Some combinatorial properties of the Hurwitz series ring

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Abstract

We study some properties and perspectives of the Hurwitz series ring $H_R[[t]]$, for a commutative ring with identity R. Specifically, we provide a closed form for the invertible elements by means of the complete ordinary Bell polynomials, we highlight some connections with well–known transforms of sequences, and we see that the Stirling transforms are automorphisms of $H_R[[t]]$. Moreover, we focus the attention on some special subgroups studying their properties. Finally, we introduce a new transform of sequences that allows to see one of this subgroup as an ultrametric dynamic space.

1 The ring of Hurwitz series, transformations of sequences and automorphisms

Given a commutative ring with identity R, let $H_R[[t]]$ denote the Hurwitz series ring whose elements are the formal series of the kind

$$A(t) := \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n,$$

equipped with the standard sum and the binomial convolution as product. Given two formal series $A(t)$ and $B(t)$, the binomial convolution is defined as follows:

$$A(t) \star B(t) := C(t)$$

where

$$c_n := \sum_{h=0}^{n} \binom{n}{h} a_h b_{n-h}.$$

The ring of Hurwitz series has been organizationally introduced by Keigher [9] and in the recent years it has been extensively studied, see, e.g., [10], [11], [7], [3], [4], [5].

The Hurwitz series ring is trivially isomorph to the ring H_R whose elements are infinite sequences of elements of R, with operations $+$ and $\star$. Moreover, fixed any integer n, we can also consider the rings $H_R^{(n)}$ whose elements are sequences of elements of R with length n.

Remark 1. *The binomial convolution is a commutative product and the identity in H_R is the sequence*

$$(1, 0, 0, ...).$$

Moreover, H_R can be also viewed as an R–algebra considering the map

$$\pi : R \to H_R, \quad \pi(r) := (r, 0, 0, ...),$$

for any $r \in R$.

Definition 1. *In the following, when we consider an element* $a \in H_R$*, we refer to a sequence* $(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, ...)$*,* $a_i \in R$ *for all* $i \geq 0$*, with exponential generating function (e.g.f.) denoted by* $A(t)$*.*

Proposition 1. *An element* $a \in H_R$ *is invertible if and only if* $a_0 \in R$ *is invertible, i.e.,*

$$H_R^* = \{a \in H_R : a_0 \in R^*\}.$$

Proof. The proof is straightforward. □

Given $a \in H_R^*$, it is possible to recursively evaluate the elements of $b = a^{-1}$. Indeed, it is easy to see that $b_0 = a_0^{-1}$ and

$$b_n = -a_0^{-1} \sum_{h=1}^{n} \binom{n}{h} a_h b_{n-h}, \quad \forall n > 0,$$

since $a \star b = (1, 0, 0, ...)$, i.e.,

$$\sum_{h=0}^{n} \binom{n}{h} a_n b_{n-h} = 0, \quad n > 0.$$

Moreover, we can write the elements of b in a closed form by means of the complete ordinary Bell polynomials [2]. Here we use the definition as given in [13].

Definition 2. *The complete ordinary Bell polynomials are defined by*

$$B_n(\boldsymbol{t}) = \sum_{k=1}^{n} B_{n,k}(\boldsymbol{t}),$$

where $\boldsymbol{t} = (t_1, t_2, \ldots)$ *and the partial ordinary Bell polynomials* $B_{n,k}(\boldsymbol{t})$ *satisfy*

$$\left(\sum_{n \geq 1} t_n z^n \right)^k = \sum_{n \geq k} B_{n,k}(\boldsymbol{t}) z^n,$$

and

$$B_{n,k}(\boldsymbol{t}) = \sum_{\substack{i_1+2i_2+\cdots+ni_n=n \\ i_1+i_2+\cdots+i_n=k}} \frac{k!}{i_1! i_2! \cdots i_n!} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} \quad .$$

We also introduce the Invert transform (see, e.g., [6] for information) that is strictly related to the Bell polynomials.

Definition 3. *The Invert transform* $\mathcal{I}$ *maps a sequence* $a = (a_n)_{n=0}^{\infty}$ *into a sequence* $b = (b_n)_{n=0}^{\infty}$ *as follows:*

$$\mathcal{I}(a) = b, \quad \sum_{n=0}^{\infty} b_n t^n = \frac{\sum_{n=0}^{\infty} a_n t^n}{1 - t \sum_{n=0}^{\infty} a_n t^n}.$$

Barbero et al. [1] highlighted that, given $a \in H_R$ and $b = \mathcal{I}(a)$, we have

$$b_n = B_{n+1}(a_0, a_1, a_2, ...), \quad n \geq 0.$$

These tools allow to obtain a nice closed form for the inverse of a sequence in the ring H_R.

Theorem 1. *Let* $a, b = a^{-1} \in H_R^*$ *be sequences with e.g.f.* $A(t)$ *and* $B(t)$*, respectively. Then*

$$b_n = \frac{n! B_n(g_0, g_1, g_2, ...)}{a_0}, \quad \forall n \geq 0,$$

where

$$g_n = -\frac{a_{n+1}}{a_0 (n+1)!}, \quad n \geq 0.$$

Proof. Since $b = a^{-1}$, we have $B(t) = \dfrac{1}{A(t)}$. It is possible to see that

$$\bar{G}(t) = \frac{1}{t}\left(1 - \frac{A(t)}{a_0}\right)$$

is the ordinary generating function of the sequence $g = (g_n)_{n=0}^{\infty}$. Indeed

$$\bar{G}(t) = \frac{1}{t}\left(1 - \frac{1}{a_0}\left(a_0 + \sum_{n=1}^{\infty}\frac{a_n}{n!}t^n\right)\right) = -\frac{1}{t}\sum_{n=1}^{\infty}\frac{a_n}{a_0 n!}t^n = \sum_{n=0}^{\infty}\left(-\frac{a_{n+1}}{a_0(n+1)!}\right)t^n = \sum_{n=0}^{\infty} g_n t^n.$$

The ordinary generating function $\bar{H}(t)$ of the sequence $h = \mathcal{I}(a)$ must satisfy

$$B(t) = \frac{1}{a_0}(1 + t\bar{H}(t)).$$

Indeed

$$\frac{1}{a_0}(1 + t\bar{H}(t)) = \frac{1}{a_0}\left(1 + \frac{t\bar{G}(t)}{1 - t\bar{G}(t)}\right) = \frac{1}{a_0(1 - t\bar{G}(t))} = \frac{1}{a_0\frac{A(t)}{a_0}} = B(t).$$

Observing that $h_n = B_{n+1}(g_0, g_1, g_2, ...)$ and conventionally $B_0(g_0, g_1, g_2, ...) = 1$, we have

$$B(t) = \frac{1}{a_0}\left(1 + \sum_{n=0}^{\infty} B_{n+1}(g_0, g_1, g_2, ...)t^{n+1}\right) = \sum_{n=0}^{\infty}\frac{B_n(g_0, g_1, g_2, ...)}{a_0}t^n$$

from which the thesis follows. □

It is interesting to observe that some well–studied transformations of sequences can be defined within the Hurwitz series ring as the product of fixed sequences. Specifically, let us consider the Binomial interpolated and the Boustrophedon transforms.

Definition 4. *The* Binomial interoplated transform $\mathcal{L}^{(y)}$*, with parameter* $y \in R$*, maps any sequence* $a \in H_R$ *into a sequence* $b = \mathcal{L}^{(y)}(a) \in H_R$*, whose terms are*

$$b_n = \sum_{h=0}^{n}\binom{n}{h}y^{n-h}a_h.$$

See [1] for some information.

Definition 5. *The* Boustrophedon transform $\mathcal{B}$ *maps any sequence* $a \in H_R$*, with e.g.f.* $A(t)$*, into a sequence* $b = \mathcal{B}(a) \in H_R$ *with e.g.f.* $B(t) = (\sec t + \tan t)A(t)$*. See [12] for some information.*

Clearly, the Binomial interpolated operator can be defined by means of the binomial convolution considering the sequence

$$\lambda = (\lambda_n)_{n=0}^{\infty} = (1, y, y^2, ..., y^n, ...),$$

i.e.,

$$\mathcal{L}^{(y)}(a) = \lambda \star a,$$

for any $a \in H_R$.

Furthermore, considering the sequence of the Euler zigzag numbers $\beta = (\beta_n)_{n=0}^{\infty}$, having e.g.f.

$$\sum_{n=0}^{\infty}\frac{\beta_n}{n!}t^n := \sec(t) + \tan(t),$$

we have

$$\mathcal{B}(a) = \beta \star a,$$

for any $a \in H_R$ (see [12] for information about the Euler zigzag numbers).

The Hurwitz series ring is strictly connected to other well–known and interesting transforms of sequences.

Definition 6. *The* Stirling transform $\mathcal{S}$ *maps any sequence $a \in H_R$ into a sequence $b = \mathcal{S}(a) \in H_R$, whose terms are*

$$b_n = \sum_{h=0}^{n} \left\{ n \atop h \right\} a_h,$$

where $\left\{ n \atop h \right\}$ are the Stirling numbers of the second kind. Let us observe that $\mathcal{S}$ is a bijection whose inverse is $\mathcal{T}$ that maps any sequence $a \in H_R$ into a sequence $b = \mathcal{T}(a) \in H_R$, whose terms are

$$b_n = \sum_{h=0}^{n} \left[n \atop h \right] a_h,$$

where $\left[n \atop h \right]$ are the Stirling numbers of the first kind. For information see [6].

Proposition 2. *Given $a \in H_R$ with e.g.f. $A(t)$, then $b = \mathcal{S}(a)$ has e.g.f. $B(t) = A(e^t - 1)$.*

Proof. Let us recall that $\left\{ n \atop h \right\} = 0$ when $h > n$ and

$$\sum_{n=0}^{\infty} \left\{ n \atop h \right\} \frac{t^n}{n!} = \frac{(e^t-1)^h}{h!},$$

see [8] pag. 337.

Thus, we have

$$\mathcal{B}(t) = \sum_{n=0}^{\infty} \left(\sum_{h=0}^{n} \left\{ n \atop h \right\} a_h \right) \frac{t^n}{n!} = \sum_{h=0}^{\infty} a_h \left(\sum_{n=0}^{\infty} \left\{ n \atop h \right\} \frac{t^n}{n!} \right) = \sum_{h=0}^{\infty} a_h \frac{(e^t-1)^h}{h!} = A(e^t - 1).$$

□

Proposition 3. *Given any sequences $a, b \in H_R$, we have*

1. *$S(a + b) = \mathcal{S}(a) + \mathcal{S}(b)$*
2. *$S(a \star b) = \mathcal{S}(a) \star \mathcal{S}(b)$*

Proof. 1. The proof is straightforward.

2. We prove that the sequences $\mathcal{S}(a \star b)$ and $\mathcal{S}(a) \star \mathcal{S}(b)$ have the same exponential generating functions. Let $A(t)$ and $B(t)$ be the e.g.f. of a and b, respectively. By Proposition 2, $A(e^t - 1)$ and $B(e^t - 1)$ are the e.g.f. of the sequences $c := \mathcal{S}(a)$ and $d := \mathcal{S}(b)$, respectively. Said $S(t)$ the e.g.f. of $\mathcal{S}(a \star b)$, we have

$$S(t) = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \binom{n}{h} c_h d_{n-h} = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{c_h t^h}{h!} \cdot \frac{d_{n-h} t^{n-h}}{(n-h)!} = A(e^t - 1) B(e^t - 1).$$

□

Hence, $\mathcal{S}$ and $\mathcal{T}$ are automorphisms of H_R. Another (more trivial) automorphism of H_R is the transform $\mathcal{E}$ defined as follows.

Definition 7. *The transform $\mathcal{E}$ maps any sequence $a \in H_R$ into a sequence $b = \mathcal{E}(a) \in H_R$, whose terms are*

$$b_n = (-1)^n a_n.$$

The transform $\mathcal{E}$ is very used for studying properties of integer sequences. In this case, it is immediate to see that, given any $a \in H_R$ with e.g.f. $A(t)$, then $\mathcal{E}(a)$ has e.g.f. $A(-t)$.

2 Special subgroups of H_R^*

In this section, we study the properties of some interesting subgroups of H_R^*.

Definition 8. *Let U_R and B_R be subgroups of H_R^* defined as*

$$U_R = \{a \in H_R^* : a_0 = 1\}, \quad B_R = \{a \in U_R : \mathcal{E}(a) = a^{-1}\}.$$

Definition 9. *The transform $\mathcal{V}$ maps any sequence $a \in H_R$ into a sequence $b = \mathcal{V}(a) \in H_R$, where $b = (1, a_0, a_{1,2}, ...)$. Let us denote $\mathcal{V}^k$ the iteration of $\mathcal{V}$ for k times.*

Trivially, given any $a \in H_R^*$, then $\mathcal{V}^k(a) \in U_R$, for any exponent $k \geq 1$. It is interesting to see in the next proposition how the e.g.f. of a sequence changes under the map $\mathcal{V}^k$.

Proposition 4. *Given $a \in H_R^*$ with e.g.f. $A(t)$, then $\mathcal{V}^k(a)$ has e.g.f. $\mathcal{J}^k(A(t))$, $k \geq 1$, where*

$$\mathcal{J}(A(t)) := 1 + \int_0^t A(u)du$$

and $\mathcal{J}^k = \underbrace{\mathcal{J} \circ ... \circ \mathcal{J}}_{k}$.

Proof. The e.g.f. $V(t)$ of $\mathcal{V}^k(a)$ is

$$V(t) = \sum_{h=0}^{k-1} \frac{t^h}{h!} + \sum_{h=0}^{\infty} a_h \frac{t^{h+1}}{(h+1)!},$$

i.e., we have

$$V(0) = V'(0) = ... = V^{(k-1)}(0) = 1, \quad V^{(k)}(t) = A(t),$$

where $V^{(k)}(t)$ is the k–th derivative of $V(t)$.
If $k = 1$, we clearly obtain that $V(t) = 1 + \int_0^t A(u)du$ and when $k > 1$ the thesis easily follows. □

Now, we see some interesting properties for the subgroup B_R. It is immediate to see that $a \in B_R$ if and only if $A(t)A(-t) = 1$. Moreover, if $f(t)$ is an odd function, then the sequence with e.g.f. $e^{f(t)}$ belongs to B_R. It is interesting to observe that the set of sequences having e.g.f. $e^{f(t)}$, with $f(t)$ odd function, is a subgroup of B_R.

Proposition 5. *The group B_R is closed with respect to the transformations $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{L}^{(y)}$, for any $y \in R$.*

Proof. By definition of B_R, it is immediate that $\mathcal{E}(B_R) = B_R$ (with this notation, we say that given any $a \in B_R$, then $\mathcal{E}(a)$ is still in B_R).

Given any $a \in B_R$, with e.g.f. $A(t)$, we have that $b = \mathcal{L}^{(y)}(a)$ has e.g.f. $e^{(kt)}A(t)$, i.e., $b \in B_R$.

Let us recall that the Euler zigzag numbers β have e.g.f. $B(t) = \sec(t) + \tan(t)$. We have

$$B(t) - \frac{1}{B(-t)} = \frac{1 + \sin(t)}{\cos(t)} - \frac{\cos(-t)}{1 + \sin(-t)} = \frac{1 - \sin^2(t) - \cos^2(t)}{\cos(t)(1 - \sin(t))} = 0,$$

i.e., $\beta \in B_R$. Hence, given any $a \in B_R$, $\mathcal{B}(a) = \beta \star a \in B_R$. □

Remark 2. *It is possible to see that the group B_R is not closed with respect to the transformation $\mathcal{S}$. It would be interesting to characterize the group $\mathcal{S}(B_R)$.*

The elements of a sequence $a \in B_R$ are constrained to severe restriction, since it holds $\mathcal{E}(a) = a$. Said $b = a^{-1}$ and $c = \mathcal{E}(a)$, we have, for instance,

$$b_0 = 1, \quad b_1 = -a_1, \quad b_2 = 2a_1^2 - a_2$$

and

$$c_0 = 1, \quad c_1 = -a_1, \quad c_2 = a_2,$$

i.e., the element a_1 of the sequence a can be arbitrary, while a_2 must satisfy

$$a_2 = 2a_1^2 - a_2,$$

i.e., $a_2 = a_1^2$. By continuing in this way, we can also see, e.g., that a_3 can be arbitrary, while $a_4 = -3a_1^4 + 4a_1 a_3$. In the following theorems, we provide a closed form for the elements of a sequence in B_R.

Theorem 2. *Given any $a \in B_R$, we have*

$$a_{2n} = (2n)! \sum_{k=0}^{n} \binom{1/2}{k} B_{n+k,2k}(x_1, ..., x_{n-k+1}), \quad \forall n \geq 1,$$

where $x_i = \dfrac{a_{2i-1}}{(2i-1)!}$, $\binom{1/2}{k} = \dfrac{\prod_{j=0}^{k-1}(1/2 - j)}{k!}$, and $B_{n+k,2k}(x_1, ..., x_{n-k+1})$ is the partial ordinary Bell polynomial as given in Definition 2.

Proof. Let $A(t)$ be the e.g.f. of a, we can write $A(t) = P(t) + D(t)$, where

$$P(t) = \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!} t^{2n}$$

and

$$D(t) = \sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n-1} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n}.$$

Moreover, we have

$$1 = A(t)A(-t) = (P(t) + D(t))(P(t) - D(t)) = P(t)^2 - D(t)^2,$$

since $A(-t) = P(-t) + D(-t) = P(t) - D(t)$ and $\mathcal{E}(a) = a^{-1}$. Since, $P(0) = a_0 = 1$ and $D(0) = 0$, we obtain from the MacLaurin series of $(1+t)^{1/2}$ that

$$P(t) = (1 + D(t)^2)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} D(t)^{2k}.$$

By definition of partial ordinary Bell polynomials we have

$$D(t)^{2k} = \sum_{m=2k}^{\infty} B_{m,2k}(x_1, ..., x_{m-2k+1}) t^{2m-2k}.$$

If we set $n = m - k$, we get

$$P(t) = \sum_{k=0}^{\infty} \binom{1/2}{k} \sum_{n=k}^{\infty} B_{n+k,2k}(x_1, ..., x_{n-k+1}) t^{2n} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{1/2}{k} B_{n+k,2k}(x_1, ..., x_{n-k+1}) \right) t^{2n}.$$

From this identity and $P(t) = \sum_{k=0}^{\infty} \dfrac{a_{2n}}{(2n)!} t^{2n}$, we finally obtain

$$a_{2n} = (2n)! \sum_{k=0}^{n} \binom{1/2}{k} B_{n+k,2k}(x_1, ..., x_{n-k+1}), \quad \forall n \geq 1.$$

□

By Definition 2 and observing that $B_{0,0} = 1$, $B_{h,0} = 0$ for $h \geq 1$, we immediately have the following corollary.

Corollary 1. *Given $a \in B_R$, we have*

$$a_{2n} = (2n)! \sum_{k=1}^{n} \binom{1/2}{k} \sum_{\substack{i_1 + 2i_2 + \cdots + (n-k+1)\, i_{n-k+1} = n+k \\ i_1 + i_2 + \cdots + i_{n-k+1} = 2k}} (2k)! \prod_{j=1}^{n-k+1} \frac{1}{i_j!\,((2j-1)!)^{i_j}} \prod_{j=1}^{n-k+1} a_{2j-1}^{i_j} \tag{1}$$

Theorem 3. *Given any $a \in B_R$, if $a_2 \in R^*$, we have*

$$a_{2n+1} = (2n+1)! \sum_{k=0}^{n} a_2^{1/2-k} \binom{1/2}{k} B_{n,k}(x_1, ..., x_{n-k+1}), \quad \forall n \geq 0,$$

where $x_i = \dfrac{1}{(2i+2)!} \sum_{k=0}^{i+1} \binom{2i+2}{k} a_{2k} a_{2(n-k+1)}$.

Proof. Let $A(t)$ be the e.g.f. of a, we can write $A(t) = P(t) + D(t)$, where

$$P(t) = \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!} t^{2n}$$

and

$$D(t) = \sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n-1} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n}.$$

We have

$$P^2(t) = 1 + a_2 t^2 + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} \binom{2n}{2k} a_{2k} a_{2n-2k} \right) \frac{t^{2n}}{(2n)!}$$

and

$$D(t) = t a_2^{\frac{1}{2}} \left(1 + \sum_{n=2}^{+\infty} \left(\frac{a_2^{-1}}{(2n)!} \sum_{k=0}^{n} \binom{2n}{2k} a_{2k} a_{2n-2k} \right) t^{2n-2} \right)^{\frac{1}{2}} = t a_2^{\frac{1}{2}} \left(1 + \sum_{m=1}^{+\infty} \left(\frac{a_2^{-1}}{(2m+2)!} \sum_{k=0}^{m+1} \binom{2m+2}{2k} a_{2k} a_{2m-2k+2} \right) t^{2m} \right)^{\frac{1}{2}}$$

Then, considering the Maclaurin series expansion of $(1+t)^{1/2}$ and by definition of partial ordinary Bell polynomials, we obtain

$$D(t) = \sum_{n=0}^{\infty} \left(a_2^{1/2-k} \binom{1/2}{k} B_{n,k}(x_1, ..., x_{n-k+1}) \right) t^{2n+1}.$$

Remembering that $D(t) = \sum_{n=1}^{\infty} \dfrac{a_{2n-1}}{(2n-1)!} t^{2n-1}$, the proof is completed. □

Remark 3. *When $R = \mathbb{Z}$, $B_{\mathbb{Z}}$ contains many well–known and important integer sequences.*

We have seen that the Euler zigzag numbers belong to $B_{\mathbb{Z}}$. They are listed in OEIS [15] as A000111. Thus, all the sequences having as e.g.f. a power of $\sec(t) + \tan(t)$ are in $B_{\mathbb{Z}}$. For instance the sequence A001250 in OEIS has e.g.f. $(\sec(t) + \tan(t))^2$, whose n–th element is the number of alternating permutations of order n.

Moreover, the sequence A000667 which is the Boustrophedon transform of all–1's sequence has e.g.f. $e^t(\sec(t) + \tan(t))$ and belongs to $B_{\mathbb{Z}}$.

Another interesting sequence in $B_{\mathbb{Z}}$ is the sequence A000831, with e.g.f. $\dfrac{1+\tan(t)}{1-\tan(t)}$.

The sequences A006229 and A002017 belong to the subgroup of $B_{\mathbb{Z}}$ assembled by the sequences having e.g.f. of the shape $e^{f(t)}$, with $f(t)$ odd function. Indeed, they have e.g.f. $e^{\tan(t)}$ and $e^{\sin(t)}$, respectively. They are very interesting sequences.

Thanks to Theorem 2 and Corollary 1, we have many new interesting identities for the elements of these sequences. Furthermore, it is quite surprising that all these (very different) sequences satisfy same limitng conditions.

In the following, we will introduce a new transform of sequences that arises from the study of B_R and it will be very useful and interesting. For example it allows to see U_R as a dynamic ultrametric space.

Given $a, b = a^{-1} \in U_R$, we know that

$$\sum_{h=0}^{n} \binom{n}{h} a_h b_{n-h} = 0, \quad \forall n > 0,$$

from which it follows that

$$a_n = -\sum_{n=0}^{n-1} \binom{n}{h} a_h b_{n-h}, \quad b_n = -\sum_{h=0}^{n-1} \binom{n}{h} b_h a_{n-h}.$$

If $a \in B_R$, i.e. $A(-t) = A(t)^{-1}$, then, for all $n > 0$, we have

$$-\sum_{h=0}^{n-1} \binom{n}{h} (-1)^h a_h a_{n-h} = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2a_n & \text{if } n \text{ even} \end{cases}.$$

Thus it is natural to define the following transform.

Definition 10. *The* autoconvolution *transform $\mathcal{A}$ maps a sequence $a \in H_R$ into a sequence $b = \mathcal{A}(a) \in H_R$, where*

$$\begin{cases} b_0 = a_0 \\ b_{2n+1} = a_{2n+1}, \quad \forall n \geq 0 \\ b_{2n} = \frac{1}{2}\left(-\sum_{h=1}^{n-1} \binom{n}{h} (-1)^h a_n a_{n-h}\right), \quad \forall n \geq 1 \end{cases}.$$

It is immediate to prove the following proposition.

Proposition 6. *Given any $a \in H_R$, we have $a \in B_R \Leftrightarrow \mathcal{A}(a) = a$.*

Finally, we introduce another transform striclty related to $\mathcal{A}$.

Definition 11. *The transform $\mathcal{U}$ maps a sequence $a \in H_R$ into a sequence $\mathcal{U}(a) = b \in H_R$ as follows:*

$$\begin{cases} b_0 = a_0 \\ b_{2n+1} = a_{2n+1}, \quad \forall n \geq 0 \\ b_{2n} = (2n)! \sum_{k=0}^{n} \binom{1/2}{k} B_{n+k,2k}(x_1, ..., x_{n-k+1}), \quad \forall n \geq 1 \end{cases},$$

where $x_i = \dfrac{a_{2i-1}}{(2i-1)!}$.

Remark 4. *Given any sequence $a \in U_R$, the transform $\mathcal{U}$ constructs a sequence in B_R where the elements in odd places are the elements of a. Clearly, we have that a sequence $a \in U_R$ is in B_R if and only if $a = \mathcal{U}(a)$.*

Proposition 7. *Given $a \in H_R$, with e.g.f. $A(t)$, then $\mathcal{U}(a)$ has e.g.f.*

$$A'(t) = \left(1 + \left(\frac{A(t) - A(-t)}{2}\right)^2\right)^{1/2} + \frac{A(t) - A(-t)}{2}.$$

Proof. We can write $A(t) = P(t) + D(t)$, where

$$P(t) = \sum_{n=0}^{\infty} \frac{a_{2n}}{(2n)!} t^{2n}, \quad D(t) = \sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n-1}.$$

We have $A(-t) = P(t) - D(t)$ and consequently $D(t) = \dfrac{A(t) - A(-t)}{2}$. By Theorem 2, the terms in the odd places of $\mathcal{U}(a)$ have e.g.f $(1 + D(t)^2)^{1/2}$. Thus, we have

$$A'(t) = (1 + D(t)^2)^{1/2} + D(t).$$

□

Given $a, b \in H_R$, let us define

$$\delta(a,b) := 2^{-k},$$

if $a_i = b_i$, for any $0 \leq i \leq k-1$. It is well–known that δ is an ultrametric in H_R. Indeed,

- $\delta(a,b) = 0 \Leftrightarrow a = b$,
- $\delta(a,b) = \delta(b,a)$,
- $\delta(a,c) \leq \max(\delta(a,b), \delta(b,c))$,

for any $a, b, c \in H_R$. Thus, (H_R, δ) is an ultrametric space.

Let us recall that we denote $H_R^{(n)}$ the ring whose elements are sequences of elements of R with length n. Similarly, $U_R^{(n)}$ and $B_R^{(n)}$ are the subgroups of $H_R^{(n)*}$ corresponding to the subgroups U_R and B_R of H_R^*, respectively.

Theorem 4. *Given any $a \in U_R$, we have*

$$\delta(\mathcal{A}^n(a), \mathcal{U}(a)) \leq \frac{1}{2^{2(n+1)}},$$

where $\mathcal{A}^n = \underbrace{\mathcal{A} \circ ... \circ \mathcal{A}}_{n}$.

Proof. We prove the thesis by induction.
Let us denote $a' = \mathcal{U}(a)$ and $b = \mathcal{A}(a)$. It is straightforward to check that

$$a' = (1, a_1, a_1^2, a_3, ...), \quad b = (1, a_1, a_1^2, a_3, ...).$$

Thus, a' and b coincide at least in the first 4 elements, i.e.,

$$\delta(\mathcal{A}(a), \mathcal{U}(a)) \leq \frac{1}{2^4}.$$

Now, let us suppose that given $b = \mathcal{A}^n(a)$, we have $\delta(\mathcal{A}^n(a), \mathcal{U}(a)) \leq \dfrac{1}{2^{2(n+1)}}$, i.e. $b_i = a'_i$ for all $i \leq 2n+1$ and consider $c = \mathcal{A}(b)$. Since $a' \in B_R$, we remember that for all $n > 0$ we have

$$-\sum_{h=0}^{n-1} \binom{n}{h} (-1)^h a'_h a'_{n-h} = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2a'_n & \text{if } n \text{ even} \end{cases}.$$

Thus, by Definition 10, we obtain $c_i = a'_i$ for all $i \leq 2n+3$, since $(b_0, ..., b_{2n+1}) = (a'_0, ..., a'_{2n+1}) \in B_R^{(2n+2)}$ by inductive hypothesis. Hence, we have proved that

$$\delta(\mathcal{A}^{n+1}(a), \mathcal{U}(a)) \leq \frac{1}{2^{2(n+2)}}.$$

□

As a consequence of Theorem 4, we can observe that $\mathcal{A}$ can be considered as an approximation of $\mathcal{U}$. Indeed, given a sequence $a \in U_R$, sequences $\mathcal{A}^n(a)$ have more elements equal to elements of $\mathcal{U}(a)$ for greater values of n.

Example 1. Given $a = (a_0, a_1, a_2, a_3, a_4, a_5) \in U_R^{(6)}$, then

$$\mathcal{U}(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_1^4, a_5)$$

and

$$\mathcal{A}(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_2^2, a_5).$$

Considering $\mathcal{A}^2$, we obtain

$$\mathcal{A}^2(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_1^4, a_5) = \mathcal{U}(a).$$

In other words, given any sequence $a \in U_R^{(6)}$, $\mathcal{A}^2(a) = \mathcal{U}(a) \in B_R^{(6)}$, i.e., in $U_R^{(6)}$ the transforms $\mathcal{A}^2$ and $\mathcal{U}$ are identical.

Frow Theorem 4 follows the next corollary.

Corollary 2. *1. Given any $a \in U_R^{(2n)}$, we have*

$$\mathcal{U}(a) = \mathcal{A}^{n-1}(a).$$

2. Given any $a \in U_R$, we have

$$\mathcal{U}(a) = \lim_{n\to\infty} \mathcal{A}^n(a)$$

Moreover, if two sequences $a, b \in H_R$ coincide in the first k places, then $\mathcal{A}(a)$ and $\mathcal{A}(b)$ coincide at least in the first k places. Thus, we have the following proposition.

Proposition 8. *Given any $a, b \in H_R$, then*

$$\delta(\mathcal{A}(a), \mathcal{A}(b)) \leq \delta(a, b).$$

Remark 5. *By the previous proposition, we have that $\mathcal{A}$ is a contraction mapping on the ultrametric space (H_R, δ). As a first interesting consequence, we can observe that $\mathcal{A}$ is a continuous function. Moreover, we have that the ultrametric group $(U_R, \star, \delta)$ with the contraction mapping $\mathcal{A}$ is an ultrametric dynamic space, where the set of fixed points is the subgroup B_R. In this way, we have found a very interesting example of ultrametric dynamic space. Ultrametric dynamics are very studied in several fields, see [14] for a good reference about dynamics on ultrametric spaces.*

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