

GOOD BEHAVIOUR OF LIE BRACKET AT A SUPERDENSITY POINT OF THE TANGENCY SET OF A SUBMANIFOLD WITH RESPECT TO A RANK 2 DISTRIBUTION

S. DELLADIO

ABSTRACT. Let H, K be a couple of vector fields of class C^1 in an open set $U \subset \mathbb{R}^{N+m}$, let \mathcal{M} be a N -dimensional C^1 submanifold of U and define $\mathcal{T} := \{z \in \mathcal{M} : H(z), K(z) \in T_z\mathcal{M}\}$. Then the following obvious property

If $z_0 \in \mathcal{M}$ is an interior point (relative to \mathcal{M}) of \mathcal{T} then $[H, K](z_0) \in T_{z_0}\mathcal{M}$

admits the following generalization:

If $z_0 \in \mathcal{M}$ is a superdensity point (relative to \mathcal{M}) of \mathcal{T} then $[H, K](z_0) \in T_{z_0}\mathcal{M}$.

As a corollary we get very easily the following result of [7]: *Let \mathcal{D} be a C^1 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and let \mathcal{M} be a N -dimensional C^1 submanifold of U . If $z_0 \in \mathcal{M}$ is a superdensity point (relative to \mathcal{M}) of the tangency set $\{z \in \mathcal{M} : T_z\mathcal{M} = \mathcal{D}(z)\}$ then \mathcal{D} is involutive at z_0 .*

1. INTRODUCTION

Let m, n, N be positive integers such that $N \geq n$. Recall that a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^{N+m}$ is a map \mathcal{D} assigning a n -dimensional vector subspace $\mathcal{D}(z)$ of \mathbb{R}^{N+m} to each point $z \in U$ and satisfying the following property: If $z \in U$ then there exists a family $\{X_1^{(z)}, \dots, X_n^{(z)}\}$ of vector fields of class C^1 in a neighbourhood $V^{(z)} \subset U$ of z such that $\{X_1^{(z)}(z'), \dots, X_n^{(z)}(z')\}$ is a basis of $\mathcal{D}(z')$ for all $z' \in V^{(z)}$. The distribution \mathcal{D} is said to be involutive at $z \in U$ if $[X_i^{(z)}, X_j^{(z)}](z) \in \mathcal{D}(z)$ for all $i, j \in \{1, \dots, n\}$.

Let \mathcal{D} be a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^{N+m}$ and let \mathcal{M} be a N -dimensional C^1 submanifold of U . Then let $\tau(\mathcal{M}, \mathcal{D})$ denote the tangency set of \mathcal{M} with respect to \mathcal{D} , that is

$$\tau(\mathcal{M}, \mathcal{D}) := \{z \in \mathcal{M} : T_z\mathcal{M} \supset \mathcal{D}(z)\}$$

where $T_z\mathcal{M}$ is the tangent space to \mathcal{M} at z . Observe that in the special case $n = N$, one obviously has $\tau(\mathcal{M}, \mathcal{D}) := \{z \in \mathcal{M} : T_z\mathcal{M} = \mathcal{D}(z)\}$. We know from the Frobenius theorem [10, Section 2.11] that if $n = N$ and \mathcal{D} is involutive at every point of U then \mathcal{D} is

2010 *Mathematics Subject Classification.* 28Axx, 58A30, 58C35, 58A17.

Key words and phrases. Tangency set of a submanifold with respect to a distribution, Distributions, Superdensity, Integral manifold, Frobenius Theorem.

completely integrable, which means that for every $z_0 \in U$ there exists a C^1 submanifold \mathcal{M} of U such that $z_0 \in \mathcal{M}$ and $\tau(\mathcal{M}, \mathcal{D}) = \mathcal{M}$.

Prior to enunciate the core result of this article, we summarize some well known main facts closely related to it. They describe the “integrability degree” of noninvolutive C^1 distributions \mathcal{D} , mainly by providing upper bounds for the Hausdorff dimension of $\tau(\mathcal{M}, \mathcal{D})$ as \mathcal{M} varies among all N -dimensional C^2 submanifolds of U . The list includes also a theorem about the density of $\tau(\mathcal{M}, \mathcal{D})$ when $\mathcal{H}^N(\tau(\mathcal{M}, \mathcal{D})) > 0$, which can actually occur for C^1 submanifolds \mathcal{M} . In this regard, for the convenience of the reader, let us recall that $z_0 \in \mathcal{M}$ is said to be a superdensity point of $\mathcal{E} \subset \mathcal{M}$ (relative to \mathcal{M}) if

$$\mathcal{H}^N(B_{\mathcal{M}}(z_0, r) \setminus \mathcal{E}) = o(r^{N+1}) \quad (\text{as } r \rightarrow 0+)$$

where $B_{\mathcal{M}}(z_0, r) \subset \mathcal{M}$ is the metric ball of radius r centered at z_0 , compare Section 2 below. In the following statements the Hausdorff dimension is denoted by \dim_H .

- (1) Let $H\mathbb{H}^k$ be the horizontal subbundle of the tangent bundle to the Heisenberg group \mathbb{H}^k , that is the distribution of rank $2k$ on \mathbb{R}^{2k+1} described by the vector fields

$$\begin{aligned} (x_1, \dots, x_{2k+1}) &\mapsto \frac{\partial}{\partial x_i} + 2x_{k+i} \frac{\partial}{\partial x_{2k+1}} \quad (i = 1, \dots, k) \\ (x_1, \dots, x_{2k+1}) &\mapsto \frac{\partial}{\partial x_{k+i}} - 2x_i \frac{\partial}{\partial x_{2k+1}} \quad (i = 1, \dots, k). \end{aligned}$$

This distribution is noninvolutive everywhere and one has

$$(1.1) \quad \dim_H(\tau(\mathcal{M}, H\mathbb{H}^k)) \leq k$$

for every $(2k)$ -dimensional C^2 submanifold \mathcal{M} of \mathbb{R}^{2k+1} (see [1, Theorem 1.2], [2, Example 6.5], [5, Corollary 4.1]). Observe that $H\mathbb{H}^k$ is translation invariant along $\mathbb{R}_{x_{2k+1}}$, that is $H\mathbb{H}^k(x_1, \dots, x_{2k+1})$ does not depend on x_{2k+1} .

- (2) If $n = N$, then [2, Theorem 1.3] provides an explicit estimate of the number

$$\sup\{\dim_H(\tau(\mathcal{M}, \mathcal{D})) : \mathcal{M} \text{ is } C^2\text{-smooth}\}$$

in terms of the involutiveness degree of \mathcal{D} . An elementary proof, based on the implicit function theorem, can be found in [6]. In [2, Example 6.5], already mentioned above, this result is used to prove the inequality (1.1).

- (3) If \mathcal{D} is of class C^∞ and fulfils the Hörmander noninvolutiveness condition (see [2, Definition 4.1]), then one has

$$\sup\{\dim_H(\tau(\mathcal{M}, \mathcal{D})) : \mathcal{M} \text{ is } C^2\text{-smooth}\} \leq N - 1$$

compare [2, Theorem 4.5]. The well-known result by Derridj [8, Theorem 1] follows immediately from this property.

- (4) Roughly speaking, C^1 smooth submanifolds \mathcal{M} are expected to produce much larger tangencies (with respect to \mathcal{D}) than those produced by C^2 smooth submanifolds. In fact, even if there are no points at which \mathcal{D} is involutive, it can well be

that a C^1 smooth \mathcal{M} exists such that $\mathcal{H}^N(\tau(\mathcal{M}, \mathcal{D})) > 0$. For instance, according to [2, Proposition 8.2], when $n = N$ this is true for a large class of distributions (including $H\mathbb{H}^k$) and there are good reasons to believe that it is true in general (compare [2, Problem 8.3]).

- (5) If \mathcal{D} is a C^1 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and \mathcal{M} is a N -dimensional C^1 submanifold of U , then \mathcal{D} must be involutive at each point $z_0 \in \mathcal{M}$ which is a superdensity point of $\tau(\mathcal{M}, \mathcal{D})$ (relative to \mathcal{M}) [7, Theorem 1.1]. In other words: (when $n = N$) despite (4), if \mathcal{D} is not involutive at a point $z_0 \in \mathcal{M}$ then z_0 cannot be a superdensity point of $\tau(\mathcal{M}, \mathcal{D})$ (relative to \mathcal{M}).

Finally, we can state our main result (namely Theorem 4.1 below).

Theorem. *Let H, K be a couple of vector fields of class C^1 in an open set $U \subset \mathbb{R}^{N+m}$. Moreover let \mathcal{M} be a N -dimensional C^1 submanifold of U and define*

$$\mathcal{T} := \{z \in \mathcal{M} : H(z), K(z) \in T_z \mathcal{M}\}.$$

If $z_0 \in \mathcal{M}$ is a superdensity point of \mathcal{T} (relative to \mathcal{M}), that is

$$\mathcal{H}^N(B_{\mathcal{M}}(z_0, r) \setminus \mathcal{T}) = o(r^{N+1}) \quad (\text{as } r \rightarrow 0+),$$

then $[H, K](z_0) \in T_{z_0} \mathcal{M}$.

Observe that such a property can be rephrased as follows: There is no N -dimensional C^1 submanifold \mathcal{M} of U , passing through z_0 , which is transversal to $[H, K](z_0)$ at z_0 and such that z_0 is a superdensity point of \mathcal{T} (relative to \mathcal{M}).

From this theorem we shall very easily obtain Corollary 4.1 below, that is a generalization of [7, Theorem 1.1]. For the reader's convenience, we report here its statement:

Corollary. *Let \mathcal{D} be a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^{N+m}$ and let \mathcal{M} be a N -dimensional C^1 submanifold of U , with $N \geq n$. If there exists a point $z_0 \in \mathcal{M}$ which is a superdensity point of $\tau(\mathcal{M}, \mathcal{D})$ (relative to \mathcal{M}), then the following facts hold:*

- (1) *For every couple H, K of vector fields of class C^1 in a neighbourhood V of z_0 such that $H(z), K(z) \in \mathcal{D}(z)$ for all $z \in \mathcal{M} \cap V$, one has $[H, K](z_0) \in T_{z_0} \mathcal{M}$.*
- (2) *If $N = n$ then \mathcal{D} is involutive at z_0 .*

We eventually note that the main theorem generalizes the following obvious statement: *If $z_0 \in \mathcal{M}$ is an interior point (relative to \mathcal{M}) of \mathcal{T} then $[H, K](z_0) \in T_{z_0} \mathcal{M}$.* Thus such a theorem, just as several other results obtained in previous works (compare [4, Introduction] and the references therein), reinforces our belief that “superdensity” provides a good category to generalize classical results where “openness” is required.

2. GENERAL NOTATION AND PRELIMINARIES

We will have to deal with maps from \mathbb{R}^N to \mathbb{R}^m . The standard basis of \mathbb{R}^{N+m} and the corresponding coordinates are denoted by e_1, \dots, e_{N+m} and $(x_1, \dots, x_N, y_1, \dots, y_m)$, respectively. We may also write \mathbb{R}_x^N in place of \mathbb{R}^N and \mathbb{R}_y^m in place of \mathbb{R}^m . If U is an open subset of $\mathbb{R}_x^N \times \mathbb{R}_y^m$ and $G \in C^1(U, \mathbb{R}^k)$, then $D_x G$ and $D_y G$ denote the Jacobian matrix of G with respect to x and the Jacobian matrix of G with respect to y , respectively, that is

$$D_x G := \left(\frac{\partial G}{\partial x_1} \middle| \dots \middle| \frac{\partial G}{\partial x_N} \right), \quad D_y G := \left(\frac{\partial G}{\partial y_1} \middle| \dots \middle| \frac{\partial G}{\partial y_m} \right).$$

For simplicity, we define

$$D_1 := \frac{\partial}{\partial x_1}, \dots, D_N := \frac{\partial}{\partial x_N}, \quad D_{N+1} := \frac{\partial}{\partial y_1}, \dots, D_{N+m} := \frac{\partial}{\partial y_m}.$$

If H and K are two C^1 vector fields on an open set $U \subset \mathbb{R}^{N+m}$, that is

$$H = \sum_{i=1}^{N+m} H_i D_i, \quad K = \sum_{i=1}^{N+m} K_i D_i$$

with $H_i, K_i \in C^1(U)$, then we recall that the Poisson bracket of H, K is the following continuous vector field on U

$$(2.1) \quad [H, K] = \sum_{j=1}^{N+m} [H, K]_j D_j, \quad [H, K]_j := \sum_{i=1}^{N+m} (H_i D_i K_j - K_i D_i H_j)$$

compare [10, Sect. 2.4.5].

Let \mathcal{M} be a N -dimensional C^1 submanifold of \mathbb{R}^{N+m} and let d denote the distance defined on each connected component of \mathcal{M} by taking the infimum over the joining paths (compare [3, Section 1.6]). Then for $z_0 \in \mathcal{M}$ and $r > 0$ we define

$$B_{\mathcal{M}}(z_0, r) := \{z \in \mathcal{M}^{(z_0)} \mid d(z, z_0) < r\}$$

where $\mathcal{M}^{(z_0)}$ is the connected component of \mathcal{M} containing z_0 . Recall that for r small enough \exp_{z_0} maps $B_{T_{z_0}\mathcal{M}}(0, r)$ diffeomorphically onto a neighbourhood of z_0 and one has

$$\exp_{z_0} \left(B_{T_{z_0}\mathcal{M}}(0, r) \right) = B_{\mathcal{M}}(z_0, r)$$

compare [3, Theorem 1.6 and Corollary 1.1]. In the special case when $m = 0$ and $\mathcal{M} = \mathbb{R}^N$ the distance d reduces to the usual Euclidean distance and we denote $B_{\mathbb{R}^N}(z_0, r)$ simply by $B(z_0, r)$.

The Lebesgue outer measure on \mathbb{R}^N and the N -dimensional Hausdorff measure on \mathbb{R}^{N+m} will be denoted by \mathcal{L}^N and \mathcal{H}^N , respectively. A point $x \in \mathbb{R}^N$ is said to be a superdensity point of $E \subset \mathbb{R}^N$ if

$$\mathcal{L}^N(B(x, r) \setminus E) = o(r^{N+1}) \quad (\text{as } r \rightarrow 0+).$$

The set of all superdensity points of E is denoted by $E^{(N+1)}$. Analogously, if \mathcal{M} is a N -dimensional C^1 submanifold of \mathbb{R}^{N+m} and $z_0 \in \mathcal{M}$, then we say that z_0 is a superdensity point of $\mathcal{E} \subset \mathcal{M}$ (relative to \mathcal{M}) if

$$\mathcal{H}^N(B_{\mathcal{M}}(z_0, r) \setminus \mathcal{E}) = o(r^{N+1}) \quad (\text{as } r \rightarrow 0+).$$

The set of all superdensity points of \mathcal{E} (relative to \mathcal{M}) is denoted by $\mathcal{E}^{(N+1)}$.

By [9, 3.2.46] and the area formula [9, Theorem 3.2.3] one can prove that C^1 embeddings preserve density-degree, namely the following property holds [7, Proposition 3.3].

Proposition 2.1. *Let \mathcal{M} be a N -dimensional C^1 submanifold of \mathbb{R}^{N+m} , let Ω be an open subset of \mathbb{R}^N and let $F : \Omega \rightarrow \mathbb{R}^{N+m}$ be an injective immersion of class C^1 such that $F(\Omega) \subset \mathcal{M}$. Moreover let E be a subset of Ω and let $x_0 \in \Omega$. Then (for $k > 0$) one has*

$$\mathcal{L}^N(B(x_0, r) \setminus E) = o(r^k) \quad (\text{as } r \rightarrow 0+)$$

if and only if

$$\mathcal{H}^n(B_{\mathcal{M}}(F(x_0), r) \setminus F(E)) = o(r^k) \quad (\text{as } r \rightarrow 0+).$$

In particular, $x_0 \in E^{(N+1)}$ if and only if $F(x_0) \in F(E)^{(N+1)}$.

3. SOME LOCALIZATION PROPERTIES AT A SUPERDENSITY POINT

For $\rho \in (0, 1)$, let us consider $\varphi_\rho \in C_c^1(\mathbb{R}^N)$ with compact support in $B(0, 1)$, such that $\text{Im}(\varphi_\rho) = [0, 1]$, $\varphi_\rho|_{B(0, \rho)} \equiv 1$ and

$$|D_i \varphi| \leq \frac{2}{1 - \rho} \quad (i = 1, \dots, N).$$

Then let $x_0 \in \mathbb{R}^N$, $r > 0$ and define

$$(3.1) \quad \varphi_{\rho, r}(x) := \varphi_\rho\left(\frac{x - x_0}{r}\right), \quad x \in \mathbb{R}^N.$$

Observe that $\varphi_{\rho, r}$ has compact support in $B(0, r)$, $\text{Im}(\varphi_{\rho, r}) = [0, 1]$, $\varphi_{\rho, r}|_{B(0, \rho r)} \equiv 1$ and

$$(3.2) \quad |D_i \varphi_{\rho, r}| \leq \frac{2}{(1 - \rho)r} \quad (i = 1, \dots, N).$$

Proposition 3.1. *Let Λ be a continuous function defined in a neighbourhood of $x_0 \in \mathbb{R}^N$, let $\varphi_{\rho, r}$ defined as in (3.1) and assume that for all $\rho \in (0, 1)$ one has*

$$(3.3) \quad \int_{B(x_0, r)} \Lambda(x) \varphi_{\rho, r}(x) dx = o(r^N)$$

as $r \rightarrow 0+$. Then $\Lambda(x_0) = 0$.

Proof. From the equality

$$\begin{aligned} \int_{B(x_0, r)} \Lambda(x) \varphi_{\rho, r}(x) dx &= \int_{B(x_0, \rho r)} \Lambda(x) \varphi_{\rho, r}(x) dx + \int_{B(x_0, r) \setminus B(x_0, \rho r)} \Lambda(x) \varphi_{\rho, r}(x) dx \\ &= \int_{B(x_0, \rho r)} \Lambda(x) dx + \int_{B(x_0, r) \setminus B(x_0, \rho r)} \Lambda(x) \varphi_{\rho, r}(x) dx \end{aligned}$$

and the assumption (3.3), it follows that for all $\rho \in (0, 1)$ one has

$$\begin{aligned} \left| \int_{B(x_0, \rho r)} \Lambda(x) dx \right| &\leq \frac{1}{\mathcal{L}^N(B(x_0, \rho r))} \left(\left| \int_{B(x_0, r) \setminus B(x_0, \rho r)} \Lambda(x) \varphi_{\rho, r}(x) dx \right| + o(r^N) \right) \\ &\leq \frac{C(1 - \rho^N) r^N}{\rho^N r^N} + \frac{o(r^N)}{\rho^N r^N} \\ &= \frac{C(1 - \rho^N)}{\rho^N} + \frac{o(r^N)}{\rho^N r^N} \end{aligned}$$

as $r \rightarrow 0+$, where C is a constant which doesn't depend on ρ and r (for r small enough). Then

$$|\Lambda(x_0)| \leq \frac{C(1 - \rho^N)}{\rho^N}$$

for all $\rho \in (0, 1)$, hence $\Lambda(x_0) = 0$. □

Definition 3.1. Let \mathcal{F} be the family of all real valued functions $r \mapsto \Phi(r)$ defined for $r \in (0, a)$, with $a > 0$ possibly depending on Φ . If $\Phi_1, \Phi_2 \in \mathcal{F}$ are such that $\Phi_1(r) - \Phi_2(r) = o(r^n)$, as $r \rightarrow 0+$, then we write $\Phi_1 \sim \Phi_2$.

This statement has a trivial proof.

Proposition 3.2. The binary relation \sim defined above is an equivalence relation. Moreover, if $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathcal{F}$ are such that

$$\Phi_1 \sim \Psi_1, \quad \Phi_2 \sim \Psi_2$$

and if $\lambda_1, \lambda_2 \in \mathbb{R}$ then

$$\lambda_1 \Phi_1 + \lambda_2 \Phi_2 \sim \lambda_1 \Psi_1 + \lambda_2 \Psi_2.$$

The following simple property will be useful in the proof of the main result.

Proposition 3.3. Let E be a measurable subset of \mathbb{R}^N and $x_0 \in E^{(N+1)}$. Moreover let Λ and Σ be a couple of continuous functions defined in a neighbourhood of x_0 such that $\Lambda|_{E \cap B(x_0, r)} = \Sigma|_{E \cap B(x_0, r)}$ (for r small enough). Then, for all $\rho \in (0, 1)$, one has

$$(3.4) \quad \int_{B(x_0, r)} \Lambda \varphi_{\rho, r} dx \sim \int_{B(x_0, r)} \Sigma \varphi_{\rho, r} dx$$

and

$$(3.5) \quad \int_{B(x_0, r)} \Lambda D_i \varphi_{\rho, r} dx \sim \int_{B(x_0, r)} \Sigma D_i \varphi_{\rho, r} dx \quad (i = 1, \dots, N).$$

Proof. A constant C , independent from $\rho \in (0, 1)$ and r (small enough), has to exist such that

$$\begin{aligned} \left| \int_{B(x_0, r)} \Lambda \varphi_{\rho, r} dx - \int_{B(x_0, r)} \Sigma \varphi_{\rho, r} dx \right| &= \left| \int_{B(x_0, r)} (\Lambda - \Sigma) \varphi_{\rho, r} dx \right| \\ &\leq \int_{B(x_0, r) \setminus E} |\Lambda - \Sigma| \varphi_{\rho, r} dx \\ &\leq C \mathcal{L}^n(B(x_0, r) \setminus E). \end{aligned}$$

Then, for all $\rho \in (0, 1)$, one has

$$\int_{B(x_0, r)} \Lambda \varphi_{\rho, r} dx - \int_{B(x_0, r)} \Sigma \varphi_{\rho, r} dx = o(r^{N+1})$$

as $r \rightarrow 0+$, which yields (3.4).

By an analogous computation and also recalling (3.2), for all $\rho \in (0, 1)$, we get

$$\begin{aligned} \left| \int_{B(x_0, r)} \Lambda D_i \varphi_{\rho, r} dx - \int_{B(x_0, r)} \Sigma D_i \varphi_{\rho, r} dx \right| &\leq \int_{B(x_0, r) \setminus E} |(\Lambda - \Sigma) D_i \varphi_{\rho, r}| dx \\ &\leq \frac{C \mathcal{L}^n(B(x_0, r) \setminus E)}{(1 - \rho)r} \end{aligned}$$

where C does not depend on $\rho \in (0, 1)$ and r (provided it is small enough). Hence, for all $\rho \in (0, 1)$

$$\int_{B(x_0, r)} \Lambda D_i \varphi_{\rho, r} dx - \int_{B(x_0, r)} \Sigma D_i \varphi_{\rho, r} dx = o(r^N)$$

as $r \rightarrow 0+$, that is (3.5). □

4. THE MAIN RESULT

Lemma 4.1. *Let $H = (H_1, \dots, H_{N+m})$ and $K = (K_1, \dots, K_{N+m})$ be a couple of vector fields of class C^1 in an open set $U \subset \mathbb{R}_x^N \times \mathbb{R}_y^m$. Moreover let Ω be an open subset of \mathbb{R}_x^N and $f = (f_1, \dots, f_m)^t \in C^1(\Omega, \mathbb{R}_y^m)$. Denote by Γ the graph of f , that is $\Gamma := F(\Omega)$ with $F : \Omega \rightarrow \mathbb{R}_x^N \times \mathbb{R}_y^m$ defined as $F(x) := (x, f(x))$. Assume that $\Gamma \subset U$ and that the set*

$$\mathcal{E} := \left\{ z \in \Gamma : H(z), K(z) \in T_z \Gamma \right\}$$

is nonempty. If we define

$$\begin{aligned} H_* &:= (H_1, \dots, H_N), \quad H_{\#} := (H_{N+1}, \dots, H_{N+m}) \\ K_* &:= (K_1, \dots, K_N), \quad K_{\#} := (K_{N+1}, \dots, K_{N+m}) \end{aligned}$$

then, for all $x \in E := F^{-1}(\mathcal{E})$, one has

$$(4.1) \quad \begin{aligned} Df(x) H_*(F(x)) &= H_{\#}(F(x)) \\ Df(x) K_*(F(x)) &= K_{\#}(F(x)) \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} DK(F(x))H(F(x)) &= D(K \circ F)(x)H_*(F(x)) \\ DH(F(x))K(F(x)) &= D(H \circ F)(x)K_*(F(x)). \end{aligned}$$

Proof. Let $x \in E$. Since $F(x) \in U$ and

$$H(F(x)) \in T_{F(x)}\Gamma = \text{Im}DF(x) = \text{Im} \begin{pmatrix} I \\ Df(x) \end{pmatrix}$$

then $v \in \mathbb{R}^N$ has to exist such that

$$H_*(F(x)) = v, \quad H_{\#}(F(x)) = Df(x)v.$$

Hence the first identity of (4.1) follows at once. By the same argument we get also the second identity of (4.1).

Moreover, from

$$\begin{aligned} D(K \circ F)(x) &= DK(F(x))DF(x) = DK(F(x)) \begin{pmatrix} I \\ Df(x) \end{pmatrix} \\ &= D_x K(F(x)) + D_y K(F(x))Df(x) \end{aligned}$$

and from the first identity of (4.1) it follows that

$$\begin{aligned} D(K \circ F)(x)H_*(F(x)) &= D_x K(F(x))H_*(F(x)) + D_y K(F(x))Df(x)H_*(F(x)) \\ &= D_x K(F(x))H_*(F(x)) + D_y K(F(x))H_{\#}(F(x)) \\ &= DK(F(x))H(F(x)) \end{aligned}$$

that is the first identity of (4.2). The second one follows analogously. \square

Theorem 4.1. *Let H, K be a couple of vector fields of class C^1 in an open set $U \subset \mathbb{R}^{N+m}$. Moreover let \mathcal{M} be a N -dimensional C^1 submanifold of U and define*

$$\mathcal{T} := \{z \in \mathcal{M} : H(z), K(z) \in T_z \mathcal{M}\}.$$

If $z_0 \in \mathcal{M}$ is a superdensity point of \mathcal{T} (relative to \mathcal{M}) then $[H, K](z_0) \in T_{z_0} \mathcal{M}$.

Proof. Since \mathcal{M} is locally the graph of a C^1 function, we can assume that there exist an open set $\Omega \subset \mathbb{R}_x^N$ and $f = (f_1, \dots, f_m) \in C^1(\Omega, \mathbb{R}_y^m)$ such that

$$x_0 \in \Omega, \quad (x_0, f(x_0)) = z_0, \quad \Gamma := \{(x, f(x)) : x \in \Omega\} \subset \mathcal{M}.$$

Define $F \in C^1(\Omega, \mathbb{R}^{N+m})$ and $\Lambda \in C(\Omega, \mathbb{R}^m)$ by

$$F(x) := (x, f(x)), \quad x \in \Omega$$

and

$$\Lambda(x) := Df(x)[H, K]_*(F(x)) - [H, K]_{\#}(F(x)), \quad x \in \Omega$$

where the subscript notation has to be intended as in Lemma 4.1. Moreover let $T := F^{-1}(\mathcal{T})$ and observe that

$$x_0 \in \Omega \cap T^{(N+1)}$$

by Proposition 2.1. If A, B are functions defined in Ω such that $A|_T = B|_T$, then we write $A \underline{T} B$. This notation will be useful to simplify the formulas in the computation below.

By (2.1) and (4.2) we obtain

$$\begin{aligned} \Lambda &= Df[(DK_* \circ F)(H \circ F) - (DH_* \circ F)(K \circ F)] + \\ &\quad + (DH_{\#} \circ F)(K \circ F) - (DK_{\#} \circ F)(H \circ F) \\ &\underline{T} Df[D(K_* \circ F)(H_* \circ F) - D(H_* \circ F)(K_* \circ F)] + \\ &\quad + D(H_{\#} \circ F)(K_* \circ F) - D(K_{\#} \circ F)(H_* \circ F) \end{aligned}$$

hence (for $h = 1, \dots, m$)

$$\Lambda_h := \Lambda \cdot e_{n+h} \underline{T} \Lambda_h^{(1)} + \Lambda_h^{(2)}$$

where

$$\begin{aligned} \Lambda_h^{(1)} &:= [D(K_* \circ F)(H_* \circ F) - D(H_* \circ F)(K_* \circ F)] \cdot Df_h \\ \Lambda_h^{(2)} &:= (K_* \circ F) \cdot D(H_{n+h} \circ F) - (H_* \circ F) \cdot D(K_{n+h} \circ F). \end{aligned}$$

One has

$$\begin{aligned} \Lambda_h^{(1)} &= \sum_{i=1}^N D_i f_h [D(K_i \circ F) \cdot (H_* \circ F) - D(H_i \circ F) \cdot (K_* \circ F)] \\ &= \sum_{i=1}^N D_i f_h \operatorname{div}[(K_i \circ F)(H_* \circ F) - (H_i \circ F)(K_* \circ F)] + \\ &\quad + \sum_{i=1}^N D_i f_h [(H_i \circ F) \operatorname{div}(K_* \circ F) - (K_i \circ F) \operatorname{div}(H_* \circ F)] \\ &= \sum_{i=1}^N D_i f_h \operatorname{div}[(K_i \circ F)(H_* \circ F) - (H_i \circ F)(K_* \circ F)] + \\ &\quad + [(H_* \circ F) \cdot Df_h] \operatorname{div}(K_* \circ F) - [(K_* \circ F) \cdot Df_h] \operatorname{div}(H_* \circ F). \end{aligned}$$

Thus, by (4.1)

$$\begin{aligned} \Lambda_h^{(1)} &\underline{T} \sum_{i=1}^N D_i f_h \operatorname{div}[(K_i \circ F)(H_* \circ F) - (H_i \circ F)(K_* \circ F)] + \\ &\quad + (H_{N+h} \circ F) \operatorname{div}(K_* \circ F) - (K_{N+h} \circ F) \operatorname{div}(H_* \circ F). \end{aligned}$$

By also recalling Proposition 3.3, it follows that (for $r > 0$ small enough and for all $\rho \in (0, 1)$)

$$\int_{B(x_0, r)} \Lambda_h^{(1)} \varphi_{\rho, r} dx \sim I_{\rho}(r) + J_{\rho}(r)$$

where

$$I_\rho(r) := \sum_{i=1}^N \int_{B(x_0, r)} \varphi_{\rho, r} D_i f_h \operatorname{div}[(K_i \circ F)(H_* \circ F) - (H_i \circ F)(K_* \circ F)] dx$$

$$J_\rho(r) := \int_{B(x_0, r)} \varphi_{\rho, r} [(H_{N+h} \circ F) \operatorname{div}(K_* \circ F) - (K_{N+h} \circ F) \operatorname{div}(H_* \circ F)] dx.$$

In order to compute $I_\rho(r)$, let us consider a sequence of functions $g^{(j)} = (g_1^{(j)}, \dots, g_m^{(j)}) \in C^2(\Omega, \mathbb{R}_y^m)$ such that $g^{(j)} \rightarrow f$ in $C^1(\Omega, \mathbb{R}_y^m)$ (as $j \rightarrow \infty$). Denote $B(x_0, r)$ by B_r and observe that

$$\begin{aligned} & \sum_{i=1}^N \int_{B_r} \varphi_{\rho, r} D_i g_h^{(j)} \operatorname{div}[(K_i \circ F)(H_* \circ F) - (H_i \circ F)(K_* \circ F)] dx = \\ &= \sum_{i=1}^N \int_{B_r} \varphi_{\rho, r} \operatorname{div} [D_i g_h^{(j)} (K_i \circ F)(H_* \circ F) - D_i g_h^{(j)} (H_i \circ F)(K_* \circ F)] dx + \\ & \quad - \int_{B_r} \varphi_{\rho, r} \underbrace{\sum_{i,k=1}^N [D_{ik}^2 g_h^{(j)} (H_k \circ F)(K_i \circ F) - D_{ik}^2 g_h^{(j)} (K_k \circ F)(H_i \circ F)]}_{=0} dx \\ &= \underbrace{\int_{B_r} \operatorname{div} (\varphi_{\rho, r} \{ [(Dg_h^{(j)}) \cdot (K_* \circ F)](H_* \circ F) - [(Dg_h^{(j)}) \cdot (H_* \circ F)](K_* \circ F) \})}_{=0 \text{ (by the divergence theorem)}} dx + \\ & \quad + \int_{B_r} \{ [(Dg_h^{(j)}) \cdot (H_* \circ F)](K_* \circ F) - [(Dg_h^{(j)}) \cdot (K_* \circ F)](H_* \circ F) \} \cdot D\varphi_{\rho, r} dx \end{aligned}$$

Passing to the limit ($j \rightarrow \infty$) we get

$$I_\rho(r) = \int_{B_r} \{ [(Df_h) \cdot (H_* \circ F)](K_* \circ F) - [(Df_h) \cdot (K_* \circ F)](H_* \circ F) \} \cdot D\varphi_{\rho, r} dx$$

hence, by (4.1) and Proposition 3.3

$$\begin{aligned} I_\rho(r) &\sim \int_{B_r} [(H_{N+h} \circ F)(K_* \circ F) - (K_{N+h} \circ F)(H_* \circ F)] \cdot D\varphi_{\rho, r} dx \\ &= \underbrace{\int_{B_r} \operatorname{div} (\varphi_{\rho, r} [(H_{N+h} \circ F)(K_* \circ F) - (K_{N+h} \circ F)(H_* \circ F)])}_{=0 \text{ (by the divergence theorem)}} dx + \\ & \quad + \int_{B_r} \varphi_{\rho, r} \operatorname{div} [(K_{N+h} \circ F)(H_* \circ F) - (H_{N+h} \circ F)(K_* \circ F)] dx \\ &= \int_{B_r} \varphi_{\rho, r} [(K_{N+h} \circ F) \operatorname{div}(H_* \circ F) - (H_{N+h} \circ F) \operatorname{div}(K_* \circ F)] dx + \\ & \quad + \int_{B_r} \varphi_{\rho, r} [(H_* \circ F) \cdot D(K_{N+h} \circ F) - (K_* \circ F) \cdot D(H_{N+h} \circ F)] dx \\ &= -J_\rho(r) - \int_{B_r} \Lambda_h^{(2)} \varphi_{\rho, r} dx. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{B_r} \Lambda_h \varphi_{\rho,r} dx &= \int_{B_r} \Lambda_h^{(1)} \varphi_{\rho,r} dx + \int_{B_r} \Lambda_h^{(2)} \varphi_{\rho,r} dx \\ &\sim I_\rho(r) + J_\rho(r) + \int_{B_r} \Lambda_h^{(2)} \varphi_{\rho,r} dx \\ &\sim 0. \end{aligned}$$

From Proposition 3.1 we conclude that $\Lambda_h(x_0) = 0$ for all $h = 1, \dots, m$, that is

$$Df(x_0)[H, K]_*(z_0) - [H, K]_\#(z_0) = 0$$

which yields

$$[H, K](z_0) = \begin{pmatrix} [H, K]_*(z_0) \\ [H, K]_\#(z_0) \end{pmatrix} = \begin{pmatrix} [H, K]_*(z_0) \\ Df(x_0)[H, K]_*(z_0) \end{pmatrix} = \begin{pmatrix} I \\ Df(x_0) \end{pmatrix} [H, K]_*(z_0).$$

Thus

$$[H, K](z_0) \in \text{Im } DF(x_0) = T_{z_0}\Gamma = T_{z_0}\mathcal{M}.$$

□

Example 4.1. Let k be a positive integer and consider the distribution $H\mathbb{H}^k$ mentioned in the introduction, that is the horizontal subbundle of the tangent bundle to the Heisenberg group \mathbb{H}^k . It is the distribution of rank $2k$ on $\mathbb{R}_x^{2k} \times \mathbb{R}_y$ described by the following vector fields:

$$\begin{aligned} X_i &:= D_i + 2x_{k+i}D_{2k+1} \quad (i = 1, \dots, k) \\ Y_i &:= D_{k+i} - 2x_iD_{2k+1} \quad (i = 1, \dots, k) \end{aligned}$$

compare [1]. As we recalled in the introduction, $H\mathbb{H}^k$ is translation invariant along \mathbb{R}_y . Hence a $(2k)$ -dimensional C^1 -submanifold \mathcal{M}_0 of \mathbb{R}^{2k+1} has to exist such that the tangency set of \mathcal{M}_0 with respect to $H\mathbb{H}^k$ has positive measure, namely

$$\mathcal{H}^{2k}(\tau(\mathcal{M}_0, H\mathbb{H}^k)) > 0$$

compare [2, Proposition 8.2]. Nevertheless, according to [7], the noninvolutivity of $H\mathbb{H}^k$ prevents the existence of superdensity points of $\tau(\mathcal{M}_0, H\mathbb{H}^k)$, that is, $\tau(\mathcal{M}_0, H\mathbb{H}^k)^{(2k+1)} = \emptyset$. Theorem 4.1 makes it possible to improve this result about the “low degree of tangency” of \mathcal{M}_0 with respect to $H\mathbb{H}^k$, showing that (for all $i \in \{1, \dots, k\}$) it even has to be $\mathcal{T}_i^{(2k+1)} = \emptyset$, where

$$\mathcal{T}_i := \{z \in \mathcal{M}_0 : X_i(z), Y_i(z) \in T_z\mathcal{M}_0\} \quad (\text{note that } \mathcal{T}_i \supset \tau(\mathcal{M}_0, H\mathbb{H}^k)).$$

In fact, if $z_0 \in \mathcal{T}_i^{(2k+1)}$ then (by Theorem 4.1) one has $[X_i, Y_i](z_0) \in T_{z_0}\mathcal{M}_0$, which contradicts the well known identity $[X_i, Y_i](z_0) = -4D_{2k+1}$.

Eventually, as a corollary of Theorem 4.1, we can easily obtain the following result which in the case $n = N$ has been proved in [7, Theorem 1.1] directly (compare point (5) in the introduction).

Corollary 4.1. *Let \mathcal{D} be a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^{N+m}$ and let \mathcal{M} be a N -dimensional C^1 submanifold of U , with $N \geq n$. Given $z_0 \in \mathcal{M} \cap \tau(\mathcal{M}, \mathcal{D})^{(N+1)}$, the following facts hold:*

- (1) *For every couple H, K of vector fields of class C^1 in a neighbourhood V of z_0 such that $H(z), K(z) \in \mathcal{D}(z)$ for all $z \in \mathcal{M} \cap V$, one has $[H, K](z_0) \in T_{z_0}\mathcal{M}$.*
- (2) *If $N = n$ then \mathcal{D} is involutive at z_0 .*

Proof. (1) Let us define

$$\mathcal{T} := \{z \in \mathcal{M} \cap V : H(z), K(z) \in T_z\mathcal{M}\}$$

and observe that $V \cap \tau(\mathcal{M}, \mathcal{D}) \subset \mathcal{T}$. It follows that $V \cap \tau(\mathcal{M}, \mathcal{D})^{(N+1)} \subset \mathcal{T}^{(N+1)}$, hence $z_0 \in \mathcal{M} \cap \mathcal{T}^{(N+1)}$ and the conclusion follows from Theorem 4.1.

(2) Since $N = n$, by continuity one has $T_{z_0}\mathcal{M} = \mathcal{D}(z_0)$. Then the conclusion follows at once from (1). \square

REFERENCES

- [1] Z.M. Balogh: Size of characteristic sets and functions with prescribed gradient. J. reine angew. Math. 564, 63-83 (2003).
- [2] Z.M. Balogh, C. Pinte, H. Rohner: Size of tangencies to non-involutive distributions. Indiana Univ. Math. J. 60, no. 6, 2061-2092 (2011).
- [3] I. Chavel: Riemannian Geometry: a Modern Introduction. (Cambridge Tracts in Mathematics 108) Cambridge University Press 1995.
- [4] S. Delladio: A note on a generalization of the Schwarz theorem about the equality of mixed partial derivatives. Math. Nachr. 290 (2017), n. 11-12, 1630-1636.
- [5] S. Delladio: Structure of prescribed gradient domains for non-integrable vector fields. Ann. Mat. Pura ed Appl. (1923 -) 198 (2019), n.3, 685-691.
- [6] S. Delladio: Structure of tangencies to distributions via the implicit function theorem. Rev. Mat. Iberoam. 34 (2018), n. 3, 1387-1400.
- [7] S. Delladio: The tangency of a C^1 smooth submanifold with respect to a non-involutive C^1 distribution has no superdensity points. Indiana Univ. Math. J. 68 (2019), n. 2, 393-412.
- [8] M. Derridj: Sur un théorème de traces. Ann. Inst. Fourier (Grenoble) 22 (1972), n. 2, 73-83.
- [9] H. Federer: Geometric Measure Theory. Springer-Verlag 1969.
- [10] R. Narasimhan : Analysis on real and complex manifolds. North-Holland Math. Library 35, North-Holland 1985.