

An Itô calculus for a class of limit processes arising from random walks on the complex plane

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Abstract

Within the framework of the previous paper [8], we develop a generalized stochastic calculus for processes associated to higher order diffusion operators. Applications to the study of a Cauchy problem, a Feynman-Kac formula and a representation formula for higher derivatives of analytic functions are also given.

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1. Introduction

One of the main instances of the fruitful interplay between analysis and probability is the connection between parabolic equations associated to second-order elliptic operators and the theory of Markov processes. The main consequence of this extensively studied topic is the famous *Feynman-Kac formula*, providing a representation of the solution of the heat equation with potential $V \in C_0^\infty(\mathbb{R}^d)$ (the continuous functions vanishing at infinity)

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) - V(x)u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d \\ u(0, x) = f(x) \end{cases} \quad (1)$$

in terms of an integral with respect to the distribution of the Wiener process, the mathematical model of the Brownian motion:

$$u(t, x) = \int_{C_t} e^{-\int_0^t V(\omega(s)+x) ds} f(\omega(t) + x) dW(\omega). \quad (2)$$

In fact a probabilistic representation of this form cannot be written in the case of semigroups whose generator does not satisfy the maximum principle. In particular if the Laplacian in Eq (1) is replaced with an higher order differential operator, i.e. if we consider a Cauchy problem of the form

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = (-1)^{n+1} \Delta^n u(t, x) - V(x)u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d, \\ u(0, x) = f(x), \end{cases} \quad (3)$$

with $n \in \mathbb{N}$, $n \geq 2$, then a formula analogous to (2), giving the solution of (3) in terms of the expectation with respect to the measure associated to a Markov process, is lacking and it is not possible to find a stochastic process which plays for the parabolic equation (3) the same role that the Wiener process plays for the heat equation. The problem of how to overcome this limitation has been studied by means of different techniques and two main approaches have been proposed. The first one was introduced by V. Yu. Krylov in 1960 [18] and

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further developed by K. Hochberg in 1978 [14]. The solution of (3) is constructed in terms of the expectation with respect to a *signed* measure with infinite total variation on a space of paths on the interval $[0, t]$. This approach is related to the theory of *pseudoprocesses*, i.e. processes associated to signed instead of probability measures. It is important to recall that due to the particular conditions necessary for the generalization of the Kolmogorov existence theorem for the limit of a projective system of complex measures (see [30]), in the case of Krylov-Hochberg process, a well defined signed measure on $\mathbb{R}^{[0,t]}$ cannot exist and the "integrals" realizing the Feynman-Kac formula for equation (3) are just formal expressions which cannot make sense in the framework of Lebesgue integration theory but are to be meant as limit of a particular approximating sequence. However, even taking into account these technical problems, an analog of the arc-sine law [14, 16, 19], of the central limit theorem [15, 29] and of Itô formula and Ito stochastic calculus [14, 25] have been developed for the (finite additive) Krylov-Hochberg signed measure. For a extensive discussion of these problems in the framework of a generalized integration theory on infinite dimensional spaces as well as for a unified view of probabilistic and complex integration see [1, 2]. It is worthwhile to mention the work by D. Levin and T. Lyons [22] on rough paths, conjecturing that the above mentioned signed measure could be finite if defined on the quotient space of equivalence classes of paths corresponding to different parametrization of the same path.

A different approach, introduced by T. Funaki [13] for the case where $n = 2$, is based on the construction of a stochastic process (with dependent increments) on the complex plane. Funaki's process is obtained by composing two independent Brownian motions and has some relations with the *iterated Brownian motion* [9, 3]. Furthermore this approach is related with the theory of Bochner subordination [7] and can be applied to partial differential equations of even and odd order 2^n , by multiple iterations of suitable processes [13, 17, 27, 26]. Complex valued processes, connected to PDE of the form (3) have been also proposed by other authors by means of different techniques [21]. In [23, 11] K. Burdzy and A. Madrecki consider equation (26) with $n = 2$ and $V \equiv 0$, constructing a probabilistic representation for its solution in terms of a stable probabilistic Borel measure m on the space $\Omega = C([0, t], \mathbb{C}^\infty)$ of continuous mappings on $[0, t]$ with values in the set \mathbb{C}^∞ of complex valued sequences, endowed with the product topology. In this setting a Feynman-Kac type formula is proved for the fourth order heat-type equation with linear potential. $\frac{\partial u}{\partial t} = \frac{1}{8} \frac{\partial^4 u}{\partial x^4} + (iax + b)u$. By means of the theory of infinite dimensional Fresnel integral, in [24] a Feynman-Kac-type formula for equation (3) has been proved for potentials V which are Fourier transform of complex bounded measures on the real line.

In a recent paper [8], two of the authors have introduced a different approach which is rather simple and elegant, providing the solution of the parabolic Cauchy problem

$$\begin{cases} \partial_t u(t, x) = \frac{\alpha}{N!} \partial_x^N u(t, x), \\ u(0, x) = f(x), \quad x \in \mathbb{R}, \end{cases} \quad (4)$$

(where $N > 2$ is an integer constant, $\alpha \in \mathbb{C}$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ is the initial datum) in terms of the expectation with respect to a the law of a sequence of random walks $\{W_n^N, n \in \mathbb{N}\}$ on the complex plane. The main idea in [8] is the construction of a sequence of processes which play for the PDE (4) of order N the same role that the Wiener process plays for the heat equation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\alpha \in \mathbb{C} \setminus \{0\}$ be a complex number and $N \geq 1$ a given integer. Let

$$R(N) = \{e^{2i\pi k/N}, k = 0, 1, \dots, N-1\}$$

be the roots of unity. Throughout the paper ξ will denote a uniformly distributed random variable on the set $\alpha^{1/N} R(N)$. Given a sequence $\{\xi_k\}$ of i.i.d. random variables $\xi_k \sim \xi$, we define the complex stochastic process

$$W^n(t) = \frac{1}{n^{1/N}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k. \quad (5)$$

For each n , the process W^n is a random walk on the complex plane, which is geometrically aligned and precisely scaled to extract information about the N -th complex derivative of analytic functions [8]. This claim will be made clear in the course of the paper.

If $\alpha = 1$ and $N = 2$ the process W_n converges weakly [6] to the Wiener process and the Feynman-Kac formula (2) (for $V \equiv 0$) can be written as

$$u(t, x) = \lim_{n \rightarrow \infty} \mathbb{E}[f(W_n(t) + x)]. \quad (6)$$

For $N > 2$ the sequence of processes W_n cannot converge because of the particular scaling exponent $1/N$ (directly related with the order of the PDE (4)) appearing in the denominator on the right hand side of (5). However for a restricted class of functions $f : \mathbb{C} \rightarrow \mathbb{C}$ the limit in (6) still exists providing a representation for the solution of (4) (see [8]).

Notice that W^n are in fact pre-Brownian motions, rescaled so they diverge in the limit due to the factor $n^{-1/N}$ when $N > 2$. Asymptotically their paths have the geometric properties of Brownian motion. To see this, we calculate the real 2-dimensional variance

$$\mathbb{E}(|W^n(t) - \mathbb{E}W^n(t)|^2) = \mathbb{E}|W^n(t)|^2 = \mathbb{E}\left(\left|n^{-1/N} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k\right|^2\right) = n^{-2/N} \left(\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}|\xi_k|^2\right) = \frac{\lfloor nt \rfloor}{n^{2/N}}$$

Thus if we reparametrize to the process $\bar{W}^n(t) = W^n(t/n^{1-2/N})$ the random walk $\bar{W}^n(t)$ has real 2-dimensional variance $\frac{\lfloor n^{2/N}t \rfloor}{n^{2/N}}$ and so, by Donsker's Theorem, converges to a 2-dimensional Brownian motion as $n \rightarrow \infty$ for $N > 2$. In this sense (since paths do not depend on the parametrisation of the curves) the paths of W^n have the geometric properties of paths of Brownian motion in the limit as $n \rightarrow \infty$.

In this paper, we improve the construction of [8] with the development of an Itô's calculus for the limit law of the random walks W^n . To achieve our aim, we first analyze the behaviour of the random walks W^n , with particular regard to the estimates on the moments and Fourier's transform of W^n .

Then, we introduce the analog of Itô's calculus. Heuristically, the rescaling $n^{1/N}$ in the construction of W^n (by comparing with the classical construction, that correspond to the case $N = 2$) implies that only the N -th moment behaves like dt in the limit, thus implying that all moments of lesser order shall be considered in the Itô formula, while we can neglect moments of higher order in the limit.

Moreover, integrals with respect to the increments of the random walk W^n and their moments lesser than N shall be analogous to *stochastic* integrals and, consequently, shall have zero mean. In Theorem 18 we prove that this intuition is indeed true, and we prove the following Itô formula (compare (25))

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(z + W^n(t))] - f(z) = \frac{\alpha}{N!} \int_0^t \lim_{n \rightarrow \infty} \mathbb{E}[\partial^N f(z + W^n(s))] ds.$$

By setting

$$u(t, z) = \lim_{n \rightarrow \infty} \mathbb{E}[f(z + W^n(t))],$$

this is equivalent to saying $u(t, z)$ is a classical solution of the N -th order Cauchy problem (4).

We shall then extend this result in two directions. First, through a rather straightforward extension of previous computations, we allow for a time-dependent coefficient in front of the diffusion operator, constructing the probabilistic representation for the solution of

$$\begin{cases} \partial_t u(t, x) = \frac{\alpha}{N!} a(t) \partial_x^N u(t, x), \\ u(0, x) = f(x), \quad x \in \mathbb{R}. \end{cases} \quad (7)$$

Then we begin the analysis of a Feynman-Kac formula, and we show that for a time dependent potential which is linear in the space variable, the classical solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} u(t, z) = \frac{\alpha}{N!} \partial_x^N u(t, z) + V(t, x) u(t, x), \\ u(0, x) = f(x), \quad x \in \mathbb{R}. \end{cases}$$

is given by

$$u(t, x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[f(x + W^n(t)) e^{\int_0^t V(t-s, x + W^n(s)) ds} \right]$$

All these results require to choose a suitable class of functions for f . The first, obvious remark, is f must be extensible to the complex plane. Moreover, it is necessary to have estimates on the function and all its derivatives in \mathbb{C} . For the sake of simplicity, we shall limit ourselves to the classical case of analytic functions of exponential type, see for instance [12] and definition 8.

Finally we study the properties of stopping time for the process W^n , proving a suggestive formula for the N -th order derivative of an analytic function

$$f^N(z) = \frac{N!}{\alpha} \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{\tau_n} (f(z + W^n(\tau_n)) - f(z)) \right]$$

where τ_n is the exit time of W^n from the ball $B(0, R) \subset \mathbb{C}$.

2. Random walk on the complex plane

The present section is devoted to the proof of some properties of the sequence of complex random walks $W^n(t)$.

Let $\alpha \in \mathbb{C}$ and $N \in \mathbb{N}$ with $N > 2$ and let us consider the complex random variable ξ uniformly distributed on the set $\alpha R(N)$, where $R(N) = \{e^{i2\pi k/N}, k = 0, 1, \dots, N-1\}$ is the set of N -roots of unity. We have

$$\mathbb{E}[f(\xi)] = \frac{1}{N} \sum_{k=0}^{N-1} f(\alpha^{1/N} e^{2i\pi k/N}). \quad (8)$$

Lemma 1. *The random variable ξ has finite powers of every order*

$$\mathbb{E}[\xi^m] = \begin{cases} \alpha^{m/N}, & m = nN, \ n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

and finite moments of every order

$$\mathbb{E}[|\xi|^m] = |\alpha|^{m/N}.$$

Proof. We compute

$$\mathbb{E}[\xi^m] = \frac{1}{N} \sum_{k=0}^{N-1} \alpha^{m/N} e^{2i\pi mk/N};$$

if $m/N = n \in \mathbb{N}$, since $e^{2i\pi n} = 1$ then each term in the sum is equal to 1; in the other case, we employ the trigonometric sum

$$\sum_{k=0}^{N-1} e^{2i\pi mk/N} = \frac{1 - e^{2i\pi m}}{1 - e^{2i\pi m/N}} = 0.$$

□

For a complex random variable X we define its *characteristic function*¹ as $\psi_X(\lambda) := \mathbb{E}[e^{i\lambda X}]$. We have

$$\psi_\xi(\lambda) := \mathbb{E}[e^{i\lambda \xi}] = \frac{1}{N} \sum_{k=0}^{N-1} \exp(i\alpha^{1/N} e^{2i\pi k/N} \lambda).$$

The next lemma, which will be applied in section 5, provides an estimate of the characteristic function of ξ .

Lemma 2. *Let $|\lambda| \leq R$. Then there exists a constant $C \in \mathbb{R}$ such that*

$$\left| \mathbb{E}[e^{\lambda\xi}] - e^{\frac{\alpha\lambda^N}{N!}} \right| \leq C|\alpha|^2|\lambda|^{2N}.$$

Proof. For the properties of the random variable ξ , setting $z_j := e^{i2\pi j/N}$, the function $\chi : \mathbb{C} \rightarrow \mathbb{C}$ defined as $\chi(\lambda) := \mathbb{E}[e^{\lambda\xi}]$ is an entire analytic function with the following power series expansion:

$$\begin{aligned} \chi(\lambda) &= \frac{1}{N} \sum_{j=0}^{N-1} e^{\alpha^{1/N} \lambda z_j} = \frac{1}{N} \sum_{j=0}^{N-1} \sum_k \frac{\lambda^k \alpha^{k/N}}{k!} z_j^k \\ &= \sum_k \frac{\lambda^k \alpha^{k/N}}{k!} \sum_{j=0}^{N-1} \frac{z_j^k}{N} \\ &= \sum_m \frac{\lambda^{mN} \alpha^m}{(mN)!}. \end{aligned}$$

The difference between $\chi(\lambda)$ and $e^{\frac{\alpha\lambda^N}{N!}}$ can be estimated as:

$$\begin{aligned} \chi(\lambda) - e^{\frac{\alpha\lambda^N}{N!}} &= \sum_{m=0}^{\infty} \lambda^{mN} \alpha^m \left(\frac{1}{(mN)!} - \frac{1}{m!(N!)^m} \right) \\ &= \sum_{m=2}^{\infty} \lambda^{mN} \alpha^m \left(\frac{1}{(mN)!} - \frac{1}{m!(N!)^m} \right) \\ &= \alpha^2 \lambda^{2N} \sum_{m=0}^{\infty} \lambda^{mN} \alpha^m \left(\frac{1}{((m+2)N)!} - \frac{1}{(m+2)!(N!)^{m+2}} \right) \\ &= \alpha^2 \lambda^{2N} g(\lambda) \end{aligned}$$

where $g : \mathbb{C} \rightarrow \mathbb{C}$ is the entire analytic function defined by the power series (with infinite radius of convergence)

$$g(\lambda) := \sum_{m=0}^{\infty} \lambda^{mN} \alpha^m \left(\frac{1}{((m+2)N)!} - \frac{1}{(m+2)!(N!)^{m+2}} \right)$$

The thesis follows by the continuity of g and the assumption of the boundedness of $|\lambda|$, by putting

$$C := \sup_{|\lambda| < R} |g(\lambda)|.$$

□

Next we proceed to analyze the random walk W^n on the complex plane defined by formula (5). The main issue here is the analysis of the complex moments of the random walk and their asymptotic behavior as $n \rightarrow \infty$. The result that we obtain in Theorem 3, is necessary in order to handle the Itô's formula introduced in the next section.

¹ Though it is more common to use $\mathbb{E}[\exp(i\text{Re}\lambda X)]$ as the characteristic function of a complex-valued random variable due to the connection with the 2-dimensional Fourier transform, the complication would not improve the results in this paper

Theorem 3. Fix $k \in \mathbb{N}$, $t \in \mathbb{R}^+$. The k -moment of $W_n(t)$ satisfies

$$\mathbb{E}[(W_n(t))^k] = \begin{cases} \left(\frac{\alpha t}{N!}\right)^{k/N} \frac{k!}{(k/N)!} \mathbb{1}_{[0, \lfloor nt \rfloor]}(k/N) + R(n, k), & k = hN, h \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

($\mathbb{1}_{[0, \lfloor nt \rfloor]}$ being the indicator function of the interval $[0, \lfloor nt \rfloor]$). For $k = 0$ and $k = N$ then $R(n, hN) = 0$, while $k = hN$, $h \in \mathbb{N}$, $h \geq 2$, then the remainder term satisfies the inequality

$$|R(n, hN)| \leq \frac{|\alpha|^h t^{h-1} (h^2 + h)}{2n} + \frac{|\alpha|^h}{n} \left(\frac{0.792hN}{\log(hN + 1)} \right)^{hN}. \quad (10)$$

Proof. Let $W_n(t) = \frac{1}{n^{1/N}} \sum_j^{\lfloor nt \rfloor} \xi_j$ and ψ_n be its characteristic function, namely:

$$\psi_n(\lambda) := \mathbb{E}[e^{i\lambda W_n(t)}]$$

We have that

$$\mathbb{E}[(W_n(t))^k] = (-i)^k \frac{d^k}{d\lambda^k} \psi_n(0),$$

where ψ_n is equal to

$$\psi_n(\lambda) = \left(\mathbb{E}[\exp(\frac{1}{n^{1/N}} i\lambda \xi)] \right)^{\lfloor nt \rfloor} = \left(\psi_\xi\left(\frac{\lambda}{n^{1/N}}\right) \right)^{\lfloor nt \rfloor},$$

where ψ_ξ is the characteristic function of ξ .

By Faá di Bruno's formula

$$\frac{d^k}{d\lambda^k} \psi_n(\lambda) = \sum_{\pi \in \Pi} C(|\pi|, \lambda) \prod_{B \in \pi} \left(\frac{\psi_\xi^{(|B|)}(\lambda/n^{1/N})}{n^{|B|/N}} \right) \quad (11)$$

where π runs over the set Π of all partitions of the set $1, \dots, k$, $B \in \pi$ means that the variable B runs through the list of the blocks of the partition π , $|\pi|$ denotes the number of blocks of the partition π and $|B|$ is the cardinality of a set B , while the function $C : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{C}$ is equal to

$$C(j, \lambda) = \begin{cases} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - j)!} (\psi_\xi(\lambda/n^{1/N}))^{\lfloor nt \rfloor - j}, & \lfloor nt \rfloor \geq j \\ 0, & \text{otherwise} \end{cases}$$

Formula (11) can be written in the equivalent form:

$$\begin{aligned} & \frac{d^k}{d\lambda^k} \psi_n(\lambda) = \\ & \sum \frac{k!}{m_1! m_2! \dots m_k!} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - (m_1 + m_2 + \dots + m_k))!} (\psi_\xi(\lambda/n^{1/N}))^{\lfloor nt \rfloor - (m_1 + m_2 + \dots + m_k)} \prod_{j=1}^k \left(\frac{\psi_\xi^{(j)}(\lambda/n^{1/N})}{j! n^{j/N}} \right)^{m_j} \end{aligned} \quad (12)$$

where the sum is over the k -ple of non-negative integers (m_1, m_2, \dots, m_k) such that $m_1 + 2m_2 + \dots + km_k = k$ and $m_1 + m_2 + \dots + m_k \leq \lfloor nt \rfloor$. In particular we have:

$$\frac{d^k}{d\lambda^k} \psi_n(0) = \sum_{\pi \in \Pi} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - |\pi|)!} \prod_{B \in \pi} \left(\frac{\psi_\xi^{(|B|)}(0)}{n^{|B|/N}} \right) \quad (13)$$

where the first sum runs over the partitions π such that $|\pi| \leq \lfloor nt \rfloor$ or equivalently

$$\frac{d^k}{d\lambda^k} \psi_n(0) = \sum \frac{k!}{m_1! m_2! \cdots m_k!} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - (m_1 + m_2 + \cdots + m_k))!} \prod_{j=1}^k \left(\frac{\psi_\xi^{(j)}(0)}{j! n^{j/N}} \right)^{m_j}. \quad (14)$$

Since $\psi_\xi^{(j)}(0) = (i)^j \mathbb{E}[\xi^j]$, and $\mathbb{E}[\xi^j] \neq 0$ iff $j = mN$, with $m \in \mathbb{N}$, then the product $\prod_{j=1}^k \left(\frac{\psi_\xi^{(j)}(0)}{j! n^{j/N}} \right)^{m_j}$ is non-vanishing iff $m_j = 0$ for $j \neq lN$ and $k = Nm_N + 2Nm_{2N} + \dots$, i.e. if k is a multiple of N . Analogously in the sum appearing in formula (13) the only terms giving a non-vanishing contribution correspond to those partitions π having blocks B with a number of elements which is a multiple of N , giving, for $k = hN$:

$$\frac{d^{hN}}{d\lambda^{hN}} \psi_n(0) = i^{hN} \frac{\alpha^h}{n^h} \sum_{\pi \in \Pi} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - |\pi|)!} \quad (15)$$

where again the sum runs over the partitions π such that $|\pi| \leq \lfloor nt \rfloor$. Equivalently:

$$\begin{aligned} \frac{d^{hN}}{d\lambda^{hN}} \psi_n(0) &= \sum \frac{(hN)!}{(m_N)! (m_{2N})! \cdots (m_{hN})!} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - (m_N + m_{2N} + \cdots + m_{hN}))!} \prod_{l=1}^h \left(\frac{\psi_\xi^{(lN)}(0)}{(lN)! n^l} \right)^{m_{lN}}, \\ &= \sum \frac{(hN)!}{(m_N)! (m_{2N})! \cdots (m_{hN})!} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - (m_N + m_{2N} + \cdots + m_{hN}))!} \prod_{l=1}^h \left(\frac{i^{lN} \alpha^l}{(lN)! n^l} \right)^{m_{lN}}, \\ &= \frac{i^{hN} \alpha^h}{n^h} \sum \frac{(hN)!}{(m_N)! (m_{2N})! \cdots (m_{hN})!} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - (m_N + m_{2N} + \cdots + m_{hN}))!} \prod_{l=1}^h \frac{1}{((lN)!)^{m_{lN}}}, \end{aligned}$$

where the sum is over the h -ple of non-negative integers $(m_N, m_{2N}, \dots, m_{hN})$ such that $m_N + 2m_{2N} + \dots + hm_{hN} = h$ and $m_N + m_{2N} + \dots + m_{hN} \leq \lfloor nt \rfloor$.

Hence, we have

$$\mathbb{E}[(W_n(t))^{hN}] = \alpha^h \sum \frac{(hN)!}{(m_N)! (N!)^{m_N} (m_{2N})! (2N)^{m_{2N}} \cdots (m_{hN})! (hN)^{m_{hN}}} \frac{\lfloor nt \rfloor!}{n^h (\lfloor nt \rfloor - (m_N + m_{2N} + \cdots + m_{hN}))!}$$

When $n \rightarrow \infty$, the leading term in the previous sum is the one corresponding to $m_N = h$ (hence $m_{2N} = \dots = m_{hN} = 0$), which is equal to

$$\alpha^h \frac{(hN)!}{(m_N)! (N!)^h} \frac{\lfloor nt \rfloor!}{n^h (\lfloor nt \rfloor - h)!} = \alpha^h t^h \frac{(hN)!}{h! (N!)^h} + \alpha^h \frac{(hN)!}{h! (N!)^h} \left(\frac{\lfloor nt \rfloor!}{n^h (\lfloor nt \rfloor - h)!} - t^h \right)$$

In the case where $\lfloor nt \rfloor < h$ then this term does not appear in the sum and we can set it equal to 0. In the case where $\lfloor nt \rfloor \geq h$, we can estimate the quantity inside the brackets as:

$$\begin{aligned} \left| \frac{\lfloor nt \rfloor!}{n^h (\lfloor nt \rfloor - h)!} - t^h \right| &= \frac{1}{n^h} \left| -(nt)^h + \prod_{j=0}^{h-1} (\lfloor nt \rfloor - j) \right| = \frac{1}{n^h} \left| \prod_{j=0}^{h-1} ((\lfloor nt \rfloor - j) + (\{nt\} + j)) - \prod_{j=0}^{h-1} (\lfloor nt \rfloor - j) \right| \\ &\leq \frac{1}{n^h} \sum_{j=0}^{h-1} (\{nt\} + j) \prod_{k \neq j} nt = \frac{(nt)^{h-1}}{n^h} \sum_{j=0}^{h-1} (\{nt\} + j) \leq \frac{(nt)^{h-1}}{n^h} \sum_{j=0}^{h-1} (1 + j) = \frac{t^{h-1} (h^2 + h)}{2n} \end{aligned}$$

where in the second line we have used that if $a_j, b_j \in \mathbb{R}$, with $a_j, b_j \geq 0$ for all $j = 0, \dots, m$, then

$$\prod_{j=0}^m (a_j + b_j) - \prod_{j=0}^m a_j \leq \sum_{j=0}^m b_j \prod_{k \neq j} (a_k + b_k).$$

Hence

$$|R_1(n, h)| = \left| \alpha^h \frac{(hN)!}{(m_N)!(N!)^h} \frac{[nt]!}{n^h([nt] - h)!} - \alpha^{ht} \frac{(hN)!}{h!(N!)^h} \right| \leq \frac{|\alpha|^h t^{h-1} (h^2 + h)}{2n}$$

By using formula (15), the remaining terms in the sum (corresponding to the h -ple $(m_N, m_{2N}, \dots, m_{hN})$ with $m_N < h$) are bounded by

$$\begin{aligned} R_2(n, h) &= \frac{\alpha^h}{n^h} \sum_{\pi \in \Pi} \frac{[nt]!}{([nt] - |\pi|)!} - \alpha^h \frac{(hN)!}{(m_N)!(N!)^h} \frac{[nt]!}{n^h([nt] - h)!} \\ &\leq \frac{\alpha^h}{n^h} \sum_{\pi \in \Pi} n^{h-1} = \frac{\alpha^h}{n} B_{hN} \end{aligned}$$

where B_{hN} is the Bell number, i.e. the number of partitions of the set $\{1, \dots, hN\}$. In particular, for $h \rightarrow \infty$ (see [5]) $B_{hN} < \left(\frac{0.792hN}{\log(hN+1)} \right)^{hN}$, hence

$$|R_2(n, h)| \leq \frac{|\alpha|^h}{n} \left(\frac{0.792hN}{\log(hN+1)} \right)^{hN}.$$

□

Remark 1. *These statistics are interesting when we consider how the processes are related to the 2-dimensional Wiener process, which has vanishing complex moments of all orders. The difference here is that the processes W^n have unbounded variance as $n \rightarrow \infty$. If we rescale to \bar{W} as in the introduction, then all moments vanish.*

To give some intuition for why the jN moments might be nonzero, consider the case $N = 4$. Roughly, the idea is that the process is more likely to be near one of the 4 rays in the directions $\{1, i, -1, -i\}$ than near the rays rotated by $\pi/4$. To see this imagine a time when $\text{Re } W^n(t) \gg 1$. Then since $\text{Im } W^n(t)$ is independent with mean 0, it is much more likely to be near 0 than it is to be near $\pm \text{Re } W^n(t)$. Thus we see the underlying geometry of the processes is not statistically symmetric, as revealed in the moments; this is despite the fact that the paths converge to the fractal curves of the Wiener process, which are statistically symmetric.

The next results show why the random walk W_n can be regarded in a very weak sense as an N -stable process, in the sense of Theorem 5.

Lemma 4. *For any $\lambda, \alpha \in \mathbb{C}$, $t \geq 0$*

$$\mathbb{E}[\exp(i\lambda W_n(t))] = \exp\left(\frac{i^N \alpha t}{N!} \lambda^N\right) + \mathbf{R}_n(\lambda),$$

where the remainder term $\mathbf{R}_n(\lambda)$ satisfies the following estimate

$$|\mathbf{R}_n(\lambda)| \leq \left| \sum_{h=[nt]+1}^{+\infty} \frac{i^{hN} \lambda^{hN}}{h!} \left(\frac{\alpha t}{N!}\right)^h \right| + \frac{1}{n} \sum_{h=2}^{\infty} \frac{|\alpha|^h |\lambda|^{hN}}{(hN)!} \left(\frac{t^{h-1} (h^2 + h)}{2} + \left(\frac{0.792hN}{\log(hN+1)} \right)^{hN} \right). \quad (16)$$

Proof.

$$\begin{aligned} \mathbb{E}[\exp(i\lambda W_n(t))] &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} i^k \lambda^k \mathbb{E}[(W_n(t))^k] \\ &= \sum_{h=0}^{[nt]} \frac{i^{hN} \lambda^{hN}}{(hN)!} \left(\frac{\alpha t}{N!}\right)^h \frac{(hN)!}{h!} + \lim_{m \rightarrow \infty} \sum_{h=2}^m \frac{i^{hN} \lambda^{hN}}{(hN)!} R(n, hN) \\ &= \exp\left(\frac{i^N \alpha t}{N!} \lambda^N\right) - \sum_{h=[nt]+1}^{+\infty} \frac{i^{hN} \lambda^{hN}}{h!} \left(\frac{\alpha t}{N!}\right)^h + \lim_{m \rightarrow \infty} \sum_{h=2}^m \frac{i^{hN} \lambda^{hN}}{(hN)!} R(n, hN) \end{aligned}$$

where $R(n, hN)$ stands for remainder term in Theorem 3. The estimate (16) follows directly from the inequality (10). \square

A direct consequence of Lemma 4 is the following result

Theorem 5. *The characteristic function $\psi_n(\lambda)$ of $W^n(t)$ satisfies*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\exp(i\lambda W^n(t))] = \exp\left(i^N \frac{\lambda^N}{N!} \alpha t\right). \quad (17)$$

Proof. The result, namely $\lim_{n \rightarrow \infty} \mathbf{R}_n(\lambda) = 0$, follows by the convergence of the series $\sum \frac{i^{hN} \lambda^{hN}}{h!} \left(\frac{\alpha t}{N!}\right)^h$ and the estimate $\sum_{h=2}^{\infty} \frac{|\alpha|^h |\lambda|^{hN}}{(hN)!} \left(\frac{t^{h-1}(h^2+h)}{2} + \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}\right) < \infty$. \square

Theorem 6. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire analytic function with the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, such that the coefficients $\{a_k\}_{k \in \mathbb{N}}$ satisfy the following assumption:*

$$\sum_{h=0}^{\infty} |a_{hN}| \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN} < \infty \quad (18)$$

Then

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(W_n(t))] = \sum_{h=0}^{\infty} a_{hN} \frac{(hN)!}{h!} \left(\frac{\alpha t}{N!}\right)^h = \sum_{h=0}^{\infty} \frac{f^{(hN)}(0)}{h!} \left(\frac{\alpha t}{N!}\right)^h.$$

Proof. We directly compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[f(W_n(t))] &= \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_k \mathbb{E}[(W_n(t))^k] \\ &= \lim_{n \rightarrow \infty} \sum_{h=0}^{\lfloor nt \rfloor} a_{hN} \left(\frac{\alpha t}{N!}\right)^h \frac{(hN)!}{h!} + \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{h=0}^m a_{hN} R(n, hN) = \sum_{h=0}^{\infty} a_{hN} \left(\frac{\alpha t}{N!}\right)^h \frac{(hN)!}{h!} \end{aligned}$$

Indeed, by assumption (18), we have

$$\left| \sum_{h=0}^m a_{hN} R(n, hN) \right| \leq \frac{C}{n}, \quad C := \sum_{h=0}^{\infty} a_{hN} |\alpha|^h \left(\frac{t^{h-1}(h^2+h)}{2} + \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}\right) < \infty.$$

\square

Lemma 7. *If there exist $C_1, C_2 \in \mathbb{R}$ such that for all k the coefficients a_k satisfy the inequality $|a_k| \leq \frac{C_1 C_2^k}{k!}$, then they satisfy assumption (18).*

Definition 8. *An analytic function f is of exponential type c if*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k$$

where $|b_k|^{1/k} \rightarrow c$ as $k \rightarrow \infty$. Clearly if $c < \infty$ then f is entire analytic.

Roughly, the idea is that a function of exponential type c is asymptotically bounded by $e^{c|z|}$. Alternatively, a necessary and sufficient condition is that $c = \limsup_{n \rightarrow \infty} |f^{(n)}(x)|$ for one (and hence all) $x \in \mathbb{C}$.

Lemma 9. *If f is of exponential type, then it satisfies assumption (18).*

Lemma 10. *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is the Fourier transform of a complex bounded variation measure μ on \mathbb{R} with compact support, then f satisfies the assumptions of Lemma 7.*

Corollary 11. *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is the Fourier transform of a complex bounded variation measure μ on \mathbb{R} with compact support, then for all $t \in \mathbb{R}, x \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(x + W_n(t))] = \int e^{iyx} e^{i^N \alpha t \frac{y^N}{N!}} d\mu(y)$$

Remark 2. *By the Paley-Wiener Theorem [28], any function $f \in L^2(\mathbb{R})$ of exponential type is the Fourier transform of a function $\hat{f} \in L^2(\mathbb{R})$ with compact support.*

More generally, any function $f : \mathbb{C} \rightarrow \mathbb{C}$ which is the Fourier transform of a complex bounded variation measure μ on \mathbb{R} with compact support is of exponential type, and furthermore it is bounded on the real line.

3. Itô calculus

We shall consider a sort of Itô calculus for suitable regular functions of the random walk W^n that mimics the development of classical stochastic differential equations. We begin by considering integrals with respect to the processes $(W^n)^k$ for arbitrary $k \in \mathbb{N}$. Even if Theorem 6 and the estimate (10) would allow the development of the theory for a more general class of analytic functions g , as stated in the introduction, we shall restrict to the case where g is an analytic function of exponential type. This is sufficient for our purposes and will simplify the notation.

We state here, for later use, the following theorem, which follows directly from Theorem 6 and Lemma 9.

Theorem 12. *Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function of exponential type c . Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(W^n(t))] = \sum_{h=0}^{\infty} a_{hN} \frac{(hN)!}{h!} \left(\frac{\alpha t}{N!} \right)^h = \sum_{h=0}^{\infty} \frac{g^{(hN)}(0)}{h!} \left(\frac{\alpha t}{N!} \right)^h. \quad (19)$$

We define an **Itô integral** for the process $g(W^n(t))$ as

$$\int_0^t g(W^n(s)) dW^n(s) = \sum_{\tau=0}^{\lfloor nt \rfloor - 1} g(W^n(\frac{\tau}{n})) (W^n(\frac{\tau+1}{n}) - W^n(\frac{\tau}{n})) = \frac{1}{n^{1/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} g(W^n(\frac{\tau}{n})) (\xi_{\tau+1})$$

and, for any $k \in \mathbb{N}$,

$$\int_0^t g(W^n(s)) d(W^n(s))^k = \frac{1}{n^{k/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} g(W^n(\frac{\tau}{n})) (\xi_{\tau+1})^k. \quad (20)$$

Our next step is the analysis of expectations. Taking the mean in both sides of (20) and recalling that $W^n(\frac{\tau}{n})$ is independent from $\xi_{\tau+1}$, we get

$$\mathbb{E} \left[\int_0^t g(W^n(s)) d(W^n(s))^k \right] = \frac{1}{n^{k/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \mathbb{E}[g(W^n(\frac{\tau}{n}))] \mathbb{E}[(\xi_{\tau+1})^k]$$

and recalling (9) we get

$$\mathbb{E} \left[\int_0^t g(W^n(s)) d(W^n(s))^k \right] = 0 \quad \text{for all } k \neq mN, m \in \mathbb{N}; \quad (21)$$

for $k = N$ we have

$$\mathbb{E} \left[\int_0^t g(W^n(s)) d(W^n(s))^N \right] = \alpha \int_0^t \mathbb{E}[g(z + W^n(s))] ds$$

and finally for $k = mN$, $m \in \mathbb{N}$, $m > 1$:

$$\mathbb{E} \left[\int_0^t g(W^n(s)) d(W^n(s))^{mN} \right] = \left(\frac{\alpha}{n}\right)^m \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \mathbb{E}[g(W^n(\frac{\tau}{n}))] = \alpha \left(\frac{\alpha}{n}\right)^{m-1} \int_0^t \mathbb{E}[g(W^n(s))] ds \quad (22)$$

so we expect this to vanish to zero as $n \rightarrow \infty$ when $m > 1$.

Lemma 13. *Assume that g is an analytic function of exponential type c . Then*

$$\lim_{n \rightarrow \infty} \int_0^t \mathbb{E}[g(W^n(s))] ds = \int_0^t \lim_{n \rightarrow \infty} \mathbb{E}[g(W^n(s))] ds.$$

Proof. From Theorem 12 we have

$$\mathbb{E}[g(W_n(t))] = g_1(t) + R(n, t)$$

where

$$g_1(t) = \sum_{h=0}^{\infty} \frac{b_{hN}}{h!} \left(\frac{\alpha t}{N!}\right)^h$$

and $|R_n(t)| \leq \frac{1}{n}C(T, \alpha)$. The claim now follows from an application of Dominated Convergence Theorem. \square

In particular, we may record the following identity, concerning the limit behavior for polynomials. It follows from Lemma 13 and a direct application of Theorem 3.

Corollary 14. *Let $g(x) = x^{mN}$. Then*

$$\lim_{n \rightarrow \infty} \int_0^t \mathbb{E}[g(W^n(s))] ds = \frac{(mN)!}{(m+1)!} \left(\frac{\alpha}{N!}\right)^m t^{m+1}.$$

Proof. From Theorem 3 we have

$$\mathbb{E}[(W_n(t))^{mN}] = \left(\frac{\alpha t}{N!}\right)^m \frac{(mN)!}{m!} + R_n(t)$$

where $|R_n(t)| \leq \frac{1}{n}C(T, m, \alpha)$. By Lemma 13 we get

$$\lim_{n \rightarrow \infty} \int_0^t \mathbb{E}[g(W^n(s))] ds = \int_0^t \lim_{n \rightarrow \infty} \mathbb{E}[g(W^n(s))] ds$$

and the right hand side is equal to

$$\frac{(mN)!}{m!} \int_0^t \left(\frac{\alpha s}{N!}\right)^m ds = \frac{(mN)!}{(m+1)!} \left(\frac{\alpha}{N!}\right)^m t^{m+1}$$

as required. \square

Corollary 15. *Assume that g is an analytic function of exponential type c . Then*

$$\lim_{n \rightarrow \infty} \int_0^t \mathbb{E}[g(W^n(s))] ds = \sum_{h=0}^{\infty} \frac{b_{hN}}{(h+1)!} \left(\frac{\alpha}{N!}\right)^h t^{h+1}.$$

Corollary 16. *For any $k \neq N$ it follows that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t g(W^n(s)) d(W^n(s))^k \right] = 0.$$

4. Diffusions

Theorem 17. *Let $g : \mathbb{C} \rightarrow \mathbb{C}$ be of exponential type $c < \infty$. Then the following Itô formula holds*

$$g(z + W^n(t)) - g(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^t \partial^k g(z + W^n(s)) d(W^n(s))^k. \quad (23)$$

Proof. For simplicity, we let $t = \theta/n$, $\theta \in \mathbb{N}$. By exploiting a telescopic sum and Taylor's expansion of g we get

$$g(z + W^n(t)) - g(z) = \sum_{\tau=0}^{\theta-1} g(z + W^n(\frac{\tau+1}{n})) - g(z + W^n(\frac{\tau}{n})) = \sum_{\tau=0}^{\theta-1} \sum_{k=1}^{\infty} \frac{1}{k!} \partial^k g(z + W^n(\frac{\tau}{n})) \frac{(\xi_{\tau+1})^k}{n^{k/N}}$$

and the representation (23) follows by the interchange of sums which is allowed due to the absolute convergence of the Taylor series. \square

Remark 3. *The Itô's formula for Brownian motion satisfies*

$$\mathbb{E}[g(z + B(t))] - g(z) = \frac{1}{2} \int_0^t \mathbb{E}[\partial^2 g(z + B(s))] ds$$

which we can restate in terms of the random walk, as

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(t))] - g(z) = \frac{1}{2} \int_0^t \lim_{n \rightarrow \infty} \mathbb{E}[\partial^2 g(z + W^n(s))] ds. \quad (24)$$

We aim to prove that (a suitable extension of) (24) holds for the N -th order differential operator with respect to the random walk on the complex plane defined on the lattice generated by $R(N)$.

Let us state the aim of our construction.

Theorem 18. *Assume that g is an analytic function of exponential type c , i.e., $|g^{(k)}(z)|^{1/k} \rightarrow c$ as $k \rightarrow \infty$ for every $z \in \mathbb{C}$. Then we have*

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(t))] - g(z) = \frac{\alpha}{N!} \int_0^t \lim_{n \rightarrow \infty} \mathbb{E}[\partial^N g(z + W^n(s))] ds. \quad (25)$$

Setting

$$u(t, z) = \lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(t))],$$

this is equivalent to say that $u(t, z)$ is a classical solution of the N -th order Cauchy problem

$$\begin{aligned} \partial_t u(t, z) &= \frac{\alpha}{N!} \partial^N u(t, z), \\ u(0, z) &= g(z), \end{aligned} \quad (26)$$

Proof. By assumption, we can start from the Itô's formula (23); by taking the expectation in both sides we get

$$\mathbb{E}[g(z + W^n(t)) - g(z)] = \sum_{m=1}^{\infty} \frac{\alpha}{(mN)!} \left(\frac{\alpha}{n}\right)^{m-1} \int_0^t \mathbb{E}[\partial^{mN} g(z + W^n(s))] ds \quad (27)$$

then we take the limit as $n \rightarrow \infty$ and apply Corollary 16 to obtain (25).

The second claim of the theorem follows once we prove that

$$\lim_{n \rightarrow \infty} \mathbb{E}[\partial^N g(z + W^n(s))] = \partial^N \lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(s))]$$

Notice that, by Theorem 12, we can write

$$\lim_{n \rightarrow \infty} \mathbb{E}[\partial^N g(z + W^n(s))] = \sum_{h=0}^{\infty} \frac{\partial^{hN} g^{(N)}(z)}{h!} \left(\frac{\alpha t}{N!}\right)^h = \partial^N \sum_{h=0}^{\infty} \frac{\partial^{hN} g(z)}{h!} \left(\frac{\alpha t}{N!}\right)^h = \partial^N \lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(s))]$$

so (26) follows. \square

For a generalization of these results to the case of the boundary value problem associated with (26) on a bounded interval with Dirichelet, resp. Neumann, resp. periodic boundary conditions see also [8].

4.1. Itô formula for polynomials

Assume for this section that $g(z) = c_\beta z^\beta$ is a polynomial of order $\beta = bN$. Then we can calculate some examples using the extended Itô's formula (25). Actually, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(t))] - g(z) &= c_\beta \frac{(bN)!}{((b-1)N)!} \frac{\alpha}{N!} \int_0^t \lim_{n \rightarrow \infty} \sum_{j=0}^{b-1} \binom{(b-1)N}{jN} z^{(b-1-j)N} \mathbb{E}[W^n(s)^{jN}] ds \\ &= c_\beta \sum_{j=1}^b \frac{(bN)!}{[(b-j)N]!} z^{(b-j)N} \left(\frac{\alpha}{N!}\right)^j \frac{t^j}{j!}. \end{aligned}$$

In the special case $\beta = N$ the Itô's formula (23) gives

$$(W^n(t))^N = \sum_{k=1}^N \binom{N}{k} \int_0^t (W^n(s))^{N-k} d(W^n(s))^k.$$

4.2. Itô formula for Wiener-type integrals

Let $\varphi : [0, T] \rightarrow \mathbb{C}$ be a continuous and bounded function. Then we can define the Wiener integral

$$\int_0^t \varphi(s) dW_n(s) = \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \varphi\left(\frac{\tau}{n}\right) (W_n\left(\frac{\tau+1}{n}\right) - W_n\left(\frac{\tau}{n}\right)) = \frac{1}{n^{1/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \varphi\left(\frac{\tau}{n}\right) \xi_{\tau+1} \quad (28)$$

and, for any $k \geq 1$,

$$\int_0^t \varphi(s) d(W_n(s))^k = \frac{1}{n^{k/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \varphi\left(\frac{\tau}{n}\right) (\xi_{\tau+1})^k. \quad (29)$$

These formulas define, for “sufficiently regular” $\varphi : \mathbb{R} \rightarrow \mathbb{C}$, a family of stochastic processes $X_{k,n}(t) = \int_0^t \varphi(s) d(W_n(s))^k$ which extend the construction of the previous section.

We have the following application, which generalizes Theorem 11 in [8].

Theorem 19. *Let us consider the initial value problem*

$$\begin{aligned}\partial_t u(t, x) &= \frac{\alpha}{N!} (\varphi(t))^N \partial_x^N u(t, x), & t \in [0, \infty), \\ u(0, x) &= f(x), & x \in \mathbb{R}.\end{aligned}\tag{30}$$

where $\varphi \in C_b([0, T]; \mathbb{C})$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ is an analytic function of exponential type. Then the function $u(t, x)$ defined by

$$u(t, x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[f \left(x + \int_0^t \varphi(s) dW_n^N(s) \right) \right]\tag{31}$$

for $t \in [0, T]$ is a classical solution of the parabolic problem (30).

Proof. We may proceed as in the proof of Theorem 18. Let us consider the process $X_n(t) = \int_0^t \varphi(s) dW_n(s)$; we search for an Itô formula for the process $f(z + X_n(t))$. Notice that

$$f(z + X_n(t)) - f(z) = \sum_{\tau=0}^{\theta-1} f(z + X_n(\frac{\tau+1}{n})) - f(z + X_n(\frac{\tau}{n})) = \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \sum_{k=1}^{\infty} \frac{1}{k!} \partial^k f(z + X_n(\frac{\tau}{n})) \frac{(\varphi(\frac{\tau}{n}) \xi_{\tau+1})^k}{n^{k/N}}$$

and exchanging the order of the sums, we get

$$f(z + X_n(t)) - f(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^t (\varphi(s))^k \partial^k f(z + X_n(s)) d(W^n(s))^k.\tag{32}$$

Taking the mean in both sides of previous formula we get

$$\mathbb{E}[f(z + X_n(t)) - f(z)] = \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E} \left[\int_0^t (\varphi(s))^k \partial^k f(z + X_n(s)) d(W^n(s))^k \right]\tag{33}$$

and proceeding as in the analysis of (20), and recalling (9), we get

$$\mathbb{E} \left[\int_0^t (\varphi(s))^k \partial^k f(z + X_n(s)) d(W_n(s))^k \right] = \frac{1}{n^{k/N}} \sum_{\tau=0}^{\lfloor nt \rfloor - 1} \varphi(s) \mathbb{E}[\partial^k f(z + X_n(s))] \mathbb{E}[(\xi_{\tau+1})^k]$$

hence we get

$$\mathbb{E} \left[\int_0^t (\varphi(s))^N \partial^k f(z + X_n(s)) d(W_n(s))^k \right] = 0 \quad \text{for all } k \neq mN, m \in \mathbb{N};\tag{34}$$

for $k = N$ we have

$$\mathbb{E} \left[\int_0^t \varphi(s) \partial^N f(z + X_n(s)) d(W_n(s))^N \right] = \alpha \int_0^t (\varphi(s))^N \mathbb{E}[\partial^N f(z + X_n(s))] ds$$

and finally for $k = mN$, $m \in \mathbb{N}$, $m > 1$:

$$\mathbb{E} \left[\int_0^t (\varphi(s))^{mN} \partial^k f(z + X_n(s)) d(W_n(s))^{mN} \right] = \alpha \left(\frac{\alpha}{n} \right)^{m-1} \int_0^t (\varphi(s))^{mN} \mathbb{E}[\partial^{mN} f(z + X_n(s))] ds.\tag{35}$$

We notice that this term is asymptotically dominated in m by $\left(\frac{\alpha}{n}\right)^{m-1} t \|\varphi\|_\infty^{mN} c^{mN}$, c being the exponential type of f .

We then see that the series on the right-hand side of (33) is dominated by a convergent series

$$\sum_{m=1}^{\infty} \frac{1}{(mN)!} \left(\frac{\alpha}{n}\right)^{m-1} t \|\varphi\|_\infty^{mN} (c + \varepsilon)^{mN}$$

where c is the exponential type of f and $\varepsilon > 0$; hence it is possible to pass to the limit as $n \rightarrow \infty$ inside the sum, to get

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(z + X_n(t)) - f(z)] = \alpha \int_0^t (\varphi(s))^N \lim_{n \rightarrow \infty} \mathbb{E}[\partial^N f(z + X_n(s))] ds.$$

□

5. The Feynman-Kac formula

The results of the previous section, in particular Theorem 18 and Theorem 19 allow the proof of a probabilistic representation for the solution of heat-type equation of order $N > 2$. In this section we generalize the second statement of Theorem 18 to the case where a time dependent potential is added, proving a Feynman-Kac type formula for the perturbed problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \frac{\alpha}{N!} \frac{\partial^N}{\partial x^N} u(t, x) + V(t, x) u(t, x), & t \in [0, \infty), \\ u(0, x) &= f(x), & x \in \mathbb{R}. \end{aligned} \tag{36}$$

i.e. a probabilistic representation of the form

$$u(t, x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[f(x + W_n(t)) e^{\int_0^t V(t-s, x+W_n(s)) ds} \right] \tag{37}$$

Since the sequence of processes $\{W_n(t)\}$ does not properly converge, the existence of the limit on the right hand side of Eq. (37) is not assured, even for smooth and bounded potential function V . Indeed the result of section 2 allow to prove the "asymptotic integrability" of cylinder functions of the form $W_n \mapsto f(W_n(t))$, i.e. the existence of the limit $\lim_{n \rightarrow \infty} \mathbb{E}[f(W_n(t))]$, for f entire analytic satisfying the assumptions of Theorem 6. The construction of formula (37) requires in fact the proof of the integrability of more general (non-cylinder) functions of the form $W_n \mapsto \exp\left(\int_0^t f(s, W_n(s)) ds\right)$.

Theorem 20. *Let $a \in L^1([0, t])$. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[e^{\int_0^t a(s) W_n(s) ds} \right] = \exp \left(\frac{\alpha}{N!} \int_0^t \left(\int_s^t a(u) du \right)^N ds \right)$$

Proof. Set $t_j := \frac{j}{n}$, for $j = 0, \dots, [nt]$

$$\begin{aligned} \mathbb{E} \left[e^{\int_0^t a(s) W_n(s) ds} \right] &= \mathbb{E} \left[e^{\sum_{j=0}^{[nt]-1} W_n(t_j) \int_{t_j}^{t_{j+1}} a(s) ds} \right] \\ &= \mathbb{E} \left[e^{\sum_{j=1}^{[nt]} \alpha^{1/N} n^{-1/N} \xi_j \int_{t_{j-1}}^t a(s) ds} \right] = \prod_{j=1}^{[nt]} \mathbb{E} \left[\exp \left(\alpha^{1/N} n^{-1/N} \xi_j \int_{t_{j-1}}^t a(s) ds \right) \right] \\ &= \prod_{j=1}^{[nt]} \left(e^{\frac{\alpha}{N!n} \left(\int_{t_{j-1}}^t a(s) ds \right)^N} + R(n, j) \right) \\ &= e^{\sum_{j=1}^{[nt]} \frac{\alpha}{N!n} \left(\int_{t_{j-1}}^t a(s) ds \right)^N} + \mathcal{R}_n \end{aligned}$$

where, by the estimate in Lemma 2 and the boundedness of the function $s \mapsto \int_s^t a(\tau) d\tau$ ², we have that there exists a constant $C \in \mathbb{R}$ such that for all $j = 1, \dots, \lfloor nt \rfloor$ the inequality $|R(n, j)| \leq C/n^2$ holds. Hence, consequently,

$$|\mathcal{R}_n| \leq \left(1 + \frac{C'}{n^2}\right)^n - 1,$$

where $C' = Ce^{\frac{|a|}{N^{1/n}} \|a\|_{L^1([0,t])}^N}$. By taking the limit for $n \rightarrow \infty$ we get the thesis. \square

Let us consider now the initial value problem (36) with a time dependent potential which is linear in the space variable, i.e. $V : [0, t] \times \mathbb{R} \rightarrow \mathbb{R}$ is of the form $V(\tau, x) = A(\tau)x$, where $A : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous function. In this case the results of Theorem 20 allow us to prove the Feynman-Kac formula (37).

Theorem 21. *Let $A : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous function and $f : \mathbb{R} \rightarrow \mathbb{C}$ the Fourier transform of a complex Borel measure on \mathbb{R} with compact support. Then the classical solution of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \frac{\alpha}{N!} \frac{\partial^N}{\partial x^N} u(t, x) + A(t)xu(t, x), & t \in [0, \infty), \\ u(0, x) &= f(x), & x \in \mathbb{R}. \end{aligned} \quad (38)$$

is given by

$$u(t, x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[f(x + W_n(t)) e^{\int_0^t A(t-s)(W_n(s)+x) ds} \right].$$

Proof. By considering the function $a : [0, t] \rightarrow \mathbb{R}$ defined as $a(s) := A(t-s)$, $s \in [0, t]$ and by representing the function $f : \mathbb{R} \rightarrow \mathbb{C}$ in the form $f(x) = \int_{\mathbb{R}} e^{iyx} d\mu_f(y)$, with μ Borel measure on \mathbb{R} with compact support, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \left[f(x + W_n(t)) e^{\int_0^t A(t-s)(W_n(s)+x) ds} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\mathbb{R}} e^{iyx+iyW_n(t)} e^{\int_0^t A(t-s)W_n(s) ds} e^{x \int_0^t A(t-s) ds} d\mu(y) \right] \\ &= e^{x \int_0^t A(t-s) ds} \lim_{n \rightarrow \infty} \int_{\mathbb{R}} e^{iyx} \mathbb{E} \left[e^{iyW_n(t)} e^{\int_0^t A(t-s)W_n(s) ds} \right] d\mu(y) \end{aligned}$$

By Theorem 20 and dominated convergence, the latter line converges to

$$e^{x \int_0^t A(t-s) ds} \int_{\mathbb{R}} e^{iyx} e^{\frac{\alpha}{N!} \int_0^t (iy + \int_s^t A(t-u) du)^N ds} d\mu(y),$$

which is, as one can directly verify, the classical solution of the Cauchy problem (38). \square

6. Stopping times

Now replace t with a stopping time τ_n for the stochastic process W^n in the above formulas, with τ_n finite a.s. Clearly

$$H_t^{n,k} := \int_0^t g(W^n(s)) d(W^n(s))^k = \sum_{j=0}^{\lfloor nt \rfloor - 1} g(W^n(\frac{j}{n})) \left(\frac{\xi_{j+1}}{n^{1/N}} \right)^k$$

²Indeed $|\int_s^t a(\tau) d\tau| \leq \|a\|_{L^1([0,t])}$

is a martingale for $k \neq mN$. To see this calculate $\mathbb{E}(H_t^{n,k} | \mathcal{F}_s) = H_s^{n,k}$ using the fact that $H_t^{n,k} = (H_t^{n,k} - H_s^{n,k}) + H_s^{n,k}$ and $\mathbb{E}[H_t^{n,k} - H_s^{n,k}] = 0$ due to (21).

Thus the stopped process $H_{\tau_n \wedge t}^{n,k}$ is a martingale, and by the Optional Stopping Theorem (also called Doob's Optional Sampling Theorem) $\mathbb{E}(H_{\tau_n \wedge t}^{n,k}) = \mathbb{E}(H_0^{n,k})$. Thus

$$\mathbb{E} \int_0^{\tau_n} g(z + W^n(s)) d(W^n(s))^k = 0, \quad \text{for } k \neq mN. \quad (39)$$

According to Itô's formula (23) we have

$$g(z + W^n(\tau_n)) - g(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\tau_n} \partial^k g(z + W^n(s)) d(W^n(s))^k.$$

Assume that τ_n is bounded \mathbb{P} -a.s. (for instance, $\tau_n \leq T$). Taking the expectation of both sides of the previous identity we get

$$\mathbb{E}[g(z + W^n(\tau_n))] - g(z) = \mathbb{E} \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\tau_n} \partial^k g(z + W^n(s)) d(W^n(s))^k$$

and since \mathbb{E} is a finite sum

$$= \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E} \int_0^{\tau_n} \partial^k g(z + W^n(s)) d(W^n(s))^k$$

which thanks to (39) is

$$\begin{aligned} &= \sum_{k=1}^{\infty} \frac{1}{(kN)!} \mathbb{E} \int_0^{\tau_n} \partial^{kN} g(z + W^n(s)) d(W^n(s))^{kN} \\ &= \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)!} \frac{1}{n^{k-1}} \mathbb{E} \int_0^{\tau_n} \partial^{kN} g(z + W^n(s)) ds \end{aligned}$$

and passing to the limit as $n \rightarrow \infty$ we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[g(z + W^n(\tau_n))] - g(z) &= \lim_{n \rightarrow \infty} \sum_{h=1}^{\infty} \frac{\alpha^h}{(hN)!} \frac{1}{n^{h-1}} \mathbb{E} \int_0^{\tau_n} \partial^{hN} g(z + W^n(s)) ds \\ &= \frac{\alpha}{N!} \lim_{n \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} [\partial^N g(z + W^n(s))] ds^{(3)}. \end{aligned} \quad (40)$$

Remark 4. Formula (40) is a generalization of the fundamental theorem of calculus. For $N = 1$ and $\alpha = 1$ the W^n are deterministic and converge to the line $W_t^\infty = t$. Then (40) reads

$$g(z + t) - g(z) = \int_0^t g'(z + s) ds$$

³by using the boundedness of τ and Corollary 16, just like in the proof of Theorem 18

for $z + t$ being the first hitting point to the boundary along the line to the right of z .

When $N = 2$ the process W^n converges to the 1-dimensional Wiener process W and τ_n converges to the Wiener process hitting time τ and we have

$$\mathbb{E}[g(z + W_\tau)] - g(z) = \frac{1}{2} \mathbb{E} \int_0^\tau g''(z + W_s) ds$$

so the average value of the integral of g'' from z to the boundary is the difference of the second antiderivative averaged on the boundary and at z . Notice W_τ has only 2 values, the hitting points to the right and left of z , since W is the 1-dimensional Wiener process. E.g., in the particular, symmetric case of U being the disc centered at z of radius r , we get

$$\frac{[g(z + r) - g(z)] + [g(z - r) - g(z)]}{2} = \frac{1}{2} \mathbb{E} \int_0^\tau g''(z + W_s) ds.$$

Now we specialize the previous formula to the case τ_n is the exit time of $W^n(t)$ from the ball $B(0, R)$. We shall see that an analytic function $g(z)$ is determined by the average on the boundary of any ball $B(z, R)$. This is a generalization of Gauss' mean value theorem for analytic functions, a classical result in complex analysis for the case of Brownian motion.

Theorem 22. *Let τ_n be the exit time of $W^n(t)$ from the ball $B(0, R)$. Then*

$$\lim_{n \rightarrow \infty} \tau_n = 0 \quad a.s. \quad (41)$$

Proof. Let $S(t)$, $t \in \mathbb{N}$, be the random walk defined by the ξ_j 's: $S(t) = \sum_{j=1}^t \xi_j$. It is immediate to see that $S(t)$

is a martingale as well as $|S(t)|^2 - \alpha t$:

$$\mathbb{E}[|S(t+1)|^2 - |S(t)|^2] = \mathbb{E} \left[\sum_{i,j=1}^{t+1} \xi_i \bar{\xi}_j - \sum_{i,j=1}^t \xi_i \bar{\xi}_j \right] = \mathbb{E} \left[|\xi_{t+1}|^2 + \sum_{i=1}^t \xi_i \bar{\xi}_{t+1} + \sum_{j=1}^t \xi_{t+1} \bar{\xi}_j \right] = \mathbb{E}[|\xi_{t+1}|^2]$$

where in the last equality we used the independence of ξ_i 's. Let $A_n = \{z \in \mathbb{C} \text{ s. th. } |z| \leq Rn^{1/N}\}$ and

$$T_{A_n} = \inf\{t \geq 0 \text{ s. th. } S(t) \notin A_n\}.$$

It is $\mathbb{P}(T_{A_n} < \infty) = 1$ (see e.g. [Lawler 2010]); then the stopped martingale $M^n(t) = |S(t \wedge T_{A_n})|^2 - (t \wedge T_{A_n})$ satisfies

$$0 = \mathbb{E}[M^n(0)] = \lim_{t \rightarrow \infty} \mathbb{E}[|S(t \wedge T_{A_n})|^2 - (t \wedge T_{A_n})] = \mathbb{E}[|S(T_{A_n})|^2 - (T_{A_n})]$$

and using the estimate $n^{2/N} R^2 \leq |S(T_{A_n})|^2 \leq n^{2/N} R^2 + 1$ it follows that

$$n^{2/N} R^2 \leq \mathbb{E}[T_{A_n}] \leq n^{2/N} R^2 + 1.$$

Notice that

$$\tau_n = \inf \left\{ s = j/n \text{ s. th. } \frac{1}{n^{1/N}} S(j) \notin B(0, R) \right\}$$

i.e., $\tau_n = \frac{1}{n} T_{A_n}$, so we finally get

$$n^{\frac{2}{N}-1} R^2 \leq \mathbb{E}[\tau_n] \leq n^{\frac{2}{N}-1} R^2 + \frac{1}{n}$$

which converges to 0 as $n \rightarrow \infty$. Since τ_n is non-negative, this concludes the proof. \square

Finally, we have a stochastic characterization of higher order complex derivatives.

Theorem 23. *Assume that g is an analytic function of exponential type c . Define τ_n be the exit time of W^n from $B(0, R)$. Then*

$$\frac{\alpha}{N!}g^{(N)}(z) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{\tau_n} [g(z + W^n(\tau_n)) - g(z)] \right].$$

Proof. With no loss of generality we set $z = 0$. As in the proof of formula (40) we calculate

$$\mathbb{E} \left[\frac{1}{\tau_n} [g(W^n(\tau_n)) - g(0)] \right] = \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E} \left[\frac{1}{\tau_n} \int_0^{\tau_n} g^{(k)}(W^n(s)) d(W^n(s))^k \right]$$

since $\tau_n \leq T$ we take conditional expectation to get

$$\begin{aligned} &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^T \mathbb{E} \left[\frac{1}{t} \int_0^t g^{(k)}(W^n(s)) d(W^n(s))^k \right] \mathbb{P}(\tau_n \in dt) \\ &= \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} \int_0^T \left[\frac{1}{t} \int_0^t \mathbb{E}[g^{(kN)}(W^n(s))] ds \right] \mathbb{P}(\tau_n \in dt). \end{aligned}$$

From Theorem 12 we know that

$$\begin{aligned} \mathbb{E}[g^{(kN)}(W^n(s))] &= \gamma_k(s) + R_k(n, s), \\ \gamma_k(s) &= \sum_{h=0}^{\infty} \frac{\partial^{(h+k)N} g(0)}{h!} \left(\frac{\alpha s}{N!} \right)^h \quad \text{and } R_k(n, s) \leq \frac{1}{n} C(\alpha, k, T). \end{aligned}$$

By using the bound $|\partial^{(n)} g(0)| \leq (c + \varepsilon)^n$ for some $\varepsilon > 0$, where c is the exponential type of g , we get also

$$|\gamma_k(s)| \leq (c + \varepsilon)^{kN} \exp \left(\frac{\alpha s (c + \varepsilon)^N}{N!} \right).$$

Then we get

$$\begin{aligned} \mathbb{E} \left[\frac{1}{\tau_n} [g(W^n(\tau_n)) - g(0)] \right] &= \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} \int_0^T \left[\frac{1}{t} \int_0^t \gamma_k(s) + R_k(n, s) ds \right] \mathbb{P}(\tau_n \in dt) \\ &= \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} \int_0^T \left[\frac{1}{t} \int_0^t \gamma_k(s) ds \right] \mathbb{P}(\tau_n \in dt) + \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} \int_0^T \left[\frac{1}{t} \int_0^t R_k(n, s) ds \right] \mathbb{P}(\tau_n \in dt) \end{aligned}$$

By using the integral mean value theorem and the estimate on R_k we get

$$\mathbb{E} \left[\frac{1}{\tau_n} [g(W^n(\tau_n)) - g(0)] \right] \approx \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} \mathbb{E}[\gamma_k(\varepsilon \tau_n)] + \frac{1}{n} \sum_{k=1}^{\infty} \frac{\alpha^k}{(kN)! n^{k-1}} C(\alpha, k, T)$$

for some $\varepsilon \in (0, 1)$. Recall that $C(\alpha, k, T) \approx c_1 \left(\frac{c_2 k}{\log(1+kN)} \right)^{kN}$, so that the second series in previous formula converges uniformly in n ; we write again

$$\mathbb{E} \left[\frac{1}{\tau_n} [g(W^n(\tau_n)) - g(0)] \right] \approx \frac{\alpha}{N!} \mathbb{E}[\gamma_1(\varepsilon \tau_n)] + \frac{1}{n} \sum_{k=0}^{\infty} \frac{\alpha^{k+2}}{((k+2)N)! n^k} \mathbb{E}[\gamma_{k+2}(\varepsilon \tau_n)] + \frac{1}{n} C(\alpha, T). \quad (42)$$

We notice that

$$\sum_{k=0}^{\infty} \left| \frac{\alpha^{k+2}}{((k+2)N)!} \frac{1}{n^k} \mathbb{E}[\gamma_{k+2}(\varepsilon\tau_n)] \right| \leq \sum_{k=0}^{\infty} \frac{\alpha^{k+2}}{((k+2)N)!} (c+\varepsilon)^{(k+2)N} \exp\left(\frac{\alpha T(c+\varepsilon)^N}{N!}\right) < +\infty$$

hence we can pass to the limit for $n \rightarrow \infty$ in (42), recalling that $\tau_n \rightarrow 0$, and we get the thesis.

□

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