IMPROVED CONVERGENCE THEOREMS FOR BUBBLE CLUSTERS. I. THE PLANAR CASE

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Abstract. We describe a quantitative construction of almost-normal diffeomorphisms between embedded orientable manifolds with boundary to be used in the study of geometric variational problems with stratified singular sets. We then apply this construction to isoperimetric problems for planar bubble clusters. In this setting we develop an *improved convergence theorem*, showing that a sequence of almost-minimizing planar clusters converging in L^1 to a limit cluster has actually to converge in a strong $C^{1,\alpha}$-sense. Applications of this improved convergence result to the classification of isoperimetric clusters and the qualitative description of perturbed isoperimetric clusters are also discussed. Analogous results for three-dimensional clusters are presented in part two, while further applications are discussed in some companion papers.

Contents

1. Introduction

1.1. **Overview.** The aim of this two-part paper is developing a basic technique in the Calculus of Variations, that we call *improved convergence*, in the case of geometric variational problems where minimizers can exhibit stratified singularities. Here we think in particular to variational problems where the minimization takes place over families of generalized surfaces.

Stratified singularities appear in many problems of physical and geometrical interest. The term stratified indicates the possibility of decomposing minimizing surfaces into a hierarchy of manifolds with boundary meeting in specific optimal ways along lower dimensional manifolds of singular points. Although this behavior is well documented from the experimental point of view, its mathematical description is a quite challenging problem, which has been satisfactorily addressed only in a few specific cases. The most celebrated example of this is probably the isoperimetric problem for bubble clusters (and, more generally, any other variational problem whose minimizers can be shown to be $(\mathbf{M},\xi,\delta)$-minimal sets in the sense of Almgren [Alm76]). Indeed, Taylor [Tay76] has shown that two-dimensional $(\mathbf{M},\xi,\delta)$-minimal sets in the physical space $\mathbb{R}^3$ satisfy *Plateau's laws*, that is to say, they consist of regular surfaces meeting in threes at 120 degrees angles along regular curves, which in turn meet in fours at common end-points forming tetrahedral singularities.

By improved convergence we mean the principle – usually exploited in the Calculus of Variations when showing that strict stability (positivity of the second variation) implies local minimality (in some suitable topology) – according to which a sequence of almost-minimizing surfaces converging to some limit in a rough sense has actually to converge to that same limit in a smoother sense. This is a very familiar idea in PDE theory: for a sequence of, say, harmonic functions, L^1-convergence always improves to smooth convergence. In the context of geometric variational problems this kind of result is known to hold (and has been extensively used, see section 1.2) under the assumption that the limit surface is smooth. Our main goal here is discussing improved convergence theorems when the limit surface has stratified singularities. In this setting, by smooth convergence one means the existence of diffeomorphisms between the involved surfaces which converge in C^1 to the identity map, and are almost-normal (in the sense that, at fixed distance from the singularities, the displacement happens in the normal direction to the limit surface only), stratified (in the sense that singular points of a kind are mapped to singular points of the same kind), and whose tangential displacements (which cannot be zero if the singular sets do not coincide) are quantitatively controlled by their normal displacements. Obtaining this precise structure is fundamental in order to use these maps in applications: in other words, the matter here is not just constructing global diffeomorphisms between singular surfaces, but also doing it in a quite specific way, and with quantitative bounds.

From the technical point of view, our main result is Theorem 3.1, see section 3, which provides one with a quantitative method to construct almost-normal diffeomorphisms between embedded orientable manifolds with boundary. This result is proved in arbitrary dimension and codimension, and should have enough flexibility to be applied to different variational problems. Given a specific variational problem, the starting point for deducing an improved convergence theorem from Theorem 3.1 is having at disposal a satisfactory local regularity theory around singular points. Bridging between such a local description of singularities and the global assumptions of Theorem 3.1 is in general a non-trivial problem, which needs to be addressed by some *ad hoc* arguments.

Our two-part paper, in addition to Theorem 3.1, contains exactly this kind of analysis for those variational problems involving isoperimetric clusters. After a review of what is known about isoperimetric clusters in arbitrary dimension, see section 4, we devote section 5 to address this problem in two-dimensions, thus obtaining an improved convergence theorem for almost-minimizing clusters in $\mathbb{R}^2$, see Theorem 1.5 below. In part two [LM15] we address this very same problem for isoperimetric clusters in $\mathbb{R}^3$.

The improved convergence theorem for planar clusters has various applications. Some are discussed in section 6, where we obtain structural results for planar isoperimetric clusters, see Theorem 1.9 and Theorem 1.10. Improve convergence is also used as the starting point to obtain quantitative stability inequalities for planar double bubbles [CLM12] and for periodic honeycombs [CM14]. These companion papers provide one with a clear illustration of why it is so important to formulate improved convergence to a singular limit in terms of the existence of almost-normal, stratified diffeomorphisms converging to the identity map, with a quantitative control between the tangential and normal components of the displacements.

It is also interesting to note that the applicability of Theorem 3.1 is definitely not limited to the problem of isoperimetric clusters. For example, in [MM15] we use Theorem 3.1 and the free boundary regularity theory from [DPM14a] to obtain an improved convergence theorem for capillarity droplets in containers.

This introduction is organized as follows. In section 1.2 we review some of the applications of improved convergence to smooth limit surfaces, and discuss which form an improved convergence theorem should take on singular limit surfaces. In section 1.3 we state our improved convergence theorem for planar minimizing clusters, Theorem 1.5, while section 1.4 presents the applications of Theorem 1.5 discussed in this paper.

1.2. **Improved convergence to a regular limit and applications.** A basic fact about sequences of perimeter almost-minimizing sets, which comes as a direct consequence of the classical De Giorgi's regularity theory [DG60], is that L^1-convergence improves to C^1-convergence whenever the limiting set has smooth boundary, that is to say

$$\begin{cases} \{E_k\}_{k\in\mathbb{N}} \text{ are perimeter almost-minimizing sets} \\ E_k \to E \text{ in } L^1 \text{ with } \partial E \text{ smooth} \end{cases} \qquad \Rightarrow \qquad \partial E_k \to \partial E \text{ in } C^1. \tag{1.1}$$

Referring to section 4.1 for the (standard) notation and terminology about sets of finite perimeter used here and in the rest of the paper, let us recall that given $\Lambda \ge 0$, $r_0 > 0$, and an open set $A \subset \mathbb{R}^n$ ($n \ge 2$), a set E of locally finite perimeter in A is a (Λ, r_0)*-minimizing set in* A if

$$P(E; B_{x,r}) \le P(F; B_{x,r}) + \Lambda\,|E\Delta F|\,, \tag{1.2}$$

whenever $E\Delta F \subset\subset B_{x,r} = \{y \in \mathbb{R}^n : |y-x| < r\} \subset\subset A$ and $r < r_0$. In this way, (1.1) means that if $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing sets in $\mathbb{R}^n$ with $|E_k\Delta E| \to 0$ as $k \to \infty$ and if ∂E is a smooth hypersurface, then for every $\alpha \in (0,1)$ there exist $k_0 \in \mathbb{N}$ and $\{\psi_k\}_{k\ge k_0} \subset C^{1,\alpha}(\partial E)$ such that, denoting by ν_E the outer unit normal to E and for $k \ge k_0$,

$$\partial E_k = (\mathrm{Id} + \psi_k \nu_E)(\partial E)\,, \qquad \sup_{k\ge k_0} \|\psi_k\|_{C^{1,\alpha}(\partial E)} < \infty\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1(\partial E)} = 0\,. \tag{1.3}$$

(Here we have set $(\mathrm{Id} + \psi_k\nu_E)(\partial E) = \{x + \psi_k(x)\nu_E(x) : x \in \partial E\}$.) A local version of this improved convergence result is found in [Mir67] in the case $\Lambda = 0$, but actually holds even for more general notions of almost-minimality than the one considered here; see [Tam84, Theorem 1.9]. It immediately implies a regularizing property of the sets E_k, in the sense that ∂E_k must be a C^1-hypersurface as a consequence of (1.3). Improved convergence finds numerous applications to geometric variational problems. These include:

(A) Sharp quantitative inequalities: In [CL12], (1.1) was used (with $E = B = B_{0,1}$) in combination with a selection principle and a result by Fuglede on nearly spherical sets [Fug89] to give an alternative proof of the sharp quantitative isoperimetric inequality of [FMP08], namely

$$P(E) \ge P(B)\left\{1 + c(n) \min_{x\in\mathbb{R}^n} |E\Delta(x+B)|^2\right\}, \qquad \forall E \subset \mathbb{R}^n\,, |E| = |B|\,.$$

This strategy of proof has been subsequently adopted to prove many other geometric inequalities in sharp quantitative form. Examples are the Euclidean isoperimetric inequality in higher codimension [BDF12], the isoperimetric inequalities on spheres and hyperbolic spaces [BDF12, BDF13], isoperimetric inequalities for eigenvalues [BDPV13], minimality inequalities of area minimizing hypersurfaces [DPM14b], and non-local isoperimetric inequalities [FFM$^+$]; moreover, in [FJ14] the same strategy is used to control by $P(E) - P(B)$ a more precise distance from the family of balls (see also [Neu14] for the case of the Wulff inequality).

(B) Qualitative properties (and characterization) of minimizers: Given a potential $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) \to +\infty$ as $|x| \to \infty$ and a one-homogeneous and convex integrand $\Phi : \mathbb{R}^n \to [0,\infty)$, in [FM11] the variational problems (parameterized by $m > 0$)

$$\inf\left\{ \int_{\partial^* E} \Phi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} g(x)\, dx : |E| = m \right\}, \tag{1.4}$$

are considered in the small volume regime $m \to 0^+$. Denoting by E_m a minimizer with volume m, one expects $m^{-1/n}\, E_m$ to converge to K, the unit volume Wulff shape of Φ. One of the main results proved in [FM11] is that if Φ is a smooth elliptic integrand and g is smooth, then $m^{-1/n}\, E_m \to K$ as $m \to 0^+$ in every $C^{k,\alpha}$, with explicit rates of convergence in terms of m. The improved convergence theorem (1.1), applied with $E = K$ and on (Φ, Λ, r_0)-minimizing sets, plays of course a basic role in this kind of analysis. The same circle of ideas has been exploited in the qualitative description of minimizers of the Ohta-Kawasaki energy for diblock copolymers

[CS13], and to characterize balls as minimizers in isoperimetric problems with competing nonlocal terms [KM13, KM14, BC13, FFM$^+$], and in isoperimetric problems with log-convex densities [FM12].

(C) Stability and L^1-local minimality: A classical problem in the Calculus of Variations is that of understanding whether stable critical points of a given functional are also local minimizers. This question was addressed in the case of the Plateau's problem by White [Whi94], who has proved that a *smooth* surface that is a strictly stable critical point of the area functional is automatically locally area minimizing in L^∞ (see [MR10, DPM14b] for the L^1-case). A key step in his argument is again an improved convergence theorem (for area almost-minimizing currents) towards a smooth limit. Similarly, in the case of the Ohta-Kawasaki energy, volume-constrained stable critical points with smooth boundary turn out to be volume-constrained L^1-local minimizers, see [AFM13]. Once again, (1.1) is the starting point of the analysis.

We now try to address the question of the precise meaning one should give to an assertion like

$$\begin{cases} \{E_k\}_{k\in\mathbb{N}} \text{ are perimeter almost-minimizing sets} \\ E_k \to E \text{ in } L^1 \end{cases} \qquad \Rightarrow \qquad \partial E_k \to \partial E \text{ in } C^1\,, \tag{1.5}$$

when ∂E is possibly singular. To this end we split ∂E into its regular and singular parts: precisely, recalling that the reduced boundary $\partial^* E$ of a (Λ, r_0)-minimizing set in $\mathbb{R}^n$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0,1)$ (relatively open into ∂E), we define the singular set $\Sigma(E)$ of ∂E as the closed subset of ∂E given by

$$\Sigma(E) = \partial E \setminus \partial^* E\,.$$

It turns out that $\Sigma(E)$ is empty if $2 \le n \le 7$, discrete if $n = 8$, and $\mathcal{H}^s$-negligible for every $s > n-8$ if $n \ge 9$; see, for example, [Mag12, Theorem 21.8, Theorem 28.1]. The regularity theory behind these results also leads to a weak form of (1.3), which in turn reduces to (1.3) when $\Sigma(E) = \emptyset$. More precisely, given a sequence $\{E_k\}_{k\in\mathbb{N}}$ of (Λ, r_0)-minimizing sets with $E_k \to E$ in L^1, denoting by $I_\rho(S)$ the ρ-neighborhood of $S \subset \mathbb{R}^n$, and setting

$$[\partial E]_\rho = \partial E \setminus I_\rho(\Sigma(E)) \subset \partial^* E\,, \qquad \rho > 0\,, \tag{1.6}$$

one finds (see, e.g. Theorem 4.12 below) that, for every $\alpha \in (0,1)$ and ρ small enough there exist $k_0 \in \mathbb{N}$ and $\{\psi_k\}_{k \ge k_0} \subset C^{1,\alpha}([\partial E]_\rho)$ such that

$$\partial E_k \setminus I_{2\rho}(\Sigma(E)) \subset (\mathrm{Id} + \psi_k \nu_E)([\partial E]_\rho) \subset \partial^* E_k\,, \qquad \forall k \ge k_0\,, \tag{1.7}$$

$$\sup_{k \ge k_0} \|\psi_k\|_{C^{1,\alpha}([\partial E]_\rho)} \le C\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1([\partial E]_\rho)} = 0\,. \tag{1.8}$$

Of course, if $\Sigma(E) = \emptyset$, then (1.7) and (1.8) coincide with (1.3). Moreover, we notice that to replace $\partial E_k \setminus I_{2\rho}(\Sigma(E))$ with, say, $[\partial E_k]_{3\rho}$ in the first inclusion in (1.7), one would need to prove Hausdorff convergence of $\Sigma(E_k)$ to $\Sigma(E)$. However, in this generality, one just knows that $\Sigma(E_k) \subset I_\rho(\Sigma(E))$ provided $k \ge k_0$, and actually $\Sigma(E_k)$ may not converge in Hausdorff distance to $\Sigma(E)$. Indeed, by a classical result of Bombieri, De Giorgi and Giusti [BDGG69], the Simons's cone in $\mathbb{R}^8$ is the limit of perimeter minimizing sets with smooth boundary.

Even though (1.7) and (1.8) seem to contain all the information we can extract from the classical regularity theory, this is however not sufficient, for several reasons, to address any of the above mentioned applications. The first evident gap is that we do not parameterize the whole ∂E_k on ∂E. Of course, in presence of singularities we cannot expect to do this by means of a normal diffeomorphism of ∂E; see Figure 1. Therefore, the best we can hope for is to find a sequence $\{f_k\}_{k\in\mathbb{N}}$ of $C^{1,\alpha}$-diffeomorphisms between ∂E and ∂E_k such that

$$\sup_{k\in\mathbb{N}} \|f_k\|_{C^{1,\alpha}(\partial E)} < \infty\,, \qquad \lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\partial E)} = 0\,. \tag{1.9}$$

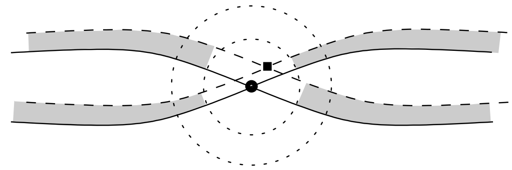

FIGURE 1. The limit boundary ∂E is depicted with continuous lines, the approximating boundaries ∂E_k by dashed lines, the singular set $\Sigma(E)$ by a black disk, and its ρ and 2ρ-neighborhoods $I_\rho(\Sigma(E))$ and $I_{2\rho}(\Sigma(E))$ by concentric balls: $I_\rho(\Sigma(E))$ contains the singular set of ∂E_k (depicted by a black square), while (1.7) says that $\partial E_k \setminus I_{2\rho}(\Sigma(E))$ can be covered by a normal deformation of $[\partial E]_\rho = \partial E \setminus I_\rho(\Sigma(E))$ (depicted as a grey region) which is C^1-close to the identity thanks to (1.8). Of course, we cannot describe ∂E_k by a normal deformation of the four components of $\partial^* E$ unless $\Sigma(E_k) = \Sigma(E)$.

A difficulty here is to specify what is meant by a $C^{1,\alpha}$-diffeomorphism between ∂E and ∂E_k, since these are singular hypersurfaces. Moreover, in passing from (1.7)–(1.8) to (1.9) we may lose the useful information that ∂E_k is actually a C^1-small normal deformation of ∂E away from the singular sets. It is therefore natural to require that

$$f_k = \mathrm{Id} + \psi_k\, \nu_E \qquad \text{on } [\partial E]_\rho\,, \tag{1.10}$$

with ψ_k as in (1.7)–(1.8). The maps f_k must have a nontrivial tangential displacement

$$u_k = (f_k - \mathrm{Id}) - \big((f_k - \mathrm{Id}) \cdot \nu_E\big)\, \nu_E\,,$$

on $[\partial E]_\rho$ if $\Sigma(E_k) \neq \Sigma(E)$: and, actually, in order for the maps f_k to be usable in addressing problems (A) and (C), it is crucial to have control of the C^1-norm of u_k in terms of the distance between $\Sigma(E_k)$ and $\Sigma(E)$. A possibility here is requiring that $f_k(\Sigma(E)) = \Sigma(E_k)$ (and this is something that makes sense only if, in the situation at hand, one has already shown the Hausdorff convergence of $\Sigma(E_k)$ to $\Sigma(E)$), with $f_k = \mathrm{Id}$ on $\Sigma(E)$ if $\Sigma(E_k) = \Sigma(E)$, and, for some constant C depending on ∂E,

$$\|u_k\|_{C^1(\partial E)} \le C\, \|f_k - \mathrm{Id}\|_{C^1(\Sigma(E))}\,. \tag{1.11}$$

Due to our limited understanding of singular sets, proving (1.7)–(1.11) seems a goal out of reach, and so the possibility of understanding improved convergence to singular limit sets. The theory of bubble clusters (partitions of the space into sets of finite perimeter) provides us with a (more complex) setting where singularities appear even in dimension $n = 2$. However, at least when $n = 2, 3$, these singularities have been classified and understood, and the corresponding local regularity theory enables one to show the Hausdorff convergence of the singular sets (see Theorem 5.5 in the case $n = 2$ and [LM15, Theorem 3.2] in the case $n = 3$). It thus makes sense to look for improved convergence theorems in this setting, and this problem is indeed addressed in this paper and in [LM15].

1.3. **An improved convergence theorem for planar clusters.** Following the ideas discussed in the previous section, we now formulate our improved convergence theorem for sequences of almost-minimizing planar clusters. Given $n, N \in \mathbb{N}$ with $n, N \ge 2$ and an open set $A \subset \mathbb{R}^n$, we let $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ be a family of Lebesgue-measurable sets in $\mathbb{R}^n$ with $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$ for $1 \le h < k \le N$, and say that $\mathcal{E}$ is an *N-cluster in A* if $\mathcal{E}(h)$ is a set of locally finite perimeter in A with $|\mathcal{E}(h) \cap A| > 0$ for every $h = 1, ..., N$. The sets $\mathcal{E}(h)$ are called the *chambers* of $\mathcal{E}$, while $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ is called the *exterior chamber* of $\mathcal{E}$. The *perimeter* of $\mathcal{E}$ relative to some $F \subset \mathbb{R}^n$ is defined by setting

$$P(\mathcal{E}; F) = \frac{1}{2} \sum_{h=0}^N P(\mathcal{E}(h); F)\,, \qquad P(\mathcal{E}) = P(\mathcal{E}; \mathbb{R}^n)\,. \tag{1.12}$$

Setting $\mathrm{vol}\,(\mathcal{E}) = (|\mathcal{E}(1)|, ..., |\mathcal{E}(N)|)$, a minimizer in the partitioning problem

$$\inf \big\{ P(\mathcal{E}) : \mathrm{vol}\,(\mathcal{E}) = m \big\} \,, \qquad m \in \mathbb{R}^N_+ \text{ given} \,, \tag{1.13}$$

where $\mathbb{R}^N_+ = \{m \in \mathbb{R}^N : m_h > 0 \,\forall h = 1, ..., N\}$, is called an *isoperimetric cluster*. It is of course natural to study partitioning problems in the presence of a potential energy term, like

$$\inf \Big\{ P(\mathcal{E}) + \sum_{h=1}^N \int_{\mathcal{E}(h)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E}) = m \Big\} \,, \tag{1.14}$$

where, say, $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) \to +\infty$ as $|x| \to \infty$. The existence of minimizers in these two problems can be proved by a careful restoration of compactness argument due to Almgren, see [Mag12, Chapter 29]. It turns out that if $\mathcal{E}$ is a minimizer either in (1.13) or in (1.14), then there exist positive constants Λ and r_0 such that $\mathcal{E}$ is *a* (Λ, r_0)*-minimizing cluster* in $\mathbb{R}^n$, that is (in analogy with (1.2))

$$P(\mathcal{E}; B_{x,r}) \le P(\mathcal{F}; B_{x,r}) + \Lambda\, \mathrm{d}(\mathcal{E}, \mathcal{F}) \,, \tag{1.15}$$

whenever $x \in \mathbb{R}^n$, $r < r_0$ and $\mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r}$ for every $h = 1, ..., N$, and where we have set

$$\mathrm{d}_F(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \sum_{h=0}^N \Big| F \cap \big(\mathcal{E}(h)\Delta\mathcal{F}(h)\big) \Big| \,, \qquad \mathrm{d}(\mathcal{E}, \mathcal{F}) = \mathrm{d}_{\mathbb{R}^n}(\mathcal{E}, \mathcal{F}) \,. \tag{1.16}$$

In this case, as a consequence of the results obtained in [Alm76] (see also [Mag12, Chapter 30] for the case $\Lambda = 0$, and section 4 below otherwise), $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0, 1)$ ($C^{1,1}$ if $n = 2$) which is relatively open into $\partial\mathcal{E}$ and $\mathcal{H}^{n-1}(\Sigma(\mathcal{E})) = 0$, where

$$\partial\mathcal{E} = \bigcup_{h=1}^N \partial\mathcal{E}(h) \,, \quad \partial^*\mathcal{E} = \bigcup_{h=1}^N \partial^*\mathcal{E}(h) \,, \quad \Sigma_F(\mathcal{E}) = F \cap (\partial\mathcal{E} \setminus \partial^*\mathcal{E}) \,, \quad \Sigma(\mathcal{E}) = \Sigma_{\mathbb{R}^n}(\mathcal{E}) \,. \tag{1.17}$$

One does not expect this almost-everywhere regularity result to be optimal in any dimension n, although the situation is clear only when $n = 2$ (by elementary arguments) and when $n = 3$ by [Tay76]. We now review the structure of singular sets when $n = 2$, and then exploit this description to formulate an improved convergence result for planar clusters. With the notation introduced in section 2.1, if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$, then one has

$$\begin{cases} \partial\mathcal{E} = \bigcup_{i\in I} \gamma_i \,, \\ \partial^*\mathcal{E} = \bigcup_{i\in I} \mathrm{int}\,(\gamma_i) \,, \end{cases} \qquad \begin{array}{l} \text{where } I \text{ is at most countable} \,, \\ \gamma_i \text{ is a closed connected } C^{1,1}\text{-curve with boundary} \,, \\ \{\gamma_i\}_{i\in I} \text{ is locally finite} \,, \end{array} \tag{1.18}$$

(see [Ble87], [Mor94], or [Mag12, Section 30.3] in the case $\Lambda = 0$, and Theorem 5.2 below in the general case – which is a simple variant of the $\Lambda = 0$ case). Moreover,

$$\Sigma(\mathcal{E}) = \bigcup_{j\in J} \{p_j\} = \bigcup_{i\in I} \mathrm{bd}\,(\gamma_i) \,, \qquad \begin{array}{l} \text{where } J \text{ is at most countable} \,, \\ \{p_j\}_{j\in J} \text{ is locally finite} \,. \end{array} \tag{1.19}$$

Finally, each $p_j \in \Sigma(\mathcal{E})$ is a common end-point to three different curves from $\{\gamma_i\}_{i\in I}$, which form three 120 degree angles at p_j.

Remark 1.1. As already noticed, if $\mathcal{E}$ is an isoperimetric cluster in $\mathbb{R}^2$, or if $\mathcal{E}$ is a minimizer in (1.14) with $n = 2$ and g is smooth, then $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $\mathbb{R}^2$ for some Λ and r_0, with the additional property of being bounded, so that I and J are finite. Moreover, if $\mathcal{E}$ is an isoperimetric cluster, then each γ_i is either a circular arc or a segment; if $\mathcal{E}$ is a minimizer in (1.14), then γ_i is a closed connected smooth curve with boundary, whose curvature is equal to (the restriction to γ_i of) g up to an additive constant.

Motivated by these examples, we now give the following definitions, and then state our improved convergence theorem for planar clusters.

Definition 1.2. Let $\mathcal{E}$ be a cluster in $\mathbb{R}^2$. One says that $\mathcal{E}$ is a $C^{k,\alpha}$*-cluster in* $\mathbb{R}^2$ if there exists a family of $C^{k,\alpha}$-curves with boundary $\{\gamma_i\}_{i\in I}$ such that (1.18) and (1.19) hold.

Definition 1.3. Let $\mathcal{E}$ be a $C^{1,\alpha}$-cluster in $\mathbb{R}^2$. Given a map $f:\partial\mathcal{E}\to\mathbb{R}^2$ one says that $f\in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^2)$ if f is continuous on $\partial\mathcal{E}$, $f\in C^{1,\alpha}(\gamma_i;\mathbb{R}^2)$ for every $i\in I$, and

$$\|f\|_{C^{1,\alpha}(\partial\mathcal{E})}:=\sup_{i\in I}\|f\|_{C^{1,\alpha}(\gamma_i)}<\infty\,.$$

If $\mathcal{E}$ and $\mathcal{E}'$ are $C^{1,\alpha}$-clusters in $\mathbb{R}^2$, then one says that f is a $C^{1,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ provided f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ with $f\in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^2)$, $f^{-1}\in C^{1,\alpha}(\partial\mathcal{E}';\mathbb{R}^2)$, and $f(\Sigma(\mathcal{E}))=\Sigma(\mathcal{E}')$.

Definition 1.4. Given a map $f:\mathbb{R}^2\to\mathbb{R}^2$ and a cluster $\mathcal{E}$ in $\mathbb{R}^2$, the tangential component of f with respect to $\mathcal{E}$ is the map $\boldsymbol{\tau}_{\mathcal{E}}f:\partial^*\mathcal{E}\to\mathbb{R}^2$ defined by

$$\boldsymbol{\tau}_{\mathcal{E}}f(x)=f(x)-(f(x)\cdot\nu_{\mathcal{E}}(x))\nu_{\mathcal{E}}(x)\,,\qquad x\in\partial^*\mathcal{E}\,,$$

where $\nu_{\mathcal{E}}:\partial^*\mathcal{E}\to\mathbb{S}^1$ is any Borel function such that either $\nu(x)=\nu_{\mathcal{E}(h)}(x)$ or $\nu(x)=\nu_{\mathcal{E}(k)}(x)$ for every $x\in\partial^*\mathcal{E}(h)\cap\partial^*\mathcal{E}(k)$, $h\neq k$.

Theorem 1.5 (Improved convergence for planar almost-minimizing clusters)**.** *Given* $\Lambda\geq 0$, $r_0>0$ *and a bounded* $C^{2,1}$*-cluster* $\mathcal{E}$ *in* $\mathbb{R}^2$*, there exist positive constants* μ_0 *and* C_0 *(depending on* Λ *and* $\mathcal{E}$*) with the following property.*

If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *is a sequence of perimeter* (Λ,r_0)*–minimizing clusters in* $\mathbb{R}^2$ *such that* $\mathrm{d}(\mathcal{E}_k,\mathcal{E})\to 0$ *as* $k\to\infty$*, then for every* $\mu<\mu_0$ *there exist* $k(\mu)\in\mathbb{N}$ *and a sequence of maps* $\{f_k\}_{k\geq k(\mu)}$ *such that each* f_k *is a* $C^{1,1}$*-diffeomorphism between* $\partial\mathcal{E}$ *and* $\partial\mathcal{E}_k$ *with*

$$\begin{aligned}\|f_k\|_{C^{1,1}(\partial\mathcal{E})}&\leq C_0\,,\\ \lim_{k\to\infty}\|f_k-\mathrm{Id}\|_{C^1(\partial\mathcal{E})}&=0\,,\\ \|\boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id})\|_{C^1(\partial^*\mathcal{E})}&\leq\frac{C_0}{\mu}\|f_k-\mathrm{Id}\|_{C^0(\Sigma(\mathcal{E}))}\,,\\ \boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id})&=0\,,\qquad \textit{on } \partial\mathcal{E}\setminus I_\mu(\Sigma(\mathcal{E}))\,.\end{aligned}$$

Remark 1.6. A natural question is of course whether the maps f_k in Theorem 1.5 can be extended to $C^{1,1}$-diffeomorphisms g_k of $\mathbb{R}^2$ with $\|g_k\|_{C^{1,1}(\mathbb{R}^2)}\leq C_0$ and $\|g_k-\mathrm{Id}\|_{C^1(\mathbb{R}^2)}\to 0$ as $k\to\infty$. The answer is yes, but at the cost of a longer proof which only employs ideas similar to the ones already used in the proof of Theorem 1.5. At the same time, in the applications of Theorem 1.5 presented in [CLM12, CM14] there seems to be no real advantage in working with the maps g_k in place of the maps f_k.

Remark 1.7. We briefly comment on the proof of Theorem 1.5. The first step consists in exploiting the interior regularity theory to show (much in the spirit of (1.7)–(1.8)) the existence of normal diffeomorphisms between those parts of $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ that are at a fixed small distance from the singular sets $\Sigma(\mathcal{E})$ and $\Sigma(\mathcal{E}_k)$. This step of the proof can be carried out in arbitrary dimension, and it is addressed in Theorem 4.12. Next, one exploits the description of singular sets of planar clusters in order to prove the Hausdorff convergence of $\Sigma(\mathcal{E}_k)$ to $\Sigma(\mathcal{E})$ (Theorem 5.5), and to prove that actually if $x_k\in\Sigma(\mathcal{E}_k)$, $x\in\Sigma(\mathcal{E})$ and $x_k\to x$, then the tangent cones to $\partial\mathcal{E}_k$ at x_k converge locally uniformly to the tangent cone to $\partial\mathcal{E}$, see step four in the proof of Theorem 5.6. In Theorem 5.6 we actually show various other preliminary convergence properties of $\partial\mathcal{E}_k$ towards $\partial\mathcal{E}$, including the fact that for k large enough, $\partial\mathcal{E}_k$ and $\partial\mathcal{E}$ share the same topological structure. Given all these preparatory facts, one is ready to extend the normal diffeomorphisms defined away from $\Sigma(\mathcal{E})$ to the whole $\partial\mathcal{E}$ by exploiting the construction of almost-normal diffeomorphism described in Theorem 3.1.

Remark 1.8. The delicate extension of Theorem 1.5 to clusters in $\mathbb{R}^3$ is discussed in [LM15].

1.4. **Some applications of Theorem 1.5.** As explained in section 1.2, a result like Theorem 1.5 opens the way to several applications. The ones given below, see Theorem 1.9 and Theorem 1.10, are inspired by a list of questions concerning partitioning problems proposed by Almgren in [Alm76, VI.1(6)], precisely "to classify in some reasonable way the different minimizing clusters corresponding to different choices of $m \in \mathbb{R}^N_+$". In this direction, let us consider the equivalence relation $\approx$ on the family of planar $C^{1,1}$-clusters such that $\mathcal{E} \approx \mathcal{F}$ if there exists a $C^{1,1}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$. Theorem 1.9 shows that isoperimetric clusters of a given volume (or with volume sufficiently close to a given one) generate only finitely many $\approx$-equivalence classes.

Theorem 1.9. *For every $m_0 \in \mathbb{R}^N_+$ there exists $\delta > 0$ with the following property. If Ω is the family of all the isoperimetric N-clusters $\mathcal{E}$ in $\mathbb{R}^2$ with $|\mathrm{vol}\,(\mathcal{E}) - m_0| < \delta$, then $\Omega/_\approx$ is a finite set.*

By an entirely analogous principle, we can describe qualitatively minimizers in (1.14) when the potential energy is small enough. (In the case of planar double bubbles $N = n = 2$ we can upgrade this description to a quantitative one in the spirit of [FM11], see [CLM12].)

Theorem 1.10. *Let $m_0 \in \mathbb{R}^N_+$ be such that there exists a unique (modulo isometries) isoperimetric cluster $\mathcal{E}_0$ in $\mathbb{R}^2$ with $\mathrm{vol}\,(\mathcal{E}_0) = m_0$, and let $g : \mathbb{R}^2 \to [0,\infty)$ be a continuous function with $g(x) \to \infty$ as $|x| \to \infty$. Then there exists $\delta_0 > 0$ (depending on $\mathcal{E}_0$ and g only) such that for every $\delta < \delta_0$ and $|m - m_0| < \delta_0$ there exist minimizers of*

$$\inf\Big\{P(\mathcal{E}) + \delta \sum_{h=1}^N \int_{\mathcal{E}(h)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E}) = m\Big\}. \tag{1.20}$$

If $\mathcal{E}$ is a minimizer in (1.20), then $\mathcal{E} \approx \mathcal{E}_0$. Moreover, if $H_{\mathcal{E}(h,k)}$ denotes the scalar mean curvature of the interface $\mathcal{E}(h,k)$ with respect to $\nu_{\mathcal{E}(h)}$, then $H_{\mathcal{E}(h,k)}$ is continuous on $\mathcal{E}(h,k)$, with

$$\max_{0\le h<k\le N} \|H_{\mathcal{E}(h,k)} - H_{\mathcal{E}_0(h,k)}\|_{C^0(\mathcal{E}(h,k))} \le C_0\,\delta\,, \tag{1.21}$$

for a constant C_0 depending on $\mathcal{E}_0$ and g only. (Notice that $H_{\mathcal{E}_0(h,k)}$ is a constant for every $0 \le h < k \le N$.)

Of course, thanks to Theorem 1.9, if the uniqueness assumption on m_0 in Theorem 1.10 is dropped, then one can still infer that minimizers in (1.20) with $\delta < \delta_0$ and $|m - m_0| < \delta_0$ generate only finitely many $\approx$-equivalence classes. Moreover, we explicitly notice that the novelty of Theorem 1.10 is not the existence part, which follows by standard arguments, but the fact that $\mathcal{E} \approx \mathcal{E}_0$.

Further applications of Theorem 1.5 are discussed elsewhere. In [CLM12], Theorem 1.5 is the starting point for obtaining a sharp stability inequality for planar double-bubbles, while in [CM14] we address a sharp quantitative version of Hales's isoperimetric theorem for the regular hexagonal tiling [Hal01].

1.5. **Organization of the paper.** The paper is essentially divided in two parts. The first part consists of sections 2–3. The goal here is to provide in a reasonable generality the construction of almost-normal diffeomorphisms between manifolds with boundary. As said, this is the key result in constructing the maps appearing in Theorem 1.5. It is considered in arbitrary dimension and co-dimension (and not just for curves in the plane) in view of its applications to the improved convergence of clusters in $\mathbb{R}^3$ [LM15] and to the description of capillarity droplets in containers [MM15]. We provide two statements of this result, see Theorem 3.1 and Theorem 3.5, the second one being more practical in applications. These results are proved in section 3, after some preliminary facts concerning the implicit function theorem and Whitney's extension theorem are gathered in section 2. In the second part of the paper, which consists of sections 4–5 we gather the various ingredients needed to deduce Theorem 1.5 from Theorem 3.1, as described in Remark 1.7. Finally, in section 6 we give the (closely related proofs of) Theorem 1.9 and Theorem 1.10.

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2. Notation and preliminaries

We gather here some basic notation and classical facts to be used here and in [LM15].

2.1. **Sets in $\mathbb{R}^n$.** Given $x \in \mathbb{R}^n$ and $r > 0$ we set $B(x,r) = B_{x,r} = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B(0,r) = B_{0,r} = B_r$, where $|v|^2 = v \cdot v$ and $v \cdot w$ is the scalar product of $v, w \in \mathbb{R}^n$. We set $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Given $S \subset \mathbb{R}^n$, we denote by $\mathring{S}$, ∂S, and $\mathrm{cl}\,(S)$ the interior, the boundary, and the closure of S respectively, while $I_\varepsilon(S)$ denotes the tubular ε-neighborhood of S in $\mathbb{R}^n$, that is $I_\varepsilon(S) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, S) < \varepsilon\}$, $\varepsilon > 0$. Given $S, T \subset \mathbb{R}^n$ we define the Hausdorff distance between S and T localized in $K \subset \mathbb{R}^n$ as

$$\mathrm{hd}_K(S,T) = \max\big\{ \sup\{\mathrm{dist}(y,S) : y \in T \cap K\}, \sup\{\mathrm{dist}(y,T) : y \in S \cap K\}\big\}\,, \tag{2.1}$$

so that $\mathrm{hd}_K(S,T) < \varepsilon$ if and only if $S \cap K \subset I_\varepsilon(T)$ and $T \cap K \subset I_\varepsilon(S)$, while

$$\mathrm{hd}_{x,r}(S,T) = \mathrm{hd}_{B_{x,r}}(S,T)\,, \qquad \mathrm{hd}(S,T) = \mathrm{hd}_{\mathbb{R}^n}(S,T)\,.$$

If S is a k-dimensional C^1-manifold in $\mathbb{R}^n$ (we always work with *embedded* manifolds), then the geodesic distance on S is given by

$$\mathrm{dist}_S(x,y) = \inf\Big\{ \int_0^1 |\gamma'(t)|\,dt : \gamma \in C^1([0,1];S)\,, \gamma(0) = x\,, \gamma(1) = y\Big\}\,, \qquad x, y \in S\,.$$

We also define the normal ε-neighborhood of S as

$$N_\varepsilon(S) = \Big\{x + \sum_{i=1}^{n-k} t_i\,\nu^{(i)}(x) : x \in S\,, \sum_{i=1}^{n-k} t_i^2 < \varepsilon^2\Big\}\,, \tag{2.2}$$

provided $\{\nu^{(i)}(x)\}_{i=1}^{n-k}$ denotes an orthonormal basis to $(T_xS)^\perp$. If S is a k-dimensional C^1-manifold with boundary in $\mathbb{R}^n$, then $\mathrm{int}\,(S)$ and $\mathrm{bd}\,(S)$ denote, respectively, the interior and the boundary points of S. If $x \in \mathrm{bd}\,(S)$, then we define T_xS as a k-dimensional space (thus, not as an half-space), and we denote by $\nu_S^{co}(x) \in T_xS$ the outer unit normal to $\mathrm{bd}\,(S)$ with respect to S. Moreover, we set

$$[S]_\rho = S \setminus I_\rho(\mathrm{bd}\,(S))\,, \qquad \forall \rho > 0\,. \tag{2.3}$$

Denoting by π_x^S the projection of $\mathbb{R}^n$ onto T_xS, for every $f : S \to \mathbb{R}^n$ we define $\pi^S f : S \to \mathbb{R}^n$ by taking

$$(\pi^S f)(x) = \pi_x^S[f(x)]\,, \qquad x \in S\,.$$

The terms curve, surface and hypersurface are used in place of 1-dimensional manifold, 2-dimensional manifold and $(n-1)$-dimensional manifold in $\mathbb{R}^n$.

2.2. **Uniform inverse and implicit function theorems.** If S is a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ ($\alpha \in (0,1]$), $x \in S$, and $f : S \to \mathbb{R}^m$, then we say that f is differentiable at x with respect to S if we can define a linear map $\nabla^S f(x) : \mathbb{R}^n \to \mathbb{R}^m$ by setting

$$\nabla^S f(x)[v] = \begin{cases} \lim\limits_{t \to 0} \frac{f(\gamma(t)) - f(x)}{t} & \text{if } v \in T_xS,\ \gamma \in C^1((-\varepsilon,\varepsilon);S),\ \gamma(0) = x,\ \gamma'(0) = v\,, \\ 0 & \text{if } v \in (T_xS)^\perp\,. \end{cases}$$

Denoting by $\|L\| = \sup\{|L[v]| : |v| = 1\}$ the operator norm of a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, we set

$$\|f\|_{C^1(S)} = \sup_{x\in S} |f(x)| + \|\nabla^S f(x)\| \,.$$

Of course, if f is differentiable on an open neighborhood of S, then $\nabla^S f(x)$ is just the restriction of the differential $\nabla f(x)$ of f at x to T_xS, extended to 0 on $(T_xS)^\perp$. For $\alpha \in (0,1]$ we set

$$\begin{aligned}
[\nabla^S f]_{C^{0,\alpha}(S)} &= \sup_{x,y\in S,\, x\neq y} \frac{\|\nabla^S f(x) - \nabla^S f(y)\|}{|x-y|^\alpha}\,,\\
\|\nabla^S f\|_{C^{0,\alpha}(S)} &= \sup_{x\in S} \|\nabla^S f(x)\| + [\nabla^S f]_{C^{0,\alpha}(S)}\,,\\
\|f\|_{C^{1,\alpha}(S)} &= \sup_{x\in S} |f(x)| + \|\nabla^S f\|_{C^{0,\alpha}(S)}\,,
\end{aligned}$$

(note the use of the Euclidean distance in the definition of $[\cdot]_{C^{0,\alpha}(S)}$). If $\{\tau_i(x)\}_{i=1}^k$ is an orthonormal basis of T_xS, then we define the tangential Jacobian of f as

$$J^S f(x) = \Big|\bigwedge_{i=1}^k \nabla^S f(x)[\tau_i(x)]\Big|\,, \qquad x \in S\,.$$

The following theorems are uniform versions of the inverse and implicit function theorems. The proof of the first result is included in Appendix A for the sake of clarity.

Theorem 2.1 (Uniform inverse function theorem)**.** *Given $n \geq 2$, $1 \leq k \leq n-1$, $\alpha \in (0,1]$, $L > 0$, and S_0 a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ with $\mathrm{diam}(S_0) \leq L$ and*

$$\mathrm{dist}_{S_0}(x,y) \leq L\,|x-y|\,, \qquad \forall x,y \in S_0\,, \tag{2.4}$$
$$|y-x| \leq 2\,|\pi_x^{S_0}(y-x)|\,, \qquad \forall x \in S_0\,, y \in B_{x,1/L} \cap S_0\,, \tag{2.5}$$
$$\|\pi_x^{S_0} - \pi_y^{S_0}\| \leq L\,|x-y|^\alpha\,, \qquad \forall x,y \in S_0\,, \tag{2.6}$$

there exist positive constants ε_0, ρ_0 and C_0, depending on n, k, α, and L only, with the following properties. If $f \in C^{1,\alpha}(S_0;\mathbb{R}^n)$ is such that

$$\inf_{S_0} J^{S_0} f \geq \frac{1}{L}\,, \qquad \|\nabla^{S_0} f\|_{C^{0,\alpha}(S_0)} \leq L\,, \tag{2.7}$$

then f is injective on $B_{x,\varepsilon_0} \cap S_0$ for every $x \in S_0$. If, moreover,

$$\|f - \mathrm{Id}\|_{C^0(S_0)} \leq \rho_0\,, \tag{2.8}$$

then $S = f(S_0)$ is a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$ and $f : S_0 \to S$ is a $C^{1,\alpha}$-diffeomorphism satisfying $\|f^{-1}\|_{C^{1,\alpha}(S)} \leq C_0$.

Theorem 2.2 (Uniform implicit function theorem)**.** *Let n, k, α, L and S_0 be as in Theorem 2.1. Then there exist positive constants C_0 and η_0 depending on n, k, α, and L only with the following property. If $x_0 \in S_0$ and $u \in C^{1,\alpha}(S_0 \times (-1,1)^{n-k};\mathbb{R}^{n-k})$ is such that*

$$u(x_0,\mathbf{0}) = \mathbf{0}\,, \qquad \Big|\bigwedge_{i=1}^{n-k} \frac{\partial u}{\partial t_i}(x_0,\mathbf{0})\Big| \geq \frac{1}{L}\,, \qquad \|u\|_{C^{1,\alpha}(S_0\times(-1,1)^{n-k})} \leq L\,, \tag{2.9}$$

where $\mathbf{0} = (0,...,0) \in \mathbb{R}^{n-k}$, then there exists a function $\zeta \in C^{1,\alpha}(S_0 \cap B_{x_0,\eta_0};\mathbb{R}^{n-k})$ such that

$$\zeta(x_0) = \mathbf{0}\,, \qquad u(z,\zeta(z)) = \mathbf{0}\,, \qquad \forall z \in S_0 \cap B_{x_0,\eta_0}\,, \qquad \|\zeta\|_{C^{1,\alpha}(S_0\cap B(x_0,\eta_0))} \leq C_0\,. \tag{2.10}$$

Proof. One applies the first conclusion of Theorem 2.1 to the manifold $S_0 \times (-1,1)^{n-k}$ and the function $f : S_0 \times (-1,1)^{n-k} \to \mathbb{R}^n$ defined by $f(x,\mathbf{t}) = (x, u(x,\mathbf{t}))$; see, e.g. [Spi65]. □

2.3. **Whitney's extension theorem.** Here we review some basic facts concerning Whitney's extension theorem. By $\mathbf{k}=(k_1,...,k_n)$ we denote the generic element of $\mathbb{N}^n$, and set

$$|\mathbf{k}|=\sum_{i=1}^n k_i\,,\qquad \mathbf{k}!=\prod_{i=1}^n k_i\,,\qquad z^{\mathbf{k}}=\prod_{i=1}^n z_i^{k_i}\,,$$

for every $\mathbf{k}\in\mathbb{N}^n$ and $z\in\mathbb{R}^n$. If f is $|\mathbf{k}|$-times differentiable at $x\in\mathbb{R}^n$, we let

$$D^{\mathbf{k}}f(x)=\frac{\partial^{|\mathbf{k}|}f}{\partial x_1^{k_1}...\partial x_n^{k_n}}(x)=\frac{\partial^{|\mathbf{k}|}f}{\partial x^{\mathbf{k}}}(x)\,,$$

denote the $\mathbf{k}$-partial derivative of f, with the convention that $D^{\mathbf{0}}f=f$ (here, $\mathbf{0}=(0,...,0)\in\mathbb{N}^n$).

Let now X be a compact set in $\mathbb{R}^n$. A *jet of order h on X* is simply a family $\mathcal{F}=\{F^{\mathbf{k}}\}_{|\mathbf{k}|\le h}$ of continuous functions on X, see [Bie80]. We denote by $J^h(X)$ the vector space of jets of order h on X, and set

$$\|\mathcal{F}\|_{J^h(X)}=\max_{|\mathbf{k}|\le h}\|F^{\mathbf{k}}\|_{C^0(X)}\,.$$

One says that $\mathcal{F}\in J^h(X)$ is a *Whitney's jet of order h on X* if, for every $|\mathbf{k}|\le h$,

$$\sup_{x,y\in X\,,0<|x-y|<r}\Big|F^{\mathbf{k}}(y)-F^{\mathbf{k}}(x)-\sum_{|j|=1}^{h-|\mathbf{k}|}F^{\mathbf{k}+\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}}\Big|=o(r^{h-|\mathbf{k}|})\,.$$

Given $\alpha\in[0,1]$, we denote by $WJ^{h,\alpha}(X)$ the space of Whitney's jets of order h on X such that

$$\begin{aligned}\|\mathcal{F}\|_{WJ^{h,\alpha}(X)}&=\max_{|\mathbf{k}|\le h}\|F^{\mathbf{k}}\|_{C^0(X)}\\&\quad+\max_{|\mathbf{k}|\le h}\sup_{x\,,y\in X\,,x\ne y}\frac{|F^{\mathbf{k}}(y)-F^{\mathbf{k}}(x)-\sum_{|j|=1}^{h-|\mathbf{k}|}F^{\mathbf{k}+\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}}|}{|x-y|^{h-|\mathbf{k}|+\alpha}}\,,\end{aligned}$$

is finite. We set $WJ^h(X)=WJ^{h,0}(X)$, and notice that $WJ^{h+1}(X)\subset WJ^{h,\alpha}(X)\subset WJ^h(X)$ for every $h\in\mathbb{N}$ and $\alpha\in(0,1]$.

Theorem 2.3 (Whitney's extension theorem)**.** *For every $n,h\ge1$, $\alpha\in[0,1]$ and $L>0$ there exists a constant C_0 depending on n, α and L only with the following property. If X is a compact set in $\mathbb{R}^n$ with $X\subset B_L$ and $\mathcal{F}\in WJ^{h,\alpha}(X)$, then there exists $f\in C^\infty(\mathbb{R}^n\setminus X)\cap C^{h,\alpha}(\mathbb{R}^n)$ such that*

$$D^{\mathbf{k}}f=F^{\mathbf{k}}\ \text{on } X \text{ for every } |\mathbf{k}|\le h\,,\tag{2.11}$$

$$\|f\|_{C^{h,\alpha}(\mathbb{R}^n)}\le C_0\,\|\mathcal{F}\|_{WJ^{h,\alpha}(X)}\,,\qquad\|f\|_{C^h(\mathbb{R}^n)}\le C_0\,\|\mathcal{F}\|_{WJ^h(X)}\,.\tag{2.12}$$

If, moreover, X is connected by rectifiable arcs and its geodesic distance dist_X *satisfies*

$$\mathrm{dist}_X(x,y)\le\omega\,|x-y|\,,\qquad\forall x,y\in X\,,\tag{2.13}$$

for some $\omega>0$, then $\|\mathcal{F}\|_{WJ^h(X)}\le2\,\omega\,\|\mathcal{F}\|_{J^h(X)}$, and thus, in particular,

$$\|f\|_{C^h(\mathbb{R}^n)}\le2\,\omega\,C_0\,\|\mathcal{F}\|_{J^h(X)}\,.\tag{2.14}$$

Proof. The classical construction introduced by Whitney (see [Ste70, Theorem 4, Chapter VI] or [Bie80, Theorem 2.3]) gives a function $g\in C^\infty(\mathbb{R}^n\setminus X)\cap C^{h,\alpha}(B_{2L})$ with

$$D^{\mathbf{k}}g=F^{\mathbf{k}}\ \text{on } X \text{ for every } |\mathbf{k}|\le h\,,\tag{2.15}$$

$$\|g\|_{C^{h,\alpha}(B_{2L})}\le C\,\|\mathcal{F}\|_{WJ^{h,\alpha}(X)}\,,\qquad\|g\|_{C^h(B_{2L})}\le C\,\|\mathcal{F}\|_{WJ^h(X)}\,,\tag{2.16}$$

where the constant C depends on n, h, α and L. If we now pick $\eta\in C^\infty_c(B_{2L};[0,1])$ with $\eta=1$ on B_L, then by setting $f=g\,\eta$ we prove the first part of the statement. The second part of the statement is [Bie80, Proposition 2.13]. For the sake of clarity, let us explain this point in the case $h=1$. If X is connected by rectifiable arcs and $x,y\in X$ with $x\neq y$, then for every $\varepsilon>0$ there exists $\phi\in C^0([0,1];X)$ such that

$$\ell(\phi)\le(1+\varepsilon)\,\mathrm{dist}_X(x,y)\le(1+\varepsilon)\,\omega\,|x-y|\,,\qquad \phi(0)=x\,,\qquad \phi(1)=y\,,\tag{2.17}$$

where $\ell(\phi)$ is the total variation of ϕ. We can re-parameterize ϕ on $[0,1]$ so to have $\phi\in\mathrm{Lip}\,([0,1];X)$ with $|\phi'(t)|=\ell(\phi)$ for every $t\in[0,1]$. By (2.17) we thus find

$$\begin{aligned}
&\Big|F^{\mathbf{0}}(y)-F^{\mathbf{0}}(x)-\sum_{i=1}^n F^{e_i}(x)(y-x)_i\Big|\\
=\;&|f(y)-f(x)-\nabla f(x)\cdot(y-x)|=\Big|\int_0^1(\nabla f(\phi(t))-\nabla f(x))\cdot\phi'(t)\,dt\Big|\\
\le\;&2\,\|\nabla f\|_{C^0(X)}\int_0^1|\phi'(t)|\,dt\le 2\,(1+\varepsilon)\,\|\mathcal{F}\|_{J^1(X)}\,\omega\,|x-y|\,.
\end{aligned}$$

□

The following two propositions are used in the proof of Theorem 3.1.

Proposition 2.4. *If $n\ge2$, $1\le k\le n-1$, $\alpha\in(0,1]$ and $L>0$, then there exist positive constants C and ε depending on n, k, α and L only, with the following property. Let X be a compact set in $\mathbb{R}^n$ with $\mathrm{diam}(X)\le L$, and assume that for every $x\in X$ one can define an orthonormal system of vectors $\{\nu^{(j)}(x)\}_{j=1}^{n-k}$ in such a way that for every $x,y\in X$ and $1\le j\le n-k$,*

$$|\nu^{(j)}(x)\cdot(y-x)|\le L\,|x-y|^{1+\alpha}\,,\qquad |\nu^{(j)}(x)-\nu^{(j)}(y)|\le L\,|x-y|^{\alpha}\,.\tag{2.18}$$

Then there exists $d\in C^\infty(\mathbb{R}^n\setminus X;\mathbb{R}^{n-k})\cap C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^{n-k})$ with

$$\begin{aligned}
&d(x)=0 \text{ and } \nabla d(x)=\textstyle\sum_{j=1}^{n-1}e_j\otimes\nu^{(j)}(x) \text{ for every } x\in X\,,\\
&I_\varepsilon(X)\cap\{d=0\} \text{ is a } k\text{-dimensional } C^{1,\alpha}\text{-manifold in } \mathbb{R}^n\,,\\
&\max\left\{\varepsilon^{-1}\,,\|d\|_{C^{1,\alpha}(\mathbb{R}^n)}\right\}\le C\,.
\end{aligned}\tag{2.19}$$

Proof. By (2.18), if one sets $F_j^{\mathbf{0}}(x)=0$ and $F_j^{e_i}(x)=\nu^{(j)}(x)\cdot e_i$ for $x\in X$ and $1\le i\le n$, then $\mathcal{F}_j\in WJ^{1,\alpha}(X)$ with $\|\mathcal{F}_j\|_{WJ^{1,\alpha}(X)}\le C$. Since $\mathrm{diam}(X)\le L$, by Theorem 2.3 one finds $d_j\in C^\infty(\mathbb{R}^n\setminus X)\cap C^{1,\alpha}(\mathbb{R}^n)$ with $d_j=0$ and $\nabla d_j=\nu^{(j)}$ on X. The function $d=\sum_{j=1}^{n-k}d_j\,e_j$ satisfies the first property in (2.19). If now $x\in I_\varepsilon(X)$, then there exists $y\in X$ such that $|y-x|<\varepsilon$, thus $\|d\|_{C^{1,\alpha}(\mathbb{R}^n)}\le C$ gives

$$\Big|\bigwedge_{j=1}^{n-k}\nabla d(x)[\nu^{(j)}(y)]\Big|\ge\Big|\bigwedge_{j=1}^{n-k}\nabla d(y)[\nu^{(j)}(y)]\Big|-C\,\varepsilon=1-C\,\varepsilon\ge\frac12\,,$$

provided ε is small enough (depending only on n, k, α and L). In particular, $\nabla d(x)$ has rank $n-k$ for every $x\in I_\varepsilon(X)$, thus $I_\varepsilon(X)\cap\{d=0\}$ is a k-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^n$. □

Proposition 2.5. *If $n\ge2$, $1\le k\le n-1$, $\alpha\in(0,1]$ and $L>0$, then there exists a constant C depending on n, k, α and L only, with the following property. If S is a compact connected k-dimensional $C^{2,1}$-manifold with boundary in $\mathbb{R}^n$ with $\mathrm{diam}(S)\le L$ and*

$$\mathrm{dist}_{\mathrm{bd}\,(S)}(x,y)\le L\,|x-y|\,,\qquad\forall x,y\in\mathrm{bd}\,(S)\,,$$

and $\bar a\in C^{1,\alpha}(\mathrm{bd}\,(S))$, then there exist $a\in C^{1,\alpha}(\mathbb{R}^n)$ with $a=\bar a$ on $\mathrm{bd}\,(S)$ and

$$\|a\|_{C^{1,\alpha}(\mathbb{R}^n)}\le C\,\|\bar a\|_{C^{1,\alpha}(\mathrm{bd}\,(S))}\,,\qquad\|a\|_{C^1(\mathbb{R}^n)}\le C\,\|\bar a\|_{C^1(\mathrm{bd}\,(S))}\,.$$

Proof. We note that, by definition of tangential gradient, $\nabla^{\mathrm{bd}\,S}\bar{a}(x) \in T_x(\mathrm{bd}\,(S))$ for every $x \in \mathrm{bd}\,(S)$. We then define $\mathcal{F} \in J^1(\mathrm{bd}\,(S))$ by setting $F^{\mathbf{0}}(x) = \bar{a}(x)$ and $F^{e_i}(x) = e_i \cdot \nabla^{\mathrm{bd}\,S}\bar{a}(x)$ for $x \in \mathrm{bd}\,(S)$, and note that $\mathcal{F} \in WJ^{1,\alpha}(\mathrm{bd}\,(S))$ with

$$\|\mathcal{F}\|_{WJ^{1,\alpha}(\mathrm{bd}\,(S))} \le \|\bar{a}\|_{C^{1,\alpha}(\mathrm{bd}\,(S))}\,, \qquad \|\mathcal{F}\|_{WJ^{1}(\mathrm{bd}\,(S))} \le \|\bar{a}\|_{C^{1}(\mathrm{bd}\,(S))}\,.$$

We conclude by Theorem 2.3. □

3. Almost-normal diffeomorphisms between manifolds with boundary

The main result of this section is Theorem 3.1, where we address the following problem. We are given two k-dimensional manifolds with boundary S_0 and S, which are known to be close in Hausdorff distance. Moreover, we are given a diffeomorphism f_0 (close to the identity map) between the boundaries of S_0 and S, and we know that S is a small normal deformation of S_0 up to some small distance from its boundary. (The motivation for considering this scenario is that – by interior and boundary/free-boundary regularity theorems – this is the typical starting point in addressing the improved convergence problem in presence of singularities). Then we would like to extend f_0 into a diffeomorphism f between S_0 and S while keeping the size of the tangential displacement $\pi_{S_0}(f - \mathrm{Id})$ of f as small as possible.

In section 3.1 we state and prove Theorem 3.1, while in section 3.2 we provide an alternative formulation of this result in terms of sequences of manifolds converging to a limit manifold S_0 which is more natural to invoke when addressing applications.

3.1. Construction of the diffeomorphisms.

Before stating the theorem we premise the following definition, which in turn is motivated by Proposition 2.4. Given an orientable k-dimensional $C^{1,\alpha}$-manifold S in $\mathbb{R}^n$ which admits a global normal frame of class $C^{1,\alpha}$ (i.e., such that for every $x \in S$ there exists an orthonormal basis $\{\nu_S^{(i)}(x)\}_{i=1}^{n-k}$ of $(T_xS)^\perp$ with the property $\nu^{(i)} \in C^{1,\alpha}(S)$ for each i) then one writes

$$\|S\|_{C^{1,\alpha}} \le L\,,$$

if

$$\begin{cases} |\nu_S^{(i)}(x) - \nu_S^{(i)}(y)| \le L\,|x-y|^\alpha\,, \\ |\nu_S^{(i)}(x)\cdot(y-x)| \le L|y-x|^{1+\alpha}\,, \end{cases} \qquad \forall x,y \in S\,, i = 1,...,n-k\,. \tag{3.1}$$

We are now ready to state the main result of this section (see Remark 3.4 below for some clarifications about the cumbersome assumption (a)).

Theorem 3.1. *If $n \ge 2$, $1 \le k \le n-1$, $\alpha \in (0,1]$, and $L > 0$, then there exist $\mu_0 \in (0,1)$ and $C_0 > 0$ (depending on n, k, α, and L only) with the following property.*
(a) Let S_0 be a compact connected k-dimensional $C^{2,1}$-manifold with boundary in $\mathbb{R}^n$, let $\widetilde{S}_0$ be a k-dimensional $C^{2,1}$-manifold in $\mathbb{R}^n$, and assume that

$$\mathrm{bd}\,(S_0) \ne \emptyset\,, \qquad S_0 \subset \widetilde{S}_0\,, \qquad \mathrm{diam}(\widetilde{S}_0) \le L\,, \tag{3.2}$$

$$\mathrm{dist}_{\mathrm{bd}\,(S_0)}(x,y) \le L\,|x-y|\,, \qquad \forall x,y \in \mathrm{bd}\,(S_0) \quad (\textit{if } k \ge 2)\,, \tag{3.3}$$

$$\mathrm{dist}_{S_0}(x,y) \le L\,|x-y|\,, \qquad \forall x,y \in S_0\,, \tag{3.4}$$

$$\mathrm{dist}_{\widetilde{S}_0}(x,y) \le L\,|x-y|\,, \qquad \forall x,y \in \widetilde{S}_0\,. \tag{3.5}$$

Moreover, let $\{\nu_0^{(i)}\}_{i=1}^{n-k} \subset C^{1,1}(\widetilde{S}_0;\mathbb{S}^{n-1})$ be such that $\{\nu_0^{(i)}(x)\}_{i=1}^{n-k}$ is an orthonormal basis of $(T_x\widetilde{S}_0)^\perp$ for every $x \in \widetilde{S}_0$, and

$$\max_{1\le i\le n-k} \|\nu_0^{(i)}\|_{C^{1,1}(\widetilde{S}_0)} \le L\,. \tag{3.6}$$

(b) Let S be a compact connected k-dimensional $C^{1,\alpha}$-manifold with boundary such that, for some $\rho \in (0,\mu_0^2)$, one has

$$\mathrm{bd}\,(S) \neq \emptyset\,, \qquad \|S\|_{C^{1,\alpha}} \leq L\,, \qquad \mathrm{hd}(S,S_0) \leq \rho\,. \tag{3.7}$$

In addition:

(i) if $k=1$, assume that, setting $\mathrm{bd}\,(S_0)=\{p_0,q_0\}$, $\mathrm{bd}\,(S)=\{p,q\}$, $f_0(p_0)=p$ and $f_0(q_0)=q$,

$$\begin{aligned} \frac{1}{L} &\leq |p_0-q_0|\,,\\ \|f_0-\mathrm{Id}\|_{C^0(\mathrm{bd}\,(S_0))} + \|\nu_S^{co}(f_0)-\nu_{S_0}^{co}\|_{C^0(\mathrm{bd}\,(S_0))} &\leq \rho\,; \end{aligned} \tag{3.8}$$

if $k\geq 2$, assume that there exists a $C^{1,\alpha}$-diffeomorphism f_0 between $\mathrm{bd}\,(S_0)$ and $\mathrm{bd}\,(S)$ with

$$\begin{aligned} \|f_0\|_{C^{1,\alpha}(\mathrm{bd}\,(S_0))} &\leq L\,,\\ \|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))} &\leq \rho\,,\\ \max_{1\leq i\leq n-k} \|\nu_S^{(i)}(f_0)-\nu_0^{(i)}\|_{C^0(\mathrm{bd}\,(S_0))} &\leq \rho\,,\\ \|\nu_S^{co}(f_0)-\nu_{S_0}^{co}\|_{C^0(\mathrm{bd}\,(S_0))} &\leq \rho\,, \end{aligned} \tag{3.9}$$

where $\{\nu_S^{(i)}\}_{i=1}^{n-k}$ is as in (3.1).

(ii) there exists $\{\psi_i\}_{i=1}^{n-k}\subset C^{1,\alpha}([S_0]_\rho)$ such that, setting $\psi=\sum_{i=1}^{n-k}\psi_i\,\nu_{S_0}^{(i)}$, one has

$$\begin{aligned} &[S]_{3\rho}\subset (\mathrm{Id}+\psi)([S_0]_\rho)\subset S\,,\\ &\|\psi\|_{C^{1,\alpha}([S_0]_\rho)}\leq L\,, \qquad \|\psi\|_{C^1([S_0]_\rho)}\leq\rho\,. \end{aligned} \tag{3.10}$$

Then, for every $\mu\in(\sqrt{\rho},\mu_0)$ there exists a $C^{1,\alpha}$-diffeomorphism f between S_0 and S such that

$$f = f_0\,, \qquad \text{on } \mathrm{bd}\,(S_0)\,, \tag{3.11}$$

$$f = \mathrm{Id}+\psi\,, \qquad \text{on } [S_0]_\mu\,, \tag{3.12}$$

$$\|f\|_{C^{1,\alpha}(S_0)} \leq C_0\,, \tag{3.13}$$

$$\|f-\mathrm{Id}\|_{C^0(S_0)} \leq C_0\big(\mathrm{hd}(S,S_0)+\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}+\|\psi\|_{C^0([S_0]_\rho)}\big)\,, \tag{3.14}$$

$$\|f-\mathrm{Id}\|_{C^1(S_0)} \leq \frac{C_0}{\mu}\,\rho^\alpha\,, \tag{3.15}$$

$$\|\pi^{S_0}(f-\mathrm{Id})\|_{C^1(S_0)} \leq \frac{C_0}{\mu}\begin{cases} \|(f-\mathrm{Id})\cdot\nu_{S_0}^{co}\|_{C^0(\mathrm{bd}\,(S_0))}\,, & \text{if } k=1\,,\\ \|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,, & \text{if } k\geq 2\,. \end{cases} \tag{3.16}$$

Remark 3.2. One would expect the C^0 norm of $f_0-\mathrm{Id}$, and not its C^1-norm, to appear in (3.14). When $k=1$ we indeed prove this, as in that case $\mathrm{bd}\,(S_0)$ consists of two points and thus $\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}=\|f_0-\mathrm{Id}\|_{C^0(\mathrm{bd}\,(S_0))}$. However, when $k\geq 2$, our construction of f requires a preliminary rough extension of f_0 from $\mathrm{bd}\,(S_0)$ to $\mathbb{R}^n$ by means of Whitney's theorem. Although the $C^{1,\alpha}(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$-norms of this rough extension will be controlled by the $C^{1,\alpha}(\mathrm{bd}\,(S_0))$ and $C^1(\mathrm{bd}\,(S_0))$-norms of f_0, because of how Whitney's extension procedure works, the $C^0(\mathbb{R}^n)$-norm will only be controlled by the full $C^1(\mathrm{bd}\,(S_0))$-norm of f_0.

Remark 3.3. In order to obtain (in the spirit of (3.14)) a more precise estimate than (3.15), that is, in order to replace ρ^α by some function of $\mathrm{hd}(S,S_0)$, $\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}$, $\|\psi\|_{C^1([S_0]_\rho)}$ etc., one would need to relate to these quantities the smallest value of ρ which makes the inclusion $[S]_{3\rho}\subset(\mathrm{Id}+\psi)([S_0]_\rho)$ in (3.10) hold. More precisely, with such a control at hand one could prove such a strengthened form of (3.15) by the same argument used below.

Remark 3.4. We claim that assumption (a) can be replaced by

$$\begin{aligned}&S_0 \text{ is a compact connected } k\text{-dimensional } C^{2,1}\text{-manifold with boundary in } \mathbb{R}^n\\&\text{and there exists } \{\nu_{S_0}^{(i)}\}_{i=1}^{n-k} \subset C^{1,1}(S_0;\mathbb{S}^{n-1}) \text{ such that}\\&\{\nu_{S_0}^{(i)}\}_{i=1}^{n-k} \text{ is an orthonormal basis of } (T_xS_0)^\perp \text{ for every } x\in S_0\,.\end{aligned} \tag{3.17}$$

(In the case $k=1$, (3.17) simply amounts in requiring that S_0 is a compact connected $C^{2,1}$-curve with boundary in $\mathbb{R}^n$.) More precisely, we claim that (3.17) implies the existence of an extension $\widetilde{S}_0$ of S_0 and of a normal frame $\{\nu_0^{(i)}\}_{i=1}^{n-k}$ to $\widetilde{S}_0$ such that assumption (a) holds for a suitable value of L: correspondingly, the constants C_0 and μ_0 given by the theorem will depend on the particular extension $\widetilde{S}_0$ we have considered. We now prove the claim. By compactness of S_0 one immediately finds a constant L' such that (3.3) and (3.4) hold with L' in place of L, $\mathrm{diam}(S_0)\le L'$, and $\|\nu_{S_0}^{(i)}\|_{C^{1,1}(S_0)}\le L'$. Now let us fix $\ell=1,...,n-k$, and for $x\in S_0$ set

$$F^{\mathbf{0}}(x)=0\,,\qquad F^{e_i}(x)=\nu_{S_0}^{(\ell)}(x)\cdot e_i\,,\qquad F^{e_i+e_j}(x)=e_i\cdot\nabla^{S_0}\nu_{S_0}^{(\ell)}(x)[e_j]\,.$$

By compactness of S_0 we find that $\mathcal{F}=\{F^{\mathbf{k}}\}_{|\mathbf{k}|\le 2}\in WJ^{2,1}(S_0)$. Hence, by arguing as in the proof of Proposition 2.4, there exist $d_{S_0}\in C^{2,1}(\mathbb{R}^n;\mathbb{R}^{n-k})$ and $\varepsilon_0>0$ such that

$$\begin{aligned}&d_{S_0}(x)=\mathbf{0} \text{ and } \nabla d_{S_0}(x)=\textstyle\sum_{i=1}^{n-k} e_i\otimes\nu_{S_0}^{(i)}(x) \text{ for every } x\in S_0\,,\\&I_{\varepsilon_0}(S_0)\cap\{d_{S_0}=\mathbf{0}\} \text{ is a } k\text{-dimensional } C^{2,1}\text{-manifold in } \mathbb{R}^n\,,\\&\max\big\{\varepsilon_0^{-1}\,,\|d_{S_0}\|_{C^{2,1}(\mathbb{R}^n)}\big\}\le C\,,\end{aligned} \tag{3.18}$$

where C depends on n, k and S_0 only. Let us set $\widetilde{S}_0=I_{\varepsilon_0}(S_0)\cap\{d_{S_0}=\mathbf{0}\}$. Up to further decreasing the value of ε_0 one immediately deduces (3.2) and (3.5) for some value of L. Moreover, by construction, for every $i=1,...,n-k$ there exists $\{h_{i,j}\}_{j=1}^n\subset C^{1,1}(\mathbb{R}^n)$ such that

$$\nabla d_{S_0}(x)=\sum_{i=1}^{n-k} e_i\otimes\big(\sum_{j=1}^n h_{i,j}(x)\,e_j\big)\,,\qquad \forall x\in\mathbb{R}^n\,.$$

Up to further decreasing the value of ε_0 we can define $\{\nu_0^{(i)}\}_{i=1}^{n-k}\in C^{1,1}(\widetilde{S}_0;\mathbb{S}^{n-1})$ in such a way that (3.6) holds by simply applying the Gram-Schmidt orthogonalization process to the vectors $\{\sum_{j=1}^n h_{i,j}(x)\,e_j\}_{i=1}^{n-k}$.

Proof of Theorem 3.1. In the following we denote by C a constant which may depend on n, k, α and L only. We start our argument by extending S into a larger manifold $\widetilde{S}$. More precisely, by $\|S\|_{C^{1,\alpha}}\le L$ and Proposition 2.4, there exist $d_S\in C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^{n-k})$ and $\varepsilon>0$ such that

$$\begin{aligned}&d_S(x)=\mathbf{0} \text{ and } \nabla d_S(x)=\textstyle\sum_{i=1}^{n-k} e_i\otimes\nu_S^{(i)}(x) \text{ for every } x\in S\,,\\&I_\varepsilon(S)\cap\{d_S=\mathbf{0}\} \text{ is a } k\text{-dimensional } C^{1,\alpha}\text{-manifold in } \mathbb{R}^n\,,\\&\max\big\{\varepsilon^{-1}\,,\|d_S\|_{C^{1,\alpha}(\mathbb{R}^n)}\big\}\le C\,,\end{aligned} \tag{3.19}$$

where $\mathbf{0}=(0,...,0)\in\mathbb{R}^{n-k}$. We shall use d_S to locate the position of S in $\mathbb{R}^n$ (see the proof of the claim below). We set

$$\widetilde{S}=I_\varepsilon(S)\cap\{d_S=\mathbf{0}\}\,,$$

and we record for future use that, by (3.19), if $v\in\mathbb{S}^{n-1}$, $\delta>0$, and $x\in S$, then

$$|\nabla d_S(x)v|\le C\,\delta\,,\qquad \text{if } |\pi_x^S(v)|\ge 1-\delta\,, \tag{3.20}$$

$$|\nabla d_S(x)v|\ge 1-C\,\delta\,,\qquad \text{if } |\pi_x^S(v)|\le\delta\,, \tag{3.21}$$

Next, we note that

$$\max_{1\le i\le n-k}|\nu_0^{(i)}(x)\cdot(y-x)|\le C\,|\pi_x^{\widetilde{S}_0}(y-x)|^2\,,\qquad \forall x\in\widetilde{S}_0\,,y\in B_{x,1/C}\cap\widetilde{S}_0\,. \tag{3.22}$$

$$\begin{aligned}|y-x|&\le 2\,|\pi_x^{\widetilde{S}_0}(y-x)|\,,\qquad \forall x\in\widetilde{S}_0\,,y\in B_{x,1/C}\cap\widetilde{S}_0\,,\\ \|\pi_x^{\widetilde{S}_0}-\pi_y^{\widetilde{S}_0}\|&\le C\,|x-y|\,,\qquad \forall x,y\in\widetilde{S}_0\,.\end{aligned} \tag{3.23}$$

Indeed, (3.22) follows from (3.6) and the fact that $\{\nu^{(i)}(x)\}_{i=1}^{n-k}$ is an orthonormal basis of $(T_x\widetilde{S}_0)^\perp$, the first condition in (3.23) follows from (3.22), and the second condition in (3.23) is an immediate consequence of $[\nu^{(i)}]_{C^{0,1}(\widetilde{S}_0)}\le L$. We now set

$$U_{x,\delta}=\widetilde{S}_0\cap B_{x,\delta}\,,\quad K_\delta=I_\delta(\mathrm{bd}\,(S_0))\cap\widetilde{S}_0\,,\quad K_\delta^+=I_\delta(\mathrm{bd}\,(S_0))\cap S_0\,,\qquad x\in\widetilde{S}_0\,,\delta>0\,,$$

and then we make the following claim:

Claim: There exists η_0 depending on n, k, α and L only such that, if μ_0 is small enough with respect to η_0, then one can construct $f:K_{\eta_0}\to\widetilde{S}$ with

$$f = f_0\,,\qquad \text{on }\mathrm{bd}\,(S_0)\,, \tag{3.24}$$

$$f = \mathrm{Id}+\psi\,,\qquad \text{on }K_{\eta_0}^+\setminus K_\mu\,, \tag{3.25}$$

$$\|f\|_{C^{1,\alpha}(K_{\eta_0})} \le C\,, \tag{3.26}$$

$$\|f-\mathrm{Id}\|_{C^0(K_{\eta_0}^+)} \le C\left(\mathrm{hd}(S,S_0)+\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right), \tag{3.27}$$

$$\|f-\mathrm{Id}\|_{C^1(K_{\eta_0}^+)} \le \frac{C}{\mu}\,\rho^\alpha\,, \tag{3.28}$$

$$\|\pi^{\widetilde{S}_0}(f-\mathrm{Id})\|_{C^1(K_{\eta_0})} \le \frac{C}{\mu}\begin{cases}\|(f-\mathrm{Id})\cdot\nu_{S_0}^{co}\|_{C^0(\mathrm{bd}\,(S_0))}\,, & \text{if }k=1\,,\\ \|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,, & \text{if }k\ge 2\,,\end{cases} \tag{3.29}$$

$$J^{\widetilde{S}_0}f \ge \frac{1}{2}\,,\qquad \text{on }K_{\eta_0}\,, \tag{3.30}$$

$$\pi^{\widetilde{S}_0}(f-\mathrm{Id}) = 0\,,\qquad \text{on }K_{\eta_0}\setminus K_\mu\,, \tag{3.31}$$

$$f(K_{\eta_0}^+) \subset S\,. \tag{3.32}$$

Given the claim, the theorem follows: Indeed, if one extends f from K_{η_0} to $K_{\eta_0}\cup S_0$ by setting $f=\mathrm{Id}+\psi$ on $S_0\setminus K_{\eta_0}$, then thanks to (3.25), (3.10) and (3.26) we find that $f\in C^{1,\alpha}(K_{\eta_0}\cup S_0;\mathbb{R}^n)$ and that (3.11), (3.12) and (3.13) hold; similarly, (3.14) and (3.15) follow by (3.27) and (3.28), while (3.29) and (3.31) imply (3.16). By Theorem 2.1, (3.4), (3.23), (3.26) and (3.30) there exists $r_0>0$ (depending on n, k, α and L only) such that if $\|f-\mathrm{Id}\|_{C^0(S_0)}\le r_0$ (as we can entail thanks to (3.27), (3.7), (3.9), and (3.10) provided we take $\mu_0^2\le r_0$), then f is a $C^{1,\alpha}$-diffeomorphism between $\mathrm{int}\,(S_0)$ and $f(\mathrm{int}\,(S_0))$. Let us set

$$S^*=\mathrm{cl}\,(f(\mathrm{int}\,(S_0)))\,,$$

so that $S^*\subset S$ by (3.10) and (3.32). Moreover, S^* is a compact connected k-dimensional $C^{1,\alpha}$-manifold with boundary in $\mathbb{R}^n$ with

$$\mathrm{int}\,(S^*)=f(\mathrm{int}\,(S_0))\,,\qquad \mathrm{bd}\,(S^*)=S^*\setminus f(\mathrm{int}\,(S_0))=f(\mathrm{bd}\,(S_0))=\mathrm{bd}\,(S)\,,$$

thus, by connectedness of S, one has $S=S^*=f(S_0)$. Indeed, in order to obtain a contradiction it suffices to consider $y\in\mathrm{int}\,(S)\setminus S^*$, together with a curve γ with $\mathrm{int}\,(\gamma)\subset\mathrm{int}\,(S)\setminus S^*$, i.e. which lives in the connected component of $\mathrm{int}\,(S)\setminus S^*$ determined by y, such that $\mathrm{bd}\,(\gamma)=\{y,x\}$ with $x\in\mathrm{bd}\,(S)$.

Proof of the claim: We first describe the case $k \ge 2$, and then explain the minor variants needed when $k=1$. We fix $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty);[0,1])$ such that, setting $\phi_\mu = \phi(\cdot,\mu)$ for $\mu > 0$,

$$\phi_\mu \in C^\infty_c(I_\mu(\mathrm{bd}\,(S_0)))\,, \qquad \phi_\mu = 1 \text{ on } I_{\mu/2}(\mathrm{bd}\,(S_0))\,, \tag{3.33}$$

$$|\nabla\phi_\mu(x)| \le \frac{C}{\mu}\,, \qquad |\nabla^2\phi_\mu(x)| \le \frac{C}{\mu^2}\,, \qquad \forall (x,\mu) \in \mathbb{R}^n \times (0,\infty)\,. \tag{3.34}$$

Let us define $\bar a_i : \mathrm{bd}\,(S_0) \to \mathbb{R}$, $i = 1,...,n-k$, and $\bar b : \mathrm{bd}\,(S_0) \to \mathbb{R}^n$ by setting

$$\bar a_i(x) = (f_0(x) - x)\cdot \nu_0^{(i)}(x)\,, \qquad \bar b(x) = f_0(x) - x - \sum_{i=1}^{n-k} a_i(x)\,\nu_0^{(i)}(x)\,, \qquad x \in \mathrm{bd}\,(S_0)\,, \tag{3.35}$$

so that, trivially,

$$f_0(x) = x + \bar b(x) + \sum_{i=1}^{n-k} \bar a_i(x)\,\nu_0^{(i)}(x)\,, \qquad \forall x \in \mathrm{bd}\,(S_0)\,. \tag{3.36}$$

By (3.9) one has

$$\begin{aligned}&\|\bar a_i\|_{C^{1,\alpha}(\mathrm{bd}\,(S_0))} + \|\bar b\|_{C^{1,\alpha}(\mathrm{bd}\,(S_0))} \le C\,,\\ &\|\bar a_i\|_{C^1(\mathrm{bd}\,(S_0))} + \|\bar b\|_{C^1(\mathrm{bd}\,(S_0))} \le C\,\|f_0 - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))} \le C\,\rho\,,\end{aligned} \tag{3.37}$$

By Proposition 2.5 and by (3.3) we find $a_i \in C^{1,\alpha}(\mathbb{R}^n)$, $i=1,...,n-k$, and $b \in C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\begin{gathered}a_i = \bar a_i \text{ and } b = \bar b\,, \qquad \text{on } \mathrm{bd}\,(S_0)\,,\\ \|a_i\|_{C^{1,\alpha}(\mathbb{R}^n)} + \|b\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C\,, \qquad \|a_i\|_{C^1(\mathbb{R}^n)} + \|b\|_{C^1(\mathbb{R}^n)} \le C\,\|f_0 - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,.\end{gathered} \tag{3.38}$$

Correspondingly we define $G \in C^{1,\alpha}(\widetilde S_0;\mathbb{R}^n)$ by setting

$$G(x) = \phi_\mu(x)\,b(x) + \sum_{i=1}^{n-k} a_i(x)\,\nu_0^{(i)}(x)\,, \qquad x \in \widetilde S_0\,. \tag{3.39}$$

By (3.33) and (3.36),

$$f_0(x) = x + G(x)\,, \qquad \forall x \in \mathrm{bd}\,(S_0)\,, \tag{3.40}$$

while (3.34), (3.38) and $\rho \le \mu^2$ give

$$\|G\|_{C^{1,\alpha}(\widetilde S_0)} \le C\,, \qquad \begin{cases}\|G\|_{C^0(\widetilde S_0)} \le C\,\|f_0 - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,,\\ \|G\|_{C^1(\widetilde S_0)} \le \dfrac{C}{\mu}\,\|f_0 - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))} \le \dfrac{C}{\mu}\,\rho \le C\,\mu_0\,.\end{cases} \tag{3.41}$$

We now define $F \in C^{1,\alpha}(\widetilde S_0 \times (-1,1)^{n-k};\mathbb{R}^n)$ by setting, for $(x,\mathbf{t}) \in \widetilde S_0 \times (-1,1)^{n-k}$,

$$\begin{aligned}F(x,\mathbf{t}) &= x + \phi_\mu(x)\,b(x) + \sum_{i=1}^{n-k}(a_i(x) + t_i)\,\nu_0^{(i)}(x)\\ &= x + G(x) + \sum_{i=1}^{n-k} t_i\,\nu_0^{(i)}(x)\,,\end{aligned} \tag{3.42}$$

and then exploit $d_S \in C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^{n-k})$ to define $u \in C^{1,\alpha}(\widetilde S_0 \times (-1,1)^{n-k};\mathbb{R}^{n-k})$ as

$$u(x,\mathbf{t}) = d_S(F(x,\mathbf{t}))\,, \qquad (x,\mathbf{t}) \in \widetilde S_0 \times (-1,1)^{n-k}\,.$$

By (3.40),

$$F(x,\mathbf{0}) = f_0(x)\,, \qquad \forall x \in \mathrm{bd}\,(S_0)\,, \tag{3.43}$$

which combined with $S \subset \{d_S = \mathbf{0}\}$ implies

$$u(x,\mathbf{0}) = \mathbf{0}\,, \qquad \forall x \in \mathrm{bd}\,(S_0)\,. \tag{3.44}$$

By (3.41) and by (3.19) one has

$$\|F\|_{C^{1,\alpha}(\widetilde{S}_0\times(-1,1)^{n-k})} \le C\,, \qquad \|u\|_{C^{1,\alpha}(\widetilde{S}_0\times(-1,1)^{n-k})} \le C\,. \tag{3.45}$$

We claim that if μ_0 is small enough (and up to identify $(n-k)$-vectors in $\mathbb{R}^{n-k}$ with real numbers, with the convention that $e_1\wedge\cdots\wedge e_{n-k}=1$), then

$$\bigwedge_{i=1}^{n-k}\frac{\partial u}{\partial t_i}(x,\mathbf{0}) \ge \frac{1}{2}\,, \qquad \forall x\in \mathrm{bd}\,(S_0)\,. \tag{3.46}$$

Indeed, by (3.43), (3.19), and by $\partial F/\partial t_i(x,t)=\nu_0^{(i)}(x)$ we find that

$$\bigwedge_{i=1}^{n-k}\frac{\partial u}{\partial t_i}(x,\mathbf{0}) = \bigwedge_{i=1}^{n-k}\nabla d_S(f_0(x))[\nu_0^{(i)}(x)] = \prod_{i=1}^{n-k}\nu^{(i)}(f_0(x))\cdot\nu_0^{(i)}(x)\,, \qquad \forall x\in \mathrm{bd}\,(S_0)\,, \tag{3.47}$$

so that (3.46) follows by (3.9) provided μ_0 is small enough (recall that $\rho<\mu_0^2$). By (3.44), (3.45), (3.46) and Theorem 2.2 (that can be applied thanks to (3.5) and (3.23)) there exists a positive constant $\eta_0>0$ (depending on n, k, α, and L) such that for each $x_0\in\mathrm{bd}\,(S_0)$ one can find $\zeta_{x_0}\in C^{1,\alpha}(U_{x_0,\eta_0};\mathbb{R}^{n-k})$ with

$$u(x,\zeta_{x_0}(x))=\mathbf{0}\,, \qquad \forall x\in U_{x_0,\eta_0}\,, \tag{3.48}$$

$$\zeta_{x_0}(x_0)=\mathbf{0}\,, \qquad \|\zeta_{x_0}\|_{C^{1,\alpha}(U_{x_0,\eta_0})}\le C\,. \tag{3.49}$$

Note that we had to put constraint on the smallness of μ_0 to assert the existence of η_0. We are of course free to decrease the value of μ_0 without affecting the value of η_0. We shall require that μ_0 is suitably smaller than η_0, precisely that $\mu_0\le\eta_0/C_*$ for some suitable $C_*=C_*(n,k,\alpha,L)$, and we shall further decrease the value of η_0 depending on n, k, α and L only.

Let us now prove that if $x_0,x_1\in\mathrm{bd}\,(S_0)$, then

$$\zeta_{x_0}(x)=\zeta_{x_1}(x)\,, \qquad \forall x\in U_{x_0,\eta_0}\cap U_{x_1,\eta_0}\,. \tag{3.50}$$

Indeed, by $[\zeta_{x_0}]_{C^{0,1}(U_{x_0,\eta_0})}\le C$ and $\zeta_{x_0}(x_0)=\mathbf{0}$ one has

$$\|\zeta_{x_0}\|_{C^0(U_{x_0,\eta_0})}\le C_1\,\eta_0\,, \tag{3.51}$$

for some constant C_1 depending on n, k, α and L only. In particular, up to further decreasing the value of η_0 in dependence of the $C^{1,\alpha}$-bound on u in (3.45) and of C_1, we can entail

$$\bigwedge_{i=1}^{n-k}\frac{\partial u}{\partial t_i}(x,\mathbf{t}) \ge \frac{1}{3}\,, \qquad \forall (x,\mathbf{t})\in U_{x_0,\eta_0}\times(-C_1\eta_0,C_1\,\eta_0)^{n-k}\,. \tag{3.52}$$

Now, if $x\in U_{x_0,\eta_0}\cap U_{x_1,\eta_0}$ and we set $A_0=(-C_1\eta_0,C_1\,\eta_0)^{n-k}$, then by (3.45) and (3.52) one has $u(x,\cdot)\in C^{1,\alpha}(A_0;\mathbb{R}^{n-k})$ with

$$\|u(x,\cdot)\|_{C^{1,\alpha}(A_0)}\le C\,, \qquad J^{A_0}u(x,\cdot)\ge\frac{1}{3} \qquad \text{on } A_0\,.$$

By Theorem 2.1, there exists ε_0 (depending on n, k, α and L only) such that $u(x,\cdot)$ is invertible on $A_0^*=(-\varepsilon_0,\varepsilon_0)^{n-k}$. By requiring that $C_1\,\eta_0<\varepsilon_0$, we thus find that $u(x,\cdot)$ is invertible on A_0, and since $\zeta_{x_0}(x),\zeta_{x_1}(x)\in A_0$ with $u(x,\zeta_{x_0}(x))=u(x,\zeta_{x_1}(x))$ by (3.48), we deduce (3.50). Moreover, by an entirely analogous argument, we deduce from (3.44) and (3.48) that

$$\zeta_{x_0}(x)=\mathbf{0}\,, \qquad \forall x\in\mathrm{bd}\,(S_0)\cap U_{x_0,\eta_0}\,. \tag{3.53}$$

By (3.48), (3.49) (3.50), and (3.53), if we define $\zeta\in C^{1,\alpha}(K_{\eta_0};\mathbb{R}^{n-k})$ (recall that $K_{\eta_0}=I_{\eta_0}(\mathrm{bd}\,(S_0))\cap\widetilde{S}_0$) by setting $\zeta=\zeta_{x_0}$ on U_{x_0,η_0} for each $x_0\in\mathrm{bd}\,(S_0)$, then

$$u(x,\zeta(x))=\mathbf{0} \quad \forall x\in K_{\eta_0}\,, \qquad \zeta(x)=\mathbf{0} \quad \forall x\in\mathrm{bd}\,(S_0)\,, \tag{3.54}$$

$$\|\zeta\|_{C^0(K_{\eta_0})}\le C_1\,\eta_0\,, \qquad \|\zeta\|_{C^{1,\alpha}(K_{\eta_0})}\le C\,. \tag{3.55}$$

We finally set

$$f(x)=F(x,\zeta(x))=x+G(x)+\sum_{i=1}^{n-k}\zeta_i(x)\,\nu_0^{(i)}(x)\,,\qquad x\in K_{\eta_0}\,,\tag{3.56}$$

where $\zeta_i=e_i\cdot\zeta$, and show that f has the required properties. By (3.43) and (3.54) we prove (3.24), while (3.26) follows from (3.45) and (3.55). Similarly, (3.54) and the definition of u give

$$f(K_{\eta_0})\subset\{d_S=\mathbf{0}\}\,.\tag{3.57}$$

By (3.24), (3.26), and $f(\mathrm{bd}\,(S_0))=f_0(\mathrm{bd}\,(S_0))=\mathrm{bd}\,(S)$, we find that $f(K_{\eta_0})\subset I_{C\,\eta_0}(\mathrm{bd}\,(S))$, so that, up to decrease η_0 and thanks to $\varepsilon>C^{-1}$ (recall (3.19)), we can entail $f(K_{\eta_0})\subset I_\varepsilon(S)$. In particular (3.57) gives

$$f(K_{\eta_0})\subset\widetilde{S}\,.\tag{3.58}$$

By (3.56) and (3.39),

$$\pi^{\widetilde{S}_0}(f-\mathrm{Id})(x)=\phi_\mu(x)\,b(x)\,,\qquad\forall x\in K_{\eta_0}\,,\tag{3.59}$$

so that (3.31) follows by $\mathrm{spt}\,\phi_\mu\subset\subset I_\mu(\mathrm{bd}\,(S_0))$. By differentiating (3.59) along $\tau\in T_x\widetilde{S}_0$ we find

$$\nabla^{\widetilde{S}_0}[\pi^{\widetilde{S}_0}(f-\mathrm{Id})](x)\,[\tau]=\Big(\nabla\phi_\mu(x)\cdot\tau\Big)\,b(x)+\phi_\mu(x)\nabla^{\widetilde{S}_0}b(x)[\tau]\,,$$

which implies (3.29) (recall we are addressing the case $k\ge 2$) once combined with (3.38) and (3.59). By differentiating (3.56) along $\tau\in T_x\widetilde{S}_0$ we find that

$$\begin{aligned}\nabla^{\widetilde{S}_0}f(x)[\tau]\;&=\;\tau+\sum_{i=1}^{n-k}\nabla^{\widetilde{S}_0}\zeta_i(x)[\tau]\,\nu_0^{(i)}(x)\\&\qquad+\nabla^{\widetilde{S}_0}G(x)[\tau]+\sum_{i=1}^{n-k}\zeta_i(x)\nabla^{\widetilde{S}_0}\nu_0^{(i)}(x)[\tau]\,.\end{aligned}\tag{3.60}$$

The first term on the second line is bounded by $C\mu_0$ thanks to (3.41), while the second term on the second line is bounded by $C\,\eta_0$ thanks to (3.6) and (3.55), so that, as we are requiring $\mu_0\le\eta_0/C_*\le\eta_0$,

$$\Big|\nabla^{\widetilde{S}_0}f(x)[\tau]-\Big(\tau+\sum_{i=1}^{n-k}\nabla^{\widetilde{S}_0}\zeta_i(x)[\tau]\,\nu_0^{(i)}(x)\Big)\Big|\le C\,\eta_0\,.\tag{3.61}$$

Thus, if $\{\tau_i\}_{i=1}^k$ is an orthonormal basis of $T_x\widetilde{S}_0$, then

$$J^{\widetilde{S}_0}f(x)\ge\Big|\bigwedge_{i=1}^k\Big(\tau_i+\sum_{j=1}^{n-k}\nabla^{\widetilde{S}_0}\zeta_j(x)[\tau_i]\,\nu_0^{(j)}(x)\Big)\Big|-C\,\eta_0$$

Since $\bigwedge_{i=1}^k\tau_i$ is orthogonal to $\bigwedge_{i\in I}\tau_i\wedge\bigwedge_{j\in J}\nu_0^{(j)}(x)$ for every $I\subset\{1,...,k\}$ and $J\subset\{1,...,n-k\}$ with $\#I+\#J=k$ and $\#I<k$, by projecting over $\bigwedge_{i=1}^k\tau_i$ one finds

$$J^{\widetilde{S}_0}f(x)\;\ge\;\Big|\bigwedge_{i=1}^k\Big(\tau_i+\sum_{j=1}^{n-k}\nabla^{\widetilde{S}_0}\zeta_j(x)[\tau_i]\,\nu_0^{(j)}(x)\Big)\cdot\bigwedge_{i=1}^k\tau_i\Big|-C\,\eta_0=1-C\,\eta_0\ge\frac12\,,\tag{3.62}$$

provided η_0 is small enough; this proves (3.30). Again by (3.61) we find that if $x\in\mathrm{bd}\,(S_0)$, then

$$\nabla^{\widetilde{S}_0}f(x)[\nu^{co}_{S_0}(x)]\cdot\nu^{co}_S(f(x))\ge\nu^{co}_{S_0}(x)\cdot\nu^{co}_S(f(x))-C\max_{1\le i\le n-k}|\nu_0^{(i)}(x)\cdot\nu^{co}_S(f(x))|-C\,\eta_0\,.$$

By (3.9), $\nu^{co}_{S_0}(x)\cdot\nu^{co}_S(f(x))\ge 1-C\,\rho$ and $|\nu_0^{(i)}(x)\cdot\nu^{co}_S(f(x))|\le C\,\rho$, so that

$$\nabla^{\widetilde{S}_0}f(x)[\nu^{co}_{S_0}(x)]\cdot\nu^{co}_S(f(x))\ge\frac{1}{2}\,,\qquad\forall x\in\mathrm{bd}\,(S_0)\,,\tag{3.63}$$

provided η_0 (thus $\rho\le\mu_0^2$) is small enough. By (3.24), (3.58), and (3.30), for every $x\in\mathrm{bd}\,(S_0)$ one has

$$\nabla^{\widetilde{S}_0}f(x)[T_x\widetilde{S}_0]=T_{f(x)}\widetilde{S}\,,\qquad\nabla^{\widetilde{S}_0}f(x)[T_x(\mathrm{bd}\,(S_0))]=T_{f(x)}(\mathrm{bd}\,(S))\,,$$

so that (3.63) gives

$$\nabla^{\widetilde{S}_0}f(x)\Big[\big\{v\in T_x\widetilde{S}_0:v\cdot\nu^{co}_{S_0}(x)\le 0\big\}\Big]=\big\{w\in T_{f(x)}\widetilde{S}:w\cdot\nu^{co}_S(f(x))\le 0\big\}\,.$$

By combining this fact with (3.58) we deduce (3.32) (up to possibly further decreasing η_0 in dependence of the bound in (3.26)). We are thus left to prove (3.25), (3.27) and (3.28).

We first prove (3.27). By (3.32) one has

$$\mathrm{hd}(S,S_0)\ge\mathrm{dist}(f(x),S_0)\,,\qquad\forall x\in K^+_{\eta_0}\,.\tag{3.64}$$

Let $\varepsilon_0>0$ be the inverse of the maximum of the largest principal curvature of S_0, so that, by (3.6), ε_0 depends on L only. Then

$$\mathrm{dist}\Big(x+\sum_{i=1}^{n-k}t_i\,\nu_0^{(i)}(x),S_0\Big)=|\mathbf{t}|\,,\qquad\forall x\in S_0\,,|\mathbf{t}|<\varepsilon_0\,.\tag{3.65}$$

By $\mathrm{spt}\phi_\mu\subset\subset I_\mu(\mathrm{bd}\,(S_0))$ and by (3.56)

$$f(x)=x+\sum_{i=1}^{n-k}(a_i(x)+\zeta_i(x))\,\nu_0^{(i)}(x)\,,\qquad\forall x\in K_{\eta_0}\setminus K_\mu\,,$$

where $\|a_i+\zeta_i\|_{C^0(K_{\eta_0})}\le C\,\eta_0$ by (3.38) and (3.55). Up to decrease η_0 in order to obtain $\|a_i+\zeta_i\|_{C^0(K_{\eta_0})}\le\varepsilon_0$, we can apply (3.65), (3.64) and $\|a_i\|_{C^0(\mathbb{R}^n)}\le C\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}$ to find

$$\|\zeta\|_{C^0(K^+_{\eta_0}\setminus K_\mu)}\le C\left(\mathrm{hd}(S,S_0)+\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right).\tag{3.66}$$

In order to estimate $\|\zeta\|_{C^0(K^+_\mu)}$ we consider, for every $x\in K_{\eta_0}$, a point $g(x)\in S_0$ such that $|f(x)-g(x)|=\mathrm{dist}(f(x),S_0)$: we claim that then one must have

$$|g(x)-x|\le\mathrm{hd}(S,S_0)+C\,\mu\,,\qquad\forall x\in K^+_\mu\,.\tag{3.67}$$

Indeed, let $x\in K^+_\mu$ so that there exists $y\in\mathrm{bd}\,(S_0)$ with $|x-y|\le\mu$: since $f(x)\in S$ implies $|f(x)-g(x)|=\mathrm{dist}(f(x),S_0)\le\mathrm{hd}(S,S_0)$, by (3.26) we find

$$|g(x)-x|\le|g(x)-f(x)|+|f(x)-f(y)|+|x-y|\le\mathrm{hd}(S,S_0)+C\,|x-y|$$

that is (3.67). By (3.67), provided μ_0 is small enough with respect to the constant $1/C$ appearing in (3.22), we find that

$$\max_{1\le i\le n-k}|(g(x)-x)\cdot\nu_0^{(i)}(x)|\le C|\pi^{S_0}_x(g(x)-x)|^2\,,\qquad\forall x\in K^+_\mu\,.\tag{3.68}$$

Now, by (3.64) and (3.59) we find that, if $x\in K^+_\mu$, then

$$\begin{aligned}\mathrm{hd}(S,S_0)&\ge\mathrm{dist}(f(x),S_0)=|f(x)-g(x)|\ge|\pi^{S_0}_x(f(x)-g(x))|\\&=|\pi^{S_0}_x(x-g(x))|-|b(x)|\,\phi_\mu(x)\end{aligned}$$

so that (3.68) and (3.38) give

$$\max_{1\le i\le n-k}|(g(x)-x)\cdot\nu_0^{(i)}(x)|\le C\left(\mathrm{hd}(S,S_0)+\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right)^2,\qquad\forall x\in K^+_\mu\,.$$

By exploiting this last inequality, (3.56) and (3.38) we deduce that if $x \in K_\mu^+$, then

$$\begin{aligned}
\mathrm{hd}(S,S_0) &\geq \mathrm{dist}(f(x),S_0) = |f(x)-g(x)| \geq |(f(x)-g(x))\cdot \nu_0^{(i)}(x)| \\
&\geq |(x-g(x))\cdot\nu_0^{(i)}(x) + (a_i(x)+\zeta_i(x))| - |b(x)|\,\phi_\mu(x) \\
&\geq |\zeta_i(x)| - C\left(\mathrm{hd}(S,S_0) + \|f_0 - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right).
\end{aligned}$$

By combining this estimate with (3.66) we thus conclude that

$$\|\zeta\|_{C^0(K_{\eta_0}^+)} \leq C\left(\mathrm{hd}(S,S_0) + \|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right). \tag{3.69}$$

By combining (3.69), (3.56) and (3.41) we prove (3.27).

We now prove (3.25). First, we claim that there exists a constant M depending on n, α, k and L only such that

$$f(x) \in [S]_{3\rho}\,, \qquad \forall x \in K_{\eta_0}^+ \setminus K_{M\rho}\,. \tag{3.70}$$

Indeed, let $x \in K_{\eta_0}^+ \setminus K_{M\rho}$ and let $y \in \mathrm{bd}\,(S_0)$ be such that $|f(x)-f(y)| = \mathrm{dist}(f(x),\mathrm{bd}\,(S))$ (we can find such a point y as f_0 is a bijection between $\mathrm{bd}\,(S_0)$ and $\mathrm{bd}\,(S)$ and since $f = f_0$ on $\mathrm{bd}\,(S_0)$). By (3.27), we have

$$\begin{aligned}
\mathrm{dist}(f(x),\mathrm{bd}\,(S)) &= |f(x)-f(y)| \geq |x-y| - |f(x)-x| - |f(y)-y| \\
&\geq \mathrm{dist}(x,\mathrm{bd}\,(S_0)) - C\,\rho \geq (M-C)\,\rho \geq 3\rho\,,
\end{aligned}$$

provided M is large enough. This proves (3.70), which, combined with assumption (ii) and (3.65), gives in particular

$$f(x) = g(x) + \psi(g(x)) \qquad g(x) \in [S_0]_\rho\,, \qquad \forall x \in K_{\eta_0}^+ \setminus K_{M\,\rho}\,. \tag{3.71}$$

By (3.31) and (3.71), we find $g(x) = x$ for every $x \in K_{\eta_0}^+ \setminus K_\mu$, so that, in particular,

$$f(x) = x + \psi(x)\,, \qquad \forall x \in K_{\eta_0}^+ \setminus K_\mu\,, \tag{3.72}$$

that is (3.25). Note that this argument also gives $\psi_i = a_i + \zeta_i$ on $K_{\eta_0}^+ \setminus K_\mu$, so that (3.38) gives us

$$\|\zeta\|_{C^1(K_{\eta_0}^+ \setminus K_\mu)} \leq C\left(\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))} + \|\psi\|_{C^1([S_0]_\mu)}\right),$$

and thus, by (3.56) and (3.41)

$$\|f-\mathrm{Id}\|_{C^1(K_{\eta_0}^+\setminus K_\mu)} \leq \frac{C}{\mu}\,\rho \leq C\,\mu_0\,, \tag{3.73}$$

which will be useful in proving (3.28), as we are now going to do. We first note that by (3.60), (3.41), (3.27), and (3.6), it is enough to show that

$$\|\nabla^{\widetilde{S}_0}\zeta\|_{C^0(K_{\eta_0}^+)} \leq \frac{C}{\mu}\,\rho^\alpha\,. \tag{3.74}$$

To this end, the natural starting point is differentiating $d_S(f) = 0$ on K_{η_0} at some fixed $x \in K_{\eta_0}$ along $\tau \in T_x\widetilde{S}_0$. By combining the resulting identity $\nabla d_S(f(x))[\nabla^{\widetilde{S}_0} f(x)[\tau]] = 0$ with (3.60), (3.41) and (3.69) one finds that, if $x \in K_{\eta_0}^+$ and $\tau \in T_xS_0$ with $|\tau| = 1$, then

$$\Big|\nabla d_S(f(x))\Big[\tau + \sum_{i=1}^{n-k}\Big(\nabla^{\widetilde{S}_0}\zeta_i(x)[\tau]\Big)\,\nu_0^{(i)}(x)\Big]\Big| \leq \frac{C}{\mu}\left(\mathrm{hd}(S,S_0) + \|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\right) \leq \frac{C}{\mu}\,\rho\,,$$

that is

$$\Big|\nabla d_S(f(x))\Big[\sum_{i=1}^{n-k}\Big(\nabla^{\widetilde{S}_0}\zeta_i(x)[\tau]\Big)\,\nu_0^{(i)}(x)\Big]\Big| \leq \frac{C}{\mu}\left(\rho + \big|\nabla d_S(f(x))[\tau]\big|\right). \tag{3.75}$$

We claim that

$$|\nabla d_S(f(x))[v]| \geq \frac{|v|}{2}\,, \qquad \forall x \in K_{\eta_0}^+\,, v \in (T_xS_0)^\perp\,. \tag{3.76}$$

Indeed, if $x\in\mathrm{bd}\,(S_0)$, then, by (3.9), $\nu_0^{(i)}(x)\cdot\nu_S^{(i)}(f(x))\ge 1-C\,\rho$ for every $i=1,...,n-k$, that is, $|\pi^S_{f(x)}[v]|\le C\,\rho\,|v|$: thus by (3.21) and provided μ_0 is small enough

$$|\nabla d_S(f(x))[v]|\ge\frac{2}{3}\,|v|\,,\qquad\forall x\in\mathrm{bd}\,(S_0)\,,v\in(T_xS_0)^\perp\,,\tag{3.77}$$

which immediately gives us (3.76) for $x\in K_\mu^+$ provided μ_0 is small enough depending on $C\ge\|d_S\|_{C^{1,\alpha}(\mathbb{R}^n)}$. If instead $x\in K^+_{\eta_0}\setminus K_\mu$, then by (3.73) we find that $|\pi^S_{f(x)}[v]|=|\pi^S_{f(x)}[v]-\pi^{S_0}_x[v]|\le C\,\mu_0\,|v|$. Thus we deduce that (3.76) holds for $x\in K^+_{\eta_0}\setminus K_\mu$ too, once again, thanks to (3.21) and provided μ_0 is small enough. By combining (3.76) with (3.75) we thus find

$$\big|\nabla^{\widetilde S_0}\zeta(x)[\tau]\big|\le\frac{C}{\mu}\big(\rho+|\nabla d_S(f(x))[\tau]|\big)\,,\qquad\forall x\in K^+_{\eta_0}\,,\tau\in T_xS_0\cap\mathbb{S}^{n-1}\,.\tag{3.78}$$

We are now going to show that

$$|\nabla d_S(f(x))[\tau]|\le C\,\rho^\alpha\,,\qquad\forall x\in K^+_{\eta_0},\tau\in T_xS_0\cap\mathbb{S}^{n-1}\,.\tag{3.79}$$

Indeed, if $x\in\mathrm{bd}\,(S_0)$, then (3.79) follows by exactly the same argument used to prove (3.77) (with ρ in place of ρ^α). By exploiting $\|\nabla d_S\|_{C^{1,\alpha}(\mathbb{R}^n)}\le C$, one deduces the validity of (3.79) for every $x\in K^+_{M\rho}$ (here is the point where ρ^α appears in place of ρ). In order to prove (3.79) on $K^+_{\eta_0}\setminus K_{M\,\rho}$ we first notice that if $x\in K^+_{\eta_0}$, then $|g(x)-f(x)|=\mathrm{dist}(f(x),S_0)\le|f(x)-x|$, so that (3.27) implies the following improvement of (3.67):

$$|g(x)-x|\le C\big(\mathrm{hd}(S,S_0)+\|f_0-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\big)\le C\,\rho\,,\qquad\forall x\in K^+_{\eta_0}\,.\tag{3.80}$$

At the same time, by $(\mathrm{Id}+\psi)([S_0]_\rho)\subset S$ and $\|\psi\|_{C^1([S_0]_\rho)}\le\rho$ one finds

$$|\pi^S_{x+\psi(x)}[\tau]|\ge(1-C\,\rho)\,|\tau|,\qquad\forall x\in[S_0]_\rho\,,\tau\in T_xS_0\,,$$

which, by (3.21), gives

$$|\nabla d_S(x+\psi(x))[\tau]|\le C\,\rho\,|\tau|\,,\qquad\forall x\in[S_0]_\rho\,,\tau\in T_xS_0\,.$$

By (3.19), (3.10) and (3.80)

$$|\nabla d_S(g(x)+\psi(g(x)))[\tau]|\le C\,\rho^\alpha\,|\tau|\,,\qquad\forall x\in K^+_{\eta_0}\setminus K_\rho\,,\tau\in T_xS_0\,,$$

which implies (3.79) for $x\in K^+_{\eta_0}\setminus K_{M\rho}$ thanks to (3.71). This completes the proof of (3.79), which combined with (3.78) gives us (3.74). The claim, thus theorem, is then proved in the case $k\ge2$. Concerning the case $k=1$, the main difference is that the extensions a_i and b of $\bar a_i$ and $\bar b$ satisfying (3.38) can now be defined by elementary means by exploiting the assumption $|p_0-q_0|\ge1/L$, with their $C^1(\mathbb{R}^n)$-norms controlled in terms of $\|f_0-\mathrm{Id}\|_{C^0(\mathrm{bd}\,(S_0))}$ (see also Remark 3.2). The rest of the proof carries on almost *verbatim*, and we thus omit the details. □

3.2. **A reformulation of Theorem 3.1.** In the situations in which we plan to apply Theorem 3.1 we are usually given a sequence of manifolds $\{S_j\}_j$ converging to a limit manifold S_0 rather than a pair of nearby manifolds S and S_0. In order to apply Theorem 3.1 one thus needs to pass from the former situation to the latter, and this can indeed be done by a simple argument. Instead of having to repeat this argument at each application of Theorem 3.1, it seems preferable to prove once and for all an alternative version Theorem 3.1 which is already tailored for the case of sequences.

Theorem 3.5. *Let $n\ge2$, $1\le k\le n-1$, $\alpha\in(0,1]$, and $L>0$. Let S_0 be a compact connected k-dimensional $C^{2,1}$-manifold with boundary in $\mathbb{R}^n$ and let $\{\nu_{S_0}^{(i)}\}_{i=1}^{n-k}\subset C^{1,1}(S_0;\mathbb{S}^{n-1})$ be such that $\{\nu_{S_0}^{(i)}\}_{i=1}^{n-k}$ is an orthonormal basis of $(T_xS_0)^\perp$ for every $x\in S_0$. Then there exist $\mu_0\in(0,1)$ and $C_0>0$ (depending on n, k, α, L and S_0 only) with the following property.*

Let $\{S_j\}_{j\in\mathbb{N}}$ be a sequence of a compact connected k-dimensional $C^{1,\alpha}$-manifold with boundary in $\mathbb{R}^n$ such that

$$\mathrm{bd}\,(S_j)\neq\emptyset\,,\qquad \|S_j\|_{C^{1,\alpha}}\le L\,,\qquad \lim_{j\to\infty}\mathrm{hd}(S_j,S_0)=0\,,\tag{3.81}$$

and assume in addition that:

(i) if $k=1$, then, setting $\mathrm{bd}\,(S_0)=\{p_0,q_0\}$, $\mathrm{bd}\,(S_j)=\{p_j,q_j\}$, $f_{0,j}(p_0)=p_j$ and $f_{0,j}(q_0)=q_j$,

$$\lim_{j\to\infty}\|f_{0,j}-\mathrm{Id}\|_{C^0(\mathrm{bd}\,(S_0))}+\|\nu^{co}_{S_j}(f_{0,j})-\nu^{co}_{S_0}\|_{C^0(\mathrm{bd}\,(S_0))}=0\,;\tag{3.82}$$

if $k\ge 2$, then there exist $C^{1,\alpha}$-diffeomorphisms $f_{0,j}$ between $\mathrm{bd}\,(S_0)$ and $\mathrm{bd}\,(S_j)$ with

$$\begin{aligned}
&\sup_{j\in\mathbb{N}}\|f_{0,j}\|_{C^{1,\alpha}(\mathrm{bd}\,(S_0))}\le L\,,\\
&\lim_{j\to\infty}\|f_{0,j}-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}=0\,,\\
&\lim_{j\to\infty}\max_{1\le i\le n-k}\|\nu^{(i)}_{S_j}(f_{0,j})-\nu^{(i)}_0\|_{C^0(\mathrm{bd}\,(S_0))}=0\,,\\
&\lim_{j\to\infty}\|\nu^{co}_{S_j}(f_{0,j})-\nu^{co}_{S_0}\|_{C^0(\mathrm{bd}\,(S_0))}=0\,,
\end{aligned}\tag{3.83}$$

where $\{\nu^{(i)}_{S_j}\}_{i=1}^{n-k}$ is satisfies (3.1) with $S=S_j$;

(ii) for every $\rho<\mu_0^2$ and $i=1,...,n-k$ there exist $j(\rho)\in\mathbb{N}$ and $\{\psi_{i,j}\}_{j\ge j(\rho)}\subset C^{1,\alpha}([S_0]_\rho)$ such that, setting $\psi_j=\sum_{i=1}^{n-k}\psi_{i,j}\,\nu^{(i)}_{S_0}$, one has

$$\begin{aligned}
&[S_j]_{3\rho}\subset(\mathrm{Id}+\psi_j)([S_0]_\rho)\subset S\,,\qquad\forall j\ge j(\rho)\,,\\
&\sup_{j\ge j(\rho)}\|\psi_j\|_{C^{1,\alpha}([S_0]_\rho)}\le L\,,\qquad \lim_{j\to\infty}\|\psi_j\|_{C^1([S_0]_\rho)}=0\,.
\end{aligned}\tag{3.84}$$

Then, for every $\mu\in(0,\mu_0)$ there exist $j(\mu)\in\mathbb{N}$ and, for each $j\ge j(\mu)$, a $C^{1,\alpha}$-diffeomorphisms f_j between S_0 and S_j such that

$$\begin{aligned}
&f_j=f_{0,j}\ \text{on}\ \mathrm{bd}\,(S_0)\,,\qquad f_j=\mathrm{Id}+\psi_j\ \text{on}\ [S_0]_\mu\,,\\
&\sup_{j\ge j(\mu)}\|f_j\|_{C^{1,\alpha}(S_0)}\le C_0\,,\qquad \lim_{j\to\infty}\|f_j-\mathrm{Id}\|_{C^1(S_0)}=0\,,\\
&\|\pi^{S_0}(f_j-\mathrm{Id})\|_{C^1(S_0)}\le\frac{C_0}{\mu}\begin{cases}\|(f_{0,j}-\mathrm{Id})\cdot\nu^{co}_{S_0}\|_{C^0(\mathrm{bd}\,(S_0))}\,, & \text{if } k=1\,,\\ \|f_{0,j}-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,, & \text{if } k\ge 2\,.\end{cases}
\end{aligned}\tag{3.85}$$

Proof. By Remark 3.4, up to increasing the value of L depending on S_0, one can entail the existence of $\widetilde{S}_0$ such that assumption (a) in Theorem 3.1 holds, and also that $|p_0-q_0|\ge 1/L$ in the case $k=1$. Now let μ_0 and C_0 be determined as in Theorem 3.1 by n, k, α and the increased S_0-depending value of L, and let us fix $\mu\in(0,\mu_0)$. Given $\rho\in(0,\mu^2)$, by (3.82), (3.83), and (3.84), and up to increasing the value of $j(\rho)$, then for each $j\ge j(\rho)$, S_j, $f_{0,j}$ and ψ_j satisfy assumption (b) of Theorem 3.1, that is, referring from now on to the case $k\ge 2$, for every $j\ge j(\rho)$ one has $[S_j]_{3\rho}\subset(\mathrm{Id}+\psi_j)([S_0]_\rho)\subset S_j$ with

$$\begin{aligned}
&\max\big\{\mathrm{hd}(S_0,S_j),\|f_{0,j}-\mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))},\|\nu^{(i)}_S(f_{0,j})-\nu^{(i)}_0\|_{C^1(\mathrm{bd}\,(S_0))},\\
&\qquad\qquad\|\nu^{co}_S(f_{0,j})-\nu^{co}_0\|_{C^1(\mathrm{bd}\,(S_0))},\|\psi_j\|_{C^1([S_0]_\rho)}\big\}\le\rho\,,\\
&\max\big\{\|S_j\|_{C^{1,\alpha}},\|f_{0,j}\|_{C^{1,\alpha}(\mathrm{bd}\,(S_0))},\|\psi_j\|_{C^{1,\alpha}([S_0]_\rho)}\big\}\le L\,.
\end{aligned}$$

Hence, by Theorem 3.1, for each $j \geq j(\rho)$ we can construct $C^{1,\alpha}$-diffeomorphisms f_j^ρ between S_0 and S_j such that

$$f_j^\rho = f_{0,j} \text{ on } \mathrm{bd}\,(S_0)\,, \qquad f_j^\rho = \mathrm{Id} + \psi \text{ on } [S_0]_\mu\,,$$

$$\|f_j^\rho\|_{C^{1,\alpha}(S_0)} \leq C_0\,, \qquad \|f_j^\rho - \mathrm{Id}\|_{C^1(S_0)} \leq \frac{C_0}{\mu}\,\rho^\alpha\,,$$

$$\|\pi^{S_0}(f_j^\rho - \mathrm{Id})\|_{C^1(S_0)} \leq \frac{C_0}{\mu}\begin{cases} \|(f_{0,j} - \mathrm{Id})\cdot \nu_{S_0}^{co}\|_{C^0(\mathrm{bd}\,(S_0))}\,, & \text{if } k = 1\,,\\ \|f_{0,j} - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(S_0))}\,, & \text{if } k \geq 2\,.\end{cases}$$

Finally, let us set, for $\ell \geq 2$, $\rho_\ell = \mu^{2/\alpha}/(2+\ell)$. For each $\ell \geq 2$, $\rho_\ell \in (0, \mu^2)$. By iteratively applying the construction above we can find a strictly increasing sequence $\{j_\ell\}_{\ell \geq 2} \subset \mathbb{N}$ such that if $j_\ell \leq j < j_{\ell+1}$, then $f_j = f_j^{\rho_\ell}$ defines a $C^{1,\alpha}$-diffeomorphism between S_0 and S_j such that (3.85) holds with

$$\|f_j - \mathrm{Id}\|_{C^1(S_0)} \leq \frac{C_0}{\mu}\,\rho_\ell^\alpha = \frac{C_0 \mu}{(2+\ell)^\alpha}\,.$$

This completes the proof of the theorem. □

4. Perimeter almost-minimizing clusters in $\mathbb{R}^n$

The goal of this section is preparing the ground for the application of Theorem 3.1 to the proof of Theorem 1.5. Specifically, in this section we discuss those preliminary facts that we can prove in arbitrary dimension n. (In particular, these results shall also be used in part two [LM15].) For the most part the arguments of this section should be familiar to some readers, but we have nevertheless included some details of most of the proofs for the sake of clarity. In section 4.1 we gather some relevant definitions from Geometric Measure Theory. In section 4.2 we recall the classic regularity criterion for almost-minimizing sets (Theorem 4.1) and derive from it a very useful technical statement (Lemma 4.4 – which is well-known to experts, although, apparently, not explicitly stated in the literature). In section 4.3 we exploit a simple "infiltration lemma" to construct normal diffeomorphisms away from the singular sets (Theorem 4.12) and to prove Hausdorff convergence of the boundaries (Theorem 4.9). Finally, in section 4.4 we briefly discuss blow-up limits of clusters.

4.1. Basic definitions and terminology.

Here we gather various definitions from Geometric Measure Theory needed in the sequel.

Rectifiable sets. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is locally k-rectifiable in $A \subset \mathbb{R}^n$ open, if $\mathcal{H}^k \llcorner S$ is a Radon measure on A and S is contained, modulo an $\mathcal{H}^k$-null set, into a countable union of k-dimensional C^1-surfaces. If S is locally $\mathcal{H}^k$-rectifiable in A then for $\mathcal{H}^k$-a.e. $x \in S \cap A$ there exists a k-plane $T_x S$ in $\mathbb{R}^n$, the *approximate tangent space to S at x*, with $\mathcal{H}^k \llcorner (S - x)/r \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner T_x S$ when $r \to 0^+$ as Radon measures; see [Mag12, Theorem 10.2]. Given such $x \in S$, $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, and $\{\tau_i(x)\}_{i=1}^k$ an orthonormal basis of $T_x S$, the *tangential divergence* $\mathrm{div}_S T$ of T over S at x is defined by $\mathrm{div}_S\, T(x) = \sum_{i=1}^k \tau_i(x) \cdot (\nabla T(x) \tau_i(x))$. One says that S has *generalized mean curvature* $\mathbf{H}_S \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner (A \cap S); \mathbb{R}^n)$ in A, if

$$\int_S \mathrm{div}_S\, T\, d\mathcal{H}^k = \int_S T \cdot \mathbf{H}_S\, d\mathcal{H}^k\,, \qquad \forall T \in C^1_c(A; \mathbb{R}^n)\,. \tag{4.1}$$

If $\mathbf{H}_S \in L^\infty(\mathcal{H}^k \llcorner (A \cap S); \mathbb{R}^n)$ one says that S has bounded generalized mean curvature.

Sets of finite perimeter. A Lebesgue-measurable set $E \subset \mathbb{R}^n$ is a *set of locally finite perimeter* in an open set $A \subset \mathbb{R}^n$ if $\sup\{\int_E \mathrm{div}\, T : T \in C^1_c(A; B)\} < \infty$, or, equivalently, if there exists a

$\mathbb{R}^n$-valued Radon measure μ on A with

$$\int_E \nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)\,, \qquad \forall \varphi \in C^1_c(A)\,. \tag{4.2}$$

The Gauss–Green measure μ_E of E is defined as the Radon measure appearing in (4.2) for the largest open set A such that E is of locally finite perimeter in A. The *reduced boundary* $\partial^* E$ of E is defined as the set of those $x \in \mathrm{spt}\,\mu_E \subset A$ such that

$$\nu_E(x) = \lim_{r\to 0^+} \frac{\mu_E(B_{x,r})}{|\mu_E|(B_{x,r})} \qquad \text{exists and belongs to } \mathbb{S}^{n-1}\,. \tag{4.3}$$

It turns out that $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in A, and the Borel vector field $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ (called the *measure-theoretic outer unit normal to* E) is such that $\mu_E = \nu_E\,\mathcal{H}^{n-1} \llcorner \partial^* E$ on bounded Borel subsets of A. If $F \subset A$ is a Borel set, then the *perimeter of E relative to the Borel set F* is defined as $P(E;F) = |\mu_E|(F) = \mathcal{H}^{n-1}(F \cap \partial^* E)$, and we set $P(E) = P(E;\mathbb{R}^n)$. One always has

$$A \cap \mathrm{cl}\,(\partial^* E) = \mathrm{spt}\,\mu_E = \big\{x \in A : 0 < |E \cap B_{x,r}| < \omega_n\, r^n \quad \forall r > 0\big\} \subset A \cap \partial E\,,$$

where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$; moreover, μ_E is invariant by modifications of $E \cap A$ on and by a set of volume zero, and up to such modifications (see, for example, [Mag12, Proposition 12.19]) we can assume that

$$A \cap \mathrm{cl}\,(\partial^* E) = \mathrm{spt}\,\mu_E = A \cap \partial E\,. \tag{4.4}$$

Throughout this paper, **all sets of finite perimeter shall be normalized so to have identity** (4.4) **in force** (where A denotes the largest open set such that E is of locally finite perimeter in A).

Let us now recall from the introduction that a family $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ of Lebesgue-measurable sets in $\mathbb{R}^n$ with $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$ for $1 \le h < k \le N$ is an *N-cluster in A* if each $\mathcal{E}(h)$ is a set of locally finite perimeter in A and $|\mathcal{E}(h) \cap A| > 0$ for every $h = 1, ..., N$. If A is the largest open set such that $\mathcal{E}$ is a cluster in A, then, according to (4.3), $\partial^*\mathcal{E}(h)$ is well-defined as a subset of A and so are the *interfaces* $\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k)$; thus $\partial^*\mathcal{E}$, as defined in (1.17), is automatically a subset of A, with

$$\partial^*\mathcal{E} = \bigcup_{0 \le h < k \le N} \mathcal{E}(h,k)\,.$$

It will be useful to keep in mind that, by (4.4), one has

$$\mathrm{cl}\,(\partial^*\mathcal{E}) = A \cap \bigcup_{h=1}^N \mathrm{spt}\mu_{\mathcal{E}(h)} = \bigcup_{h=1}^N \Big\{x \in A : 0 < |\mathcal{E}(h) \cap B_{x,r}| < \omega_n\, r^n \ \forall r > 0\Big\} = A \cap \partial\mathcal{E}\,,$$

4.2. **A regularity criterion for** (Λ, r_0)**-minimizing sets.** Given $x \in \mathbb{R}^n$, $r > 0$ and $\nu \in \mathbb{S}^{n-1}$, let us set

$$\mathbf{C}^\nu_{x,r} = \big\{y \in \mathbb{R}^n : |(y-x)\cdot\nu| < r\,, |(y-x) - ((y-x)\cdot\nu)\nu| < r\big\}\,,$$
$$\mathbf{D}^\nu_{x,r} = \big\{y \in \mathbb{R}^n : |(y-x)\cdot\nu| = 0\,, |(y-x) - ((y-x)\cdot\nu)\nu| < r\big\}\,,$$

and define the *cylindrical excess* of $E \subset \mathbb{R}^n$ at x, in direction ν, and at scale r, as

$$\mathbf{exc}^\nu_{x,r}(E) = \frac{1}{r^{n-1}} \int_{\mathbf{C}^\nu_{x,r} \cap \partial^* E} |\nu_E - \nu|^2\, d\mathcal{H}^{n-1}\,,$$

provided E is of finite perimeter on $\mathbf{C}^\nu_{x,r}$. When $\nu = e_n$ and $x = 0$ we simply set

$$\mathbf{C}_r = \mathbf{C}^{e_n}_{0,r}\,, \qquad \mathbf{D}_r = \mathbf{D}^{e_n}_{0,r}\,, \qquad \mathbf{exc}_r(E) = \mathbf{exc}^{e_n}_{0,r}(E)\,.$$

The next result is a classical local regularity criterion for (Λ, r_0)-minimizing sets.

Theorem 4.1 (Small excess regularity criterion)**.** *For every $n \ge 2$ and $\alpha \in (0,1)$ there exist positive constants $\varepsilon_*(n)$, $C(n)$ and $C(n,\alpha)$ with the following property. If E is a (Λ, r_0)-minimizing set in $\mathbf{C}^\nu_{x_0,r}$ with $x_0 \in \partial E$, $r < r_0$, and*

$$\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r \le \varepsilon_*(n)\,, \tag{4.5}$$

then there exists a Lipschitz function $v : \mathbf{D}^\nu_{x_0,r/2} \to \mathbb{R}$ with $v(x_0) = 0$,

$$\|v\|_{C^0(\mathbf{D}^\nu_{x_0,r/2})} \le C(n)\, r\, \mathbf{exc}^\nu_{x_0,r}(E)^{1/2(n-1)}\,, \tag{4.6}$$

$$\|\nabla v\|_{C^0(\mathbf{D}^\nu_{x_0,r/2})} \le C(n)\, \big(\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r\big)^{1/2(n-1)}\,, \tag{4.7}$$

$$r^\alpha\, [\nabla v]_{C^{0,\alpha}(\mathbf{D}^\nu_{x_0,r/2})} \le C(n,\alpha)\, \big(\mathbf{exc}^\nu_{x_0,r}(E) + \Lambda\, r\big)^{1/2(n-1)}\,, \qquad \forall \alpha \in (0,1)\,, \tag{4.8}$$

and such that

$$\mathbf{C}^\nu_{x_0,r/2} \cap \partial E = (\mathrm{Id} + v\,\nu)(\mathbf{D}^\nu_{x_0,r/2})\,. \tag{4.9}$$

Moreover, if $n = 2$ then one can replace (4.8) with $\|v''\|_{L^\infty(\mathbf{D}^\nu_{x_0,r/2})} \le C\,\Lambda$.

Proof. Without loss of generality we set $x_0 = 0$ and $\nu = e_n$. By [Mag12, Theorem 26.3] (applied, in the notation of that theorem, with $\gamma = 1/4$) there exist positive constants $\varepsilon_*(n)$ and $C(n)$ such that if (4.5) holds then (4.9) holds for a Lipschitz function $v : \mathbf{D}_{2r/3} \to \mathbb{R}$ with $v(0) = 0$ and

$$\frac{|v(x)|}{r} + |\nabla v(x)| + r^{1/4}\, \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1/4}} \le C(n)\, \big(\mathbf{exc}_r(E) + \Lambda\, r\big)^{1/2(n-1)}\,, \tag{4.10}$$

for every $x \ne y \in \mathbf{D}_{2r/3}$. We now prove (4.8). By (1.2) and (4.9) one finds that

$$\int_{\mathbf{D}_{2r/3}} \sqrt{1 + |\nabla v|^2} \le \int_{\mathbf{D}_{2r/3}} \sqrt{1 + |\nabla (v+\varphi)|^2} + \Lambda \int_{\mathbf{D}_{2r/3}} |\varphi|\,, \tag{4.11}$$

for every $\varphi \in C^1_c(\mathbf{D}_{2r/3})$. In particular, there exists $g \in L^\infty(\mathbf{D}_{2r/3})$ such that

$$\|g\|_{L^\infty(\mathbf{D}_{2r/3})} \le \Lambda\,, \qquad -\int_{\mathbf{D}_{2r/3}} \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla \varphi = \int_{\mathbf{D}_{2r/3}} g\,\varphi \qquad \forall \varphi \in C^1_c(\mathbf{D}_{2r/3})\,.$$

By taking incremental ratios one sees that $v \in W^{2,2}_{loc}(\mathbf{D}_{2r/3})$ with

$$\mathrm{tr}(A(x)\nabla^2 v(x)) = g(x)\,, \qquad \text{for a.e. } x \in \mathbf{D}_{2r/3}\,,$$

where $A = (1+|\nabla v|^2)^{-3/2}\,[(1+|\nabla v|^2)\mathrm{Id} - \nabla v \otimes \nabla v] = F(\nabla v)$ for a Lipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$. Thanks to (4.10),

$$\frac{\mathrm{Id}}{C(n)} \le A(x) \le C(n)\,\mathrm{Id}\,, \qquad r^{1/4}\,|A(x) - A(y)| \le C(n)\,|x-y|^{1/4}\,, \qquad \forall x, y \in \mathbf{D}_{2r/3}\,.$$

If we set $A^*(x) = A(r\,x)$, $v^*(x) = v(r\,x)$ and $g^*(x) = g(r\,x)$ for $x \in \mathbf{D}_{2/3}$, then

$$\mathrm{tr}(A^*(x)\nabla^2 v^*(x)) = r^2\, g^*(x)\,, \qquad \text{for a.e. } x \in \mathbf{D}_{2/3}\,,$$

with $|A^*(x) - A^*(y)| \le C(n)\,|x-y|^{1/4}$ for every $x, y \in \mathbf{D}_{2/3}$. If $n \ge 3$, then by [GT01, Theorem 9.11], for every $p \in (1,\infty)$ one has

$$\begin{aligned}
\|v^*\|_{W^{2,p}(\mathbf{D}_{1/2})} &\le C(n,p)\big(\|v^*\|_{L^p(\mathbf{D}_{2/3})} + \|r^2\, g^*\|_{L^p(\mathbf{D}_{2/3})}\big)\\
&\le C(n,p)\, r\, \big(\mathbf{exc}_r(E) + \Lambda\, r\big)^{1/2(n-1)}
\end{aligned}$$

thanks to (4.10). At the same time, by Morrey's inequality, if we pick $p > n-1$ such that $\alpha = 1 - (n-1)/p$ then

$$C(n,p)\|v^*\|_{W^{2,p}(\mathbf{D}_{1/2})} \ge \|v^*\|_{C^{1,\alpha}(\mathbf{D}_{1/2})} \ge [\nabla v^*]_{C^{0,\alpha}(\mathbf{D}_{1/2})} = r^{1+\alpha}\, [\nabla v]_{C^{0,\alpha}(\mathbf{D}_{r/2})}\,,$$

which gives us (4.8) in the case $n \geq 3$. If $n = 2$, then (4.11) directly implies that $(1+(v')^2)^{-1/2}v'$ has a bounded distributional derivative g on the interval $\mathbf{D}_{2r/3}$. By the chain rule we immediately find $\|v''\|_{L^\infty(\mathbf{D}_{2r/3})} \leq (1 + \|v'\|^2_{C^0(\mathbf{D}_{2r/3})})^{3/2}\Lambda \leq C\,\Lambda$. □

Remark 4.2. Recall that $\lim_{r\to 0^+} \inf_{\nu\in\mathbb{S}^{n-1}} \mathbf{exc}^\nu_{x,r}(E) = 0$ for every $x \in \partial^* E$; see, for example, [Mag12, Proposition 22.3]. In particular, if E is a (Λ, r_0)-minimizing set in A, then $A \cap \partial^* E$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0,1)$ ($C^{1,1}$ if $n = 2$).

Theorem 4.1 can be used to locally represent the boundaries of (Λ, r_0)-minimizing sets E_k converging to a set E as graphs with respect to ∂E, at least provided ∂E is smooth enough. This basic idea is made precise in Lemma 4.4 below. Before stating this lemma, we prove the following technical statement where, given $u \in C^{k,\alpha}(\mathbf{D}_r)$ we set

$$\|u\|^*_{C^{k,\alpha}(\mathbf{D}_r)} = \sum_{j=0}^k r^{j-1}\,\|D^j u\|_{C^0(\mathbf{D}_r)} + r^{k-1+\alpha}\,[D^k u]_{C^{0,\alpha}(\mathbf{D}_r)}\,.$$

In this way, if we set $\lambda_r(u)(x) = r^{-1}\,u(r\,x)$ for $x \in \mathbf{D}$, then

$$\|\lambda_r(u)\|_{C^{k,\alpha}(\mathbf{D})} = \|\lambda_r(u)\|^*_{C^{k,\alpha}(\mathbf{D})} = \|u\|^*_{C^{k,\alpha}(\mathbf{D}_r)}\,, \qquad \forall r > 0\,.$$

Moreover, given $u : \mathbf{D}_{4r} \to \mathbb{R}$ with $|u| < 4r$ on $\mathbf{D}_{4r}$ we set

$$\Gamma_r(u) = (\mathrm{Id} + u\,e_n)(\mathbf{D}_{4r}) \subset \mathbf{C}_{4r}\,,$$

and let $\alpha \wedge \beta = \min\{\alpha, \beta\}$.

Lemma 4.3. *Given $n \geq 2$, $L > 0$ and $\alpha, \beta \in [0,1]$ there exist positive constants $\sigma_0 < 1$ and C_0 with the following property. If $u_1 \in C^{2,\alpha}(\mathbf{D}_{4r})$, $u_2 \in C^{1,\beta}(\mathbf{D}_{4r})$, and*

$$\max_{i=1,2} \|u_i\|^*_{C^1(\mathbf{D}_{4r})} \leq \sigma_0\,, \qquad \max\left\{\|u_1\|^*_{C^{2,\alpha}(\mathbf{D}_{4r})}, \|u_2\|^*_{C^{1,\beta}(\mathbf{D}_{4r})}\right\} \leq L\,, \tag{4.12}$$

then there exists $\psi \in C^{1,\alpha\wedge\beta}(\mathbf{C}_{2r} \cap \Gamma_r(u_1))$ such that

$$\mathbf{C}_r \cap \Gamma_r(u_2) \subset (\mathrm{Id} + \psi\nu)(\mathbf{C}_{2r} \cap \Gamma_r(u_1)) \subset \Gamma_r(u_2)\,, \tag{4.13}$$

$$\frac{\|\psi\|_{C^0(\mathbf{C}_{2r}\cap\Gamma_r(u_1))}}{r} + \|\nabla\psi\|_{C^0(\mathbf{C}_{2r}\cap\Gamma_r(u_1))} + r^{\alpha\wedge\beta}\,[\nabla\psi]_{C^{0,\alpha\wedge\beta}(\mathbf{C}_{2r}\cap\Gamma_r(u_1))} \leq C_0\,, \tag{4.14}$$

$$\frac{\|\psi\|_{C^0(\mathbf{C}_{2r}\cap\Gamma_r(u_1))}}{r} + \|\nabla\psi\|_{C^0(\mathbf{C}_{2r}\cap\Gamma_r(u_1))} \leq C_0\,\|u_1 - u_2\|_{C^1(\mathbf{D}_{4r})}\,. \tag{4.15}$$

Here, $\nu \in C^{1,\alpha}(\Gamma_r(u_1); \mathbb{S}^{n-1})$ is the normal unit vector field to $\Gamma_r(u_1)$ defined by

$$\nu(z, u_1(z)) = \frac{(-\nabla u_1(z), 1)}{\sqrt{1 + |\nabla u_1(z)|^2}}\,, \qquad \forall z \in \mathbf{D}_{4r}\,. \tag{4.16}$$

Proof. Up to replacing u_i with $\lambda_r(u_i)$ we may directly assume that $r = 1$. Correspondingly, we write $\Gamma(u_i)$ in place of $\Gamma_1(u_i)$ for the sake of simplicity. We define $F : \mathbf{D}_4 \times \mathbb{R} \to \mathbb{R}^n$ and $\phi : \mathbf{D}_4 \times \mathbb{R} \to \mathbb{R}$ by setting

$$F(z,t) = \left(z - t\,\frac{\nabla u_1(z)}{\sqrt{1 + |\nabla u_1(z)|^2}}, u_1(z) + \frac{t}{\sqrt{1 + |\nabla u_1(z)|^2}}\right), \tag{4.17}$$

$$\phi(z,t) = u_2(z) - t\,, \tag{4.18}$$

for $(z,t) \in \mathbf{D}_4 \times \mathbb{R}$. Notice that $F \in C^{1,\alpha}(\mathbf{C}_4)$ and $\phi \in C^{1,\beta}(\mathbf{C}_4)$ with

$$\|F\|_{C^{1,\alpha}(\mathbf{C}_4)} \leq C\,, \qquad \|\phi\|_{C^{1,\beta}(\mathbf{C}_4)} \leq C\,, \tag{4.19}$$

where C is a constant depending on n, α, β and L only. Provided σ_0 is small enough we also find $F(\mathbf{C}_2)\subset\mathbf{C}_4$, so that we can define $\Phi:\mathbf{C}_2\to\mathbb{R}$ by setting

$$\Phi(z,t)=\phi(F(z,t))=u_2\Big(z-t\,\frac{\nabla u_1(z)}{\sqrt{1+|\nabla u_1(z)|^2}}\Big)-u_1(z)-\frac{t}{\sqrt{1+|\nabla u_1(z)|^2}}\,.$$

By exploiting (4.12) and (4.19) we find that, provided σ_0 is small enough,

$$\|\Phi\|_{C^{1,\alpha\wedge\beta}(\mathbf{C}_2)}\le C\,,\quad \Phi(z,2)\le -1\,,\quad \Phi(z,-2)\ge 1\,,\quad \frac{\partial\Phi}{\partial t}(z,t)\le-\frac{1}{2}\,,$$

for every $(z,t)\in\mathbf{C}_2$; hence there exists $\zeta\in C^{1,\alpha\wedge\beta}(\mathbf{D}_2;(-1,1))$ with

$$\|\zeta\|_{C^{1,\alpha\wedge\beta}(\mathbf{D}_2)}\le C\,,\qquad \Phi(z,\zeta(z))=0\,,\qquad \forall z\in\mathbf{D}_2\,. \tag{4.20}$$

By (4.16) and (4.20) we find

$$\big\{(z,u_1(z))+\zeta(z)\,\nu(z,u_1(z)):z\in\mathbf{D}_2\big\}\subset\Gamma(u_2)\,. \tag{4.21}$$

Again by $\Phi(z,\zeta(z))=0$ we deduce that

$$\zeta(z)=\sqrt{1+|\nabla u_1(z)|^2}\bigg(u_2\Big(z-\zeta(z)\,\frac{\nabla u_1(z)}{\sqrt{1+|\nabla u_1(z)|^2}}\Big)-u_1(z)\bigg)\,, \tag{4.22}$$

so that, by (4.12),

$$\|\zeta\|_{C^0(\mathbf{D}_2)}\le\sqrt{1+\sigma_0^2}\,\Big(\|u_2-u_1\|_{C^0(\mathbf{D}_2)}+\sigma_0^2\,\|\zeta\|_{C^0(\mathbf{D}_2)}\Big)$$

and thus $\|\zeta\|_{C^0(\mathbf{D}_2)}\le C\,\|u_1-u_2\|_{C^0(\mathbf{D}_2)}$. Similarly, by differentiating (4.22), by exploiting the fact that $u_1\in C^{2,\alpha}(\mathbf{D}_2)$ and thanks to (4.12), one finds that

$$\|\zeta\|_{C^1(\mathbf{D}_2)}\le C\,\|u_1-u_2\|_{C^1(\mathbf{D}_2)}\,. \tag{4.23}$$

We finally define $\psi\in C^{1,\alpha\wedge\beta}(\mathbf{C}_2\cap\Gamma(u_1))$ by the identity $\psi(z,u_1(z))=\zeta(z)$, $z\in\mathbf{D}_2$. In this way (4.14) and (4.15) follow immediately from (4.12), (4.20) and (4.23), whereas (4.21) gives the second inclusion in (4.13). The first inclusion in (4.13) is obtained by noticing that: (i) up to further decreasing the value of σ_0 we have

$$\begin{cases} x\in\mathbf{C}_2\cap\Gamma(u_1)\,,\\ x+t\,\nu(x)\,,x+s\,\nu(x)\in\Gamma(u_2)\end{cases}\qquad\Rightarrow\qquad t=s\,; \tag{4.24}$$

(ii) there exists $\eta>0$ (depending on L only) such that every $y\in N_\eta(\mathbf{C}_2\cap\Gamma(u_1))$ has a unique projection over $\mathbf{C}_2\cap\Gamma(u_1)$. Since (by (4.12) and provided σ_0 is small enough) we can entail

$$\mathbf{C}_1\cap\Gamma(u_2)\subset N_\eta(\mathbf{C}_2\cap\Gamma(u_1))\,,$$

by (ii) we find that for every $y\in\mathbf{C}_1\cap\Gamma(u_2)$ there exists a unique $\hat y\in\mathbf{C}_2\cap\Gamma(u_1)$ such that

$$y=\hat y+\mathrm{dist}(y,\mathbf{C}_2\cap\Gamma(u_1))\,\nu(\hat y)\,.$$

By the second inclusion in (4.13), $\hat y\in\mathbf{C}_2\cap\Gamma(u_1)$ implies that $\hat y+\psi(\hat y)\,\nu(\hat y)\in\Gamma(u_2)$. By (4.24) we thus find $\mathrm{dist}(y,\mathbf{C}_2\cap\Gamma(u_1))=\psi(\hat y)$, and thus $y=\hat y+\psi(\hat y)\,\nu(\hat y)$. The first inclusion in (4.13) is thus proved. □

Lemma 4.4. *If $n\ge2$, $\alpha\in[0,1]$, $\beta\in(0,1)$, $\Lambda\ge0$, and E is an open set with $0\in\partial E$ and*

$$\mathbf{C}_1\cap E=\big\{z+s\,e_n:z\in\mathbf{D}_1\,,v(z)<s<1\big\}\,, \tag{4.25}$$

for some $v\in C^{2,\alpha}(\mathbf{D}_1)$ with $v(0)=0$ and $\nabla v(0)=0$, then there exists $r\in(0,1/64)$ (depending on n, α, β, Λ and $\|v\|_{C^{2,\alpha}(\mathbf{D}_1)}$) with the following property.

If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing sets in $B_{32\,r}$ with $|B_{32\,r}\cap(E_k\Delta E)|\to 0$ as $k\to\infty$, then there exist $k_0\in\mathbb{N}$ and $\{\psi_k\}_{k\ge k_0}\subset C^{1,\alpha\wedge\beta}(\mathbf{C}_{2\,r}\cap\partial E)$ such that

$$\mathbf{C}_r\cap\partial E_k\subset(\mathrm{Id}+\psi_k\nu_E)(\mathbf{C}_{2r}\cap\partial E)\subset\mathbf{C}_{4\,r}\cap\partial E_k\,,\qquad\forall k\ge k_0\,, \tag{4.26}$$

$$\sup_{k\ge k_0}\|\psi_k\|_{C^{1,\alpha\wedge\beta}(\mathbf{C}_{2r}\cap\partial E)}\le C\,,\qquad\lim_{k\to\infty}\|\psi_k\|_{C^1(\mathbf{C}_{2r}\cap\partial E)}=0\,, \tag{4.27}$$

where $C=C(n,\alpha,\beta,\Lambda,\|v\|_{C^{2,\alpha}(\mathbf{D}_1)})$. Moreover, when $n=2$, one can take $\beta=1$.

Proof. First, we note that by (4.25) one has

$$\mathbf{C}_1\cap\partial E=\big\{z+v(z)\,e_n:z\in\mathbf{D}_1\big\}\,. \tag{4.28}$$

Second, we set $M=\|v\|_{C^{2,\alpha}(\mathbf{D}_1)}$, and exploit $v(0)=0$ and $\nabla v(0)=0$ to find $r\in(0,1/64)$ (depending on n, Λ, and M) in such a way that

$$\mathbf{exc}_{64\,r}(E)+\Lambda\,(64r)\le\sigma\,,\qquad\|v\|^*_{C^1(\mathbf{D}_{4r})}\le\sigma\,, \tag{4.29}$$

for a positive constant σ to be chosen later depending on n, α, β, Λ and M. Since $0\in\partial E$, E_k is a (Λ,r_0)-minimizing set in $B_{32\,r}$, and $|(E_k\Delta E)\cap B_{32r}|\to 0$ as $k\to\infty$, by [Mag12, Theorem 21.14-(ii)] there exists $\{x_k\}_{k\in\mathbb{N}}$ with $x_k\in\partial E_k$ and $x_k\to 0$ as $k\to\infty$. By [Mag12, Proposition 22.6], for a.e. $t\in(16\,r,32\,r)$,

$$\lim_{k\to\infty}\mathbf{exc}_{x_k,t}(E_k)=\lim_{k\to\infty}\mathbf{exc}_t(E_k-x_k)=\mathbf{exc}_t(E)\le C(n)\,\mathbf{exc}_{64\,r}(E)\,.$$

By (4.29) there exists $k_0\in\mathbb{N}$ such that

$$\mathbf{exc}_{x_k,t}(E_k)+\Lambda\,t\le C(n)\,\sigma\,,\qquad\forall k\ge k_0\,. \tag{4.30}$$

Provided σ is suitably small with respect to the constant $\varepsilon_*(n)$ introduced in Theorem 4.1, one finds that for every $k\ge k_0$ there exists $w_k:\mathbf{D}_{x_k,t/2}\to\mathbb{R}$ such that

$$\mathbf{C}_{x_k,t/2}\cap E_k\;=\;\big\{z+s\,e_n:z\in\mathbf{D}_{x_k,t/2}\,,w_k(z)\le s\le\frac{t}{2}\big\}\,, \tag{4.31}$$

$$\mathbf{C}_{x_k,t/2}\cap\partial E_k\;=\;\big\{z+w_k(z)\,e_n:z\in\mathbf{D}_{x_k,t/2}\big\}\,, \tag{4.32}$$

$$\|w_k\|^*_{C^{1,\beta}(\mathbf{D}_{x_k,t/2})}\;\le\;C(n,\beta)\,. \tag{4.33}$$

(Note that (4.31) follows by (4.32), (4.25) and the fact that $|B_{32\,r}\cap(E_k\Delta E)|\to 0$.) By composing the functions w_k with vanishing horizontal and vertical translations, and since $t/2>8\,r$, we actually find that, up to further increasing the value of k_0, then for every $k\ge k_0$ there exists $v_k:\mathbf{D}_{8\,r}\to\mathbb{R}$ such that

$$\mathbf{C}_{8\,r}\cap E_k\;=\;\big\{z+s\,e_n:z\in\mathbf{D}_{8\,r}\,,v_k(z)\le s\le 8\,r\big\}\,, \tag{4.34}$$

$$\mathbf{C}_{8\,r}\cap\partial E_k\;=\;\big\{z+v_k(z)\,e_n:z\in\mathbf{D}_{8\,r}\big\}\,, \tag{4.35}$$

$$\|v_k\|^*_{C^{1,\beta}(\mathbf{D}_{8\,r})}\;\le\;C(n,\beta)\,. \tag{4.36}$$

If we set $L=\max\{M/r,C(n,\beta)\}$ with $C(n,\beta)$ as in (4.36), then by (4.36) and by definition of M we have

$$\max\big\{\|v\|^*_{C^{2,\alpha}(\mathbf{D}_{4\,r})},\|v\|^*_{C^{1,\beta}(\mathbf{D}_{4\,r})}\big\}\le L\,,\qquad\forall k\ge k_0\,,$$

Let $\sigma_0=\sigma_0(n,\alpha,\beta,L)=\sigma_0(n,\alpha,\beta,\Lambda,M)$ be determined as in Lemma 4.3. By (4.25), (4.34) and $|B_{32\,r}\cap(E_k\Delta E)|\to 0$ we have $v_k\to v$ in $L^1(\mathbf{D}_{8\,r})$, thus by (4.36) we find $v_k\to v$ in $C^1(\mathbf{D}_{8\,r})$, so that, up to further increasing k_0, decreasing σ in terms of σ_0, and thanks to (4.29),

$$\max\big\{\|v\|^*_{C^1(\mathbf{D}_{4\,r})},\|v_k\|^*_{C^1(\mathbf{D}_{4\,r})}\big\}\le\sigma_0\,,\qquad\forall k\ge k_0\,.$$

We thus apply Lemma 4.3 and find $\psi_k\in C^{1,\alpha\wedge\beta}(\mathbf{C}_{2\,r}\cap\partial E)$ with the required properties. □

4.3. **Infiltration lemma and consequences.** In this section we exploit an infiltration lemma (Lemma 4.5 – which is a special case of [LT02, Lemma 4.6], see also [Leo01, Theorem 3.1] for a similar result in the context of immiscible fluids) together with Theorem 4.1 to address various regularity properties of (Λ, r_0)-minimizing clusters, and to prove some basic convergence properties, see Theorem 4.9 and Theorem 4.12.

Lemma 4.5 (Infiltration lemma)**.** *There exists a positive constant $\eta_0 = \eta_0(n) < \omega_n$ with the following property: if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, then there exists a positive constant $r_1 \le r_0$ (depending on Λ and r_0 only) such that, if*

$$\sum_{h\in H} |\,\mathcal{E}(h)\cap B_{x,r}| \le \eta_0\, r^n\,, \tag{4.37}$$

for some $r \le r_1$, $H \subset \{0, \dots, N\}$, and $x \in \mathbb{R}^n$ with $B_{x,r} \subset\subset A$, then

$$\sum_{h\in H} |\,\mathcal{E}(h)\cap B_{x,r/2}| = 0\,. \tag{4.38}$$

Proof. By arguing as in [Mag12, Lemma 30.2] one sees that if $\mathcal{E}$ is a N-cluster in A such that

$$P(\mathcal{E}; B_{x,r}) \le P(\mathcal{F}; B_{x,r}) + C_0\,|\,\mathrm{vol}\,(\mathcal{E}) - \mathrm{vol}\,(\mathcal{F})|\,, \tag{4.39}$$

whenever $\mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r} \subset\subset A$ for some $x \in \mathbb{R}^n$, $r < r_0$ and every $h = 1, ..., N$, then (4.37) implies (4.38) with $r_1 = \min\{r_0, 1/8C_0\}$. This is achieved by exploiting the perturbed minimality inequality (4.39) on comparison clusters $\mathcal{F}$ having the property that, if $0 \le h \le N$, then either $\mathcal{F}(h) \subset \mathcal{E}(h)$ or $\mathcal{E}(h) \subset \mathcal{F}(h)$. We now notice that, on such clusters $\mathcal{F}$ one has

$$\mathrm{d}(\mathcal{E},\mathcal{F}) = \sum_{h=1}^{N} |\,|\mathcal{E}(h)| - |\mathcal{F}(h)|\,| \le \sqrt{N}\,|\,\mathrm{vol}\,(\mathcal{E}) - \mathrm{vol}\,(\mathcal{F})\,|\,.$$

Therefore, if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, then (4.39) holds on every comparison cluster $\mathcal{F}$ as above with $C_0 = \sqrt{N}\Lambda$, and we can argue as in [Mag12, Lemma 30.2] to prove the lemma (with $r_1 = \min\{r_0, 1/8\sqrt{N}\Lambda\}$). □

Corollary 4.6 (Almost everywhere regularity)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, then $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0,1)$ ($C^{1,1}$ if $n = 2$), it is relatively open inside $A\cap\partial\mathcal{E}$, and $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E})) = 0$. Moreover, if $n = 2$, then we can replace $C^{1,\alpha}$ with $C^{1,1}$.*

Proof. Step one: We prove that there exists $c(n) \in (0,1)$ and $r_1 \le r_0$ (depending on $\mathcal{E}$), such that, if $0 \le h \le N$, $x \in \partial\mathcal{E}(h)$, and $r < r_1$ is such that $B_{x,r} \subset\subset A$, then

$$c(n) \le \frac{|\mathcal{E}(h)\cap B_{x,r}|}{\omega_n\, r^n} \le (1 - c(n))\,, \tag{4.40}$$

$$c(n) \le \frac{P(\mathcal{E}(h); B_{x,r})}{r^{n-1}} \le C(n,\Lambda)\,(1 + r)\,. \tag{4.41}$$

Indeed, Lemma 4.5 implies (4.40) with $c(n) = \eta_0(n)/\omega_n$; see [Mag12, Section 30.2]. Up to further decreasing the value of $c(n)$, the lower bound in (4.41) follows from (4.40) and the relative isoperimetric inequality on balls, see [Mag12, Proposition 12.37]. Finally, by testing (1.15) on $\mathcal{F}(h) = \mathcal{E}(h) \setminus B_{x,r}$, $1 \le h \le N$, we find that $P(\mathcal{E}(h); B_{x,r}) \le n\omega_n\, r^{n-1} + \Lambda\, \omega_n\, r^n$, whence the upper bound in (4.41).

Step two: We show that if $x \in \mathcal{E}(h,k) = \partial^*\mathcal{E}(h)\cap\partial^*\mathcal{E}(k)$, then there exists $r_x \in (0, r_0)$ such that $|\mathcal{E}(j)\cap B_{x,r_x}| = 0$ if $j \ne h, k$ and $B_{x,r_x} \subset\subset A$. Indeed, by standard density estimates (see [Mag12, Exercise 29.6]), we have

$$\lim_{r\to 0^+} \frac{|\,\mathcal{E}(h)\cap B_{x,r}|}{\omega_n\, r^n} + \frac{|\,\mathcal{E}(k)\cap B_{x,r}|}{\omega_n\, r^n} = 1\,,$$

so that the existence of r_x follows from Lemma 4.5. As a consequence, (1.15) implies that both $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are (Λ, r_0)-minimizing sets on B_{x,r_x}. By Theorem 4.1 and Remark 4.2, $\partial^*\mathcal{E}$ is a $C^{1,\alpha}$-hypersurface for every $\alpha \in (0,1)$ ($C^{1,1}$ if $n = 2$) and it is relatively open inside $A \cap \partial\mathcal{E}$. The lower $(n-1)$-dimensional estimate in (4.41) implies $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E})) = 0$ by a classical argument (see for example [Mag12, Theorem 16.14]). □

Corollary 4.7 (Local finiteness away from the singular set)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing N-cluster in A, $\rho > 0$, and $A' \subset\subset A$ is open, then $(A' \cap \partial\mathcal{E}) \setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ is the union of finitely many disjoint connected hypersurfaces.*

Proof. By Corollary 4.6, we can directly assume that $\partial^*\mathcal{E} = \bigcup_{i\in\mathbb{N}} S_i$, where each S_i is a nonempty connected C^1-hypersurface with $S_i \cap S_j = \emptyset$ for $i \neq j$. If we set $S_i^\rho = (A' \cap S_i) \setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ then $\{S_i^\rho\}_{i\in\mathbb{N}}$ is a disjoint family of connected C^1-hypersurfaces whose union is equal to $(A' \cap \partial\mathcal{E}) \setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$. We claim that only finitely many elements of $\{S_i^\rho\}_{i\in\mathbb{N}}$ are nonempty. If this were not the case, then, up to extracting subsequences, we could find $\{x_i\}_{i\in\mathbb{N}} \subset (A' \cap \partial\mathcal{E}) \setminus \mathrm{cl}\,(I_\rho(\Sigma_A(\mathcal{E})))$ with $x_i \in S_i$ for every $i \in \mathbb{N}$ and $x_i \to x$ for some $x \in (\mathrm{cl}\,(A') \cap \partial\mathcal{E}) \setminus I_\rho(\Sigma_A(\mathcal{E}))$. Since $x \in \partial^*\mathcal{E}$, by Theorem 4.1 and step two in the proof of Corollary 4.6, there exists $r_x > 0$ and $\nu \in \mathbb{S}^{n-1}$ such that $\partial\mathcal{E} \cap \mathbf{C}^\nu_{x,r_x} = \partial^*\mathcal{E} \cap \mathbf{C}^\nu_{x,r_x} = (\mathrm{Id} + v\,\nu)(\mathbf{D}^\nu_{x,r_x})$ for some $v \in C^1(\mathbf{D}^\nu_{x,r_x})$. By connectedness, we infer that $S_i \cap \mathbf{C}^\nu_{x,r_x} = S_j \cap \mathbf{C}^\nu_{x,r_x}$, which contradicts the assumption on S_i and S_j. □

Corollary 4.8 (Bounded mean curvature)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, then $A \cap \partial\mathcal{E}$ is a locally $\mathcal{H}^{n-1}$-rectifiable set with bounded mean curvature in A, and*

$$\|\mathbf{H}_{\partial\mathcal{E}}\|_{L^\infty(\mathcal{H}^{n-1} \llcorner (A\cap\partial\mathcal{E}))} \le \Lambda\,. \tag{4.42}$$

Proof. Since $\partial^*\mathcal{E}$ is locally $\mathcal{H}^{n-1}$-rectifiable in A and $\mathcal{H}^{n-1}(\Sigma_A(\mathcal{E})) = 0$, one finds immediately that $A \cap \partial\mathcal{E}$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in A. By Riesz theorem and Lebesgue–Besicovitch differentiation theorem, in order to prove (4.42) it suffices to show that

$$\int_{\partial\mathcal{E}} \mathrm{div}_{\partial\mathcal{E}} T\, d\mathcal{H}^{n-1} \le (1+\eta)\,\Lambda\, P(\mathcal{E}; B_{x,r})\,, \tag{4.43}$$

whenever $B_{x,r} \subset\subset A$, $r < r_0$, $T \in C^1_c(B_{x,r};\mathbb{R}^n)$ with $|T| \le 1$. To this end, let $\{f_t\}_{|t|<\varepsilon}$ be the flow with initial velocity T, so that (see, e.g., [Mag12, Theorem 17.5])

$$P(f_t(E); B_{x,r}) = P(E; B_{x,r}) + t \int_{\partial^* E} \mathrm{div}_{\partial E} T\, d\mathcal{H}^{n-1} + O(t^2)\,,$$

for every set E of finite perimeter in $B_{x,r}$. By Lemma B.2 (see Appendix B) one sees that for every $\eta > 0$ it is possible to decrease $\varepsilon > 0$ in such a way that

$$|f_t(E)\Delta E| \le (1+\eta)\, P(E; B_{x,r})\, |t|\,, \qquad \forall |t| < \varepsilon\,,$$

for every Borel set $E \subset \mathbb{R}^n$. Up to further decreasing the value of ε we have $f_t(\mathcal{E}(h))\Delta\mathcal{E}(h) \subset\subset B_{x,r}$ for every $h = 1, ..., N$, so that by (1.15) one finds

$$\begin{aligned} P(\mathcal{E}; B_{x,r}) &\le P(f_t(\mathcal{E}); B_{x,r}) + \frac{\Lambda}{2} \sum_{h=0}^{N} |\mathcal{E}(h)\Delta f_t(\mathcal{E}(h))| \\ &= P(\mathcal{E}; B_{x,r}) + t \int_{\partial\mathcal{E}} \mathrm{div}_{\partial\mathcal{E}} T\, d\mathcal{H}^{n-1} + O(t^2) + (1+\eta)\,\Lambda\,|t|\, P(\mathcal{E}; B_{x,r})\,, \end{aligned}$$

and immediately deduces (4.43). □

We now start to consider the situation when

$$\begin{aligned} &\{\mathcal{E}_k\}_{k\in\mathbb{N}} \text{ are } (\Lambda, r_0)\text{-minimizing } N\text{-clusters in } A \\ &\text{and } \mathcal{E} \text{ is a } N\text{-cluster in } A \text{ with } \mathrm{d}_A(\mathcal{E}_k, \mathcal{E}) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{4.44}$$

Note that in this situation, by arguing exactly, say, as in the proof of [Mag12, Theorem 21.14], one has that $\mathcal{E}$ is also a (Λ, r_0)-minimizing cluster in A. As a further corollary of the infiltration lemma and of Theorem 4.1 we have the following theorem.

Theorem 4.9 (Hausdorff convergence of boundaries)**.** *If* (4.44) *holds, then for every* $A' \subset\subset A$ *one has* $\mathrm{hd}_{A'}(\partial\mathcal{E}_k, \partial\mathcal{E}) \to 0$ *as* $k \to \infty$*, and actually*

$$\lim_{k\to\infty} \mathrm{hd}_{A'}\Big(\partial\mathcal{E}_k(i) \cap \partial\mathcal{E}_k(j), \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j)\Big) = 0\,, \qquad \text{for every } 0 \le i < j \le N\,. \tag{4.45}$$

Moreover, for every $\varepsilon > 0$ *there exist* $k_0 \in \mathbb{N}$ *such that*

$$\Sigma_{A'}(\mathcal{E}_k) \subset I_\varepsilon(\Sigma_A(\mathcal{E}))\,, \qquad \forall k \ge k_0\,. \tag{4.46}$$

Remark 4.10. We are not able, in general, to prove the inclusion $\Sigma_{A'}(\mathcal{E}) \subset I_\varepsilon(\Sigma_A(\mathcal{E}_k))$ for k large, and thus infer the full Hausdorff convergence $\Sigma_A(\mathcal{E}_k)$ to $\Sigma_A(\mathcal{E})$ in every $A' \subset\subset A$. We can achieve this if $n = 2$, see Theorem 5.5 below, and if $n = 3$, see [LM15].

Remark 4.11. Note that if $A' \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) = \emptyset$, then (4.45) forces $A' \cap \partial\mathcal{E}_k(i) \cap \partial\mathcal{E}_k(j) = \emptyset$ for every k large enough. Indeed, $\mathrm{hd}_{A'}(\emptyset, T) = 0$ if $T \cap A' = \emptyset$, with $\mathrm{hd}_{A'}(\emptyset, T) = +\infty$ whenever $T \cap A' \neq \emptyset$.

Proof of Theorem 4.9. Step one: We prove (4.45). To this end, let us fix $0 \le i < j \le N$, set

$$S^k_{i,j} = \partial\mathcal{E}_k(i) \cap \partial\mathcal{E}_k(j)\,, \qquad S_{i,j} = \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j)\,,$$

and show that for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$A' \cap S^k_{i,j} \subset I_\varepsilon(S_{i,j})\,, \qquad A' \cap S_{i,j} \subset I_\varepsilon(S^k_{i,j})\,, \qquad \forall k \ge k_0\,. \tag{4.47}$$

We prove the first inclusion in (4.47) by contradiction. Let us consider $x_k \in A' \cap S^k_{i,j}$ with $\mathrm{dist}(x_k, S_{i,j}) > \varepsilon$ for every $k \in \mathbb{N}$. (Note that if $S_{i,j} = \emptyset$, then $\mathrm{dist}(x, S_{i,j}) = +\infty$ for every $x \in \mathbb{R}^n$ and contradicting the first inclusion in (4.47) exactly amounts in saying that $A' \cap S^k_{i,j} \neq \emptyset$.) Up to extracting subsequences, we may assume that $x_k \to x$ for some $x \in \mathrm{cl}\,(A') \subset A$. Since $\mathrm{dist}(x, S_{i,j}) \ge \varepsilon$, by (4.4) there exists $r_x < \mathrm{dist}(x, \partial A)$ such that

$$\begin{aligned} &\text{either} \quad |B_{x,r_x} \cap \mathcal{E}(i)| = 0\,, \quad \text{or} \quad |B_{x,r_x} \cap \mathcal{E}(i)| = \omega_n\, r_x^n\,,\\ &\text{or} \quad |B_{x,r_x} \cap \mathcal{E}(j)| = 0\,, \quad \text{or} \quad |B_{x,r_x} \cap \mathcal{E}(j)| = \omega_n\, r_x^n\,. \end{aligned}$$

For r_1 as in Lemma 4.5, let $s_x = \min\{r_x, r_1\}/2$, then for $k \ge k_0$ one has

$$\begin{aligned} &\text{either} \quad |B_{x_k,2s_x} \cap \mathcal{E}_k(i)| < \eta_0\,(2s_x)^n\,, \quad \text{or} \quad |B_{x_k,2s_x} \cap \mathcal{E}_k(i)| > (\omega_n - \eta_0)\,(2s_x)^n\,,\\ &\text{or} \quad |B_{x_k,2s_x} \cap \mathcal{E}_k(j)| < \eta_0\,(2s_x)^n\,, \quad \text{or} \quad |B_{x_k,2s_x} \cap \mathcal{E}_k(j)| > (\omega_n - \eta_0)\,(2s_x)^n\,, \end{aligned}$$

and thus, by Lemma 4.5,

$$\begin{aligned} &\text{either} \quad |B_{x_k,s_x} \cap \mathcal{E}_k(i)| = 0\,, \quad \text{or} \quad |B_{x_k,s_x} \cap \mathcal{E}_k(i)| = \omega_n\, s_x^n\,,\\ &\text{or} \quad |B_{x_k,s_x} \cap \mathcal{E}_k(j)| = 0\,, \quad \text{or} \quad |B_{x_k,s_x} \cap \mathcal{E}_k(j)| = \omega_n\, s_x^n\,. \end{aligned}$$

By (4.4), $x_k \in A' \setminus S^k_{i,j}$ for k large, a contradiction. We now prove the second inclusion in (4.47): by contradiction, there exist $x \in A' \cap S_{i,j}$ and $\varepsilon > 0$ such that $B_{x,\varepsilon} \cap S^k_{i,j} = \emptyset$, i.e., by (4.4),

$$\begin{aligned} &\text{either} \quad |B_{x,\varepsilon} \cap \mathcal{E}_k(i)| = 0\,, \quad \text{or} \quad |B_{x,\varepsilon} \cap \mathcal{E}_k(i)| = \omega_n\, \varepsilon^n\,,\\ &\text{or} \quad |B_{x,\varepsilon} \cap \mathcal{E}_k(j)| = 0\,, \quad \text{or} \quad |B_{x,\varepsilon} \cap \mathcal{E}_k(j)| = \omega_n\, \varepsilon^n\,, \end{aligned}$$

for infinitely many values of k; by letting $k \to \infty$ along such values we thus find that $x \notin S_{i,j}$.

Step two: We prove (4.46). Should (4.46) fail, we could find $\varepsilon > 0$ and $x_k \in \Sigma_{A'}(\mathcal{E}_k)$ with $\mathrm{dist}(x_k, \Sigma(\mathcal{E})) > \varepsilon$ for infinitely many $k \in \mathbb{N}$. By step one, up to extracting subsequences, $x_k \to x$ as $k \to \infty$ for some $x \in A \cap \partial\mathcal{E}$. Since $\mathrm{dist}(x, \Sigma(\mathcal{E})) \ge \varepsilon$, we have $x \in \partial^*\mathcal{E}$. By step two in the proof of Corollary 4.6, there exist $0 \le h < h' \le N$ and $2\,r_* < \min\{r_1, \mathrm{dist}(x,\partial A)\}$ such that $x \in \mathcal{E}(h,h')$ and $B_{x,2\,r_*} \subset \mathcal{E}(h) \cup \mathcal{E}(h')$. Hence, for some $k_0 \in \mathbb{N}$ we have

$$|\mathcal{E}_k(h) \cap B_{x_k,2\,r_*}| + |\mathcal{E}_k(h') \cap B_{x_k,2\,r_*}| \ge (\omega_n - \eta_0)\, r_*^n\,, \qquad \forall k \ge k_0\,.$$

By Lemma 4.5, $\mathcal{E}_k(j) \cap B_{x_k,r_*} = \emptyset$ for every $k \ge k_0$ and $j \ne h, h'$, so that $\mathcal{E}_k(h)$ is a (Λ, r_0)-minimizing set in B_{x_k,r_*}. By arguing as in Lemma 4.4 we find that

$$\mathbf{exc}^\nu_{x,r}(\mathcal{E}(h)) = \lim_{k\to\infty} \mathbf{exc}^\nu_{x_k,r}(\mathcal{E}_k(h))\,, \qquad \text{for a.e. } r < r_*\,. \tag{4.48}$$

Since $x \in \mathcal{E}(h,h')$, by Remark 4.2 there exist $r_{**} < \min\{r_*, r_0\}$ and $\nu \in \mathbb{S}^{n-1}$ such that

$$\mathbf{exc}^\nu_{x,r_{**}}(\mathcal{E}(h)) + \Lambda\, r_{**} \le \frac{\varepsilon_*(n)}{2^n}\,, \tag{4.49}$$

where $\varepsilon_*(n)$ is defined as in Theorem 4.1. By (4.48) and (4.49) we conclude that, for some $r \in (r_{**}/2, r_{**})$ and up to increasing k_0, $\mathbf{exc}^\nu_{x_k,r}(\mathcal{E}_k(h)) + \Lambda\, r < \varepsilon_*$ for every $k \ge k_0$. By Theorem 4.1, $B_{x_k,r/2} \cap \partial^*\mathcal{E}_k(h)$ is a $C^{1,\alpha}$-hypersurface, against $x_k \in \Sigma_{A'}(\mathcal{E}_k)$. □

We now set

$$[\partial\mathcal{E}]_\rho = (A \cap \partial\mathcal{E}) \setminus I_\rho(\Sigma_A(\mathcal{E}))\,,$$

recall the definition (2.2) of normal ε-neighborhood $N_\varepsilon(S)$ of a manifold $S \subset \mathbb{R}^n$, and then combine Theorem 4.1 and Theorem 4.9 to obtain the following weak improved convergence theorem.

Theorem 4.12 (Normal representation theorem)**.** *If $\Lambda \ge 0$, $\alpha \in (0,1)$, and $\mathcal{E}$ is a N-cluster in $A \subset \mathbb{R}^n$ such that $\partial^*\mathcal{E}$ is a $C^{2,1}$-hypersurface, then there exist positive constants ρ_0 (depending on $\mathcal{E}$) and C (depending on α, Λ, and $\mathcal{E}$) with the following property.*

If (4.44) holds, then for every $A' \subset\subset A$ and $\rho < \rho_0$ there exist $k_0 \in \mathbb{N}$, $\varepsilon \in (0,\rho)$, Ω open with $A' \subset\subset \Omega \subset\subset A$, and $\{\psi_k\}_{k\ge k_0} \subset C^{1,\alpha}(\Omega \cap [\partial\mathcal{E}]_\rho)$ such that

$$(A' \cap \partial\mathcal{E}_k) \setminus I_{2\rho}(\Sigma_A(\mathcal{E})) \subset (\mathrm{Id} + \psi_k \nu_{\mathcal{E}})(\Omega \cap [\partial\mathcal{E}]_\rho) \subset \partial^*\mathcal{E}_k\,, \tag{4.50}$$

$$N_\varepsilon(A' \cap [\partial\mathcal{E}]_\rho) \cap \partial\mathcal{E}_k = (\mathrm{Id} + \psi_k\, \nu_{\mathcal{E}})(A' \cap [\partial\mathcal{E}]_\rho)\,, \tag{4.51}$$

with

$$\lim_{k\to\infty} \|\psi_k\|_{C^1(\Omega\cap[\partial\mathcal{E}]_\rho)} = 0\,, \qquad \sup_{k\ge k_0} \|\psi_k\|_{C^{1,\alpha}(\Omega\cap[\partial\mathcal{E}]_\rho)} \le C\,. \tag{4.52}$$

Moreover, when $n = 2$ one can set $\alpha = 1$ in this statement.

Proof. Since $\partial^*\mathcal{E}$ is a $C^{2,1}$-hypersurface, for every $x \in \partial^*\mathcal{E}$ there exist $r_x > 0$, $\nu_x \in \mathbb{S}^{n-1}$ and $v_x \in C^{2,1}(\mathbf{D}^{\nu_x}_{x,64\,r_x})$ with $v_x(x) = 0$, $\nabla v_x(x) = 0$, and

$$\partial\mathcal{E} \cap \mathbf{C}^{\nu_x}_{x,64\,r_x} = (\mathrm{Id} + v_x\, \nu_x)(\mathbf{D}^{\nu_x}_{x,64\,r_x})\,, \qquad \mathbf{C}^{\nu_x}_{x,64\,r_x} \subset\subset A\,. \tag{4.53}$$

By Theorem 4.9, $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in A, so that by step two in the proof of Corollary 4.6 there also exist $0 \le h_x < h'_x \le N$ such that, up to further decreasing r_x, one has

$$|\mathcal{E}(j) \cap \mathbf{C}^{\nu_x}_{x,64\,r_x}| = 0\,, \qquad \forall j \ne h_x, h'_x\,, \tag{4.54}$$

and thus, taking (4.53) into account and without loss of generality,

$$\mathbf{C}^{\nu_x}_{x,64\,r_x} \cap \mathcal{E}(h_x) = \Big\{ z + s\,\nu_x : z \in \mathbf{D}^{\nu_x}_{x,64\,r_x}\,, v_x(z) < s < 64\,r_x \Big\}\,. \tag{4.55}$$

By Lemma 4.5 and by (4.54) there exists $k_x \in \mathbb{N}$ such that

$$|\mathcal{E}_k(j) \cap B_{x,32\,r_x}| = 0\,, \qquad \forall j \ne h_x, h'_x\,, \qquad \forall k \ge k_x\,, \tag{4.56}$$

so that $\mathcal{E}_k(h_x)$ is a (Λ, r_0)-minimizing set in $B_{x,32\,r_x}$ for $k \ge k_x$. By Lemma 4.4 there exist $s_x \in (0, r_x)$ and, up to increasing k_x, functions $\psi_{x,k} \in C^{1,\alpha}(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}(h_x))$ with

$$\mathbf{C}^{\nu_x}_{x,s_x} \cap \partial\mathcal{E}_k(h_x) \subset (\mathrm{Id} + \psi_{x,k}\, \nu_{\mathcal{E}_k(h_x)})(\mathbf{C}^{\nu_x}_{x,2\,s_x} \cap \partial\mathcal{E}(h_x)) \subset \mathbf{C}^{\nu_x}_{x,4\,s_x} \cap \partial\mathcal{E}_k(h_x) \tag{4.57}$$

$$\sup_{k\geq k_0}\|\psi_{x,k}\|_{C^{1,\alpha}(\mathbf{C}^{\nu_x}_{x,2\,s_x}\cap\partial\mathcal{E}(h_x))}\leq C\,,\qquad\lim_{k\to\infty}\|\psi_{x,k}\|_{C^1(\mathbf{C}^{\nu_x}_{x,2\,s_x}\cap\partial\mathcal{E}(h_x))}=0\,,\tag{4.58}$$

where C depends on α, Λ and $\mathcal{E}$.

Let $\rho_0>0$ be such that $[\partial\mathcal{E}]_{\rho_0}\neq\emptyset$. For every $\rho\in(0,\rho_0)$ we can find $\{x_i\}_{i=1}^M\subset A'\cap[\partial\mathcal{E}]_\rho\subset\partial^*\mathcal{E}$ such that (for $s_i=s_{x_i}$ and $\nu_i=\nu_{x_i}$) one has

$$A'\cap[\partial\mathcal{E}]_\rho\subset\subset\bigcup_{i=1}^M\mathbf{C}^{\nu_i}_{x_i,s_i}\,,\qquad\mathbf{C}^{\nu_i}_{x_i,64\,s_i}\subset\subset A\,.\tag{4.59}$$

Since $\partial^*\mathcal{E}$ is a C^2-hypersurface we can find $\varepsilon(\rho)\in(0,\rho)$ such that every point in $N_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)$ has a unique projection onto $A'\cap[\partial\mathcal{E}]_\rho$ and

$$N_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\subset I_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\subset\bigcup_{i=1}^M\mathbf{C}^{\nu_i}_{x_i,s_i}\,.\tag{4.60}$$

By arguing as in the proof of Lemma 4.3, we see that $\psi_{i,k}=\psi_{j,k}$ on $\mathbf{C}^{\nu_i}_{x_i,2\,s_i}\cap\mathbf{C}^{\nu_j}_{x_j,2\,s_j}\cap\partial\mathcal{E}$ for every i,j. In particular, if we set

$$\Omega=\bigcup_{i=1}^M\mathbf{C}^{\nu_i}_{x_i,2\,s_i}\,,$$

then we can define $\psi_k\in C^{1,\alpha}(\Omega\cap\partial\mathcal{E})$ for every $k\geq k_0=\max\{k_{x_i}:1\leq i\leq M\}$ by letting $\psi_k=\psi_{x_i,k}$ on $\mathbf{C}^{\nu_i}_{x_i,2\,s_i}\cap\partial\mathcal{E}$. In this way,

$$\partial\mathcal{E}_k\cap\bigcup_{i=1}^M\mathbf{C}^{\nu_i}_{x_i,s_i}\subset(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})(\Omega\cap\partial\mathcal{E})\subset\partial^*\mathcal{E}_k\,,\qquad\forall k\geq k_0\,,\tag{4.61}$$

$$\sup_{k\geq k_0}\|\psi_k\|_{C^{1,\alpha}(\Omega\cap\partial\mathcal{E})}\leq C\,,\qquad\lim_{k\to\infty}\|\psi_k\|_{C^1(\Omega\cap\partial\mathcal{E})}=0\,.\tag{4.62}$$

By (4.60), (4.61), $A'\cap[\partial\mathcal{E}]_\rho\subset\Omega$, and since $\mathrm{Id}+\psi_k\nu_{\mathcal{E}}$ is a normal deformation of $\Omega\cap\partial\mathcal{E}$,

$$\begin{aligned}N_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\cap\partial\mathcal{E}_k&\subset(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})(\Omega\cap\partial\mathcal{E})\cap N_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\\&=(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})(A'\cap[\partial\mathcal{E}]_\rho)\subset N_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\cap\partial\mathcal{E}_k\,,\end{aligned}$$

where the last inclusion follows by the second inclusion in (4.61) provided $\|\psi_k\|_{C^0(\Omega\cap\partial\mathcal{E})}<\varepsilon(\rho)$ for every $k\geq k_0$. This proves (4.51). Finally, by Theorem 4.9, up to increasing k_0, $A'\cap\partial\mathcal{E}_k\subset I_{\varepsilon(\rho)}(\partial\mathcal{E})$ for every $k\geq k_0$, so that $\varepsilon(\rho)<\rho$ gives us

$$(A'\cap\partial\mathcal{E}_k)\setminus I_{2\rho}(\Sigma_A(\mathcal{E}))\subset A'\cap\Big(I_{\varepsilon(\rho)}(\partial\mathcal{E})\setminus I_{2\rho}(\Sigma_A(\mathcal{E}))\Big)\subset A'\cap I_{\varepsilon(\rho)}([\partial\mathcal{E}]_\rho)\subset I_{\varepsilon(\rho)}(A'\cap[\partial\mathcal{E}]_\rho)\,.$$

By combining this last inclusion with (4.60) we find that

$$(A'\cap\partial\mathcal{E}_k)\setminus I_{2\rho}(\Sigma_A(\mathcal{E}))\subset\partial\mathcal{E}_k\cap\bigcup_{i=1}^M\mathbf{C}^{\nu_i}_{x_i,s_i}\,,$$

and thus deduce (4.50) from (4.61). □

4.4. **Blow-ups of (Λ,r_0)-minimizing clusters.** If $\mathcal{E}$ is a N-cluster in A and $x\in A$, then the blow-up of $\mathcal{E}$ at x at scale $r>0$ is the N-cluster $\mathcal{E}_{x,r}$ in $(A-x)/r$ defined by setting

$$\mathcal{E}_{x,r}(h)=\frac{\mathcal{E}(h)-x}{r}\,,\qquad1\leq h\leq N\,.$$

We set

$$\theta(\partial\mathcal{E},x,r)=\frac{P(\mathcal{E};B_{x,r})}{r^{n-1}}=\theta(\partial\mathcal{E}_{x,r},0,1)\,,\qquad\theta(\partial\mathcal{E},x)=\lim_{r\to0^+}\theta(\partial\mathcal{E},x,r)\,,$$

provided this last limit exists. By a classical argument based on comparison with cones (see, for example [Mag12, Theorem 28.4]), one sees that if $\mathcal{E}$ is a (Λ, r_0)-minimizing N-cluster in A, $x \in A \cap \partial\mathcal{E}$, and $r_* \in (0, r_0)$ is such that $\omega_n\, r_*^n < \min\{|\mathcal{E}(h) \cap A| : 1 \le h \le N\}$, then

$$\theta(\partial\mathcal{E}, x, r)\, e^{(n-1)\,\omega_n\,\Lambda\, r} \qquad \text{is increasing on } (0, r_*)\,, \tag{4.63}$$

so that $\theta(\partial\mathcal{E}, x)$ is defined for every $x \in A \cap \partial\mathcal{E}$. Moreover, the same argument shows that if $\Lambda = 0$ and $\theta(\partial\mathcal{E}, x, r)$ is constant on $r \in (0, r_*)$, then $B_{x,r_*} \cap \partial\mathcal{E}$ is a cone with vertex at x. Now let us say that a M-cluster $\mathcal{K}$ in $\mathbb{R}^n$ is a *cone-like minimizing cluster* if $\mathcal{K}(i)$ is an open cone with vertex at the origin for each $i = 1, ..., M$, $|\mathcal{K}(0)| = |\mathbb{R}^n \setminus \bigcup_{i=1}^M \mathcal{K}(i)| = 0$, and

$$P(\mathcal{K}; B_R) \le P(\mathcal{F}; B_R)\,, \tag{4.64}$$

whenever $R > 0$ and $\mathcal{F}$ is an M-cluster in $\mathbb{R}^n$ with $\mathcal{F}(i)\Delta\mathcal{K}(i) \subset\subset B_R$ for every $i = 1, ..., M$. Moreover, given a N-cluster $\mathcal{E}$ in A and an injective map $\sigma : \{1, ..., M\} \to \{0, ..., N\}$, let us denote by $\sigma(\mathcal{E})$ the M-cluster in A defined by setting

$$\sigma\mathcal{E}(i) = \mathcal{E}(\sigma(i))\,, \qquad i = 1, ..., M\,.$$

Theorem 4.13 (Tangent cone-like minimizing clusters)**.** *If $\mathcal{E}$ is a (Λ, r_0)-minimizing N-cluster in A, $x \in A \cap \partial\mathcal{E}$, and $s_j \to 0$ as $j \to \infty$, then there exist a subsequence $\{s_j'\}_{j\in\mathbb{N}}$ and a cone-like minimizing M-cluster $\mathcal{K}$ (with $2 \le M \le N$) such that $\theta(\partial\mathcal{E}, x) = \theta(\partial\mathcal{K}, 0)$ and*

$$\lim_{j\to\infty} \mathrm{d}_{B_R}\big(\sigma\mathcal{E}_{x,s_j'}, \mathcal{K}\big) = 0\,, \qquad \forall R > 0\,. \tag{4.65}$$

for some injective map $\sigma : \{1, \dots, M\} \to \{0, \dots, N\}$. (Note that given $R > 0$ one has $B_R \subset\subset (A - x)/r$ as soon as r is small enough.) Moreover, $x \in \Sigma_A(\mathcal{E})$ if and only if $0 \in \Sigma(\mathcal{K})$.

Proof. Once again this follows by a classical argument. We refer to [Mag12, Theorem 28.6] for a proof in the case of (Λ, r_0)-minimizing sets. □

We conclude this section with a technical lemma, which is the starting point in showing (under the situation described in (4.44)) the Hausdorff convergence of $\Sigma(\mathcal{E}_k)$ to $\Sigma(\mathcal{E})$ when $n = 2, 3$.

Lemma 4.14. *Let $n \ge 2$ be fixed. Either $\mathrm{hd}_{A'}(\Sigma(\mathcal{E}_k), \Sigma(\mathcal{E})) \to 0$ as $k \to \infty$ whenever (4.44) holds and $A' \subset\subset A$, or there exist a cone-like minimizing M-cluster $\mathcal{K}$ in $\mathbb{R}^n$ and a sequence $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ of $(\delta_j, \delta_j^{-1})$-minimizing M-clusters $\mathcal{F}_j$ in B_2 with*

$$0 \in \Sigma(\mathcal{K})\,, \qquad \Sigma_{B_2}(\mathcal{F}_j) = \emptyset \qquad \forall j \in \mathbb{N}\,, \qquad \lim_{j\to\infty} \max\big\{\delta_j, \mathrm{d}_{B_2}(\mathcal{F}_j, \mathcal{K})\big\} = 0\,.$$

Proof. Let us assume that for some $\mathcal{E}_k$, $\mathcal{E}$ and A as in (4.44) there exists $A' \subset A$ such that $\limsup_{k\to\infty} \mathrm{hd}_{A'}(\Sigma(\mathcal{E}_k), \Sigma(\mathcal{E})) > 0$. By Theorem 4.9 and by (4.46) in Theorem 4.9 and up to extracting subsequences, we may directly assume the existence of $x \in \Sigma_{A'}(\mathcal{E})$ and $\varepsilon > 0$ such that $B_{x,\varepsilon} \subset\subset A$,

$$B_{x,\varepsilon} \cap \Sigma_A(\mathcal{E}_k) = \emptyset \qquad \forall k \in \mathbb{N}\,, \tag{4.66}$$

and such that $x_k \to x$ for some $x_k \in A \cap \partial\mathcal{E}_k$. In particular, up to discarding finitely many values of k, we may assume that $x_k \in A' \cap \partial^*\mathcal{E}_k$ for every k, and finally, up to translating $\mathcal{E}_k$, that $x_k = x$ for every k. Summarizing, we have $\mathcal{E}_k$ and $\mathcal{E}$ as in (4.44) such that there exists

$$x \in \Sigma_A(\mathcal{E}) \cap \bigcap_{k\in\mathbb{N}} \partial^*\mathcal{E}_k\,.$$

By Theorem 4.13 we can find a cone-like minimizing M-cluster $\mathcal{K}$ in $\mathbb{R}^n$ with $\theta(\partial\mathcal{E}, x) = \theta(\partial\mathcal{K}, 0)$ (so that $0 \in \Sigma(\mathcal{K})$ by $x \in \Sigma_A(\mathcal{E})$), an injective map $\sigma : \{1, ..., M\} \to \{0, ..., N\}$, and a sequence $s_j \to 0^+$ as $j \to \infty$ such that (4.65) holds (with s_j directly in place of s_j'). Correspondingly, we consider $\{k(j)\}_{j\in\mathbb{N}}$ such that

$$\mathrm{d}_{B_{x,\varepsilon}}(\mathcal{E}_{k(j)}, \mathcal{E}) = o(s_j^n) \qquad \text{as } j \to \infty\,, \tag{4.67}$$

and finally define $(s_j\,\Lambda, r_0/s_j)$-minimizing M-clusters $\mathcal{F}_j$ in $(A-x)/s_j$ by setting

$$\mathcal{F}_j(i) = \frac{\mathcal{E}_{k(j)}(\sigma(i)) - x}{s_j}\,,\qquad \text{that is}\qquad \mathcal{F}_j = \sigma\big((\mathcal{E}_{k(j)})_{x,s_j}\big)\,.$$

By (4.65) and (4.67) for every fixed $R>0$ one has $\mathrm{d}_{B_R}(\mathcal{F}_j,\mathcal{K})\to 0$, while (4.66) implies that $\Sigma_{B_R}(\mathcal{F}_j)=\emptyset$ provided j is large enough. □

5. Improved convergence for planar clusters

In this section we finally prove Theorem 1.5. First, in section 5.1, we address the structure of (Λ, r_0)-minimizing clusters in $\mathbb{R}^2$, and deduce from this structure result and Lemma 4.14 the Hausdorff convergence of singular sets. Next, in section 5.2, and specifically in Theorem 5.6, we complete the preparations needed to exploit Theorem 3.5 in the proof of Theorem 1.5. This last argument is then presented at the end of the section.

5.1. (Λ, r_0)**-minimizing clusters in** $\mathbb{R}^2$**.** In view of Theorem 4.13, the starting point in the analysis of almost-minimizing clusters near their singular sets is the classification of cone-like minimizing clusters. Such a classification is currently known only in $\mathbb{R}^2$ and $\mathbb{R}^3$. Referring to [LM15] for the latter case, we work from now on in $\mathbb{R}^2$. Let us denote by $\mathcal{Y}_2$ the cone-like minimizing 3-cluster in $\mathbb{R}^2$ defined by

$$\mathcal{Y}_2(i) = \Big\{(t\cos\theta, t\sin\theta):\ t>0\,,(i-1)\,\frac{2\pi}{3}<\theta<i\,\frac{2\pi}{3}\Big\}\,,\qquad i=1,2,3\,. \tag{5.1}$$

Up to rotations around the origin, $\mathcal{Y}_2$ is the only cone-like minimizing cluster in $\mathbb{R}^2$ (other than the one defined by a pair of complementary half-planes, of course); see, for example, [Mag12, Proposition 30.9]. As a consequence, by Theorem 4.13 one has that if $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $A\subset\mathbb{R}^2$, then $\partial^*\mathcal{E}=\{x\in A\cap\partial\mathcal{E}:\theta(\partial\mathcal{E},x)=2\}$ and

$$\Sigma_A(\mathcal{E}) = \Big\{x\in A\cap\partial\mathcal{E}:\theta(\partial\mathcal{E},x)=\theta(\mathcal{Y}_2,0)=3\Big\}\,. \tag{5.2}$$

We now localize Definition 1.2, and then, in Theorem 5.2, describe the structure of planar almost-minimizing clusters.

Definition 5.1. Let $\mathcal{E}$ be a cluster in $A\subset\mathbb{R}^2$ open. One says that $\mathcal{E}$ is a $C^{k,\alpha}$*-cluster in* A if there exist at most countable families $\{\gamma_i\}_{i\in I}$ of connected $C^{k,\alpha}$-curves with boundary relatively closed in A, and $\{p_j\}_{j\in J}$ of points of A, which are both locally finite in A (that is, given $A'\subset\subset A$ we have $\gamma_i\cap A'\neq\emptyset$ and $p_j\in A'$ only for finitely many $i\in I$ and $j\in J$), and such that

$$\begin{aligned} A\cap\partial\mathcal{E} &= \bigcup_{i\in I}\gamma_i\,,\qquad \partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\\ \Sigma_A(\mathcal{E}) &= A\cap\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i) = A\cap\bigcup_{j\in J}\{p_j\}\,. \end{aligned} \tag{5.3}$$

Theorem 5.2. *If $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $A\subset\mathbb{R}^2$, then $\mathcal{E}$ is a $C^{1,1}$-cluster in A. Moreover, each γ_i has distributional curvature bounded by Λ and each p_j is a common boundary point of exactly three different curves from $\{\gamma_i\}_{i\in I}$ which form three 120 degrees angles at p_j. Finally,* $\mathrm{diam}(\gamma_i)\geq 1/2\Lambda$ *for every $i\in I$ such that $\gamma_i\subset\subset A$ and $\mathrm{bd}\,(\gamma_i)=\emptyset$. (In particular, if $\Lambda=0$, then $\mathrm{bd}\,(\gamma_i)\neq\emptyset$ for every $i\in I$.)*

Proof. By exploiting the argument of [Mag12, Theorem 30.7] (which addresses the case of planar isoperimetric clusters, but actually uses only a minimality condition of the form (1.15), and that can be easily localized to a given open set) we just need to prove that the curves γ_i have distributional curvature bounded by Λ and the diameter lower bound when $\gamma_i\subset\subset A$ with $\mathrm{bd}\,(\gamma_i)=\emptyset$. By Corollary 4.8 we have that

$$\int_{\partial\mathcal{E}}\mathrm{div}_{\partial\mathcal{E}}T\,d\mathcal{H}^1=\int_{\partial\mathcal{E}}T\cdot\mathbf{H}_{\partial\mathcal{E}}\,d\mathcal{H}^1\,,\qquad\forall T\in C^1_c(A;\mathbb{R}^2)\,,\tag{5.4}$$

where $|\mathbf{H}_{\partial\mathcal{E}}|\le\Lambda$. In particular,

$$\int_{\gamma_i}\mathrm{div}_{\gamma_i}T\,d\mathcal{H}^1=\int_{\gamma_i}T\cdot\mathbf{H}_{\partial\mathcal{E}}\,d\mathcal{H}^1\,,\tag{5.5}$$

for every $T\in C^1_c(A';\mathbb{R}^2)$ such that $\mathrm{spt}\,T\cap\partial\mathcal{E}=\mathrm{spt}\,T\cap\mathrm{int}\,(\gamma_i)$. Since $|\mathbf{H}_{\partial\mathcal{E}}|\le\Lambda$ this proves that each $A'\cap\gamma_i$ has distributional mean curvature bounded by Λ. If, in addition, $\gamma_i\subset\subset A'\subset\subset A$ and $\mathrm{bd}\,(\gamma_i)=\emptyset$, then we can test (5.5) with $T(x)=\zeta(x)(x-x_0)$ where $x_0\in\mathbb{R}^2$ is such that $\gamma_i\subset B_{x_0,2\,\mathrm{diam}(\gamma_i)}$ and $\zeta\in C^1_c(A')$ with $\zeta=1$ on γ_i and $\mathrm{spt}\zeta\cap\partial\mathcal{E}=\mathrm{spt}\zeta\cap\gamma_i$, to find that $\mathcal{H}^1(\gamma_i)\le2\Lambda\,\mathrm{diam}(\gamma_i)\,\mathcal{H}^1(\gamma_i)$, as required. □

Remark 5.3 (Topology of boundaries of planar (Λ,r_0)-minimizing clusters)**.** If $\mathcal{E}$ is a bounded (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$, then Theorem 5.2 implies the existence of *finite* families of closed connected $C^{1,1}$-curves with boundary $\{\gamma_i\}_{i\in I}$ (whose distributional curvature is bounded by Λ) and of finitely many points $\{p_j\}_{j\in J}$ such that each p_j is the common end-point of three different curves from $\{\gamma_i\}_{i\in I}$, which form three 120 degrees angles at p_j. Moreover, (5.3) takes the form

$$\partial\mathcal{E}=\bigcup_{i\in I}\gamma_i\,,\qquad\partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\qquad\Sigma(\mathcal{E})=\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i)=\bigcup_{j\in J}\{p_j\}\,.\tag{5.6}$$

Let I'' denotes the set of those $i\in I$ such that γ_i is diffeomorphic to $[0,1]$ (so that γ_i is diffeomorphic to $\mathbb{S}^1$ for every $i\in I'=I\setminus I''$, this will be the notation used in the proof of Theorem 5.6). For each $i\in I''$, γ_i has exactly two end-points, both belonging to $\Sigma(\mathcal{E})$, and for every $x\in\Sigma(\mathcal{E})$ there exist three curves from $\{\gamma_i\}_{i\in I''}$ sharing x as a common end-point: therefore we find that

$$\#(I'')=\frac{3}{2}\,\mathcal{H}^0(\Sigma(\mathcal{E}))\,.$$

Remark 5.4. With the notation of the previous remark, we claim that $I''=I$ whenever $\mathcal{E}$ is a planar isoperimetric cluster (that is, $\mathcal{E}$ is a minimizer in (1.13) with $N\ge2$ and $n=2$; notice that $\mathcal{E}$ is necessarily bounded). Indeed, arguing by contradiction, let us assume there exists $i\in I$ such that γ_i is C^1-diffeomorphic to $\mathbb{S}^1$. Since $\gamma_i\cap\Sigma(\mathcal{E})=\emptyset$, the constant curvature condition on interfaces of $\mathcal{E}$ implies that γ_i is, in fact, a circle. Moreover, since $N\ge2$, we must have $\#(I)\ge2$. Since $\#(I)\ge2$, we can translate γ_i along a suitable direction until it intersects for the first time $\partial\mathcal{E}\setminus\gamma_i$ at some point x. Denoting by $\mathcal{E}'$ the resulting cluster, we have that $P(\mathcal{E}')=P(\mathcal{E})$ and $\mathrm{vol}\,(\mathcal{E}')=\mathrm{vol}\,(\mathcal{E})$, so that $\mathcal{E}'$ is a minimizing cluster in $\mathbb{R}^2$. The fact that, in a neighborhood of x, $\partial\mathcal{E}'$ is the union of two tangent circular arcs, leads to a contradiction with Theorem 5.2 (applied to $\mathcal{E}'$).

We now upgrade (4.46) to the full Hausdorff convergence of singular sets.

Theorem 5.5 (Hausdorff convergence of singular sets)**.** *If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ,r_0)-minimizing clusters in $A\subset\mathbb{R}^2$ with $\mathrm{d}_A(\mathcal{E}_k,\mathcal{E})\to0$ as $k\to\infty$, then*

$$\lim_{k\to\infty}\mathrm{hd}_{A'}(\Sigma_A(\mathcal{E}_k),\Sigma_A(\mathcal{E}))=0\qquad\forall A'\subset\subset A\,.$$

Proof. We argue by contradiction. In this way, by Lemma 4.14 there exists a sequence $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ of (δ_j,δ_j^{-1})-minimizing M-clusters in $B_2\subset\mathbb{R}^2$ such that

$$\Sigma_{B_2}(\mathcal{F}_j)=\emptyset\qquad\forall j\in\mathbb{N}\,,\qquad\lim_{j\to\infty}\max\big\{\delta_j,\mathrm{d}_{B_2}(\mathcal{F}_j,\mathcal{Y}_2)\big\}=0\,,\tag{5.7}$$

where $\mathcal{Y}_2$ is defined as in (5.1). By Theorem 4.9,

$$\lim_{j\to\infty}\max_{1\le i<\ell\le 3}\mathrm{hd}_B\Big(\partial\mathcal{F}_j(i)\cap\partial\mathcal{F}_j(\ell),\partial\mathcal{Y}_2(i)\cap\partial\mathcal{Y}_2(\ell)\Big)=0\,, \tag{5.8}$$

while, by Theorem 4.12, for every δ small enough one can find $\{\psi_j\}_{j\ge j_0}\subset C^1(B\cap[\partial\mathcal{Y}_2]_\delta)$ such that (on taking into account that $I_{2\delta}(\Sigma(\mathcal{Y}_2))=B_{2\delta}$)

$$\partial\mathcal{F}_j\cap(B\setminus B_{2\delta})\subset(\mathrm{Id}+\psi_j\nu)(B\cap[\partial\mathcal{Y}_2]_\delta)\,,\qquad\forall j\ge j_0\,, \tag{5.9}$$

where ν denotes a continuous normal vector field to $\partial^*\mathcal{Y}_2$. By Theorem 5.2 there exists a finite family of connected $C^{1,1}$-curves with boundary $\{\gamma_i\}_{i\in I}$, relatively closed in B, such that $B\cap\partial\mathcal{F}_j=B\cap\bigcup_{i\in I}\gamma_i$ and $\Sigma_B(\mathcal{F}_j)=\bigcup_{i\in I}B\cap\mathrm{bd}\,(\gamma_i)$, so that, by (5.7), $B\cap\mathrm{bd}\,(\gamma_i)=\emptyset$ for every $i\in I$. Let $\gamma_{i\ell}$ denote the connected curve in $\partial\mathcal{F}_j$ that contains $(\mathrm{Id}+\psi_j\nu)(B\cap[\partial\mathcal{Y}_2(i)\cap\partial\mathcal{Y}_2(\ell)]_\delta)$, for $1\le i<\ell\le 3$. By (5.9) we notice that

$$\partial\mathcal{F}_j\cap(B\setminus B_{2\delta})=\bigcup_{1\le i<\ell\le 3}\gamma_{i\ell}\cap(B\setminus B_{2\delta}) \tag{5.10}$$

while by (5.8) we get $\gamma_{i\ell}\cap B\subset I_\delta(\partial\mathcal{Y}_2(i)\cap\partial\mathcal{Y}_2(\ell))$ for all $1\le i<\ell\le 3$. By combining this last fact with (5.10), we deduce that $\mathrm{bd}\,(\gamma_{i\ell})\cap B_{2\delta}\ne\emptyset$, against the fact that $B\cap\mathrm{bd}\,(\gamma_i)=\emptyset$ for every $i\in I$. □

5.2. **Proof of the improved convergence theorem for planar clusters.** We now prove Theorem 1.5. We start by setting some notation. Let us consider Λ, r_0, $\mathcal{E}$ and $\mathcal{E}_k$ as in Theorem 1.5. Since $\partial\mathcal{E}$ is bounded, by Theorem 4.9 also $\partial\mathcal{E}_k$ is bounded, and thus according to (5.6) there exist *finite* families of $C^{2,1}$-curves $\{\gamma_i\}_{i\in I}$ and $C^{1,1}$-curves $\{\gamma_i^k\}_{i\in I_k}$, and finite families of points $\{p_j\}_{j\in J}$ and $\{p_j^k\}_{j\in J_k}$ such that

$$\partial\mathcal{E}=\bigcup_{i\in I}\gamma_i\,,\qquad\partial^*\mathcal{E}=\bigcup_{i\in I}\mathrm{int}\,(\gamma_i)\,,\qquad\Sigma(\mathcal{E})=\bigcup_{i\in I}\mathrm{bd}\,(\gamma_i)=\bigcup_{j\in J}\{p_j\}\,,$$

$$\partial\mathcal{E}_k=\bigcup_{i\in I_k}\gamma_i^k\,,\qquad\partial^*\mathcal{E}_k=\bigcup_{i\in I_k}\mathrm{int}\,(\gamma_i^k)\,,\qquad\Sigma(\mathcal{E}_k)=\bigcup_{i\in I_k}\mathrm{bd}\,(\gamma_i^k)=\bigcup_{j\in J_k}\{p_j^k\}\,.$$

Moreover, each p_j is the common boundary point of exactly three curves from $\{\gamma_i\}_{i\in I}$, and an analogous assertion holds for p_j^k and $\{\gamma_i^k\}_{i\in I}$.

Theorem 5.6. *Under the assumptions of Theorem 1.5, there exist positive constants ρ_0 and L, depending on Λ and $\mathcal{E}$ only, such that the following properties hold:*

(i) *there exists $k_0\in\mathbb{N}$ such that for each $k\ge k_0$, up to a relabeling of I_k and J_k, one has $I=I_k$ and $J=J_k$, with $\mathrm{bd}\,(\gamma_i)\ne\emptyset$ if and only if $\mathrm{bd}\,(\gamma_i^k)\ne\emptyset$ for every $i\in I$, and*

$$\lim_{k\to\infty}|p_j^k-p_j|+\mathrm{hd}(\gamma_i^k,\gamma_i)=0\,,\qquad\forall i\in I,j\in J\,; \tag{5.11}$$

moreover

$$\|\gamma_i^k\|_{C^{1,1}}\le L\,,\qquad\forall i\in I_k\,, \tag{5.12}$$

and if $p_j\in\mathrm{bd}\,(\gamma_i)$ then $p_j^k\in\mathrm{bd}\,(\gamma_i^k)$ with

$$\lim_{k\to\infty}|\nu^{co}_{\gamma_i}(p_j)-\nu^{co}_{\gamma_i^k}(p_j^k)|=0\,; \tag{5.13}$$

(ii) *for every $\rho<\rho_0$ there exist $k(\rho)\in\mathbb{N}$ and $\{\psi_k\}_{k\ge k(\rho)}\subset C^{1,1}([\partial\mathcal{E}]_\rho)$ such that*

$$[\partial\mathcal{E}_k]_{3\rho}\subset(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k\,,\qquad\forall k\ge k(\rho)\,, \tag{5.14}$$

where ν is a $C^{1,1}$-normal unit vector field to $\partial^*\mathcal{E}$ and

$$\lim_{k\to\infty}\|\psi_k\|_{C^1([\partial\mathcal{E}]_\rho)}=0\,,\qquad\sup_{k\ge k(\rho)}\|\psi_k\|_{C^{1,1}([\partial\mathcal{E}]_\rho)}\le L\,. \tag{5.15}$$

Proof. Step one: We prove statement (ii). By Theorem 4.12 (applied with $A=\mathbb{R}^2$ and A' equal to an open ball such that $\mathcal{E}(h)\subset\subset A'$ for every $h=1,...,N$) there exist $\rho_0,L>0$ such that for every $\rho\le\rho_0$ one can find $k(\rho)\in\mathbb{N}$, $\varepsilon(\rho)>0$ and $\{\psi_k\}_{k\ge k(\rho)}\subset C^{1,1}([\partial\mathcal{E}]_\rho)$ such that (5.15) holds, with

$$\partial\mathcal{E}_k\setminus I_{2\rho}(\Sigma(\mathcal{E}))\subset(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\subset\partial^*\mathcal{E}_k\,,\tag{5.16}$$

$$N_{\varepsilon(\rho)}([\partial\mathcal{E}]_\rho)\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\nu)([\partial\mathcal{E}]_\rho)\,,\tag{5.17}$$

for all $k\ge k(\rho)$. In turn, by Theorem 5.5 (applied with $A=\mathbb{R}^2$ and A' as above), we have $\mathrm{hd}(\Sigma(\mathcal{E}_k),\Sigma(\mathcal{E}))\to0$ as $k\to\infty$. Hence, up to increasing the value of $k(\rho)$ we find $\Sigma(\mathcal{E})\subset I_\rho(\Sigma(\mathcal{E}_k))$ for $k\ge k(\rho)$, and thus $[\partial\mathcal{E}_k]_{3\rho}=\partial\mathcal{E}_k\setminus I_{3\rho}(\Sigma(\mathcal{E}_k))\subset\partial\mathcal{E}_k\setminus I_{2\rho}(\Sigma(\mathcal{E}))$. Thus (5.14) follows from (5.16).

Step two: We prove (i) up to (5.11). We first note that if $x_k,y_k\in\Sigma(\mathcal{E}_k)$ with $x_k\ne y_k$ and $x_k\to x$ and $y_k\to y$ (so that $x,y\in\Sigma(\mathcal{E})$ by $\mathrm{hd}(\Sigma(\mathcal{E}_k),\Sigma(\mathcal{E}))\to0$), then it must be $x\ne y$. Indeed, if $x=y$, then $\varepsilon_k=|x_k-y_k|\to0$, and the sequence of clusters $\mathcal{F}_k=(\mathcal{E}_k-x_k)/\varepsilon_k$ would converge (up to subsequences and in the sense explained in Theorem 4.13) to a Steiner partition of $\mathbb{R}^2$. At the same time, this Steiner partition should have a singular point at unit distance from the origin, arising as the limit of of some subsequence of $(y_k-x_k)/\varepsilon_k$. This contradiction proves our remark, which coupled with the Hausdorff convergence of $\Sigma(\mathcal{E}_k)$ to $\Sigma(\mathcal{E})$ allows us to assume without loss of generality that $J=J_k$ with

$$\lim_{k\to\infty}|p_j^k-p_j|=0\,,\qquad\forall j\in J\,.\tag{5.18}$$

Let now I' and I'' be the sets of those $i\in I$ such that γ_i is homeomorphic, respectively, either to $\mathbb{S}^1$ or to $[0,1]$, and similarly define I'_k and I''_k starting from I_k. By intersecting with $N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})$ in (5.17) and by directly assuming that $\|\psi_k\|_{C^0([\partial\mathcal{E}]_{\rho_0})}\le\varepsilon(\rho_0)\le\rho_0$ for every $k\ge k_0=k(\rho_0)$ we find

$$N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\nu)([\gamma_i]_{\rho_0})\,,\qquad\forall i\in I\,,k\ge k_0\,.$$

In particular, by exploiting the connectedness of the curves $\{\gamma_k^i\}_{i\in I_k}$, one defines for every $k\ge k_0$ a map $\sigma_k:I\to I_k$ in such a way that

$$\begin{aligned}&(\mathrm{Id}+\psi_k\nu)([\gamma_i]_{\rho_0})\subset\gamma^k_{\sigma_k(i)}\,,\\&(\mathrm{Id}+\psi_k\nu)([\gamma_i]_{\rho_0})\cap\gamma^k_{i'}=\emptyset\,,\qquad\forall i\in I\,,\forall i'\in I_k\setminus\{\sigma_k(i)\}\,;\end{aligned}$$

hence,

$$(\mathrm{Id}+\psi_k\nu)([\gamma_i]_{\rho_0})=N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\cap\partial\mathcal{E}_k=N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\cap\gamma^k_{\sigma_k(i)}\,,\qquad\forall k\ge k_0\,,i\in I\,.\tag{5.19}$$

To complete the proof of (5.11) it will suffice to show that

$$\sigma_k\text{ is a bijection with }\sigma_k(I')=I'_k\text{ and }\sigma_k(I'')=I''_k\,,\tag{5.20}$$

$$\lim_{k\to\infty}\mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)})=0\,,\qquad\forall i\in I\,.\tag{5.21}$$

We start by choosing $\eta>0$ such that

$$I_\eta(\gamma_i)\cap I_\eta(\gamma_{i'})=\emptyset\,,\qquad\forall i,i'\in I'\,.\tag{5.22}$$

If $i\in I'$, then $[\gamma_i]_\rho=\gamma_i$ and $N_{\varepsilon(\rho)}(\gamma_i)=I_{\varepsilon(\rho)}(\gamma_i)$ for every $\rho>0$, so that (5.19) gives

$$(\mathrm{Id}+\psi_k\nu)(\gamma_i)=I_{\varepsilon(\rho_0)}(\gamma_i)\cap\partial\mathcal{E}_k=I_{\varepsilon(\rho_0)}(\gamma_i)\cap\gamma^k_{\sigma_k(i)}\,,\qquad\forall k\ge k_0\,,i\in I'\,.\tag{5.23}$$

Since $(\mathrm{Id}+\psi_k\nu)(\gamma_i)$ is homeomorphic to $\mathbb{S}^1$ and is contained in $\gamma^k_{\sigma_k(i)}$, by connectedness of $\gamma^k_{\sigma_k(i)}$ we conclude that $\sigma_k(i)\in I'_k$ with

$$(\mathrm{Id}+\psi_k\nu)(\gamma_i)=I_{\varepsilon(\rho_0)}(\gamma_i)\cap\partial\mathcal{E}_k=\gamma^k_{\sigma_k(i)}\,,\tag{5.24}$$

$$\mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)})\le\|\psi_k\|_{C^0([\partial\mathcal{E}]_{\rho_0})}\le\rho_0\,,\qquad\forall k\ge k_0\,,i\in I'\,.\tag{5.25}$$

By combining (5.22), (5.24) and (5.15) and up to requiring $\rho_0\le\eta$ we conclude that

$$(5.21)\text{ holds for every } i\in I',\ \sigma_k(I')\subset I'_k,\ \sigma_k \text{ is injective on } I'\,. \tag{5.26}$$

Before showing that $\sigma_k(I')=I'_k$, we first prove that

$$\sigma_k \text{ is a bijection between } I'' \text{ and } I''_k\,. \tag{5.27}$$

To this end, we shall first need to prove (5.28) and (5.32) below. In order to formulate (5.28) we introduce the following notation: given $j\in J$, let us denote by $a_j(1)$, $a_j(2)$, and $a_j(3)$ the three distinct elements in I'' such that the curves $\{\gamma_{a_j(\ell)}\}_{\ell=1}^3$ share p_j as a common boundary point (as described in Theorem 5.2), and let $\{a_j^k(\ell)\}_{\ell=1}^3\subset I''_k$ be defined analogously starting from p_j^k. We claim that, up to permutations in the index $\ell\in\{1,2,3\}$, one has

$$a_j^k(\ell)=\sigma_k(a_j(\ell))\,,\qquad \forall j\in J\,,k\ge k(\rho)\,,\ell\in\{1,2,3\}\,. \tag{5.28}$$

Indeed, by Theorem 5.2, up to decrease the value of $\eta>0$, we find that, for every $j\in J$,

$$\partial\mathcal{E}\cap B_{p_j,\eta}=\bigcup_{\ell=1}^3\gamma_{a_j(\ell)}\cap B_{p_j,\eta}\,,\qquad \{p_j\}=\Sigma(\mathcal{E})\cap B_{p_j,\eta}=\bigcup_{\ell=1}^3\mathrm{bd}\,(\gamma_{a_j(\ell)})\cap B_{p_j,\eta}\,. \tag{5.29}$$

Since $\varepsilon(\rho_0)\le\rho_0$, up to further decreasing ρ_0 depending on η, we can entail by Theorem 4.9 and (5.18) that

$$\partial\mathcal{E}_k\subset I_{\varepsilon(\rho_0)}(\partial\mathcal{E})\,,\qquad \Sigma(\mathcal{E}_k)\cap B_{p_j,\eta}=\{p_j^k\}\subset B_{p_j,\varepsilon(\rho_0)}\,,\qquad \forall j\in J\,,k\ge k_0\,. \tag{5.30}$$

By (5.29) and provided ρ_0 is small enough,

$$\begin{aligned} I_{\varepsilon(\rho_0)}(\partial\mathcal{E})\cap B_{p_j,\eta} &= \bigcup_{\ell=1}^3 I_{\varepsilon(\rho_0)}(\gamma_{a_j(\ell)})\cap B_{p_j,\eta}\\ &\subset B_{p_j,2\rho_0}\cup\bigcup_{\ell=1}^3\Big(N_{\varepsilon(\rho_0)}([\gamma_{a_j(\ell)}]_{\rho_0})\cap B_{p_j,\eta}\Big)\,,\qquad \forall j\in J\,. \end{aligned}$$

By $\partial\mathcal{E}_k\subset I_{\varepsilon(\rho_0)}(\partial\mathcal{E})$ and by (5.19) one thus finds

$$\partial\mathcal{E}_k\cap B_{p_j,\eta}\subset\Big(\partial\mathcal{E}_k\cap B_{p_j,2\rho_0}\Big)\cup\bigcup_{\ell=1}^3\Big(\gamma^k_{\sigma_k(a_j(\ell))}\cap B_{p_j,\eta}\Big)\,. \tag{5.31}$$

Let now ω be the connected component of $\gamma^k_{a_j^k(1)}\cap\mathrm{cl}\,(B_{p_j,\eta})$ which contains p_j^k. In this way, ω is a connected $C^{1,1}$-curve with boundary, homeomorphic to $[0,1]$, with $p_j^k\in\mathrm{bd}\,(\omega)\cap B_{p_j,\eta}$. It cannot be $\omega\subset\subset B_{p_j,\eta}$, because otherwise it would be $\omega=\gamma^k_{a_j^k(1)}\subset\subset B_{p_j,\eta}$, and thus $\Sigma(\mathcal{E}_k)\cap B_{p_j,\eta}\setminus\{p_j^k\}\ne\emptyset$, against (5.30). Hence $\omega\cap\partial B_{p_j,\eta}\ne\emptyset$. At the same time, by (5.31),

$$\omega\cap B_{p_j,\eta}\subset\Big(\omega\cap B_{p_j,2\rho_0}\Big)\cup\bigcup_{\ell=1}^3\Big(\omega\cap\gamma^k_{\sigma_k(a_j(\ell))}\cap B_{p_j,\eta}\Big)\,,$$

and since ω is connected with $\omega\cap\partial B_{p_j,\eta}\ne\emptyset$, it must be $\omega\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset$ for some $\ell\in\{1,2,3\}$, thus $\gamma^k_{a_j^k(1)}\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset$. Up to relabeling $\ell\in\{1,2,3\}$, we have thus proved that

$$\gamma^k_{a_j^k(\ell)}\cap\gamma^k_{\sigma_k(a_j(\ell))}\ne\emptyset\,,\qquad \forall j\in J\,,k\ge k_0\,,\ell\in\{1,2,3\}\,,$$

from which (5.28) follows by the connectedness of the curves $\{\gamma_i^k\}_{i\in I}$. Having proved (5.28), we now introduce the notation needed to formulate (5.32): given $i\in I''$, let $b_i(1)$ and $b_i(2)$ denote the two distinct elements of J such that $\mathrm{bd}\,(\gamma_i)=\{b_i(1),b_i(2)\}$, and define similarly $b_i^k(m)$ $(m=1,2)$ for each $i\in I''_k$. Then, up to permutations in the index $m\in\{1,2\}$,

$$b^k_{\sigma_k(i)}(m)=b_i(m)\,,\qquad \forall i\in I''\,,k\ge k_0\,,m=1,2\,. \tag{5.32}$$

Indeed, if $i\in I''$ then $i=a_{b_i(1)}(\ell)$ for some $\ell\in\{1,2,3\}$, therefore, by (5.28),

$$\sigma_k(i)=\sigma_k(a_{b_i(1)}(\ell))=a^k_{b_i(1)}(\ell)\,,$$

that is,

$$p^k_{b_i(1)}\in\mathrm{bd}\,(\gamma_{\sigma_k(i)})=\{p_{b^k_{\sigma_k(i)}(1)},p_{b^k_{\sigma_k(i)}(2)}\}\,,\qquad\text{thus}\qquad b_i(1)\in\{b^k_{\sigma_k(i)}(1),b^k_{\sigma_k(i)}(2)\}\,,$$

as required. With (5.28) and (5.32) in force, we now prove (5.27). The fact that $\sigma_k(I'')\subset I''_k$ is immediate from $I''=\{a_j(\ell):j\in J\,,\ell\in\{1,2,3\}\}$ and (5.28). If now $i,i'\in I''$ are such that $\sigma_k(i)=\sigma_k(i')$ then by (5.32)

$$\begin{aligned}\{j\in J:p_j\in\mathrm{bd}\,(\gamma_i)\}&=\{b_i(m)\}_{m=1}^2=\{b^k_{\sigma_k(i)}(m)\}_{m=1}^2=\{b^k_{\sigma_k(i')}(m)\}_{m=1}^2\\&=\{b_{i'}(m)\}_{m=1}^2=\{j\in J:p_j\in\mathrm{bd}\,(\gamma_{i'})\}\,,\end{aligned}$$

so that $\mathrm{bd}\,(\gamma_i)=\mathrm{bd}\,(\gamma_{i'})$, and thus $i=i'$; this proves that σ_k is injective on I''. Finally, by Remark 5.3, it must be $\#\,I''=(3/2)\,\#\,J=(3/2)\,\#\,J_k=\#\,I''_k$, so that σ_k is actually a bijection between I'' and I''_k, and (5.27) is proved.

Let us now show that

$$\lim_{k\to\infty}\mathrm{hd}(\gamma_i,\gamma^k_{\sigma_k(i)})=0\,,\qquad\forall i\in I''\,. \tag{5.33}$$

We first notice that, by (5.32),

$$\{j\in J:p^k_j\in\mathrm{bd}\,(\gamma^k_{\sigma_k(i)})\}=\{b^k_{\sigma_k(i)}(m)\}_{m=1}^2=\{b_i(m)\}_{m=1}^2=\{j\in J:p_j\in\mathrm{bd}\,(\gamma_i)\}\,,$$

so that (5.18) gives

$$\lim_{k\to\infty}\mathrm{hd}(\mathrm{bd}\,(\gamma_i),\mathrm{bd}\,(\gamma^k_{\sigma_k(i)}))=0\,,\qquad\forall i\in I''\,. \tag{5.34}$$

Next, if $i\in I''$, then by (5.23) one has $\gamma^k_{\sigma_k(i)}\cap I_{\varepsilon(\rho_0)}(\gamma_{i'})=\emptyset$ for every $i'\in I'$, while (5.19) gives $\gamma^k_{\sigma_k(i)}\cap N_{\varepsilon(\rho_0)}([\gamma_{i'}]_{\rho_0})=\emptyset$ for every $i'\in I''\setminus\{i\}$; since $\partial\mathcal{E}_k\subset I_{\varepsilon(\rho_0)}(\partial\mathcal{E})$ for $k\ge k_0$, we thus find

$$\gamma^k_{\sigma_k(i)}\subset I_{2\,\rho_0}(\gamma_i)\cup\bigcup_{i'\in I''}I_{2\rho_0}(\mathrm{bd}\,(\gamma_{i'}))\,,\qquad\forall i\in I''\,,k\ge k_0\,.$$

Since $I_{2\,\rho_0}(\gamma_i)$ is disjoint from $\bigcup_{i'\in I''}I_{2\rho_0}(\mathrm{bd}\,(\gamma_{i'}))$ thanks to (5.29), we conclude that $\gamma^k_{\sigma_k(i)}\subset I_{2\,\rho_0}(\gamma_i)$ for every $i\in I''$ and $k\ge k_0$. At the same time, by (5.19), (5.15), and (5.34)

$$[\gamma_i]_{\rho_0}\subset I_{\rho_0}(\gamma^k_{\sigma_k(i)})\,,\qquad I_{\rho_0}(\mathrm{bd}\,(\gamma_i))\subset I_{2\rho_0}(\gamma^k_{\sigma_k(i)})\,,\qquad\forall i\in I''\,,k\ge k_0\,,$$

that is, $\gamma_i\subset I_{2\rho_0}(\gamma^k_{\sigma_k(i)})$ for every $i\in I''$ and $k\ge k_0$. We have thus proved (5.33).

In order to complete the proof of (5.20) and (5.21) we are thus left to show that $\sigma_k(I')=I'_k$. We argue by contradiction, and assume the existence of $i_*\in I'_k\setminus\sigma_k(I')$. Since $I_{\varepsilon(\rho_0)}(\gamma_i)\cap\partial\mathcal{E}_k=\gamma^k_{\sigma_k(i)}$ for every $i\in I'$ (recall (5.24)), by connectedness we deduce that

$$\gamma^k_{i_*}\cap\bigcup_{i\in I'}I_{\varepsilon(\rho_0)}(\gamma_i)=\emptyset\,. \tag{5.35}$$

Since $N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\cap\partial\mathcal{E}_k=N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\cap\gamma^k_{\sigma_k(i)}$ for every $i\in I$ (recall (5.19)), if $\gamma^k_{i_*}\cap N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})\ne\emptyset$, then, by connectedness of $\gamma^k_{\sigma_k(i)}$, one finds $i_*=\sigma_k(i)\in\sigma_k(I)$, a contradiction: hence,

$$\gamma^k_{i_*}\cap\bigcup_{i\in I''}N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0})=\emptyset\,. \tag{5.36}$$

Since $\partial\mathcal{E}_k \subset I_{\varepsilon(\rho_0)}(\partial\mathcal{E}) = \bigcup_{i\in I} I_{\varepsilon(\rho_0)}(\gamma_i)$, by (5.35) and (5.36) we find

$$\gamma^k_{i_*} \subset \bigcup_{i\in I''} I_{\varepsilon(\rho_0)}(\gamma_i) \setminus \bigcup_{i\in I''} N_{\varepsilon(\rho_0)}([\gamma_i]_{\rho_0}) \subset \bigcup_{j\in J} B_{p_j,\eta}\,,$$

and since the balls $\{B_{p_j,\eta}\}_{j\in J}$ are disjoint by (5.29), we conclude that for every $i_* \in I'_k \setminus \sigma_k(I')$ there exists a unique $j \in J$ such that $\gamma^k_{i_*} \subset B_{p_j,2\rho_0}$; however, by Theorem 5.2,

$$\frac{1}{\Lambda} \le \mathrm{diam}(\gamma^k_{i_*}) < 2\rho_0\,,$$

which leads to a contradiction if ρ_0 is sufficiently small.

Step three: We prove (5.12). We directly consider the case when $\mathrm{bd}\,(\gamma^k_i) \ne \emptyset$, and omit the (analogous) details for the case $\mathrm{bd}\,(\gamma^k_i) = \emptyset$. Let us set $\ell^k_i = \mathcal{H}^1(\gamma^k_i)$, consider $\alpha^k_i \in C^{1,1}([0,\ell^k_i];\mathbb{R}^2)$ to be an arc-length parametrization of γ^k_i, and define unit normal vector fields $\nu^k_i \in C^{0,1}(\gamma_k;\mathbb{S}^1)$ by setting $\nu^k_i(\alpha^k_i(t)) = (\alpha^k_i)'(t)^\perp$, with the convention that $v^\perp = (v_2,-v_1)$ for every $v = (v_1,v_2) \in \mathbb{R}^2$. According to Definition 3.1, we just need to show that, up to further increasing the value of L

$$|\nu^k_i(x)\cdot(y-x)| \le L\,|x-y|^2\,, \qquad |\nu^k_i(x) - \nu^k_i(y)| \le L\,|x-y|\,, \qquad \forall x,y \in \gamma^k_i\,. \tag{5.37}$$

Indeed, if $x,y \in \gamma^k_i$ with $s,t \in [0,\ell^k_i]$ such that $x = \alpha^k_i(s)$ and $y = \alpha^k_i(t)$, then, by $\mathrm{Lip}\,((\alpha^k_i)') \le \Lambda$,

$$|\nu^k_i(x)\cdot(y-x)| \le C\,|s-t|^2\,, \qquad |\nu^k_i(x) - \nu^k_i(y)| \le C\,|s-t|\,;$$

we are thus left to show that

$$|s-t| \le C\,|\alpha^k_i(s) - \alpha^k_i(t)|\,, \qquad \forall s,t \in [0,\ell^k_i]\,. \tag{5.38}$$

If $|s-t| \le 1/\Lambda$, then (5.38) follows with $C \ge 2$ by noticing that

$$|\alpha^k_i(s) - \alpha^k_i(t)| = \Big|\int_t^s (\alpha^k_i)'(r)\,dr\Big| \ge |t-s| - \Lambda\,\frac{|t-s|^2}{2}\,,$$

once again thanks to $\mathrm{Lip}\,((\alpha^k_i)') \le \Lambda$. If $\gamma_i[x,y]$ denote the arc of γ_i with end-points $x,y \in \gamma_i$, then by compactness

$$\min_{i\in I}\,\inf\Big\{|x-y| : x,y \in \gamma_i\,, \mathcal{H}^1(\gamma_i[x,y]) \ge \frac{1}{2\Lambda}\Big\} \ge c\,,$$

where $c > 0$ depends on $\mathcal{E}$ and Λ only. Since for every $i \in I$ we have $\mathrm{hd}(\gamma^k_i,\gamma_i) \to 0$ as $k \to \infty$, we can thus entail

$$\min_{i\in I}\,\inf\Big\{|x-y| : x = \alpha^k_i(s)\,, y = \alpha^k_i(t)\,, |s-t| \ge \frac{1}{\Lambda}\Big\} \ge \frac{c}{2}\,,$$

so that (5.38) holds on $|s-t| > 1/\Lambda$ provided $C \ge 2\Lambda/c$. This completes the proof of (5.37), thus of (5.12).

Step four: We prove (5.13). Let us fix $j \in J$, and consider $p^k_j \in \Sigma(\mathcal{E}_k)$ and $i_1,i_2,i_3 \in I$ such that $\{p^k_j\} = \mathrm{bd}\,(\gamma^k_{i_1}) \cap \mathrm{bd}\,(\gamma^k_{i_2}) \cap \mathrm{bd}\,(\gamma^k_{i_3})$. Since each γ^k_i is a compact connected $C^{1,1}$-curve with distributional curvature bounded by Λ one finds that, for every $i = i_1,i_2,i_3$,

$$\lim_{r\to 0^+}\sup_{k\in\mathbb{N}} \mathrm{hd}_B\Big(\frac{\gamma^k_i - p^k_j}{r}, \mathbb{R}_+\,[\tau^k_i(p^k_j)]\Big) = 0\,, \tag{5.39}$$

where we have set $\mathbb{R}_+[\tau] = \{t\,\tau : t \ge 0\}$ for every $\tau \in \mathbb{S}^1$, and we have set

$$\tau_i = -\nu^{co}_{\gamma_i}\,, \qquad \tau^k_i = -\nu^{co}_{\gamma^k_i}\,,$$

for the sake of brevity. We thus find

$$\begin{aligned}\mathrm{hd}_B\Big(\mathbb{R}_+\,[\tau_i(p_j)],\mathbb{R}_+\,[\tau_i^k(p_j^k)]\Big) &\le \sup_{k\in\mathbb{N}}\mathrm{hd}_B\Big(\frac{\gamma_i^k-p_j^k}{r},\mathbb{R}_+\,[\tau_i^k(p_j^k)]\Big) \\ &\quad +\mathrm{hd}_B\Big(\frac{\gamma_i-p_j}{r},\mathbb{R}_+\,[\tau_i(p_j)]\Big)+2\,\frac{\mathrm{hd}(\gamma_i^k,\gamma_i+(p_j^k-p_j))}{r}\,,\end{aligned} \tag{5.40}$$

where we have also used the fact that, for k large enough,

$$\mathrm{hd}_B\Big(\frac{\gamma_i^k-p_j^k}{r},\frac{\gamma_i-p_j}{r}\Big)\le 2\,\frac{\mathrm{hd}(\gamma_i^k,\gamma_i+(p_j^k-p_j))}{r}\,. \tag{5.41}$$

At this point we can choose a sequence $r_k\to 0^+$, such that the right-hand side of (5.41) with $r=r_k$ is infinitesimal as $k\to\infty$. By also exploiting (5.11) and (5.39), this gives $\mathrm{hd}_B(\mathbb{R}_+\,[\tau_i(p_j)],\mathbb{R}_+\,[\tau_i^k(p_j^k)])\to 0$ as $k\to\infty$, that is (5.13). □

Proof of Theorem 1.5. Let $\mathcal{E}$ be a $C^{2,1}$-cluster in $\mathbb{R}^2$, $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a sequence of (Λ,r_0)-minimizing clusters such that $\mathrm{d}(\mathcal{E}_k,\mathcal{E})\to 0$ as $k\to\infty$, and let L, ρ_0 and, for each $\rho\le\rho_0$, $k(\rho)\in\mathbb{N}$, be the constants given by Theorem 5.6. Denote by μ_0 and C_0 the smallest and the largest constants, respectively, associated by Theorem 3.5 to some γ_i such that $\mathrm{bd}\,(\gamma_i)\ne\emptyset$. In this way, μ_0 and C_0 depend on Λ and $\mathcal{E}$ only. Up to further decreasing the value of μ_0, we can also assume that $\mu_0^2\le\rho_0$. Given $\mu<\mu_0$, we now want to find $k(\mu)\in\mathbb{N}$ such that for every $k\ge k(\mu)$ there exists a $C^{1,1}$-diffeomorphism f_k between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ with

$$\|f_k\|_{C^{1,1}(\partial\mathcal{E})} \le C_0\,, \tag{5.42}$$

$$\lim_{k\to\infty}\|f_k-\mathrm{Id}\|_{C^1(\partial\mathcal{E})} = 0\,, \tag{5.43}$$

$$\|\boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id})\|_{C^1(\partial^*\mathcal{E})} \le \frac{C_0}{\mu}\,\|f_k-\mathrm{Id}\|_{C^0(\Sigma(\mathcal{E}))}\,, \tag{5.44}$$

$$\boldsymbol{\tau}_{\mathcal{E}}(f_k-\mathrm{Id}) = 0\,,\quad\text{on }[\partial\mathcal{E}]_\mu\,. \tag{5.45}$$

Let us fix $i\in I$ such that $\mathrm{bd}\,(\gamma_i)\ne\emptyset$*.* Since $\mu^2<\mu_0^2\le\rho_0$, Theorem 5.6 ensures that $\{\gamma_i^k\}_{k\ge k_0}$ satisfies the assumptions (i) and (ii) of Theorem 3.5. By Theorem 3.5 for every $k\ge k(\mu)$ one finds a $C^{1,1}$-diffeomorphism f_i^k between γ_i and γ_i^k with $f_i^k(p_j)=p_j^k$, $f_i^k(p_{j'})=p_{j'}^k$ (j and j' as in statement (ii) of Theorem 5.6) and

$$\|f_i^k\|_{C^{1,1}(\gamma_i)} \le C_0\,, \tag{5.46}$$

$$\|(f_i^k-\mathrm{Id})\cdot\tau_i\|_{C^1(\gamma_i)} \le \frac{C_0}{\mu}\,\|f_i^k-\mathrm{Id}\|_{C^0(\mathrm{bd}\,(\gamma_i))}\,, \tag{5.47}$$

$$(f_i^k-\mathrm{Id})\cdot\tau_i = 0 \qquad\text{on }[\gamma_i]_\mu\,; \tag{5.48}$$

$$\lim_{k\to\infty}\|f_i^k-\mathrm{Id}\|_{C^1(\gamma_i)} = 0\,. \tag{5.49}$$

Let us now fix $i\in I$ such that $\mathrm{bd}\,(\gamma_i)=\emptyset$*.* Up to further decreasing μ_0, γ_i is a connected component of $[\partial\mathcal{E}]_\mu$, and thus by statement (ii) in Theorem 5.6, $\{\psi_k\}_{k\ge k(\rho)}\subset C^{1,1}([\partial\mathcal{E}_0]_\rho)$ are such that

$$\gamma_i^k=(\mathrm{Id}+\psi_k\nu)(\gamma_i)\,,\qquad \lim_{k\to\infty}\|\psi_k\|_{C^1(\gamma_i)}=0\,,\qquad \sup_{k\in\mathbb{N}}\|\psi_k\|_{C^{1,1}(\gamma_i)}\le C_0\,. \tag{5.50}$$

We set $f_i^k=\mathrm{Id}+\psi_k\,\nu$ for every $i\in I$ such that $\mathrm{bd}\,(\gamma_i)=\emptyset$, and finally define $f_k(x)=f_i^k(x)$ for $x\in\gamma_i$. The resulting map f_k defines a $C^{1,1}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ (see Definition 1.3) with (5.42)–(5.45) in force. □

6. Some applications of the improved convergence theorem

We now prove Theorem 1.9 and Theorem 1.10. To this end, let us notice that if $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of planar isoperimetric clusters with $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k)<\infty$, then there exist $x_k\in\mathbb{R}^2$ and a planar N-cluster $\mathcal{E}_0$ such that, up to extracting subsequences, $x_k+\mathcal{E}_k\to\mathcal{E}_0$. This is a simple consequence of (i) the inequality $2\,\mathrm{diam}(E)\le P(E)$, which holds for every indecomposable set of finite perimeter E in $\mathbb{R}^2$ (this, of course, after the normalization (4.4)); (ii) the fact that $\mathbb{R}^2\setminus\mathcal{E}(0)$ is indecomposable whenever $\mathcal{E}$ is an isoperimetric cluster (as it can be easily inferred by arguing as in Remark 5.4).

Proof of Theorem 1.9. We argue by contradiction, and assume that there exists a sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ of isoperimetric N-clusters with $\mathrm{vol}\,(\mathcal{E}_k)\to m_0$ such that $[\mathcal{E}_k]_\approx\ne[\mathcal{E}_j]_\approx$ whenever $k\ne j$. Let $\phi:\mathbb{R}^N_+\to(0,\infty)$ denote the infimum in (1.13), then it is easily seen that ϕ is locally bounded. In particular, $\sup_{k\in\mathbb{N}}P(\mathcal{E}_k)<\infty$, and thus there exists a N-cluster $\mathcal{E}_0$ and $x_k\in\mathbb{R}^2$ such that, up to extracting subsequences, $x_k+\mathcal{E}_k\to\mathcal{E}_0$ as $k\to\infty$. We claim that, for k large enough, $x_k+\mathcal{E}_k$ is a (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$, where Λ and r_0 are independent from k. To this end, let ε_0, r_0, and C_0 be the constants associated with $\mathcal{E}_0$ by Theorem B.1 and let k_0 be such that $\mathrm{d}(x_k+\mathcal{E}_k,\mathcal{E}_0)<\varepsilon_0$ for $k\ge k_0$. Given $\mathcal{F}$ with $\mathcal{F}(h)\Delta(x_k+\mathcal{E}_k(h))\subset\subset B_{x,r_0}$ for $h=1,...,N$, by applying Theorem B.1 with $\mathcal{E}=x_k+\mathcal{E}_k$ we find $\mathcal{F}_k'$ such that

$$\mathrm{vol}\,(\mathcal{F}_k')=\mathrm{vol}\,(x_k+\mathcal{E}_k)=\mathrm{vol}\,(\mathcal{E}_k)\,,\qquad P(\mathcal{F}_k')\le P(\mathcal{F})+C_0\,\mathrm{d}(x_k+\mathcal{E}_k,\mathcal{F})\,.$$

so that, by the isoperimetric property of $\mathcal{E}_k$, $P(x_k+\mathcal{E}_k)\le P(\mathcal{F}_k')\le P(\mathcal{F})+C_0\,\mathrm{d}(x_k+\mathcal{E}_k,\mathcal{F})$. Thus $x_k+\mathcal{E}_k$ is a (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$ for k large enough. By Theorem 4.9 we infer that $\mathcal{E}$ is also a (Λ,r_0)-minimizing cluster in $\mathbb{R}^2$, and thus conclude by Theorem 1.5 that $x_k+\mathcal{E}_k\approx\mathcal{E}$ for k large enough. Since $x_k+\mathcal{E}_k\approx\mathcal{E}_k$, we have found a contradiction to $[\mathcal{E}_k]_\approx\ne[\mathcal{E}_j]_\approx$ for $k\ne j$. □

Proof of Theorem 1.10. Step one: We first prove that, if $\mathcal{E}$ is a minimizer in (1.20) with $\delta\in(0,\delta_0)$ and $|m-m_0|<\delta_0$, then $\mathcal{E}\approx\mathcal{E}_0$. We argue by contradiction, and consider a sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ of minimizers in

$$\lambda_k=\inf\Big\{P(\mathcal{E})+\delta_k\sum_{h=1}^N\int_{\mathcal{E}(h)}g(x)\,dx:\mathrm{vol}\,(\mathcal{E})=m_k\Big\}\,,\qquad k\in\mathbb{N}\,,\tag{6.1}$$

where $\delta_k\to0$ and $m_k\to m_0$ as $k\to\infty$, and $[\mathcal{E}_k]_\approx\ne[\mathcal{E}_0]_\approx$ for every $k\in\mathbb{N}$. Let $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ be a sequence of isoperimetric clusters with $\mathrm{vol}\,(\mathcal{F}_k)=m_k$. Since $m_k\to m_0$ implies $\sup_{k\in\mathbb{N}}P(\mathcal{F}_k)<\infty$, by the argument presented at the beginning of this section there exists $R>0$ such that, up to translations, $\mathcal{F}_k(h)\subset\subset B_R$ for every $h=1,...,N$ and $k\in\mathbb{N}$. By comparing $\mathcal{E}_k$ and $\mathcal{F}_k$ in (6.1) we find

$$P(\mathcal{E}_k)+\delta_k\sum_{h=1}^N\int_{\mathcal{E}_k(h)}g\le P(\mathcal{F}_k)+\delta_k\sum_{h=1}^N\int_{\mathcal{F}_k(h)}g\le P(\mathcal{F}_k)+\delta_k\,|m_k|\sup_{B_R}g\tag{6.2}$$

and since $P(\mathcal{F}_k)\le P(\mathcal{E}_k)$ we thus find that for every $r>0$

$$\inf_{\mathbb{R}^2\setminus B_r}g\sum_{h=1}^N|\mathcal{E}_k(h)\setminus B_r|\le|m_k|\sup_{B_R}g\,.$$

By $g(x)\to\infty$ as $|x|\to\infty$, we conclude that

$$\lim_{r\to\infty}\sup_{k\in\mathbb{N}}\sum_{h=1}^N|\mathcal{E}_k(h)\setminus B_r|=0\,.\tag{6.3}$$

Since (6.2) also implies $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < \infty$, by (6.3) we conclude that up to extracting subsequences, $\mathrm{d}(\mathcal{E}_k,\mathcal{E})\to 0$ as $k\to\infty$, where $\mathcal{E}$ is a planar cluster with $\mathrm{vol}\,(\mathcal{E}) = m_0$. In particular, recalling that $\mathcal{E}_0$ denotes the unique isoperimetric cluster with $\mathrm{vol}\,(\mathcal{E}_0) = m_0$, we have

$$P(\mathcal{E}_0) \le P(\mathcal{E}) \le \liminf_{k\to\infty} P(\mathcal{E}_k)\,. \tag{6.4}$$

Now, by [Mag12, Theorem 29.14] there exist positive constants ε, η and C, a smooth map $\Phi \in C^1((-\eta,\eta)^N\times\mathbb{R}^2;\mathbb{R}^2)$, and a disjoint family of balls $\{B_{z_i,\varepsilon}\}_{i=1}^M$ such that, for every $v\in(-\eta,\eta)^N$, the N-cluster defined by $\mathcal{E}_{0,v}(h) := \Phi(v,\mathcal{E}_0(h))$, $h = 1,...,N$, satisfies

$$\mathcal{E}_{0,v}(h)\Delta\mathcal{E}_0(h) \subset\subset A = \bigcup_{i=1}^M B_{z_i,\varepsilon}\,,\quad P(\mathcal{E}_{0,v}) \le P(\mathcal{E}_0) + C\,|v|\,,\quad \mathrm{vol}\,(\mathcal{E}_{0,v}) = \mathrm{vol}\,(\mathcal{E}_0) + v\,.$$

For k large, $v_k = \mathrm{vol}\,(\mathcal{E}_k) - \mathrm{vol}\,(\mathcal{E}_0) \in (-\eta,\eta)^N$, so that $\mathrm{vol}\,(\mathcal{E}_{0,v_k}) = m_k$ and, by $g\ge 0$

$$P(\mathcal{E}_k) + \delta_k\sum_{h=1}^N\int_{\mathcal{E}_k(h)} g \le P(\mathcal{E}_{0,v_k}) + \delta_k\sum_{h=1}^N\int_{\mathcal{E}_{0,v_k}(h)} g \le P(\mathcal{E}_0) + C\,|v_k| + \delta_k\sup_{B_{2S}} g$$

where S is such that $\bigcup_{h=1}^N \mathcal{E}_0(h)\cup A\subset\subset B_S$. Letting $k\to\infty$ we find that

$$\limsup_{k\to\infty} P(\mathcal{E}_k) \le P(\mathcal{E}_0)\,,$$

so that, by (6.4), $P(\mathcal{E}) = P(\mathcal{E}_0)$. Since $\mathrm{vol}\,(\mathcal{E}) = m_0$, we find $\mathcal{E}\approx\mathcal{E}_0$ (through an isometry), and we may thus assume, without loss of generality, that $\mathcal{E} = \mathcal{E}_0$. By arguing as in the previous proof (with some minor modification because of the presence of the potential), we see that, for k large enough, $\mathcal{E}_k$ is a (Λ, r_0)-minimizer with Λ and r_0 uniform in k. Since $\mathrm{d}(\mathcal{E}_k,\mathcal{E}_0)\to 0$ as $k\to\infty$, by Theorem 1.5 we find that $\mathcal{E}_k\approx\mathcal{E}_0$ for k large enough, a contradiction.

Step two: The argument of step one can be easily adapted to show the existence of minimizers in (1.20), together with the existence of R_0 (depending on $\mathcal{E}_0$, δ_0 and g only) such that $\mathcal{E}(h)\subset B_{R_0}$ for every $h = 1,...,N$ and every minimizer $\mathcal{E}$. In particular, there exists C_0 depending on g and R_0 only such that

$$P(\mathcal{E}) \le P(\mathcal{F}) + C_0\,\delta\,\mathrm{d}(\mathcal{E},\mathcal{F})\,, \tag{6.5}$$

whenever $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{F})$ and $\mathcal{F}(h)\subset B_{2R_0}$. Let us fix $x_1, x_2\in\mathcal{E}(h,k)$, $T_i\in C^1_c(B_{x_i,r};\mathbb{R}^n)$ $(i = 1,2)$ with $|\mathcal{E}(j)\cap B_{x_i,r}| = 0$ if $i\ne h,k$ and $r < |x_1 - x_2|$, and with

$$\int_{\partial^*\mathcal{E}(h)} T_i\cdot\nu_{\mathcal{E}(h)}\,d\mathcal{H}^{n-1} = \eta_i > 0\,,\qquad \sup_{\mathbb{R}^n}|T_i| \le 1\,.$$

By a standard argument we can construct a one-parameter family of diffeomorphisms f_t with $f_t(x) = x + t\,(T_1(x) - (\eta_1/\eta_2)T_2(x)) + O(t^2)$ such that $\mathrm{vol}\,(f_t(\mathcal{E})) = \mathrm{vol}\,(\mathcal{E})$. For t small enough $\mathcal{F} = f_t(\mathcal{E})$ is admissible in (6.5), with

$$\mathrm{d}(\mathcal{E}, f_t(\mathcal{E})) \le 2|f_t(\mathcal{E}(h))\Delta\mathcal{E}(h)| \le 2\,P(\mathcal{E}(h); B_{x_1,r}\cup B_{x_2,r})\,|t|\,,$$

by Lemma B.2. Since

$$P(f_t(\mathcal{E})) = P(\mathcal{E}) + t\int_{\partial^*\mathcal{E}(h)} (T_1 - (\eta_1/\eta_2)T_2)\cdot\nu_{\mathcal{E}(h)}\,H_{\mathcal{E}(h,k)} + O(t^2)\,,$$

and $P(\mathcal{E}(h); B_{x_1,s}\cup B_{x_2,s}) = \omega_{n-1}\,s^{n-1}(1 + O(1))$ as $s\to 0^+$, by (6.5) we conclude that

$$\int_{\partial^*\mathcal{E}(h)} (T_1 - (\eta_1/\eta_2)T_2)\cdot\nu_{\mathcal{E}(h)}\,H_{\mathcal{E}(h,k)} \le 2\,C_0\,\delta\,\omega_{n-1}r^{n-1}(1 + O(1))\,.$$

Let now $T_i = T_i^j \to 1_{B_{x_i,r}}\nu_{\mathcal{E}(h)}$ in $L^1(\mathcal{H}^1 \llcorner \partial\mathcal{E}(h))$ as $j \to \infty$, so that

$$\int_{B_{x_1,r}\cap\partial^*\mathcal{E}(h)} H_{\mathcal{E}(h,k)} - \frac{\eta_1}{\eta_2}\int_{B_{x_1,r}\cap\partial^*\mathcal{E}(h)} H_{\mathcal{E}(h,k)} \le 2\,C_0\,\delta\,\omega_{n-1}r^{n-1}(1+O(1))\,.$$

By the mean value theorem, as $r \to 0^+$, we find that $H_{\mathcal{E}(h,k)}(x_1) - H_{\mathcal{E}(h,k)}(x_2) \le 2\,C_0\,\delta$, that is,

$$\max_{0\le h<k\le N} \|H_{\mathcal{E}(h,k)} - H^\delta_{h,k}\|_{C^0(H_{\mathcal{E}(h,k)})} \le C\,\delta\,,$$

for some $H^\delta_{h,k} \in \mathbb{R}$. At the same time, by arguing for example as in [CL12, Lemma 3.7(ii)], one see that $H_{\mathcal{E}(h,k)}$ has to converge in the sense of distributions to $H_{\mathcal{E}_0(h,k)}$ as $\delta \to 0^+$, and thus prove (1.21). □

Appendix A. Proof of Theorem 2.1

Proof of Theorem 2.1. In the following, we denote by C a generic constant depending on n, k, α, and L only. Let us set $\lambda_{\min}, \lambda_{\max} : S_0 \to \mathbb{R}$ as $\lambda_{\min}(x) = \inf\{|\nabla^{S_0} f(x)v| : v \in T_xS_0, |v| = 1\}$ and $\lambda_{\max}(x) = \|\nabla^{S_0} f(x)\|$. By (2.7) we find that

$$\frac{1}{L} \le J^{S_0} f(x) \le \lambda_{\min}(x)\,\lambda_{\max}(x)^{k-1} \le \lambda_{\min}(x)\,L^{k-1}\,,$$

that is $\lambda_{\min}(x) \ge L^{-k}$ for every $x \in S_0$. In particular, by also using (2.5) we find that

$$|\nabla^{S_0} f(x)(y-x)| = |\nabla^{S_0} f(x)\pi^{S_0}_x(y-x)| \ge \frac{|\pi^{S_0}_x(y-x)|}{L^k} \ge \frac{|y-x|}{2L^k}\,, \qquad \forall y \in B_{x,1/L}\cap S_0\,. \tag{A.1}$$

We now assume $\varepsilon_0 < 1/L$ and fix $y \in B_{x,\varepsilon_0} \cap S_0 \setminus \{x\}$. Since $\mathrm{dist}_{S_0}(x, y) > 0$ we can find $\gamma \in C^1([0, 1]; S_0)$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$\mathrm{dist}_{S_0}(x, y) \le \int_0^1 |\dot\gamma(t)|\,dt \le 2\ \mathrm{dist}_{S_0}(x, y)\,. \tag{A.2}$$

By (A.1),

$$\begin{aligned}|f(y) - f(x)| &= \Big|\nabla^{S_0} f(x)(y - x) - \int_0^1 (\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x))\dot\gamma(t)\,dt\Big|\\ &\ge \frac{|y-x|}{2L^k} - \int_0^1 \|\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x)\||\dot\gamma(t)|\,dt\,.\end{aligned}$$

By (2.7), (A.2), and (2.4)

$$\begin{aligned}\int_0^1 \|\nabla^{S_0} f(\gamma(t)) - \nabla^{S_0} f(x)\||\dot\gamma(t)|\,dt &\le L\int_0^1 |x-\gamma(t)|^\alpha\,|\dot\gamma(t)|\,dt\\ &\le L\int_0^1 \Big(\int_0^t |\dot\gamma(s)|\,ds\Big)^\alpha |\dot\gamma(t)|\,dt\\ &\le L\,2^{1+\alpha}\,\mathrm{dist}_{S_0}(x, y)^{1+\alpha} \le L\,(2L)^{1+\alpha}\,|x-y|^{1+\alpha}\,.\end{aligned}$$

We thus conclude (up to further decreasing the value of ε_0) that if $x \in S_0$ and $y \in B_{x,\varepsilon_0} \cap S_0$, then

$$|f(x) - f(y)| \ge |y-x|\left(\frac{1}{2L^k} - L\,(2L)^{1+\alpha}\varepsilon_0^\alpha\right) \ge \frac{|y-x|}{4L^k}\,. \tag{A.3}$$

This shows that f is injective on $B_{x,\varepsilon_0} \cap S_0$ for every $x \in S_0$. If now (2.8) is in force with $\rho_0 \le \varepsilon_0/4$, then by $\mathrm{diam}(S_0) \le L$ one finds that for every $x, y \in S_0$ with $|x - y| \ge \varepsilon_0$

$$|f(x) - f(y)| \ge |x-y| - |f(x) - x| - |f(y) - y| \ge \varepsilon_0 - 2\,\rho_0 \ge \frac{\varepsilon_0}{2} \ge \frac{\varepsilon_0}{2\,L}\,|x-y|\,,$$

so that, in conclusion, f is injective on S_0 with

$$|f^{-1}(p_1)-f^{-1}(p_2)|\le C\,|p_1-p_2|\,,\qquad \forall p_1,p_2\in S=f(S_0)\,. \tag{A.4}$$

We are thus left to prove that

$$\|\nabla^S f^{-1}(p_1)-\nabla^S f^{-1}(p_2)\|\le C\,|p_1-p_2|^\alpha\,,\qquad \forall p_1,p_2\in S\,. \tag{A.5}$$

Indeed, by (2.6), (2.7) and (A.4) we can entail

$$\|\pi^S_p-\pi^S_q\|\le C\,|p-q|^\alpha\,,\qquad \forall p,q\in S\,. \tag{A.6}$$

Let us now fix $p_1,p_2\in S$ and set

$$M_i=\nabla^S f^{-1}(p_i)\,,\quad \pi_i=\pi^S_{p_i}\,,\quad x_i=f^{-1}(p_i)\,,\quad N_i=\nabla^{S_0}f(x_i)\,,\quad \pi_i^0=\pi^{S_0}_{x_i}\,.$$

By exploiting the relations

$$\pi_i^0M_i=M_i=M_i\pi_i\,,\qquad \pi_iN_i=N_i=N_i\pi_i^0\,, \tag{A.7}$$

$$N_1M_1\pi_1=\pi_1\,,\qquad N_2M_2\pi_2=\pi_2\,,\qquad M_1N_1\pi_1^0=\pi_1^0\,,\qquad M_2N_2\pi_2^0=\pi_2^0\,, \tag{A.8}$$

one finds that

$$\begin{aligned}
&M_1(N_2-N_1)M_2+M_2(N_2-N_1)M_1\\
=\;&M_1N_2M_2-M_1N_1M_2+M_2N_2M_1-M_2N_1M_1\\
=\;&M_1N_2M_2\pi_2-M_1N_1\pi_1^0M_2+M_2N_2\pi_2^0M_1-M_2N_1M_1\pi_1\\
=\;&M_1\pi_2-\pi_1^0M_2+\pi_2^0M_1-M_2\pi_1\\
=\;&2(M_1-M_2)+(M_1+M_2)(\pi_2-\pi_1)+(\pi_2^0-\pi_1^0)(M_1+M_2)\,.
\end{aligned}$$

By (2.6) and (A.6), and since $\|M_i\|\le C$ by (A.4), we thus find

$$\begin{aligned}
2\|M_2-M_1\|&\le 2\,\|M_1\|\|M_2\|\|N_2-N_1\|+\|M_1+M_2\|\Big(\|\pi_2-\pi_1\|+\|\pi_2^0-\pi_1^0\|\Big)\\
&\le C\,\Big(\|N_2-N_1\|+|p_2-p_1|^\alpha+|x_2-x_1|^\alpha\Big)\\
&\le C\,\Big((1+L)\,|x_2-x_1|^\alpha+|p_2-p_1|^\alpha\Big)\le C\,|p_2-p_1|^\alpha\,,
\end{aligned}$$

where in the last line we have first used $[\nabla^{S_0}f]_{C^{0,\alpha}(S_0)}\le L$ and then (A.4). This completes the proof of (A.5), thus of the theorem. □

Appendix B. Volume-fixing variations

Comparison sets used in variational arguments usually arise as compactly supported perturbations of the considered minimizer. In order to use these constructions in volume constrained variational problems, one needs to restore changes in volume due to such local variations. In the study of minimizing clusters, this kind of tool is provided in [Alm76, Proposition VI.12]; see also [Mag12, Section 29.6]. The following theorem is a version of Almgren's result which is suitably adapted to the problems considered in here. In particular, it adds to [Mag12, Corollary 29.17] the conclusions (B.6) and (B.7).

Theorem B.1 (Volume-fixing variations)**.** *If $\mathcal{E}_0$ is a N-cluster in $\mathbb{R}^n$, then there exist positive constants r_0, ε_0, R_0 and C_0 (depending on $\mathcal{E}_0$) with the following property: if $\mathcal{E}$ and $\mathcal{F}$ are N-clusters in $\mathbb{R}^n$ with*

$$\mathrm{d}(\mathcal{E},\mathcal{E}_0)\;\le\;\varepsilon_0\,, \tag{B.1}$$

$$\mathcal{F}(h)\Delta\mathcal{E}(h)\;\subset\subset\;B_{x,r_0}\,,\qquad \forall h=1,...,N\,, \tag{B.2}$$

for some $x \in \mathbb{R}^n$, then there exists a N-cluster $\mathcal{F}'$ such that

$$\begin{aligned} \mathcal{F}'(h)\Delta\mathcal{F}(h) &\subset\subset B_{R_0} \setminus \overline{B_{x,r_0}}\,, \qquad \forall h = 1, ..., N\,, & \text{(B.3)}\\ \mathrm{vol}\,(\mathcal{F}') &= \mathrm{vol}\,(\mathcal{E})\,, & \text{(B.4)}\\ |P(\mathcal{F}') - P(\mathcal{F})| &\le C_0\, P(\mathcal{E})\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,, & \text{(B.5)}\\ |\mathrm{d}(\mathcal{F}', \mathcal{E}) - \mathrm{d}(\mathcal{F}, \mathcal{E})| &\le C_0\, P(\mathcal{E})\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,. & \text{(B.6)} \end{aligned}$$

Moreover, if $g : \mathbb{R}^n \to [0, \infty)$ is locally bounded, then

$$\sum_{h=0}^{N} \int_{\mathcal{F}'(h)\Delta\mathcal{F}(h)} g \le C_0\, \|g\|_{L^\infty(B_R)}\, P(\mathcal{E})\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E})|\,. \tag{B.7}$$

We shall need the following slight refinement of [Mag12, Lemma 17.9].

Lemma B.2. *If $g : \mathbb{R}^n \to [0, \infty)$ is locally bounded, E is a set of locally finite perimeter in an open set A and $T \in C^1_c(A; \mathbb{R}^n)$, then for every $\eta > 0$ there exist $K \subset A$ compact and $\varepsilon > 0$ (depending on T) such that if $\{f_t\}_{|t|<\varepsilon}$ is a flow with initial velocity T, then*

$$\int_{f_t(E)\Delta E} g \le (1 + \eta)\, \|T\|_{C^0(\mathbb{R}^n)}\, \|g\|_{L^\infty(K)}\, P(E; K)\, |t|\,, \qquad \forall |t| < \varepsilon\,. \tag{B.8}$$

Proof. Since $(d(f_t)^{-1}/dt)|_{t=0} = -T$, if we set $\Phi_{s,t}(x) = s\,x + (1 - s)\,(f_t)^{-1}(x)$ for $x \in \mathbb{R}^n$ and $s \in (0, 1)$, then for every $\eta > 0$ there exists $\varepsilon > 0$ such that $\{\Phi_{s,t}\}_{|t|<\varepsilon}$ is a family of diffeomorphism on $\mathbb{R}^n$ with

$$\inf_{x\in\mathbb{R}^n} J\Phi_{s,t}(x) \ge 1 - \eta\,, \qquad \|\mathrm{Id} - (f_t)^{-1}\|_{C^0(\mathbb{R}^n)} \le (1 + \eta)\, |t|\, \|T\|_{C^0(\mathbb{R}^n)}\,, \qquad \forall |t| < \varepsilon\,.$$

Let $K \subset A$ compact be such that $\{f_t \neq \mathrm{Id}\} \subset K$ for every $|t| < \varepsilon$. By Fubini's theorem and by the area formula, if $u \in C^1(\mathbb{R}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} g\, |u - u((f_t)^{-1})| &\le (1 + \eta)\, |t|\, \|T\|_{C^0(\mathbb{R}^n)} \int_K g(x)\, dx \int_0^1 |\nabla u(\Phi_{s,t}(x))|\, ds\\ &= (1 + \eta)\, |t|\, \|T\|_{C^0(\mathbb{R}^n)} \|g\|_{L^\infty(K)} \int_0^1 ds \int_K \frac{|\nabla u(y)|}{J\Phi_{s,t}(\Phi^{-1}_{s,t}(y))}\, dy\\ &\le \frac{1 + \eta}{1 - \eta}\, |t|\, \|T\|_{C^0(\mathbb{R}^n)} \|g\|_{L^\infty(K)} \int_K |\nabla u|\,. \end{aligned}$$

By [Mag12, Theorem 13.8] there exists $\{u_h\}_{h\in\mathbb{N}} \subset C^1(\mathbb{R}^n)$ such that $u_h \to 1_E$ a.e. on A and $\limsup_{h\to\infty} \int_K |\nabla u_h| \le P(E; \overline{K})$. Since $|u_h - u_h((f_t)^{-1})| \to 1_{E\Delta f_t(E)}$ a.e. on A, we conclude the proof by Fatou's lemma. □

Proof of Theorem B.1. One repeats the proof of [Mag12, Corollary 29.17], exploiting Lemma B.2 in place of [Mag12, Lemma 17.9] in order to obtain (B.6) and (B.7). We thus omit the details. □

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