Semi-Analytical Minimum Time Solutions for a Vehicle Following Clothoid-Based Trajectory Subject to Velocity Constraints

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***Abstract*— A possible strategy for finding the optimal path that connects two different configurations of a car-like vehicle can be addressed by solving two different sub-problems: 1. identifying a curve that connects the two points respecting the geometric constraints, 2. finding an optimal control strategy that drives the vehicle along the trajectory accounting for its dynamic constraints. In this paper, we focus on the second problem. Assuming that the vehicle moves along a specified clothoid, we find a semi-analytical solution for the optimal profile of the longitudinal acceleration (the control variable). Our technique explicitly considers non–linear dynamics, aerodynamic drag effect and bounds on the longitudinal and on the lateral acceleration.**

I. Introduction and State of the Art

One of the most active research lines in mobile robotics is how to plan optimal trajectories to move a car–like robot from an initial configuration to a final one, where the robot configuration is given by both its position and its velocity. The problem arises in the context of Advanced Driving Assistance Systems, e.g., collision avoidance, lane keeping systems, etc., and in self-driving vehicles. Figure 1 shows two example scenarios where the generation of feasible and efficient trajectories (for example, minimum time, safest, etc.) in a short time frame is a fundamental building block of successful automotive applications.

A large number of research papers have addressed the problem of path generation in presence of obstacles (see [1] for a comprehensive survey). The problem is multifaceted. Some authors have focused on its geometric implications. For instance, Fraichard et a al. [2] revised the problem of generating an optimal geometrical path using curves made of circular arcs and tangent straight lines or continuous curvature (such as polynomials or B-splines). Generally speaking, the geometrical approaches do not consider the vehicle dynamics and produce discontinuous curvature when path is made of arc of circle and straight lines. Other authors have explicitly accounted for kinematic models and curvature constraints. The research in this area has been pioneered by Dubins [3], who showed that for a vehicle moving at constant speed the optimal maneouvre to steer a vehicle with bounded curvature between two position consists of a concatenation

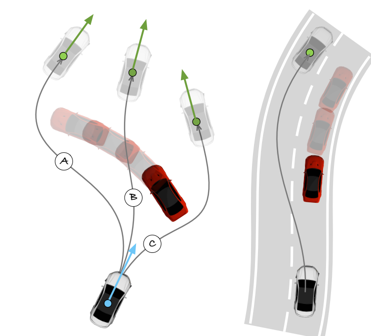

Fig. 1. Example of manoeuvres for autonomous vehicles: generation of manoeuvres for collision avoidance (left), generation of manoeuvres for overtake (right)

of straight lines and circles. In a more recent work, Sanfelice et al. [4] consider the approach of Pontryagin Maximum Principle to extend the Dubins results to changing speed and space dependent maximum curvature.

When high-speed vehicles are considered, trajectory planners necessarily need to account for the vehicle dynamics and its related constraints [5], [6]. This problem can be solved by the application of complex optimisation tools, which produce trajectories that can realistically be tracked by a vehicle. However, the incurred computational cost hardly permits their applications to reactive real–time planning to unexpected situation such as in Figure 1 where guarantee to convergence in short time frame is mandatory. An alternative approach for on-line trajectory optimisation of ground vehicles is based on the fast generation and selection of feasible kinematic trajectories with direct search and subsequent improvement of the obtained solution incorporating dynamic properties [7], [8], [9].

A different idea is to break down the problem into two different sub-problems that can be solved separately [10]. The first one is the geometric problem of finding the best curve that joins two extreme points with assigned tangent, which has been proved to have an efficient solution using clothoid splines [11]. The second one is to find the minimum time control strategy that accounts for the vehicle dynamics, speed and control constraints. As long as the solution to this problem can be found efficiently, the result can be directly employed in direct search algorithms [7], [8], [9] or in global planning algorithms [12]. In [10], the semi-analytical solution have been used to compute the speed profile for a

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racing vehicle within a receding horizon approach. In [13] semi-analytical solutions represent the motor-primitives of more complex manoeuvres embodied into a hierarchical structure used to infere the driver intention for warning and intervention strategies. However, there are a number of limitations that make the semi-analytical solutions presented so far inadequate to apply some practical cases. Some of the most important are: 1. the use of linearised dynamic models, 2. the vehicle is constrained to follow straight or circular paths, 3. the aerodynamic drag effects are neglected or their inclusions requires the numerical solution of differential equations.

In this work we show a semi-analytical solution to the minimum time control for a vehicle between two given configurations. The vehicle is assumed to follow a clothoid. The clothoid is frequently used in road design and, more generally, in the automotive domain. However, our approach can be generalised to any smooth trajectory. Our technique explicitly considers non–linear dynamics and aerodynamic drag effect, control bounds and acceleration and speed constraints. Therefore the semi-analytical solution presented here does not need a numerical forward and backward integration as in [10] but it uses pure analytical speed profiles to numerically find the switching points. A set of numeric experiments presented at the end of the paper show the accuracy and the efficiency of our solution.

The paper organises as follows: Section II describes the vehicle models adopted formulated in curvilinear coordinates with simplification assumptions. Section III introduces the optimal control formulation. Section IV discusses the lateral acceleration constraints and its implication in the analytical solution of the speed profile which is derived in Section V. Section VI describes the solution method and numerical results are presented in Section VII. Conclusions draw reader attention to advantages of the proposed method and further developments and extensions to more general cases.

II. Vehicle Model and Curvilinear Coordinates

In this article it is assumed that the vehicle follows a reference trajectory $(x_R(s), y_R(s))$, with a lateral displacement n. This trajectory is possibly the outcome of a geometric trajectory planning step [10], [11]. The model is parametrised with the curvilinear abscissa $s(t)$, a customary choice for path planning problems. The reference trajectory is given by a clothoid arc that connects two points with assigned tangent vectors (see Figure 2). In curvilinear coordinate representation the absolute velocity of vehicle's centre of mass $V(t)$ is tangent to the trajectory and in general it is not aligned with the vehicle x_v-axis forming an angle $\beta(t)$, named chassis slip angle. Additionally, the x_v-axis of the vehicle has an orientation $\alpha(t)$ with respect to the reference line (i.e. clothoid) local tangent. The clothoid is a natural solution for many geometric optimisation problems involving vehicles since it has continuous curvature change rate and it may conveniently represent complex trajectories on a mid length planning horizon. However, the discussion below is easily generalised to different types of curves.

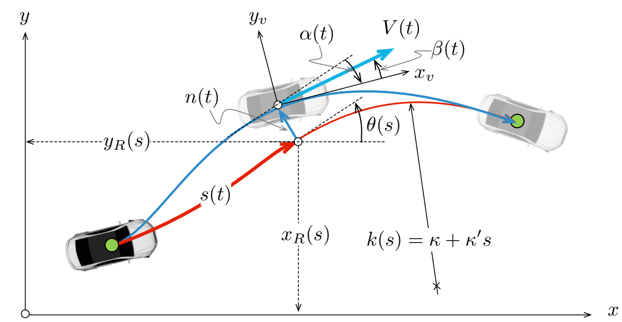

Fig. 2. Curvilinear coordinates (s, n, α) defined with respect to a clothoid reference trajectory. x and y represent the absolute frame axis, $V(t)$ is the absolute velocity of vehicle centre of mass, which is tangent to the trajectory, and $\beta(t)$ is the chassis slip angle. Finally, $\alpha(t)$ is the orientation of the vehicle w.r.t. reference line (i.e. clothoid) local tangent.

Let the initial and final point on the clothoid be $(x_R(0), y_R(0))$ and $(x_R(L), y_R(L))$ and the corresponding angles be ϑ_0 and ϑ_L. In order to uniquely specify the clothoid between the two given points, the length L and the two curvature parameters $\kappa, \kappa' \in \mathrm{R}$ are needed. Finding these parameters is equivalent to solving the G^1 Hermite interpolation problem. Then the clothoid can be easily evaluated using the Fresnel generalised integrals [11] $X_k(a, b, c)$, $Y_k(a, b, c)$ for $s \in [0, L]$ as:

$$\begin{aligned} x_R(s) &= x_R(0) + sX_0(\kappa' s^2, \kappa s, \vartheta_0), \\ y_R(s) &= y_R(0) + sY_0(\kappa' s^2, \kappa s, \vartheta_0). \end{aligned}$$

where

$$\begin{aligned} X_n(a, b, c) &= \int_0^1 \tau^n \cos\left(\frac{a}{2}\tau^2 + b\tau + c\right) \mathrm{d}\tau, \\ Y_n(a, b, c) &= \int_0^1 \tau^n \sin\left(\frac{a}{2}\tau^2 + b\tau + c\right) \mathrm{d}\tau. \end{aligned}$$

The derivative of the trajectory is usually expressed with trigonometric functions of the angle locally tangent to the reference trajectory $\theta(s)$,

$$\frac{\mathrm{d}}{\mathrm{d}s} x_R(s) = \cos\theta(s), \qquad \frac{\mathrm{d}}{\mathrm{d}s} y_R(s) = \sin\theta(s). \tag{1}$$

It is worthwhile to note that for a clothoid the angle θ takes the form $\theta(s) = \frac{1}{2}\kappa' s^2 + \kappa s + \vartheta_0$, while the curvature, which is linear with the arc length, is $k(s) = \kappa + \kappa' s$. The relation between the angle θ and the curvature k is differential, i.e., $\theta'(s) = k(s) = \kappa + \kappa' s$, where $\theta'(s)$ is the notation adopted for the derivative $\mathrm{d}\theta(s)/\mathrm{d}s$ in this paper.

Due to the availability of a reference trajectory, the vehicle kinematics can be given in terms of time and (x, y) coordinates defined as follows:

$$\begin{aligned} x(t) &= x_R(s(t)) - n(t)\sin\theta(s(t)), \\ y(t) &= y_R(s(t)) + n(t)\cos\theta(s(t)), \\ \psi(t) &= \alpha(t) + \theta(s(t)) \end{aligned} \tag{2}$$

where n is the normal displacement of the vehicle with respect to the reference trajectory at the abscissa $s(t)$ and $\psi(t)$ is the vehicle absolute orientation with respect to the $x-$axis, named yaw angle. Differentiating (2) with respect to the time variable t and substituting (1), yields:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}x_R}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} - n'(t)\sin\theta(s) - n(t)k(s)s'(t)\cos\theta(s),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y_R}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} + n'(t)\cos\theta(s) - n(t)k(s)s'(t)\sin\theta(s),\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\mathrm{d}\alpha}{\mathrm{d}t} + k(s)s'(t).\end{aligned}$$

Since $v(t)$ is the component of the velocity directed along x_v (see Figure 2), the previous time derivatives can be expressed as a function of the angle α, that is the angle between the tangent to the reference trajectory and the axis x_v, and the angle θ. We assume here $\beta \equiv 0$, as justified later on, that states the vehicle direction x_v-axis being tangent to the trajectory, i.e., $x' = v\cos(\alpha+\theta)$ and $y' = v\sin(\alpha+\theta)$. By combining the two equations, the dependence to the angle θ can be removed. Hence, by further applying trigonometric identities, one gets

$$s'(t) = \frac{v(t)\cos\alpha(t)}{1-n(t)k(s)}, \quad n'(t) = v(t)\sin\alpha(t).$$

In order to complete the model of the vehicle, the longitudinal velocity $v(t)$ is modelled taking into account the longitudinal acceleration $a(t)$, which is a control input, and both the laminar friction $c_0 > 0$ and the aerodynamic drag $c_1 > 0$. With the assumption that the side slip angle β is zero (or negligible), the vehicle is aligned with the velocity vector $V(t)$ and thus $v(t) = |V(t)|$ and the steering angle of the vehicle $\delta(t)$ can be expressed equal to $\psi(t) = \alpha(t)+\theta(s(t))$. Therefore the vehicle yaw rate $\psi(t)'$ is:

$$\alpha'(t) + k(s)s'(t) = \frac{v(t)}{W_b}\tan\delta(t), \qquad \delta'(t) = u,$$

where u is the bounded control on the angle δ which models the steering angle, and $W_b > 0$ is the wheelbase. Thus the dynamic system modelling the car, dropping the time dependence for simplicity, is

$$\begin{aligned} v' &= a - c_0 v - c_1 v^2\\ \delta' &= u\\ n' &= v\sin\alpha\\ s' &= \frac{v\cos\alpha}{1-n(\kappa+\kappa' s)}\\ \alpha' &= \frac{v}{W_b}\tan\delta - \frac{v(\kappa+\kappa' s)\cos\alpha}{1-n(\kappa+\kappa' s)}. \end{aligned} \tag{3}$$

This model is in general impossible to solve analytically. Indeed, while for specific classes of functions $a(t)$ a closed form solution can be found for the first equation describing the evolution of v (which has the form of a Riccati Differential Equation), this is obviously impossible for the remaining four non-linear and coupled ODEs.

However, we assume that: 1. the vehicle is initially on the clothoid and with orientation aligned with the tangent of the clothoid ($n(0) = 0, \alpha(0) = 0$), 2. the ideal controller $\delta = \tan^{-1}(W_b k(s))$ is applied for δ that allows a perfect tracking of the path. Such assumptions guarantee that $n(t) = 0, \alpha(t) = 0$ for all t. Thus the system of equations (3) simplifies to:

$$v' = a - c_0 v - c_1 v^2, \qquad s' = v, \tag{4}$$

where the only control variable left is the acceleration a (*i.e.* normalised control longitudinal force). Finally, it is worth it to mention that the point mass vehicle model is subject to a constraint on the acceleration (*i.e.* $-a_{\mathrm{brake}} \le a(t) \le a_{\mathrm{push}}$), which we will refer to as *longitudinal constraint*. Moreover, we have an additional constraint on the lateral acceleration (*i.e.* $|a_y| \le A_{\max}$), which we will refer to as *lateral constraint*. Considering the model (3), the lateral acceleration:

$$a_y = \frac{\mathrm{d}}{\mathrm{d}t}\left(V(t)\sin(\beta(t)) + \psi(t)'v(t)\right.$$

which simplifies to $a_y = \psi(t)'v(t) = k(s)v(t)^2$ under the assumption of perfect trajectory tracking above and negligible chassis sideslip angle β. Indeed, the body slip angle β is, in general, close to zero and quantifies the steering behaviour of the vehicle: that is the ability of the vehicle to follow the intended trajectory of the driver or the one planned by the autonomous system. Here, we assume that a low level controller is available that keeps β as close as possible to zero. Many different algorithms have been proposed to control body slip angle β exploiting the vehicle architecture, for instance differential braking, torque vectoring, active front steering (see the review [14]). Under the assumption of $\beta \to 0$ the lateral acceleration constraint becomes: $|k(s)|v(t)^2 \le A_{\max}$. Following the same logic, it is possible to include more complex acceleration constraints, such as friction ellipse and GG diagrams. More details on the last issue are given in the conclusion section.

III. Optimal control problem (OCP)

The optimal control problem can be formulated as follows

find control $a(t)$ that minimize T subject to:

$$\begin{aligned} &\text{ODE}: && \begin{cases} v'(t) = a(t) - c_0 v(t) - c_1 v^2(t),\\ s'(t) = v(t) \end{cases}\\ &\text{boundary conditions} && \begin{cases} v(0) = v_0, \quad s(0) = 0, \quad \text{(initial)}\\ v(T) = v_T, \quad s(T) = L \quad \text{(final)} \end{cases}\\ &\text{constraints} && \begin{cases} -a_{\mathrm{brake}} \le a(t) \le a_{\mathrm{push}},\\ |k(s)|v(t)^2 \le A_{\max}, \end{cases} \end{aligned} \tag{5}$$

As a preliminary remark, since the bound of a lateral acceleration is a function of the state only (and not of the control variable), we can solve the problem in two phases. In the first phase, we solve the OCP accounting only for the longitudinal constraint. For the sake of simplicity, this solution will be referred to as *Bang Bang solution*. In the second one, we take into consideration also the lateral constraint.

A. Solution without lateral acceleration constraint

The Hamiltonian of the minimum time optimal control problem (OCP) (5) (neglecting the lateral constraint) is

$$\mathcal{H} = 1 + \lambda_1(a - c_0 v - c_1 v^2) + \lambda_2 v$$

and the control a appears linearly, its optimal synthesis is obtained from Pontryagin's Maximum (minimum) Principle (PMP), and is

$$a(t) = \arg\min \mathcal{H} = \begin{cases} a_{\text{push}} & \text{if } \lambda_1 < 0, \\ -a_{\text{brake}} & \text{if } \lambda_1 > 0, \\ a_{\text{sing}} & \text{if } \lambda_1 \equiv 0. \end{cases}$$

The control $a(t)$ is bounded in the interval $[-a_{\text{brake}}, a_{\text{push}}]$. Hence the solution of the PMP produces a typical Bang Bang controller. The term a_{sing} represents a possible singular control when λ_1 is identically zero on an interval, however, in our specific case, it can be seen that singular control are not present in the solution.

According to a well known result in Optimal Control Theory [15], the solution of problem (5), if exists, has at most one switching instant (denoted as t_s) when the control value changes. This implies that the optimal control has to be chosen from a family of four candidate controls. The first and second correspond to pure acceleration (i.e., the $a(t) \equiv a_{\text{push}}$) or braking manoeuvre (i.e., $a(t) \equiv -a_{\text{brake}}$), the third and fourth are a combination of the two. Having explicit expressions for the optimal states, with the complete boundary conditions, the solution of each case is obtained by the solution of a nonlinear system in two unknowns (see (15) in Section VI): the switching time t_s and the final time t_{bb}. It is possible to rule out the case of a braking manoeuvre followed by an acceleration because it is not optimal. Moreover, the cases of pure acceleration or braking are very unlikely, because the boundary conditions should exactly satisfy the boundary value problem (5). Hence, only the case of acceleration and braking with a switching time (possibly degenerate at the extrema of the time interval) is herein considered. It follows that the solution is given by the intersection at the switching point t_s of the two curves of the velocity and the space $v(t)$ and $s(t)$. The closed form solution of $v(t)$ and $s(t)$ is a difficult and important part, therefore it is postponed to section V. For the sequel, assume to know such solutions.

B. Solution with lateral acceleration constraint

The composition between the Bang Bang solution described earlier in the section and the lateral constraint $|k(s)|v(t)^2 \leq A_{\max}$ is made identifying intervals in which the solution is found without considering the lateral constraint, intervals in which the constraint is active and "gluing" such intervals in an optimal way.

When the constraint is active the evolution of the system has to respect the equation $|k(s(t))|v(t)^2 = A_{\max}$. We have to translate this relation into an analytic expression for $a(t)$, $v(t)$ and $s(t)$. The control a appears only implicitly in the bound. In order determine its value, it is necessary to differentiate it and substitute the corresponding value of the differential equations (4) for the state variable v. Mathematically this gives

$$a(t) = c_0 v(t) + c_1 v^2(t) - \frac{\kappa' v^2(t)}{2(\kappa + \kappa' s(t))}. \tag{6}$$

To determine the state variables on the bound, consider the bound written as

$$s(t) = \frac{A_{\max}}{\kappa' v^2(t)} - \frac{\kappa}{\kappa'}, \tag{7}$$

which is an explicit expression for the state s once the velocity has been determined. To find a differential equation for the velocity derive the previous expression with respect to time and use the substitution $s' = v$, which gives

$$s'(t) = v(t) = -\frac{2A_{\max} v'(t)}{\kappa' v^3(t)} \;\Rightarrow\; v'(t) = -\frac{1}{2}\frac{\kappa' v^4(t)}{A_{\max}}.$$

This first order nonlinear ordinary differential equation of Bernoulli has solution (for $v(t_b) = v_b$)

$$v(t) = \frac{\sqrt[3]{2A_{\max}}}{\sqrt[3]{3\kappa'(t - t_b) + 2A_{\max}/v_b^3}}. \tag{8}$$

An important point is to determine when the lateral constraint is active. A first condition for it is that the velocity resulting from the Bang Bang solution violates the constraint. However, when the lateral constrain is active, the acceleration evolves as in Equation (6) and is no longer constrained to be equal to a_{push} or $-a_{\text{brake}}$. Therefore, it could violate the longitudinal constraint. When this happens, the constraint is no longer active.

This discussion is applicable to any curve and has not yet exploited the geometric properties of the clothoid. Since the clothoid has a linear relation between curvature and arc length s, three cases can occur: 1. $|k(s)|$ is strictly positive decreasing, 2. $|k(s)|$ is strictly positive increasing or 3. $|k(s)|$ is V shaped and reaches zero for $s^\star := -\kappa/\kappa' \in [0, L]$. The first two cases represent a C shaped clothoid, while the last one is an S shaped curve [11]. Therefore the decision on which case should be considered can be taken *a priori*, once the clothoid is decided. The complete analysis of the constraint on the lateral acceleration is offered in the next section.

IV. Analysis of the lateral acceleration constraint

The analysis of the lateral constraint and the equations describing it are significantly simplified if $\kappa' = 0$. Indeed, a first consequence is that the reference curve is no more a general clothoid, but a straight line (if $\kappa = 0$) or a circle (when $\kappa \neq 0$). When the trajectory is a line, there is no lateral acceleration and the only bound that applies is the one on the maximum acceleration a, that is the Bang Bang solution. When $\kappa \neq 0$, the maximum velocity is thus given by the constant value $\sqrt{k_{\max}/|\kappa|}$. The second consequence is that the differential equation for v on the bound becomes simply $v' = 0$ and the velocity will be constant to the previously obtained maximum velocity $\sqrt{A_{\max}/|\kappa|}$. The corresponding optimal control becomes $a = c_0 v + c_1 v^2$, and the space variable s is a linear function $s(t) = \sqrt{A_{\max}/|\kappa|}t + s_0$ for a certain initial value s_0.

On the contrary, when both κ, κ' are different from zero, the curve is a proper clothoid. In particular, If $|k(s)|$ is monotone increasing or decreasing, then the bound is not active at the beginning/end of the interval, provided that the initial/final condition is admissible.

Curvature modulus strictly decreasing. In this case the bounded velocity is increasing with a lower value of $\underline{v} := \sqrt{A_{\max}/\kappa}$ and an upper value of $\overline{v} := \sqrt{A_{\max}/(\kappa + \kappa' L)}$. If $v_0 > \underline{v}$ or if $v_1 > \overline{v}$ then the problem does not admit a solution. If the bound is never active the solution reduces to the previously seen Bang Bang situation. Otherwise, the bound becomes active at $t_b < t_s$, the time instant when the velocity is equal to

$$v(t_b) = \sqrt{A_{\max}}/\sqrt{|\kappa + \kappa' s(t_b)|}. \tag{9}$$

Before this time instant, the velocity is determined by the Bang Bang control a, after t_b, the bang-bang control is no more optimal because v does not satisfy the lateral bound, hence the control law (6) must be used. After entering the bound, the velocity must stay adherent to the bound until the exit instant t_e, the velocity and space are determined from the bound itself, e.g., from Equations (8) and (7). The exit time t_e can be determined by three events: the end of the bound (intersection with the Bang Bang solution), the saturation of the control (6), $a(t_e) = a_{\text{push}}$ or $a(t_e) = a_{\text{brake}}$ or the need to brake for the matching of the final condition. These three possibilities must be controlled and the correct one is the case that happens earlier.

Curvature modulus strictly increasing. In this case the bound on the velocity is decreasing, with maximum value at the beginning and minimum at the end of the interval, that is $[0, L]$ or $[s^\star, L]$. The bound becomes active for t_b given by equation (9) and also the other equations of the previous case apply here.

V - shaped Curvature Modulus. This case is the most structured but can be recast to the previous two separately. Indeed there can be two lateral constraints, one for the interval $[0, s^\star)$ with decreasing curvature and one for the interval $(s^\star, L]$ where the curvature is increasing. The important fact is that the two constraints cannot overlap, because for an interval around $s^\star$ the lateral constraint is inactive.

V. Analitical solution of speed profile

The ODE for the dynamic model and the corresponding initial conditions are given in (4). It is possible to transform the first equation so that its solution becomes easier to compute. Consider the transformation $v(t) = a\tilde{v}(t + t_0)$, for a shift t_0 where the control is $a = a_{\text{push}}$ for acceleration or $a = -a_{\text{brake}}$ for braking, then the ODE associated to the new function $\tilde{v}$ is

$$\tilde{v}' = \operatorname{sign} a - c_0 \tilde{v}(t + t_0) - c_1 |a| \tilde{v}^2(t + t_0), \tag{10}$$

with $\tilde{v}(0) = 0$. Depending on the coefficients c_0, c_1 and on the value of the control variable a, there are different solutions of the ODE.

A. *Positive* a

Suppose $a > 0$, then reduce (10) to:

$$\tilde{v}'(t) = 1 - c_0 \tilde{v}(t) - \tilde{c}_1 \tilde{v}^2(t), \qquad \tilde{v}(0) = 0,$$

where $\tilde{c}_1 = c_1 a_{\text{push}}$. It is convenient to define a new constant $w_p := \sqrt{c_0^2 + 4\tilde{c}_1}$. When the initial condition v_0 is less than the limit value $v_\infty = 2a_{\text{push}}/(w_p + c_0)$, the solution of the previous ODE is

$$\tilde{v}(t) = \frac{2(1 - e^{-t w_p})}{w_p + c_0 + e^{-t w_p}(w_p - c_0)}, \qquad t \geq 0.$$

The corresponding translation t_0 such that the initial condition $\tilde{v}(t + t_0)\big|_{t=0} = v_0/a_{\text{push}}$ is satisfied, is given explicitly by

$$t_0 = \frac{1}{w_p} \log\left(\frac{2a + v_0(w_p - c_0)}{2a - v_0(w_p + c_0)}\right). \tag{11}$$

It is easy to check that v is a monotone growing function, which takes values in the interval $[0, v_\infty)$ for $t \geq 0$. The integration of v yields the expression for the space variable s, after some manipulation the explicit expression is

$$\tilde{s}(t) = \frac{1}{\tilde{c}_1} \log\left(1 - \tilde{c}_1 \tilde{v}_\infty \frac{1 - \mathrm{e}^{-t w_p}}{w_p}\right) + t\, \tilde{v}_\infty. \tag{12}$$

where $\tilde{v}_\infty = 2/(w_p + c_0)$ is the asymptotic speed of the scaled function. The case of $v_0 > v_\infty$, that is, the initial velocity is higher than the maximum possible velocity has an expression that resembles the previous, but is here omitted because of limited application.

B. *Negative* a

Here there are a couple of subcases to consider depending on the sign of the radicand of $w_n = \sqrt{c_0^2 - 4a_{\text{brake}}\tilde{c}_1}$. If it is positive, then consider $w_n \geq 0$ and the differential equation

$\tilde{v}' = -1 - c_0\tilde{v}(t) - \tilde{c}_1\tilde{v}^2(t)$ with $\tilde{v}(0) = 0$ as in the previous case. Its solution is

$$\tilde{v}(t) = \frac{2\,(1 - \mathrm{e}^{tw_n})}{(w_n + c_0)\mathrm{e}^{tw_n} + (w_n - c_0)}, \qquad t_{\min} < t \le 0.$$

The solution is positive and goes backwards in time until the time instant $t_{\min} = \frac{1}{w_n}\log\left(4c_1/(w_n + c_0)^2\right) < 0$ where it blows up. The integration of $\tilde{v}$ gives the space variable s, which has an expression close to (12), omitted. If $c_0^2 - 4a_{\mathrm{brake}}\tilde{c}_1 < 0$ so that w_n is a complex number, the solution for $\tilde{v}$ is

$$\tilde{v}(t) = -\frac{\sin(t\alpha)}{\beta\sin(t\alpha) + \alpha\cos(t\alpha)}, \qquad \begin{cases} \beta = 2c_0, \\ \beta^2 + \alpha^2 = \tilde{c}_1, \end{cases} \tag{13}$$

where α and β come from solving the system in bracket. The solution is positive and is defined in the interval

$$0 \le \tilde{v} < \infty, \qquad -\frac{2}{c_0} \le -\frac{1}{\alpha}\arctan\left(\frac{2\alpha}{c_0}\right) < t \le 0.$$

The corresponding translation t_0 such that the initial condition $\tilde{v}(t+t_0)\big|_{t=0} = v_0/a_{\mathrm{brake}}$ is satisfied, is given explicitly by

$$t_0 = \frac{1}{\alpha}\arctan\left(\frac{\alpha v_0}{\beta v_0 + a_{\mathrm{brake}}}\right). \tag{14}$$

The corresponding expression for $\tilde{s}(t)$ is the integral of (13), here omitted.

VI. Solution Method

From the discussion above, we have seen that the solution of the OCP is found by first solving the problem with only longitudinal constraints and then by combining this solution with lateral constraint. In this section, we show the concrete steps to take in order to implement the idea.

The solution of the OCP (4) with longitudinal constraints can be found by using the analytic expression of the integrated velocity and space discussed in Section V. This is done by going backward from the final (unknown) time and forward from the initial time in order to meet in the middle. This leads to the detection of the switching point t_s and of the final Bang Bang time t_{bb}, which are the solutions of the system:

$$\begin{aligned} a_{\mathrm{push}}\tilde{v}_{\mathrm{push}}(t_s + t_l) &= a_{\mathrm{brake}}\tilde{v}_{\mathrm{brake}}(t_s + t_r - t_{bb}), \\ a_{\mathrm{push}}[\tilde{s}_{\mathrm{push}}(t_s + t_l) &- \tilde{s}_{\mathrm{push}}(t_l)] \\ +a_{\mathrm{brake}}[\tilde{s}_{\mathrm{brake}}(t_r) &- \tilde{s}_{\mathrm{brake}}(t_s + t_r - t_{bb})] = L. \end{aligned} \tag{15}$$

Clearly, the quantities labelled by "push" refer to the expressions of the velocity and space corresponding to positive a, while the quantities labelled by "brake" refer to the appropriate case (depending on the correct w_n) for negative a. The shifts t_l and t_r (left and right) are given respectively by equations (11) and (14). The solution of this system can be reduced to the solution of a single equation for t_s, because it is possible to analytically solve the final time t_{bb} from the first equation and plug it into the second. This equation has a V-shaped graph and can be solved with high precision using a few iterations of the Newton method.

The second step is the combination of the solution with lateral bound and longitudinal bound. In the simplest case, this combination is simply done by taking the point-wise minimum between the solution found in the first step and the expression of the lateral bound. However, there could be some cases in which by moving along the lateral bound the vehicle longitudinal acceleration could fall out of the admissible range $[-a_{\mathrm{brake}},\, a_{\mathrm{push}}]$ (see the example in the next section). The analytical condition to check to make the lateral constraint in-active is when either (6) saturates to a_{push} or a_{brake} or the trajectory obtained following the bound (given by equations (8) and (7))intersects the Bang Bang solution. In practice, it is required to compute the two instants in which each condition is verified and take the minimum. This computation has to be done numerically, however the expressions are well behaved and only few Newton iterations are required.

With this strategy the sequence of switching times of the optimal control has been determined and the problem is solved. For an example of the solution method a numerical example is provided in the next section.

VII. Numerical Results

For a numerical test, consider a case with the maximum number of switching, i.e. an S shaped manoeuvre like Case A of Figure 1 having five switches. The values of the various constants are taken from characteristic parameters of a high performance car, e.g. a Formula 1 racing car: $c_0 = 0.00002\,\mathrm{m}^{-1}$, $c_1 = 0.0015\,\mathrm{s}^{-1}$, $v(0) = v(T) = 50\,{}^{\mathrm{km}}/_{\mathrm{h}}$ (implicitly $v(T) = v(L)$ because $s(T) = L$) , $a_{\mathrm{push}} = 5\,{}^{\mathrm{m}}/_{\mathrm{s}^2}$, $a_{\mathrm{brake}} = 5\,{}^{\mathrm{m}}/_{\mathrm{s}^2}$, $A_{\max} = 5\,{}^{\mathrm{m}}/_{\mathrm{s}^2}$, the trajectory is given by the clothoid of parameters $\kappa = 0.01\,\mathrm{m}^{-1}$, $\kappa' = -0.00002\,/$ and a length of $L = 1000\,\mathrm{m}$. In Figure 3 is shown the optimal velocity of both the non saturated and the saturated case, together with the plot of the bound. The switching time $ts = 19.157376$ and the final time is $t_{bb} = 25.243209$, for the Bang Bang solution. The time of first activation of the bound is $t_{b_1} = 2.107096$, the bound is no more active from $t_{e_1} = 11.512500$ to the switching of the control from a_{push} to $-a_{\mathrm{brake}}$ at time $t_w = 18.315002$, the bound is again active for $t_{b_2} = 18.922907$ until $t_{e_2} = 30.613425$ (see also Figure 4). The final minimum time for this manoeuvre is $T = 32.278542$. Notice that approximate computational time for the present method is around 15 microseconds, while a pure numeric solution, resulting in similar results, takes around 1.5 seconds with Pins [16] or GPOPS-II [17].

VIII. Conclusions and Future Developments

We have shown the synthesis of a minimum time trajectory that joins two different state space configurations of a car–like robot. The vehicle is assumed to move along a specified path, which for the purposes of the paper is a *given*. We have

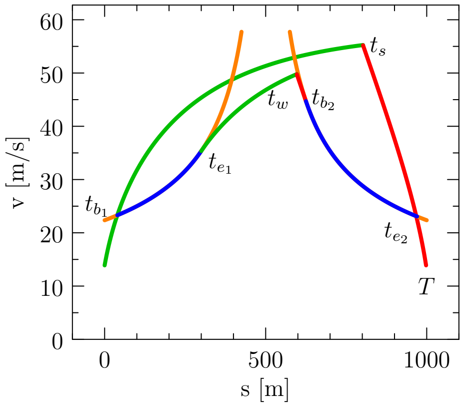

Fig. 3. The optimal velocity w.r.t. space s. In orange the lateral bound, the upper curve is the Bang Bang solution, the lower curve is the optimal solution of the problem. In green the arcs where the control has value a_{push}, in red the control is a_{brake} and in blue the velocity is equal to the bound.

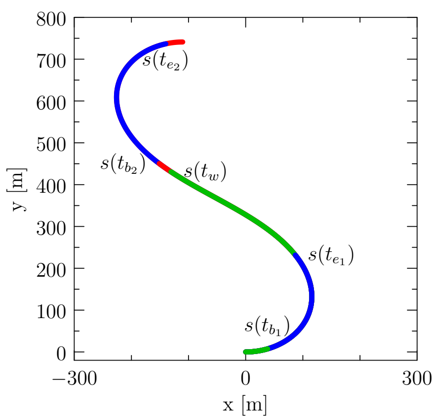

Fig. 4. The trajectory is the given clothoid, the colors represent the maximum acceleration a_{push} in green, the maximum deceleration a_{brake} in red, and the bounded velocity in blue.

restricted to clothoids, but most of the ideas developed in the paper are applicable to any smooth path.

The optimal control formulation assumes a non linear model for the longitudinal dynamics of the problem, which considers the aerodynamic drag and the presence of constraints on the longitudinal and on the lateral acceleration. The key point of the paper is that the proposed solution is semi-analytical, and the numeric computations are limited to a few Newton iterations required to find the unique zero of a well-behaved equation. Therefore, our solution is amenable to real–time implementation on low cost hardware. The proposed method renders itself naturally applicable into a Receding Horizon scheme, currently under investigation, where the optimal speed profile over a finite horizon can be fast regererated to adapt to vehicle states and environment changes such as friction, presence obstacles, etc.

There are at least three open points that will require future investigations. The first open problem is the extension of the analysis to paths different from a clothoid. The only real difference that we expect is in the analytic expression of the contraint and in the way the Bang Bang solution is composed with the lateral constraint (Section IV). The second open problem is how to synthesise an optimal path in the presence of obstacles by solving a geometric optimisation problem (which is a pre-requisite fot this paper). Finally, the third problem is how to account for complex constraints for lateral acceleration, such as friction ellipse and GG diagrams. We expect that the inclusion of GG diagrams may be cast into a speed constraint that could be treated similarly to what has been done here for lateral constraint. Proof of this will be objective of future work.

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