Interpolations and Fractional Sobolev Spaces in Carnot Groups

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Abstract

In this paper we present an interpolation approach to the fractional Sobolev spaces in Carnot groups using the K-method. This approach provides us with a different characterization of these Sobolev spaces, moreover, it provides us with the limiting behavior of the fractional Sobolev norms at the end-points. This allows us to deduce results similar to the Bourgain-Brezis-Mironescu and Maz'ya-Shaposhnikova in the case $p > 1$ and Dávila's result in the case $p = 1$. Also, this allows us to deduce the limiting behavior of the fractional perimeter in Carnot groups.

1 Introduction

Carnot groups appear as the first level extension of the classical Euclidean spaces, in the sense that they are modeled over $\mathbb{R}^n$ but with a different group structure. Nevertheless, they share many analytical properties with the Euclidean case. The typical example of Carnot group is the classical Heisenberg group. Lately, there have been a lot of interest in PDEs and fractional PDEs in this group coming from a geometric background since it is the flat context of CR-geometry, see for instance [22, 23, 34, 27] and the references therein. Moreover, Carnot groups have also been largely studied in several respects, such as differential geometry [13], subelliptic differential equations [11, 18, 17, 36] and complex variables [39]. For a general introduction to Carnot groups from the point of view of the present paper and for further examples, we refer, e.g., to [11, 18, 39].

It is natural then to investigate to which extent one can generalize to Carnot groups the analytical tools that are well understood in the Euclidean case, see for instance [16, 28].

In this setting, we propose to study fractional Sobolev spaces from an interpolation point of view. Fractional Sobolev spaces in the literature, are also called Aronszajn, Gagliardo or Slobodeckij spaces, by the name of the ones who introduced them, almost simultaneously [2, 25, 37]. In Carnot groups fractional Sobolev spaces have been introduced and studied in [18, 17] and many different characterizations are now present, such as the ones in [35]. In the present paper we use the K-method for real interpolation, see for instance [4], to give an alternative characterization of fractional Sobolev spaces in Carnot groups. As a consequence, we derive a Bourgain-Brezis-Mironescu [5, 6, 7] (Theorem 5.1) and Ma'zya-Shapashnikova type limiting behavior (Theorem 5.2) of the Sobolev norms similarly to the approach developed in [29]. We point out that the exact limit of the fractional Sobolev norm (as the fractional parameter goes to

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1) was investigated in [3] using exact and technical computations. We also bring to the reader's attention the extensions of these type of results to other settings and to different functionals as in [1, 7, 8, 9, 10, 30, 31, 32]. For $p = 1$ we provide a limiting behavior leading to the space of BV-functions that are of great interest in geometric measure theory in the setting of Carnot groups, see [19, 20, 21]. This will allow us to characterize the fractional perimeter in Carnot groups and understand its limiting behavior when the fractional parameter goes to 1, as it was done in the Euclidean setting in [15, 33].

This manuscript is structured as follows: First, in Section 2, we present the structure of Carnot groups and define Sobolev Spaces and BV-Spaces in this setting. In Section 3, we provide the necessary notations, definitions and properties of the K-interpolation, which will be the main tool in our investigation. In Section 4, we provide another characterization of the K function in Carnot groups. This allows us to deduce an alternative characterization of the fractional Sobolev spaces. Finally, in Section 5, we provide applications of the characterizations given in Section 3. Namely, we present the limiting behavior of the Fractional Sobolev norms in the two end points, allowing us to obtain results similar to the ones already proved by Bourgain-Brezis-Mironescu and by Ma'zya-Shaposhnikova in [5, 6, 7]for the case $p > 1$ and by Davila in [15] for the case $p = 1$. Also, we provide an alternative definition and characterization to the fractional perimeter and its limiting behavior at the end-points as in [15, 33].

2 Carnot groups

A connected and simply connected stratified nilpotent Lie group $(\mathbb{G}, \cdot)$ is said to be a *Carnot group of step k* if its Lie algebra $\mathfrak{g}$ admits a *step k stratification*, i.e., there exist linear subspaces $V_1, ..., V_k$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

Set $m_i = \dim(V_i)$, for $i = 1, \ldots, k$ and $h_i = m_1 + \cdots + m_i$, so that $h_k = n$. For sake of simplicity, we write also $h_0 = 0$, $m := m_1$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e., we set

$$Q := \sum_{i=1}^{k} i \dim(V_i).$$

We choose now a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e., such that $e_{h_{j-1}+1}, \ldots, e_{h_j}$ is a basis of V_j for each $j = 1, \ldots, k$. Moreover, let $X = \{X_1, \ldots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \ldots, n$. The exponential mapping $\exp\colon \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism. Given a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ adapted to the stratification, any $x \in \mathbb{G}$ can be written in a unique way as

$$x = \exp(x_1 X_1 + \ldots + x_n X_n) = e^{x_1 X_1 + \ldots + x_n X_n}.$$

We identify x with $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and hence $\mathbb{G}$ with $\mathbb{R}^n$. This is known as *exponential coordinates of the first kind*

The sub-bundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \ldots, X_m$ is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x), \ldots, X_m(x)\}, \qquad x \in \mathbb{G}.$$

We can endow each fiber of $H\mathbb{G}$ with an inner product $\langle \cdot, \cdot \rangle$ and with a norm $|\cdot|$ that make the basis $X_1(x), \dots, X_m(x)$ an orthonormal basis. For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda \xi_1, \dots, \lambda^k \xi_k), \tag{2.2}$$

where $x = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k} \equiv \mathbb{G}$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is Lebesgue measurable, we write $|A|$ to denote its Lebesgue measure.

Once an orthonormal basis $X_1, \dots, X_m$ of the horizontal layer is fixed, we define, for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist, the horizontal gradient of f, denoted by $\nabla_H f$, as the horizontal section

$$\nabla_H f := \sum_{i=1}^{m} (X_i f) X_i,$$

whose coordinates are $(X_1 f, ..., X_m f)$. If $\varphi = (\varphi_1, \dots, \varphi_m) \in C_c^1(\mathbb{G}, \mathbb{R}^m)$ we put

$$\mathrm{div}_{\mathbb{G}} \varphi = \sum_{i=1}^{m} X_i \varphi_i.$$

Let $|\cdot| : \mathbb{G} \to [0, \infty)$ denote a symmetric homogeneous norm on $\mathbb{G}$ [11]. Since any two continuous homogeneous norm are equivalent [11], from now on we denote by $|\cdot|$ any one of them. We denote by

$$B_r(x) = \{y \in \mathbb{G} : |y^{-1} \cdot x| < r\}$$

the ball centered at $x \in \mathbb{G}$ with radius $r > 0$ and by $B_r = B(0, r)$.

We are now in position to introduce Sobolev and BV functions in Carnot groups.

Definition 2.1 *Let $1 \le p < \infty$. We define the horizontal Sobolev space $W^{1,p}(\mathbb{G})$ as*

$$W^{1,p}(\mathbb{G}) = \{f \in L^p(\mathbb{G}) \mid X_i f \in L^p(\mathbb{G}),\ i = 1, \dots, m\}.$$

endowed with the norm

$$\|f\|_{W^{1,p}} = \|f\|_{L^p(\mathbb{G})} + \|\nabla_H f\|_{L^p(\mathbb{G})}.$$

We also define $\dot{W}^{1,p}(\mathbb{G})$ as the closure of $C_0^\infty(\mathbb{G})$ in the norm

$$\|f\|_{1,p} = \|\nabla_H f\|_{L^p(\mathbb{G})}.$$

Finally, $BV(\mathbb{G})$ denotes the set of functions $f \in L^1(\mathbb{G})$ such that

$$|D_{\mathbb{G}} f|(\mathbb{G}) := \sup \left\{ \int_{\mathbb{G}} f \mathrm{div}_{\mathbb{G}} \varphi \, dx \mid \varphi \in C_c^1(\mathbb{G}, \mathbb{R}^m),\ \|\varphi\|_\infty \le 1 \right\}$$

is finite. Moreover, if the characteristic function χ_E of the measurable set $E \subset \mathbb{G}$ belongs to $BV(\mathbb{G})$ we say that E has finite intrinsic perimeter and we write $Per_{\mathbb{G}}(E)$ instead of $|D_{\mathbb{G}} \chi_E|(\mathbb{G})$,

The following results are well-known, we refer to [21, 24] for a proof.

Theorem 2.1 *For any $f \in BV(\mathbb{G})$, the following identity holds:*

$$|D_{\mathbb{G}} f|(\mathbb{G}) = \int_{\mathbb{R}} Per_{\mathbb{G}}(\{x \in \Omega \mid f(x) > t\}) \, dt$$

Theorem 2.2 *Let $f \in BV(\mathbb{G})$. Then there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{G})$ such that*

1. *$f_k \to f$ in $L^1(\mathbb{G})$;*
2. *$|D_{\mathbb{G}} f_k|(\mathbb{G}) \to |D_{\mathbb{G}} f|(\mathbb{G})$.*

We conclude this section recalling the definition of horizontal fractional Sobolev spaces

Definition 2.2 *Let $0 < s < 1$ and $1 \le p < \infty$. We define $\dot{W}^{s,p}(\mathbb{G})$ as the closure of $C_c^\infty(\mathbb{G})$ under the norm*

$$\|f\|_{\dot{W}^{s,p}} = \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|^p}{|y^{-1}x|^{Q+sp}} dxdxy \right)^{\frac{1}{p}}$$

3 Real Interpolation Theory and the K-method

In this section we recall some important notions in interpolation theory, we refer the interested reader to [4] for a nice introduction to the subject.
We will use the following notations: given two quantities f and g,

- We say that $f \lesssim g$ if there exists $C > 0$ such that $f \le Cg$.
- We say that $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

For the rest of the paper, the constant C with only depend of the data of $\mathbb{G}$ and on $1 \le p < \infty$.

Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be Banach spaces both continuously embedded in some Banach space C. We refer to the couple (A, B) as an interpolation pair. Let us consider the Banach spaces $(A \cap B, \|\cdot\|_{A\cap B})$ and $(A + B, \|\cdot\|_{A+B})$ where

$$\|a\|_{A\cap B} := \max\{\|a\|_A,\ \|a\|_B\}$$

and

$$\|a\|_{A+B} := \inf_{a=a_1+a_2} \left(\|a_1\|_A + \|a_2\|_B\right).$$

The space $A + B$ can be equivalently renormed by Peetre's K-functional,

$$K(t, a) = K(t, a, A, B) := \inf_{a=a_1+a_2} \left(\|a_1\|_A + t\|a_2\|_B\right).$$

for any $t > 0$. A vector space D is called intermediate if $A \cap B \subset D \subset A + B$ and the inclusions are continuous embeddings if D is topologized. An intermediate space D is an interpolation space is all linear operator on $A \cap B$ which map A continuously into itself and B continuously into itself also map D into itself.
Let (A, B) be an interpolation pair and let $0 < s < 1$ and $q \in [1, \infty)$. We define the interpolation space

$$(A, B)_{s,q} := \{f \in A + B \mid |f|_{s,q} < \infty\}$$

where

$$|f|_{s,q} := s^{\frac{1}{q}}(1 - s)^{\frac{1}{q}} q^{\frac{1}{q}} \left(\int_0^\infty \left(t^{-s} K(t, f, A, B)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Definition 3.1 *We say that an interpolation pair (A, B) is normal if the following condition holds:*

1. $\lim_{t\to 0} \frac{K(t,f,A,B)}{t} = \|f\|_B$, *for* $f \in B$;
2. $\lim_{t\to\infty} K(t,f,A,B) = \|f\|_A$, *for* $f \in A$;

The equalities above can be relaxed to $\approx$.

The following Theorem has been proved in [29, Theorem 1]

Theorem 3.1 *Let* (A, B) *be an interpolation pair. Then*

1. *For* $1 \le q < \infty$ *and* $f \in A \cap B$ *we have* $\lim_{s\to 1}[f]_{s,q} = \|f\|_B$;
2. *For* $1 \le q < \infty$ *and* $f \in A \cap B$ *we have* $\lim_{s\to 0}[f]_{s,q} = \|f\|_A$;
3. *For* $1 \le q < \infty$ *and* $f \in A \cap \bigcup_{s\in(0,1)}(A,B)_{s,q}$ *we have* $\lim_{s\to 0}[f]_{s,q} = \|f\|_A$.

4 The Modulus of Continuity

Consider a Carnot group $\mathbb{G}$ of homogeneous dimension Q and let $V = V_1$ be its first layer. Given $f \in L^p(\mathbb{G})$, where $1 \le p < \infty$ is fixed, we define the modulus of continuity $\omega_p(f,t)$ by

$$\omega_p(f,t) = \sup_{X\in V;|X|\le t} \Big(\int_{\mathbb{G}} |f(e^X x) - f(x)|^p dx\Big)^{\frac{1}{p}} = \sup_{y\in\exp(V);|y|\le t} \Big(\int_{\mathbb{G}} |f(yx) - f(x)|^p dx\Big)^{\frac{1}{p}}.$$

We set $\Delta_X f(x) = f(e^X x) - f(x)$ and if $h = e^X$ then we write $\Delta_h f(x) = f(hx) - f(x)$.

Proposition 4.1 *Given* $1 \le p < \infty$, *we have*

$$K(t, f, L^p, W^{1,p}) \approx \omega_p(f,t) + \min(1,t)\|f\|_{L^p}.$$

Proof: For the sake of notation we will write $K(t,f)$ instead of $K(t,f,L^p,W^{1,p})$. Recall that

$$K(t,f) = \inf\{\|f_0\|_{L^p} + t\|f_1\|_{W^{1,p}}; f_0 \in L^p, f_1 \in W^{1,p}; f = f_0 + f_1\}.$$

It follows then that

$$\min(1,t)\|f\|_{L^p} \le \|f_0\|_{L^p} + t\|f_1\|_{L^p} \le \|f_0\|_{L^p} + t\|f_1\|_{W^{1,p}}.$$

Thus,

$$\min(1,t)\|f\|_{L^p} \le K(t,f). \tag{4.1}$$

Let $X \in V$ with $|X| \le t$. One can easily see that

$$\|\Delta_X f\|_{L^p} \le \|\Delta_X f_0\|_{L^p} + \|\Delta_X f_1\|_{L^p} \le 2\|f_0\|_{L^p} + \|\Delta_X f_1\|_{L^p}.$$

Also we have that

$$\begin{aligned}\Delta_X f_1(x) &= \int_0^1 \frac{\partial}{\partial r} f_1(e^{rX}x)dr \\ &= \int_0^1 (Xf_1)(e^{rX}x)dr.\end{aligned}$$

Hence,

$$\|\Delta_X f_1(x)\|_{L^p} \le |X|\|\nabla_H f_1\|_{L^p} \le t\|\nabla_H f_1\|_{L^p}. \tag{4.2}$$

Therefore, by (4.1) and (4.2)

$$\omega_p(f,t) + \min(1,t)\|f\|_{L^p} \lesssim K(t,f).$$

We move now to the proof of the reverse inequality:
We identify V to $\mathbb{R}^k$ via the Jacobian basis $(X_1,\ldots,X_k)$ and let U be its unit square. That is every vector X in V will be written as $X = \sum_{i=1}^k a_i X_i$ for some $a_i \in \mathbb{R}$. if X is in the unit square then $0 \le a_i \le 1$. Let $t > 0$ and consider the function

$$f_0(x) = -\int_{[0,1]^k} \Delta_{t\sum_{i=1}^k a_i X_i} f(x) da_1 \cdots da_k.$$

Then one have

$$\|f_0\|_{L^p} \le \omega_p(f,t).$$

On the other hand, we have $f_1 = f - f_0 = \int_{[0,1]^k} f(e^{t\sum_{i=1}^k a_i X_i} x) da_1 \cdots da_k$. Therefore,

$$\|f_1\|_{L^p} \le \|f\|_{L^p}.$$

Next, we write $\hat{X}_j = \sum_{i=1, h\neq j}^k a_i X_i$. Notice that

$$\frac{\partial}{\partial a_j}\Big(f(e^{\sum_{i=1}^k ta_i X_i} x)\Big) = t(X_j f)(e^{\sum_{i=1}^k ta_i X_i} x).$$

Hence,

$$\begin{aligned}
X_j f_1 &= \int_{[0,1]^k} (X_j f)(e^{tX} x) da_1 \cdots da_k \\
&= \int_{[0,1]^{k-1}} \int_0^1 \frac{1}{t}\frac{\partial}{\partial a_j}\Big(f(e^{tX} x)\Big) da_j da_1 \cdots da_{j-1} da_{j+1} \cdots da_k \\
&= \frac{1}{t}\int_{[0,1]^{k-1}} f(e^{tX} x) - f(e^{t\hat{X}_j} x) da_1 \cdots da_{j-1} da_{j+1} \cdots da_k \\
&= \frac{1}{t}\int_{[0,1]^{k-1}} \Delta_{tX_j} f(e^{t\hat{X}_j} x) da_1 \cdots da_{j-1} da_{j+1} \cdots da_k.
\end{aligned}$$

Hence,

$$\|\nabla_H f_1\|_{L^p} \lesssim \frac{1}{t}\omega_p(f,t),$$

leading to

$$K(t,f) \lesssim \min(1,t)\|f\|_{L^p} + \omega_p(f,t).$$

□

It is easy to see from the proof above that

$$\dot{K}(t,f) = K(t,f,L^p,\dot{W}^{1,p}) \approx \omega_p(f,t).$$

Here the interpolation is to be understood for the pair $(L^p, \dot{W}^{1,p})$ modulo constants, which makes them two Banach spaces.

Lemma 4.1 *Define the total modulus $\overline{\omega}_p$ by*

$$\overline{\omega}_p(f,t) = \sup_{y\in\mathbb{G};|y|\le t}\Big(\int_{\mathbb{G}}|f(yx)-f(x)|^p dx\Big)^{\frac{1}{p}}.$$

Then there exists $A>0$ and $C>0$ such that

$$\omega_p(f,t) \le \overline{\omega}_p(f,t) \le C\omega_p(f,At).$$

In particular

$$K(f,t) \approx \overline{\omega}_p(f,t) + \min(1,t)\|f\|_{L^p}.$$

Proof: First notice that

$$\omega_p(f,t) \le \overline{\omega}_p(f,t).$$

Next, we recall that, there exists $A>0$ and $N\in\mathbb{N}$ such that for all $y\in\mathbb{G}$, there exists $y_1,\cdots,y_N \in exp(V)$ such that $y = y_1y_2\cdots y_N$ and $|y_j|\le A|y|$ for $1\le j\le N$. Thus we write

$$|f(yx)-f(x)| \le |f(y_1\cdots y_N x) - f(y_2\cdots y_N x)| + \cdots + |f(y_N x) - f(x)|.$$

Hence,

$$\|f(yx)-f(x)\|_{L^p} \le \sum_{j=1}^{N}\|f(y_jx)-f(y_j)\|_{L^p}.$$

Passing to the sup in the previous inequality gives

$$\overline{\omega}_p(f,t) \le N\omega_p(f,At).$$

□

Proposition 4.2 *Given $f\in L^p(\mathbb{G})$. then*

$$\overline{\omega}_p(f,t) \approx \Big(\frac{1}{t^Q}\int_{|h|\le t}\|\Delta_h f\|_{L^p}^p dh\Big)^{\frac{1}{p}}.$$

Proof: Let $\eta\in C_c^\infty(B_1)$ such that $\int_{\mathbb{G}}\eta = 1$. We write

$$\begin{aligned} f(x) &= \frac{1}{t^Q}\Big(\int_{B_t} f(yx)\eta(\delta_{t^{-1}}y)dy + \int_{B_t}(f(x)-f(yx))\eta(\delta_{t^{-1}}y)dy\Big) \\ &= I_1(x,t) + I_2(x,t). \end{aligned}$$

It follows that

$$|f(hx)-f(x)| \le |I_1(hx,t) - I_1(x,t)| + |I_2(hx,t)| + |I_2(x,t)|.$$

Now,

$$|I_2(hx,t)| + |I_2(x,t)| \lesssim \frac{\|\eta\|_\infty}{t^Q}\int_{B_t}|\Delta_y f(x)| + |\Delta_y f(hx)|dy.$$

On the other hand, we notice that

$$\begin{aligned} \nabla_H I_1(x,t) &= -\frac{1}{t^{Q+1}}\int_{B_t(x)} f(y)\nabla_H\eta(yx^{-1})dy \\ &= -\frac{1}{t^{Q+1}}\int_{B_t(x)}(f(y)-f(x))\nabla_H\eta(yx^{-1})dy. \end{aligned}$$

But, if $h = e^X$ and $X \in V$, we have

$$\begin{aligned} I_1(hx,t) - I_1(x) &\lesssim \int_0^1 |h||\nabla_H I_1|(e^{rX}x,t)dr \\ &\lesssim |h|\frac{\|\nabla_H \eta\|_\infty}{t^{Q+1}} \int_0^1 \int_{B_t(e^{rX}x)} |f(y) - f(e^{rX}x)|dydr \\ &\lesssim |h|\frac{\|\nabla_H \eta\|_\infty}{t^{Q+1}} \int_0^1 \int_{B_t} |f(ye^{rX}x) - f(e^{rX}x)|dydr. \end{aligned}$$

Taking $|h| \leq t$, we get

$$|u(hx) - u(x)| \lesssim \frac{1}{t^Q}\Big(\int_{B_t} |\Delta_y f(x)| + |\Delta_y f(hx)|dy + \int_0^1 \int_{B_t} |\Delta_y f(e^{rX}x)|dydr\Big).$$

Using Minkowski's inequality and Hölder inequality, we have

$$\begin{aligned} \Big(\int_G |u(hx) - u(x)|^p dx\Big)^{\frac{1}{p}} &\lesssim \frac{1}{t^Q} \int_{B_t} \|\Delta_y f\|_{L^p} dy \\ &\lesssim \frac{t^{\frac{Q}{p'}}}{t^Q}\Big(\int_{B_t} \|\Delta_y f\|_{L^p}^p dy\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\frac{1}{t^Q}\int_{B_t} \|\Delta_y f\|_{L^p}^p dy\Big)^{\frac{1}{p}}. \end{aligned}$$

This yields to

$$\omega_p(f,t) \lesssim \Big(\frac{1}{t^Q}\int_{|h|\leq t} \|\Delta_h f\|_{L^p}^p dh\Big)^{\frac{1}{p}}.$$

Thus,

$$\overline{\omega}_p(f,t) \lesssim \Big(\frac{1}{t^Q}\int_{|h|\leq t} \|\Delta_h f\|_{L^p}^p dh\Big)^{\frac{1}{p}}.$$

The reverse inequality is straight forward. $\square$

Corollary 4.1 *Given $0 < s < 1$ and $1 \leq p < \infty$, then*

$$(L^p(\mathbb{G}), \dot{W}^{1,p}(\mathbb{G}))_{s,p} = \dot{W}^{s,p}(\mathbb{G}).$$

Proof: Recall that the norm in $(L^p(\mathbb{G}), \dot{W}^{1,p}(\mathbb{G}))_{s,p}$ is defined by

$$\|f\|_{s,p} = \Big(\int_0^\infty \Big(t^{-s}\dot{K}(t,f)\Big)^p \frac{dt}{t}\Big)^{\frac{1}{p}}.$$

Therefore, from the previous propositions, we have that

$$\begin{aligned} \|f\|_{s,p} &\approx \Big(\int_0^\infty \frac{1}{t^{Q+sp}} \int_{|h|\leq t}\int_{\mathbb{G}} |f(hx) - f(x)|^p dxdh\frac{dt}{t}\Big)^{\frac{1}{p}} \\ &\approx \Big(\int_{\mathbb{G}}\int_{\mathbb{G}}\int_{t\geq|h|} \frac{1}{t^{Q+sp+1}}|f(hx) - f(x)|^p dtdhdx\Big)^{\frac{1}{p}} \\ &\approx \Big(\frac{1}{Q+sp}\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|f(hx) - f(x)|^p}{|h|^{Q+sp}} dhdx\Big)^{\frac{1}{p}} = \frac{1}{(Q+sp)^{\frac{1}{p}}}\|f\|_{\dot{W}^{s,p}}. \end{aligned}$$

$\square$

The following Corollary is well known in the Euclidean case [14, p. 216-217] or [29, Remark 2]. We add it here in our setting.

Corollary 4.2 *Given* $0 < s < 1$*, then*

$$(L^1(\mathbb{G}), \dot{BV}(\mathbb{G}))_{s,1} = \dot{W}^{s,1}(\mathbb{G}).$$

Proof: Let $A_0 = \dot{W}^{1,1}(\mathbb{G})$ and $A_1 = L^1(\mathbb{G})$. Following [14], we define

$$A_0 + \infty \cdot A_1 = \{a \in A_0 + A_1 \mid \|a\|_{A_0+\infty\cdot A_1} = \lim_{t\to\infty} K(t, a; A_0, A_1) < \infty\}$$

and

$$A_1 + \infty \cdot A_0 = \{a \in A_0 + A_1 \mid \|a\|_{A_1+\infty\cdot A_0} = \lim_{t\to 0} \frac{1}{t} K(t, a; A_0, A_1) < \infty\}.$$

By [14, Lemma 1] and Theorem 2.2 we immediately get

$$A_0 + \infty \cdot A_1 = BV(\mathbb{G}) \cap L^1(\mathbb{G}) \qquad \text{and} \qquad A_1 + \infty \cdot A_0 = L^1(\mathbb{G}),$$

and by [14, Lemma 2] we have

$$K(t, f, L^1, \dot{W}^{1,1}) = K(t, f, L^1, \dot{BV})$$

and the thesis follows by Corollary 4.1. □

Lemma 4.2 *The pair* $(L^p(\mathbb{G}), \dot{W}^{1,p}(\mathbb{G}))$ *is normal.*

Proof: First we show that

$$\lim_{t\to\infty} \dot{K}(f, t) \approx \|f\|_{L^p}.$$

Indeed, consider a function f_ε compactly supported such that $\|f - f_\varepsilon\|_{L^p} < \varepsilon$. Clearly for t large enough, we have that

$$\overline{\omega}_p(f_\varepsilon, t) = 2\|f_\varepsilon\|_{L^p} \geq 2\|f\|_{L^p} - 2\varepsilon.$$

Also, for all $f \in L^p$,

$$\overline{\omega}_p(f, t) \leq 2\|f\|_{L^p}.$$

Now notice that for h outside the support of f_ε

$$\begin{aligned}\|\Delta_h f\|_{L^p} &\geq \|\Delta_h f_\varepsilon\|_{L^p} - \|\Delta_h(f - f_\varepsilon)\|_{L^p}\\ &\geq 2\|f\|_{L^p} - 4\varepsilon,\end{aligned}$$

which yields to our conclusion.
Next we need to show that

$$\lim_{t\to 0} \frac{\dot{K}(f, t)}{t} \approx \|\nabla_H f\|_{L^p}.$$

First, we have, if $h = e^{tX}$, $X \in V$ then

$$\Delta_h f(x) = \int_0^1 (Xf)(e^{rX}x)dr.$$

It follows that

$$\omega_p(f, t) \leq t\|\nabla_H f\|_{L^p}.$$

Now, let $f_\varepsilon \in C_c^\infty(\mathbb{G})$ such that $\|\nabla_H(f - f_\varepsilon)\|_{L^p} < \varepsilon$. Notice that

$$f_\varepsilon(hx) - f_\varepsilon(x) - Xf_\varepsilon(x) = \int_0^1 (Xf_\varepsilon)(e^{rX}x) - (Xf_\varepsilon)(x)dr.$$

Hence, one can see that

$$\mu_\varepsilon(t) = \sup_{|X|=t, h=e^X} \frac{\|f_\varepsilon(hx) - f_\varepsilon(x) - (Xf_\varepsilon)(x)\|_{L^p}}{t} \to 0, \quad t \to 0.$$

In particular, there exists t_ε small, so that $\mu_\varepsilon(t) < \varepsilon$ for $0 < t < t_\varepsilon$. Moreover, we have that

$$\omega_p(f_\varepsilon, t) \leq \omega_p(f_\varepsilon) + t\varepsilon.$$

Hence, for $0 < t < t_\varepsilon$ we have that

$$\|\nabla_H f\|_{L^p} \leq \|\nabla_H f_\varepsilon\|_{L^p} + \varepsilon \leq \mu_\varepsilon(t) + \frac{\omega_p(f_\varepsilon, t)}{t} + \varepsilon \leq \frac{\omega_p(f, t)}{t} + 2\varepsilon.$$

and this finishes the proof. $\square$

The following result immediately follows from Lemma 4.2 and the fact that $K(t, f, L^1, \dot{W}^{1,1}) = K(t, f, L^1, \dot{BV})$.

Corollary 4.3 *The pair $(L^1(\mathbb{G}), \dot{BV}(\mathbb{G}))$ is normal.*

5 Applications

Applying Lemma 4.2 and Corollary 4.3 we get various interesting limiting formulas. Theorems 5.1 and 5.2 can be proved exactly as in [29] whereas Theorem 5.3 has its roots in [5, 15, 33].

5.1 Limit Behavior of Fractional Sobolev Spaces

Theorem 5.1 (Bourgain-Brézis-Mironescu) *Let $f \in \dot{W}^{1,p}(\mathbb{G})$, then*

$$\lim_{s\to 1}(1-s)^{\frac{1}{p}} \Big(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \Big)^{\frac{1}{p}} \approx (Q+p)^{\frac{1}{p}} p^{-\frac{1}{p}} \|\nabla_H f\|_{L^p}.$$

Theorem 5.2 (Maz'ya-Shaposhnikova) *Let $f \in \cup_{s\in(0,1)} \dot{W}^{s,p}$, then*

$$\lim_{s\to 0} s^{\frac{1}{p}} \Big(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \Big)^{\frac{1}{p}} \approx Q^{\frac{1}{p}} p^{-\frac{1}{p}} \|f\|_{L^p}.$$

Theorem 5.3 (Davila) *Let $f \in \dot{BV}(\mathbb{G})$, then*

$$\lim_{s\to 1}(1-s) \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|}{|y^{-1}x|^{Q+s}} dxdy \approx (Q+1)|D_{\mathbb{G}} f|(\mathbb{G}).$$

5.2 Fractional Perimeter and its Limiting Behavior

We recall here the definition of the fractional perimeter Per_s in a Carnot group $\mathbb{G}$, namely

$$\mathrm{Per}_s(A) = \int_A \int_{\mathbb{G}\setminus A} \frac{1}{|y^{-1}x|^{Q+s}} dydx \quad \text{ for } A \subset \mathbb{G}.$$

A similar definition has been proposed in [16], moreover if $\mathbb{G} = \mathbb{R}^n$ the previous definition boils to the one proposed by Caffarelli-Roquejoffre and Savin in [12].

Clearly, $\mathrm{Per}_s(A) = \|\chi_A\|_{\dot{W}^{1,s}}$ and from the results above, we have that

$$\lim_{s\to 1}(1-s)\mathrm{Per}_s(A) \approx Per_{\mathbb{G}}(A)$$

and

$$\lim_{s\to 0} s\mathrm{Per}_s(A) \approx |A|.$$

In order to provide a more exact convergence, as in [33], we define the following perimeter:

$$\widetilde{\mathrm{Per}_s}(A) = s(1-s)\int_0^\infty t^{-s}K(t,\chi_A,L^1,\dot{BV})\frac{dt}{t}.$$

Then we have

Theorem 5.4 *Let $A \subset \mathbb{G}$, then the we have*

$$\widetilde{\mathrm{Per}_s}(A) \le |A|^{1-s}Per_{\mathbb{G}}(A)^s.$$

Moreover,

i) $\lim_{s\to 0}\widetilde{\mathrm{Per}_s}(A) = |A|$.

ii) $\lim_{s\to 1}\widetilde{\mathrm{Per}_s}(A) = Per_{\mathbb{G}}(A)$.

Proof: For short we will write $K(t,\chi_A,L^1,BV) = K(t,A)$. Notice that $i)$ and $ii)$ follow from the interpolation limits as above. So we propose to establish the first statement. By the definition of K, we have that

$$K(t,A) \le |A| \quad \text{and} \quad K(t,A) \le tPer_{\mathbb{G}}(A).$$

Therefore,

$$\begin{aligned}\int_0^\infty t^{-s}K(t,A)\frac{dt}{t} &= \int_0^r t^{-s}K(t,A)\frac{dt}{t} + \int_r^\infty t^{-s}K(t,A)\frac{dt}{t}\\ &\le Per_{\mathbb{G}}(A)\int_0^r t^{-s}ds + |A|\int_r^\infty t^{-s-1}dt\\ &\le Per_{\mathbb{G}}(A)\frac{r^{1-s}}{1-s} + |A|r^{-s}s.\end{aligned}$$

Minimizing over $r \in (0,\infty)$ we get the desired inequality. $\square$.

Following [33], we provide another quantitative form of K using the symmetric difference of sets.

Proposition 5.1 *Let $A \subset \mathbb{G}$, then*

$$K(t,A) = \inf_{U\subset\mathbb{G}} |A\Delta U| + tPer_{\mathbb{G}}(U).$$

Proof: Let $g(t,A) = \inf_{U\subset\mathbb{G}}|A\Delta U| + tPer_{\mathbb{G}}(U)$. First, we write $\chi_A = \chi_A - \chi_U + \chi(U)$. Since $\|\chi_A - \chi_U\|_{L^1} = |A\Delta U|$ and $|D_G\chi_U|(\mathbb{G}) = Per_{\mathbb{G}}(U)$, it follows from the definition of K that

$$K(t,A) \le g(t,A).$$

On the other hand, we claim first that

$$K(t,f)=\inf\{\|f_1\|_{L^1}+t\|\nabla_H f_2\|_{L^1}; f=f_1+f_2, f_1\in L^1(\mathbb{G}); f_2\in C_c^\infty(\mathbb{G})\}.$$

Indeed, we already have from the definition itself that

$$K(t,f)\leq\inf\{\|f_1\|_{L^1}+t\|\nabla_H f_2\|_{L^1}; f=f_1+f_2, f_1\in L^1(\mathbb{G}); f_2\in C_c^\infty(\mathbb{G})\}.$$

Now given $f=f_1+f_2$ such that $f_1\in L^1$ and $f_2\in \dot{BV}$ with

$$\|f_1\|_{L^1}+t|D_{\mathbb{G}}f_2|(\mathbb{G})\leq K(t,f)+\varepsilon,$$

then by Theorem 1.2, we can always find $g_\varepsilon\in C_c^\infty(\mathbb{G})$ such that $|D_{\mathbb{G}}(f_2-g_\varepsilon)|(\mathbb{G})<\frac{\varepsilon}{t}$ and $\|f_2-g_\varepsilon\|_{L^1}<\varepsilon$. Hence, writing $f=f_1+f_2-g_\varepsilon+g_\varepsilon$ we get

$$\|f_1+f_2-g_\varepsilon\|_{L^1}+t|D_{\mathbb{G}}g_\varepsilon|(\mathbb{G})\leq\|f_1\|_{L^1}+t|D_{\mathbb{G}}f_2|(\mathbb{G})+2\varepsilon\leq K(t,f)+3\varepsilon.$$

Therefore,

$$\inf\{\|f_1\|_{L^1}+t\|\nabla_H f_2\|_{L^1}; f=f_1+f_2, f_1\in L^1(\mathbb{G}); f_2\in C_c^\infty(\mathbb{G})\}\leq K(t,f).$$

Now, we can decompose the characteristic function of A as follows: $\chi_A=\chi_A-f+f$ where $f\in C_c^\infty(\mathbb{G})$. Let $U=\{|f|>r\}$, then we have

$$A\Delta U\subset\Big(\{|\chi_A-f|\geq r\}\setminus A\Big)\cup\Big(\{|\chi_A-f|\geq 1-r\}\cap A\Big).$$

Using Theorem 1.1, we have

$$\int_0^\infty|\{|\chi_A-f|\geq r\}|dr+t\int_0^\infty Per_{\mathbb{G}}(\{|f|>r\})dr=\|\chi_A-f\|_{L^1}+t|D_{\mathbb{G}}f|(\mathbb{G}).$$

Next, we notice that

$$\int_0^1\Big(|\{|\chi_A-f|\geq r\}\setminus A|+|\{|\chi_A-f|\geq 1-r\}\cap A|\Big)dr=\int_0^1|\{|\chi_A-f|\geq r\}|dr\leq\int_0^\infty|\{|\chi_A-f|\geq r\}|dr.$$

Hence,

$$\int_0^1\Big(|\{|\chi_A-f|\geq r\}\setminus A|+|\{|\chi_A-f|\geq 1-r\}\cap A|+tPer_{\mathbb{G}}(\{|f|\geq r\})\Big)dr\leq\|\chi_A-f\|_{L^1}+t|D_{\mathbb{G}}f|(\mathbb{G}).$$

Therefore, by the mean value theorem, there exists $r\in[0,1]$, depending on t such that

$$\Big(|\{|\chi_A-f|\geq r\}\setminus A|+|\{|\chi_A-f|\geq 1-r\}\cap A|+tPer_{\mathbb{G}}(\{|f|\geq r\})\Big)\leq\|\chi_A-f\|_{L^1}+t|D_{\mathbb{G}}f|(\mathbb{G}).$$

In particular,

$$\begin{aligned}g(t,A)\leq|A\Delta U|+tPer_{\mathbb{G}}(U)&\leq\Big(|\{|\chi_A-f|\geq r\}\setminus A|+|\{|\chi_A-f|\geq 1-r\}\cap A|+tPer_{\mathbb{G}}(\{|f|\geq r\})\Big)\\&\leq\|\chi_A-f\|_{L^1}+t|D_{\mathbb{G}}f|(\mathbb{G}).\end{aligned}$$

Leading to

$$g(t,A)\leq K(t,A).$$

$\square$.

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