamic analysis. The method builds on the recently developed tail-equivalent linearization method (TELM) and it is designed for the class of evolutionary processes. The original TELM employs a tail-equivalent linear system (TELS) by equating the tail probability of a linear system response for a specified threshold to the first-order approximation of the tail probability of the nonlinear system response. For stationary problems, the TELS is time-independent and only one linear system needs to be defined for the specified threshold. However, for a transient input, the TELS is time dependent and an evolutionary tail-equivalent linear system (ETELS) must be defined to study the entire transient response. Algorithms are developed to determine a discrete-time ETELS based on a sequence of linearization points, and a continuous-time approximation based on Priestley's evolutionary theory. The linearized evolutionary system is used to compute the response statistics of interest, including the first-passage probability, in first-order approximation. Numerical examples demonstrate the accuracy and limitations of the proposed method.

Keywords: Dynamics, equivalent linearization, evolutionary process, first-order methods, nonlinear, stochastic process, non-stationary process, metamodeling.

1 Prologue

It is a privilege to contribute to this special volume of *Structural Safety* honoring the distinguished career and seminal contributions of Bruce Ellingwood. The second author recalls how he came to know Bruce during his first year at the University of Illinois at Urbana Champaign: "Seeking a research assistantship, I interviewed with Professor Alfredo H-S. Ang, who suggested that I join his research group in structural reliability. I knew nothing about the field. To get acquainted, Professor Ang gave me a copy of Bruce's doctoral thesis. I remember reading parts of it that evening, not understanding much as I didn't have the background, but being intrigued, nevertheless. Two days later I accepted the offer. So, in a sense, Bruce's thesis provided the motivation for me to get into this field. I thank him for that." Bruce's contributions to the field of structural reliability and probabilistic codified design are enormous. We wish him good health and continued productivity.

2 Introduction

Nonlinear stochastic dynamic analysis of structures and mechanical systems has been a topic of intense study in the last few decades and different approaches are available in the scientific

literature. These include now classical methods, such as the Fokker-Plank equation, stochastic averaging, the perturbation method and moment closure, which however are all limited to specialized systems or excitations (see, e.g., Roberts and Spanos 1990, Lin and Cai 1995, Soong and Grigoriu 1993, Lutes and Sarkani 2004). The simulation method is another broad family, which has few theoretical limitations, but large computational cost when low probability values are of interest (Shinozuka 1972). Another approach is the equivalent (or statistical) linearization method (ELM), which is more broadly applicable to multi-degree-of-freedom (MDoF) systems subjected to stationary or non-stationary excitations (Atalik and Utku 1976, Roberts and Spanos 1990). This method has become popular because of its general applicability and computational efficiency; however, it suffers from lack of accuracy for predicting the distribution of the nonlinear response, particularly in the tail region. More recently, the tail-equivalent linearization method (TELM) has been developed, which defines the equivalent linear system non-parametrically by equating its tail probability to the first-order approximation of the tail probability of the nonlinear response. Previous applications of TELM have focused mostly on stationary excitation and response processes. The present paper extends TELM for non-stationary problems by use of Priestley's (1967) definition of evolutionary processes.

TELM was originally developed by Fujimura and Der Kiureghian (2007) in the time domain and later extended to the frequency domain by Garrè and Der Kiureghian (2009). Further developments and applications include a gradient-free formulation by Alibrandi and Der Kiureghian (2011), multi-component excitation analysis by Broccardo and Der Kiureghian (2016), and multi-support structures subjected to spatially varying ground motions by Wang and Der Kiureghian (2016). The first attempt to extend TELM to nonstationary problems is by Broccardo and Der Kiureghian (2013). Departing from that short study, the focus of this paper is on the development of the *evolutionary* tail-equivalent linearization method (ETELM) for non-stationary excitations.

TELM requires discretization of the input process and its representation in terms of a vector of standard normal random variables. In all cases, the discretized process is defined in terms of a set of deterministic time-varying basis functions and a set of standard normal random coefficients. A limit-state function, formulated for a given response threshold and a point in time, defines the tail probability of the nonlinear response. Linearization of the limit-state surface at the point of maximum likelihood in the standard normal space, also known as the “design point,” defines the tail-equivalent linear system (TELS), which is characterized non-parametrically in terms of its impulse-response or frequency-response function. Due to this definition, the tail probability of the nonlinear response is approximated, in first order, by the tail probability of the TELS. With the TELS determined, linear random vibration analysis is carried out to compute the response statistics of interest, including the mean rate of threshold crossings and the first-passage probability. For stationary random vibration problems, the TELS only depends on the selected threshold. Thus, one TELM analysis provides all the needed information to compute the required statistics for the specified threshold. However, for a transient input, the TELS is time dependent. Thus, for this case, we define an *evolutionary* tail-equivalent linear system (ETELS) to study the statistics of the non-stationary response for a specified threshold as a function of time.

We propose two computational schemes (Algorithm 1 and Algorithm 2) to derive the ETELS and calculate the associated statistics. A straightforward strategy to compute a time-discretized ETELS is to compute a series of TELSs for an ordered sequence of time instances. This requires solution of multiple nonlinear optimization problems to find the corresponding design points, which can be computationally costly. Algorithm 1 uses neighboring design points to formulate a

“warm” starting point for the iterative solution, thus drastically reducing the amount of required computations.

Algorithm 2 takes advantage of Priestley’s theory to formulate a continuous-time approximation of the ETELS. We show that linearizations at a few critical time points can provide accurate estimates of the ETELS and the statistics of the evolutionary response process, including the first-passage probability. Since Algorithm 2 uses only a few design points, its computational cost is significantly less than that of Algorithm 1. Example applications to a hysteretic system subjected to broad- and narrow-band evolutionary excitations demonstrate the accuracy and limitations of the proposed method.

3 Review of TELM

3.1 Discrete representation of the excitation process

TELM was originally developed in the time domain, using a discretized filtered white-noise representation of a Gaussian excitation process in the form (Fujimura and Der Kiureghian 2007)

$$\hat{F}(t) = \mathbf{s}(t)\mathbf{u}, \quad (1)$$

where $\mathbf{s}(t) = [s_1(t), \dots, s_N(t)]$ is a row vector of basis functions involving the unit impulse-response function (IRF) of a filter and $\mathbf{u} = [u_1, \dots, u_N]^T$ is a vector of standard normal random variables representing a series of random pulses in discrete time points. Following a formulation pioneered by Shinozuka (1972) and Shinozuka and Deodatis (1991), the same format can be used to represent a stationary Gaussian process defined by a power spectral density (PSD), $S_{FF}(\omega)$, in the frequency domain. For selected equally spaced frequency points ω_k , $k = 1, \dots, K$, with frequency increment $\Delta\omega = \omega_k - \omega_{k-1}$, the discretized process $\hat{F}(t)$ is written as

$$\hat{F}(t) = \sum_{k=1}^K \sigma_{\hat{F},k} [u_k \sin(\omega_k t) + u_{K+k} \cos(\omega_k t)], \quad (2)$$

where $\sigma_{\hat{F},k} = \sqrt{2S_{FF}(\omega_k)\Delta\omega}$. Eq. (2) has the form of Eq. (1) with $\mathbf{u} = [u_1, \dots, u_K; u_{K+1}, \dots, u_{2K}]^T$ and $\mathbf{s}(t) = [s_1(t), \dots, s_K(t); s_{K+1}(t), \dots, s_{2K}(t)]$, in which $s_k(t) = \sigma_{\hat{F},k} \sin(\omega_k t)$ and $s_{K+k}(t) = \sigma_{\hat{F},k} \cos(\omega_k t)$. It is noted that the frequency-domain discretization leads to a periodic process with period $T = 2\pi/\Delta\omega$. Hence, $\Delta\omega$ should be sufficiently small so that T is longer than the duration of the excitation.

The above approaches can be used for non-Gaussian excitation process that are defined through translation of a Gaussian process (Grigoriu 1995). One simply discretizes the underlying Gaussian process and translates it to the original space to obtain the discretized non-Gaussian process. An example is given in Der Kiureghian (2000).

3.2 Frequency-domain TELM

The stationary response of a linear system to the discretized excitation in Eq. (2) is given by (Garre and Der Kiureghian 2009)

$$\begin{aligned} \hat{X}(t) &= \sum_{k=1}^K \sigma_{\hat{X},k} [\sin(\theta_k t)u_k + \cos(\theta_k t)u_{K+k}] \\ &= \mathbf{a}(t)\mathbf{u}, \end{aligned} \quad (3)$$

where $\sigma_{\hat{X},k} = \sigma_{\hat{F},k} |H(\omega_k)|$ and $\mathbf{a}(t) = [a_1(t), \dots, a_K(t); a_{K+1}(t), \dots, a_{2K}(t)]$, in which $a_k(t) = \sigma_{\hat{X},k} \sin(\theta_k t)$, $a_{K+k}(t) = \sigma_{\hat{X},k} \cos(\theta_k t)$, and $\theta_k = \omega_k t + \varphi_k$. In these expressions, $|H(\omega_k)|$ and φ_k respectively represent the modulus and the phase of the frequency-response function (FRF) of the linear system at frequency ω_k for the selected input-output pair. We recall that the FRF is the Fourier transform of the IRF of the linear system for the selected input-output pair.

We see that the response of the linear system in Eq. (3) has the same form as the discretized excitation in Eq. (2) with $a_n(t)$ replacing $s_n(t)$. Garrè and Der Kiureghian (2009) have shown that the following relations hold between the elements of the gradient vector $\mathbf{a}(t)$ and the FRF of the linear system:

$$|H(\omega_k)| = \frac{1}{\sigma_{\hat{F},k}} \sqrt{a_k(t)^2 + a_{K+k}(t)^2}, \quad (4)$$

$$\varphi_k = -\omega_k t + \tan^{-1} \left[\frac{a_k(t)}{a_{K+k}(t)} \right]. \quad (5)$$

Thus, if $\mathbf{a}(t)$ is known, the FRF of the linear system can be determined from the above equations.

For a nonlinear system, the response naturally is a function of the standard normal random variables $\mathbf{u}$ and we denote it as $\hat{X}(t, \mathbf{u})$. Given a response threshold x and a specific point in time t_x , we define the tail probability of the response as $\Pr[x \leq \hat{X}(t_x, \mathbf{u})]$. Following the convention in structural reliability theory, this probability is written as $\Pr[G(x, t_x, \mathbf{u}) \leq 0]$, where $G(x, t_x, \mathbf{u}) = x - \hat{X}(t_x, \mathbf{u})$ denotes the limit-state function. TELM employs the first-order reliability method (FORM) to obtain an approximation of the tail probability by linearizing the limit-state function at the so-called design point $\mathbf{u}^*$, which belongs to the limit state surface $G(x, t_x, \mathbf{u}) = 0$ and has minimum distance from the origin in the outcome space of $\mathbf{u}$. This distance is known as the reliability index. If the system is linear, the limit-state surface is $x - \mathbf{a}(t)\mathbf{u} = 0$, which is a hyperplane having the gradient vector $\mathbf{a}(t)$. In that case, the design point and the reliability index are given in closed form as

$$\mathbf{u}^*(x, t_x) = \frac{x}{\|\mathbf{a}(t_x)\|} \frac{\mathbf{a}(t_x)^T}{\|\mathbf{a}(t_x)\|}, \quad (6)$$

$$\beta(x, t_x) = \frac{x}{\|\mathbf{a}(t_x)\|}, \quad (7)$$

respectively. Furthermore, using geometric relations, Fujimura and Der Kiureghian (2007) have shown that the gradient vector $\mathbf{a}(t_x)$ is given in terms of the design point as

$$\mathbf{a}(t_x) = \frac{x}{\|\mathbf{u}^*\|} \frac{\mathbf{u}^{*T}}{\|\mathbf{u}^*\|}. \quad (8)$$

It is noted that, for a linear system, $\mathbf{a}(t_x)$ is independent of the threshold x even though x appears in the above expression. The reason has to do with the fact that the limit-state surfaces for different response thresholds of a linear system to a Gaussian excitation are parallel hyperplanes with identical gradient vectors. The tail probability of the response at time t_x is given by $\Pr[x \leq \hat{X}(t_x, \mathbf{u})] = \Phi[-\beta(x, t_x)]$, where $\Phi[\cdot]$ is the standard normal cumulative probability function. This expression is exact for a linear system.

For TELM analysis, given a general nonlinear system and a stationary stochastic input described by Eq. (2), the design point $\mathbf{u}^*$ is first determined for a specified threshold x and a sufficiently long time t_x so that the response has reached the stationary state. The gradient vector of the tangent

hyperplane, $\mathbf{a}(t)$, is determined from Eq. (8). The latter is then used in Eq. (4) and Eq. (5) to uniquely identify the tail-equivalent linear system (TELS) in terms of its FRF.

With the FRF of the TELS available, methods of linear random vibration analysis are used to compute first-order approximations of the statistics of interest for the nonlinear system response for threshold x and time t_x . These include the tail probability, the variance, the crossing rate and other point-in-time statistics (see Fujimura and Der Kiureghian 2007 and Garrè and Der Kiureghian 2009). For a stationary process, the TELS for a given threshold is independent of time and, hence, statistics over an interval of time, such as the first-passage probability, are also determined in terms of the TELS obtained for one design point.

4 Evolutionary TELM

4.1 Input representation

An important class of nonstationary processes is defined in terms of the evolutionary power spectral density (EPSD) (Priestly 1967)

$$S_{FF}(\omega, t) = |A(\omega, t)|^2 S(\omega), \quad (9)$$

where $A(\omega, t)$ is a possibly complex-valued, time-frequency modulating function and $S(\omega)$ is the PSD of an underlying stationary process. Both $S(\omega)$ and $A(\omega, t)$ must be symmetric functions over the frequency domain ($-\infty < \omega < +\infty$). The variance of the process at time t is given by

$$\sigma_F^2(t) = 2 \int_0^\infty |A(\omega, t)|^2 S(\omega) d\omega. \quad (10)$$

Observe that not only the variance but also the frequency content of the process varies with time.

One can always select $S(\omega) = 1$ so that $A(\omega, t)$ completely defines the evolutionary process. This is equivalent to employing a white noise of unit intensity as the underlying stationary process. In that case, Eq. (3) can be used to represent the evolutionary processes in a discretized form by replacing $\sigma_{\hat{F},k}$ with $\sigma_{\hat{F},k}(t) = \sqrt{2|A(\omega_k, t)|^2 \Delta\omega}$. It follows that the variance of the frequency-discretized process is

$$\sigma_{\hat{F}}^2(t) = \sum_{k=1}^K \sigma_{\hat{F},k}^2(t) = 2 \sum_{k=1}^K |A(\omega_k, t)|^2 \Delta\omega. \quad (11)$$

Moreover, following Vlachos et al. (2016) and Broccardo and Dabaghi (2017), it is possible to select the time-frequency modulating function in the form

$$|A(\omega, t)|^2 = q^2(t) \frac{|\phi(\omega, t)|^2}{2 \int_0^\infty |\phi(\omega, t)|^2 d\omega}, \quad (12)$$

where $q(t)$ is a time-modulating function and $\phi(\omega, t)$ is a frequency-time modulating function. Observe that, due to the normalization of the latter function, substitution of Eq. (12) in Eq. (10) with $S(\omega) = 1$ yields $\sigma_{\hat{F}}^2(t) = q^2(t)$, while the frequency content of the process is allowed to vary with time. We use this formulation in the example applications later in this paper.

The discretized evolutionary input process preserves the form in Eq. (1) with $\mathbf{s}(t) = [s_1(t), \dots, s_K(t); s_{K+1}(t), \dots, s_{2K}(t)]$, in which $s_k(t) = \sigma_{\hat{F},k}(t) \sin(\omega_k t)$ and $s_{K+k}(t) = \sigma_{\hat{F},k}(t) \cos(\omega_k t)$. We use this discrete form in the following to define the evolutionary tail-equivalent linear system.

4.2 Evolutionary frequency-response function

It is well known (Priestley, 1967) that the response $X(t)$ of a linear system to an evolutionary input process is also evolutionary with the response EPSD given by

$$S_{XX}(\omega, t) = |M(\omega, t)|^2 S(\omega), \quad (13)$$

where

$$M(\omega, t) = \int_0^t h(\tau) A(\omega, t - \tau) \exp(-i\omega\tau) d\tau \quad (14)$$

is the new time-frequency modulating function with $h(t)$ being the IRF of the linear system for the particular input-output pair, which is identical to the inverse Fourier transform of the FRF. Note that $M(\omega, t)$ in general is complex-valued, even when $A(\omega, t)$ is real-valued. Since $A(\omega, t - \tau) = 0$ for $t < \tau$ and $h(\tau) = 0$ for $\tau < 0$, the limits of the above integral can be changed to $-\infty$ and $+\infty$. It follows that $M(\omega, t)$ is the Fourier transform of $h(\tau)A(\omega, t - \tau)$. In this light, $M(\omega, t)$ is interpreted as a generalization of the FRF of the linear system. For that reason, we name this function *evolutionary* FRF (EFRF). Note, however, that while the FRF purely characterizes the linear system for the particular input-output pair, the EFRF also depends on the input excitation through the modulating function $A(\omega, t)$. As mentioned earlier, without loss of generality, one can select $S(\omega) = 1$. In that case, $M(\omega, t)$ completely characterizes the EPSD of the response.

The response of a linear system to a discretized evolutionary process preserves the form in the second line of Eq. (3) with $a_k(t) = \sigma_{\tilde{X},k}(t) \sin(\theta_k t)$, $a_{K+k}(t) = \sigma_{\tilde{X},k}(t) \cos(\theta_k t)$, $\sigma_{\tilde{X},k}^2(t) = 2|M(\omega_k, t)|^2 \Delta\omega$ and $\theta_k = \omega_k t + \varphi_k(t)$. In these expressions, $|M(\omega_k, t)|$ and $\varphi_k(t)$ respectively represent the modulus and phase of the frequency-discretized EFRF of the linear system. In analogy with Eqs. (4) and (5), we have

$$|M(\omega_k, t)| = \sqrt{\frac{a_k(t)^2 + a_{K+k}(t)^2}{2\Delta\omega}}, \quad (15)$$

$$\varphi_k(t) = -\omega_k t + \tan^{-1} \left[\frac{a_k(t)}{a_{K+k}(t)} \right]. \quad (16)$$

Thus, having the gradient vector $\mathbf{a}(t)$, one can determine the EFRF of the linear system at time t .

4.3 Sequence of evolutionary tail-equivalent linear systems

One approach in TELM analysis with a nonstationary input is to find the design points for a sequence of time instances over the duration of the excitation, leading to a series of TELSs and corresponding first-order approximations of the response statistics. Provided a sequence of time instances $[t_1, \dots, t_N]$, the N design points $[\mathbf{u}^*(x, t_1), \dots, \mathbf{u}^*(x, t_N)]$ are determined by solving N constrained optimization problems. A discrete-frequency, discrete-time EFRF is then obtained as

$$|M(\omega_k, t_n)| = \sqrt{\frac{a_k(t_n)^2 + a_{K+k}(t_n)^2}{2\Delta\omega}}, \quad k = 1, \dots, K, \quad n = 1, \dots, N, \quad (17)$$

$$\varphi_k(t_n) = -\omega_k t_n + \tan^{-1} \left[\frac{a_k(t_n)}{a_{K+k}(t_n)} \right], \quad k = 1, \dots, K, \quad n = 1, \dots, N, \quad (18)$$

where the elements of $\mathbf{a}(t_n)$ are obtained from Eq. (8) in terms of the design points $\mathbf{u}^*(x, t_n)$. Observe that $M(\omega_k, t_n)$ completely defines a *linear metamodel* for the threshold x of the nonlinear system. We name this metamodel *evolutionary tail-equivalent linear system* (ETELS) since it represents a sequence of N TELSs. Observe that Eqs. (17) and (18) define a discrete-time version of the “true” ETELS. In fact, the “true” ETELS is defined by a continuous-time varying hyperplane, which can be viewed as a generalization of the TELS. For stationary excitations, the time-trajectory of the gradient converges to a vector, which defines a linear subspace (i.e., leading to parallel hyperplanes and losing the time-dependency). For non-stationary excitations, the gradient forms a time-dependent curve, which is a set but not a subspace. See Der Kiureghian (2000) and Broccardo and Der Kiureghian (2018) for more geometrical insights. The response of the ETELS is

$$\begin{aligned}\hat{X}_{ETELS}(t_n) &= \sum_{k=1}^K \sigma_{\hat{X},k}(t_n) \{\sin[\theta_k(t_n) \cdot t_n]u_k + \cos[\theta_k(t_n) \cdot t_n]u_{K+k}\} \\ &= \mathbf{a}_{ETELS}(t_n)\mathbf{u}, \quad n = 1, \dots, N,\end{aligned}\tag{19}$$

where $\sigma_{\hat{X},k}^2(t_n) = 2|M(\omega_k, t_n)|^2\Delta\omega$, $\theta_k(t_n) = -\omega_k t_n + \hat{\phi}_k(t_n)$, and

$$a_{ETELS,k}(t_n) = \sigma_{\hat{X},k}(t_n)\sin[\theta_k(t_n)t_n],\tag{20a}$$

$$a_{ETELS,K+k}(t_n) = \sigma_{\hat{X},k}(t_n)\cos[\theta_k(t_n)t_n].\tag{20b}$$

The set of equations in Eq. (19) can be rewritten in a matrix form as $\hat{\mathbf{X}}_{ETELS} = \mathbf{A}_{ETELS}\mathbf{u}$, where $\mathbf{A}_{ETELS} = [\mathbf{a}(t_1); \dots; \mathbf{a}(t_N)]^T$ is an $N \times 2K$ matrix and $\hat{\mathbf{X}}_{ETELS} = [\hat{X}_{ETELS}(t_1), \dots, \hat{X}_{ETELS}(t_N)]^T$ is the N -vector of discrete stochastic responses of the linear system.

Given the ETELS for a series of time points, the corresponding response statistics of interest, such as the tail probability, the mean crossing rate and the first-passage probability, can be computed directly using Monte Carlo simulation (MCS). Specifically, provided with a large test set, $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_L]$, the corresponding output set $\mathbf{X}_{ETELS} = [\mathbf{A}_{ETELS}\mathbf{u}_1, \dots, \mathbf{A}_{ETELS}\mathbf{u}_L]$ is directly available for statistical analysis. Note that this operation has a negligible computational cost because the system is linear and completely vectorizable. In addition, variance reduction techniques can be used to improve the efficiency of the computations. In the context of the first-passage probability, the most suitable variance reduction technique is the Au and Beck (2001a) importance sampling algorithm. This algorithm is here customized for the frequency domain. As an alternative to simulation-based approaches, analytical solutions available for the non-stationary stochastic response of linear systems can be used.

The accuracy and efficiency of this approach is strongly related to the choice of the time step sequence $[t_1, \dots, t_N]$. In this study, we focus on excitations that show a build-up phase, followed by a (non-stationary) strong-motion phase, and ending with a gradually diminishing phase. These phases are typical of classical models used to describe seismic and wind loads. The system response usually has similar three phases, with the peak response variance exhibiting a phase delay relative to the peak variance of the excitation. For linear systems, the phase delay in the peak response is of the order of the fundamental period of the system, i.e., $t_{X_p} \approx t_{F_p} + T_s$, where t_{X_p} and t_{F_p} are the times at which the variances of the response and excitation take their peak values, respectively, and T_s is the fundamental period of the system. For nonlinear softening systems, the phase delay is expected to be slightly longer than T_s . Given this, t_1 and t_N should be chosen to include the peak response phase and to exclude the early build-up and late descent phases. This

reduces the computational cost and avoids convergence problems of the optimization algorithm for finding design points with extremely large reliability indices.

The time step $\Delta t = t_{n+1} - t_n$ should be selected with due consideration of the bandwidth of the input excitation. The optimal computational choice clearly is $\Delta t = \pi/\Omega$, where Ω denotes the upper frequency of the EPSD. Furthermore, to enhance the computational efficiency, it is possible to use a simple time-gradient-based sequential algorithm to determine a “warm” starting point for the optimization algorithm. Specifically, provided design points $\mathbf{u}^*(x, t_{n-1})$ and $\mathbf{u}^*(x, t_n)$, the initial trial point at time step t_{n+1} is determined as

$$\mathbf{u}^0(x, t_{n+1}) = 2\mathbf{u}^*(x, t_n) - \mathbf{u}^*(x, t_{n-1}). \quad (21)$$

Observe that Eq. (21) can be used also as a backward scheme simply by swapping t_{n+1} and t_{n-1} . An effective strategy is as follows: Compute three consecutive design points around the peak of the response; then, use the forward and backward versions of Eq. (21) to determine warm starting points and compute the sequence of design points in time. A stopping criterion is set such that $\beta(t_1)$ and $\beta(t_N) \geq \kappa \min \beta(t_n)$, where κ is a constant. We recommend selecting $\kappa = -\Phi^{-1}\{10^{-5}\Phi[-\min(\beta_n)]\} / \min(\beta_n)$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Below we present the algorithm for determining the sequence of design points and construction of the ETELS. The iHL-RF algorithm (Zhang and Der Kiureghian 1994) is used for solving the optimization problem. In the following section, we present a more efficient formulation that uses only a small number of time points to construct the entire ETELS.

Algorithm 1. ETELM with sequence of design points

- Step 1.** Select the threshold of interest, x , and estimate the time of the peak response variance, t_{x_p} . Formulate the limit state function as $G(x, t_{x_p}, \mathbf{u}) = x - \hat{X}(t_{x_p}, \mathbf{u})$ and determine the design point by use of, e.g., the iHL-RF algorithm. Determine $M(\omega_k, t_{x_p})$ and its effective bandwidth, Ω .
- Step 2.** Starting from t_{x_p} , compute forward and backward in time a sequence of design points $[\mathbf{u}^*(x, t_1), \dots, \mathbf{u}^*(x, t_N)]$ with $\Delta t = \pi/\Omega$, such that $\beta(t_1)$ and $\beta(t_N) \geq (\kappa \min \beta(t_n))$, where $\kappa = -\Phi^{-1}(10^{-5}\Phi(-\min(\beta_n))) / \min(\beta_n)$. Use Eq. (21) to determine “warm” starting points for the search algorithm.
- Step 3.** Use Eqs. (8), (17) and (18) to determine, $\mathbf{A}_{ETELS} = [\mathbf{a}(t_1); \dots; \mathbf{a}(t_N)]$ and $[M(\omega_k, t_1), \dots, M(\omega_k, t_N)]$, $k = 1, \dots, K$.
- Step 4.** Compute the statistics of ETELS, e.g., point-in-time tail probabilities, $\{\Phi[-\beta(t_1)], \dots, \Phi[-\beta(t_N)]\}$, and the first-passage probability by MCS. For the latter, a variance-reduction technique can be used, e.g., the importance sampling method by Au Beck (2011) that employs the sequence of design points.

4.4 Approximate ETELM

In Section 3.3, we showed that a sequence of design points in time defines a discrete-time ETELS. This section defines an approximate continuous-time ETELS, which is based on the computation of one or only a few design points in time. The basic idea is to determine the IRF, $\tilde{h}_{ETELS}(t)$,

associated with an “equivalent stationary response process” and use it to obtain a discrete-frequency, continuous-time EFRF using

$$\tilde{M}(\omega_k, t) = \int_0^t \tilde{h}_{TELS}(\tau) A(\omega_k, t - \tau) \exp(-i\omega\tau) d\tau. \quad (22)$$

Observe that we have introduced the tilde accent in the definition of the continuous-time EFRF approximation to highlight the difference with the true EFRF. Knowledge of $\tilde{M}(\omega_k, t)$ is sufficient to completely determine the response of the ETELS from Eq. (19) by use of $\sigma_{\tilde{x},k}^2(t_n) = 2|\tilde{M}(\omega_k, t_n)|^2 \Delta\omega$. In this case, however, time is a continuous variable whose discretization does not affect the computational cost.

The central point of this approximation is the definition of $\tilde{h}_{TELS}(t)$. A logical choice is to consider the EFRF at the peak of the response as a surrogate for the FRF of the equivalent system, i.e.,

$$\tilde{H}_{TELS}(\omega_k) \approx M(\omega_k, t_{x_p}). \quad (23)$$

Next, the IRF is determined by the inverse Fourier transform of $\tilde{H}_{TELS}(\omega_k)$, i.e.,

$$\tilde{h}_{TELS}(t) = \frac{1}{\pi} \sum_{k=1}^K \tilde{H}_{TELS}(\omega_k) \exp(i\omega t) \Delta\omega. \quad (24)$$

Eq. (22) is now used to determine the EFRF, $\tilde{M}(\omega_k, t_{x_p})$. In essence, Eq. (24) is used to define a linear system and the associated subspace of the stationary response processes; then, Eqs. (22) and (20) are used to define a surrogate of the time trajectory of the gradient of the ETELS. At the peak of the excitation, the error in using $\tilde{M}(\omega_k, t_{x_p})$ in place of $M(\omega_k, t_{x_p})$ should be minimized. Observe that the scalings of $\tilde{M}(\omega_k, t_{x_p})$ and $M(\omega_k, t_{x_p})$ might differ as they depend on the normalization of $A(\omega, t - \tau)$. Therefore, to obtain a better fit, we introduce a constant $C > 0$ so that the least square error between $C\tilde{M}(\omega_k, t_{x_p})$ and $M(\omega_k, t_{x_p})$ is minimized. Specifically, the constant C is determined by solving the least-square optimization problem

$$\arg \min_C \sum_{k=1}^K \left(C |\tilde{M}(\omega_k, t_{x_p})| - |M(\omega_k, t_{x_p})| \right)^2. \quad (25)$$

The solution of the above quadratic problem is

$$C = \frac{\sum_{k=1}^K |M(\omega_k, t_{x_p}) \tilde{M}(\omega_k, t_{x_p})|}{\sum_{k=1}^K |\tilde{M}(\omega_k, t_{x_p})|^2}, \quad (26)$$

and the corrected ETELS approximation is denoted as $\tilde{M}_C(\omega_k, t_{x_p}) = C\tilde{M}(\omega_k, t_{x_p})$. This correction ensures that the mean square error between $\tilde{M}_C(\omega_k, t)$ and $M(\omega_k, t)$ is minimized at the peak of the structural response.

There are two main limitations to the proposed approach. First, the location of the peak response variance is not known exactly. Second, one design point might be too crude an approximation for defining $\tilde{H}_{TELS}(\omega_k)$. The second limitation is critical for non-uniformly modulated processes, i.e., processes with time-varying frequency content. To address these limitations, we propose the following solution: Compute a sequence of design points in time, $[\mathbf{u}^*(x, t_1), \dots, \mathbf{u}^*(x, t_M)]$, starting from the peak of the excitation, here denoted $t_1 = t_{F_p}$, up to t_M so that $\beta(t_M) \gtrsim \beta(t_1)$. In practice, 3-5 design points are sufficient and $\Delta t \approx T_s$ works in most cases. Note

that by construction $t_1 \leq t_{x_p} \leq t_M$. Next, $\tilde{H}_{TELS}(\omega_k)$ is computed as the weighted average of the EFRFs at the selected time points

$$\tilde{H}_{TELS}(\omega_k) \approx \sum_{m=1}^M w_m M(\omega_k, t_m), \quad (27)$$

with

$$w_m = \frac{\sum_{k=1}^K |M(\omega_k, t_m)|}{\sum_{m=1}^M \sum_{k=1}^K |M(\omega_k, t_m)|}. \quad (28)$$

Provided with $\tilde{H}_{TELS}(\omega_k)$, Eqs. (24) and (22) are applied while Eq. (25) is replaced by

$$C = \arg \min_C \sum_{k=1}^K \left(C |\tilde{M}(\omega_k, t_{x_p})| - \left| \sum_{m=1}^M w_m M(\omega_k, t_m) \right| \right)^2. \quad (29)$$

Once $[\tilde{M}(\omega_k, t_1), \dots, \tilde{M}(\omega_k, t_N)]$ are computed from Eq. (22) and corrected by C , the gradient of the ETELS is completely defined by determining the vectors $\mathbf{a}(t_n)$, and the matrix $\mathbf{A}_{TELS}$ from Eqs. (20). Next, the statistic of the response are determined by MCS with variance reduction techniques, or linear non-stationary random vibration analysis. Below, a summary of these steps is listed.

Algorithm 2. Approximate ETELM solution

- Step 1** Select the threshold of interest, x , and compute a sequence of 3-5 design points $[\mathbf{u}^*(x, t_1), \dots, \mathbf{u}^*(x, t_M)]$, where $t_1 = t_{F_p}$ and t_M is selected such that $\beta(t_M) \gtrsim \beta(t_1)$. Select $\Delta t \approx T_s$.
- Step 2** Use Eqs. (27), (24) and (22) to determine $\tilde{H}_{TELS}(\omega_k)$, $\tilde{h}_{TELS}(t_n)$ and $\tilde{M}(\omega_k, t_n)$, respectively, for $[t_1, \dots, t_N]$ with $\Delta t \leq \pi/\Omega$, where Ω is the effective bandwidth of $\tilde{M}(\omega_k, t_{x_p})$.
- Step 3** Determine the constant C by solving Eq. (29) and set $\tilde{M}_C(\omega_k, t_n) = C \tilde{M}(\omega_k, t_n)$.
- Step 4** Use Eqs. (20) with $\tilde{M}_C(\omega_k, t_n)$ to determine $\mathbf{A}_{ETELM}$, and Eq. (6) to determine the sequence of design points in time $[\mathbf{u}^*(x, t_1), \dots, \mathbf{u}^*(x, t_N)]$.
- Step 5** Compute the statistics of ETELS.

4.5 Approximate ETELM for uniformly modulated processes

A uniformly modulated process is obtained from Eq. (12) by choosing a time-independent frequency modulating function $\phi(\omega)$. In this case, only the intensity of the motion changes with time, while the frequency content remains constant. For this special case, we modify Eq. (22) to read

$$\tilde{M}(\omega_k, t) = \int_0^t \tilde{h}_{TELS}(\tau) q(t - \tau) \exp(-i\omega\tau) d\tau. \quad (30)$$

The change is introduced because $\tilde{h}_{TELS}(t)$ already includes the frequency content of the ETELS response by its construction (see Eqs. 23 and 24), and thus also the frequency content of the excitation process. Therefore, using $A(\omega, t)$ would distort the frequency content of $\tilde{M}(\omega_k, t)$, as it considers “twice” the frequency content of the input (one in the definition of $\tilde{h}_{TELS}(t)$, and one in

the definition of $\tilde{M}(\omega_k, t)$). It turns out that this distortion is not critical for broadband processes, while it becomes more important for narrowband processes. The reason is that for broad-band processes, the modulating function $\phi(\omega, t)$ is flat and the important frequency content of $\tilde{M}(\omega_k, t)$ is determined more by the system properties (i.e., the fundamental frequency of the linear system and the extent of softening of the system). Eq. (30) can also be used for processes with slowly varying frequency content; however, for processes with medium and high temporal variations Eq. (22) should be used. In this case, the distortion is less important than the global evolution of the frequency content. For this approximation, Algorithm 2 remains the same provided that $\tilde{M}(\omega_k, t_n)$ at Step 2 is computed according to Eq. (30) instead of Eq. (22).

5 Numerical Investigation

5.1 System definition

This section presents a numerical investigation to explore the effectiveness and limitations of the ETELM. For simplicity and without loss of generality, we focus on a single-degree-of-freedom (SDOF) oscillator. We select a softening system with a gradual transition from the elastic to the inelastic domain. Abrupt changes between the two domains are not allowed in TELM, as they result in nondifferentiable limit state surfaces (which is a requirement for FORM approximation, see Fujimura and Der Kiureghian, 2007). The generalization to multiple-degrees-of-freedom systems is straightforward.

The SDOF oscillator is described by the non-degrading hysteretic Bouc-Wen model (Bouc 1963, Wen 1989). Specifically, the nonlinear response of the system generated by the input $F(t)$ is governed by the differential equation

$$m\ddot{X}(t) + c\dot{X}(t) + k[\alpha X(t) + (1 - \alpha)Z(t)] = -mF(t), \quad (31)$$

where m is the mass, c is the damping coefficient, k is the stiffness, and α is a parameter controlling the degree of hysteresis with $\alpha = 1$ corresponding to the linear case. $Z(t)$ follows the Bouc-Wen hysteresis law

$$\dot{Z}(t) = -\gamma|\dot{X}(t)||Z(t)|^{(n-1)}Z(t) - \eta|Z(t)|^n \dot{X}(t) + A\dot{X}(t), \quad (32)$$

where γ, n, η and A are parameters that shape the hysteresis loop.

The system parameters m and k are selected such that the fundamental period of the linear system is $T_s = 1[s]$, and c is set to match a viscous damping ratio of 5%. The hysteretic parameters γ and η are given in terms of the mean-square response of the linear oscillator when it is subjected to a white-noise excitation with unit spectral density, which we denote as $\sigma_0^2 = \pi m^2/(ck)$. Table 1 lists the system parameters and Figure 1 shows the associated hysteresis loop.

5.2 Input excitation

Four input excitations are considered: i) $\hat{F}_1(t)$, a uniformly modulated, broad-band process, ii) $\hat{F}_2(t)$, a uniformly modulated, narrow-band process, iii) $\hat{F}_3(t)$, a fully non-stationary process with slowly varying broadband frequency content, and iv) $\hat{F}_4(t)$, a fully non-stationary process with rapidly varying narrowband frequency content. All four excitations are filtered Gaussian noises with the time-frequency modulating function $A(\omega, t)$ defined by Eq. (12), where the frequency-modulating function $\phi(\omega, t)$ is defined by a band-limited Kanai-Tajimi filter with a time-varying frequency

$$|\phi(\omega, t)|^2 = \frac{\omega_f^4(t) + 4\zeta_f^2\omega_f^2(t)\omega^2}{(\omega_f^2(t) - \omega^2)^2 + 4\zeta_f^2\omega_f^2(t)\omega^2} \Pi(-\Omega_f, \Omega_f), \quad (33)$$

where $\omega_f(t)$ is the filter frequency, ζ_f the filter damping ratio, $\Pi(-\Omega_f, \Omega_f) = 1$ for $-\Omega_f \leq \omega \leq \Omega_f$ and $\Pi(-\Omega_f, \Omega_f) = 0$ elsewhere is the rectangular function, in which Ω_f is the bandwidth of the excitation. Moreover, $\omega_f(t) = \omega_0 - \dot{\omega}(t - t_{F_p})$ is assumed, where ω_0 is the predominant frequency at $t = t_{F_p}$ and $\dot{\omega}$ is the rate of change in that frequency. The time-modulating function in Eq. (12) is defined as

$$q^2(t) = A_F \frac{1 + \sin\left[\pi\left(t/t_{F_p} - 1/2\right)\right]}{2} \Pi\left(0, 2t_{F_p}\right), \quad (34)$$

where A_F is a scale parameter. Observe that $\max q^2(t) = q^2(t_{F_p}) = A_F$ is the maximum variance of the input excitation occurring at t_{F_p} .

The model parameters used in this study are listed in Table 2. Figure 2 shows $q^2(t)$ together with normalized plots of $|\phi(\omega, t)|^2$. Figure 3 shows the complete modulating functions $|A(\omega, t)|^2$ together with a simulated realization of each process. The same realization of the underlying vector of standard normal random variables $[u_1, \dots, u_K; u_{K+1}, \dots, u_{2K}]$ is used to generate these simulations so that a direct comparison across different modulating functions can be made. The frequency increment used is $\Delta\omega = 0.05[\text{Hz}]$ (i.e., $T = 20[\text{s}]$) for both Algorithms. The effective bandwidth of the response is fixed at $\Omega = 5[\text{Hz}]$ since, as shown below, the contribution of higher frequencies is negligible. This allows us to cut by half the number of random variables and, therefore, the computational time of a gradient evaluation. The time increment for Algorithm 1 is $\Delta t = 1/(2\Omega) = 0.1$. For Algorithm 2 it can be chosen arbitrarily small without significantly affecting the computation demand.

5.3 Results of ETELM analysis

5.3.1 Uniformly modulated, broad-band process

Algorithms 1 and 2 are used to compute the design points, point-in-time reliability indices and the first-passage probability. In Algorithm 1, first the design point at the peak of the excitation ($t_{x_p} \approx 10[\text{s}] + 1[\text{s}] = 11[\text{s}]$) is determined using the iHL-RF algorithm with a zero vector as the starting point. Then, the forward and backward scheme (step 2, Algorithm 1) is used to produce a sequence of design points spaced at $\Delta t = 0.1[\text{s}]$. Figure 4 shows in red and orange¹ dots the point-in-time reliability indices of the system for two thresholds, $x_1 = 2.5\sigma_0$ and $x_2 = 3.5\sigma_0$. The stopping criterion results in $\kappa_1 = 1.84$ (i.e., $\beta_{ETELS}(x_1, t) = 5.58$) and $\kappa_2 = 1.56$ (i.e., $\beta_{ETELS}(x_2, t) = 6.15$). These results are compared with the results of the nonlinear response obtained by a crude MCS with 10^5 simulations, shown in Figure 4 as a black line with light gray lines indicating the three-sigma confidence bounds.

Algorithm 2 uses four design points at $t = [10, 11, 12, 13] [\text{s}]$ to determine $\tilde{H}_{TELS}(\omega_k)$, $\tilde{h}_{TELS}(t_n)$ and $\tilde{M}(\omega_k, t_n)$. In this approach, the point-in-time reliability index is a time-continuous

¹ In black and white reproductions dark red is distinguishable as dark gray and orange and green colors are distinguishable as light gray

function and, therefore, reported with a continuous line in Figure 4. Specifically, the Figure shows in dark red and green lines the point-in-time reliability indices of the system, for the two thresholds of interest. There is near perfect agreement between the results of Algorithm 1 and Algorithm 2, and a good agreement with the MCS results of the nonlinear response.

Figure 5a-b shows the EFRFs $|M(\omega, t_n)|$ and $|\tilde{M}_C(\omega, t)|$ for $x = 3.5\sigma_0$, and figure 5c-d shows their cross-sections for selected time instances. These figures show a critical advantage of ETELM over analysis that focuses only on estimating failure probabilities. They provide a *structuralist view of the problem* (Rickart, 1995, Breitung 2018), which in engineering analyses is more important than a highly accurate estimation of probability values, which in any case are strongly dependent on the assumptions inherent in the underlying models. These figures show how the energy is driven into the nonlinear system as time evolves. Moreover, they identify the frequency ranges that contribute most to the failure of the system. The moduli of the two EFRFs show excellent agreement at the peak of the excitation, while at earlier and later times they show a reasonable matching.

Next, the first-passage probability is computed using the linear metamodel represented by $\mathbf{A}_{ETELS}$. Specifically, the metamodel is used together with a crude MCS and then the customized importance sampling technique introduced by Au and Beck (2001a). Since the results are substantially identical, we report only the importance sampling estimates obtained with a sample size of 10^4 . As suggested earlier, a negligible computational cost is associated with this approach. The results are reported in the first two lines of Table 3. Specifically, $\beta_{ETELS}^{(IS)}(x) = -\Phi^{-1}(\Pr[\max |\hat{X}(t)| > x])$ denotes the reliability index associated with the first-passage probability computed with $\mathbf{A}_{ETELS}$ and important-sampling, and $\beta_{MCS}(x)$ denotes the same quantity computed for the nonlinear response with a crude MCS with 10^5 samples. In all cases, ETELM provides an excellent estimate of the first-passage probability compared to the MCS benchmark.

In ETELM analysis, the computational cost is associated with the number of evaluations of the limit-state and its gradient in the search for the design points. Therefore, it is critical to implement efficient direct differentiation algorithms (Zhang and Der Kiureghian 1993) to minimize the cost associated with the gradient evaluation. As indicated earlier, Algorithm 1 is computationally more expensive than Algorithm 2 as it requires many design points. To appreciate the difference between the two algorithms, Table 3 reports the computational cost in terms of the number of limit-state (NG) and gradient (N grad) evaluations. Algorithm 2 for this input excitation has an excellent computational performance. Algorithm 1 is computationally expensive; however, its computational cost is on the same order as that of the subset simulation method (Au and Beck 2001b, Wang et al. 2019), which is often used to solve this type of problem.

A critical advantage of Algorithms 1 and 2 compared to the standard sampling method is that the computational cost is not affected by the reliability level. As a matter of fact, it is expected to decrease slightly with the amplitude of the thresholds, as shown in Table 3. In Algorithm 1, this is due to the stopping criterion, since κ is decreasing for higher thresholds. In Algorithm 2, this is less clear, but one reason is that the design points of the first threshold have been used as warm starting points for the second threshold. A disadvantage of ETELM compared to the classical sampling method is that the computational time depends on the number of random variables. Therefore, it is critical to use a parsimonious number of random variables. This is partly achieved by reducing the dimensionality based on the effective bandwidth of the problem.

5.3.2 Uniformly modulated, narrow-band process

For the uniformly modulated, narrowband excitation $\hat{F}_2(t)$, we examine results for the same thresholds as above. Figure 6 confirms an excellent agreement between the results obtained from Algorithms 1 and 2; however, the accuracy of both approximations has degraded when compared with the MCS benchmark. This is not a direct limitation of ETELM, but of the FORM approximation. The EPSD of this excitation is extremely sharp with a predominant frequency close to the fundamental frequency of the linear system. The resulting “resonance” effect produces a highly nonlinear limit-state surface, which deteriorates the accuracy of the FORM approximation. (See Igusa and Der Kiureghian 1988 for another example of strong nonlinearity due to resonance effects.) However, even for this extreme case, the error is not unacceptably large, i.e., it is less than 5% in the point-in-time reliability index.

Figure 7a-b shows the EFRFs $|M(\omega, t_n)|$ and $|\tilde{M}_C(\omega, t)|$ for the threshold $x = 3.5\sigma_0$, and figure 7c-d shows their cross-sections for selected time points. These Figures show a bimodal EFRF, where one peak corresponds to the frequency of the system and the other to the predominant frequency of the excitation. The moduli of two EFRFs show good agreement at the peak of the excitation, while at earlier and later times they show similar trends, though the matching is not perfect.

The results for the first-passage probability are summarized in the third and fourth rows of Table 3. Compared to the case of the broad-band excitation, the accuracy has deteriorated. However, the error remains below 13% for x_1 and below 6% for x_2 —with a decreasing trend due to the asymptotic nature of FORM analysis. The computational cost in terms of the number of limit-state and gradient evaluations is higher in this case due to the increased number of iterations required to find the design points. However, the higher number of gradient evaluations is compensated by a reduced cost of gradient computations due to the reduced number of random variables in account of a smaller bandwidth Ω_f .

5.3.3 Fully non-stationary, broad-band process

Figure 8 shows the estimates of the point-in-time reliability indices for the response to excitation $\hat{F}_3(t)$, which has a broadband, time-varying frequency content. Good agreements between the estimates based on Algorithm 1 and 2 and the MCS benchmark are observed. Figure 9a-b shows the EFRFs $|M(\omega, t_n)|$ and $|\tilde{M}_C(\omega, t)|$ for the threshold $x = 3.5\sigma_0$, and figure 9c-d shows their cross-sections for selected time points. The moduli of the EFRFs computed by Algorithms 1 and 2 are in excellent agreement at the peak of the excitation, while at earlier and later times they show reasonable matching (Figure 9c-d). Finally, the fifth and sixth rows of Table 3 list the estimates for the first-passage probability. Both Algorithms 1 and 2 produce results in close agreement with the MCS result. Moreover, the computational cost is of the same order as that associated with excitation $\hat{F}_1(t)$. These results suggest that the ETELM is accurate and efficient for fully nonstationary, broad-band excitations.

5.3.4 Fully non-stationary, narrow-band process

The excitation process $\hat{F}_4(t)$ is particularly challenging for ETELM since it is both narrowband and has a rapidly evolving frequency content. To maintain a comparable level of reliability with the previous cases, the thresholds are selected as $x_1 = 2.1\sigma_0$, and $x_2 = 3.0\sigma_0$. Figure 10 shows the results for the point-in-time reliability index based on Algorithms 1 and 2 and the MCS. Results based on Algorithms 1 and 2 are somewhat divergent with the estimates based on Algorithm 1 being closer to the MCS benchmark.

Figure 11a-b shows the EFRFs $|M(\omega, t_n)|$ and $|\tilde{M}_c(\omega, t)|$ for $x = 3.0\sigma_0$. The decreasing trend of the frequency content of the input EPSD is clearly evident in the EFRFs of both algorithms. A close look at the contours of the EFRFs reveals the presence of a bimodal EFRF at early times, when the input EPSD is concentrated at higher frequencies. This pattern is clearly visible in Figure 11a, where the cross-section of the EFRF at $t = 6[s]$ shows two peaks. At $t = 8[s]$, the gap between the two peaks is closed, and for later times the EFRF is unimodal. Interestingly, at later times ($t > 11[s]$), the peak of the EFRF is no more at the original fundamental frequency of the linear system, but it is shifted to a lower frequency. This leads to an interesting physical observation: For softening systems, if a narrow-band process has a frequency content higher than the original fundamental frequency of the system, then the nonlinear response of the system has two main frequencies (two peaks in the FRF, or EFRF). Conversely, if a narrow-band process has a frequency content lower than the original fundamental frequency of the system, then the response process has one predominant frequency, which is lower than the original fundamental frequency of the system. The moduli of the EFRFs based on Algorithm 1 and 2 show similar trends, but the matching is the least accurate among the examined excitations.

The last two rows of Table 3 list the reliability indices for the first-passage probability and the computational efforts of Algorithms 1 and 2. Even though the point-in-time reliability indices are not perfectly matching, estimates of the first-passage probability are close and are in reasonable agreement with the MCS results. The errors associated with Algorithm 2 are below 9% and 3% for thresholds x_1 and x_2 , respectively.

The computational cost associated with this excitation is remarkably lower than that for $\hat{F}_2(t)$. It turns out that the iHL-RF algorithm requires more iterations to converge when the EFRF is bimodal, i.e., when the narrow-band process has a higher frequency content than the fundamental frequency of the linear system. Conversely, it requires fewer steps to converge when the EFRF is unimodal, i.e., when the narrow-band process has a lower frequency content than the fundamental frequency of the linear system. Since the EFRF associated with excitation $\hat{F}_4(t)$ is bimodal only at earlier times, it is not surprising to observe a lower computational cost for this case.

6 Conclusions

The tail-equivalent linearization method (TELM) is extended to stochastic dynamic analysis of nonlinear systems subjected to nonstationary excitations defined by Priestley's evolutionary process. Named evolutionary TELM or ETELM, the method introduces an evolutionary tail equivalent system, ETELS, defined through an evolutionary frequency-response function, EFRF, for a selected response threshold. The EFRF depends on the considered threshold and captures not only the dynamic characteristics of the nonlinear system, but also the evolving nature of the intensity and frequency content of the excitation. The approximation is based on linearizations of the response function at the selected threshold at a sequence of time points.

In Algorithm 1, the sequence of linearization points are solved exactly leading to a discrete-time ETELS. This defines a linear metamodel with respect to the point-in-time tail probabilities of the response. This metamodel can be used in conjunction with sampling methods to compute the first-passage probability with minimal computational effort. The main computational effort is in finding the linearization points, which requires solution of a sequence of constraint optimization problems. To reduce the computational effort, Algorithm 2 uses Priestley's evolutionary theory to define a continuous-time approximation of the ETELS. The Algorithm employs only a few linearization points and approximates the EFRF at other time instances.

An extensive numerical investigation is performed using input processes that are broad- or narrow-band with slowly or rapidly varying frequency contents. The investigation shows that ETELM produces excellent results for broadband excitations, and a fairly good approximation for narrowband processes. Furthermore, through the EFRF, ETELM provides valuable information about the physical characteristics of the response process, such as the evolution of the frequency content with time and the frequency range dominantly contributing to the event of the response exceeding the specified threshold.

Acknowledgements

The first author was supported by the Italian Ministry of Education, University and Research (MIUR) in the frame of the “Departments of Excellence” (grant L 232/2016).

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Tables

Table 1: System and hysteresis parameters

System parameters				Hysteresis parameters			
m	k	c	α	n	γ	η	A
1	$(2\pi)^2 m$	$0.2\pi m/T_s$	0.1	1	$1/(2\sigma_0^n)$	$1/(2\sigma_0^n)$	1

Table 2 Input excitation parameters. NRV denotes number of random variables. The value on the parenthesis indicates the NRV *without* using the effective bandwidth $\Omega = 5[\text{Hz}]$.

Excitation process	Frequency modulating function				Time modulating function		
	$\omega_0/2\pi$ [Hz]	$\omega/2\pi$ [Hz/s]	ζ_f	NRV	$\Omega_f/2\pi$ [Hz]	t_{F_p} [s]	A_F [g ²]
$\hat{F}_1(t)$	1	0	0.8	100 (200)	10	10	2.5
$\hat{F}_2(t)$	1.3	0	0.1	60	3	10	3.5
$\hat{F}_3(t)$	1	0.06	0.8	100 (200)	10	10	2.5
$\hat{F}_4(t)$	1	0.08	0.1	100 (200)	10	10	1

Table 3 ETELM and MCS results for first-passage probability and computational effort

Excitation	Threshold	Algorithm 1			Algorithm 2			MCS		
		$\beta_{ETELM}^{(IS)}(x)$	NG	N Grad	$\beta_{ETELM}^{(IS)}(x)$	NG	N Grad	$\beta_{MCS}(x)$	c.o.v. P_f	NG
$\hat{F}_1(t)$	$x_1 = 2.5\sigma_0$	1.52	755	442	1.52	32	18	1.59	1%	10^5
	$x_2 = 3.5\sigma_0$	2.85	678	395	2.86	20	11	2.88	7%	10^5
$\hat{F}_2(t)$	$x_1 = 2.5\sigma_0$	1.31	1886	1009	1.41	40	22	1.49	1%	10^5
	$x_2 = 3.5\sigma_0$	2.89	2113	1113	2.99	82	43	3.06	9%	10^5
$\hat{F}_3(t)$	$x_1 = 2.5\sigma_0$	1.52	757	444	1.53	32	18	1.56	1%	10^5
	$x_2 = 3.5\sigma_0$	2.80	754	433	2.81	14	9	2.83	7%	10^5
$\hat{F}_4(t)$	$x_1 = 2.1\sigma_0$	1.70	510	302	1.64	49	27	1.79	2%	10^5
	$x_2 = 30\sigma_0$	3.24	478	279	3.19	37	21	3.28	14%	10^5

Figures

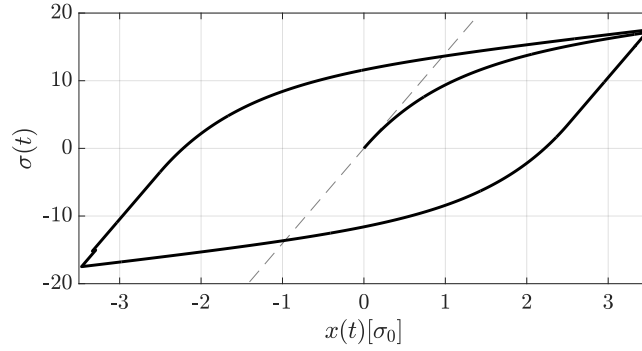


Figure 1 Hysteresis loop of SDOF oscillator; dashed line indicates the original linear system.

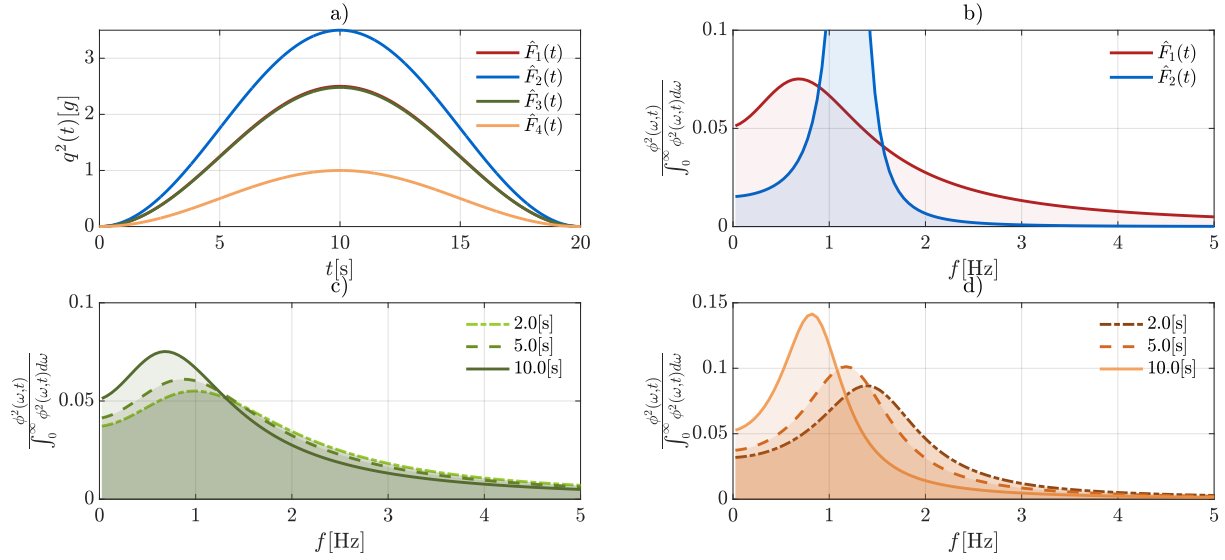


Figure 2. Characterization of input excitations: a) Time-modulating functions, b) normalized frequency-modulating functions for $\hat{F}_1(t)$ and $\hat{F}_2(t)$, c) normalized frequency-modulating functions for $\hat{F}_3(t)$, d) normalized frequency-modulating functions for $\hat{F}_4(t)$.

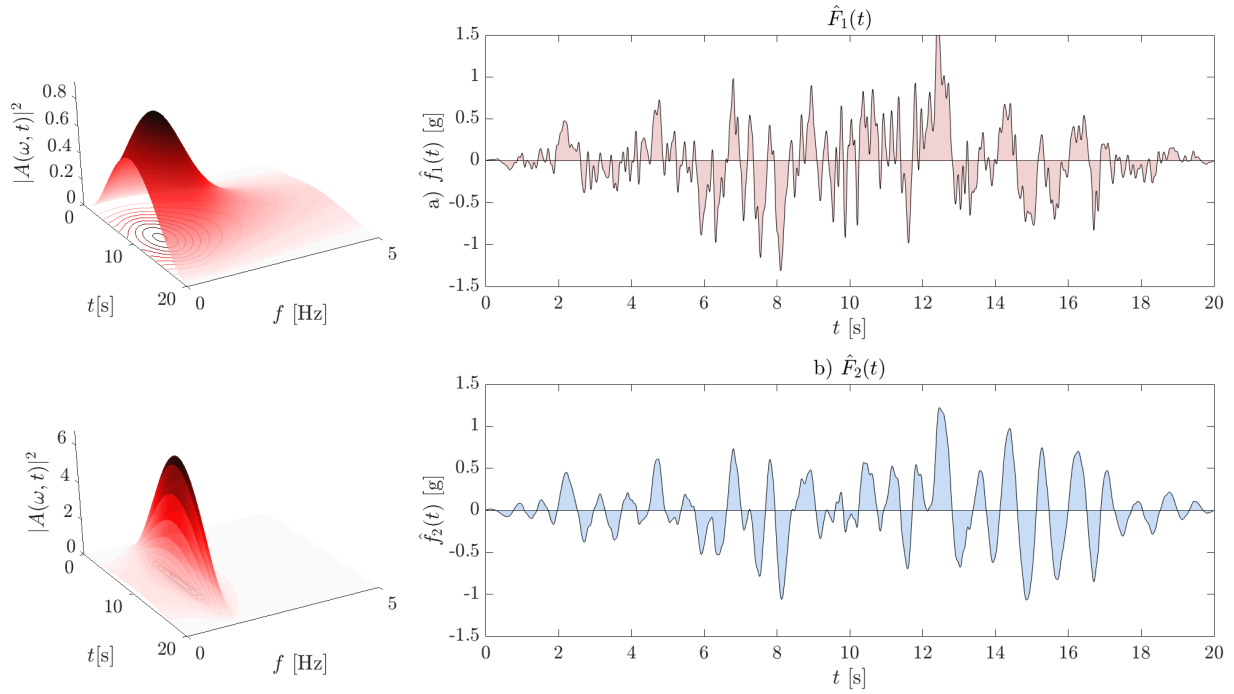


Figure 3-a EPSDs and sample realizations of $\hat{F}_1(t)$ and $\hat{F}_2(t)$.

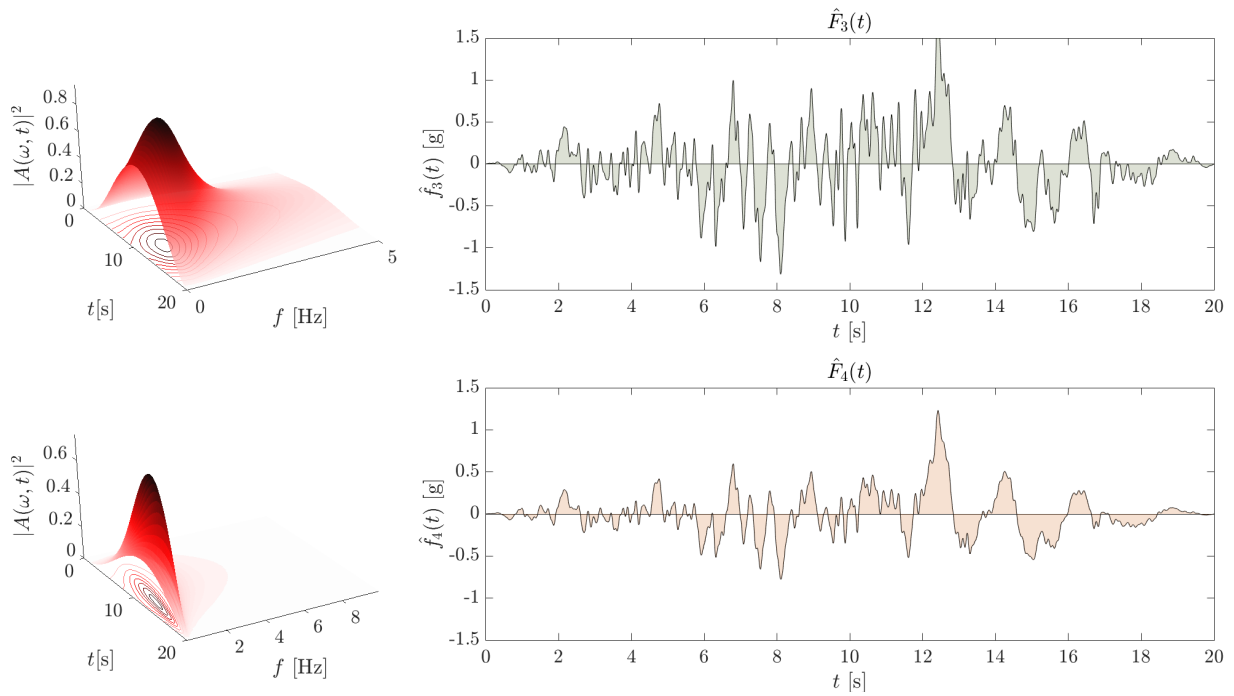


Figure 3-b. EPSDs and sample realizations of $\hat{F}_3(t)$ and $\hat{F}_4(t)$.

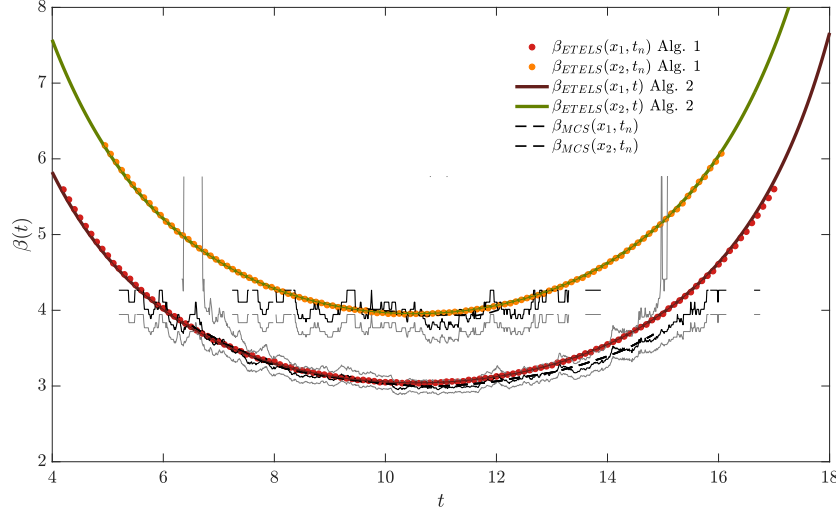


Figure 4. Point-in-time reliability indices for excitation $\hat{F}_1(t)$ and response thresholds $x_1 = 2.5\sigma_0$ and $x_2 = 3.5\sigma_0$: Dotted lines are ETELM using Algorithm 1; solid lines are ETELM using Algorithm 2; dashed lines are MCS and 3 sigma confidence bounds for the nonlinear response. (The upper confidence bounds for x_2 is not reported as 10^5 samples are not sufficient).

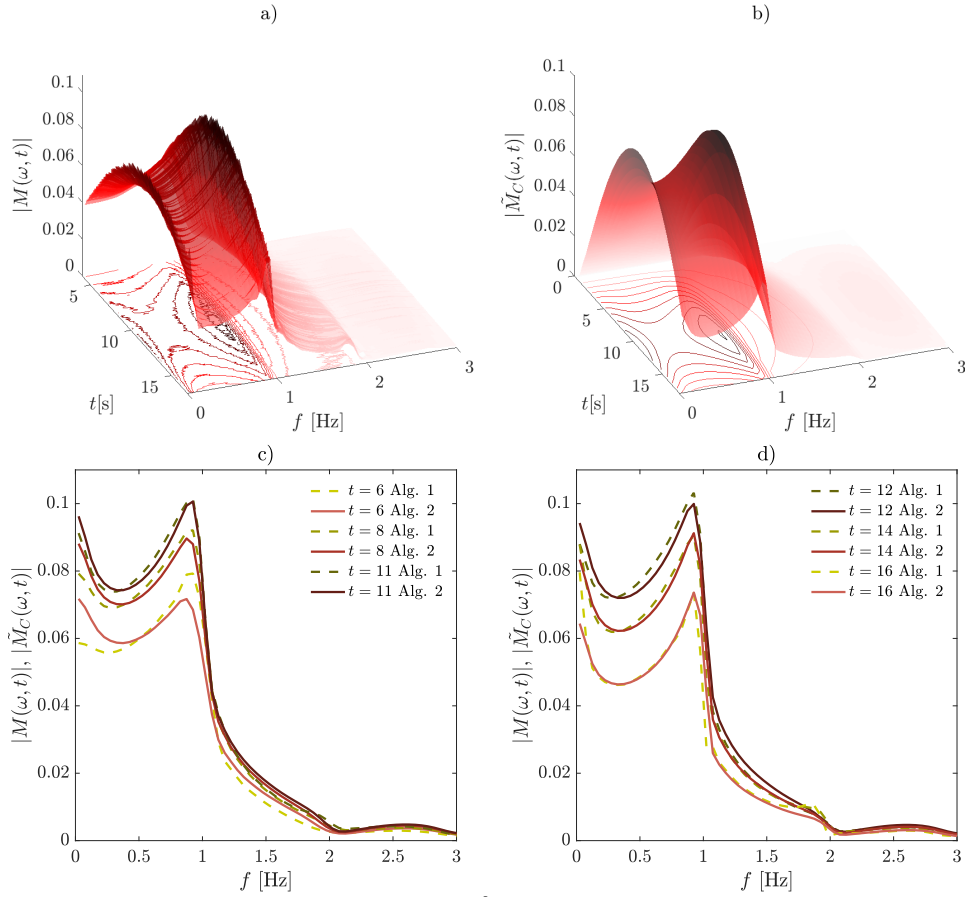


Figure 5. EFRFs for response to excitation $\hat{F}_1(t)$ at threshold $x_2 = 3.5\sigma_0$: a) Algorithm 1; b) Algorithm 2; c) and d) comparison of EFRFs for Algorithms 1 and 2 at selected times.

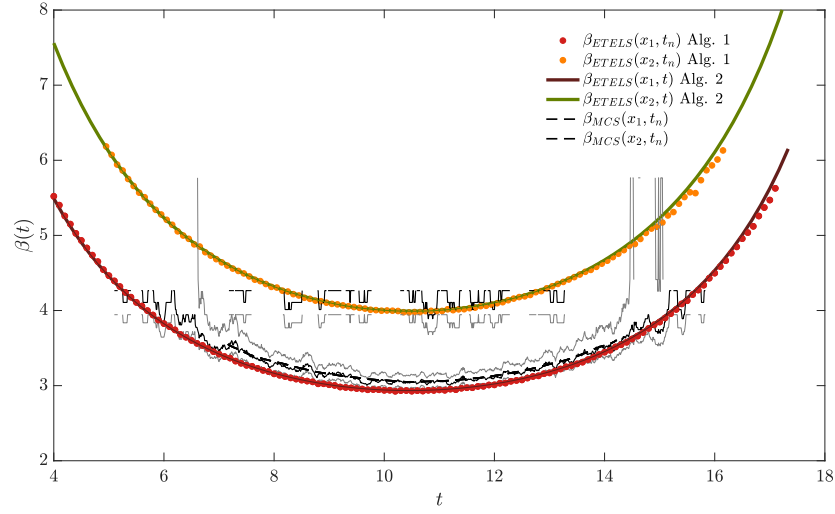


Figure 6. Point-in-time reliability indices for excitation $\hat{F}_2(t)$ and response thresholds $x_1 = 2.5\sigma_0$ and $x_2 = 3.5\sigma_0$: Dotted lines are ETELM using Algorithm 1; solid lines are ETELM using Algorithm 2; dashed lines are MCS and 3 sigma confidence bounds for the nonlinear response. (The upper confidence bounds for x_2 is not reported as 10^5 samples are not sufficient)

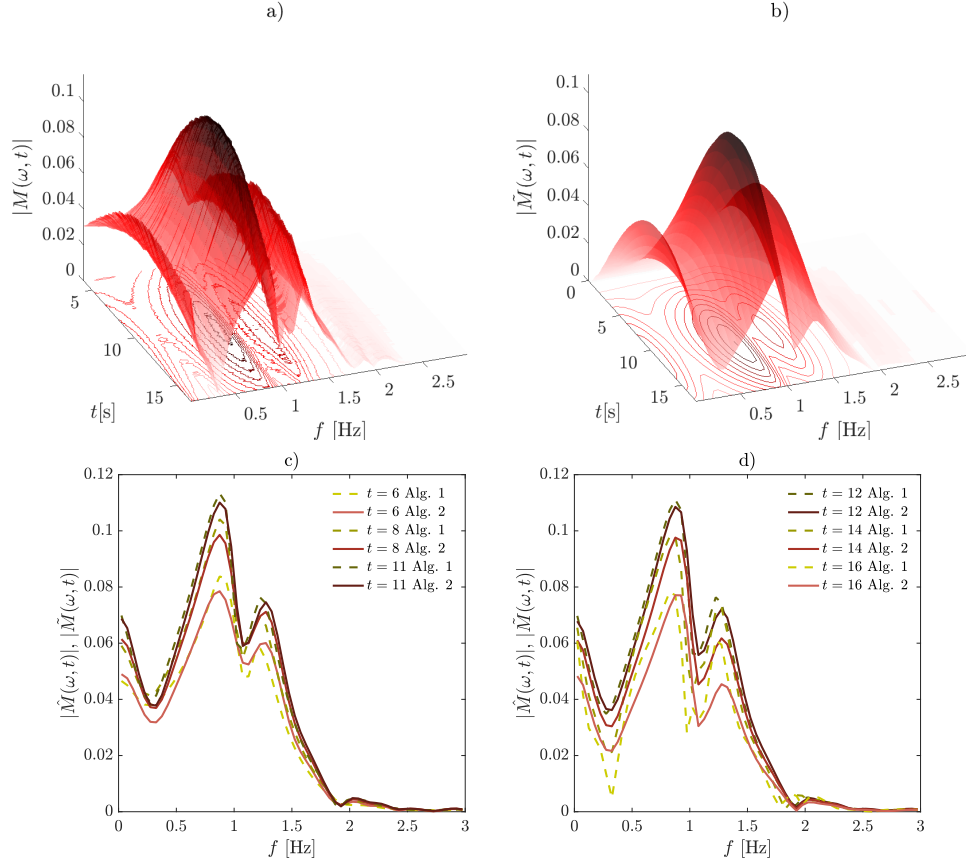


Figure 7. EFRFs for response to excitation $\hat{F}_2(t)$ at threshold $x_2 = 3.5\sigma_0$: a) Algorithm 1; b) Algorithm 2; c) and d) comparison of EFRFs for Algorithms 1 and 2 at selected times.

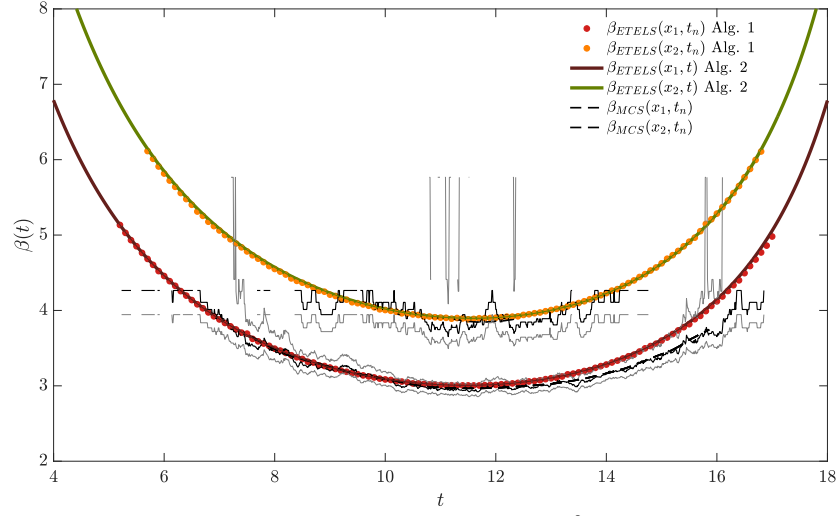


Figure 8. Point-in-time reliability indices for excitation $\hat{F}_3(t)$ and response thresholds $x_1 = 2.5\sigma_0$ and $x_2 = 3.5\sigma_0$: Dotted lines are ETELM using Algorithm 1; solid lines are ETELM using Algorithm 2; dashed lines are MCS and 3 sigma confidence bounds for the nonlinear response.

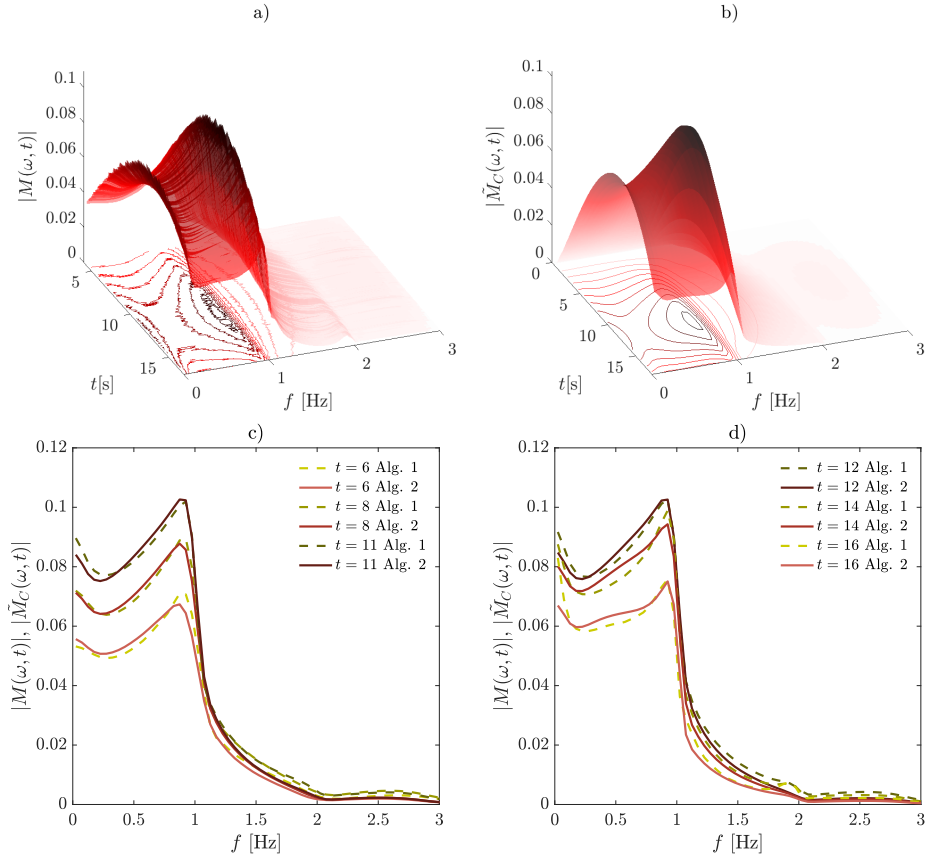


Figure 9. EFRFs for response to excitation $\hat{F}_3(t)$ at threshold $x_2 = 3.5\sigma_0$: a) Algorithm 1; b) Algorithm 2; c) and d) comparison of EFRFs for Algorithms 1 and 2 at selected times.

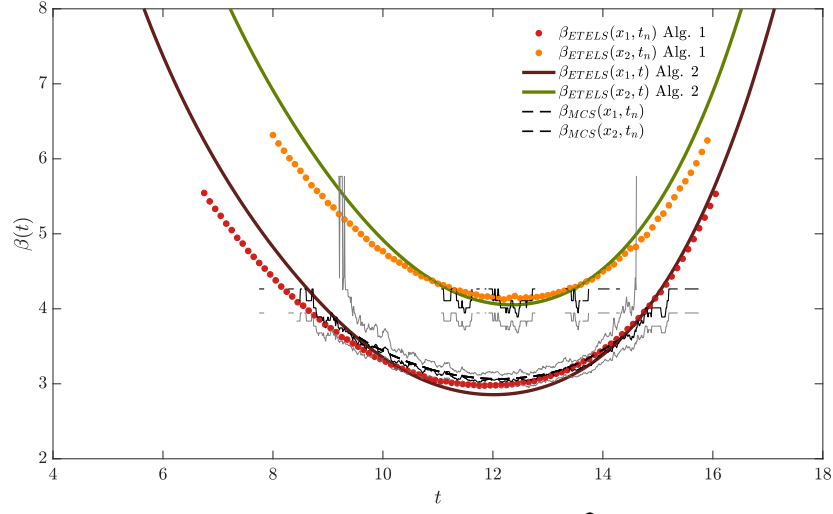


Figure 10. Point-in-time reliability indices for excitation $\hat{\mathbf{F}}_3(t)$ and response thresholds $\mathbf{x}_1 = 2.1\sigma_0$ and $\mathbf{x}_2 = 3.0\sigma_0$: Dotted lines are ETELM using Algorithm 1; solid lines are ETELM using Algorithm 2; dashed lines are MCS and 3 sigma confidence bounds for the nonlinear response.

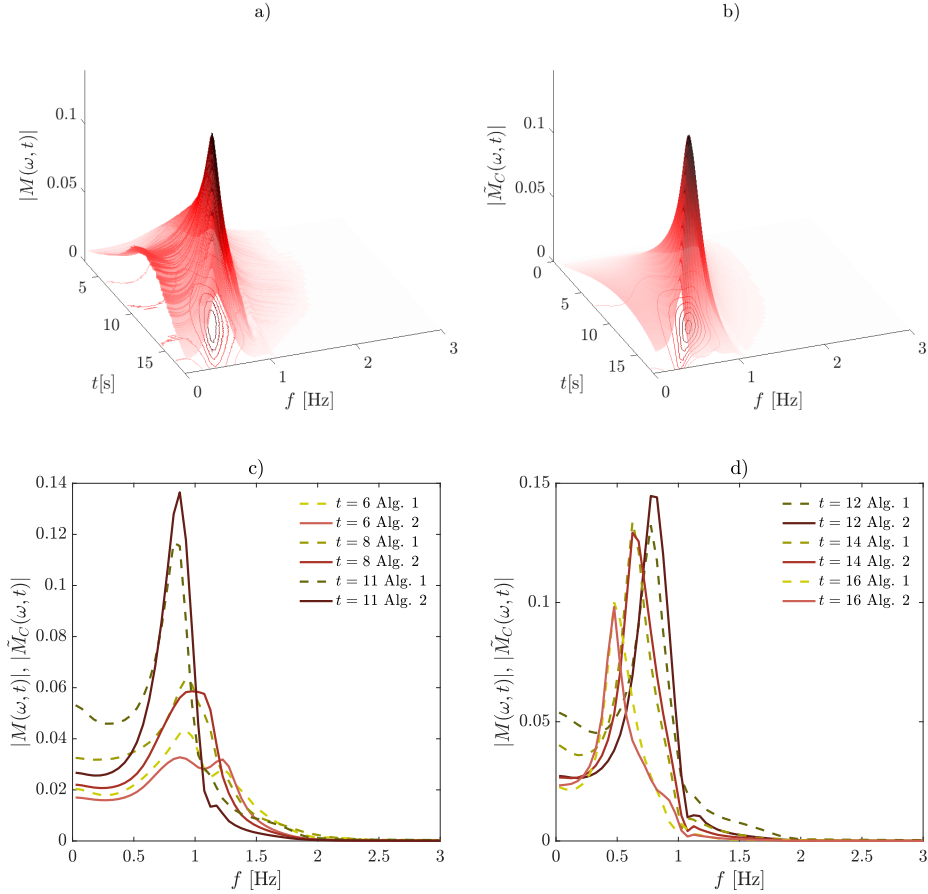


Figure 11. EFRFs for response to excitation $\hat{\mathbf{F}}_3(t)$ at threshold $\mathbf{x}_2 = 3.0\sigma_0$: a) Algorithm 1; b) Algorithm 2; c) and d) comparison of EFRFs for Algorithms 1 and 2 at selected times.