Computing interval weights for incomplete pairwise-comparison matrices of large dimension - a weak consistency based approach

Věra Jandová, Jana Krejčí, Jan Stoklasa, and Michele Fedrizzi

Abstract

Multiple-criteria decision making (MCDM) and evaluation problems dealing with a large number of objects are very demanding. Particularly when the use of pairwise-comparison (PC) techniques is required. A major drawback arises when it is not possible to obtain all the PCs, due to time or cost limitations, or to split the given problem into smaller subproblems. In such cases, two tools are needed to find acceptable weights of objects: an efficient method for partially filling a pairwise-comparison matrix (PCM) and a suitable method for deriving weights from this incomplete PCM. This paper presents a novel interactive algorithm for large-dimensional problems guided by two main ideas: the sequential optimal choice of the PCs to be performed and the concept of weak consistency. The proposed solution significantly reduces the number of needed PCs by adding information implied by the weak consistency after the input of each PC (providing sets of feasible values for all missing PCs). Interval weights of objects are computed from the resulting incomplete weakly consistent PCM adapting the methodology for calculating fuzzy weights from fuzzy PCMs. The computed weight intervals thus cover all possible weakly consistent completions of the incomplete PCM. The algorithm works both with Saaty's PCMs and fuzzy preference relations (FPRs). The performance of the algorithm is illustrated by a numerical example and a real-life case study. The performed simulation demonstrates that the proposed algorithm is capable of reducing the number of PCs required in PCMs of dimension 15 and greater by more than 60% on average.

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Index Terms

incomplete matrix, weak consistency, fuzzy preference relation, interval weights, multiple-criteria decision making, pairwise comparison, large-dimensional problems, guided decision support.

I. Introduction

Decision-making methods based on PCs (such as Saaty's Analytic Hierarchy Process - AHP or methods using FPRs) are a popular tool in MCDM. Weights of objects (i.e. weights of criteria or evaluations of alternatives) and information concerning the preference ordering of objects can be expressed by the decision makers (DMs) by comparing objects pairwise and by expressing the strength of preferences between them. These methods also usually require means for the assessment of the consistency of the DMs' preferences and mechanisms for achieving sufficient consistency of the information provided by the DMs.

When the methods of the AHP type are confronted with real-life problems where a large amount, say n, of objects needs to be mutually compared, the $n(n-1)/2$ PCs needed to get a complete PCM are not easy to obtain in sufficient quality. In fact, the more PCs need to be made, the less reliable the information expressed by the DMs might be (due to fatigue, due to time constraints and similar factors). In these cases, Saaty suggests (see e.g. [39], [40], [42]) to split the large problems into several subproblems of smaller dimensions. This may imply creating supercategories of the objects being compared (or creating the so-called rating categories for criteria [26], [39]), and usually requires to provide also the mutual comparison of these supercategories. The objects are then mutually compared only within the defined supercategories (or in case of rating categories, the objects are not directly compared at all but instead, the more abstract rating categories are compared). This results in the reduction of the complexity of the problem and in making the information requirements (the number of PCs needed) feasible. However we can also observe a slight loss of information (some objects - e.g. from different supercategories - are not directly mutually compared). This loss of information can be to some extent compensated for by introducing a strong enough consistency condition on the preferences expressed by the experts, which provides means of calculating the missing values in the PCM.

For some real-life problems, the above described approach works fine. There are, however, situations when splitting the problem into several smaller ones renders parts of the problem too abstract and hence intractable for the experts providing the information on the preferences among objects. Stoklasa, Jandová and Talašová [46] provide an up-to-date real-life example of such a problem in the area of arts evaluation. Paper [46] refers to the development of the evaluation model for the Registry of Artistic Performances that has been used in the Czech Republic (CZ) within the ‘principles and rules of financing public universities’ [34]. This model has been used since 2012 to provide a basis for the distribution of a part of the subsidy from the state budget among public universities in the CZ. The mathematical model presented in [46] is designed to compute evaluations (weights) for different categories of works of art (currently 27 categories) based on a combination of expert assessment of the significance of the respective work of art and two more objective criteria (extent and institutional reception). The authors deal with a 27x27 PCM that represents a problem that could not be split into several smaller ones due to partial dependencies among the evaluation criteria and due to the necessity of providing real-life examples to all the compared objects for the experts to be able to express their intensities of preference.

In large-dimensional problems (a large number of objects to be compared pairwise) which cannot be split into smaller subproblems, the required weights of objects may be obtained from incomplete PCMs. This however raises issues connected with not providing all the PCs in the PCM by the DM. Focusing on an appropriate reduction of the number of PCs which have to be provided by the DM, and obtaining enough information in the incomplete PCM to be able to compute the weights are of paramount importance. When using incomplete PCMs, we have to deal with two key tasks adequately: finding a method for efficiently selecting the subset of the $n(n-1)/2$ PCs to be provided by the DM and finding a method for deriving the weight vector from the incomplete PCM.

Harker [18], [19], [20] and later Harker and Millet [21] were the first authors who dealt with the problem of reducing the number of PCs. They proposed to perform only a part of the $n(n-1)/2$ PCs by means of an algorithm which iteratively selects the ‘next’ PC to be submitted to the DM. This selection is made according to the largest modification in the weight vector. The process of inputting PCs is stopped when the given PC changes the weight vector by less than a fixed threshold. Wedley, Schoner and Tang [49] focused on the choice of the $n-1$ PCs, which is the minimum number required for comparing n elements, and they compared and discussed several methods of entering them. Sanchez and Soyer [43] proposed to use entropy-based measures of the information content to evaluate judgment accuracy and to state a stopping rule of the process of filling in PCs. Ra [37] worked with n PCs which form a closed chain. Fedrizzi and Giove [17] proposed a method which takes into account both the robustness of the collected data and the consistency of the expressed preferences.

For what concerns the methods to derive the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ for the objects from incomplete PCMs, several different approaches have been proposed; see e.g. [2], [9], [16], [17], [18], [19], [21], [30], [31], [35], [38], [45], [50], [51]. Some of these methods are aimed at determining the missing PCs in order to complete the incomplete PCM. Once the PCM is filled in, one of the known methods for deriving the weights from a complete PCM can be used. Conversely, other methods compute the weights from the incomplete PCM directly. Clearly, having first computed the weights $w_1, w_2, ..., w_n$, every missing PCs can then be determined accordingly, thus completing the PCM.

In this paper, we aim to propose an algorithm for obtaining weights of objects for large incomplete PCMs where the consistency preservation plays a crucial role. The aim of this paper is to merge into an interactive algorithm the sequencing method for inputting incomplete PCMs suggested in [17] with the concept of weak consistency (see [27] and [46]) as a minimum requirement for the consistency of expert's preferences (expressed either using Saaty's PCM or a FPR). Since the weak consistency is defined in [27] for Saaty's PCMs, we introduce in this paper the additive version of this condition to be used with FPRs. This way we strive to contribute to the existing body of research on the use of incomplete FPRs in MCDM (see e.g. [16], [17], [38], [36], [50]) and group MCDM (see e.g. [10], [23], [24], [32]), by providing tools to efficiently deal with large incomplete FPRs. We show that combining a sequencing method for inputting incomplete Saaty's PCMs and FPRs with the weak consistency can dramatically reduce the number of PCs needed to provide sufficient information for the ranking of objects and also to obtain reliable final weights.

The objective of our paper is not simply the reduction of the number of required PCs to be provided by the DM. It is known that this number could be radically reduced to $n-1$, as proposed in [25] and in other papers. Such choice completely fulfills the requirement of maximally reducing the number of needed PCs but, in our view, it gives up a fundamental characterizing property of the PC methods. This property is the ability to use the redundancy of information contained in a PCM in order to suitably manage the unavoidable inconsistency of human judgements. Thus, a better accuracy and reliability of the output, by means of the compensation of possible contradictory elicited judgements [49] is guaranteed.

In the numerical example in section III, it is demonstrated that methods requiring only initial $n-1$ PCs do not always result in reliable outcomes. In cases when the DM is capable to provide only minimal amount of information, a better choice than relying on a PC-based method and providing only $n-1$ PCs could be the application of a method from a completely different family. For instance DRSA [44], a method which establishes a set of decision rules from the given set of criteria and which leads to the classification of objects (e.g. acceptable, uncertain, non-acceptable), can be used.

The PC-based methods discussed in this paper assume independent criteria. In case of dependencies among criteria, a different class of methods should be considered, e.g. the Hybrid Multiple Attribute Decision-Making (MADM) Model [22] which is based on these steps: first the relationships among criteria are determined by the DEMATEL technique, next the influential weights are derived through the DANP (DEMATEL-based ANP) and finally objects are ranked by a modified VIKOR method. The evaluations of alternatives are not obtained by PCs, they are determined based on their distance from an aspiration level (unlike traditional models, where the distance from the best alternative of set of considered alternatives is used). These methods analyse the relationships among criteria and introduce aspiration levels to the MADM context.

We, however, restrict ourselves to the traditional and frequently used basic approach to MCDM via PCs. This paper strives to find an ideal compromise between requiring as little information from the DM as possible and still obtaining enough information to calculate weights of objects that are close to the (hypothetical) full-information case. In our view, there are many real-world problems where both the reduction of the number of PCs and suitable accuracy of the output are crucial issues that must be taken into account. This was a central concern that guided the development of our method.

Our approach differs from the PC-based ones mentioned above since in all steps of our method the weak consistency of the incomplete PCM is preserved. A similar property is not required in any other known method. Moreover, the final weights of objects provided by the algorithm proposed in this paper are computed in such a way that they contain information concerning the uncertainty which stems from the fact that some PCs are not provided by the DM. The weights of objects are computed as intervals in order to reflect the missing information in the incomplete PCM and to provide ranges for the values of the real weights of objects obtainable for any weakly consistent completion of the incomplete PCM. The range of the interval weights depends on the amount of information that is missing in the PCM. The formulas for calculating fuzzy weights from a fuzzy PCM proposed in [28] and applied on a practical example in [29] are utilized for the computation of these interval weights.

The paper is organized as follows. In Section II, preliminary notions on Saaty's PCMs and FPRs are provided, the form of the weak-consistency property introduced in [27] is defined for FPRs, and the iterative algorithm for the optimal choice of the PC performed by the DM introduced in [17] is recalled. In Section III, a new algorithm for optimal choice of PCs performed by the DM, based on the combination of the algorithm proposed by Fedrizzi and Giove [17] and the weak-consistency condition, is introduced. The method of obtaining an incomplete weakly consistent PCM and deriving the interval weights is also described on an illustrative example. The same example is solved using the method proposed by Herrera-Viedma et al. in [25] in order to compare the results and to demonstrate the effectiveness of our method. In Section IV, a case study of a real-life problem, that might benefit from the use of the proposed algorithm, is provided. The model for evaluating artistic production in the CZ studied in [46] is described here, and the numerical results obtained in [46] are compared with the results obtained by the novel algorithm proposed in this paper. The conclusion with some final remarks and a discussion of the performance of the proposed algorithm are given in Section V.

II. Preliminaries

A. Pairwise-comparison methods

Let us consider n objects $A_1, A_2, \dots, A_n$ which need to be compared or to which weights need to be assigned. Let us also assume that the PC methods can be applied. These methods are based on the construction of PCMs which express the preference intensities between pairs of objects. Saaty's PCM or FPR can be used.

One of the most frequently used PC methods is *Saaty's Analytic Hierarchy Process* (AHP); see [41]. Saaty's PCM is represented by a square matrix $S = \{s_{ij}\}_{i,j=1}^{n}$ with elements s_{ij} from Saaty's scale where the meaning of s_{ij} is given by Table I. If an object A_i is preferred over A_j, an appropriate value of s_{ij} is chosen by the DM from Table I. Otherwise, a reciprocal value is used for s_{ji}. Hence, the reciprocity condition for Saaty's PCM (i.e. multiplicative reciprocity) is required, i.e. $s_{ji} = \frac{1}{s_{ij}}$ for all $i, j = 1, 2, \dots, n.$ As s consequence, it is $s_{ii} = 1$ for all $i = 1, 2, \dots, n.$

In order to obtain a useful vector of weights of objects from Saaty's PCM, the given preference intensities s_{ij}, $i, j = 1, 2, \dots, n$, must be entered by the DM in S in a reasonable way. This means that Saaty's PCM should be close to consistency. The consistency condition in Saaty's

TABLE I: Elements of Saaty's scale and corresponding linguistic labels

Preference intensity s_{ij}	Linguistic labels
1	object A_i is equally preferred as object A_j
3	object A_i is moderately preferred over object A_j
5	object A_i is strongly preferred over object A_j
7	object A_i is very strongly preferred over object A_j
9	object A_i is extremely preferred over object A_j
2,4,6,8	Intermediate values between the two adjacent judgements

sense (i.e. multiplicative consistency) means that for all $i, j, k = 1, 2, \ldots, n$ the following must hold:

$$s_{ij}s_{jk} = s_{ik}. \tag{1}$$

The elements of S express the relation between multiplicative weights w_i and w_j in the form of $s_{ij} = \frac{w_i}{w_j}$ for all $i, j = 1, 2, \ldots, n$. There are several methods for deriving the weights of objects $A_1, A_2, \ldots, A_n$ from Saaty's PCM $S = \{s_{ij}\}_{i,j=1}^n$ that can be used; see e.g. [7] for an overview. Saaty suggests to derive the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ as the principal eigenvector of S associated with the principal eigenvalue λ_{max}, i.e. $S\mathbf{w} = \lambda_{max}\mathbf{w}$. Another commonly used method for computing the weights of objects from S is the geometric mean method, where the weights w_i are computed as

$$w_i = \sqrt[n]{\prod_{j=1}^{n} s_{ij}} \tag{2}$$

for all $i = 1, 2, \ldots, n$. If required, the normalization of the weight vector $\mathbf{w}$ is performed dividing each weight by $\alpha = \sum_{i=1}^n w_i$, which, in the case when the geometric mean method is used to compute weights, translates into $\alpha = \sum_{i=1}^n (\prod_{j=1}^n s_{ij})^{1/n}$.

Another option to compare objects pairwise is to use FPRs; see e.g. [36], [48]. Given a non-empty finite set of objects $X = \{A_1, A_2, ..., A_n\}$, a fuzzy (binary) preference relation on X is determined by a fuzzy set on the Cartesian product $X \times X$, that is, by a membership function $\mu_R : X \times X \to [0, 1]$ over the set $X \times X$. A FPR can be represented by a square PCM $R = \{r_{ij}\}_{i,j=1}^n$ where $r_{ij} = \mu_R(A_i, A_j)$, $i, j = 1, 2, \ldots, n$. The meaning of r_{ij} is given by Table II.

Similarly to Saaty's PCM, a reciprocity condition for FPRs (i.e. additive reciprocity) is required, $r_{ij} + r_{ji} = 1$ for all $i, j = 1, 2, \ldots, n$. As a consequence, it is $r_{ii} = 0.5$, $i = 1, 2, \ldots, n$.

TABLE II: Elements of a FPR and corresponding linguistic labels

Preference intensity r_{ij}	Linguistic labels
1	object A_i is extremely preferred over object A_j
$r_{ij} \in (0.5, 1)$	object A_i is preferred over object A_j
0.5	object A_i is equally preferred as object A_j
$r_{ij} \in (0, 0.5)$	object A_j is preferred over object A_i
0	object A_j is extremely preferred over object A_i

The consistency condition for FPRs (i.e. additive consistency) was defined such that for all $i, j, k = 1, 2, \ldots, n$ it holds

$$r_{ij} = r_{ik} + r_{kj} - 0.5. \tag{3}$$

The elements of R express the relation between additive weights w_i^R and w_j^R in the form of $r_{ij} = 0.5 + 0.5\,(w_i^R - w_j^R)$ for all $i, j = 1, 2, \ldots, n$; see Tanino [48]. Then, obviously $r_{ij} - r_{ji} = w_i^R - w_j^R$ for all $i, j = 1, 2, \ldots, n$. This way, if $r_{ij} = 0.5$ (which corresponds with the DM's indifference between objects A_i and A_j), then the weights of these two objects must be identical, i.e. $w_i^R = w_j^R$. If $r_{ij} = 0.7$, then $r_{ji} = 0.3$ and the difference of the respective weights, that is $w_i^R - w_j^R$, is equal to the difference of the preference intensities $r_{ij} - r_{ji} = 0.4$.

The weight vector $\mathbf{w^R} = (w_1^R, w_2^R, \ldots, w_n^R)$ corresponding to an additively consistent FPR R is unique up to addition of a constant. If required, normalization of such a weight vector is then performed by adding a constant β to this vector. Fedrizzi and Brunelli, for example, suggest in [15] that $\beta = -\min\{w_1^R, w_2^R, \ldots, w_n^R\}$. The additive weights $w_1^R, w_2^R, \ldots, w_n^R$ of the objects $A_1, A_2, \ldots, A_n$ can be derived from a FPR $R = \{r_{ij}\}_{i,j=1}^n$ using the formula

$$w_i^R = \frac{2}{n} \sum_{j=1}^{n} r_{ij} \tag{4}$$

for all $i = 1, 2, \ldots, n$; see [14].

FPRs are also often called *additive preference relations* and the terms are often used as synonyms (which is the view adopted in this paper), although some authors prefer to distinguish between these terms. Other authors, as De Baets et al. [12], prefer to use the term *reciprocal relation*. Different types of consistency have also been studied. For a more deep insight into this topic, the interested reader can refer to [7], [8] and [12].

Approaches based on Saaty's PCMs and on FPRs are equivalent as was demonstrated in [13]; Saaty's PCM $S = \{s_{ij}\}_{i,j=1}^{n}$ can be transformed into a FPR $R = \{r_{ij}\}_{i,j=1}^{n}$ using the following formula for all $i, j = 1, 2, \ldots, n$:

$$r_{ij} = \frac{1}{2}\left(1 + \log_9 s_{ij}\right). \tag{5}$$

Analogously, a FPR R can be transformed into Saaty's PCM S using the inverse function for all $i, j = 1, 2, \ldots, n$:

$$s_{ij} = 9^{2r_{ij}-1}. \tag{6}$$

Because the approaches based on Saaty's PCMs and on FPRs are equivalent (using transformation formulas (5) and (6)), this means that the matrices S and R carry analogous information. Moreover, if R is additively reciprocal or additively consistent, then S is multiplicatively reciprocal or multiplicatively consistent as well, and vice versa. There is also an apparent relationship between weights obtained from a FPR and from Saaty's PCM computed using formulas (2) and (4):

$$w_i^R = \frac{2}{n}\sum_{j=1}^{n} r_{ij} = \frac{2}{n}\sum_{j=1}^{n}\frac{1}{2}(1 + \log_9 s_{ij}) = 1 + \frac{1}{n}\sum_{j=1}^{n}\log_9 s_{ij} = 1 + \log_9 \sqrt[n]{\prod_{j=1}^{n} s_{ij}} = 1 + \log_9 w_i$$

for all $i = 1, 2, \ldots, n$; analogously $w_i = 9^{w_i^R - 1}$ for all $i = 1, 2, \ldots, n$.

B. *Weak consistency*

In both Saaty's PCM approach and FPR approach, maintaining consistency represented by (1) and (3), respectively, can be problematic. These consistency conditions are not achievable in many real situations because the evaluation scales defined in Tables I and II are restricted.

Let us consider three objects A, B and C and focus on Saaty's PCM case where Saaty's scale is restricted from $\frac{1}{9}$ to 9. If $s_{AB} = 5$ and $s_{BC} = 7$, then from the consistency condition (1) it should follow that $s_{AC} = 35$. However s_{AC} can take maximally the value 9 which means that the requirement (1) is not achievable in this situation. Since some level of violation of the consistency condition (1) is to be expected (and low levels of inconsistency might not prevent us from determining useful weights from S), Saaty [41] proposed the *consistency index* $CI = \frac{\lambda_{max} - n}{n-1}$ to measure the level of inconsistency for the PCM of type $n \times n$. This index is compared with the *random index* RI which is the average consistency index of randomly generated Saaty's PCMs of the dimension n. The PCM S is considered consistent enough if CI

reaches at most 10% of RI, i.e. S is consistent enough if $CR = \frac{CI}{RI} \leq 0.1$; CR is called the *consistency ratio*.

Beside Saaty also various other authors tried to construct alternative consistency measures. Alonso and Lamata [1] and Lamata and Pelaez [33] suggested an alternative consistency measure in the way similar to Saaty. They created a coefficient which is computed after Saaty's PCM is completed and it is compared with a given number. Therefore, it is not possible to verify the satisfaction of this kind of consistency during the process of entering the preference intensities. If fully completed Saaty's PCM is not consistent, the DM has to create a new one. And this might be a significant setback when the PCM is large.

Basile and D'Apuzzo [5] suggested a *weak-consistency condition* as a more suitable alternative to the classic consistency in Saaty's sense, particularly when qualitative criteria (attributes) are considered. Their condition is based on the idea that if A is preferred over B and B is preferred over C, then the preference intensity of A over B must be greater than both, the preference intensity of A over B and the preference intensity of B over C. This condition can be controlled during the process of filling in the preference intensities, nevertheless, it is not achievable in certain situations. If one of the considered preference intensities of A over B or B over C is equal to 9, then the preference intensity of A over C cannot be greater than number 9 which is the maximal value of Saaty's scale.

Stoklasa et al. [46] and Jandová and Talašová [27] proposed their own version of the *weak-consistency condition* for Saaty's PCM (although the name is the same, the concepts and their use are different in [5] and [27]); it was introduced with the intention of specifying a minimum consistency requirement for Saaty's PCM. Their approach is similar to the consistency condition of Basile and D'Apuzzo (see e.g. [3], [4]), however, it is not that strict and it is defined in the way which is easily achievable in real situations. The idea of the weak consistency as introduced in [27] is to require the preference intensity s_{AC} to be at least the maximal value of preference intensities s_{AB} and s_{BC} if A is preferred over B and B is preferred over C. Again, this requirement can be checked during the process of filling the preference intensities in Saaty's PCM. Moreover, if the categories are ordered according to the preference, there is even no need for a software solution to check its fulfillment; see [27] for more details. The weak consistency according to Jandová and Talašová [27] is defined for Saaty's PCM in the following way.

Definition 1: Let $S = \{s_{ij}\}_{i,j=1}^{n}$ be Saaty's PCM where the elements s_{ij}, $i, j = 1, 2, \ldots, n$, are given by the scale described in Table I. We say that S is *weakly consistent* if the following holds for all $i, j, k \in \{1, 2, \dots, n\}$:

$$s_{ij} > 1 \wedge s_{jk} > 1 \quad \Longrightarrow \quad s_{ik} \geq \max\{s_{ij}, s_{jk}\}, \tag{7}$$

$$(s_{ij} = 1 \wedge s_{jk} \geq 1) \vee (s_{ij} \geq 1 \wedge s_{jk} = 1) \quad \Longrightarrow \quad s_{ik} = \max\{s_{ij}, s_{jk}\}. \tag{8}$$

The form of the weak-consistency condition for a FPR will be introduced in this paper as an analogy to its form for Saaty's PCM. Since the weak consistency defined by (7) and (8) found its use with Saaty's PCMs, reformulation of this condition for FPR and presentation of its properties is a reasonable step.

Definition 2: Let $R = \{r_{ij}\}_{i,j=1}^{n}$ be a FPR where the elements r_{ij}, $i, j = 1, 2, \dots, n$, are given by the scale described in Table II. We say that R is *weakly consistent* if the following holds for all $i, j, k \in \{1, 2, \dots, n\}$:

$$r_{ij} > 0.5 \wedge r_{jk} > 0.5 \quad \Longrightarrow \quad r_{ik} \geq \max\{r_{ij}, r_{jk}\}, \tag{9}$$

$$(r_{ij} = 0.5 \wedge r_{jk} \geq 0.5) \vee (r_{ij} \geq 0.5 \wedge r_{jk} = 0.5) \quad \Longrightarrow \quad r_{ik} = \max\{r_{ij}, r_{jk}\}. \tag{10}$$

Applying the transformation formulas (5) and (6), it is obvious that the weak-consistency condition for Saaty's PCM (represented by (7) and (8)) and the weak-consistency condition for a FPR (represented by (9) and (10)) are equivalent. It can be demonstrated that every FPR consistent according to (3) is also weakly consistent and that the weak-consistency condition keeps the transitivity of the preferences; see [27]. If we order the categories from the most preferred one to the least preferred one, it is very easy to control the fulfillment of the weak consistency. Then the upper triangle of the PCM R contains only numbers greater than or equal to 0.5. In such a PCM, the weak consistency means that the sequence of numbers must be non-decreasing in every row of the upper triangle and non-increasing in every column of the upper triangle of R. Moreover, if $r_{ij} = 1$, $i \neq j$, then the rows i and j must be identical, and the same holds for columns i and j, $i, j \in \{1, 2, \dots, n\}$. The proofs of these properties for a FPR are analogous to the proofs for Saaty's PCM demonstrated in [27].

In the weakly consistent FPR $R = \{r_{ij}\}_{i,j}^{n}$, the following properties hold (again this can be proven analogously to the proof for Saaty's PCM presented in [27]):

1) If $r_{ij} \leq 0.5$ and $r_{jk} \leq 0.5$ for $i, j, k \in \{1, 2, \dots, n\}$, then

$$r_{ij} < 0.5 \wedge r_{jk} < 0.5 \quad \Longrightarrow \quad r_{ik} \leq \min\{r_{ij}, r_{jk}\}, \tag{11}$$

$$(r_{ij} = 0.5 \wedge r_{jk} \leq 0.5) \vee (r_{ij} \leq 0.5 \wedge r_{jk} = 0.5) \quad \Longrightarrow \quad r_{ik} = \min\{r_{ij}, r_{jk}\}. \tag{12}$$

2) If $r_{ij} > 0.5$ and $r_{jk} < 0.5$ for $i, j, k \in \{1, 2, \ldots, n\}$, then

$$0.5 < r_{ik} \leq r_{ij}, \quad \text{if } r_{ij} > 1 - r_{jk} = r_{kj}, \tag{13}$$

$$0.5 > r_{ik} \geq r_{jk}, \quad \text{if } r_{ij} < r_{kj}, \tag{14}$$

$$r_{ji} \leq r_{ik} \leq r_{ij}, \quad \text{if } r_{ij} = r_{kj}. \tag{15}$$

3) If $r_{ij} < 0.5$ and $r_{jk} > 0.5$ for $i, j \in \{1, 2, \ldots, n\}$, then

$$0.5 < r_{ik} \leq r_{jk}, \quad \text{if } r_{jk} > 1 - r_{ij} = r_{ji}, \tag{16}$$

$$r_{ij} \leq r_{ik} < 0.5, \quad \text{if } r_{jk} < r_{ji}, \tag{17}$$

$$r_{kj} \leq r_{ik} \leq r_{jk}, \quad \text{if } r_{jk} = r_{ji}. \tag{18}$$

The application of these rules for keeping the weak consistency can be seen in the following example:

1) If $r_{AB} = 0.3$ and $r_{BC} = 0.2$, then from (11) follows $r_{AC} \in [0, 0.2]$.
2) If $r_{AB} = 0.8$ and $r_{BC} = 0.3$, then from (13) follows $r_{AC} \in (0.5, 0.8]$.
3) If $r_{AB} = 0.3$ and $r_{BC} = 0.6$, then from (17) follows $r_{AC} \in [0.3, 0.5)$.
4) If $r_{AB} = 0.4$ and $r_{BC} = 0.6$, then from (18) follows $r_{AC} \in [0.4, 0.6]$.

C. Algorithm of Fedrizzi and Giove for the optimal choice of PCs in large PCMs

In this section, we summarize the algorithm for the optimal choice of PCs as it was described in [17]. Let us consider n objects $A_1, A_2, \ldots, A_n$. To fully complete the PCM in order to derive weights, we require the DM to provide $n(n-1)/2$ PCs. When n is large, providing all these PCs is time demanding and their consistency might fluctuate. In these situations, methods focusing on the reduction of the number of PCs needed can be applied to make the input phase manageable for the DM and even to manage the consistency of the preferences provided by the DM.

One of these methods for reducing number of PCs needed in large FPRs was introduced by Fedrizzi and Giove [17]. The authors proposed an algorithm for determining which element (that has not yet been provided by the DM) of the incomplete FPR should be filled in by the DM in each step. In the following, we describe briefly the proposed method. The interested reader may refer to [17] for a more detailed description.

The algorithm proposed in [17] uses a selection rule based on two criteria. The first criterion, quantified by z_{ij}, is used to achieve enough indirect PCs for missing elements of the PCM.

The second criterion, quantified by p_{ij}, is used to reduce the inconsistency of judgements (or to be more specific to resolve possible consistency issues). A scoring function F is defined to determine the usefulness of selecting a particular pair of not yet mutually compared objects $\{A_i, A_j\}$, $i, j \in \{1, 2, \dots n\}$. A high value of the scoring function indicates high necessity to compare A_i with A_j, $i, j \in \{1, 2, \dots n\}$. Thus, at each step, the pair of objects with the maximal value of F is selected. The scoring function is defined as

$$F(z_{ij}, p_{ij}) = \lambda z_{ij} + (1 - \lambda) p_{ij}, \tag{19}$$

where $\lambda \in [0, 1]$ is the parameter quantifying the importance of the first criterion z_{ij} over the second criterion p_{ij}. If we use the notation $f(i, j) := F(z_{ij}, p_{ij})$ to refer to the indices of the objects, the selection rule is defined as

$$(i, j) = \arg \max_{(k,l) \in \Omega \setminus Q} f(k, l), \tag{20}$$

where Q is the set of PCs that were already performed during the questioning process and $\Omega = \{\{A_i, A_j\}; i < j; i, j = 1, 2, \dots, n\}$ is the set of all PCs between the n objects. The criteria used in the scoring function (19) are defined by the following formulas:

$$z_{ij} = 1 - \frac{|q_i| + |q_j|}{2(n - 2)}, \tag{21}$$

$$p_{ij} = \frac{\varphi_{ij}}{|q_i \cap q_j| + 1} \frac{1}{3} = \frac{3}{|q_i \cap q_j| + 1} \varphi_{ij}. \tag{22}$$

Let us consider the first expression (21), where $q_i = \{k; \{A_i, A_k\} \in Q\}$. Then $|q_i| + |q_j|$ is the number of PCs involving object A_i or A_j. Maximum value of $|q_i|$ is $n - 2$, since $\{A_i, A_j\}$ were not yet compared and $\{A_i, A_i\}$ is excluded. Thus, maximum value of $|q_i| + |q_j|$ is $2(n - 2)$, and $\frac{|q_i|+|q_j|}{2(n-2)}$ represents the normalized number of PCs involving A_i and A_j. Criterion z_{ij} is defined by (21) in order to have a scoring function F increasing in both variables. The criterion z_{ij} determines the lack of PCs suffered by objects A_i and A_j.

Let us consider the second expression (22), where φ_{ij} is the mean inconsistency of indirect PCs of objects A_i and A_j. First, let us define the variable μ_{ij} which expresses the mean value of all indirect PCs of A_i and A_j, which is based on additive consistency condition (3):

$$\mu_{ij} = \begin{cases} 0 & \text{if } q_i \cap q_j = \emptyset, \\ \frac{1}{|q_i \cap q_j|} \sum\limits_{k \in q_i \cap q_j} (r_{ik} + r_{kj} - 0.5) & \text{if } q_i \cap q_j \neq \emptyset. \end{cases} \tag{23}$$

Because indirect PCs of A_i and A_j are usually not completely consistent, the mean inconsistency φ_{ij} of indirect PCs of A_i and A_j is defined as

$$\varphi_{ij} = \begin{cases} 0 & \text{if } q_i \cap q_j = \emptyset, \\ \frac{1}{|q_i \cap q_j|} \sum_{k \in q_i \cap q_j} (r_{ik} + r_{kj} - 0.5 - \mu_{ij})^2 & \text{if } q_i \cap q_j \neq \emptyset. \end{cases} \tag{24}$$

Note that for $q_i \cap q_j \neq \emptyset$, φ_{ij} is the variance of $(r_{ih} + r_{hj} - 0.5)$, and it holds that $\varphi_{ij} = 0$ if and only if all the indirect PCs of A_i and A_j are additively consistent.

The maximum achievable reduction $\Delta\varphi_{ij}$ of φ_{ij} is obtained if the direct PC is $r_{ij} = \mu_{ij}$ and it holds $\Delta\varphi_{ij} = \frac{\varphi_{ij}}{|q_i \cap q_j|+1}$. In the formula (22), $\Delta\varphi_{ij}$ is normalized, i.e. it is divided by $\frac{1}{3}$ as it is the maximum achievable value of $\Delta\varphi_{ij}$; see [17]. The criterion p_{ij} expresses the normalized maximum achievable reduction of the inconsistency φ_{ij} which can be reached by means of the direct PCs of A_i and A_j.

The algorithm of Fedrizzi and Giove for selecting the PCs to fill in the FPR consists of the following steps:

1) At the beginning, no PCs are performed and $Q = \emptyset$. Thus, $z_{ij} = 1$, $p_{ij} = 0$ and $f(i,j) = \lambda$ for all $i, j = 1, 2, \ldots, n$. Instead of a random selection, recommended initial PCs are $\{(2i-1, 2i); i = 1, 2, \ldots, n/2\}$ if n is even and $\{(2i-1, 2i); i = 1, 2, \ldots, (n-1)/2\}$ if n is odd.
2) In each step of the selection process, the value of the scoring function f is quantified for each missing PC by using the formula (19). According to (20), the suitable PC (i,j) is selected. In case of equal values of $f(i,j)$, indexes (i^*, j^*) such that $i^* + j^*$ minimizes $i + j$ are selected. In case of equal values of $i + j$, the pair containing the minimum index is selected.
3) The selection is stopped when the value of scoring function becomes lower than the threshold $\delta \in [0,1]$ which is subjectively defined by the DM, i.e.

$$\max_{\{i,j\} \in \Omega \setminus Q} f(i,j) \leq \delta. \tag{25}$$

III. Proposed algorithm

A. Description of the method

In this section, we propose an algorithm for inputting preferences in large PCMs and for computing interval weights from incomplete PCMs. The algorithm utilizes the concept of weak consistency (7)–(10) as well as the PC selection process proposed in [17]. The proposed interactive algorithm guides the DM through the PCs input phase by identifying which pair of objects should be compared next. This way, the information increase, added by the PC, is maximized and the compliance with the weak-consistency condition in each step of the algorithm is ensured. This results in a weakly consistent incomplete PCM after each input. Moreover, information on all feasible values of a given element of a PCM (that is such values, that would not violate the weak consistency when put in the PCM) is available in each step of the algorithm. Values that are unambiguous are input automatically and the DM is not bothered to provide these. This way, the amount of information contained in the PCM can increase after each step without the effort of the DM. When enough information is provided by the DM, the algorithm stops asking the DM for inputs and determines the preference ordering of the objects and their weights, that are in this case in the form of intervals. It is shown on a practical example, that the interval weights are computed in such a way, that the crisp weights, which would be obtained if the DM filled in all the necessary $n(n-1)/2$ PCs, lie in the respective interval weights.

Let us consider objects $A_1, A_2, \ldots, A_n$ to which weights need to be assigned. The PC of a pair of objects A_i and A_j will be denoted as (A_i, A_j), and where no ambiguity is possible, we will allow a simpler notation (i, j), $i, j \in \{1, 2, \ldots, n\}$. Considering Saaty's PCM approach and the FPR approach are equivalent (transformation of one representation into the other can be done using formulas (5) and (6)), the DM can express the preference intensities either with Saaty's PCM or a FPR. For the sake of the algorithm presentation, we will assume that the DM chose to input preference intensities in Saaty's PCM. We do so also because the practical application of large PCMs discussed in Section IV was done using Saaty's PCM. We want to be able to confront the outputs of our algorithm with the practical result of a method using PCMs and full information directly. Since Saaty's PCM can be easily transformed into a FPR, there is no loss of information in presenting the algorithm for Saaty's PCM. The derived weights of the objects should be expressed in the form corresponding to the scale which was used by the DM for data input. For the purpose of computations in this algorithm, we will use Saaty's PCM representation. Saaty's PCM will be denoted $S = \{s_{ij}\}_{i,j=1}^{n}$, where $s_{ij} \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9\}$, $i, j = 1, 2, \ldots, n$, with the meanings described in Table I.

Since Saaty's PCM S is reciprocal, only the PCs in the upper triangle have to be put in by the DM. Hence, the set Ω of all PCs required to complete the PCM is $\Omega = \{(A_i, A_j); i < j, i, j = 1, 2. \ldots, n\}$, the cardinality of Ω being $Card(\Omega) = n(n-1)/2$. The objective of this algorithm is to find such a set $\bar{\Omega} \subset \Omega$ that its cardinality (i.e. the number of the PCs required from the DM) allows for the computation of all the weights of the objects and to propose a way of generating the elements of this set in such order that minimizes the cardinality of $\bar{\Omega}$.

The set of all PCs already performed will be denoted by Q, and the set of PCs not yet entered in the PCM will be denoted $\Omega \backslash Q$. For each $(i,j) \in \Omega \backslash Q$, the set $FV_{ij} \subseteq \{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9\}$ of feasible values which corresponds to the weak-consistency requirement will be always given. For simplicity and in the figures, the notation $[\min FV_{ij}, \max FV_{ij}]$ will be used where there is no risk of ambiguity. The notation $[\min FV_{ij}, \max FV_{ij}]$ represents a range of the values from Saaty's scale from $\min FV_{ij}$ to $\max FV_{ij}$ for a given $(i,j) \in \Omega \setminus Q$. For example, the set $\{6, 7, 8, 9\}$ will be denoted as $[6, 9]$ and interpreted as a range of values of Saaty's scale from 6 to 9. Incomplete Saaty's PCM will be denoted $\tilde{S} = \{\tilde{s}_{ij}\}_{i,j=1}^{n}$, where

$$\tilde{s}_{ij} = \begin{cases} [s_{ij}^L, s_{ij}^U] & \text{for } (i,j) \in \Omega \setminus Q, \\ s_{ij} & \text{for } (i,j) \in Q. \end{cases}$$

It is obvious that $[s_{ij}^L, s_{ij}^U] = [\min FV_{ij}, \max FV_{ij}]$ for each $(i,j) \in \Omega \setminus Q$.

The process of guided input of preferences and computation of weights of the n compared objects can be briefly summarized in the following steps:

1) The DM chooses which PCM will be used to express the preference intensities (we consider Saaty's PCM for the purpose of the description of the algorithm). The diagonal elements (i,i) of Saaty's PCM $\tilde{S} = \{\tilde{s}_{ij}\}_{i,j=1}^{n}$ are set, i.e. $\tilde{s}_{ii} = 1$ for all $i = 1, 2, \dots, n$. The sets of feasible values (FV sets) FV_{ij} are established for $(i,j) \in \Omega \setminus Q$. At the beginning, $FV_{ij} = [\frac{1}{9}, 9]$ for $(i,j) \in \Omega$.
2) The DM provides initial PCs. In this algorithm, the setting from [17] is used, i.e. the initial PCs $(2i-1, 2i)$, $i = 1, 2, \dots, \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the integer value of $n/2$, are required from the DM. However, also a different initial PCs can be selected. The only restriction is that these initial PCs cannot violate the weak consistency.

The following steps 3 - 5 are repeated until the stopping criterion is met:

3) Based on the algorithm of Fedrizzi & Giove [17], we determine which $(i,j) \in \Omega \setminus Q$ is to be provided next by the DM. The PC (i,j) that maximizes the scoring function (19) is selected, and the DM is asked to provide the corresponding preference intensity into the PCM. The DM selects the value of the PC (i,j) from its FV set FV_{ij}.

4) Based on the weak-consistency requirement, the FV set FV_{ij} is recalculated for each element $(i,j) \in \Omega \setminus Q$ that is still missing. The form of the rules (9)–(18) for Saaty's PCM as introduced in [27] is now used in order to determine $[\min FV_{ij}, \max FV_{ij}]$.
   Obviously, the FV set is restricted only when an indirect PC exists. That is when for a PC (i,j) not yet entered in the PCM there exists at least one object with an index $k, k \neq i, j$, such that (i,k) and (k,j) are already entered in the PCM or a restricted FV set is determined for them.
5) The elements $(i,j) \in \Omega \setminus Q$, for which FV_{ij} contains just a single element, are entered in the PCM automatically. Obviously, the occurrence of such single-element FV_{ij} sets is far more frequent when a discrete scale is used for making PCs of objects. In real-life applications, the requirement of a discrete scale rather than a continuous scale is not a constraint of the decision-making problem. That is because in real-life applications discrete scales of numbers with assigned linguistic terms expressing the intensities of preference are used far more frequently than continuous scales. Discrete scales are more natural for DMs as they provide the required simplifying granularity for continuous universes similar to the common language. The algorithm, however, remains valid also for continuous scales. The choice of the scale is out of the scope of this paper and is left with the users of the algorithm; a discrete scale is assumed for the description of the algorithm. The sets FV_{ij} are recalculated (step 4 is performed) after each such input and step 5 is performed again. Steps 4 and 5 are repeated until there are no elements of the PCM that could be entered automatically this way.
6) Stopping criterion: For every missing PC in the PCM, there exists at least one indirect PC. This condition requires us to be able to determine for each missing (i,j) the set of feasible intensities of preference (restricted) which can be entered in order to preserve the weak consistency of the PCM. Once the stopping criterion is met, we know for each element (i,j) of the PCM either its value or its FV set FV_{ij} restricted by the weak consistency if the PC (i,j) was not entered in yet.
   This stopping criterion varies from the stopping criterion proposed by Fedrizzi & Giove [17]. Since our scope is to be able to compute interval weights of the objects, we require that for every missing PC in the PCM there exists at least one indirect PC. It means that, for any missing PC in the PCM, we are able to determine a (restricted) set FV_{ij} of feasible intensities of preference which can be entered in order to preserve the weak consistency.

7) The so-called ***reciprocal FV sets*** are identified, i.e. such FV_{ij}, $(i, j) \in V \subseteq \Omega \setminus Q$, that contain at least one of the values of the respective scale along with its reciprocal value. As an example, a set containing the two numbers 3 and $\frac{1}{3}$ is a reciprocal FV set. From a reciprocal FV set FV_{ij}, it is not possible to derive which object from the pair (i, j) is preferred over the other one automatically. This ambiguity is not desired. Thus, all reciprocal FV sets need to be replaced by a specific value provided by the DM or restrected by a non-reciprocal FV set (as a consequence of filling in a value from another reciprocal FV set), so that $V = \emptyset$. The DM is asked to provide a PC $(k, l) \in V$ such that $(k, l) = \arg \max_{(i,j) \in V} Card(FV_{ij})$. In case that there are more pairs of objects with the same maximal cardinality of their reciprocal FV set, one of them is chosen randomly. After inputting a value instead of the reciprocal FV set by the DM, FV_{ij}, $(i, j) \in \Omega \setminus Q$, are recalculated using steps 4 and 5. This step is repeated until there are no reciprocal FV sets left.

   Described technique enables us to reduce the amount of information required from the DM as much as possible since providing the PC of the pair of objects with the maximal cardinality of the problematic set adds the most information to the PCM.

8) The preference ordering of objects from the incomplete PCM is derived. For each object (represented by the corresponding row of the PCM), we determine the number of elements that are present in the PCM and are greater than or equal to the indifference value or for which the elements of the FV set are all greater than or equal to the indifference value, which is 1 for Saaty's PCM $\tilde{S}$. Based on this information, the objects $A_1, A_2, \ldots, A_n$ can be ordered from the most preferred one to the least preferred one, i.e. $A_{(1)} \succeq A_{(2)} \succeq \cdots \succeq A_{(n)}$. The respectively reordered PCM with rows and columns ordered from the most preferred object to the least preferred one will be denoted $\tilde{S}_O$.

9) In order to obtain the weights of the objects from the PCM, the sets FV_{ij} of feasible intensities of preference for all missing PCs are considered to be intervals given by the minimal and the maximal value in the set (for example, the set $\{2, 3, 4, 5\}$ is now considered to represent the interval $[2, 5]$). This allows us to obtain the weights of the objects in the form of intervals. The weights can be obtained either from preference ordered PCM $\tilde{S}_O$ or non-preference-ordered PCM $\tilde{S}$. For simplicity, the formulas for calculating weights will be given for the initial ordering of objects without the permutation of indices. To obtain the weights of objects, the formulas for the fuzzified geometric mean method proposed in

[28] are used here. Specifically, the formulas for obtaining the lower and the upper value of a triangular fuzzy weight are applied in order to obtain the lower and the upper value of an interval weight. According to [28], the formulas for computing the lower and the upper value w_i^L, w_i^U of an interval weight $\widetilde{w}_i = \left[w_i^L, w_i^U\right]$, $i = 1, 2, \ldots, n$, for incomplete Saaty's PCM $\tilde{S}$ are given as

$$w_i^L = \frac{\sqrt[n]{\prod_{j=1}^{n} s_{ij}^L}}{\sqrt[p]{\prod_{j=1}^{n} s_{ij}^L} + \max\left\{ \sum_{\substack{k=1\\k\neq i}}^{n} \sqrt[n]{s_{ki}^U \prod_{\substack{l=1\\l\neq i}}^{k-1} \frac{1}{s_{lk}} \prod_{\substack{l=k+1\\l\neq i}}^{n} s_{kl}} \; ; \begin{array}{l} s_{kl} \in \left[s_{kl}^L, s_{kl}^U\right], \\ k, l = 1, 2, \ldots, n, \\ k, l \neq i, k < l \end{array} \right\}}, \tag{26}$$

$$w_i^U = \frac{\sqrt[n]{\prod_{j=1}^{n} s_{ij}^U}}{\sqrt[n]{\prod_{j=1}^{n} s_{ij}^U} + \min\left\{ \sum_{\substack{k=1\\k\neq i}}^{n} \sqrt[n]{s_{ki}^L \prod_{\substack{l=1\\l\neq i}}^{k-1} \frac{1}{s_{lk}} \prod_{\substack{l=k+1\\l\neq i}}^{n} s_{kl}} \; ; \begin{array}{l} s_{kl} \in \left[s_{kl}^L, s_{kl}^U\right], \\ k, l = 1, 2, \ldots, n, \\ k, l \neq i, k < l \end{array} \right\}}. \tag{27}$$

The computed interval weights contain all the weights that would be computed for any particular selection of real values from the sets FV_{ij} corresponding to the missing values in $\tilde{S}$ (that is if $\tilde{S}$ was completed) preserving the weak-consistency condition. This means that if the DM provided all the missing PCs preserving the weak consistency, the real weights computed from such a PCM would lie within the computed interval weights.

The interval weights obtained from the PCM $\tilde{S}_O$ by formulas (26) and (27) have the following property. From the weak consistency and particularly from the property of non-decreasing sequence of numbers in every row and non-increasing sequence of numbers in every column of an ordered PCM, it follows, that any two interval weights $\widetilde{w}_i, \widetilde{w}_j, i, j \in \{1, 2, \ldots, n\}$, obtained by formulas (26) and (27) can be ordered according to the standard partial order $\leq$ on intervals; $[a, b] \leq [c, d]$ if $a \leq c, b \leq d$. Therefore, on the set of all interval weights $\widetilde{w}_i, i = 1, 2, \ldots, n,$ $\leq$ is a total order. According to step 8, the preference ordering of objects is derived immediately from the preference information in $\tilde{S}_O$ without the need of computing the interval weights. Moreover, for any two objects A_i, A_j such that $A_i \succ A_j$, it holds that $\tilde{w}_i > \tilde{w}_j$; for the case when $A_i \succeq A_j$ and $A_j \succeq A_i$ it holds that $\tilde{w}_i = \tilde{w}_j$.

To evaluate the benefit of the proposed method from the point of view of sparing the number of PCs required from the DM, the following simulations were performed. One hundred weakly consistent Saaty's PCMs were randomly generated for each fixed dimension $n = 5, 10, \ldots, 30$. The proposed algorithm was applied to each of them, and the average number of PCs which the DM did not have to fill in the PCM was calculated for each dimension; see Table III. According to the obtained results, the percentage of the spared PCs increases on average with the dimension of the PCM, and for PCMs of dimension 15 and greater, more than 60% of PCs are spared on average.

TABLE III: Average number of spared PCs needed from the DM

	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$
number of PCs required in the full-information case	10	45	105	190	300	435
average number of spared PCs	4	24	64	123	207	312
average $\%$ of spared PCs	42%	53%	61%	65%	69%	72%

B. Illustrative example

For better understanding, the algorithm is demonstrated step-by-step on a simple illustrative example of weakly consistent Saaty's PCM for seven objects. Obviously, applying our algorithm to a PCM of just several (in our case seven) objects has only limited significance as such a PCM does not require many PCs provided by the DM in the first place. However, for better visual illustration of each step of the proposed algorithm, an example with just several objects is more suitable.

Let $A_1, A_2, \ldots, A_7$ be objects which need to be compared and $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_7$ be the weights of the respective objects that the DM needs to determine. The preferences will be entered in Saaty's PCM form. Let us also consider that Fig. 1 presents Saaty's PCM as it would look like if the DM provided all the PCs. For the sake of simplicity, only the elements above the main diagonal are given since the elements below the main diagonal are the reciprocals of the corresponding elements above the main diagonal. The weights $w_1, w_2, \ldots, w_7$ of objects $A_1, A_2, \ldots, A_7$ that would be computed from Saaty's PCM S by the geometric mean method are given in the second column of Tab. IV. For better illustration, easier understanding and an easy check of the compliance with the weak-consistency condition, the objects in Saaty's PCM in Fig. 1 are ordered from the most preferred one to the least preferred one.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	1	9	9	9	9	9	9
A_2		1	2	2	9	9	9
A_3			1	1	5	8	8
A_4				1	5	8	8
A_5					1	7	8
A_6						1	7
A_7							1

Fig. 1: Saaty's PCM with all the preference intensities provided by the DM

The method proposed in this paper is designed to be applicable on the most general PC problems with no information about the preference ordering of the objects which are to be compared pairwise, i.e. we suppose that the ordering of the objects from the most preferred one to the least preferred one is not known. It means that the method can be applied on any random initial ordering of objects in Saaty's PCM. Let us therefore assume that the preference ordering of objects was not known in advance and instead the objects were ordered in random order (we can assume that Saaty's PCM, presented in Fig. 2 without any PCs entered, was the initial starting point).

At the beginning, the diagonal elements are set to the value 1 as in step 1 of the proposed algorithm. Then the DM has to provide initial PCs $\{(2i-1,2i)\,;i=1,2,\ldots,\lfloor 7/2\rfloor\}=\{(1,2),(3,4),(5,6)\}$ as is required in step 2. Any values from Saaty's scale can be chosen in this step. It means that FV sets for all missing elements are $[\frac{1}{9},9]$. For easier orientation in the figures, the initial FV sets $[\frac{1}{9},9]$ are replaced by empty fields. Only FV sets calculated from indirect PCs in the following steps will be entered in Saaty's PCM.F

Next, the steps from 3 to 5 are repeated until the stopping criterion is met. In step 3, we apply the algorithm based on searching for a missing PC (i,j) with the maximum value of the scoring function (19) to select the next PC (i,j) which is to be given by the DM. Let us remark here that i and j in (i,j) are the coordinates of the element in the PCM, not actual indices of objects. The same scoring function is used as in the method proposed by Fedrizzi & Giove in [17]. In this illustrative example, both criteria of the scoring function (19) are considered to have the same importance, therefore the parameter $\lambda=0.5$ is set.

As was already mentioned in the previous section, in contrast to the method proposed by Fedrizzi & Giove [17], we require Saaty's PCM to be weakly consistent; it has to satisfy the equivalent of properties of (9)–(18) for Saaty's PCM as introduced in [27]. According to this requirement, in step 4, we are able to restrict the sets FV_{ij} of feasible intensities of preference for some missing PCs in order to preserve the weak consistency. If any set FV_{ij} consists of only one value, then this value is entered in Saaty's PCM as is suggested in step 5.

Fig. 2 demonstrates incomplete Saaty's PCM $\tilde{S}$ after the initial PCs $(1,2) = 1$, $(3,4) = \frac{1}{9}$ and $(5,6) = 9$ and after the first iteration of the algorithm. The first PC chosen in the first iteration and provided by the DM is $(1,7) = 8$. As can be seen from Saaty's PCM, the PC $(2,7) = 8$ was filled in automatically according to the weak consistency since $(1,2) = 1$ and $(1,7) = 8$. The legend explaining the notation in the figures is provided in Fig. 3.

	A_3	A_4	A_6	A_1	A_2	A_5	A_7
A_3	1	1					8
A_4		1					8
A_6			1	1/9			
A_1				1			
A_2					1	9	
A_5						1	
A_7							1

Fig. 2: Incomplete Saaty's PCM after the first iteration

x	initial PC given by the DM
x	PC given by the DM during the algorithm
x	PC filled in automatically according to the weak consistency
[1/x,x]	reciprocal FV set
[x,y]	FV set
x	value from reciprocal FV set filled in by the DM

Fig. 3: Legend

Fig. 4 shows incomplete Saaty's PCM $\tilde{S}$ after two iterations of the algorithm. The PC $(2,3) = 8$ was provided by the DM, and according to the weak consistency, one missing PC and ranges for other three missing PCs were added automatically. For example, the range $[1/9, 1/2]$ for the missing PC $(1,4)$ was derived from the PCs $(1,3) = 8$ and $(3,4) = 1/9$ according to the property (13). The DM continues providing the missing PCs until the stopping criterion is met. Fig. 5 represents incomplete Saaty' PCM obtained so far.

	A_3	A_4	A_6	A_1	A_2	A_5	A_7
A_3	1	1	8	[1/9,1/2]			8
A_4		1	8	[1/9,1/2]			8
A_6			1	1/9			[1/8,8]
A_1				1			
A_2					1	9	
A_5						1	
A_7							1

Fig. 4: Incomplete Saaty's PCM after the second iteration

	A_3	A_4	A_6	A_1	A_2	A_5	A_7
A_3	1	1	8	1/9	[1/9,1/2]	5	8
A_4		1	8	1/9	[1/9,1/2]	5	8
A_6			1	1/9	[1/9,9]	[1/8,1/2]	[1/8,8]
A_1				1	9	9	9
A_2					1	9	9
A_5						1	[2,8]
A_7							1

Fig. 5: Incomplete Saaty's PCM after the stopping criterion is met

In our Saaty's PCM $\tilde{S}$, two reciprocal FV sets are present; see the PCs $(3,5) = [\frac{1}{9}, 9]$ and $(3,7) = [\frac{1}{8}, 8]$ in Fig. 5. It means that for these pairs of objects we are not even able to decide which one is preferred over the other; the information obtained from indirect PCs is too vague. Therefore, according to step 7, we have to ask the DM to determine the intensities of preference for these pairs of objects.

First the DM is asked to provide the value $(3,5) = \frac{1}{9}$, its reciprocal FV set having the biggest cardinality 17. In this case, no restriction of the other FV sets occurs. Then the DM provides the value $(3,7) = 7$, and as a consequence, the FV set of $(6,7)$ is reduced from $[2,8]$ to $[7,8]$. Fig. 6 shows incomplete Saaty's PCM after step 7.

Once the reciprocal FV sets are removed, we are able to order the compared objects from the most preferred one to the least preferred one according to step 8 and reorder the whole incomplete PCM accordingly. Fig. 7 demonstrates ordered incomplete Saaty's PCM which is

	A_3	A_4	A_6	A_1	A_2	A_5	A_7
A_3	1	1	8	1/9	[1/9,1/2]	5	8
A_4		1	8	1/9	[1/9,1/2]	5	8
A_6			1	1/9	1/9	[1/8,1/2]	7
A_1				1	9	9	9
A_2					1	9	9
A_5						1	[7,8]
A_7							1

Fig. 6: Incomplete Saaty's PCM without reciprocal FV sets

weakly consistent.The resulting PCM is incomplete Saaty's PCM with FV sets for every missing PC such that by choosing an arbitrary element from FV set of an arbitrary missing PC the weak consistency is not violated. If the DM wanted only information about preference ordering of the objects, then the process would be stopped here. Otherwise, from such Saaty's PCM, the interval weights of objects are obtained using the formulas presented in the step 9 of the algorithm. The computed interval weights are summarized in Tab. IV along with the weights computed from the full Saaty's PCM.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	1	9	9	9	9	9	9
A_2		1	[2,9]	[2,9]	9	9	9
A_3			1	1	5	8	8
A_4				1	5	8	8
A_5					1	[2,8]	[7,8]
A_6						1	7
A_7							1

Fig. 7: Final incomplete Saaty's PCM

To summarize the results of this illustrative example, to have complete information, the DM woud have to fill 21 PCs in Saaty's PCM. Using the algorithm proposed in this paper for incomplete PCMs, the DM provided 10 PCs (approx. 48%), 7 PCs (approx. 33%) were added automatically from the properties of the weak consistency and 4 PCs (approx. 19%) left empty having the FV set which will not violate the weak consistency of Saaty's PCM. The calculated interval weights are very narrow and contain the original weights; see Tab. IV.

To emphasize the advantage and the significant contribution of the method proposed in this

TABLE IV: Weights of the objects

Objects	Real weights	Interval weights	Weights computed according to [25]
A_1	0.5083	$[0.4840, 0.5102]$	0.2377
A_2	0.1765	$[0.1765, 0.2594]$	0.1128
A_3	0.1166	$[0.0896, 0.1170]$	0.2284
A_4	0.1166	$[0.0896, 0.1170]$	0.2284
A_5	0.0463	$[0.0363, 0.0472]$	0.0535
A_6	0.0228	$[0.0213, 0.0273]$	0.1128
A_7	0.0128	$[0.0122, 0.0131]$	0.0264

paper to the decision-making theory, we will compare this method with another known method for incomplete Saaty's PCMs. Particularly, the method proposed by Herrera-Viedma et al. in [25] will be applied on this illustrative example for the comparison (paper [25] has been cited over 560-times which suggests wide recognition of the method). In [25], only $n-1$ PCs above the main diagonal, i.e. $\{(i, i+1); i = 1, 2, \ldots, n-1\}$, are required from the DM. After, the missing PCs are completed automatically so that resulting Saaty's PCM $A = \{a_{ij}\}_{i,j=1}^n$ is consistent according to (1). Clearly, in most of the cases, the missing PCs completed by this procedure exceed Saaty's scale $\left[\frac{1}{9}, 9\right]$. That is why it is suggested in [25] to transform obtained Saaty's PCM A given on scale $\left[\frac{1}{c}, c\right]$, $c > 9$, to Saaty's PCM $B = \{b_{ij}\}_{i,j=1}^n$ given on scale $\left[\frac{1}{9}, 9\right]$ by using transformation formula

$$b_{ij} = a_{ij}^{1/\log_9 c}, \qquad i, j = 1, 2, \ldots, n. \tag{28}$$

In Fig. 8, completed, transformed and ordered Saaty's PCM B after providing the 6 initial PCs above the main diagonal is given. The 6 PCs provided by the DM are highlighted in bold. Obviously, unlike Saaty's PCM in Fig. 7, Saaty's PCM in Fig. 8 differs substantially from original Saaty's PCM in Fig. 1. Thus, also the weights obtained from this Saaty's PCM that are given in the fourth column of Tab. IV vary essentially from the original weights given in the second column. Even the ranking of the objects based on these weights varies.

To demonstrate how far Saaty's PCM in Fig. 8 obtained by the method proposed in [25] is from original Saaty's PCM in comparison to Saaty's PCM obtained by the method proposed in this paper, we will measure their distances. We will apply the distance for Saaty's PCMs defined in [11]. Since Saaty's PCM in Fig. 7 contains intervals, we need to generalize the distance from [11] to interval Saaty's PCMs. For two interval Saaty's PCMs $\overline{A} = \{\overline{a}_{ij}\}_{i,j=1}^n$, $\overline{a} = \left[a_{ij}^L, a_{ij}^U\right]$, $\overline{B} =$

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	1	**2.1080**	1.0408	1.0408	4.4436	**2.1080**	9
A_2		1	0.4937	0.4937	**2.1080**	1	4.2695
A_3			1	**1**	4.2695	2.0254	8.6473
A_4				1	4.2695	**2.0254**	8.6473
A_5					1	0.4742	**2.0254**
A_6						1	4.2695
A_7							1

Fig. 8: The PCM obtained by Herrera's approach

$\left\{\overline{b}_{ij}\right\}_{i,j=1}^{n}, \overline{b} = \left[b_{ij}^L, b_{ij}^U\right]$, the interval distance based on the distance defined by [11] is given as $\overline{D}(\overline{A}, \overline{B}) = \left[d^L, d^U\right]$:

$$d^L = \min_{\substack{a_{ij} \in \left[a_{ij}^L, a_{ij}^U\right] \\ b_{ij} \in \left[b_{ij}^L, b_{ij}^U\right]}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln(a_{ij}/b_{ij})| = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \min_{\substack{a_{ij} \in \left[a_{ij}^L, a_{ij}^U\right] \\ b_{ij} \in \left[b_{ij}^L, b_{ij}^U\right]}} |\ln(a_{ij}/b_{ij})|, \tag{29}$$

$$d^U = \max_{\substack{a_{ij} \in \left[a_{ij}^L, a_{ij}^U\right] \\ b_{ij} \in \left[b_{ij}^L, b_{ij}^U\right]}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln(a_{ij}/b_{ij})| = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \max_{\substack{a_{ij} \in \left[a_{ij}^L, a_{ij}^U\right] \\ b_{ij} \in \left[b_{ij}^L, b_{ij}^U\right]}} |\ln(a_{ij}/b_{ij})|. \tag{30}$$

For crisp Saaty' PCMs, the interval distance (29), (30) is identical to the distance defined in [11]. By applying the formulas (29) and (30), the distance of Saaty's PCM obtained by the method proposed in [25] given in Fig. 8 and original Saaty's PCM given in Tab. 1 is $D = 22.8941$. The interval distance of interval Saaty's PCM in Fig. 7 from original Saaty's PCM in Fig. 1 is $\overline{D} = [0, 4.3934]$. Clearly, $\overline{D} = [0, 4.3934]$ is significantly smaller than $D = 22.8941$, which demonstrates better performance of the method proposed in this paper.

Notice that the lower boundary value d^L of the distance of any Saaty's PCM with intervals obtained by the method proposed in this paper from original complete Saaty's PCM is always 0. This follows from the fact that the original PCM is contained in the interval PCM, which is the substance and the main advantage of the method.

IV. Case study of evaluating artistic performance in the CZ

A. Description of the problem

As a real-life case study of the proposed interactive algorithm, we will use the Registry of Artistic Performances from the CZ and the data gathered for the determination of scores for various categories of artistic production; see [46]. The outputs of artistic performance are currently evaluated in the CZ against the following three criteria, for each of which there are three levels distinguished (denoted by capital letters that are then used for the description of categories):

Criterion 1 - Relevance or significance of the piece of art

A - a new piece of art or a performance of crucial significance;

B - a new piece of art or a performance containing numerous important innovations;

C - a new piece of art or a performance pushing forward modern trends.

This is an expertly assessed criterion that brings a peer-review element into the evaluation. Each segment of art provided a general linguistic specification for each level of this criterion to be made available for the expert evaluators, real-life (historical) examples for levels A, B and C are also available.

Criterion 2 - Extent of the piece of art

K - a piece of art or a performance of large extent;

L - a piece of art or a performance of medium extent;

M - a piece of art or a performance of limited extent.

The levels of this criterion are again specified linguistically. This criterion was however intended to be measurable for each segment on such a level of accuracy that most of the ambiguity in categorizing works of art according to this criterion is removed.

Criterion 3 - Institutional and media reception/impact of the piece of art

X - international reception/impact;

Y - national reception/impact;

Z - regional reception/impact.

For this criterion, lists of institutions corresponding to level X, Y and Z are provided. Hence, there is no subjectivity in evaluation against this criterion in the process.

By combining the letters representing various levels of the three criteria, 27 categories of works of art can be defined. These categories are represented by triplets of capital letters in the model (e.g. AKY, BLZ, or CMZ). Each of these 27 categories needs to be assigned a score. The original idea was to obtain all PCs of the well represented (that is represented by real-life examples) 27 categories of works of art (351 PCs in total) using Saaty's scale (see Table I), and based on these, to compute the score for each category using Saaty's approach in AHP.

As AHP was not intended for PCMs of large dimensions, Saaty proposed to approach these problems by splitting them into problems of a lower dimension - that is to determine the weights of each criterion and the weights of each level within each criterion and based on these to calculate the evaluations (weights) of the categories. This approach was however not applicable in this case for the following reasons: a) there are some dependencies among the criteria which are not easy to describe or capture, b) to compare various levels of one criterion - for example big, medium and small - without any real-life representatives (good representatives for such broad categories proved to be difficult to find) is not easy for the experts, c) the experts were not able to express their preferences between the criteria (these too proved to be too abstract to provide enough representation for the experts to be able to express their preferences). For these reasons, all 27 categories were compared pairwise. To adapt Saaty's method for this dimension of Saaty's PCM and to facilitate the process for the experts, the consistency condition in Saaty's sense that is almost impossible to achieve with large Saaty's PCMs, was replaced by the much more relaxed weak-consistency condition. The weak consistency was considered as a minimum requirement on the consistency of Saaty's PCM.

As the weak-consistency condition is easy to check within the process of inputting preferences, and even more so when the rows and columns of Saaty's PCM are ordered in accordance with the preference ordering of the categories (from the most preferred to the least preferred one), the 27 categories were preference ordered first using the PC method; see [46] for more details.

After two years of using the described mathematical model and the resulting evaluations, minor adjustments to the evaluation methodology proved to be necessary. Adding one more level to one of the criteria and changing the initial preference ordering of the categories were considered [47]. All such changes would result in the need of inputting large Saaty's PCM again (and in the case of adding one level of one of the criteria the dimension of Saaty's PCM would increase, thus dramatically increasing the number of PCs needed). Generally, we should be prepared that any model based on large PCMs might need to be adapted to meet new requirements in the future. If such an adaptation results in the need of doing all the PCs again (or even in doing more of them), functioning algorithms that are capable of reducing the number of PCs that need to be provided, and thus reducing the time consumption for the experts (this all without substantial loss of information), are most needed. The algorithm proposed in this paper aims to provide an assistance and a solution to these types of problems.

B. Numerical results

In this section, we approach the large-dimensional problem of evaluating outcomes of artistic performance in the CZ solved in [46]. The method for the construction of incomplete Saaty's PCM and obtaining interval weights of the categories of artistic production introduced in Section III-A is applied, and the outcome is compared with the outcome in [46]. Again, we draw from the knowledge of complete Saaty's PCM and conduct a numerical experiment - we start with empty Saaty's PCM, utilize the algorithm proposed in this paper and, whenever a preference intensity is required from the DM, we find the appropriate value in complete Saaty's PCM presented in [46, p. 75].

We assume the randomly generated initial order of the categories (i.e. categories are not ordered according to their preference but randomly) given in the heading of Fig. 9 and start with the execution of the algorithm. First, the DM was asked to provide 13 initial PCs $\{(2i-1, 2i)\,;$ $i = 1, 2, \ldots, 13\}$. Subsequently, the algorithm for selecting the missing PCs $(i, j)\,, i < j, i, j \in \{1, 2, \ldots, 27\}$ which are to be provided by the DM was applied. The parameter $\lambda = 0.5$ was used in the scoring function (19) as both its criteria were considered having the same importance. The algorithm was stopped after just 109 PCs provided by the DM. Because missing PCs with reciprocal FV sets were present in Saaty's PCM, it was not possible to order the categories from the most preferred one to the least preferred one immediately. First, all reciprocal FV sets FV_{ij} needed to be removed. This was done one by one, and after the replacement of each single reciprocal FV set FV_{ij} by the value specified by the DM or by a non-reciprocal FV set, all the remaining missing elements were recalculated (23 inputs were required from the DM in the process). Obtained incomplete Saaty's PCM is given in Fig. 9. Finally, categories creating incomplete PCM were ordered from the most preferred category to the least preferred one. Preference ordered weakly consistent incomplete Saaty's PCM is given in Fig. 10.

Overall, from the total number of 351 PCs, the DM provided 145 PCs (approx. 41%), 153 PCs (approx. 44%) were added automatically according to the weak consistency, and for remaining 53 PCs (approx. 15%), sets of feasible intensities of preference were derived from the weak-consistency properties. These FV sets were relatively narrow containing at most 4 values.

From Saaty's PCM in Fig. 10, interval weights of the categories were obtained using the formulas (26) and (27). The interval weights together with the weights of the categories, obtained from complete Saaty's PCM in [46] by the geometric mean method, are given in Tab. V. The

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
		ALZ	ALX	AMX	CLX	AMZ	CLY	BLX	CLZ	CMX	CKY	CKX	AKZ	AKY	BLZ	AMY	BMY	BLY	ALY	AKX	BKZ	BMZ	CMZ	BKY	BKX	CKZ	CMY	BMX
1	ALZ	1	1/5	3	5	5	[5,9]	5	[5,9]	[7,9]	5	5	1/5	1/5	5	[3,5]	5	5	[1/5,1/2]	1/5	5	5	9	5	5	5	9	5
2	ALX		1	5	7	5	[7,9]	5	[7,9]	[7,9]	[5,7]	[5,7]	1/3	1/5	5	5	5	5	[2,3]	1/5	5	[5,7]	9	5	5	7	9	5
3	AMX			1	5	5	[5,7]	5	[5,7]	7	5	5	1/5	1/5	5	3	5	5	[1/5,1/2]	1/5	5	5	[7,9]	5	5	5	7	5
4	CLX				1	1/5	3	1/5	5	5	1/5	1/5	[1/9,1/7]	[1/9,1/7]	1/5	1/5	1/5	1/5	[1/9,1/5]	1/7	1/5	1/5	5	1/5	1/5	1/3	5	1/5
5	AMZ					1	5	5	[5,7]	7	5	5	1/5	1/5	5	[1/5,1/2]	5	[2,5]	[1/5,1/2]	1/5	5	5	[7,9]	5	3	5	7	5
6	CLY						1	1/5	5	5	1/5	1/5	[1/9,1/7]	[1/9,1/7]	1/5	1/5	1/5	1/5	[1/9,1/5]	1/7	1/5	1/5	5	1/5	1/5	[1/5,1/3]	5	1/5
7	BLX							1	5	7	5	5	1/5	1/5	5	[1/5,1/2]	5	[2,5]	1/5	[1/7,1/5]	1/3	5	7	1/5	1/5	5	7	5
8	CLZ								1	3	1/5	1/5	[1/9,1/7]	[1/9,1/7]	1/5	1/5	1/5	1/5	[1/9,1/5]	1/7	[1/7,1/5]	1/5	[3,5]	[1/7,1/5]	[1/7,1/5]	1/5	[3,5]	1/5
9	CMX									1	1/5	1/5	[1/9,1/7]	[1/9,1/7]	1/7	1/7	1/5	1/7	[1/9,1/7]	1/7	1/7	1/5	3	1/7	1/7	1/5	3	[1/7,1/5]
10	CKY										1	1/3	1/7	[1/9,1/7]	1/5	1/5	1/5	1/5	[1/7,1/5]	1/7	1/5	1/5	5	1/5	1/5	[2,5]	5	1/5
11	CKX											1	1/7	1/7	[1/5,1/2]	[1/5,1/2]	[1/5,1/2]	[1/5,1/2]	[1/7,1/5]	[1/9,1/7]	1/5	[1/5,1/2]	[5,7]	1/5	1/5	[2,5]	[5,7]	[1/5,1/2]
12	AKZ												1	1/5	5	5	5	5	3	1/5	5	[5,7]	9	5	5	[7,9]	9	5
13	AKY													1	5	5	5	5	5	1/5	5	[7,9]	9	5	5	[7,9]	9	5
14	BLZ														1	1/5	3	1/3	1/5	[1/7,1/5]	1/5	5	7	1/5	1/5	5	7	3
15	AMY															1	5	5	[1/5,1/2]	1/5	[2,5]	5	[7,9]	[2,5]	[2,5]	5	7	5
16	BMY																1	1/3	1/5	[1/9,1/7]	1/5	[2,5]	[5,7]	1/5	1/5	[2,5]	[5,7]	1/3
17	BLY																	1	1/5	1/5	[1/5,1/2]	5	[7,9]	[1/5,1/2]	[1/5,1/2]	5	7	3
18	ALY																		1	1/5	5	5	9	5	5	5	[7,9]	5
19	AKX																			1	5	9	9	5	5	9	9	7
20	BKZ																				1	5	7	1/5	1/5	5	7	5
21	BMZ																					1	5	1/5	1/5	[2,5]	5	1/5
22	CMZ																						1	1/7	1/7	[1/7,1/5]	1/3	1/7
23	BKY																							1	1/5	5	7	5
24	BKX																								1	5	7	5
25	CKZ																									1	5	1/5
26	CMY																										1	1/7
27	BMX																											1

Fig. 9: Saaty's PCM after filling in the PCs by the DM

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
		AKX	AKY	AKZ	ALX	ALY	ALZ	AMX	AMY	AMZ	BKX	BKY	BKZ	BLX	BLY	BLZ	BMX	BMY	BMZ	CKX	CKY	CKZ	CLX	CLY	CLZ	CMX	CMY	CMZ
1	AKX	1	5	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	7	[7,9]	9	9	9	9	9	9	9	9	9	9
2	AKY		1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	7	[7,9]	[7,9]	[7,9]	[7,9]	[7,9]	[7,9]	9	9
3	AKZ			1	3	3	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	7	7	[7,9]	[7,9]	[7,9]	[7,9]	[7,9]	9	9
4	ALX				1	[2,3]	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	[5,7]	[5,7]	7	7	[7,9]	[7,9]	[7,9]	9	9
5	ALY					1	[2,5]	[3,5]	[3,5]	5	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	[5,9]	[5,9]	[7,9]	9	9
6	ALZ						1	3	[3,5]	5	5	5	5	5	5	5	5	5	5	5	5	5	5	[5,9]	[5,9]	[7,9]	9	9
7	AMX							1	3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	[5,7]	[5,7]	7	7	[7,9]
8	AMY								1	[2,5]	[3,5]	5	5	5	5	5	5	5	5	5	5	5	5	5	5	7	7	[7,9]
9	AMZ									1	3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	7	7	[7,9]
10	BKX										1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	7	7	7
11	BKY											1	5	5	5	5	5	5	5	5	5	5	5	5	5	7	7	7
12	BKZ												1	3	[3,5]	5	5	5	5	5	5	5	5	5	5	7	7	7
13	BLX													1	[2,5]	5	5	5	5	5	5	5	5	5	5	7	7	7
14	BLY														1	3	3	3	5	5	5	5	5	5	5	7	7	7
15	BLZ															1	3	3	5	5	5	5	5	5	5	7	7	7
16	BMX																1	3	5	5	5	5	5	5	5	[5,7]	7	7
17	BMY																	1	[2,5]	[2,5]	5	5	5	5	5	5	[5,7]	[5,7]
18	BMZ																		1	[2,5]	5	5	5	5	5	5	5	5
19	CKX																			1	3	[3,5]	5	5	5	5	5	5
20	CKY																				1	[2,5]	5	5	5	5	5	5
21	CKZ																					1	3	[3,5]	5	5	5	5
22	CLX																						1	3	5	5	5	5
23	CLY																							1	5	5	5	5
24	CLZ																								1	3	[3,5]	[3,5]
25	CMX																									1	3	3
26	CMY																										1	3
27	CMZ																											1

Fig. 10: Saaty’s PCM ordered according to the preferences

resulting weights are rounded to four decimal places. Obviously, the weights of the categories lie in the intervals delimited by the interval weights. This result is natural since FV sets for missing PCs in the incomplete PCM obtain all feasible intensities of preference which preserve the weak consistency. Therefore, actual complete Saaty’s PCM from [46] can be obtained from incomplete weakly consistent Saaty’s PCM in Fig. 10 by a particular combination of the values from the FV sets.

Using the algorithm proposed in this paper, we obtained the interval weights of the categories which represent very well the actual weights obtained from complete Saaty’s PCM (compare the results in Tab. V). In contrast to the original method, however, the DM provided only 145 PCs instead of 351. It means that the amount of the information required from the DM was reduced to just over 40% of the information required in [46]. This is a very significant reduction of the information required from the DM that reduces considerably the time demands and raises the quality of the information provided.

V. Conclusion

The presented paper contributes to the current pool of knowledge on MCDM with FPRs and Saaty’s PCMs. It presents a novel approach to dealing with large incomplete PCMs (both represented by Saaty’s PCMs or FPRs) that strives to identify the tradeoff between decreasing the

TABLE V: Interval weights of the categories

Categories		Real weights	Interval weights
1	AKX	0.1357	$[0.1314, 0.1370]$
2	AKY	0.1132	$[0.1126, 0.1166]$
3	AKZ	0.0967	$[0.0917, 0.0995]$
4	ALX	0.0862	$[0.0829, 0.0895]$
5	ALY	0.0761	$[0.0687, 0.0799]$
6	ALZ	0.0612	$[0.0593, 0.0660]$
7	AMX	0.0552	$[0.0542, 0.0573]$
8	AMY	0.0498	$[0.0495, 0.0509]$
9	AMZ	0.0418	$[0.0415, 0.0423]$
10	BKX	0.0385	$[0.0382, 0.0390]$
11	BKY	0.0335	$[0.0333, 0.0340]$
12	BKZ	0.0292	$[0.0280, 0.0296]$
13	BLX	0.0269	$[0.0258, 0.0273]$
14	BLY	0.0222	$[0.0211, 0.0249]$

Categories		Real weights	Interval weights
15	BLZ	0.0204	$[0.0194, 0.0215]$
16	BMX	0.0184	$[0.0176, 0.0192]$
17	BMY	0.0167	$[0.0160, 0.0175]$
18	BMZ	0.0134	$[0.0133, 0.0140]$
19	CKX	0.0117	$[0.0114, 0.0125]$
20	CKY	0.0106	$[0.0102, 0.0112]$
21	CKZ	0.0088	$[0.0088, 0.0092]$
22	CLX	0.0080	$[0.0077, 0.0082]$
23	CLY	0.0072	$[0.0067, 0.0074]$
24	CLZ	0.0057	$[0.0053, 0.0066]$
25	CMX	0.0047	$[0.0045, 0.0051]$
26	CMY	0.0042	$[0.0040, 0.0045]$
27	CMZ	0.0038	$[0.0035, 0.0040]$

number of PCs required from the DM and obtainig sufficient amount of information to compute relevant weights of objects. The algorithm is suggested as a possible solution to large-dimensional problems where the complete information (all PCs) is either costly, too time consuming or infeasible to obtain or where the preference intensities in the large PCMs require (frequent) revisions.

The algorithm is based on the combination of the weak-consistency conditions (7)–(10), introduced in this paper for FPRs, with the modified version of the optimal PC selection algorithm [17]. The weak consistency of the incomplete PCM is required during the whole process as a minimal consistency requirement. As a consequence, certain PCs are added automatically on the base of PCs previously filled in by the DM. The resulting algorithm provides 1) means of effectively inputting large incomplete weakly consistent PCMs, 2) means of computing interval weights from these incomplete PCMs whose range depends on the amount of the missing information and that include the real weights of objects obtainable for any weakly consistent completion of the incomplete PCM.

The numerical results of performed simulations demonstrate that the application of the proposed algorithm can significantly reduce the number of PCs required and thus results in significant resource savings. At the same time, a high accuracy of the output is guaranteed by the algorithm as the resulting interval weights contain the weights that would be obtained from a complete PCM. For randomly generated Saaty's PCMs of dimension $n \geq 15$, even more than 60% of PCs needed (with respect to the full-information case) were spared on average. The numerical example of a small 7x7 PCM exhibited a reduction of ca. 50% in the number of PCs required (from 21 to 10). In the real-life case study of a works-of-art evaluation model utilising a 27x27 PCM, the requirement on the number of PCs obtained from the DMs was reduced by ca. 60% (from 351 PCs only 145 were required). The obtained interval weights of the 27 categories of works of art contained the real-number weights that would be obtained from a complete PCM and were reasonably narrow.

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