Good components of curves in projective spaces outside the Brill–Noether range

Edoardo BALLICO
Deptartment of Mathematics, University of Trento, Povo, Italy

Received: 29.11.2019 • Accepted/Published Online: 09.12.2020 • Final Version: 21.01.2021

Abstract: For all integers \( n, d, g \) such that \( n \geq 4 \), \( d \geq n + 1 \), and \( (n + 2)(d − n − 1) \geq n(g − 1) \), we define a good (i.e. generically smooth of dimension \( (n + 1)d + (3 − n)(g − 1) \) and with the expected number of moduli) irreducible component \( A(d, g; n) \) of the Hilbert scheme of smooth and nondegenerate curves in \( \mathbb{P}^n \) with degree \( d \) and genus \( g \). For most of these \( (d, g) \), we prove that a general \( X \in A(d, g; n) \) has maximal rank. We cover, in this way, a range of \((d, g, n)\) outside the Brill–Noether range.

Key words: Curve in projective spaces, normal bundle, Hilbert scheme, Hilbert function

1. Introduction
Let \( X \subset \mathbb{P}^n \) be any closed subscheme. We recall that \( X \) is said to have maximal rank if for all \( t \in \mathbb{N} \) the restriction map \( H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_X(t)) \) has maximal rank, i.e. it is either injective or surjective. Note that \( X \) has maximal rank if and only if for each \( t \in \mathbb{N} \) either \( h^0(\mathcal{I}_X(t)) = 0 \) or \( h^1(\mathcal{I}_X(t)) = 0 \). Assume that \( X \) has maximal rank. We know that \( h^0(\mathcal{O}_{\mathbb{P}^n}(t)) = \binom{n + t}{n} \). Hence, \( h^0(\mathcal{I}_X(t)) = \max\{0, \binom{n + t}{n} − h^0(\mathcal{O}_X(t))\} \) and \( h^1(\mathcal{I}_X(t)) = \max\{0, h^0(\mathcal{O}_X(t)) − \binom{n + t}{n}\} \). If \( X \) is a curve of degree \( d \) and genus \( g \) with maximal rank and if \( h^1(\mathcal{O}_X(t)) = 0 \), then \( h^0(\mathcal{I}_X(t)) = \max\{0, (\binom{n + t}{n} − td − 1 + g)\} \) and \( h^1(\mathcal{I}_X(t)) = \max\{0, td + 1 − g − \binom{n + t}{n}\} \). In this paper we always consider nondegenerate curves \( X \subset \mathbb{P}^n \) (hence with \( h^0(\mathcal{I}_X(1)) = 0 \)) with \( h^1(\mathcal{O}_X(2)) = 0 \) (and so with \( h^1(\mathcal{O}_X(t)) = 0 \) for all \( t \geq 2 \)). For these curves, if we know that \( X \) has maximal rank then we know its Hilbert function. In the range \( d < g + n \), i.e. in the range of degrees and genera not covered by [4–6], we always consider linearly normal curves, i.e. curves \( X \) with \( h^1(\mathcal{I}_X(1)) = 0 \).

The problem of the existence of curves in projective space with maximal rank and the related problem of the existence of good components of the Hilbert scheme of curves in \( \mathbb{P}^n \) have been considered in a huge number of papers ([1, 2, 4–15, 17, 18, 20–26, 28, 29, 31]). From these papers, it became clear that there is a fundamental difference between the case \( n = 3 \) and the case \( n > 3 \).

In this paper for all integers \( d, g, n \) such that \( n \geq 4 \), \( d \geq n + 1 \), \( g > 0 \) and \( (n + 2)(d − n − 1) \geq n(g − 1) \) we construct an irreducible component \( A(d, g; n) \) of the Hilbert scheme \( \text{Hilb}(\mathbb{P}^n) \) of \( \mathbb{P}^n \) whose general element is a smooth and nondegenerate curve \( X \subset \mathbb{P}^n \) with \( \text{deg}(X) = d \) and \( p_a(X) = g \). We prove that \( A(d, g; n) \) is generically smooth and with the expected dimension \( (n + 1)d + (n − 3)(1 − g) \) and that (if \( g \geq 2 \)) it has the expected number of moduli in the sense of Sernesi ([30]), i.e. the natural map \( A(d, g; n) \to \mathcal{M}_g \) is either

*Correspondence: ballico@science.unitn.it
2010 AMS Mathematics Subject Classification: 14H50

© TÜBİTAK
doi:10.3906/mat-1911-106

This work is licensed under a Creative Commons Attribution 4.0 International License.
dominant or its generic fiber has dimension $n^2 + 2n = \dim \text{Aut}(\mathbb{P}^n)$ (Proposition 3.1). For $d \geq g + n$ we just take as $A(d, g; n)$ the irreducible component of Hilb($\mathbb{P}^n$) with as general elements the nonspecial nondegenerate smooth curves of degree $d$ and genus $g$. In the next theorem, $N_X$ denotes the normal bundle of the smooth curve $X \subset \mathbb{P}^n$

**Theorem 1.1** Fix an integer $n \geq 4$ and a real number $\epsilon > 0$. Then there exists an integer $d(n, \epsilon)$ such that for all integers $d$, $g$ with $d \geq d(n, \epsilon)$ and $0 \leq g \leq (\frac{n+2}{n} - \epsilon)d$ there is a smooth, connected, and nondegenerate curve $X \subset \mathbb{P}^n$ with $\deg(X) = d$, $p_a(X) = g$, $h^1(N_X) = 0$, $h^1(O_X(2)) = 0$, $X \in A(d, g; n)$ and $X$ has maximal rank.

The fact that we get a nice curve with maximal rank in a specific component of Hilb($\mathbb{P}^n$) is (in our opinion) interesting, but it is also useful to prove the corresponding statement without the mention of the component. Indeed, to prove the existence of some $X \subset \mathbb{P}^n$ with degree $d$, genus $g$, $h^1(O_X(2)) = 0$ we will need (for a certain integer $k$, the critical value of the triple $(d, g, n))$ to prove the existence of some $X', X'' \in A(n, d; g)$ with degree $d$, genus $g$, $h^1(O_{X'}(2)) = h^1(O_{X''}(2)) = 0$, $h^0(I_{X'}(k - 1)) = 0$ and $h^1(I_{X''}(t)) = 0$ for all $t \geq k$. Since $\dim X'' = 1$, a standard exact sequence gives $h^1(I_{X''}(t)) = 0$ for all $t \geq 0$ and all $i \geq 2$. Since in all cases we need $k \geq 3$, the Castelnuovo–Mumford Lemma shows that to prove $h^1(I_X(t)) = 0$ for all $t \geq k$ it is sufficient to prove $h^1(I_X(k)) = 0$. Thus, if we know that $X, X'$, and $X''$ are in the same irreducible component of the Hilbert scheme of $\mathbb{P}^n$, then by the semicontinuity theorem for cohomology we have $h^0(I_X(t)) = 0$ for all $t < k$ and $h^1(I_X(t)) = 0$ for all $t \geq k$; hence, $X$ has maximal rank.

**Question 1.2** Describe the triples $(d, g, n)$, $n \geq 4$, such that there is a smooth, connected, and nondegenerate curve $X \subset \mathbb{P}^n$ such that $\deg(X) = d$, $p_a(X) = g$ and $h^1(N_X) = 0$.

**Question 1.3** Do we obtain the same triples as in Question 1.2 if we allow integral and nondegenerate curves with mild singularities (e.g., locally complete intersection singularities or nodal singularities)?

**Question 1.4** Are there (for many of the triples $(d, g, n)$ as in Theorem 1.1) other irreducible components of Hilb($\mathbb{P}^n$) whose general element is a smooth curve $X$ with maximal rank? And with the additional restriction that $h^1(O_X(2)) = 0$? Or with the restriction that $X$ is linearly normal?

In section 2 we define the component $A(d, g; n)$ and prove its existence. In Section 3 we prove many properties of $A(d, g; n)$. We prove that it has the expected number of moduli in the sense of Sernesi [30] (Proposition 3.1). We also prove some statements on the intersection of some elements of $A(d, g; n)$ with a hyperplane (in the spirit of [2, 4, 8, 10, 23, 26]) (see Section 4). The last 5 sections are devoted to the proof of Theorem 1.1, the last one containing the numerical lemmas used in the proof.

2. The definition and existence of the component $A(d, g; n)$

For any nodal curve $X \subset \mathbb{P}^n$, let $N_X$ denote its normal sheaf in $\mathbb{P}^n$. The sheaf $N_X$ is locally free, $\text{rank}(N_X) = n - 1$ and $\deg(N_X) = (n + 1)\deg(X) + (n - 3)\chi(O_X)$.

Let $A \subset \mathbb{P}^n$ be a reduced curve. A line $L \subset \mathbb{P}^n$ is said to be $k$-secant to $A$ if $|A \cap L| = k$, $\text{Sing}(A) \cap L = \emptyset$, and $L$ is not tangent to $A$. We give the definition of the component $A(d, g; n)$ of the Hilbert scheme Hilb($\mathbb{P}^n$) of $\mathbb{P}^n$ quoting inside the definition the lemmas used to prove that $A(d, g; n)$ is well-defined.
Definition 2.1 Fix an integer \( n \geq 3 \).

(a) For all integers \( g \geq 0 \) and \( d \geq g + n \), let \( A(d, g; n) \) be the irreducible component of \( \text{Hilb}(\mathbb{P}^n) \) whose general element is a smooth, connected, and nondegenerate curve \( X \subset \mathbb{P}^n \) such that \( \deg(X) = d \), \( p_a(X) = g \), and \( h^1(\mathcal{O}_X(1)) = 0 \).

(b) Fix an integer \( g \geq n + 1 \). Let \( A(g + n - 1, g; n) \) denote the irreducible component of \( \text{Hilb}(\mathbb{P}^n) \) whose general element is a smooth, connected, and nondegenerate curve \( X \subset \mathbb{P}^n \) such that \( \deg(X) = g + n - 1 \) and \( p_a(X) = g \), i.e. whose general element is a linear projection of a general canonically embedded smooth curve \( C \subset \mathbb{P}^{g-1} \) from \( g - n \) general points of \( C \).

(c) Assume \( g \geq n + 3 \) and fix an integer \( d \) such that \( d \leq g + n - 2 \) and

\[
(n + 2)(d - n - 1) \geq n(g - 1).
\] (2.1)

There are uniquely determined integers \( t, y, x \) such that \( g = 1 + t(n + 2) + y \), \( d = n + 1 + tn + x \) and \( 0 \leq y \leq n + 1 \).

By the inequality \( g \geq n + 3 \) and (2.1) \( t > 0 \) and \( x \geq y \). Let \( A(d - x, g - y; n) \) be the only component of \( \text{Hilb}(\mathbb{P}^n) \) containing the nodal curve \( K \cup D_1 \cup \cdots \cup D_t \), where \( K \subset \mathbb{P}^n \) is a linearly normal elliptic curve and \( D_1, \ldots, D_t \) are \( t \) general rational normal curves with \( \#(D_i \cap K) = n + 3 \) for all \( i \) (Lemma 2.6). Let \( A(d, g; n) \) be the irreducible component of \( \text{Hilb}(\mathbb{P}^n) \) containing the nodal curve \( Y \cup L_1 \cup \cdots \cup L_y \cup R_{y+1} \cup \cdots \cup R_{x-y} \), where \( Y \) is a general element of \( A(d - x, g - y; n) \), each \( L_i \) is a general \( 2 \)-secant line of \( Y \) and, if \( x > y \), each \( R_i \) is a general \( 1 \)-secant line of \( Y \) (Lemma 2.7).

Remark 2.2 Fix \( (d, g, n) \) for which Definition 2.1 defines \( A(d, g; n) \). In case (a) \( A(d, g; n) \) is the component of the Hilbert scheme of \( \mathbb{P}^n \) containing the nonspecial and nondegenerate smooth curves of degree \( d \) and genus \( g \). In case (b) \( A(d, g; n) \) is a linear projection of a canonically embedded smooth curve \( C \subset \mathbb{P}^{g-1} \) from \( g - n \) general points of \( C \). These are the triples \( (d, g, n) \) such that \( h^1(\mathcal{O}_X(1)) = 1 \) for a general \( X \in A(d, g; n) \). Case (c) covers all cases with \( h^1(\mathcal{O}_X(1)) \geq 2 \) for a general \( X \in A(d, g; n) \) for which we are able to prove that \( A(d, g; n) \) has good properties. Its definition using a linearly normal elliptic curve \( K \subset \mathbb{P}^n \) is given for its use in Remark 4.4.

Notation 2.3 For any hyperplane \( H \subset \mathbb{P}^n \), \( n \geq 4 \), we write \( A(d, g; H) \) instead of \( A(d, g; n - 1) \) to emphasize that all elements of \( A(d, g; H) \) are contained in \( H \).

A general element of \( A(d, g; H) \) is smooth. When \( n = 4 \) for \( A(d, g; H) \) we may take the irreducible component of \( \text{Hilb}(\mathbb{P}^3) \) defined in [6]. We only need \( A(d, g; H) \) when \( d \geq g + n - 1 \) and in this case \( A(d, g; H) \) is the irreducible component of \( \text{Hilb}(\mathbb{P}^{n-1}) \) whose general element is a nonspecial smooth curve spanning \( H \).

Among the extremal components \( A(d, g; n) \) (i.e. the ones for which \( A(d - 1, g; n) \) and \( A(d, g + 1; n) \) are not defined) there are the ones with \( d = 1 + n + tn \) and \( g = 1 + t(n + 2) \), \( t > 0 \), (resp. \( d = n + 1 + tn \) and \( g = t(n + 2) \), resp. \( d = 2n + tn \) and \( g = n + 1 + t(n + 2) \)) obtained from a smooth degree \( n + 1 \) curve of genus 1 (resp. a smooth rational curve of degree \( n + 1 \), resp. a canonically embedded smooth curve of degree \( 2n \) and genus \( n + 1 \)) adding \( t \) times an \((n + 3)\)-secant rational normal curve and applying each time Lemma 2.5.

Remark 2.4 Fix a finite set \( S \subset \mathbb{P}^n \), \( n \geq 3 \), such that \( |S| = n + 3 \) and \( S \) is in linear general position, i.e. any \( S' \subset S \) with \( |S'| = n + 1 \) spans \( \mathbb{P}^n \). Let \( D \) be the unique rational normal curve of \( \mathbb{P}^n \) containing \( S \). Since \( N_D \)
is a direct sum of \(n - 1\) line bundles of degree \(n + 2\), we have \(h^1(N_D) = 0\) and \(h^1(N_D(-S)) = 0\); hence, the restriction map \(H^0(N_D) \to H^0(N_D|_S)\) is surjective. Thus, for each rank \(n - 1\) vector bundle \(E\) on \(D\) with an injective map \(N_D \to E\) we have \(h^1(E(-S)) = 0\); hence, the restriction map \(H^0(E) \to H^0(E|_S)\) is surjective.

**Lemma 2.5** Let \(Y \subset \mathbb{P}^n\) be an integral and nondegenerate curve with \(\deg(Y) \neq n\). Fix \(S \subset Y_{\text{reg}}\) such that \(|S| = n + 3\) and \(S\) in linear general position. Let \(D\) be the only rational normal curve containing \(S\). Assume \(Y \cap D = S\) and that \(X := Y \cup D\) is nodal. Then \(X\) is a connected nodal curve, \(\deg(X) = \deg(Y) + n\), \(p_a(X) = p_a(Y) + n + 2\), \(h^1(N_X) = 0\) and \(X\) is smoothable.

**Proof** By assumption the curve \(X\) is connected, nodal and \(p_a(X) = p_a(Y) + n + 2\). Thus, \(N_X\) is a rank \(n - 1\) vector bundle on \(X\) with degree \((n + 1)\deg(X) + 2 - 2p_a(X)\). By [16, \S 2] the vector bundle \(N_{X|Y}\) on \(Y\) is obtained from \(N_Y\) making \(n + 3\) positive elementary transformations. Thus, \(h^1(N_{X|Y}) \leq h^1(N_Y) = 0\). By [16, \S 2] the vector bundle \(N_{X|D}\) on \(D\) is obtained from \(N_D\) making \(n + 3\) positive elementary transformations. By Remark 2.4 we have \(h^1(N_{X|Y}) = 0\) and the restriction map \(H^0(N_{X|D}) \to H^0(N_{X|S})\) is surjective. Thus, the Mayer–Vietoris exact sequence

\[
0 \to N_X \to N_{X|Y} \oplus N_{X|D} \to N_{X|S} \to 0
\]  

(2.2)
gives \(h^1(N_X) = 0\). Since \(h^1(N_Y) = 0\), we have \(h^1(F) = 0\) for each vector bundle \(F\) on \(Y\) obtained from \(N_Y\) making \(n + 2\) positive elementary transformations. Since \(h^1(N_D(-S)) = 0\) (Remark 2.4), we have \(h^1(A) = 0\) for each vector bundle \(A\) on \(D\) obtained from \(N_{X|D}\) making \(\# S\) negative elementary transformation, one for each \(q \in S\) (with the language of [2] for each \(q \in S\) take \(q \in T_qY \setminus \{q\}\); the relevant vector bundle is \(N_D[q_1 \to p_{q_1}] \cdots [q_{n+3} \to p_{q_{n+3}}]\), where \(S = \{q_1, \ldots, q_{n+3}\}\). Thus, \(X\) is smoothable. \(\square\)

**Lemma 2.6** Take a smooth \(Y \in A(d, g; n)\) with \(h^1(N_Y) = 0\) and a rational normal curve \(D \subset \mathbb{P}^n\) such that \(Y \cup D\) is nodal and \(1 \leq |Y \cap D| \leq n + 3\). Then \(Y \cup D \in A(d + n, g + |D \cap Y| - 1; n)\).

**Proof** As in the proof of Lemma 2.5 we get \(h^1(N_{Y \cup D}) = 0\). Thus, we may assume that \(Y\) is a general element of \(A(d, g; n)\). Set \(x := |D \cap Y|\).

(a) Assume for the moment \(d \leq g + n - 2\). It is easy to check that \(h^1(N_{Y \cup D}) = 0\) and so \(Y \cup D\) is a smooth point of the Hilbert scheme. Thus, it is sufficient to do it for one \(D\) to get it for a general \(D\) intersecting \(Y\) with the prescribed cardinality. We degenerate \(Y\) to a defining curve \(Y_1 \cup D_1 \cup \cdots \cup D_i \cup R_1 \cup \cdots \cup R_a \cup L_1 \cup \cdots \cup L_b\) of \(A(d, g; n)\) with \(Y_1\) a linearly normal elliptic curve. The case \(|D \cap Y| = n + 3\) is done taking a rational normal curve \(D_{i+1}\) with \(|D_{i+1} \cap Y_1| = n + 3\). Thus, it is sufficient to do the cases with \(1 \leq |D \cap Y| \leq n + 2\). By the case just done it is sufficient to do it for the curve \(Y_1 \cup R_1 \cup \cdots \cup R_a \cup L_1 \cup \cdots \cup L_b\). If either \(x \leq n + 1\) or \(b > 0\), then we see with another degeneration that we land in a nonspecial case. Now assume \(b = 0\) and \(x = n + 2\). It is sufficient to show that \(Y_1 \cup D \in A(2n, n + 2; n)\). This is obvious, because \(h^1(N_{Y_1 \cup D}) = 0\), \(Y_1 \cup D\) is smoothable and in this range the Hilbert scheme of smooth and nondegenerate curves is irreducible.

(b) Now assume \(d = g + n - 1\) and \(d \geq 2n\).

(b1) If \(x \leq n + 1\) we land again in a smoothable curve with \(h^1(O_{Y \cup D}(1)) = 1\).

(b2) Assume \(x = n + 2\). We degenerate \(Y \cup D\) to a nodal curve \(Y_2 \cup L_1 \cup \cdots \cup L_{g-n-1} \cup D_1\) with \(Y_2\) a canonically embedded curve of degree \(2n\) and genus \(n + 1\), \(D_1\) a rational normal curve, \(#(Y_2 \cap D_1) = n + 3\) and
The nodal curve $Y_1 \cup D_1$ is smoothable and $h^1(N_{Y_1 \cup D_1}) = 0$ (Lemma 2.5). In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible.

(b3) Assume $r = n + 3$. We degenerate $Y \cup D$ to a nodal curve $Y_2 \cup L_1 \cup \cdots \cup L_{g-n-1} \cup D_1$ with $Y_2$ a canonically embedded curve of degree $2n$ and genus $n + 1$, $D_1$ a rational normal curve, $(Y_2 \cap D_1) = n + 3$ and $L_1, \ldots, L_{g-n-1}$ general secant lines of $Y_2$. By step (n) it is sufficient to prove that $Y_1 \cup D_1 \in A(3n, 2n + 3; n)$. The nodal curve $Y_1 \cup D_1$ is smoothable and $h^1(N_{Y_1 \cup D_1}) = 0$ (Lemma 2.5). In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible.

(c) Now assume $d \geq g + n$. If $(Y \cap D) \leq n + 1$, then $h^1(O_{Y \cup D}(1)) = 0$, because $h^1(O_Y(1)) = 0$ by the generality of $Y$. Now assume $(Y \cap D) = n + 2$. We get $h^1(O_{Y \cup D}(1)) \leq 1$. Since any $n + 3$ points in linear general position in $\mathbb{P}^n$ are contained in a unique rational normal curve, we may degenerate $Y \cup D$ to the nodal curve $E := Y_1 \cup D_1 \cup L_1 \cup \cdots \cup L_g \cup R_1 \cup \cdots \cup R_{d-g-n}$, where $Y_1$ and $D_1$ are rational normal curves, $(Y_1 \cap D_1) = n + 2$, each $L_i$ is a 2-secant line of $Y_1$, and each $R_i$ is a 1-secant line of $Y_1$. Since $Y_1 \cup D_1$ is a limit of canonically embedded curves, we reduce to a case $(d', g')$ with $d' = g' + n - 1$ and $d \geq 2n$ done in step (b). Now assume $(Y \cap D) = n + 3$. Assume $g > 0$. We degenerate $Y \cup D$ to the nodal curve $K \cup D_1 \cup L_1 \cup \cdots \cup L_{g-1} \cup R_1 \cup \cdots \cup R_{d-g}$, where $K$ is a linearly normal elliptic curve, $D_1$ is a rational normal curve, $(K \cap Y_1) = n + 3$, each $L_i$ is a general 2-secant line of $K$ and each $R_i$ is a general 1-secant line of $K$.

We apply the case $t = 1$ part (c) of Definition 2.1 and then we add the 2-secant and 1-secant lines (a case fully proved). Now assume $g = 0$. Since $(Y \cap D) = n + 3$ and $D$ is a rational normal curve, deg$(Y) > n$. As above it is sufficient to do the case $d = n + 1$. We degenerate $Y \cup D$ to the nodal curve $C \cup D_1 \cup L$, where $C$ and $D$ are rational normal curves, $(C \cap D_1) = n + 2$ and $L$ is a line intersecting both $C$ and $D_1$ at a unique point, which are not in $C \cap D_1$. In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible.

Lemma 2.7 Fix integers $d, g, n$ such that $A(d, g; n)$ is defined. Fix integers $a > 0$ and $1 \leq b \leq a + 1$. Take a smooth $Y \in A(d, g; n)$ such that $h^1(N_Y) = 0$ and a smooth rational curve $D$ with $Y \cup D$ nodal, deg$(D) = a$ and $|Y \cap D| = b$. Then $h^1(N_{Y \cup D}) = 0$ and $Y \cup D \in A(d + a, g + b - 1; n)$.

Proof The assertions that $h^1(N_{Y \cup D}) = 0$ and that $Y \cup D$ is smoothable are well-known ([2], [16, Theorem 4.1, Remark 4.1.1]) and easier than the proof of Lemma 2.5. To prove that $Y \cup D \in A(d + a, g + b - 1; n)$ we may assume (moving if necessary $D$) that $Y$ is general in $A(d, g; n)$ and that $D$ is a general rational curve of degree $a$ intersecting $Y$ at $b$ points and quasitransversally (the set of all such $D$’s is an irreducible variety, because $a \geq b - 1$). We distinguish the following cases.

First assume $d \geq g + n$. We have $h^1(O_Y(1)) = 0$. A Mayer–Vietoris exact sequence gives $h^1(O_{Y \cup D}(1)) = 0$; hence (since $Y \cup D$ is smoothable), $Y \cup D \in A(d + a, g + b - 1; n)$.

Now assume $d = g + n - 1$. Let $C \subset \mathbb{P}^{g-1}$ be a general canonically embedded curve and $T \subset \mathbb{P}^{g-1}$ a general degree $a$ smooth rational curve such that $C \cup T$ is nodal and $|C \cap T| = b$. Since $h^1(N_C) = 0$ and $T$ is a rational normal curve in its linear span, it is easy to check that $h^1(N_{C \cup T}) = 0$ and that $C \cup T$ is smoothable (to a nonspecial curve if $b \leq a$, to a curve $E$ with $h^1(O_E(1)) = 1$ if $b = a + 1$). Then we use a family of inner projections in the fiber of this smoothing to get that $Y \cup D \in A(d + a, g + b - 1; n)$.

Now assume $d \leq g + n - 2$. Take a general $Y_1 \in A(d - n, g - n - 2; n)$. Let $D_2$ be a general rational
normal curve such that \( Y_1 \cup D_2 \) is nodal and \(|D_2 \cap Y_1| = n + 3\). By the definition of \( A(d,g;n) \) we may deform \( Y_1 \cup D_2 \) to the general \( Y \). By induction on \( g \) we have \( Y_1 \cup D_1 \in A(d - n + a, g - n + b; n) \), where \( D_1 \) is a general rational curve with \( Y_1 \cup D_1 \) nodal and \(|Y_1 \cap D_1| = b\). In the deformation \( \beta : X \to \Delta \) of \( Y_1 \cup D_2 \) to \( Y \) (i.e. with \( \Delta \) irreducible and \( Y \) and \( Y_1 \cup D_2 \) fibers of \( \beta \), say \( Y_1 \cup D_2 = \beta^{-1}(o) \)) as we may find (up to a covering of \( \Delta \)) \( b \) sections \( s_1, \ldots, s_b \) of \( \beta \) such that \(|s_1(o), \ldots, s_b(o)| = Y_1 \cap D_1\). In this way in the deformation of \( Y_1 \cup D_2 \) to \( Y \) we get a deformation of \( Y_1 \cup D_1 \cup D_2 \) to \( Y \). We have \( h^1(N_{Y_1 \cup D_1 \cup D_2}) = 0 \). Since \( Y_1 \cup D_1 \in A(d - n + a, g - n + 2 + b - 1; n) \) by the inductive assumption, we get \( Y_1 \cup D_1 \cup D_2 \in A(d + a, g + b - 1; n) \). 

**Lemma 2.8** Fix integers \( d, g, n \) such that \( A(d,g;n) \) is defined. Fix an integer \( b \in \{ n + 2, n + 3 \} \). If \( b = n + 3 \) assume \( d > n \). Fix a smooth \( Y \in A(d,g;n) \) such that \( h^1(N_Y) = 0 \). Let \( D \subset \mathbb{P}^n \) be a rational normal curve such that \( Y \cup D \) is nodal and \(|Y \cap D| = b\). Then \( h^1(N_{Y \cup D}) = 0 \) and \( Y \cup D \in A(d + n, g + b - 1; n) \).

**Proof** As in Lemma 2.5 we see that \( h^1(N_{Y \cup D}) = 0 \) and that \( Y \cup D \) is smoothable. We use induction on the integer \( d \), the starting point of the induction being the case \((d,g) = (n,0)\) when \( b = n + 2 \) and the cases \((d,g) = (n + 1,0)\) and \((d,g) = (n + 1,1)\) when \( b = n + 3 \). When \( b = n + 3 \) to start the induction it is sufficient to use the definition of the varieties \( A(x,y;n) \) when \( x \leq y - 2 \). If \((d,g) = (n,0)\) and \( b = n + 2 \) we use that a general union of 2 rational normal curves with \( n + 2 \) common points is a flat limit of canonically embedded smooth curves of \( \mathbb{P}^n \).

(a) Assume \( d > g + n \). If \( b = n + 3 \), we may assume \((d,g) \neq (n + 1,0)\), since we did the case \((d,g) = (n + 1,0, n + 3)\) as an initial case. We degenerate \( Y \) to \( Y_1 \cup L \) with \( Y_1 \) a general element of \( A(d - 1, g; n) \) and \( L \) a general 1-secant line of \( Y_1 \). We add to \( Y_1 \) a general rational normal curve containing \( b \) points of \( Y_1 \) and then we apply the case \( a = b = 1 \) of Lemma 2.7 to \( A(d + n - 1, g + b - 1; n) \).

(b) Assume \( d = g + n \). The case \( b = n + 3 \) follows from the definition of the varieties \( A(x,y;n) \); hence, we may assume \( b = n + 2 \). If \( g = 0 \), i.e. if \((d,g) = (n,0)\) we use again the starting case of the induction. If \( g > 0 \) we degenerate \( Y \) to \( Y_1 \cup L \) with \( Y_1 \) a general element of \( A(d - 1, g - 1; n) \), \( L \) a general 2-secant line of \( Y_1 \), then we add a general rational normal curve containing \( b \) points of \( Y_1 \) and then apply Lemma 2.7 with \( a = 1 \) and \( b = 2 \).

(c) Assume \( d = g + n - 1 \); hence, \( g \geq n + 1 \). Since the case \( b = n + 3 \) follows from the definition of \( A(g + 2n - 1, g + n + 2; n) \), it is sufficient to do the case \( b = n + 2 \).

First assume \( g = n + 1 \), i.e. assume that \( Y \) is canonically embedded. We have \( h^1(N_{Y \cup D}) = 0 \) and \( Y \cup D \) is smoothable ([10, Lemma 2.3]); these facts are easier than the proof of Lemma 2.5.

If \( g \geq n + 2 \), we degenerate \( Y \) to \( Y_1 \cup L \) with \( Y_1 \) of degree \( d - 1 \) and genus \( g - 1 \) and \( L \) a 2-secant line of \( Y \).

(d) Now assume \( d \leq g + n - 2 \). By the definition of \( A(d+n, g + n + 2; n) \) we conclude when \( b = n + 3 \). Now assume \( b = n + 2 \). Take a general \( Y_1 \in A(d - n, g - n - 2; n) \). Let \( D_2 \) be a general rational normal curve such that \( Y_1 \cup D_2 \) is nodal and \(|D_2 \cap Y_1| = n + 3 \). By the definition of \( A(d,g;n) \) we may deform \( Y_1 \cup D_2 \) to the general \( Y \). By induction on \( g \) we have \( Y_1 \cup D_1 \in A(d,g - 1) \), where \( D_1 \) is a general rational normal curve with \( Y_1 \cup D_1 \) nodal and \(|Y_1 \cap D_1| = n + 2 \). In the deformation \( \beta : X \to \Delta \) of \( Y_1 \cup D_2 \) to \( Y \) (with \( \beta^{-1}(o) = Y_1 \cup D_1 \) for some \( o \in \Delta \)) we may find (up to a covering) \( b \) sections \( s_1, \ldots, s_b \) of \( \beta \) such that \(|s_1(o), \ldots, s_b(o)| = Y_1 \cap D_1\). In this way in the deformation of \( Y_1 \cup D_2 \) to \( Y \) we get (after a finite covering of the base of the deformation) a deformation of \( Y_1 \cup D_1 \cup D_2 \) to \( Y \). We have \( h^1(N_{Y_1 \cup D_1 \cup D_2}) = 0 \). Since \( Y_1 \cup D_1 \in A(d,g-1; n) \) by the
Lemma 2.9 Take $Y \in A(x, y; n)$ and a nonspecial curve $D \subset H$ meeting quasitransversally and at a unique point. Set $z := \deg(D)$ and $w := p_n(D)$. Then $Y \cap D \in A(x + z, y + w; n)$.

Proof We degenerate $D$ to a union $D_1 \cup \cdots \cup D_r \subset H$ of smooth rational curves such that from $D_1 \cup \cdots \cup D_i$ to $D_1 \cup \cdots \cup D_i \cup D_{i+1}$ we may use either Lemma 2.7 or Lemma 2.8 in $H$. Then we apply $k$ times Lemmas 2.7 or 2.8 first to $Y \cup D_1$ and then adding each time a curve $D_i$.

3. The right number of moduli in the sense of Sernesi
We adapt the proof of [10, Proposition 3.1] to prove the following result.

Proposition 3.1 The irreducible component $A(d, g; n)$, $g \geq 2$, of $\text{Hilb}(\mathbb{P}^n)$ has the expected number of moduli.

Proof If $d \geq g + n - 1$ we use that in these cases there is a unique irreducible component of $\text{Hilb}(\mathbb{P}^n)$ which dominates $\mathcal{M}_g$. Now assume $d \leq g + n - 2$. Set $r := \rho(g, n, d) := (n + 1)d - ng - n(n + 1)$ (the Brill–Noether number). Let $A'(d, g; n) := \{X \in A(d, g; n) \mid X$ is nodal and semistable} and let $p_{d,g} : A(d, g; n) \rightarrow \overline{\mathcal{M}}_g$ be the moduli map.

(a) First assume $r \geq 0$. Let $x$ be the only integer such that $d = n + r + xn$ and $g = r + x(n + 1)$. Since $d < g + n$, we have $x > 0$. We have $\rho(g, d, n) = \rho(g - n - 1, d - n, n)$. By induction on $x$ starting with the case $x = 0$ we may assume that $p_{d-n,g-n-1}$ is dominant. We want to prove that the general fibers of $p_{d-n,g-n-1}$ have the same dimension. Take a general $C \in A(d-n, g-n-1; n)$. In particular $C$ is smooth and $h^1(N_C) = 0$. Take a general $B \subset C$ with $|B| = n + 2$ and let $D$ be a general rational normal curve containing $C$. By Lemma 2.8 the curve $C \cup D$ is nodal, $p_n(C \cup D) = g$ and $C \cup D \in A(d, g; n)$. It is sufficient to prove that the fiber $F$ of $p_{d-n,g-n-1}$ containing $C$ has the same dimension as the fiber $F'$ of $p_{d,g}$ containing $C \cup D$. Let $G$ be the set of all rational normal curves containing $B$. We have $\dim G = n - 1$ and $G$ is an irreducible variety. Fix any ordering of the set $B = \{q_1, \ldots, q_{n+2}\}$. To show that $\dim_{[C]} F = \dim_{[C \cup D]} F'$ it is sufficient to check that there are only finitely many $D', D'' \in G$ such that $p_{d,g}(C \cup D') = p_{d,g}(C \cup D'')$. It is sufficient to show that if $h : D \rightarrow D'$ is an isomorphism, then $h$ is uniquely determined by the element of $S_{n+2}$ induced by the permutation $\sigma$ such that $\sigma(i)$ is the only element of $\{1, \ldots, n+2\}$ such that $q_{\sigma(i)} = h(q_i)$. Since $D, D'$ are rational normal curves, $h$ is induced by an element $h'$ of $\text{Aut}(\mathbb{P}^n)$. The projective transformation $h'$ fixes the set $B$ in linear general position and with $|B| = n + 2$. Thus, $h'$ is uniquely determined.

(b) Now assume $r < 0$. We need to prove that each irreducible component of a general fiber of $p_{d,g}$ parametrizes projectively equivalent elements of $A(d, g; n)$, i.e. (since in this range a general element of $A(d, g; n)$ has finitely many automorphisms) it is sufficient to prove that a general fiber of $p_{d,g}$ has dimension $n^2 + 2n$. We use induction on the integer $d$. Since $r < 0$, by the definition of $A(d, g; n)$ there is an integer $t > 0$ and a pair $(d', g')$ such that $d = d' + tn$, $g = g' + t(n + 2)$ and either $d' \geq g' + n$ or $g' \geq n + 1$ and $d' \geq g + n - 1$. Hence, $A(d-n, g-n-2; n)$ is defined. By the inductive assumption the irreducible component $\Gamma$ of the fiber of $p_{d-n,g-n-2}$ over $p_{d-n,g-n-2}(Y)$ has dimension $\max\{n^2 + 2n, r + 3n + n^2\}$. We fix a general $S \subset Y$ with $|S| = n + 3$ and let $D \subset \mathbb{P}^n$ be the only rational normal curve containing $S$. Moving $Y$ among the elements of $A(d-n, g-n-2)$ containing $S$ we may assume that $Y \cup D$ is nodal and $Y \cap D = S$. Thus,
Y ∪ D ∈ A(d, g; n). Since |S| ≥ 3, Y ∪ D is stable. Take a general element W of an irreducible component Γ of \(\mathcal{P}_{d,g}(\mathbb{P}^n, Y ∪ D)\) containing Y ∪ D. W is a 2-component nodal curve, say W = W_1 ∪ W_2 with W_1, W_2 smooth, W_1 of genus \(g − n − 2\), W_2 of genus \(0\) and |W_1 ∩ W_2| = n + 3. Assume \(\dim Γ > 0.\) Since Y ∪ D is a limit of a family of curves like W_1 ∪ W_2, we get \(\deg(W_1) = d − n, \ \deg(W_2) = n\) and W_1 ∩ W_2 in linear general position, there is a nonempty open subset Γ′ of Γ such that all W′ ∈ Γ′ are nodal 2-component curves, say W′ = A ∪ B with A, B smooth, A of genus \(g − n − 2\) and isomorphic to Y as an abstract curve, B rational and |A ∩ B| = n + 3. Since h^1(N_Y) = 0, Y is a smooth point of \(\text{Hilb}(\mathbb{P}^n)\). Hence, all A appearing as a nonrational normal curve of an element of Γ′ are in \(A(d − n, g − n − 2; n)\). Since S is in linear general position A ∩ B is in linear general position for a general A ∪ B ∈ Γ′. If \(r ≤ −n\) we get that \(\dim Γ′ = n^2 + 2n\).

From now on we assume \(r ≥ −n + 1\).

Assume for the moment \(g − n − 2 ≥ 2\). For a fixed (but general) \(Y ∈ \mathcal{M}_{g−n−2}\) (seen as an abstract curve) there are \(∞ + n\) degree \(d − n\) nondegenerate embeddings of Y in \(\mathbb{P}^n\), we have \(\deg(W_2) ≥ n\). Call \(\{u_t\}, \ t ∈ Λ\) the family of embeddings and for each \(t ∈ Λ\) call B_t the unique rational normal curve containing u_t(S). Note that B_t is uniquely determined by the set B_t ∩ u_t(Y) with cardinality \(n + 3\). We need to prove that for each \(t ∈ Λ\) the set of all \(s ∈ Λ\) such that \(u_s(Y) ∩ B_s ∼ u_t(Y) ∩ B_t\) (as abstract curves) have finitely many orbits for the action of \(\text{Aut}(\mathbb{P}^n)\). Since \(g − n − 2 ≥ 2\), \(\text{Aut}(Y)\) is finite and there are only finitely many \(S_t′ ∈ Y\), such that \((Y, S_t′)\) and \((Y, S_t′)\) give the same element of \(\mathcal{M}_{g−n−2,n+3}\). Consider the forgetful map \(φ : \mathcal{M}_{g,n−2,n+3} → \mathcal{M}_{g−n−2}\) and set \(Δ := φ^{-1}(Y)\). For a general \(S ∈ Y\) we get an \((n + r)\)-dimensional family \(\{u_t(S)\}_{t ∈ Λ'}\) of subschemes of \(\mathbb{P}^n\) and, taking a Zariski dense open subset Λ′ of Λ instead of Λ, we may assume that each set \(\{u_t(S)\}_{t ∈ Λ'}\) has cardinality \(n + 3\) and it is in linear general position in \(\mathbb{P}^n\).

Now assume \(g − n − 2 ≤ 1\). Since \(ρ(d − n, g − n − 2, n) ≤ n − 1\) and \(d > n\), we have either \(g − n − 2 ≥ 2\) or \((d − n, g − n − 2) = (n + 1, 1)\). Ordering the points \(q_1, \ldots, q_{n+3}\) we may see \((Y, S)\) as an element of \(\mathcal{M}_{1,n+3}\).

We may use \((Y, q_1)\) as an element of the moduli space \(\mathcal{M}_{1,1}\).

\[\square\]

4. Intersection with a hyperplane

**Lemma 4.1** There is a smooth linearly normal elliptic curve \(Y ⊂ \mathbb{P}^n, \ n ≥ 3\), and a rational normal curve \(D ⊂ \mathbb{P}^n\) such that |D ∩ Y| = n + 3 and Y ∪ D is nodal.

**Proof** First assume \(n\) is odd. Set \(e := (n − 1)/2\). The line bundle \(O_{\mathbb{P}^1} \times \mathbb{P}^1(1, e)\) is very ample and \(h^0(O_{\mathbb{P}^1} \times \mathbb{P}^1(1, e)) = 2e + 2 = n + 1\). Let \(j : \mathbb{P}^1 × \mathbb{P}^1 → \mathbb{P}^n\) denote the linearly normal embedding induced by the complete linear system \(|O_{\mathbb{P}^1} \times \mathbb{P}^1(1, e)|\). Fix a general \(A ∈ |O_{\mathbb{P}^1} \times \mathbb{P}^1(1, e + 1)|\) and a general \(B ∈ |O_{\mathbb{P}^1} \times \mathbb{P}^1(2, 2)|\). A is a smooth rational curve and B is a smooth elliptic curve. For a general \((A, B)\) the curve A ∪ B is nodal and \(|A ∩ B| = O_{\mathbb{P}^1} \times \mathbb{P}^1(1, e + 1) · O_{\mathbb{P}^1} \times \mathbb{P}^1(2, 2) = 2 + 2e + 2 = n + 3\). Take \(Y := j(B)\) and \(D := j(A)\).

Now assume \(n ≥ 4\) even and set \(a = n/2 ≥ 2\). Let \(F_1\) be the Hirzebruch surface with a minimal self-intersection curve \(h\) with \(h^2 = −1\). Take as a \(Z\)-basis of Pic(\(F_1\)) the curve \(h\) and a fiber \(f\) of the ruling of \(F_1\). The line bundle \(O_{F_1}(h + af)\) is very ample and \(h^0(O_{F_1}(h + af)) = 2a + 1 = n + 1\). Let \(u : F_1 → \mathbb{P}^n\) be the linearly normal embedding induced by the complete linear system \(|O_{F_1}(h + af)|\). The adjunction formula gives \(ω_{F_1} ∼ O_{F_1}(−2h − 3f)\). Fix a general \(E ∈ |O_{F_1}(h + (a + 1)f)|\) and a general \(F ∈ |O_{F_1}(2h + 3f)|\). The adjunction formula gives that \(E\) is a smooth rational curve and that \(F\) is an elliptic curve. For a general \((E, F)\), the curve \(E ∪ F\) is nodal and \(|E ∩ F| = O_{F_1}(h + (a + 1)f) · O_{F_1}(2h + 3f) = −2 + 2a + 2 + 3 = n + 3\).
Take $Y := u(F)$ and $D := u(E)$. 

Lemma 4.2 For every positive integer $t > 0$ there are a smooth linearly normal elliptic curve $Y \subset \mathbb{P}^n$, $n \geq 3$, and rational normal curves $D_1, \ldots, D_t \subset \mathbb{P}^n$ such that $D_i \cap D_j = \emptyset$ for all $i \neq j$, $|Y \cap D_i| = n + 3$ for all $i$ and $Y \cup D_1 \cup \cdots \cup D_t$ is nodal.

Proof The case $t = 1$ is true by Lemma 4.1, which gives the existence of a linear normal elliptic curve $Y$ such that for a general $S \subset Y$ with $|S| = n + 3$ the only smooth rational normal curve $D_S$ containing $S$ is quasi-transversal to $Y$ and $Y \cap D_S = S$. We only need to prove that for a general $(A, B) \subset Y \times Y$ with $|A| = |B| = n + 3$ we have $D_A \cap D_B = \emptyset$. Call $S$ the set of all rational normal curves $D \subset \mathbb{P}^n$ intersecting quasi-transversally $Y$ and with $|Y \cap D| = n + 3$. We just observed that $S \neq \emptyset$. Since $\dim Y = 1$, $Y$ is irreducible, a general $A \subset Y$ with $|A| = n + 3$ is in linear general position and any $n + 3$ points of $\mathbb{P}^n$ in linear general position are contained in a unique rational normal curve, $S$ is an irreducible variety of dimension $n + 3$. Thus, we only need to prove that $D \cap D' = \emptyset$ for a general $(D, D') \in S \times S$.

Claim 1: $\bigcup_{D \in S} D$ is dense in $\mathbb{P}^n$.

Proof of Claim 1: We use induction on $n$ starting the induction with the case $n = 2$ in which the result is obvious. Assume $n > 2$. Fix a general $(a, b) \in Y \times \mathbb{P}^n$ and call $\ell_a : \mathbb{P}^n \setminus \{a\} \to \mathbb{P}^{n-1}$ the linear projection from $a$. Let $Y_a$ be the closure of $\ell_a(Y \setminus \{a\})$ in $\mathbb{P}^{n-1}$. The curve $Y_a$ is a linearly normal elliptic curve of $\mathbb{P}^{n-1}$. Fix an open neighborhood $\mathcal{U}$ of $b$ in $\mathbb{P}^n$ such that $a \notin \mathcal{U}$ and call $\mathcal{V} \subset \mathbb{P}^{n-1}$ an open neighborhood of $\ell_a(b)$ contained in the dense set $\ell_a(\mathcal{U})$. Restricting if necessary $\mathcal{U}$ we may assume $\mathcal{U} = \ell_a^{-1}(\mathcal{V})$. By the inductive assumption there is a rational normal curve $D' \subset \mathbb{P}^{n-1}$ such that $|D' \cap Y_a| = n + 2$, $D'$ intersects transversally $Y_a$ and $\mathcal{V} \cap D' \neq \emptyset$. Let $T$ be the cone of $\mathbb{P}^n$ with vertex $a$ and base $D'$. For a general $(b, D')$ we may assume that $T$ is quasi-transversal to $Y$ outside $a$. Thus, it is sufficient to find a rational normal curve $D \subset \mathbb{P}^n$ such that $a \in D$ and $\ell_a(D \setminus \{a\}) \subset D'$. Let $\pi : W \to T$ denote minimal desingularization of $T$. The surface $W$ is isomorphic to the Hirzebruch surface $F_{n-1}$ and $\pi$ is induced by the complete linear system $|O_{F_{n-1}}(h + (n-1)f)|$, where $h = \pi^{-1}(a)$ and $f$ is a fiber of the ruling of $F_{n-1}$. A general $K \in |O_{F_{n-1}}(h+nf)|$ is smooth and rational and $\pi(K)$ is a rational normal curve. Take $D := \pi(K)$.

Fix a large integer $k \gg 0$. Since $Y$ is a curve, its Hilbert polynomial has degree 1. Thus, for large $k$ we have $h^0(I_Y(k)) \geq (n+k) - k^2 > (kn-n-1)(kn-n-2)/2$. Set $x := kn-n-2$ and take a general $(D_1, \ldots, D_x) \in S^x$. Since $(D_1, \cup D_x)$ is general, we have $|Y \cup D_1 \cup \cdots \cup D_x| \geq n+3+i$ for $i = 1, \ldots, x-1$. Since $h^0(O_{P^1}(t)(-Z)) = \max\{0, t+1-\deg(Z)\}$ for any finite set $Z \subset P^1$ and $\deg(O_{D_{i+1}}(k)) = kn+1$, we have $h^0(I_{Y \cup \cdots \cup D_{i+1}}(k)) \geq h^0(I_{Y \cup \cdots \cup D_i}(k)) - kn - n - 3 - i$ for all $i = 1, \ldots, x-1$. We get that a general $D \subset S$ is in the base locus of $|I_{Y \cup D_1 \cup \cdots \cup D_x}(k)|$. Claim 1 implies $h^0(I_{Y \cup D_1 \cup \cdots \cup D_x}(k)) = 0$. Thus, $h^0(I_Y(k)) \leq (kn-n-1)(kn-n-2)/2$, a contradiction. 

Remark 4.3 Let $S \subset \mathbb{P}^n$, $n \geq 2$, be a subset in linear general position and with $|S| = n + 3$. It is well-known that $S$ is contained in a unique rational normal curve of $\mathbb{P}^n$. This is obvious for $n = 2$, while if $n > 2$ the rational normal curve $C$ and its uniqueness is obtained in the following way. Fix $o \in S$ and a hyperplane $M \subset \mathbb{P}^n$ such that $o \notin M$. Set $S' := S \setminus \{o\}$ and $B := \ell_o(S')$. Since $S$ is in linear general position, the set $B$ is a finite subset of $M$ with cardinality $n + 2$ and in linear general position in $M$. By the inductive assumption there is a unique rational normal curve $D$ of $M$ containing $B$. Let $T \subset \mathbb{P}^n$ be the cone with
vertex $o$ and base $D$. It is obvious that if $C$ exists, then $C \subset T$. The minimal desingularization of $T$ is isomorphic to the Hirzebruch surface $F_{n-1}$, i.e. (calling $h$ the section of the ruling of $F_{n-1}$ and $f$ a fiber of its ruling) the complete linear system $|O_{F_{n-1}}(h + (n - 1)f)|$ has no base points and it induces a birational morphism $\pi : F_{n-1} \to T$ with $\pi(h) = \{o\}$ and $\pi$ inducing an isomorphism between $F_{n-1} \setminus h$ and $T \setminus \{o\}$. Since $S' \subset T \setminus \{o\}$, $A := \pi^{-1}(S')$ has cardinality $n + 2$. No two of the points of $S'$ are contained in a line of $T$, because $S$ is in linearly normal position. The rational normal curves $C \subset T$ are the images of the irreducible elements of $|O_{F_{n-1}}(f)|$ and each such element contains $o$. Thus, it is sufficient to prove that $|I_A(h + nf)|$ is a singleton, $\{Y\}$, and that $Y$ is irreducible. Since $h^0(|O_{F_{n-1}}(h + nf)|) = h^0(|O_Y^n(n)|) + h^0(|O_Y^n(1)|) = n + 3$, we have $|I_A(h + nf)| \neq \emptyset$. Since $S$ is in linear general position, no two of the points of $A$ are contained in the same element of $|O_{F_{n-1}}(f)|$.

Claim: A general $M \in |I_A(h + nf)|$ is irreducible.

Proof of the claim: Assume that a general $M \in |I_A(h + nf)|$ is reducible. There is $F \in |O_{F_{n-1}}(f)|$ such that $A = G + F$ for some $G \in |O_{F_{n-1}}(h + (n - 1)f)|$ containing a subset $A' \subseteq A$ with $|A'| \geq n + 1$ and no two of the points of $A$ are contained in the same element of $|O_{F_{n-1}}(h + (n - 1)f)|$. Since $|I_{A'}(h + nf)| = \emptyset$, we get a contradiction.

Remark 4.4 Let $K \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Let $S$ be the set of all $S \subset K$ such that $|S| = n + 3$ and $S$ is in linear general position. For each $S \in S$ let $D_S$ be the unique rational normal curve containing $S$. For any $q \in \mathbb{P}^n \setminus K$ set $\mathcal{A}(q) := \{S \in S \mid q \in D_S\}$. Let $\mathcal{B}$ denote the set of all $q \in \mathbb{P}^n \setminus K$ such that $\mathcal{A}(q) = \emptyset$. Let $E$ denote the set of all $q \in \mathbb{P}^n \setminus K$ such that $\dim \mathcal{A}(q) \geq 5$. For any $o \in \mathbb{P}^n$ let $\pi : \mathbb{P}^n \setminus \{o\} \to \mathbb{P}^{n-1}$ denote the linear projection from $o$. We identify $\mathbb{P}^{n-1}$ with a hyperplane $M_o \subset \mathbb{P}^n$ such that $o \notin M_o$.

Claim 1: Assume $n = 3$. The set $\mathcal{B}$ is formed by the 4 points $\{o_1, o_2, o_3, o_4\}$ of $\mathbb{P}^3 \setminus K$ which are the vertices of the quadric cones containing $K$ and $\dim \mathcal{A}(q) = 4$ for all $q \in \mathbb{P}^3 \setminus (K \cup \mathcal{B})$.

Proof of Claim 1: There are exactly 4 points $q \in \mathbb{P}^3 \setminus K$ (call it $o_1, \ldots, o_4$) such that the linear projection from $q$ induces a $2 : 1$ map onto a smooth conic. Call $T_1, \ldots, T_4$ the quadric cones containing $K$ and with vertex $o_1, \ldots, o_4$, respectively. Fix $q \in \mathbb{P}^3 \setminus (T_1 \cup T_2 \cup T_3 \cup T_4)$. Since $K$ is the complete intersection of 2 quadrics and $q \notin K$, there is a unique quadric, $Q$, containing $K \cup \{q\}$. Since $q \notin T_1 \cup T_2 \cup T_3 \cup T_4$, $Q$ is smooth. We have $K \subset |O_Q(2,2)|$. Since $Q$ is homogeneous and there is a smooth $C \in |O_Q(2,1)|$, a general $C \in |I_Q(2,1)|$ is smooth. By Bertini’s theorem and the assumption $q \notin K$ for a general $C \in |I_Q(2,1)|$ the scheme $K \cap C$ is smooth, i.e. it is formed by 6 points. Since $C$ is a rational normal curve, $K \cap C$ is formed by 6 points in linear general position. Thus, $C \in \mathcal{A}(q)$. We get an irreducible subset of $\mathcal{A}(q)$ with dimension 4 and another one is obtained taking the elements of $|I_Q(1,2)|$. To prove that $\dim \mathcal{A}(q) = 4$ it is sufficient to prove that each element of $\mathcal{A}(q)$ is contained in $Q$. Now take an arbitrary $D \in \mathcal{A}(q)$. If $\deg(C \cap K) > 6$, then $C \subset Q$ by Bezout. Thus, we may assume $\deg(K \cap C) = 6$, i.e. $p_a(K \cup C) = 6$ and $K \cup C$ is nodal. Since $\omega_{K \cup C}^1 \cong O_K(K \cap C)$ and $\deg(\omega_{K \cup C}^1) = 4$, duality gives $h^1(|O_{K \cup C}^1(2)|) = 0$. Since $p_a(C \cup D) \geq 1 + 6 - 1 = 6$, $\deg(D \cup C) = 7$ and $h^0(|O_D^1(2)|) = 10$, Riemann–Roch gives $h^0(|I_{C \cup K}(2)|) \neq 0$. Since $Q$ is the only quadric containing $K \cup \{q\}$, we get $C \subset Q$. Now assume $q \in T_i$ for some $i$, but $q \notin o_i$. $T_i$ is the only quadric surface containing $K \cup \{q\}$. We conclude using the desingularization $F_2$ of $T_i$ discussed in Remark 4.3; with the notation of Remark 4.3 it is sufficient to observe that a general element of $|I_p(h + 3f)|$, $p \in F_2 \setminus h$, is
irreducible. First, use the inductive assumption and then use Lemma 4.3.

Claim 2: We have \( \dim \mathcal{E} \leq n - 2 \).

Proof of Claim 2: Assume \( \dim \mathcal{E} \geq n - 1 \). Since \( \dim S = n + 3 \) and \( S \) is irreducible, we get \( B = \mathbb{P}^n \setminus (K \cup \mathcal{E}) \) contradicting the fact that \( \dim B \leq n - 1 \) proved as Claim 1 during the proof of Lemma 4.2.

This is the key lemma (an adaptation of [10, Lemma 5.2]) for the proof of Theorem 1.1.

Lemma 4.5 Fix an integer \( t \geq 2 \), a hyperplane \( H \subset \mathbb{P}^n \), a linearly normal elliptic curve \( K \subset \mathbb{P}^n \) and a reduced scheme \( Y \subset H \) such that \( \dim Y \leq 1 \) and \( h^0(H, \mathcal{I}_{Y,H}(t - 1)) > n \). Then there exists a rational normal curve \( D \subset \mathbb{P}^n \) such that \( D \) intersects transversally \( H \) and quasitransversally \( K \), \( Y \cap D = \emptyset \), \( |K \cap D| = n + 3 \) and \( h^0(H, \mathcal{I}_{Y \cup (H \cap D),H}(t)) = h^0(H, \mathcal{I}_{Y,H}(t)) - n \).

Proof Fix a degree \( t - 1 \) hypersurface \( T \) of \( H \). Let \( S \) be the set of all \( S \subset K \) such that \( |S| = n + 3 \) and \( S \) is in linear general position. The algebraic set \( S \) is irreducible and \( \dim S = n + 3 \). For each \( S \in S \) let \( D_S \) be the only rational normal curve of \( \mathbb{P}^n \) containing \( S \). We know that \( K \cup D_S \) is nodal and with arithmetic genus \( n + 3 \) for a general \( S \in S \). Thus, \( D_S \cap K = S \) for a general \( S \in S \). Taking \( S \) not containing any point of \( K \cap H \) we get \( D \cap H \cap K = \emptyset \).

Claim: For a general \( S \in S \) the curve \( D_S \) is transversal to \( H \).

Proof of the claim: We allow the case \( n = 2 \) and use induction on \( n \) to prove the claim. In the case \( n = 2 \) Claim 1 is true, because the pencil of conics through 4 points of \( \mathbb{P}^2 \), no 3 of them collinear, has the 4 points as scheme-theoretic base locus. Now assume \( n > 2 \) and that for a general \( S \subset K \) there is \( q \in S \) such that \( D_S \) and \( K \) are tangent at \( q \). Since \( S \) is irreducible, a monodromy argument shows that \( K \) and \( D_S \) are tangent at all points of \( S \). Since \( K \) is a linearly normal elliptic curve, the linear projection of \( K \) from \( q \) maps \( K \) isomorphically onto a linearly normal elliptic curve. Apply the inductive assumption to get a contradiction.

By Claim 2 of Remark 4.4 the set of all \( S \in S \) containing at least one point of \( T \setminus T \cap K \cap H \) has dimension at most \( n - 2 \). Thus, for a general \( S \in S \) the set \( D_S \cap H \) is formed by \( n \) points (Claim 1), say \( p_1, \ldots, p_n \), none of them contained in \( T \). Since \( D_S \) is a rational normal curve, the points \( p_1, \ldots, p_n \) span \( H \). Call \( A \) the \( n \)-dimensional linear subspace of \( |\mathcal{I}_Y(n)| \) formed by the hypersurfaces \( T \cup M \) with \( M \) a hyperplane of \( H \). It is sufficient to prove that \( p_1, \ldots, p_n \) gives \( n \) independent conditions to \( A \). This is true, because \( p_1, \ldots, p_n \) are linearly independent.

\[ \square \]

5. Preliminaries for the proof of Theorem 1.1

We recall the following result ([2, 25, 31]).

Lemma 5.1 Fix integers \( m, d, g, s \) such that \( m \geq 3 \), \( g \geq 0 \), \( d \geq g + m \) and \( 0 \leq s(m - 1) \leq (m + 1)d + (m - 1)(1 - g) \). Exclude the following cases:

\[ (d, g, s) \in \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\}. \]

Let \( S \subset \mathbb{P}^m \) be a general subset with cardinality \( s \). Then there exists a smooth, connected, and nondegenerate curve \( X \subset \mathbb{P}^m \) such that \( S \subset X \), \( \deg(X) = d \), \( p_a(X) = g \), and \( h^1(\mathcal{O}_X(1)) = 0 \).

For more in the Brill–Noether range (instead of just the nonspecial range), see [25].
Lemma 5.2 Fix integers $m, d, g, s$ such that $s \geq 0$ and $A(d-s, g; m)$ is defined. Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix a general $S \subset H$ such that $|S| = s$. Then there exists $X \in A(d, g; m)$ such that $X$ is smooth, $h^1(N_X) = 0$, $X$ intersects transversally $H$ and $S \subset X$.

Proof Fix $Y \in A(d-s, g; m)$ such that $Y$ is smooth and $h^1(N_Y) = 0$. By Bertini’s theorem (or the definition of $A(d-s, g; m)$) a general hyperplane section of $Y$ is formed by $d-s$ points. Instead of $Y$ we take the curve $h(Y)$ with $h$ a general element of $\text{Aut}(\mathbb{P}^n)$. Thus, we may assume that $H$ is transversal to $Y$. Moving $S$ to another general subset of $H$ with cardinality $s$ we may assume $S \cap (Y \cap H) = \emptyset$. We order the points $p_1, \ldots, p_s$ of $S$. Let $D_i, 1 \leq i \leq s$, be a general line containing $p_i$ and 1-secant to $Y$. Set $W := Y \cup D_1 \cup \cdots \cup D_s$. Each $D_i$ meets $H$ only at $p_i$; hence, $\deg(W) = d$. For general $S$ and general $D_1, \ldots, D_s$ we have $p_s(W) = g$. Applying $s$ times Lemma 2.7 we get $W \in A(d, g; m)$. Since $W$ contains $S$ and intersects transversally $H$, a general element of $A(d, g; m)$ contains $s$ general points of $H$ and intersects transversally $H$. $$
abla$$

We often use the following lemma, called lemme d’Horace in the original source ([18]) and sometimes called the Horace Lemma.

Lemma 5.3 Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix reduced schemes $Y, D \subset \mathbb{P}^n$ such that $D \subset H$ and no irreducible component of $Y$ is contained in $H$. Fix any $k \in \mathbb{N}$. We have an exact sequence

$$0 \rightarrow \mathcal{I}_Y(k-1) \rightarrow \mathcal{I}_{Y \cup D}(k) \rightarrow \mathcal{I}_{(Y \cap H) \cup D, H}(k) \rightarrow 0$$

(called the residual exact sequence of $H$). Thus,

$$h^0(\mathcal{I}_{Y \cup D}(k)) \leq h^0(\mathcal{I}_Y(k-1)) + h^0(H, \mathcal{I}_{(Y \cap H) \cup D, H}(k)),$$

$$h^1(\mathcal{I}_{Y \cup D}(k)) \leq h^1(\mathcal{I}_Y(k-1)) + h^0(H, \mathcal{I}_{(Y \cap H) \cup D, H}(k)).$$

Lemma 5.4 Fix nonnegative integers $n, d, d', g', x, s, t, g, e, k, t, t', \delta, w$ satisfying the following conditions:

(i) $n \geq 4$, $x > 0$, $k \geq 4$, $0 \leq w \leq \delta$, $0 \leq s \leq n-2$;

(ii) $d' \geq g' + n - 1$;

(iii) $0 \leq s \leq [(nd + (n-2)(1-g))/(n-1)]$;

(iv) $(w+x)(n-2) \leq nd' + (n-1)(1-g')$;

(v) $g = 1 + t(n+2) + s$.

Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix $T \subset H$ such that $T = T' \cup T''$, $T' \cap T'' = \emptyset$, $T'$ is a union of $t'$ disjoint lines $T_1, \ldots, T_{t'}$ and $T''$ is a closed subscheme with $\dim T'' \leq 1$. Assume the existence of $D \in A(d', g'; H)$ such that $D \cap T = \emptyset$, $h^1(H, \mathcal{I}_{D \cup T, H}(k-2)) = 0$ and $h^0(H, \mathcal{I}_{D \cup T, H}(k-2)) \geq n + 1 + (t-1)n$. Then there exist $Y \in A(d, g; n)$, $\delta$ disjoint lines $D_1, \ldots, D_\delta$, and $D' \in W(d', g'; H)$ such that

1. $D_i \cap D' \neq \emptyset$ if and only if $1 \leq i \leq w$;
2. $|T'_i \cap Y| = 1$ for all $i = 1, \ldots, t'$.
Let 

\[ h^0(H, I_{D' \cup T \cup Y \cup D_1 \cup \cdots \cup D_t}) \cap H, H(k)) = \max(0, h^0(H, I_{D' \cup T \cup (k)} - (d - x - t') - (\delta - w)); \]

4. For all integers \( k, m \) we will have \( h^1(C, T, H) = 0 \) for all \( k, m \leq 19 \).

5. \( Y \cup D' \cup T' \in A(d + d' + t', g + g' + x - 1; n) \).

**Proof** Let \( P \subset H \) be a general subset with cardinality \( w + x \). By (iv) and Lemma 5.1 there is a nonspecial \( Y' \in A(d', g'; H) \) such that \( P \subset Y' \) and \( h^1(N_{Y' / H}) = 0 \). We may assume \( Y' \cap T = \emptyset \) (use \( h(T) \), with \( h \) a general element of \( \text{Aut}(H) \) instead of \( T \) and apply Kleiman’s Bertini theorem [19]). By semicontinuity we may assume \( h^1(I_{D' \cup T}(a)) = 0 \) for all \( a \geq k - 1 \). Fix \( p \in P \). Let \( C \subset \mathbb{P}^n \) be a general linearly normal elliptic curve such that \( p \in C \) and \( C \) intersects transversally \( H \). Take \( s + 2 \) general 2-secant lines to \( C \). Take \( t \) general rational normal curves \( D_1, \ldots, D_t \) such that \( |D_i \cap C| = n + 3 \) for all \( i \) and \( D_i \cap D_j = \emptyset \) for all \( i \neq j \) (Lemma 4.2). We have \( E := C \cup D_1 \cup \cdots \cup D_t \in A(n + 1 + tn, 1 + t(n + 2); n) \). Applying \( t \) times Lemma 4.5 we get \( h^1(H, I_{D' \cup T \cup (Y \cup H)}(k)) = 0 \). Then we add \( x - 1 \) lines 1-secant to \( C \), each of them containing a different point of \( P \setminus \{ p \} \). Then we add \( \delta \) further 1-secants to \( C \); we add \( w \) lines \( D_1, \ldots, D_w \) through the remaining points of \( P \) and \( \delta - w \) general lines \( D_{w+1}, \ldots, D_{\delta} \). Note that (5) follows from Lemmas 2.7 and 2.8. \( \square \)

**Remark 5.5** In all quotations of Lemma 5.4 we will have

\[
\chi(I_{Y \cup T, H}(k)) + d + \delta - x + I = \binom{n + k - 1}{n - 1} \tag{5.1}
\]

with \( I = 0 \), except in Section 9. Thus, to check that \( h^0(H, I_{Y \cup T, H}(k)) \geq 2n + t(n - 1) - 1 \) to apply Lemma 5.4 it is sufficient to check the following inequality:

\[
d + \delta + 2n + t(n - 1) \leq \binom{n + k - 1}{n - 1} + x - I \tag{5.2}
\]

6. The assertion \( B(k) \)

For all integers \( k \geq 2 \) set

\[
b_k := n! k^{n-2} \tag{6.1}
\]

For all integer \( m \geq 3 \) and \( k \geq 2 \) define the integers \( g_{k, m} \) and \( f_{k, m} \) by the following relations

\[
k(g_{k, m} + m) + f_{k, m} = \binom{m + k}{m}, \quad 0 \leq f_{k, m} \leq k - 2 \tag{6.2}
\]

Note that

\[
-1 - m + \binom{m + k}{m} / k \leq g_{k, m} \leq \binom{m + k}{m} / (k - 2) \tag{6.3}
\]

Since \( f_{k, m} \leq k - 2 \), (6.3) gives \( g_{k, m} \geq f_{k, m} \); hence, \( A(g_{k, m} + m, g_{k, m} - f_{k, m}; m) \) is well-defined and its general element is nonspecial.

**Remark 6.1** By \([4, 5, 8]\) (respectively for \( m = 4 \), \( m = 3 \) and \( m > 4 \)) a general \( T \in A(g_{k, m} + m, g_{k, m} - f_{k, m}; m) \) satisfies \( h^i(I_{T}(k)) = 0 \), \( i = 0, 1 \).
Fix an integer \( a \geq 2n + 6 \) (depending on \( n \)) such that for all \( k > a - 2 \) the following inequalities hold:

\[
g_{k,n} \geq n(n + 2 + k) + (n + 2)b_k
\]  
(6.4)

\[
\left(\frac{n + k - 1}{n}\right)/(k - 3) \geq 1 + \left(\frac{n + k - 1}{n - 1}\right) - \left(\frac{n + k - 3}{n - 1}\right) + b_k
\]  
(6.5)

\[
k(k - b_{k-1} + 2k) \leq \left(\frac{n + k - 1}{n - 1}\right) - \left(\frac{n + k - 1}{n}\right)/(k - 2) - b_k + b_{k-1}
\]  
(6.6)

\[
2\left(\frac{n + a - 1}{n}\right)/(a - 2) \geq n^2 + n + 1 + b_{a-1}
\]  
(6.7)

The integer \( a \) exists because \( \binom{t}{m} \) (as a function of \( t \)) is a degree \( m \) polynomial with \( t^m/m! \) as its leading term. For instance, the left hand side of (6.5) is a degree \( n - 1 \) polynomial in \( k \) with \( 1/n! \) as its leading coefficient, while the right hand side is a degree \( n - 2 \) polynomial in \( k \).

For all integers \( k \geq 2 \) we define the integers \( d_k \) and \( g_k \) satisfying the following relation

\[
kd_k + 1 - g_k = \left(\frac{n + k}{n}\right)
\]  
(6.8)

in the following way. For \( k \leq a - 1 \) set \( d_k := g_{k,n} + n \) and \( g_k := g_{k,n} - f_{k,n} \) (we have \( g_{k,n} \geq f_{k,n} \) by Lemma 9.1). Set \( c_{a-1} := (n + 2)(d_{a-1} - n - 1) - n(g_{a-1} - 1) \). We have \( c_{a-1} \geq b_{a-1} \) (Lemma 9.2). For all integers \( k \geq a \) set \( c_k := b_k - b_{a-1} + c_{a-1} \). Note that \( b_k - b_{k-1} = c_k - c_{k-1} \) for all \( k \geq a \). Fix an integer \( k \geq a \) and assume defined the pairs \((d_i, g_i) \in \mathbb{N}^2\) for all \( i \leq k - 1 \). We also assume that for \( a - 1 \leq i \leq k - 1 \) (6.8) for \( i \) instead of \( k \) is satisfied and that for \( a - 1 \leq i \leq k - 1 \) the following inequalities are satisfied

\[
c_i \leq (n + 2)(d_i - n - 1) - n(g_i - 1) \leq c_i + (i - a + 1)(i - a + 2)/2
\]  
(6.9)

By the definition of \( c_{a-1} \) for \( i = a - 1 \) the two inequalities in (6.9) are equalities for \( i = a - 1 \).

**Notation 6.2** Let \( z \) be the maximal integer such that:

\[
(k - \frac{n + 2}{n})z \leq \left(\frac{n + k - 1}{n - 1}\right) - d_{k-1} - b_k + b_{k-1}
\]  
(6.10)

Set \( d_k := d_{k-1} + z \) and \( g_k := kd_k + 1 - \binom{n + k}{n} \).

The maximality of the integer \( z \) in (6.10) gives

\[
0 \leq \left(\frac{n + k - 1}{n - 1}\right) - c_k + c_{k-1} - (k - \frac{n + 2}{n})z \leq k
\]  
(6.11)

By the definition of the integer \( g_k \) (6.8) is satisfied. Taking the difference between (6.8) and the same equation for the integer \( k - 1 \) we get

\[
d_{k-1} + k(d_k - d_{k-1}) + g_{k-1} - g_k = \left(\frac{n + k - 1}{n - 1}\right)
\]  
(6.12)
Claim 1: The inequalities in (6.9) are satisfied for \( i = k \).

Proof of Claim 1: Since the inequalities in (6.9) are satisfied for \( i = k - 1 \), it is sufficient to prove that

\[
c_k - c_{k-1} \leq (n+2)(d_k - d_{k-1}) - n(g_k - g_{k-1}) \leq c_k - c_{k-1} + kn
\]  

(6.13)

This is true by Lemma 9.3.

For \( i = k \) the first inequality in (6.9) shows that the variety \( A(d_k, g_k; n) \) is defined for all \( k \geq 2 \).

For each integer \( k > 0 \) we define the Assertion \( B(k) \) in the following way.

Assertion \( B(k) \): We have \( h^i(I_X(k)) = 0, \ i = 0, 1 \), for a general \( X \in A(d_k, g_k; n) \).

Lemma 6.3 Assertion \( B(k) \) is true for all positive integers \( k \).

Proof If \( k < a \), then use [4, 5] respectively for the case \( n = 4 \) and the case \( n \geq 5 \). Now assume \( k \geq a \) and that \( B(k-1) \) is true. Fix a general \( Y \in A(d_{k-1}, g_k; n) \). By \( B(k-1) \) and the semicontinuity theorem for cohomology we have \( h^i(I_Y(k-1)) = 0 \). Take \( z, d_k \) and \( g_k \) as in Notation 6.2 and set \( s := n + g_k - g_{k-1} - z \).

(a) Assume \( s > 0 \), i.e. assume \( z < n + g_k - g_{k-1} \). Fix a general \( S \subset H \) such that \( |S| = s \). Since

\[
(n+2)(d_k - d_{k-1}) - n(g_k - g_{k-1}) = (d_k - d_{k-1} - s) - n(g_k - g_{k-1} - s + 1)
\]

for \( s > 0 \), we have \( A(d_k - s, g_k; n) \) is defined. We have \( d_k - s = d_k - d_{k-1} - n(g_k - g_{k-1} - s + 1) \).

By Lemma 9.5 \( A(d_k - s; g_k; n) \) is well-defined. By Lemma 5.2 we may assume that \( Y \) intersects transversally \( H \) and that \( S \subset Y \setminus H \). We have \( s(n-2) < n(z + (n-2)(1 - 1 + n)) = 2z + n^2 - 2n \) by Lemma 9.6. Thus, by Lemma 5.2, i.e. by [2], there is a non-special \( D \in A(z, z - n + 1; H) \) containing \( S \). For a general \( S \) we may also assume that \( D \) is general in \( A(z, z - n + 1; H) \); hence, by [4–6], respectively in the cases \( n - 1 = 4 \), \( n - 1 = 3 \) and \( n - 1 > 4 \), the curve \( D \) has maximal rank. By Lemma 9.7 we have \( (k-2)z + 1 - z + n - 1 \leq \frac{n(k-3)}{n-1} \). Thus, \( h^i(H, I_D(H)(k-2)) = 0 \). By Lemmas 2.7 and 2.8 we may degenerate \( Y \) to a curve \( K \cup D_1 \cdots \cup D_s \), where \( K = E \cup T \cup L_1 \cup \cdots \cup L_{s-1} \), \( E \) is a general element of \( A(d_k - s, g_k - 1 - s(n+2); n) \), \( T \) a linearly normal elliptic curve meeting \( E \) quasitransversally at a unique point, \( L_1, \ldots, L_{s-1} \) general lines intersecting \( T \), each \( D_i \) a general rational normal curve of \( P^n \) intersecting \( T \) quasitransversally at exactly \( n + 3 \) points. For a general \( T \) we may assume that \( T \) intersects transversally \( H \) and that any of the points of \( T \cap H \) span \( H \). Since any two sets of \( n+1 \) points of \( H \) are projectively equivalent, we may assume that \( T \cap H \) are \( n+1 \) general points of \( H \).

Thus, for general lines \( L_1, \ldots, L_{s-1} \) intersecting \( H \) the \( s \) points \( (T \cup L_1 \cup \cdots \cup L_{s-1}) \cap H = (T \cup L_1 \cup \cdots \cup L_{s-1} \cup D_1 \cup \cdots \cup D_s) \cap H \) is a general union of \( s \) points of \( H \). Thus, we may take \( (T \cup L_1 \cup \cdots \cup L_{s-1}) \cap H = S \). Thus, \( (T \cup L_1 \cup \cdots \cup L_{s-1} \cup D_1 \cup \cdots \cup D_s) \cap H = D \) as schemes. Recall that \( h^i(H, I_D(k-2)) = f \). Set \( W := T \cup L_1 \cup \cdots \cup L_{s-1} \cup D_1 \cup \cdots \cup D_s \). We may smooth \( W \cup E \) to a general \( Y' \in A(d_k - s, g_k; n) \). Moving \( S \) (and the curve \( D \)) along this smoothing we get that \( Y' \cup D \) is a connected nodal curve of degree \( d_k \) and arithmetic genus \( g_k \) such that \( Y' = \text{Res}_H(Y' \cup D) \) satisfies \( h^1(I_{Y'}(k-2)) = 0, \ i = 0, 1 \). We have \( h^i(H, I_{(Y' \cup D) \cap H}(k)) = 0, \ i = 0, 1 \), by (6.5) and Lemma 5.4; to apply Lemma 5.4 we need the inequality \( (k-2)z + 1 - z + n + (n+1) + n(t-1) \leq \frac{n(k-3)}{n-1} \) which is true by Lemma 9.7. The residual exact sequence of \( H \) gives \( h^i(I_{Y' \cup D}(k)) = 0, \ i = 0, 1 \). Lemma 2.9 give \( Y' \cup D \in A(d_k, g_k; n) \), concluding the inductive proof.

(b) Assume \( s \leq 0 \), i.e. assume \( z \geq n + g_k - g_{k-1} \). We take one point, \( P \), instead of \( S \), and take as \( D \) a general \( D \in A(z, g_k - g_{k-1}; H) \) containing \( P \).
7. Assertion $A(k)$

Fix a real number $\epsilon > 0$. To prove Theorem 1.1 we need to find an integer $d_0$ (depending on $\epsilon$ and $n$) such that for all $(d, g) \in \mathbb{N}^2$ with $d \geq d_0$ and $g \leq (\frac{n+2}{n} - \epsilon)d$ the component $A(d, g; n)$ is defined and a general $X \in A(d, g; n)$ has maximal rank.

Definition 7.1 For any $(d, g) \in \mathbb{N}^2$ such that $d < g + n$, $2d + 1 - g > \left(\frac{n+2}{2}\right)$ and $A(d, g; n)$ is defined (i.e. $(n+2)(d-n-1) \geq n(g-1)$) the critical value of $(d, g)$ is the minimal integer $x \geq 2$ such that $xd + 1 - g \leq \left(\frac{k+n}{n}\right)$.

Definition 7.2 Fix an integer $h \geq a + 2$ (depending on $n$ and $\epsilon$) such that for all integers $k \geq h - 2$ we have

$$d_k < \frac{(n + \epsilon/2)g_k}{n + 2},$$

$$g_{k+1}(k + 4 - \frac{1}{n + \epsilon/2}) \leq g_k(k + 4 - \frac{1}{n + \epsilon}).$$

The existence of $h$ is obvious, because $\lim_{k \to +\infty} (\frac{k}{n})/k^n = 1/n!$ and we have $\lim_{k \to +\infty} d_k/g_k = n/(n + 2)$ and $\lim_{k \to +\infty} g_{k+1}/g_k = 1$ by Lemma 9.4.

Definition 7.3 Set $k(\epsilon) := h + 6$.

Fix an integer $v \geq k(\epsilon)$ and $(d, g) \in A(d, n; g)$ such that $d < g + n$, $(d, g)$ has critical value $v$ and $(n+2)(d-n-1) \geq (n+\epsilon)(g-1)$. Since $h \geq a$, we have $g_{a-1} \leq g$. Let $m$ be the maximal integer $k \geq a - 1$ such that $g_k \leq g$; $g_k$ is well-defined, because $g \geq g_{a-1}$ and $g_i < g_{i+1}$ for all $i$ (Lemma 9.3). By Lemma 9.8 we have $m \leq \nu - 6$. If $g - g_m < m$ set $u := m$, $d'_u := d_u$, $v_u := g - g_u$; note that $0 \leq v_u < m$.

Assume $g - g_u \geq m$. In this case we set $u := m + 1$ and define the integers $d'_u$ and $v_u$ by the relations

$$ud'_u + 1 - g + v_u = \left(\frac{n + u}{n}\right), \quad 0 \leq v_u < u.$$ (7.3)

Remark 7.4 Since $g_{m+1} > g$, when $u = m + 1$ we have $d'_u \leq d_u$ and

$$u(d_u - d'_u) = g_u - g + v_u.$$ (7.4)

Lemma 7.5 We have $h^i(I_W(u)) = 0$, $i = 0, 1$, for a general $W \in A(d'_u, g - v_u; n)$.

Proof If $u = m$, then the lemma is the case $k = u$ of Lemma 6.3. Now assume $u = m + 1$ and set $z' := d'_m - d_m$, $s := 1 - g - v_u - g_m - (z' - n + 1)$. By [4, 5, 8] there is $D \in A(z', z' - n + 1; H)$ with maximal rank.

(a) Assume $s > 0$. By [4–6] (respectively for the case $n = 5$, $n = 4$ and $n > 5$) a general $D \in A(z', z' - n + 1; H)$ has maximal rank. We apply Lemma 5.4 with $d = d_m$, $g = g_m$, $d' = z$, $g' = z - n + 1$, $k = u$, $e = t' = \delta = w = 0$, $T = \emptyset$, $x$ and $t$ determined by the following inequalities:

$$0 \leq x \leq n + 1, \quad g_m = n + 1 + x + t(n + 2).$$

We need to check the assumptions of Lemma 5.4, i.e. $t \geq 0$ and
\(h^0(H, \mathcal{I}_{D,H}(u-2)) \geq n + 1 + (t-1)n\). We explain the numerology behind Lemma 5.4. To compare the set-up of the lemma we are proving with the one of Lemma 6.3 we write \(k := u, \mu_k := g - v_u\) and \(z' := d'_u - d_{k-1}\).

In the set-up of Lemma 6.3 we set \(z := d_k - d_{k-1}\). Thus, we have the equality

\[
d_{k-1} + kz + g_{k-1} - g_k = \binom{n + k - 1}{n - 1} \tag{7.5}
\]

We called \(Y\) a curve in \(\mathbb{P}^1\) with \(\deg(Y) = d_{k-1}, p_a(Y) = g_{k-1}\) and \(h^i(\mathcal{I}_X(k-1)) = 0\), \(i = 0, 1\). We had \(\xi(Y \cap D) = s\) and \(g_k - g_{k-1} = z - n + 1 + s - 1\). In the set-up of the lemma we need to prove that the curve \(Y\) we have for the degree \(k - 1\) is the same curve as the one in Lemma 6.3. We set \(s' := \xi(Y \cap D)\). We have \(\mu_k = g_{k-1} + z - n + 1 + s' - 1\). Thus,

\[
d_{k-1} + kz' + g_{k-1} - \mu_k = \binom{n + k - 1}{n - 1} \tag{7.6}
\]

Since \(\mu_k \leq g_k\), we have \(z' \leq z\) and \(s' \leq s\). The inequality \(s' \geq 0\) is true by the definition of \(u\). Since \(s' \leq s\), the check for \(Y\) is Lemma 9.5. Moreover, to check that \(h^0(H, \mathcal{I}_{D'}(k-2)) \geq 2n + n(t-1)\) we have the same \(t\) as the one in Lemma 6.3. Both \(D'\) and \(D\) have maximal rank with \(D' \in A(z', z-n+1; H), D \in A(z, z-n+1; H)\) and \(z' \leq z\). Thus, \(h^0(H, \mathcal{I}_{D'}(k-2)) \geq h^0(H, \mathcal{I}_{D}(k-2)) \geq n + 1 + (t-1)n\). Now we check that we may find \(D'\) passing through \(s'\) general points of \(H\). Since \(\mu_k = s' - 1 + g_{k-1} + z' - n + 1\) and \(g_k = s - 1 + g_{k-1} + z - n + 1\), subtracting (7.6) from (7.5) we get

\[
(k - 1)(z - z') = s - s' \tag{7.7}
\]

Since \(k > n\) and \(z \geq z' \geq n\) if \((z, s)\) satisfies the assumptions of Lemma 5.2 for \(m = n - 1\), then \((z', s)\) satisfies the same assumption.

(b) Assume \(s \leq 0\). In this case instead of \(D\) we add a nonspecial curve of degree \(z'\) with lower genus and/or meeting in a smaller number of points the curve \(Y\). To quote Lemma 5.4 we only need to use that \((u-2)z' + 1 + n + 1 + (t-1)n \leq \binom{u + u - 3}{n - 1}\), which is true, because \(z' \leq d_k - d_{k-1}\) and the integer \(t\) is the same as the one appearing in the proof of Lemma 6.3.

\[\square\]

**Definition 7.6** For every integer \(j \geq u\) define the integers \(a_j\) and \(x_j\) by the relations

\[
ja_j + 1 - g + x_j = \binom{n + j}{n}, 0 \leq x_j < j. \tag{7.8}
\]

In particular \(a_u = d'_u\) and \(x_u = v_u\). Taking the difference between (7.8) and the same equation for the integer \(j - 1\) we get

\[
a_j + j(a_j - a_{j-1}) + x_j - x_{j-1} = \binom{n + j - 1}{n - 1} \tag{7.9}
\]

for all \(j > u\).

Consider the following assertion \(A(j)\) defined for every integer \(j > u\).

**Assertion** \(A(j), j > u\): There is \(Y = Z \cup T\) with \(Z \in A(a_j - x_j; g; n), T\) a union of \(x_j\) disjoint lines, \(Z \cap T = \emptyset\) and \(h^i(ZY(j)) = 0, i = 0, 1\).
Note that \( A(a_j - x_j; g; n) \) is defined, because \( a_j - x_j > a_{j-1} \) for all \( j > u \) by Lemma 9.9. Lemma 9.9, induction on \( j \) and the definition of \( c_{a-1} \) give

\[
(n + 2)(a_j - j - 1) \geq n(g - 1) + c_j
\]  

(7.10)

**Lemma 7.7** \( A(u + 1) \) is true.

**Proof** Fix a hyperplane \( H \subset \mathbb{P}^n \). By Lemma 9.9 we have \( a_{u+1} - x_{u+1} - a_u \geq 0 \). By Lemma 7.5 there is \( Y \in A(d_u', g - v_u; n) \) with \( h^i(\mathcal{I}_Y(u)) = 0 \), \( i = 0, 1 \). We add in \( H \) a curve \( A \cup B \subset H \) with \( A \) smooth and rational, \( \deg(A) = a_{u+1} - x_{u+1} - a_u \), \( A \) containing exactly one point of \( Y \cap H \), \( B \) a union of \( x_{u+1} \) disjoint lines and \( A \cap B = \emptyset \). By [17] (case \( n = 3 \) and [7] (case \( n > 3 \)) we may assume that \( A \cup B \) has maximal rank in \( H \). We take \( Y \) transversal to \( H \), with \( h(A \cap Y) = 1 \) and with \( B \cap Y = \emptyset \). Write \( p_u(Y) = g - v_u = 1 + (n+2)t + w, \) \( d'_u = n + 1 + nt + w + w' \) with \( w' \geq 0 \). To apply Lemma 5.4 (in the set-up of Remark 5.5) it is sufficient to use that \( h^0(H, \mathcal{I}_{A \cup B}(u - 1)) \geq n + 1 + (t - 1)n \) (Lemma 9.10). We conclude by Lemma 5.3.

\[ \square \]

**Lemma 7.8** \( A(j) \) is true for all \( j \geq u + 2 \).

**Proof** Assume by induction that \( A(j - 1) \) is true and take \( Y = Z \cup T \) satisfying \( A(j - 1) \).

(a) Assume \( x_j \geq x_{j-1} \). In this case the only difference with respect to the proof of Lemma 7.7 is that now we take \( x = 2 \) and \( d' = a_j - a_{j-1} \). To check the condition on \( h^0 \) in Lemma 5.4 we use Remark 5.5 and the proof of Lemma 9.10, i.e. the proof of Lemma 9.5 using (7.9) instead of (6.12) and that \( n_{g_{k-1}} - (n + 2)d_{k-1} \geq ng - (n + 2)a_{j-1} \).

(b) Assume \( x_j < x_{j-1} \). By Lemma 9.10 we have \( a_j - a_{j-1} \geq n + j \geq n + 1 + x_{j-1} - x_j \). Let \( F \subset H \) be a smooth rational curve with maximal rank ([4–6, 18] passing through \( 1 + x_{j-1} + x_j \) general points of \( H \), one on \( Z \) and the remaining ones in different lines of \( T \). We may apply Lemma 5.4 for the reasons explained in step (a).

\[ \square \]

8. End of the proof of Theorem 1.1

For all integers \( k \geq 2 \) set \( \gamma_k := 1 - g_k + \lfloor (n + 2)(d_k - n - 1)/n \rfloor \). The integer \( \gamma_k \) is the maximal integer such that \( A(d_k, g_k + \gamma_k; n) \) is defined.

**Remark 8.1** By Lemma 9.4 we have \( \lim_{k \to +\infty} \gamma_k/k^{n-1} = 0. \)

Note that \( a_{u+4} \leq d - u - n \) by Lemma 9.8. Let \( \sigma \) be the maximal integer such that \( a_{\sigma} + \sigma \leq d \). Thus, \( d < a_{\sigma+1} + \sigma + 1 \).

**Remark 8.2** If \( d > a_{\sigma+1} \) the critical value \( v \) of \( (d, g) \) is \( \sigma + 2 \), while if \( d \leq a_{\sigma+1} \) we have \( v = \sigma + 1 \).

**Lemma 8.3** Assume \( d > a_{\sigma+1} \). Then there is \( X \in A(d, g; n) \) such that \( h^1(\mathcal{I}_X(\sigma + 2)) = 0 \).

**Proof** The proof is divided into two steps. We first prove an assertion similar to \( A(\sigma + 1) \) for a connected curve \( X \in A(a_{\sigma+1}, g; n) \). Then in step (b) we add a smooth rational curve \( A \subset H \) with \( \deg(A) = d - a_{\sigma+1} \), \( X \cup A \in A(d, g; n) \) and \( h^1(\mathcal{I}_{X \cup A}(\sigma + 1)) = 0 \).
(a) Take $Y = Z \cup T$ satisfying $A(\sigma)$ and intersecting transversally $H$. Take a smooth rational curve $F \subset H$ such that $\deg(F) = a_{\sigma+1} - a_\sigma$ and containing exactly one point of each connected component of $Y$. We use Lemmas 5.3 and 5.4 and Remark 5.5 with $I > 0$; to apply Remark 5.5 we observe that when $I > 0$ the inequalities used in the proof of Lemma 9.5 are better by $I$.

Deform $Y \cup F$ to a smooth $X \in A(a_{\sigma+1}, g; n)$ intersecting transversally $H$.

(b) By Lemma 9.9 we have $d < a_{\sigma+2}$. Take the union of $X$ and a smooth rational curve $A \subset H$ such that $\deg(A) = a_{\sigma+1} - a_\sigma$ containing exactly one point of $Y \cap H$. By the definition of $\sigma$ we have $d - a_{\sigma+1} \leq \sigma$. Thus, $\deg(A)$ is much smaller than the integer $\deg(F)$ used in step (a). We apply Lemma 5.4 and Remark 5.5 with $I > 0$; the inequalities needed here are easier than the ones used in step (a).

\[ \square \]

**Lemma 8.4** Assume $a_\sigma + \sigma \leq d \leq a_{\sigma+1}$. Then there is $X \in A(d, g; n)$ such that $h^1(\mathcal{I}_X(\sigma + 1)) = 0$.

**Proof** We start with a curve $C$ satisfying $A(\sigma)$ and hence with $h^0(\mathcal{I}_C(\sigma)) = 0$. We add a smooth rational curve $D$ with $\deg(D) = d - a_\sigma \geq 0$, meeting $C$ at a unique point and quasitransversally. We have $C \cup D \in A(d, g; n)$ by Lemmas 2.7 and 2.8.

\[ \square \]

**Proof** [Proof of Theorem 1.1:] By Remark 8.1 and Lemma 9.4 Theorem 1.1 follows from the irreducibility of $A(d, g; n)$ and Lemma 8.3 (case $d > a_{\sigma+1}$) and Lemma 8.4 (case $a_\sigma + \sigma \leq d \leq a_{\sigma+1}$).

\[ \square \]

9. Numerical lemmas

We will often silently use that as a polynomial in $t$ the polynomial function $\binom{t}{m}$, $m \geq 0$, has degree $m$ and $t^m/m!$ as its leading term.

**Lemma 9.1** For each integer $k \geq 2$ we have $g_{k,n} \geq f_{k,n}$.

**Proof** Since $f_{k,n} \leq k - 2$, we have $g_{k,n} = \lfloor \binom{n+k}{n} - kn \rfloor / (k-1)$. Thus, it is sufficient to check the easy inequality $\binom{n+k}{n} \geq 1 + kn + (k-1)(k-2)$.

\[ \square \]

**Lemma 9.2** We have $c_{a-1} \geq b_{a-1}$.

**Proof** We have $g_{a-1,n} \leq d_{a-1,n} - n$. Since $c_{a-1} = (n+2)(d_{a-1,n} - n - 1) - n(g_{a-1,n} - 1)$ we have $c_{a-1} \geq 2d_{a-1,n} - n^2 - 2n - 1$. Since

\[(a-1)d_{a-1,n} + 1 - g_{a-1,n} = \binom{n+a-1}{n} \]

and $g_{a-1,n} \leq d_{a-1,n} - n$, we have

\[(a-2)d_{a-1,n} + 1 - n \geq \binom{n+a-1}{n} \tag{9.1} \]

Use (6.7).

\[ \square \]

**Lemma 9.3** Fix an integer $k \geq a$. For every integer $k \geq a$ we have:
(i) \( d_k - d_{k-1} \geq n(b_k - b_{k-1} + 2k) \);

(ii) \( g_k > g_{k-1} \);

(iii) \( c_k - c_{k-1} \leq (n + 2)(d_k - d_{k-1}) - n(g_k - g_{k-1}) \leq c_k - c_{k-1} + nk \).

**Proof** We use induction on \( k \). We do not write down the initial case of the induction, i.e. the case \( k = a \), because the inductive step works verbatim for \( k = a \), just using that \( g_{a-1} \leq d_{a-1} - n \) and in particular \( g_{a-1} < 2d_{a-1} \). We also use that for the integer \( k - 1 \) the variety \( A(d_{k-1}, g_{k-1}; n) \) is defined, which is true if we assume (iii) for the integer \( k - 1 \). Set \( z := d_k - d_{k-1} \). By (6.10) and the inequality \( n \geq 4 \) it is sufficient to prove that

\[
nk(b_k - b_{k-1} + 2k) \leq \binom{n + k - 1}{n - 1} - d_{k-1} - b_k + b_{k-1} \quad (9.2)
\]

Since \( g_{k-1} < 2d_{k-1} \), (6.8) for the integer \( k - 1 \) gives \( d_{k-1} \leq \binom{n + k - 1}{n}/(k - 2) \). Hence, it is sufficient to use the inequality (6.6). Since \( d_{k-1} \leq \binom{n + k - 1}{n}/(k - 2) \leq \binom{n + k - 1}{n-1} \) and \((k - 1)d_{k-1} + 1 - g_{k-1} \leq \binom{n + k - 1}{n} \) by (6.8) for the integer \( k - 1 \), we have \( kd_{k-1} + 1 - g_{k-1} \leq \binom{n + k}{n} \). Since \( d_k > d_{k-1} \), (6.8) gives \( g_k > g_{k-1} \).

Part (iii) is the case \( i = k \) of (6.9) proved before the definition of \( B(k) \). \( \square \)

**Lemma 9.4** We have

\[
\lim_{k \to +\infty} \frac{d_k}{g_k} = \frac{n}{n + 2}, \quad (9.3)
\]

\[
\lim_{k \to +\infty} \frac{d_{k+1}/d_k}{g_{k+1}/g_k} = 1. \quad (9.4)
\]

\[
\lim_{k \to +\infty} k^{n-1}/d_k = k! .
\]

**Proof** By part (ii) of Lemma 9.3 we have \( \lim_{k \to +\infty} g_k = +\infty \). Part (iii) of Lemma 9.3 gives (9.3). By (9.3) the two equalities in (9.4) are equivalent. We also see that \( \lim_{k \to +\infty} g_k/k^{n-1} = (n + 2)/(n!)n \), which implies the second equality in (9.4). \( \square \)

**Lemma 9.5** In the set-up of the proof of Lemma 6.3 we have \((n + 2)(d_{k-1} - n - g_k + g_{k-1} + z) \geq n(g_{k-1} - 1)\), i.e. \( A(d_{k-1} - n - g_k + g_{k-1} + z, g_{k-1}; n) \) is well-defined.

**Proof** We have \( n + g_k - g_{k-1} - z \leq 2z/n \). Since \((n + 2)(d_{k-1} - n - 1) \geq n(g_{k-1} - 1) + c_{k-1}\), it is sufficient to observe that \( 2z/n \leq c_{k-1} \) by (6.10) and the inequality \( c_{k-1} \geq b_{k-1} \). \( \square \)

**Lemma 9.6** In the set-up of the proof of Lemma 6.3 we have \( s(n - 2) \leq 2z + n(n - 2) \), where \( s := n + g_k - g_{k-1} - z \).

**Proof** By the definition of \( s \) the lemma is true if and only if

\[
(n - 2)(g_k - g_{k-1}) \leq nz \quad (9.5)
\]

Since \((n + 2)z \geq n(g_k - g_{k-1}) + (n + 2)(b_k - b_{k-1})\) and \( n/(n - 2) \geq (n + 2)/n \), (9.5) is true. \( \square \)
Lemma 9.7 In the notation of the proof of Lemma 6.3 we have \((k-2)z+1-z+n+(n+1)+n(t-1) \leq \binom{n+k-3}{n-1}\).

Proof Look at (6.12). We have \(g_k - g_{k-1} \leq (n+2)(d_k - d_{k-1})/n - (b_k + b_{k-1})/n\); hence, \((k - \frac{n+2}{n})z \leq (\binom{n+k-1}{n-1}) - (b_k - b_{k-1})/n\). We have \(d_{k-1} \geq 1+nt+b_{k-1}/n\). Thus, it is sufficient to have \(b_k/n \geq (\binom{n+k-1}{n-1}) - (\binom{n+k-3}{n-1})\), which is true by the definition of \(b_k\) and the inequality \(a \geq n + 7\).

\(\square\)

Lemma 9.8 We have \(m \leq v - 6\).

Proof Assume \(m \geq v - 5\). Thus, \(m \geq h\) and \(g_{v-5} \leq g\). Since \((d, g)\) has critical value \(v\), we have
\[vd + 1 - g \leq vd_v + 1 - g_v.\]  
(9.6)

By (9.6), (7.1), and (7.2) we get a contradiction.

\(\square\)

Lemma 9.9 For every integer \(j > u\) we have \(a_j - x_j > a_{j-1}\) and (7.10) is true.

Proof Assume \(a_j - a_{j-1} \leq x_j\). Since \(x_j \leq j - 1\) and \(x_{j-1} \geq 0\), (7.9) gives \(a_j + j^2 - 1 \leq (\binom{n+j-1}{n-1})\), contradicting the inequality \(a_j \leq d_j\). Since \(j(d_j - a_j) \leq g_j - j\) and \((n+2)(d_j - d_{a-1}) \geq n(g_j - g_{j-1}) + b_j - b_{a-1}\), we get (7.10).

\(\square\)

Lemma 9.10 In the set-up of the proof of Lemma 7.7 we have \(h^0(H, \mathcal{I}_{A \cup B}(u-1)) \leq n + 1 + (t - 1)n\).

Proof We mimic the proof of Lemma 9.7 using (7.9) instead of (6.12). In the set-up of Lemma 7.7 we have \(z := \deg(A \cup B) = a_{u+1} - a_u\). Since \(x_{u+1} \leq u\) and \(x_u \geq 0\), (7.9) gives \(kz \leq \binom{n+u}{n-1} - u - a_u\). We use that \(a_u \geq n + 1 + nt + c_u/n\) as in the proof of Lemma 9.5.

\(\square\)

Acknowledgments
The author was partially supported by Ministero Istruzione Università Ricerca (Italy) and Gruppo Nazionale Geometria Algebra (Istituto di Alta Matematica, Italy).

References


[29] Perrin D. Courbes passant par \( m \) points généraux de \( \mathbb{P}^3 \). Bulletin de la Société Mathématique de France, Mémoire 1985; 28/29 (in French).
